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Some explicit and recursive formulas of the large and little Schröder numbers

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Abstract. In the paper, the authors analytically find some explicit formulas and recursive formulas for the large and little Schröder numbers.

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1. Introduction

In combinatorics and number theory, there are two kinds of Schröder numbers, the large Schröder numbers S_n and the little Schröder numbers s_n. They are named after the German mathematician Ernst Schröder.

A large Schröder number S_n describes the number of paths from the southwest corner $(0,0)$ of an $n\times n$ grid to the northeast corner (n,n), using only single steps north, northeast, or east, that do not rise above the southwest–northeast diagonal. The first eleven large Schröder

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numbers S_n for $0 \le n \le 10$ are

$$1, \quad 2, \quad 6, \quad 22, \quad 90, \quad 394, \quad 1806, \quad 8558, \quad 41586, \quad 206098, \quad 1037718.$$

In [1, Theorem 8.5.7], it was proved that the large Schröder numbers S_n have the generating function

$$G(x) = \frac{1 - x - \sqrt{x^2 - 6x + 1}}{2x} = \sum_{n=0}^{\infty} S_n x^n, \tag{1}$$

which can also be rearranged as

$$\mathcal{G}(x) = G(-x) = \frac{\sqrt{x^2 + 6x + 1} - 1 - x}{2x} = \sum_{n=0}^{\infty} (-1)^n S_n x^n. \tag{2}$$

The little Schröder numbers s_n form an integer sequence that can be used to count the number of plane trees with a given set of leaves, the number of ways of inserting parentheses into a sequence, and the number of ways of dissecting a convex polygon into smaller polygons by inserting diagonals. The first eleven little Schröder numbers s_n for $1 \le n \le 11$ are

$$1, \quad 1, \quad 3, \quad 11, \quad 45, \quad 197, \quad 903, \quad 4279, \quad 20793, \quad 103049, \quad 518859.$$

They are also called the small Schröder numbers, the Schröder–Hipparchus numbers, or the Schröder numbers, after Ernst Schröder and the ancient Greek mathematician Hipparchus who appears from evidence in Plutarch to have known of these numbers. They are also called the super-Catalan numbers, after Eugéne Charles Catalan, but different from a generalization of the Catalan numbers [2,10]. In [1, Theorem 8.5.6], it was proved that the little Schröder numbers s_n have the generating function

$$g(x) = \frac{1 + x - \sqrt{x^2 - 6x + 1}}{4} = \sum_{n=1}^{\infty} s_n x^n. \tag{3}$$

For more information on the large Schröder numbers S_n and the little Schröder numbers s_n, please refer to [1,7–9] and plenty of references therein.

Comparing (1) with (3), we can reveal

$$\sqrt{x^2 - 6x + 1} = 1 + x - 4\sum_{n=1}^{\infty} s_n x^n = 1 - x - 2\sum_{n=0}^{\infty} S_n x^{n+1},$$

that is,

$$1 - 2\sum_{n=1}^{\infty} s_n x^{n-1} = 1 - 2\sum_{n=0}^{\infty} s_{n+1} x^n = -\sum_{n=0}^{\infty} S_n x^n.$$

Accordingly, we acquire

$$S_n = 2s_{n+1}, \quad n \in \mathbb{N}. \tag{4}$$

See also [1, Corollary 8.5.8]. This relation tells us that it is sufficient to analytically study the large Schröder numbers S_n.

Recently, in the paper [3] and the preprints [4–6], some new conclusions, including several explicit formulas, integral representations, and some properties such as the convexity, complete monotonicity, product inequalities, and determinantal inequalities, for the large and little Schröder numbers S_n and s_{n+1} were discovered. Some of them can be reformulated as follows.

Theorem 1 *([3, Theorem 1] and [6, Theorem 1]).* *For $n \in \mathbb{N}$, the large and little Schröder numbers S_n and s_{n+1} can be computed by*

$$S_n = 2s_{n+1} = \frac{(-1)^{n+1}}{12} \frac{1}{6^n} \sum_{k=1}^{n+1} (-1)^k \frac{6^{2k}}{k!} \frac{(2k-3)!!}{2^k} \binom{k}{n-k+1},$$

where $\binom{p}{q} = 0$ for $q > p \geq 0$.

Theorem 2 *([4, Theorem 1.1]).* *For $n \geq 0$, the large and little Schröder numbers S_n and s_{n+1} can be represented by*

$$S_n = 2s_{n+1} = \frac{1}{2\pi} \int_{3-2\sqrt{2}}^{3+2\sqrt{2}} \frac{\sqrt{(u-3+2\sqrt{2})(3+2\sqrt{2}-u)}}{u^{n+2}} \, \mathrm{d}\, u.$$

Theorem 3 *([5]).* *For $n \geq 0$, the sequences S_n, s_{n+1}, $(n+1)!S_n$, and $(n+1)!s_{n+1}$ are convex. For $n \geq 0$, the sequence $n!S_n$ is logarithmically convex.*

The main aims of this paper are to analytically find two new explicit formulas and two recursive formulas for the large and little Schröder numbers S_n and s_n respectively. Our main results can be stated as the following theorems.

Theorem 4. *For $n \in \mathbb{N}$, the large and little Schröder numbers S_n and s_{n+1} can be computed by*

$$S_n = 2s_{n+1} = -\frac{(3 \mp 2\sqrt{2})^{n+1}}{2} \sum_{\ell=0}^{n+1} \frac{(2\ell-3)!!}{(2\ell)!!} \frac{[2(n-\ell+1)-3]!!}{[2(n-\ell+1)]!!} (3 \pm 2\sqrt{2})^{2\ell}. \tag{5}$$

Remark 1. Since

$$3 \pm 2\sqrt{2} = (\sqrt{2}+1)^{\pm 2} = (\sqrt{2}-1)^{\mp 2}$$

and

$$17 \pm 12\sqrt{2} = (\sqrt{2}+1)^{\pm 4} = (\sqrt{2}-1)^{\mp 4},$$

the explicit formulas in (5) can be reformulated as

$$S_n = 2s_{n+1} = -\frac{(\sqrt{2}-1)^{\pm 2(n+1)}}{2} \sum_{\ell=0}^{n+1} \frac{(2\ell-3)!!}{(2\ell)!!} \frac{[2(n-\ell+1)-3]!!}{[2(n-\ell+1)]!!} (\sqrt{2}+1)^{\pm 4\ell},$$

$$S_n = 2s_{n+1} = -\frac{\left(\sqrt{2}+1\right)^{\mp 2(n+1)}}{2}\sum_{\ell=0}^{n+1}\frac{(2\ell-3)!!}{(2\ell)!!}\frac{[2(n-\ell+1)-3]!!}{[2(n-\ell+1)]!!}\left(\sqrt{2}+1\right)^{\pm 4\ell},$$

$$S_n = 2s_{n+1} = -\frac{\left(\sqrt{2}-1\right)^{\mp 2(n+1)}}{2}\sum_{\ell=0}^{n+1}\frac{(2\ell-3)!!}{(2\ell)!!}\frac{[2(n-\ell+1)-3]!!}{[2(n-\ell+1)]!!}\left(\sqrt{2}-1\right)^{\pm 4\ell},$$

and

$$S_n = 2s_{n+1} = -\frac{\left(\sqrt{2}+1\right)^{\pm 2(n+1)}}{2}\sum_{\ell=0}^{n+1}\frac{(2\ell-3)!!}{(2\ell)!!}\frac{[2(n-\ell+1)-3]!!}{[2(n-\ell+1)]!!}\left(\sqrt{2}-1\right)^{\pm 4\ell}.$$

Theorem 5. *For $n \ge 0$, the large and little Schröder numbers S_n and s_{n+1} satisfy the recursive formulas*

$$S_{n+3} = 3S_{n+2} + \sum_{\ell=0}^{n} S_{\ell+1}S_{n-\ell+1} \tag{6}$$

and

$$s_{n+4} = 3s_{n+3} + 2\sum_{\ell=0}^{n} s_{\ell+2}s_{n-\ell+2}. \tag{7}$$

2. Proofs of Theorems 4 and 5

Now we are in a position to prove our main results.

Proof of Theorem 4. From (2), it follows that

$$\sqrt{x^2+6x+1} = 1 + x + 2\sum_{n=0}^{\infty}(-1)^n S_n x^{n+1} = 1 + 2x + 2\sum_{n=2}^{\infty}(-1)^{n-1}S_{n-1}x^n$$

which implies that, for $n \ge 2$,

$$\begin{aligned}
2(-1)^{n-1}n!S_{n-1} &= \lim_{x\to 0}\left[\sqrt{x^2+6x+1}\right]^{(n)}\\
&= \lim_{x\to 0}\left[\sqrt{\left(x+3+2\sqrt{2}\right)\left(x+3-2\sqrt{2}\right)}\right]^{(n)}\\
&= \lim_{x\to 0}\left(\sqrt{x+3+2\sqrt{2}}\,\sqrt{x+3-2\sqrt{2}}\right)^{(n)}\\
&= \lim_{x\to 0}\sum_{\ell=0}^{n}\binom{n}{\ell}\left(\sqrt{x+3+2\sqrt{2}}\right)^{(\ell)}\left(\sqrt{x+3-2\sqrt{2}}\right)^{(n-\ell)}
\end{aligned}$$

$$
\begin{aligned}
&= \lim_{x\to 0}\sum_{\ell=0}^{n}\binom{n}{\ell}\left\langle\frac{1}{2}\right\rangle_{\ell}\left(x+3+2\sqrt{2}\right)^{1/2-\ell}\left\langle\frac{1}{2}\right\rangle_{n-\ell}\left(x+3-2\sqrt{2}\right)^{1/2-n+\ell}\\
&= \lim_{x\to 0}\frac{\sqrt{x^2+6x+1}}{\left(x+3-2\sqrt{2}\right)^{n}}\sum_{\ell=0}^{n}\binom{n}{\ell}\left\langle\frac{1}{2}\right\rangle_{\ell}\left\langle\frac{1}{2}\right\rangle_{n-\ell}\left(\frac{x+3-2\sqrt{2}}{x+3+2\sqrt{2}}\right)^{\ell}\\
&= \frac{(-1)^n}{\left(3-2\sqrt{2}\right)^{n}}\sum_{\ell=0}^{n}\binom{n}{\ell}\frac{(2\ell-3)!!}{2^{\ell}}\frac{[2(n-\ell)-3]!!}{2^{n-\ell}}\left(\frac{3-2\sqrt{2}}{3+2\sqrt{2}}\right)^{\ell}\\
&= \frac{(-1)^n}{\left(3-2\sqrt{2}\right)^{n}}\sum_{\ell=0}^{n}\frac{n!}{\ell!(n-\ell)!}\frac{(2\ell-3)!!}{2^{\ell}}\frac{[2(n-\ell)-3]!!}{2^{n-\ell}}\left(\frac{3-2\sqrt{2}}{3+2\sqrt{2}}\right)^{\ell}\\
&= \frac{(-1)^n n!}{\left(3-2\sqrt{2}\right)^{n}}\sum_{\ell=0}^{n}\frac{(2\ell-3)!!}{(2\ell)!!}\frac{[2(n-\ell)-3]!!}{[2(n-\ell)]!!}\left(17-12\sqrt{2}\right)^{\ell}.
\end{aligned}
$$

Therefore, we obtain

$$
S_{n-1} = -\frac{1}{2\left(3-2\sqrt{2}\right)^{n}}\sum_{\ell=0}^{n}\frac{(2\ell-3)!!}{(2\ell)!!}\frac{[2(n-\ell)-3]!!}{[2(n-\ell)]!!}\left(17-12\sqrt{2}\right)^{\ell}, \quad n\ge 2.
$$

Similarly, we have

$$
\begin{aligned}
2(-1)^{n-1}n!S_{n-1} &= \lim_{x\to 0}\left[\sqrt{x^2+6x+1}\right]^{(n)}\\
&= \lim_{x\to 0}\left[\sqrt{\left(x+3-2\sqrt{2}\right)\left(x+3+2\sqrt{2}\right)}\right]^{(n)}\\
&= \lim_{x\to 0}\left(\sqrt{x+3-2\sqrt{2}}\,\sqrt{x+3+2\sqrt{2}}\right)^{(n)}\\
&= \lim_{x\to 0}\sum_{\ell=0}^{n}\binom{n}{\ell}\left(\sqrt{x+3-2\sqrt{2}}\right)^{(\ell)}\left(\sqrt{x+3+2\sqrt{2}}\right)^{(n-\ell)}
\end{aligned}
$$

$$
\begin{aligned}
&= \lim_{x\to 0}\sum_{\ell=0}^{n}\binom{n}{\ell}\left\langle\frac{1}{2}\right\rangle_{\ell}\left(x+3-2\sqrt{2}\right)^{1/2-\ell}\left\langle\frac{1}{2}\right\rangle_{n-\ell}\left(x+3+2\sqrt{2}\right)^{1/2-n+\ell}\\
&= \lim_{x\to 0}\frac{\sqrt{x^2+6x+1}}{\left(x+3+2\sqrt{2}\right)^{n}}\sum_{\ell=0}^{n}\binom{n}{\ell}\left\langle\frac{1}{2}\right\rangle_{\ell}\left\langle\frac{1}{2}\right\rangle_{n-\ell}\left(\frac{x+3+2\sqrt{2}}{x+3-2\sqrt{2}}\right)^{\ell}\\
&= \frac{(-1)^n}{\left(3+2\sqrt{2}\right)^{n}}\sum_{\ell=0}^{n}\binom{n}{\ell}\frac{(2\ell-3)!!}{2^{\ell}}\frac{[2(n-\ell)-3]!!}{2^{n-\ell}}\left(\frac{3+2\sqrt{2}}{3-2\sqrt{2}}\right)^{\ell}\\
&= \frac{(-1)^n}{\left(3+2\sqrt{2}\right)^{n}}\sum_{\ell=0}^{n}\frac{n!}{\ell!(n-\ell)!}\frac{(2\ell-3)!!}{2^{\ell}}\frac{[2(n-\ell)-3]!!}{2^{n-\ell}}\left(\frac{3+2\sqrt{2}}{3-2\sqrt{2}}\right)^{\ell}\\
&= \frac{(-1)^n n!}{\left(3+2\sqrt{2}\right)^{n}}\sum_{\ell=0}^{n}\frac{(2\ell-3)!!}{(2\ell)!!}\frac{[2(n-\ell)-3]!!}{[2(n-\ell)]!!}\left(17+12\sqrt{2}\right)^{\ell}.
\end{aligned}
$$

Therefore, we obtain

$$
S_{n-1} = -\frac{1}{2\left(3+2\sqrt{2}\right)^{n}}\sum_{\ell=0}^{n}\frac{(2\ell-3)!!}{(2\ell)!!}\frac{[2(n-\ell)-3]!!}{[2(n-\ell)]!!}\left(17+12\sqrt{2}\right)^{\ell}, \quad n\ge 2.
$$

The proof of Theorem 4 is complete. $\square$

Proof of Theorem 5. From (1), it follows that

$$\sqrt{x^2-6x+1} = 1-x-2\sum_{n=1}^{\infty} S_{n-1}x^n = 1-3x-2\sum_{n=2}^{\infty} S_{n-1}x^n.$$

Squaring on both sides of the above equation yields

$$\begin{aligned}
x^2-6x+1 &= \left(1-3x-2\sum_{n=2}^{\infty} S_{n-1}x^n\right)^2 \\
&= (1-3x)^2-4(1-3x)\sum_{n=2}^{\infty} S_{n-1}x^n+4\left(\sum_{n=2}^{\infty} S_{n-1}x^n\right)^2, \\
&= 1-6x+9x^2-4(1-3x)\sum_{n=2}^{\infty} S_{n-1}x^n+4\left(\sum_{n=2}^{\infty} S_{n-1}x^n\right)^2, \\
0 &= 2x^2-(1-3x)\sum_{n=2}^{\infty} S_{n-1}x^n+\left(\sum_{n=2}^{\infty} S_{n-1}x^n\right)^2, \\
0 &= -\sum_{n=3}^{\infty}(S_{n-1}-3S_{n-2})x^n+x^4\left(\sum_{n=0}^{\infty} S_{n+1}x^n\right)^2, \\
0 &= -\sum_{n=3}^{\infty}(S_{n-1}-3S_{n-2})x^n+x^4\sum_{n=0}^{\infty}\left[\sum_{\ell=0}^{n} S_{\ell+1}S_{n-\ell+1}\right]x^n, \\
0 &= -\sum_{n=3}^{\infty}(S_{n-1}-3S_{n-2})x^n+\sum_{n=4}^{\infty}\left[\sum_{\ell=0}^{n-4} S_{\ell+1}S_{n-\ell-3}\right]x^n, \\
0 &= -(S_2-3S_1)x^3+\sum_{n=4}^{\infty}\left[3S_{n-2}-S_{n-1}+\sum_{\ell=0}^{n-4} S_{\ell+1}S_{n-\ell-3}\right]x^n, \\
3S_{n-2}&-S_{n-1}+\sum_{\ell=0}^{n-4} S_{\ell+1}S_{n-\ell-3}=0.
\end{aligned}$$

Replacing n by $n+4$ in the last equality and simplifying immediately lead to the recursive formula (6).

Making use of the relation (4) in (6) gives the recursive formula (7) readily. The proof of Theorem 5 is thus complete. □

Remark 2. This paper is a companion of the paper [3] and the preprints [4–6].

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