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Skew-standard tableaux with three rows

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ABSTRACT

Let $\mathcal{T}_3$ be the three-rowed strip. Recently Regev conjectured that the number of standard Young tableaux with $n-3$ entries in the "skew three-rowed strip" $\mathcal{T}_3/(2,1,0)$ is $m_{n-1}-m_{n-3}$, a difference of two Motzkin numbers. This conjecture, together with hundreds of similar identities, were derived automatically and proved rigorously by Zeilberger via his powerful program and WZ method. It appears that each one is a linear combination of Motzkin numbers with constant coefficients. In this paper we will introduce a simple bijection between Motzkin paths and standard Young tableaux with at most three rows. With this bijection we answer Zeilberger's question affirmatively that there is a uniform way to construct bijective proofs for all of those identities.

1. Introduction

The enumeration of standard Young tableaux (SYTs) is a fundamental problem in combinatorics and representation theory. For example, it is known that the number of SYTs of a given shape $\lambda \vdash n$ is counted by the hook-length formula [3]. However, the problem of counting SYTs of bounded height is a hard one. Let $\mathcal{T}_k(n) := \{\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k) \vdash n\colon\ \lambda_1 \geqslant \cdots \geqslant \lambda_k \geqslant 0\}$ be the set of SYTs with n entries and at most k rows, and let $\mathcal{T}_k = \bigcup_{n=1}^{\infty} \mathcal{T}_k(n)$ be the k-rowed strip. In 1981, Regev proved that

$$\left|\mathcal{T}_2(n)\right| = \binom{n}{\lfloor \frac{n}{2} \rfloor} \quad \text{and} \quad \left|\mathcal{T}_3(n)\right| = \sum_{i \geqslant 0} \frac{1}{i+1} \binom{n}{2i} \binom{2i}{i}$$

in terms of symmetric functions [7]. Note that $|\mathcal{T}_3(n)|$ is exactly the Motzkin number m_n. In 1989, together with $|\mathcal{T}_2(n)|$ and $|\mathcal{T}_3(n)|$, Gouyou-Beauchamps derived that

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$$\left|\mathcal{T}_4(n)\right| = c_{\lfloor \frac{n+1}{2} \rfloor} c_{\lceil \frac{n+1}{2} \rceil} \quad \text{and} \quad \left|\mathcal{T}_5(n)\right| = 6 \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2i} c_i \frac{(2i+2)!}{(i+2)!(i+3)!}$$

combinatorially, where $c_n = \frac{1}{n+1}\binom{2n}{n}$ is the Catalan number [5]. His idea relied on the fact that the number of SYTs with n entries and at most k rows equals the number of involutions of $[n]$ with the length of a longest decreasing subsequence at most k, hence it suffices to count these restricted involutions. These are in fact all the simple formulae we have for $|\mathcal{T}_k(n)|$ so far [9]. Meanwhile, Zeilberger proved that for each k the generating function of $|\mathcal{T}_k(n)|$ is always P-recursive [11]. Gessel also pointed out this fact and derived explicitly the exponential generating function of $|\mathcal{T}_k(n)|$ in terms of hyperbolic Bessel functions of the first kind [4,10].

Recently Regev considered the following variation among others. Given $\mu = (\mu_1, \mu_2, \mu_3)$ a partition of at most three parts, let $|\mu| := \mu_1 + \mu_2 + \mu_3$ and $\mathcal{T}_3(\mu; n - |\mu|)$ be the set of SYTs with $n - |\mu|$ entries in the "skew strip" $\mathcal{T}_3/\mu$. Regev conjectured that for $\mu = (2, 1, 0)$,

$$\left|\mathcal{T}_3\big((2,1,0); n-3\big)\right| = m_{n-1} - m_{n-3},$$

a difference of two Motzkin numbers [8]. This conjecture is confirmed by Zeilberger by using the WZ method [2]. What's more, with his powerful Maple package AMITAI, Zeilberger could generate and rigorously prove many similar identities, among them are a list of formulae of $|\mathcal{T}_3(\mu; n - |\mu|)|$ for $\mu_1 \leqslant 20$, and the number of SYTs in $\mathcal{T}_3$ with the restriction that the (i, j) entry is m for $1 \leqslant m \leqslant 15$. Amazingly, each formula is a linear combination of negative shifts of the Motzkin numbers with constant coefficients.

In the remark of [2] Zeilberger then asked that, besides Regev's question of finding a combinatorial proof of the $m_{n-1} - m_{n-3}$ conjecture (now a theorem, after Zeilberger), is there a uniform way to construct combinatorial proofs to all of these results, or prove that there is no natural bijection because the identities are true 'just because'.

In this paper we answer Regev and Zeilberger's questions affirmatively. We shall present a simple bijection between $\mathcal{T}_3(n)$ and the set of Motzkin paths of length n, which gives another proof for $|\mathcal{T}_3(n)| = m_n$. With this bijection we can prove Regev's conjecture and consequently all of Zeilberger's identities for three-rowed SYTs combinatorially.

The paper is organized as follows. We introduce the bijection in Section 2. In Section 3 we give combinatorial proofs to Regev's and Zeilberger's results. In the last section we give a conjecture, regarding a relation between $|\mathcal{T}_{2\ell+1}(n)|$ and $|\mathcal{T}_{2\ell}(n)|$.

2. Motzkin paths and the three-rowed SYTs

Let m_n denote the nth Motzkin number. One way to define the Motzkin numbers is by their generating function $M = \sum_{n \geqslant 0} m_n x^n = \frac{1-x-\sqrt{1-2x-3x^2}}{2x^2}$. This function satisfies the equation

$$M = 1 + xM + x^2M^2. \tag{1}$$

One combinatorial interpretation of the Motzkin numbers is the Motzkin paths. A *Motzkin path* of length n is a lattice path from $(0, 0)$ to $(n, 0)$ using *up steps* $(1, 1)$, *down steps* $(1, -1)$, and *level steps* $(1, 0)$ that never go below the x-axis. Let U, D, and L denote an up step, a down step, and a level step, respectively.

Given a standard Young tableau T with n entries, we associate T with a word $\chi(T)$ of length n on the alphabet $\mathbb{Z}^+ = \{1, 2, 3, \ldots\}$, where $\chi(T)$ is obtained from T by letting the jth letter be the row index of the entry of T containing the number j. The words $\chi(T)$ are known as *Yamanouchi words*. For example,

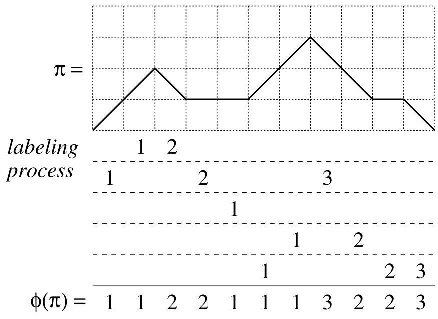


Fig. 1. A Motzkin path and the corresponding word.

$$T = \begin{array}{|c|c|c|}\hline 1 & 3 & 6 \\ \hline 2 & 5 & 7 \\ \hline 4 & 8 \\ \cline{1-2} \end{array} \quad \longleftrightarrow \quad \chi(T) = 12132123.$$

On the other hand, given a Yamanouchi word ω, it is straightforward to recover the corresponding tableau $\chi^{-1}(\omega)$, i.e., the ith row of which contains the indices of the letters of ω that are equal to i.

Now we present a bijection ϕ between Motzkin paths and the tableaux of $\mathcal{T}_3(n)$ in terms of Yamanouchi words.

Let $\mathcal{M}_n$ denote the set of Motzkin paths of length n. Let $\mathcal{W}_3(n) = \{\chi(T)\colon\ T \in \mathcal{T}_3(n)\}$. Note that $\mathcal{W}_3(n)$ is the set of Yamanouchi words of length n on the alphabet $\{1, 2, 3\}$. Given a $\pi \in \mathcal{M}_n$, let $\pi = x_1 x_2 \cdots x_n$ where x_i is the ith step of π. We shall associate π with a word $\phi(\pi)$ of length n by the following procedure.

(A1) If π starts with a level step then we label the first step by 1. Otherwise π starts with an up step. Let j be the least integer such that x_j is not an up step. There are two cases.
- x_j is a down step. Then we label the two steps x_{j-1} and x_j by 1 and 2, respectively.
- x_j is a level step. Then we find the least integer k such that $k > j$ and x_k is a down step, and label the three steps x_{j-1}, x_j, and x_k by 1, 2, and 3, respectively.

(A2) Form a new path π' from π by removing those labeled steps and concatenating the remaining segments of steps. If π' is empty then we are done, otherwise go to (A1) and proceed to process π'.

Reading the labels of $x_1 x_2 \cdots x_n$ in order, we obtain the requested word $\phi(\pi)$. Note that each step with the label 2 is preceded by a matching step with the label 1, and whenever a step is labeled by 3 there is a matching pair of steps with labels 1 and 2. Hence $\phi(\pi) \in \mathcal{W}_3(n)$.

For example, Fig. 1 shows a Motzkin path π and the corresponding Yamanouchi word $\phi(\pi)$, along with the stages of step-labeling process.

With $\phi(\pi) = 11221113223$, we have the associated standard Young tableau:

$$\begin{array}{|c|c|c|c|c|}\hline 1 & 2 & 5 & 6 & 7 \\ \hline 3 & 4 & 9 & 10 \\ \cline{1-4} 8 & 11 \\ \cline{1-2} \end{array}\ .$$

To find ϕ^{-1}, given a word $\omega \in \mathcal{W}_3(n)$, let $\omega = z_1 z_2 \cdots z_n$ where z_i is the ith letter of ω. We shall recover the Motzkin path $\phi^{-1}(\omega) \in \mathcal{M}_n$ by the following procedure.

(B1) If ω consists of letters 1 only, then each letter is associated with a level step. Otherwise, there are two cases.

- ω has no letters 3. Find the first letter 2, say z_j, and associate z_{j-1} and z_j with an up step and a down step, respectively.
- ω has letters 3. Suppose $C_1, \ldots, C_d$ are the marks of letter 3 in ω. For $k = 0, \ldots, d-1$ and from right to left, let B_{d-k} be the first unmarked letter 2 that C_{d-k} encounters. For $i = 1, \ldots, d$, suppose there are t_i unmarked letters 2 on the left of B_i, let $E_{i,j}$ mark the jth 1 from left to right. For $j = 1, \ldots, t_i$, let $D_{i,j}$ be the first unmarked letter 1 from right to left that $E_{i,j}$ encounters. We associate each pair $(D_{i,j}, E_{i,j})$ with an up step and a down step, respectively. Then let A_i be the first unmarked letter 1 on the left of B_i and associate the triple (A_i, B_i, C_i) with an up step, a level step, and a down step.

(B2) If all of the letters have been associated with steps then we are done, otherwise form a new word ω' by removing those letters with steps, and then go to (B1) and proceed to process ω'.

Note that at each stage of the process the path never goes below the x-axis and the starting point and end point of the path are always on the x-axis. Hence $\phi^{-1}(\omega) \in \mathcal{M}_n$. It is easy to see that ϕ and ϕ^{-1} are indeed inverses to each other.

Now that $|\mathcal{T}_3(n)| = |\mathcal{W}_3(n)| = |\mathcal{M}_n|$, we prove the following classic result.

Theorem 2.1. *(See Regev [7].) The number of standard Young tableaux with n entries and at most three rows is the Motzkin number m_n.*

3. Regev's and Zeilberger's results

3.1. A combinatorial proof to Regev's result

Recall that $\mathcal{T}_3(\mu; n - |\mu|)$ is the set of SYTs with $n - |\mu|$ entries in the "skew strip" $\mathcal{T}_3/\mu$, where μ is a partition of at most three parts. The following identity is conjectured by Regev and proved by Zeilberger.

Theorem 3.1. *(See Regev [8], Zeilberger [2].)*

$$\left|\mathcal{T}_3\big((2,1,0); n-3\big)\right| = m_{n-1} - m_{n-3}.$$

We present a combinatorial proof by using the bijection ϕ. First we need some enumerative results. For $0 \leqslant j \leqslant i$, let $X(i, j; n)$ be the set of lattice paths that go from the point (i, j) to the point $(n, 0)$ using steps U, D, L and never go below the x-axis. For those paths whose starting points are on the x-axis, we clearly have

$$\left|X(i, 0; n)\right| = m_{n-i} \quad (0 \leqslant i \leqslant n). \tag{2}$$

For those paths with other starting points, we have the following results.

Proposition 3.2. *For $0 \leqslant j \leqslant i \leqslant n$, the cardinality of $X(i, j; n)$ can be expressed as a linear combination of Motzkin numbers. In particular, we have*

(i) $|X(i, 1; n)| = m_{n-i+1} - m_{n-i}$.
(ii) $|X(i, 2; n)| = m_{n-i+2} - 2m_{n-i+1}$.

Proof. Given a $\pi \in X(i, j; n)$, the path π can be factorized as $\pi = \beta_1 D_1 \beta_2 D_2 \cdots \beta_j D_j \beta_{j+1}$, where D_k is the first down step that goes from the line $y = j - k + 1$ to the line $y = j - k$ $(1 \leqslant k \leqslant j)$, and β_k is a Motzkin path of certain length (possibly empty). Hence the generating function for the number $|X(i, j; n)|$ is $x^j M^{j+1}$, i.e., $|X(i, j; n)| = [x^{n-i}]\{x^j M^{j+1}\}$. With the equation $x^2M^2 = M - 1 - xM$, an equivalent form of (1), the generating function $x^j M^{j+1}$ can be reduced to a linear combination of

$\{x^d M\}_{d\in\mathbb{Z}}$. Hence $|X(i,j;n)|$ can be expressed as linear combinations of Motzkin numbers. In particular,

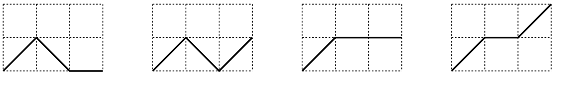
$$\left|X(i,1;n)\right| = \left[x^{n-i}\right]\left\{xM^2\right\} = \left[x^{n-i+1}\right]\left\{x^2M^2\right\} = \left[x^{n-i+1}\right]\{M-1-xM\} = m_{n-i+1} - m_{n-i}.$$

Moreover,

$$\begin{aligned}\left|X(i,2;n)\right| &= \left[x^{n-i}\right]\left\{x^2M^3\right\}\\ &= \left[x^{n-i}\right]\left\{M(M-1-xM)\right\}\\ &= \left[x^{n-i+2}\right]\{M-1-xM\} - \left[x^{n-i}\right]\{M\} - \left[x^{n-i+1}\right]\{M-1-xM\}\\ &= m_{n-i+2} - 2m_{n-i+1},\end{aligned}$$

as required. □

Proof of Theorem 3.1. Given a $T \in \mathcal{T}_3((2,1,0), n-3)$, we form a new SYT $\overline{T}$ by letting $\overline{T}(1,1)=1$, $\overline{T}(2,1)=2$, $\overline{T}(1,2)=3$, and $\overline{T}(i,j)=T(i,j)+3$ for other entries, where $\overline{T}(i,j)$ is the (i,j) entry of $\overline{T}$. It is clear that $\overline{T}$'s and T's are equinumerous. We are about to count the number of $\overline{T}$'s. The associated Yamanouchi word $\omega=\chi(\overline{T})$ starts with $1,2,1$, and according to the bijection ϕ, the initial three steps $x_1x_2x_3$ of the corresponding Motzkin path $\pi=\phi^{-1}(\omega)$ of $\overline{T}$ must be one of {UDL, UDU, ULL, ULU} (shown below).



Let $\pi = x_1x_2x_3\beta$, where β is the remaining part of π. Then π can be classified in one of the following cases.

- $x_1x_2x_3 =$ UDL. Then β is a Motzkin path from the point $(3,0)$ to the point $(n,0)$. The possibilities of π are m_{n-3}.
- $x_1x_2x_3 =$ UDU or ULL. Then β starts from the point $(3,1)$. By Proposition 3.2(i), the possibilities of π are $2(m_{n-2}-m_{n-3})$.
- $x_1x_2x_3 =$ ULU. Then β starts from the point $(3,2)$. By Proposition 3.2(ii), the possibilities of π are $m_{n-1}-2m_{n-2}$.

The assertion follows from summing up the above three quantities. □

3.2. A uniform way to prove Zeilberger's identities

Note that the method we used to prove Regev's conjecture can be applied to tableaux of various skew shapes. Let μ be a partition with at most three parts. The purpose is to compute $|\mathcal{T}_3(\mu; n-|\mu|)|$ for a fixed SYT T' of shape μ. For every $T \in \mathcal{T}_3(\mu; n-|\mu|)$, let $\overline{T}$ be the SYT defined by

$$\overline{T}(i,j) = \begin{cases} T'(i,j), & \text{if the } (i,j) \text{ entry is in } T';\\ T(i,j)+|\mu|, & \text{if the } (i,j) \text{ entry is in } T.\end{cases}$$

Converting $\overline{T}$ into a Yamanouchi word $\omega=\chi(\overline{T})$, it suffices to count the Motzkin paths $\pi=\phi^{-1}(\omega)$. Such paths π have a factorization $\pi=\alpha\beta$, where α consists of the initial $|\mu|$ steps and β is the remaining part of π. Now one can list all possible segments α whose words coincide with the initial subword of length $|\mu|$ of $\chi(\bar{T})$, and can classify these segments according to their end points, say (i, j), in the plane. Then for each class the possibilities of β can be determined by the formulae $|X(i, j; n)|$ in Lemma 3.2. Since each term $|X(i, j; n)|$ can be expressed as a linear combination of Motzkin numbers, we answer Zeilberger's question.

Theorem 3.3. *For every μ of at most three parts, the cardinality of $\mathcal{T}_3(\mu; n - |\mu|)$ can be expressed as a linear combination of negative shifts of the Motzkin numbers m_n with constant coefficients, and each formula can be proved combinatorially.*

Zeilberger also considered the problem of counting those SYTs in $\mathcal{T}_3$ with the restriction that the (i, j) entry is a fixed number m. He pointed out that the formula can be expressed as a linear combination of $|\mathcal{T}_3(\mu; n - |\mu|)|$ (hence also a linear combination of Motzkin numbers), and produced a list of formulae for $1 \leqslant m \leqslant 15$ and all feasible (i, j) [2]. Therefore, by using the same method we can also prove all these formulae combinatorially.

4. A conjecture

Although obtaining simple formulae for $|\mathcal{T}_k(n)|$, $k \geqslant 6$, seems hopeless, in this section we give a conjecture which reveals an (unexpected) relation between $|\mathcal{T}_{2\ell}(n)|$ and $|\mathcal{T}_{2\ell+1}(n)|$. The proof of the following simple fact is omitted.

Lemma 4.1. *The number of Motzkin paths of length n with the restriction that the level steps $(1, 0)$ are always on the x-axis is the central binomial number $\binom{n}{\lfloor \frac{n}{2} \rfloor}$.*

Let $\mathbb{R}_{\geqslant 0}$ denote the set of nonnegative real numbers. Hence we have the following fact:

Corollary 4.2. *$|\mathcal{T}_3(n)|$ equals the number of lattice paths in $\mathbb{R}^2_{\geqslant 0}$ from the origin to the x-axis using steps $(1, 0)$, $(1, 1)$, $(1, -1)$, and $|\mathcal{T}_2(n)|$ equals the number of these lattice paths with the restriction that the $(1, 0)$ steps appear only on the x-axis.*

This leads to our conjecture. Let $\{\mathbf{e}_1, \ldots, \mathbf{e}_{\ell+1}\}$ denote the standard basis of $\mathbb{R}^{\ell+1}$ and let $\mathcal{L}_{2\ell+1}(n)$ be the set of n-step lattice paths in $\mathbb{R}^{\ell+1}_{\geqslant 0}$ from the origin to the axis along $\mathbf{e}_1$, using $2\ell + 1$ kinds of steps $\mathbf{e}_1, \mathbf{e}_1 \pm \mathbf{e}_2, \mathbf{e}_1 \pm (\mathbf{e}_2 - \mathbf{e}_3), \mathbf{e}_1 \pm (\mathbf{e}_3 - \mathbf{e}_4), \ldots, \mathbf{e}_1 \pm (\mathbf{e}_\ell - \mathbf{e}_{\ell+1})$. By combining works of Grabiner and Magyar [6] and Gessel [4], Zeilberger proved [12], equivalently, that

$$\left|\mathcal{T}_{2\ell+1}(n)\right| = \left|\mathcal{L}_{2\ell+1}(n)\right|.$$

Conjecture 4.3. *Let $\mathcal{L}_{2\ell}(n)$ be the set of lattice paths in $\mathcal{L}_{2\ell+1}(n)$ with the restriction that the $\mathbf{e}_1$ steps appear only on the hyperplane spanned by $\{\mathbf{e}_1, \ldots, \mathbf{e}_\ell\}$. Then we have*

$$\left|\mathcal{T}_{2\ell}(n)\right| = \left|\mathcal{L}_{2\ell}(n)\right|.$$

This conjecture has been proved for $\ell = 2, 3$ and checked by computer for $\ell \leqslant 10$ up to $n = 30$ [1].

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