# On the least common multiple of q-binomial coefficients

Victor J. W. Guo

Department of Mathematics, East China Normal University, Shanghai 200062, People's Republic of China

jwguo@math.ecnu.edu.cn

**Abstract.** In this paper, we prove the following identity

$$\operatorname{lcm}\left(\begin{bmatrix} n \\ 0 \end{bmatrix}_q, \begin{bmatrix} n \\ 1 \end{bmatrix}_q, \dots, \begin{bmatrix} n \\ n \end{bmatrix}_q\right) = \frac{\operatorname{lcm}([1]_q, [2]_q, \dots, [n+1]_q)}{[n+1]_q},$$

where  $\binom{n}{k}_q$  denotes the q-binomial coefficient and  $[n]_q = \frac{1-q^n}{1-q}$ . This result is a q-analogue of an identity of Farhi [Amer. Math. Monthly, November (2009)].

Keywords: least common multiple; q-binomial coefficient; cyclotomic polynomial

AMS Subject Classifications (2000): 11A07; 05A30

### 1 Introduction

An equivalent form of the prime number theorem states that  $\log \operatorname{lcm}(1, 2, ..., n) \sim n$  as  $n \to \infty$  (see, for example, [4]). Nair [7] gave a nice proof for the well-known estimate  $\operatorname{lcm}\{1, 2, ..., n\} \geq 2^{n-1}$ , while Hanson [3] already obtained  $\operatorname{lcm}\{1, 2, ..., n\} \leq 3^n$ . Recently, Farhi [1] established the following interesting result.

**Theorem 1 (Farhi)** For any positive integer n, there holds

$$\operatorname{lcm}\left(\binom{n}{0}, \binom{n}{1}, \dots, \binom{n}{n}\right) = \frac{\operatorname{lcm}(1, 2, \dots, n+1)}{n+1}.$$
 (1)

As an application, Farhi shows that  $lcm\{1, 2, ..., n\} \ge 2^{n-1}$  follows immediately from (1).

The purpose of this note is to give a q-analogue of (1) by using cyclotomic polynomials. Recall that a natural q-analogue of the nonnegative integer n is given by  $[n]_q = \frac{1-q^n}{1-q}$ . The corresponding q-factorial is  $[n]_q! = \prod_{k=1}^n [k]_q$  and the q-binomial coefficient  $\begin{bmatrix} M \\ N \end{bmatrix}_q$  is defined as

$$\begin{bmatrix} M \\ N \end{bmatrix}_q = \begin{cases} \frac{[M]_q!}{[N]_q![M-N]_q!}, & \text{if } 0 \le N \le M, \\ 0, & \text{otherwise.} \end{cases}$$

Let lcm also denote the least common multiple of a sequence of polynomials in  $\mathbb{Z}[q]$ . Our main result can be stated as follows:

**Theorem 2** For any positive integer n, there holds

$$\operatorname{lcm}\left(\begin{bmatrix} n \\ 0 \end{bmatrix}_{q}, \begin{bmatrix} n \\ 1 \end{bmatrix}_{q}, \dots, \begin{bmatrix} n \\ n \end{bmatrix}_{q}\right) = \frac{\operatorname{lcm}([1]_{q}, [2]_{q}, \dots, [n+1]_{q})}{[n+1]_{q}}.$$
 (2)

#### 2 Proof of Theorem 2

Let  $\Phi_n(x)$  be the *n*-th *cyclotomic polynomial*. The following easily proved result can be found in [5, (10)] and [2].

**Lemma 3** The q-binomial coefficient  $\begin{bmatrix} n \\ k \end{bmatrix}_q$  can be factorized into

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \prod_d \Phi_d(q),$$

where the product is over all positive integers  $d \leq n$  such that  $\lfloor k/d \rfloor + \lfloor (n-k)/d \rfloor < \lfloor n/d \rfloor$ .

**Lemma 4** Let n and d be two positive integers with  $n \ge d$ . Then there exists at least one positive integer k such that

$$|k/d| + |(n-k)/d| < |n/d|$$
 (3)

if and only if d does not divide n + 1.

*Proof.* Suppose that (3) holds for some positive integer k. Let

$$k \equiv a \pmod{d}, \qquad (n-k) \equiv b \pmod{d}$$

for some  $1 \le a, b \le d-1$ . Then  $n \equiv a+b \pmod{d}$  and  $d \le a+b \le 2d-2$ . Namely,  $n+1 \equiv a+b+1 \not\equiv 0 \pmod{d}$ . Conversely, suppose that  $n+1 \equiv c \pmod{d}$  for some  $1 \le c \le d-1$ . Then k=c satisfies (3). This completes the proof.

Proof of Theorem 2. By Lemma 3, we have

$$\operatorname{lcm}\left(\begin{bmatrix} n \\ 0 \end{bmatrix}_{q}, \begin{bmatrix} n \\ 1 \end{bmatrix}_{q}, \dots, \begin{bmatrix} n \\ n \end{bmatrix}_{q}\right) = \prod_{d} \Phi_{d}(q), \tag{4}$$

where the product is over all positive integers  $d \leq n$  such that for some k  $(1 \leq k \leq n)$  there holds  $\lfloor k/d \rfloor + \lfloor (n-k)/d \rfloor < \lfloor n/d \rfloor$ . On the other hand, since

$$[k]_q = \frac{q^k - 1}{q - 1} = \prod_{d|k, d>1} \Phi_d(q),$$

we have

$$\frac{\operatorname{lcm}([1]_q, [2]_q, \dots, [n+1]_q)}{[n+1]_q} = \prod_{d \le n, \ d \nmid (n+1)} \Phi_d(q). \tag{5}$$

By Lemma 4, one sees that the right-hand sides of (4) and (5) are equal. This proves the theorem.

# 3 Theorem 2 is a q-analogue of Theorem 1

In this section we will show that

$$\lim_{q \to 1} \operatorname{lcm} \left( \begin{bmatrix} n \\ 0 \end{bmatrix}_q, \begin{bmatrix} n \\ 1 \end{bmatrix}_q, \dots, \begin{bmatrix} n \\ n \end{bmatrix}_q \right) = \operatorname{lcm} \left( \begin{pmatrix} n \\ 0 \end{pmatrix}, \begin{pmatrix} n \\ 1 \end{pmatrix}, \dots, \begin{pmatrix} n \\ n \end{pmatrix} \right), \tag{6}$$

and

$$\lim_{q \to 1} \frac{\operatorname{lcm}([1]_q, [2]_q, \dots, [n+1]_q)}{[n+1]_q} = \frac{\operatorname{lcm}(1, 2, \dots, n+1)}{n+1}.$$
 (7)

We need the following property.

**Lemma 5** For any positive integer n, there holds

$$\Phi_n(1) = \begin{cases} p, & \text{if } n = p^r \text{ is a prime power,} \\ 1, & \text{otherwise.} \end{cases}$$

*Proof.* See for example [6, p. 160].

In view of (4), we have

$$\lim_{q \to 1} \operatorname{lcm}\left(\begin{bmatrix} n \\ 0 \end{bmatrix}_q, \begin{bmatrix} n \\ 1 \end{bmatrix}_q, \dots, \begin{bmatrix} n \\ n \end{bmatrix}_q\right) = \prod_d \Phi_d(1), \tag{8}$$

where the product is over all positive integers  $d \leq n$  such that for some k  $(1 \leq k \leq n)$  there holds  $\lfloor k/d \rfloor + \lfloor (n-k)/d \rfloor < \lfloor n/d \rfloor$ . By Lemma 5, the right-hand side of (8) can be written as

$$\prod_{\text{primes } p \le n} p^{\sum_{r=1}^{\infty} \max_{0 \le k \le n} \{ \lfloor n/p^r \rfloor - \lfloor k/p^r \rfloor - \lfloor (n-k)/p^r \rfloor \}}. \tag{9}$$

We now claim that

$$\sum_{r=1}^{\infty} \max_{0 \le k \le n} \left\{ \lfloor n/p^r \rfloor - \lfloor k/p^r \rfloor - \lfloor (n-k)/p^r \rfloor \right\}$$

$$= \max_{0 \le k \le n} \sum_{r=1}^{\infty} \left( \lfloor n/p^r \rfloor - \lfloor k/p^r \rfloor - \lfloor (n-k)/p^r \rfloor \right). \tag{10}$$

Let  $n = \sum_{i=0}^{M} a_i p^i$ , where  $0 \le a_0, a_1, \dots, a_M \le p-1$  and  $a_M \ne 0$ . By Lemma 4, the left-hand side of (10) (denoted LHS(10)) is equal to the number of r's such that  $p^r \le n$  and  $p^r \nmid n+1$ . It follows that

$$LHS(10) = \begin{cases} 0, & \text{if } n = p^{M+1} - 1, \\ M - \min\{i : a_i \neq p - 1\}, & \text{otherwise.} \end{cases}$$

It is clear that the right-hand side of (10) (denoted RHS(10)) is less than or equal to LHS(10). If  $n = p^{M+1}-1$ , then both sides of (10) are equal to 0. Assume that  $n \neq p^{M+1}-1$  and  $i_0 = \min\{i: a_i \neq p-1\}$ . Taking  $k = p^M - 1$ , we have

$$\lfloor n/p^r \rfloor - \lfloor k/p^r \rfloor - \lfloor (n-k)/p^r \rfloor = \begin{cases} 0, & \text{if } r = 1, \dots, i_0, \\ 1, & \text{if } r = i_0 + 1, \dots, M, \end{cases}$$

and so

$$\sum_{r=1}^{\infty} \lfloor n/p^r \rfloor - \lfloor k/p^r \rfloor - \lfloor (n-k)/p^r \rfloor = M - i_0.$$

Thus (10) holds. Namely, the expression (9) is equal to

$$\prod_{\text{primes } p \leq n} p^{\max_{0 \leq k \leq n} \sum_{r=1}^{\infty} (\lfloor n/p^r \rfloor - \lfloor k/p^r \rfloor - \lfloor (n-k)/p^r \rfloor)} = \operatorname{lcm} \left( \binom{n}{0}, \binom{n}{1}, \dots, \binom{n}{n} \right).$$

This proves (6). To prove (7), we apply (5) to get

$$\lim_{q \to 1} \frac{\operatorname{lcm}([1]_q, [2]_q, \dots, [n+1]_q)}{[n+1]_q} = \prod_{d \le n, \ d\nmid (n+1)} \Phi_d(1),$$

which, by Lemma 5, is clearly equal to

$$\frac{\operatorname{lcm}(1,2,\ldots,n+1)}{n+1}.$$

Finally, we mention that (10) has the following interesting conclusion.

Corollary 6 Let p be a prime number and let  $k_1, k_2, ..., k_m \le n$ ,  $r_1 < r_2 < \cdots < r_m$  be positive integers such that

$$\lfloor n/p^{r_i} \rfloor - \lfloor k_i/p^{r_i} \rfloor - \lfloor (n-k_i)/p^{r_i} \rfloor = 1$$
 for  $i = 1, 2, \dots, m$ .

Then there exists a positive integer  $k \leq n$  such that

$$\lfloor n/p^{r_i} \rfloor - \lfloor k/p^{r_i} \rfloor - \lfloor (n-k)/p^{r_i} \rfloor = 1$$
 for  $i = 1, 2, \dots, m$ .

## References

- [1] B. Farhi, An identity involving the least common multiple of binomial coefficients and its application, Amer. Math. Monthly, November (2009).
- [2] V.J.W. Guo and J. Zeng, Some arithmetic properties of the q-Euler numbers and q-Salié numbers, European J. Combin. 27 (2006), 884–895.
- [3] D. Hanson, On the product of primes, Canad. Math. Bull. 15 (1972), 33–37.
- [4] G.H. Hardy and E.M. Wright, The Theory of Numbers, 5th Ed., Oxford University Press, London, 1979.

- [5] D. Knuth and H. Wilf, The power of a prime that divides a generalized binomial coefficient, J. Reine Angew. Math. **396** (1989), 212–219.
- [6] T. Nagell, Introduction to Number Theory, Wiley, New York, 1951.
- [7] M. Nair, On Chebyshev-type inequalities for primes, Amer. Math. Monthly 89 (1982), 126–129.