On The Coloring of Graphs and Chromatic Polynomials

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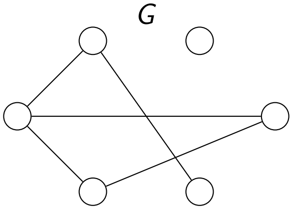
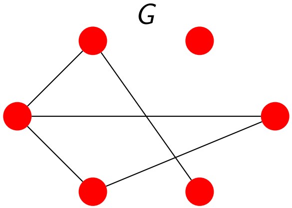
Introduction



- G is an example of a Graph

Introduction



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- Graphs are made up of:
    - Vertices

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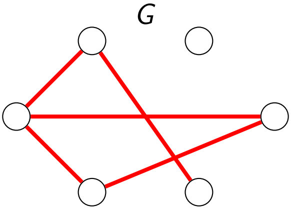

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- Graphs are made up of:
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  - Edges

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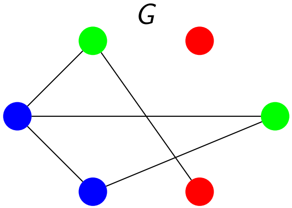

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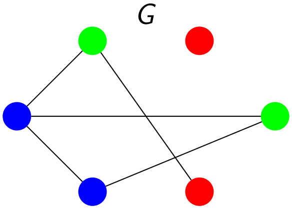

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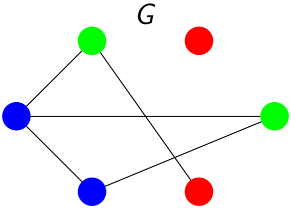

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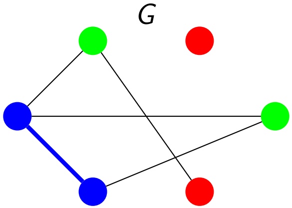

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Natural Questions

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- How many ways can a graph be properly colored with x colors?

Chromatic Polynomials and Numbers

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Chromatic Polynomials and Numbers

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- How many ways are there to properly color a graph G with x colors?
- We can create a function of a graph G and a number of colors x which is the number of ways to color G properly with x colors
- This function is actually a polynomial, called the Chromatic Polynomial, and is denoted $F(G, x)$

Reduction Algorithm

- Finding the Chromatic Polynomial for a graph is not always easy!

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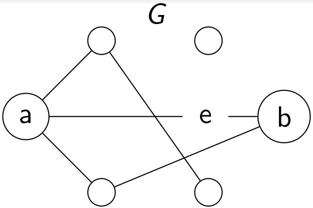
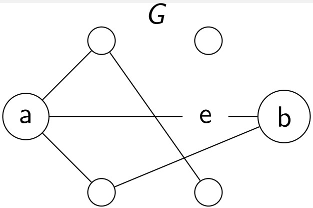
- But thanks to Birkhoff and Lewis, we have a simple algorithm for computing $F(G, x)$ given a graph G.

Reduction Algorithm



- Consider two vertices a and b that are connected by edge e.

Reduction Algorithm



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- Since we want proper colorings, *a* and *b* must be colored differently

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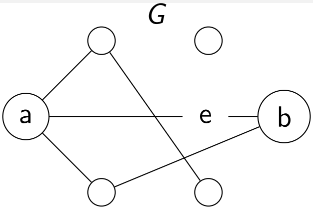

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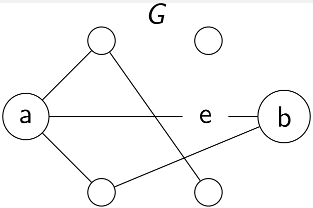

- Consider two vertices *a* and *b* that are connected by edge *e*.
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    - Find the number of colorings where *a* and *b* are colored the same or different

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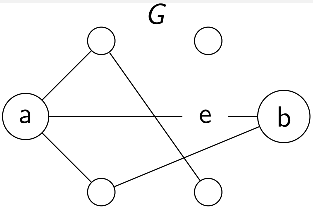
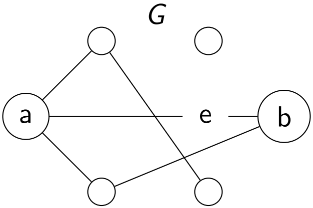


- Consider two vertices a and b that are connected by edge e.
- Since we want proper colorings, a and b must be colored differently
- To determine the number of ways in which vertices a, b can be colored differently:
  - Find the number of colorings where a and b are colored the same or different
  - Then subtract the number of ways where a and b are colored identically.

Reduction Algorithm

First find the number of colorings where a and b are colored the same or different.



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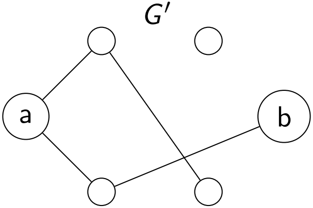

- Remove edge e

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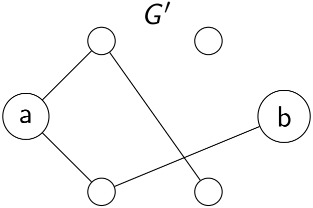

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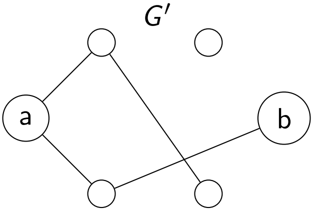
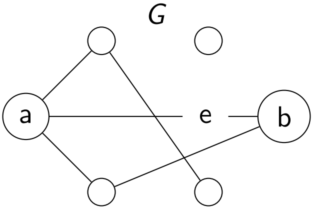


- Remove edge e
- Call this new graph G'
- $F(G', x)$ is the number of (otherwise proper) colorings of G where a and b can have the same or different colors.

Reduction Algorithm

Then subtract the number of ways where a and b are colored identically.



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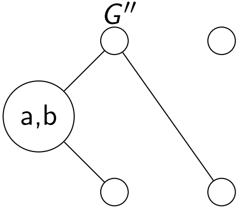

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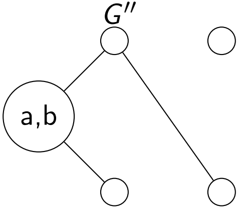

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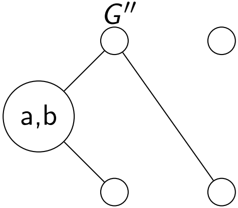

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According to the algorithm, the chromatic polynomial of G is equal to the difference of the chromatic polynomials of G' and G'',

$$F(G, x) = F(G', x) - F(G'', x).$$

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$$F(G,x) = F(G',x) - F(G'',x).$$

- This describes one iteration of the algorithm
- Continue in this manner by selecting edges until all of the obtained graphs are without edges
- For each of these *empty graphs* with n vertices and x colors there are x^n different colorings, since each of the n vertices has x possible colorings

Results

- Finally, we introduce some well known types of graphs

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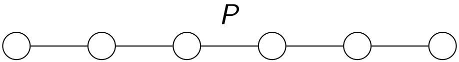
- Applying the Birkhoff-Lewis Reduction Algorithm and induction to these graphs yields general formulas for Chromatic Polynomials

Results

Path

A path graph of n vertices, denoted P_n, is a graph that can be arranged in such a way that vertices are only connected to their adjacent vertices



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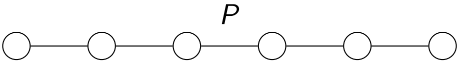
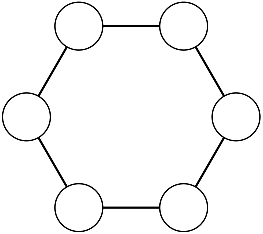


- The chromatic polynomial for a path graph has a general form, $F(P_n, x) = x(x-1)^{n-1}$

Results

Cycle

A cycle graph of n vertices, denoted C_n, is composed of an equal number of vertices and edges such that each vertex has exactly two edges connected with it, creating a closed graph.



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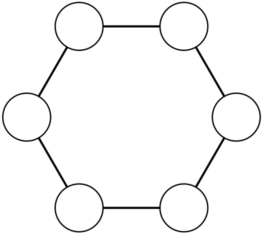
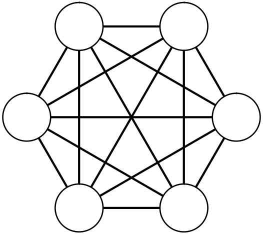


- The chromatic polynomial for a cycle graph has a general form, $F(C, n) = (x-1)^n + (-1)^{n+1}(1-x)$

Results

Complete

A graph with n vertices where any two vertices are connected by an edge is called a complete graph, denoted K_n.



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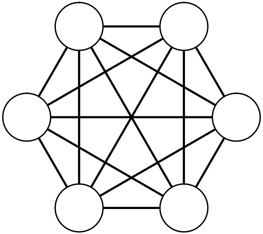

- The chromatic polynomial for a complete graph has a general form, $F(K_n, x) = x(x-1)(x-2)\cdots(x-n+1) = \frac{x!}{(x-n)!}$.

Acknowledgments

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Bibliography

Mount Holyoke College *Laboratories in Mathematical Experimentation: A Bridge to Higher Mathematics* 1997: Springer-Verlag New York, Inc.
www.sagemathcloud.com