Graphical Enumeration

B.Tech. Seminar Report

Submitted in partial fulfillment of the requirements
for the degree of

Bachelor of Technology

by

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October 1, 1998

Abstract

In this report, we discuss the labeled and unlabeled enumeration of graphs, connected graphs, trees, eulerian graphs etc. Digraphs and multigraphs are not discussed much. In the labeled case, the use of exponential generating functions is discussed. For the unlabeled case, we only consider applications of Burnside's theorem and Polya's theorem. The use of pair group, product and cartesian product of groups is also discussed.

March 24, 1998

Acknowledgment

I would like to thank my guide, **Dr. A. A. Diwan** for his help and guidance without which this report would not have been possible.

Nikhil Bansal

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Chapter 1

Introduction

To a student of graph theory, one of the first questions which may come to his mind is that how many graphs or trees are there on p vertices. One would at least like to know the number approximately, if not exactly.
We shall be concerned with determining only the exact number of graphs which satisfy a given property. Generally, the number would be a function of the appropriate parameters.
Let $\{a_n\}$ be a sequence of numbers. The *generating function* of the sequence is the series $\sum_{n=0}^{\infty} a_n x^n$ considered as a formal power series, to which algebraic operations can be applied without consideration of their convergence. Generating functions are used to reduce a combinatorial problem to an algebraic one, which is generally simpler to deal with.
Most of the enumeration problems we discuss are discussed either partially or fully in [HPa]. Our main idea would be to obtain a recurrence formula or an expression for the generating function by considering the properties of the class of graphs we wish to enumerate. In most of the cases we will leave the answer in this form and not find out the number explicitly.

1.1 Enumeration

The enumeration problems can be classified into two broad kinds

- Find the number of graphs satisfying a given property. For example, connected graphs, trees, Eulerian graphs etc.

- Given a graph, find the number of subgraphs with a given property. For example, determining the number of subgraphs, number of spanning trees, number of Eulerian trails etc. in a given graph.

Often problems of the second kind are used to solve problems of the first kind. For instance, number of graphs and trees on p vertices may be obtained by determining the subgraphs and spanning trees of K_p (complete graph on p vertices). The number of bicolored graphs by the number of subgraphs of $K_{m,n}$ (complete (m, n) bipartite graph).

Enumeration is of two types

- Labeled Enumeration

- Unlabeled Enumeration

Given a graph G, we shall denote its vertex-set by $V(G)$ and its edge-set by $X(G)$. In a graph of order p (i.e. with p vertices), the vertices will be assigned numbers 1 to p and a edge will be a 2-subset (i, j) of $V(G)$. Thus multiple edges and self loops are excluded in G.

Labeled Enumeration Two graphs G_1 and G_2 are considered the same iff $V(G_1) = V(G_2)$ and $X(G_1) = X(G_2)$.
Thus a labeled graph G is determined uniquely by $V(G)$ and $X(G)$.

Unlabeled Enumeration Two graphs G_1 and G_2 are considered the same iff there is a permutation α from $V(G_1)$ to $V(G_2)$ such that $(i, j) \in X(G_1)$ iff $(\alpha i, \alpha j) \in X(G_2))$.

For example, the number of p *cycles* in a graph with p vertices, $p \geq 3$, will be $\frac{1}{2}(p-1)!$ in the labeled case, while in the unlabeled it will be 1.

1.2 Graphs and Groups

A non-empty set A together with a binary operation (.) is called group if the following four axioms are satisfied.

1. **Closure** For all $\alpha_1, \alpha_2 \in A, \quad \alpha_1.\alpha_2 \in A$

2. **Associativity** For all $\alpha_1, \alpha_2, \alpha_3 \in A, \quad \alpha_1.(\alpha_2.\alpha_3) = (\alpha_1.\alpha_2).\alpha_3$

3. **Identity** There is an element $e \in A$ such that $e.\alpha = \alpha.e = \alpha$ for all $\alpha \in A$

4. **Inverse** For each $\alpha \in A$, there is an element denoted by α^{-1} such that $\alpha\alpha^{-1} = \alpha^{-1}\alpha = e$

Composition of mappings provides a binary operation for permutations acting on the same set. A collection of permutations, closed with respect to composition forms a group. If a permutation group A acts on an object set X, then $|A|$ is called the *order* and $|X|$ is called the *degree* of A.

A permutation α on $V(G)$ that preserves adjacency is called an *automorphism* of G. It can be seen that the collection of all automorphisms of G, denoted by $\Gamma(G)$ forms a group under composition. $\Gamma(G)$ is called the *group* of G.

It can be easily seen that given an unlabeled graph G of order p the number of different ways of labeling it will be

$$\frac{p!}{|\Gamma(G)|}. \tag{1.1}$$

This expression gives a relation between the number of labeled and unlabeled graphs.

Chapter 2

Labeled Enumeration

Labeled enumeration problems are generally simpler than corresponding unlabeled enumeration problems. In this chapter we see some labeled enumeration problems, and discuss some techniques and tricks involved in their counting. We shall assume that all graphs are labeled graphs.

2.1 Counting by first principles

To get familiar with labeled enumeration we shall state and prove some results which are obtained by direct counting.

2.1.1 Labeled (p, q) graphs and connected p graphs

Let $G_{p,q}$ denote the number of (p, q) labeled graphs. Also let G_p denote the number of graphs on p vertices.
Now, in a labeled graph each pair of vertices is adjacent or non-adjacent, since the total number of pair of vertices is $\binom{p}{2}$ thus,

$$G_{p,q} = \binom{\binom{p}{2}}{q} \tag{2.1}$$

$$G_p = 2^{\binom{p}{2}} \tag{2.2}$$

Define a *rooted* graph as one which has one of its vertices called the root. Let C_p denote the number of connected,labeled graphs on p vertices. We observe that a different rooted graph is obtained when a labeled graph is rooted at each of its vertices. Thus, the number of rooted graphs $= pG_p$ and the number of rooted connected graphs $=pC_p$. The number of rooted graphs in which the root

is in a component of exactly k vertices will be $kC_k\binom{p}{k}G_{p-k}$.
Summing from $k = 1$ to p we get the number of rooted graphs. Thus,

$$C_p = 2^{\binom{p}{2}} - \frac{1}{p}\sum_{k=1}^{p-1} k\binom{p}{k}2^{\binom{p-k}{2}}C_k. \tag{2.3}$$

2.1.2 k-coloured (p, q) graphs

Suppose the k colours are fixed and there are p_i vertices of colour i. Clearly the number of pairs of vertices of different colour will be

$$\binom{p}{2} - \sum_{i=1}^{k}\binom{p_i}{2} = (p^2 - \sum_{i=1}^{k} p_i^2)/2$$

The number of ways of selecting labels $= \left(\begin{array}{c} p \\ p_1, p_2, \cdots, p_k \end{array}\right)$. Thus the number of k-coloured (p,q) graphs is the coefficient of x^q in

$$\frac{1}{k!}\sum_{(p)} \left(\begin{array}{c} p \\ p_1, p_2, \cdots, p_k \end{array}\right) (1 + x)^{(p^2 - \sum_{i=1}^{k} p_i^2)/2} \tag{2.4}$$

2.1.3 Acyclic digraphs on p vertices

Let $a_{p,k}$ denote the number of acyclic digraphs of order p which have exactly k vertices on indegree 0. (Each acyclic graph has atleast 1 vertex of indegree 0) Given an acyclic digraph, D we define $\phi(D)$ as the digraph obtained by removing all vertices of indegree 0 in D.
If D is of the type $a_{p,k}$,then $\phi(D)$ will be of type $a_{p-k,n}$ for some n. We will determine the number of preimages of each $D' \in a_{p-k,n}$ in $a_{p,k}$.
For each D' of the type $a_{p-k,n}$ there are $\binom{p}{k}$ ways of labeling k new vertices. Each of the n vertices of indegree 0 in D' should be adjacent to atleast one of the k vertices. Thus there are $(2^k - 1)$ possible choices for each of the n vertices. For each of the rest $p - k - n$ vertices there 2^k choices. So,

$$a_{p,k} = \sum_{n=1}^{p-k} (2^k - 1)^n 2^{k(p-n-k)} a_{p-k,n} \tag{2.5}$$

So the required number is $a_p = \sum_{k=1}^{p} a_{p,k}$.

2.1.4 Even p and (p, q) graphs

Let W_p be the number of even p graphs. A graph is *even* if each of its vertices has even degree.

Theorem 2.1 $W_p = 2^{\binom{p-1}{2}}$

Proof: Any graph G of order $p-1$ must have even number of vertices of odd degree, construct a new graph by adding a new vertex v which is given the label p and is adjacent to the vertices of G of odd degree.This new graph is a labeled even p graph. It can be seen that the correspondence is 1-1. □

Let $w_p(x)$ be the polynomial which has as the coefficient of x^q the number of even (p,q) graphs. We state the following result due to R.C.Read as stated in [HPa].

Theorem 2.2

$$w_p(x) = \frac{1}{2^p}(1+x)^{\binom{p}{2}} \sum_{n=0}^{p} \binom{p}{n} \left(\frac{1-x}{1+x}\right)^{n(p-n)} \tag{2.6}$$

Proof: Consider any graph with the labels of vertices multiplied by +1 and -1 arbitrarily .The numbers +1 or -1 are then assigned to each edge as the product of the signs of its vertices. The *sign* of G, denoted $\sigma(G)$ is then defined as the product of the signs of its edges.Clearly, there are 2^p ways in which the signs can be assigned to the labels of a given graph.
It can be seen that $\sigma(G) = (-1)^a$, where a is the sum of the degrees of the negative vertices. On the other hand, $\sigma(G) = (-1)^b$, where b is the number of negative edges of G.
Let L denote the set of all (p,q) graphs. We consider the sum $\sum \sigma(G)$ where summation is over all graphs in L and for the set S of 2^p allocations of +1 or -1 to the vertices. It follows that,

$$\sum_{G\in L}\left\{\sum_{S}(-1)^a\right\} = \sum_{S}\left\{\sum_{G\in L}(-1)^b\right\}.$$

If G is even ,then a is even ,so G contributes 2^p to the left.If G is not even, at least one vertex v has odd degree.The allocations in S for which v is positive and those for which it is negative are equinumerous ,so G contributes 0 to the left.So left side $= 2^p$ times the number of even graphs in L.
Consider an allocation in S for which n vertices are positive and $m = p-n$ vertices are negative. There are $\binom{p}{n}$ such allocations. If there are k edges that join positive to negative vertices, these may occur in $\binom{mn}{k}$ different ways. The remaining $q-k$ can occur in $\begin{pmatrix} \binom{n}{2}+\binom{m}{2} \\ q-k \end{pmatrix}$ different ways.
Summing from $k=0$ to q , we obtain

$$\sum_{k=0}^{q}(-1)^k \begin{pmatrix} mn \\ k \end{pmatrix} \begin{pmatrix} n(n-1)/2 + m(m-1)/2 \\ q-k \end{pmatrix}$$

as the contribution to the right for each allocation with n and m. This is the coefficient of x^q in $(1-x)^{nm}(1+x)^{n(n-1)/2+m(m-1)/2}$
Hence the right side is the coefficient of x^q in

$$\sum_{n=0}^{p} \binom{p}{n}(1-x)^{nm}(1+x)^{n(n-1)/2+m(m-1)/2}$$

Thus the result follows. □

2.2 Labeled counting lemma

In this section we discuss the use of exponential generating functions as a natural means for handling labeled enumeration problems.
Its application in finding the number of connected graphs,connected Euler graphs, number of trees and the number of block graphs is discussed.

Definition 2.1 (Exponential generating function) *Let a_k be the number of graphs on k vertices satisfying some property $P(a)$, the power series $a(x) = \sum_{k=1}^{\infty} \frac{a_k}{k!} x^k$ is called an exponential generating function.*

Let $b(x)$ be another exponential generating function for a class of graphs with property $P(b)$.

Lemma 2.3 (Labeled counting lemma) *The coefficient of $\frac{x^k}{k!}$ in $a(x)b(x)$ is the number of ordered pairs (G_1, G_2) of two disjoint graphs, where G_1 has property $P(a)$, G_2 has property $P(b)$, k is the number of vertices in $G_1 \cup G_2$ and the labels 1 to k have been distributed over $G_1 \cup G_2$.*

2.2.1 Connected graphs and Euler graphs

Let $C(x) = \sum_{k=1}^{\infty} \frac{c_k x^k}{k!}$ be the exponential generating function for connected graphs. Let $G(x)$ denote the exponential generating function for graphs.
By Labeled counting Lemma, we observe that $G(x) = \sum_{k=1}^{\infty} \frac{C^k(x)}{k!}$
So, $1 + G(x) = e^{C(x)}$

We now state an important theorem.

Theorem 2.4 *If $\sum_{k=0}^{\infty} A_k x^k = \exp\{\sum_{k=1}^{\infty} a_k x^k\}$ then for $k \geq 1$, we have*

$$a_m = A_m - \frac{1}{m}(\sum_{k=1}^{m-1} k a_k A_{m-k}) \tag{2.7}$$

Proof: Taking formal derivative on both sides, we get,
$\sum_{k=1}^{\infty} kA_k x^{k-1} = \sum_{k=1}^{\infty} ka_k x^{k-1}(\exp\{\sum_{k=0}^{\infty} a_k x^k\})$
$= \sum_{k=1}^{\infty} ka_k x^{k-1}(\sum_{k=0}^{\infty} A_k x^k)$
comparing coefficients of x^{m-1} on both the sides, we get,

$$mA_m = \sum_{k=1}^{m} ka_k A_{m-k}$$

noting that $A_0 = 1$, the result follows.
□

Using this fact we see that 2.3 is obtained.

An *Euler graph* is an even connected graph:
Arguing as for connected graphs we see that exponential generating function for Euler graphs $U(x) = log(1 + W(x))$

2.2.2 Polya's method for counting trees

Let t_p denote the number of trees on p vertices. Let $T(x)$ be the exponential generating series for rooted trees.(i.e. $T(x) = \frac{1}{p!} pt_p x^p$)
We will obtain a functional equation for T(x)
Given a rooted tree with root r.Suppose degree of r is n.
We note the following useful fact

Lemma 2.5 *The set of rooted trees with root degree n is in 1-1 correspondence with an unordered set of n rooted trees.*

Proof: The correspondence is given by deleting the root and making each of its child as the new root.It can be easily seen to be 1-1. □

Thus we can write

$$T(x) = \sum_{n=0}^{\infty} = \frac{xT(x)^n}{n!} \tag{2.8}$$

To solve it we apply a special case of

Lemma 2.6 (Lagrange's Inversion Formula) *If $\phi(y)$ is analytic in a neighbourhood of $y = 0$ with $\phi(0) \neq 0$, then the equation $x = y/\phi(y)$ is uniquely solved by the generating function $y = \sum_{k=1}^{\infty} c_k x^k$ whose coefficients are given by* $c_k = (\frac{1}{k!})\{(d/dy)^{k-1}(\phi(y))^k\}_{y=0}$

The proof can be found in Goursat and Hedrick [GH].
Applying this lemma to 2.8 gives $T(x) = \sum_{p=1}^{\infty} p^{p-1}\frac{x^p}{p!}$ thus

$$t_p = p^{p-2} \tag{2.9}$$

2.3 Some tree counting methods

In this section we present three more proofs of 2.9, which use completely different ideas. [Mo] presents an interesting compilation of such proofs. Our third proof is a recent one due to Peter Shor. [Sh]

2.3.1 Prufer's Correspondence

This proof uses a correspondence due to Prufer between the number of trees of order p and $(p-2)$ tuples $(a_1, a_2, \ldots, a_{p-2})$, where each a_k is an integer from 1 to p, Thus giving the result.

Prufer's Correspondence Given a labeled tree T , let v be the leaf with the smallest label and let a_1 be the label adjacent to v. Now to obtain a_2 repeat this step with $T - v$, the tree obtained from T by deleting v (and edge incident with v). This procedure is terminated when only two adjacent vertices remain. It is seen by induction on the number of vertices of T that each tree yields a unique tuple.Thus $t_p \leq p^{p-2}$.
Now , given a tuple a unique labeled tree is constructed as follows:
Let b_1 be the smallest positive integer that does not occur in the tuple and let $(c_2, \ldots, c_{p-2})$ denote the $p-3$ tuple obtained from $(a_2, \ldots, a_{p-2})$ by decreasing all terms larger than b_1 by 1. Then $(c_2, \ldots, c_{p-2})$ consists of numbers 1 through $p-1$, assume there is a corresponding tree T' of order $p-1$ and relabel the vertices of T' by adding 1 to each label larger than $b_1 - 1$.Then introduce the label b_1 and join it to the vertex labeled a_1 in T.
By induction on p, it is clear that the edges are inserted in the order they were removed. Thus $t_p \geq p^{p-2}$. Thus the result follows.

It can be seen that the degree of a vertex k in T is exactly one more that number of occurrences of k in the tuple. Thus the number of trees in which k has degree d_k is

$$\binom{p-2}{d_1 - 1, d_2 - 1, \ldots, d_p - 1} \tag{2.10}$$

Let $S(n,k)$ denote the Stirling number of the second kind (i.e. the number of partitions of n distinguishable objects into k indistinguishable non-empty sets.)

Theorem 2.7 *The number of trees with t leaves is*

$$\frac{p!}{t!}S(p-2,p-t) \tag{2.11}$$

Proof: The leaves can be chosen in $\binom{p}{t}$ ways, since each of the rest $p-t$ vertices occur at least once in the $(p-2)$ tuple, if vertex k occurs at $a_{k_1}, a_{k_2}, \ldots, a_{k_{d_{k-1}}}$ we say that $a_{k_1}, \ldots, a_{k_{d_{k-1}}}$ lie in the set k, using this correspondence we see that the number of partitions $= (p-t)!S(p-2,p-t)$ thus the number of tuples is $\binom{p}{t}(p-t)!S(p-2,p-t)$. □

2.3.2 Matrix-tree theorem

Given a graph G with p vertices and q edges, then the *incidence matrix* of G, denoted by $B(G)$, is the $p \times q$ matrix such that $B_{i,j} = 1$, if line j is incident on vertex i, 0 otherwise.
A $p \times p$ matrix $M(G)$ is defined as $M_{i,j} = -1$ if i,j are adjacent in G , $M_{i,i} = deg(i)$, 0 otherwise.
We state the following theorem due to Kirchoff called the Matrix-tree theorem.

Theorem 2.8 (Matrix-tree Theorem) *For any connected labeled graph G, all cofactors of the matrix $M(G)$ are equal and their common value is the number of spanning trees of G.*

Sketch of proof Let $B(G)$ be the incidence matrix of G. Let $E(G)$ be obtained by changing either of the two $1's$ in each column to -1. Consider a submatrix of E consisting of $p-1$ of its columns, corresponding to spanning graph H of G having $p-1$ edges. Either H is a tree or is disconnected, Remove an arbitrary row to from a $p-1$ square matrix F, if it is disconnected, $det(F) = 0$. If H is a tree, its edges and vertices can be relabeled, such that the new matrix F' with permuted rows and columns is lower triangular . So, $|det(F')| = |det(F)| = 1$. Now, $EE^t = M$. Let E_1 be the $(p-1) \times q$ submatrix obtained from E by striking out its first row. Apply $Cauchy-Binet$ Theorem to E_1 and E_1^t.The first principal cofactor of M is the sum of the corresponding $p-1 \times p-1$ determinants of E_1 and E_1^t. The product is 1 if the columns from E_1 correspond to a tree of G, 0 otherwise. Thus, the sum of these products is the number of spanning trees. Also as row sums and columns sums = 0, all cofactors are equal. Thus proved.

The number of labeled trees with p vertices is found by applying the theorem to K_p. By direct evaluation, it is seen that value of each cofactor $= p^{p-2}$.

This was generalized by Austin [Au] to count the number of k-colored trees as, Suppose the colors are numbered and p_i vertices are colored by the i^{th} color and vertices with color 1 are labeled 1 to p_1, with color 2 are labeled $p_1 + 1$ to $p_1 + p_2$. Then the number of trees is obtained by applying the theorem to the *complete* $k - partite$ graph $K_{p_1} \times \ldots \times K_{p_k}$.

2.3.3 Peter Shor's Proof

This method counts rooted trees. For a node i let $\beta(i)$ denote the smallest label on any node in the subtree rooted at i. A edge e is *proper* if $\beta(child(e)) > parent(e)$ where $parent(e)$ is the vertex containing e towards the root and $child(e)$ away from the root, else the edge is *improper*.
Let $T(n, j)$ denote the number of rooted with n vertices and j improper edges.
Given a rooted tree T define $\phi(T)$ as
Take the vertex n in T with the largest label.
1 If it is a leaf remove it,
2 if not then from the children of n, say $x_1, \ldots, x_d$, choose the child x_l with the largest $\beta(\alpha)$, and contract the edge (n, x_s) .
In case 1, number of improper edges remains same, while in case 2 it decreases by one.
If we have a tree R on $n-1$ vertices we can add n as the child of any of the vertices of R, thus the number of trees in which n is a leaf has n-1 preimages.
Next, take any vertex x , put n in the place of x and make x the child of n .Let $x_1, \ldots, x_b$ be the improper edges of x in R such that $\beta(x_1) < \beta(x_2) < \ldots < \beta(x_b) < x$, choose a such that $0 \leq a \leq b$ and make $x_1, \ldots, x_a$ the children of n and $x_{a+1}, \ldots, x_b$ the children of x in T. It is easy to verify that these are the only ways of partitioning the children of x .So $b + 1$ such ways,
Summing over all x gives j+n-1 preimages in T.
So,

$$T(n, j) = (n-1)T(n-1, j) + (n+j-2)T(n-1, j-1).$$

$T(1, j) = 0$ for $j \geq 1$,$T(1, 0) = 1$ and $T(i, -1) = 0$ for $i \geq 1$.
It can be shown that $\sum_{j=0}^{i-1} T(i, j) = i^{i-1}$, thus $t_i = i^{i-2}$.

2.4 Counting Eulerian trials in digraphs

A digraph is *Eulerian* if there exists a closed spanning directed walk passing through each edge exactly once. Such a walk is a directed *Eulerian trail*. It is easy to show that a digraph is Eulerian iff it is connected and each vertex has equal indegree and outdegree.

Let D be a digraph with adjacency matrix A. Define the matrix M_{out} with i, j entry the outdegree of v_i. Then let $C_{out} = M_{out} - A$. Similarly define $C_{in} = M_{in} - A$. We note that every row sum of $C_{out} = 0$ and every column sum of $C_{in} = 0$. Thus D is Eulerian iff all row and column sums are 0.

We have the following important theorem

Theorem 2.9 (Matrix-tree theorem for digraphs) *All the cofactors of the i^{th} row of C_{out} are equal, and their common value is the number of spanning trees of D rooted into v_i. Similarly, the common value of the cofactors of the i^{th} column of C_{in} is the number of spanning trees from v_i.*

A combinatorial proof of this result was discovered by Chaiken and can be found in [SW].

Let D be a Eulerian digraph, and let d_i=indegree v_i=outdegree v_i.

Theorem 2.10 *The number $e(D)$ if eulerian trails in D is*

$$e(D) = c \prod_i (d_i - 1)! \tag{2.12}$$

where c is the common value of the common factors of $C = C_{out} = C_{in}$.

Proof: To construct the spanning tree to v_1 determined by a given eulerian trail E in D, call the *exit edge* from each each vertex $v_i \neq v_1$ the last edge out of v_i when traversing E with starting and finishing vertex v_1. Thus only v_1 has no exit edge. Now, the subgraph T consisting of exit edges will be a tree to v_1 as v_1 has outdegree 0 and every vertex other than v_1 has outdegree 1, and v_1 can be reached from any vertex v_i by a path in T.

Fix T, a spanning tree to v_1. We construct all eulerian trails E such that the exit edges of E are edges of T. Now, one edge from each $v_i \neq v_1$ is fixed for use as an exit edge, and one edge from v_1 is fixed as the first edge of E. So at each v_i there are exactly $(d_i - 1)!$ orders in which the 0 of edges in E can appear. Since these are independent, the result follows. □

Chapter 3

Counting unlabeled trees

Let A be a permutation group acting on a set X on objects. Define a relation $x \sim y$, if for some α in A, $\alpha x = y$.
It can be verified that $\sim$ is an equivalence relation. The equivalence classes of this relation are known as orbits and form a partition of X.
Unlabeled enumeration involves the calculation of the number of orbits of some permutation group. Burnside gave a formula for this. Polya developed a theorem not only for counting orbits but also determining their number with a "prescribed weight" of some sort.
In this chapter we shall state and prove his theorem and discuss its applications in determining the number of unlabeled trees and tree-like graphs of various kinds.

3.1 Polya's Theorem

Given an $x \in X$, we denote by $A(x)$ the subgroup of A which consists of all permutations $\alpha \in A$ such that $\alpha x = x$. Let $\theta(x)$ denote the orbit to which x belongs. It can be easily shown that $|A| = |A(x)|.|\theta(x)|$ for all $x \in X$.
Any permutation α in A can be written as a product of disjoint cycles, let $j_k(\alpha)$ denote the number of cycles of length k in α,
Let $|X| = d, s_1, s_2, \ldots, s_d$ be variables.

The *cycle index* of A is the polynomial $Z(A)$ defined by the equation.

$$Z(A) = \frac{1}{|A|} \sum_{\alpha \in A} \prod_{k=1}^{k=d} s_k^{j_k(\alpha)} \tag{3.1}$$

Lemma 3.1 (Burnside's lemma) *The number of orbits n determined by A acting on X is $n = \frac{1}{|A|} \sum_{\alpha \in A} j_1(\alpha)$*

Proof: It can be seen that
$n = \sum_{x\in X} \frac{1}{|\theta(x)|} = \sum_{x\in X} \frac{|A(x)|}{|A|}$
$= \frac{1}{|A|} \sum_{x\in X} \sum_{\alpha\in A(x)} 1 = \frac{1}{|A|} \sum_{\alpha\in A} \sum_{x=\alpha x} 1$
$= \sum_{\alpha\in A} j_1(\alpha)$.
□

Let w be a weight function from X into N^k such that for all α in A and all x in X, $w(x) = w(\alpha(x))$, we can thus define $w(\theta_i) = w(x)$ for any x in θ_i. Then using above argument it follows that $\sum w(\theta_i) = \frac{1}{|A|} \sum_{\alpha\in A} \sum_{x=\alpha(x)} w(x)$.

We now discuss Polya's Theorem as in [Po].
Let D be a set of elements called *places*, R be a set of elements called *figures*, and A be a permutation group acting on D. Each figure r has an associated *weight* $w(r) = (m,n)$ $m,n \geq 0$ (In general, it can be a k tuple).The *figure counting series* $c(x,y)$ is defined as $c(x,y) = \sum_{m,n=0}^{\infty} c_{m,n} x^m y^n$ where $c_{m,n}$ is the number of figures of weight (m,n).
Let R^D denote the set of functions from D to R . A function f in R^D is called a *configuration*, and it has weight $w(f)$ defined as $x^m y^n$ where $(m,n) = \sum_{d\in D} w(f(d))$. Each $\alpha \in A$ induces a permutation on the configurations of R^D as $\tilde{\alpha} f(x) = f(\alpha x)$. This set of permutations is called the *power group* induced by A and is denoted by E^A. The figure counting series $C(x,y)$ is defined by $C(x,y) = \sum_{m,n=0}^{\infty} C(m,n) x^m y^n$ where $C_{m,n}$ is the number of inequivalent configurations of weight $x^m y^n$ determined by the power group E^A.

Thus we have $C(x,y) = \frac{1}{|E^A|} \sum_{\alpha\in E^A} \sum_{f=\alpha f} w(f)$.

Theorem 3.2 (Polya's Theorem)

$$C(x,y) = Z(A, c(x,y)).$$

Proof: Let $\tilde{\alpha}$ be a permutation in E^A and let f be a configuration fixed by $\tilde{\alpha}$. Suppose z_k is any cycle of length k in the disjoint cycle decomposition of α. Then since $\tilde{\alpha} f = f$ we have $f(d) = f(z_k d)$ for each element d permuted by z_k Thus all elements permuted by z_k must have the same image under f.And conversely, if the elements of each cycle of the permutation α are taken to the same image by a configuration f, then f is fixed by $\tilde{\alpha}$.Now each configuration fixed by A is obtained by independently selecting an element $r(z_k)$ in R for each cycle z_k, and setting $f(d) = r(z_k)$ for all d permuted by z_k .Thus each cycle z_k

if α contributes $\sum_{r\in R}(x^m y^n)^k$ to $\sum_{f=\tilde{\alpha}} w(f)$.But, $\sum_{r\in R}(x^m y^n)^k = c(x^k, y^k)$, $\sum_{f=\tilde{\alpha}} w(f) = c(x,y)^{j_1(\alpha)} \ldots c(x^s, y^s)^{j_s(\alpha)}$
since $|A| = |E^A|$,we have
$C(x,y) = \frac{1}{|A|}\sum_{\alpha\in A} c(x,y)^{j_1(\alpha)} \ldots c(x^s, y^s)^{j_s(\alpha)}$
Hence the result, $C(x,y) = Z(A, c(x,y))$. □

Special Groups

Symmetric group (S_n) $Z(S_n) = \frac{1}{n!}\sum_{(j)} \frac{n!}{\prod_k k^{j_k} j_k!} s_1^{j_1} s_2^{j_2} \ldots s_n^{j_n}$
where the sum is over all partitions (j) of n. $j = (j_1, j_2, \ldots, j_n)$ such that $\sum_{k=1}^{k=n} k j_k$.
Alternating group (A_n) $Z(A_n) = \frac{1}{n!}\sum_{(j)} \frac{n!(1+(-1)^{j_2+j_4+\ldots})}{\prod_k k^{j_k} j_k!} s_1^{j_1} s_2^{j_2} \ldots s_n^{j_n}$

We shall denote the cyclic, dihedral and identity groups on n objects by C_n, D_n and $E_n$respectively.

We shall write $Z(A_n - S_n)$ for $Z(A_n) - Z(S_n)$ and $Z(S_\infty)$ for $\sum_{n=0}^{\infty} Z(S_n)$ Taking $Z(S_0) = 1$, and observing that $Z(S_n) = \frac{1}{n}(s_1 Z(S_{n-1}) + s_2 Z(S_{n-2}) + \ldots + s_{n-1} Z(S_1) + s_n)$, it can be shown that,

$$Z(S_\infty) = exp(\sum_{k=1}^{\infty} s_k/k) \tag{3.2}$$

$$Z(A_\infty - S_\infty) = exp(\sum_{k=1}^{\infty} (-1)^{k+1}(s_k/k)) \tag{3.3}$$

We note two important corollaries of Polya's Theorem as discussed in [HPa] pp. 46-49, which we state as theorems.
Two subsets X_1 and X_2 of X are called $A-equivalent$ if there is a permutation $\alpha \in A$ such that $\alpha(X_1) = X_2$.

Theorem 3.3 *The number of A-equivalence classes of r-subsets of X is the coefficient of x^r in the polynomial Z(A,1+x).*

Proof: In the figure counting series 1+x, the term 1=x^0 indicates the absence of an object in X while x stands for its presence.Thus the result follows. □

The next theorem expresses the number of weighted 1-1 functions.

Theorem 3.4 *The counting series for all S_n equivalent configurations of figures whose counting series is f(x) where each figure appears at most once in any configurations is $Z(A_n - S_n, f(x))$.*

Proof: It follows from Polya's theorem that the counting series for orbits of all functions is simply $Z(S_n, c(x))$.
Any two 1-1 functions are in the same orbit of E^{A_n} iff they are both odd or both even.Thus $Z(A_n, c(x))$ counts twice those orbits of E^{S_n} which consist of 1-1 functions.
Next we show that orbits of E^{S_n} which are not 1-1 are counted only once. Suppose $f(x) = g(\alpha x)$ for all x in X for some $\alpha \in S_n$, then f and g clearly lie in the same orbit of E^{A_n}. But if α is odd, since f is not 1-1 $f(x_1) = f(x_2)$ for some $x_1, x_2 \in X, x_1 \neq x_2$. Let $\beta = (x_1, x_2)$. As β is odd, $\alpha\beta$ will be even and for all $x \in X$, $f(x) = g(\alpha\beta x)$. Thus f,g are also in the same orbit of E^{A_n}. Thus, the result follows. □

3.2 Counting Trees

3.2.1 Rooted trees

Let $T(x) = \sum_{p=1}^{\infty} T_p x^p$ be the generating function for rooted trees, where T_p is the number of rooted trees of order p.

Theorem 3.5

$$T(x) = xZ(S_\infty, T(x)) \tag{3.4}$$

Proof: Given a rooted tree, by lemma 2.5, it is in 1-1 correspondence with some set of n rooted trees.
Since the n rooted trees are mutually interchangeable without altering the isomorphism class of the given rooted tree. So taking figure counting series as $T(x)$ and configuration group as S_n , we get that the number of rooted trees of order p where the root has degree n is the coefficient of x^p in $xZ(S_n, T(x)$
Thus $T(x) = x\sum_{n=0}^{\infty} Z(S_n, T(x)) = x(Z(S_\infty, T(x))$.
Using the expression for $T(S_\infty)$ it can be shown that,

$$T_{p+1} = \frac{1}{p}\sum_{k=1}^{p}(\sum_{d_k} dT_d)T_{p-k+1} \tag{3.5}$$

□

We call rooted tree *planted* if its root has degree 1. We shall denote it by $\bar{T}(x)$
We note that

Lemma 3.6 *The set of planted trees with* $p+1$ *vertices is in 1-1 correspondence with the set of rooted trees with* p *vertices.*

Lemma 3.7 *The set of rooted trees with p vertices with root-degree n is in 1-1 correspondence with set of unordered n tuples of planted trees with a total of $p+n-1$ vertices.*

3.2.2 Unrooted Trees

One of the standard methods to obtain the number of an unrooted structure is to enumerate corresponding rooted structure and then find an appropriate "Dissimilarity Characteristic Theorem".
Unrooted trees were first enumerated by Otter [Ot].
In this chapter,this technique is also used for enumerating triangular cacti.
We first state and prove the dissimilarity theorem for graphs.
Let G be a graph and A be some subgroup of $\Gamma(G)$, let p^* be the number of dissimilar vertices determined by the action of A, let b^* be the number of dissimilar blocks and p_1^* be the number of dissimilar vertices in the ith class of dissimilar blocks. The following theorem is due to [HN] ,

Theorem 3.8 (Dissimilarity characteristic theorem for graphs)

$$p^* - 1 = \sum_{i=1}^{b^*}(p_i^* - 1) \tag{3.6}$$

Proof: Apply induction on b^*. If $b^* = 1$ the result obviously holds. Otherwise, consider any block of G, say b_1 that has exactly one cutvertex. Delete from G the vertices of the members of this class except for the cutvertices. The graph G' so obtained has $b^* - 1$ classes of blocks and $p^* - (p_1^* - 1)$ classes of vertices. On applying induction hypothesis, the result follows. □

Let p^* and q^* denote the number similarity classes of vertices and edges respectively. A *symmetry edge* joins two similar vertices. Let s denote the number of symmetry edges.

Theorem 3.9 (Dissimilarity characteristic theorem for trees) *For any tree T,*

$$p^* - (q^* - s) = 1 \tag{3.7}$$

Proof: For trees, each block will be a edge, so $b^* = q^*$, and $p_i^* = 2$ for each class of edges other than symmetry edges, for which p_i^* =1. Thus the result follows from Thm 3.8 . □

We state a well-known theorem without proof.

Theorem 3.10 *A tree has either one or two central vertices, and if there are two they are adjacent.*

So, $s = 1$ iff the tree is bicentered and the two central vertices are similar, otherwise $s = 0$.
Let $t(x)$ denote the counting series for trees.

Theorem 3.11 $t(x) = T(x) - \frac{1}{2}(T^2(x) - T(x^2))$

Proof: Summing 3.8 over all trees with exactly p vertices, we get,
$\sum 1 = \sum p^* - \sum (q * -s)$
or $t_p = T_p - L_p$ or $t(x) = T(x) - L(x)$
where $L(x)$ is the counting series for trees rooted at a non-symmetric edge.
To obtain $L(x)$ we note that each tree rooted at a non-symmetric edge corresponds to two different rooted trees, obtained by removing the edge and making the vertices adjacent to the edge as roots.
Thus $L(x) = Z(A_2 - S_2, T(x)) = \frac{1}{2}(T^2(x) - T(x^2))$
Thus the result follows. □

3.3 Trees with given properties

In this section we obtain counting series for trees with given properties, the four examples chosen from [HPr] serve to show the wide applicability and power of the methods discussed in the previous section.

3.3.1 Trees with a given partition

Let the partition be denoted by the sequence $(i_1, i_2, \ldots, i_n)$ where i_m is the number of vertices of degree m.Let $\bar{P}(x, t_1, t_2, \ldots) = \sum_{j,t_1,t_2,\ldots=0}^{\infty} \bar{P}_{j,i_1,i_2,\ldots} x^j t_1^{i_1} t_2^{i_2} \ldots$ where $\bar{P}_{j,i_1,i_2,\ldots}$ is the number of planted trees with i_m vertices of degree m and a total of j vertices.Let $P(x, t_1, t_2, \ldots)$ and $p(x, t_1, t_2, \ldots)$ be the corresponding counting series for rooted trees and trees respectively.

Theorem 3.12 $\bar{P}(x, t_1, t_2, \ldots) = \sum_{m=0}^{\infty} x^2 t_1 t_{m+1} Z(S_m, \frac{1}{xt_1} \bar{P}(x, t_1, t_2, \ldots))$
$P(x, t_1, t_2, \ldots) = \sum_{m=0}^{\infty} x t_m Z(S_m, \frac{1}{xt_1} \bar{P}(x, t_1, t_2, \ldots))$
$p(x, t_1, t_2, \ldots) = P(x, t_1, t_2, \ldots) - \frac{1}{x^2 t_1^2} Z(A_2 - S_2, \bar{P}(x, t_1, t_2, \ldots))$

Proof: By lemma 3.7 we get, $P(x, t_1, t_2, \ldots) = \sum_{m=0}^{\infty} x^1 t_m Z(S_m, \frac{1}{xt_1} \bar{P}(x, t_1, t_2, \ldots))$ and by lemma 3.6 $\bar{P}(x, t_1, t_2, \ldots) = \sum_{m=0}^{\infty} x^2 t_1 t_{m+1} Z(S_m, \frac{1}{xt_1} \bar{P}(x, t_1, t_2, \ldots))$.

Any two planted trees determine a edge rooted tree by deleting the roots of the planted trees and joining the children of the roots by a edge, Conversely a edge rooted tree determines 2 planted trees in this way. So, counting series for edge rooted trees would be, $\frac{1}{x^2t_1^2}Z(S_2, \bar{P}(x, t_1, t_2, \ldots))$
Thus by Thm 3.9
$p(x, t_1, t_2, \ldots) = P(x, t_1, t_2, \ldots) - \frac{1}{x^2t_1^2}Z(A_2 - S_2, \bar{P}(x, t_1, t_2, \ldots))$ □

3.3.2 Trees with a given diameter

Given a rooted tree define root-diameter as the maximum distance between the root and all other vertices.
Define $T^{(n)}(x)$ as the generating function for rooted trees with root-diameter $\leq n$.
So $Z(S_2, T^{(n)}(x) - T^{(n-1)}(x))$ is the counting series for rooted trees with root-diameter n. It can be seen that $T^{(0)}(x) = x$ and $T^{(n)}(x) = xZ(S_\infty, T^{(n-1)}(x))$
It can be shown that any diameter passes through all the centers.(The proof is easy, but lengthy, so we omit it)
Trees with odd diameter are bicentered,while trees with even diameter are centered.
Let $d_m(x)$ be the counting series for trees with diameter m.
Consider a bicentered tree with odd diameter $2n+1$ and bicenters p, q. deleting the edge (p, q) and making p, q as roots we get 2 rooted trees of root-diameter n.Conversely, any two rooted trees of root diameter n uniquely determine a tree of diameter $2n + 1$. Hence,
$d_1(x) = Z(S_2, T^0(x))$
$d_{2n+1}(x) = Z(S_2, T^{(n)}(x) - T^{(n-1)}(x)), n \geq 1.$
Now, a centered tree with diameter $2n$ and center p is determined by the set of its subtrees, if we delete p , and make vertices adjacent to p roots, we have a set of rooted trees, at least two of which have root-diameter $n - 1$, and the rest root-diameter $\leq n - 2$.Therefore,

$$d_0(x) = T^{(0)}(x) \tag{3.8}$$

$$d_2(x) = x\sum_{m=2}^{\infty} Z(S_m, T^{(0)}(x)) \tag{3.9}$$

$$d_2(x) = x(Z(S_\infty, T^{(n-2)}(x)))(\sum_{m=2}^{\infty} Z(S_m, T^{(n-1)}(x) - T^{(n-2)}(x))), n > 1 \tag{3.10}$$

3.3.3 Homeomorphically irreducible trees

A *homeomorphically irreducible* tree is one with no vertices of degree 2.Let $h(x), H(x)$ and $\bar{H}(x)$ be the counting series for homeomorphically irreducible trees,rooted trees and planted trees respectively.Let $\bar{T}$ be a planted tree ,now (unless $\bar{T}$ is a single edge) $\bar{T}$ is homeomorphically irreducible iff it has atleast 2 corresponding irreducible planted trees.
Hence $\bar{H}(x)$ is determined by the equation

$$\bar{H}(x) = x^2(1 + \sum_{n=2}^{\infty} Z(S_n, \frac{\bar{H}(x)}{x})) \tag{3.11}$$

If T is a homeomorphically irreducible rooted tree, then so is $\bar{T}(x)$ (where $\bar{T}(x)$ is obtained as in lemma 3.6)unless the root of T is a leaf. However, not all homeomorphically irreducible trees are formed in this way.For if root of T has degree 2,then $\bar{T}(x)$ is homeomorphically irreducible, even though T is not. Thus,

$$\bar{H}(x) = x((H(x) - \bar{H}(x)) + Z(S_2, \bar{H}(x)))$$

$$H(x) = \frac{x+1}{x}\bar{H}(x) - \frac{1}{x}Z(S_2, \bar{H}(x)) \tag{3.12}$$

Applying Thm 3.9, we get,

$$h(x) = H(x) - \frac{1}{x^2}Z(A_2 - S_2, \bar{H}(x)). \tag{3.13}$$

3.3.4 Trees having identity group

The absolute $|T|$ of a rooted tree is the unrooted tree with the same vertices and adjacencies as in T. It is clear that if $\Gamma(|T|) = E_n$,then $\Gamma(T) = E_n$,but not conversely.

Theorem 3.13 *Let $u(x)$ and $U(x)$ be the counting series for trees and rooted trees whose automorphism group is the identity. Then*

$$U(x) = xZ(A_\infty - S_\infty, U(x)) \tag{3.14}$$

$$u(x) = U(x) - Z(S_2, U(x)). \tag{3.15}$$

Proof: It is clear that T has identity group iff each of its corresponding rooted subtrees have group identity and if they are different.Thus,

$$U(x) = xZ(A_\infty - S_\infty, U(x))$$

To find $u(x)$: We note that since there are no symmetry edges the Dissimilarity Characteristic Equation would be $1 = p^* - q^*$
So $u(x) = U_1(x) - U_2(x)$, where $U_1(x)$ and $U_2(x)$ are the respective counting series for rooted and edge-rooted trees whose absolute has the identity group.
We also define $V_1(x)$ and $V_2(x)$ as the counting series for rooted and edge-rooted trees respectively who have the identity group but their absolute does not.
So, $U_1(x) = U(x) - V_1(x)$
$U_2(x) = Z(A_2 - S_2, U(x)) - V_2(x)$
So, $u(x) = U(x) - Z(A_2 - S_2, U(x)) + V_2(x) - V_1(x)$
We will show that $V_1(x) - V_2(x) = U(x^2)$
For all trees T whose group is not the identity ,we find how many trees T' and edge rooted trees T'' with identity group have T as absolute.
Case 1: T has no symmetry edge: If there exist any T' or T'' then

Claim: $\Gamma(T)$ *has exactly one element besides the identity.*

Proof: We first note that the root in T' or T'' should not remain fixed by the non-identity permutation in T , otherwise T' or T'' will not have identity group. Suppose ρ_1, ρ_2 are the two elements besides identity.Let v_1, v_2 be vertices such that ρ_1, ρ_2 permute two subtrees of v_1, v_2 respectively.Now, either the root r and v_1 lie in the same subtree of v_2 or r and v_2 lie in the same subtree of v_1.Suppose r and v_1 do. Then we get another permutation which permutes two subtrees corresponding to $\rho_2(v_1)$; clearly this permutation leaves r fixed, thus leading to a contradiction. □

Now if the two similar subtrees of v_1 have n vertices then we have exactly n rooted trees T'.Also only the edges in one of the subtrees rooted at v_1 can be chosen to get T''.
Thus , the contribution of T' and T'' to $V_1(x) - V_2(x) = 0$.
Case 2: T has a symmetry edge.
If there exist T' or T'' with identity group, then

Claim: *The permutation permuting the two central elements is the only non-identity permutation.*

Proof: Let ρ_1 be the permutation permuting the central elements,suppose some another non-identity permutation $\rho_2 \neq \rho_1$ exists ,clearly ρ_2 does not fix root r and there exists a vertex v_2, such that ρ_2 permutes some two subtrees rooted at v_2,one of which contains r.But, then we get another permutation which permutes two subtrees rooted at $\rho_2(v_1)$, which fixes r. Thus we have a contradiction. □

Thus if the two permuted subtrees have $2n$ vertices ,then there can be n rooted trees T' and $n-1$ edge-rooted trees T''. Thus each tree of order p contributes 1 to the coefficient of x^p in $V_1(x) - V_2(x)$.
So $V_1(x) - V_2(x)$ is the counting series for trees with a symmetry edge whose group have order 2. So, $V_1(x) - V_2(x) = U(x^2)$. Thus the result follows.
□

3.4 Counting tree-like graphs

3.4.1 Unicyclic Graphs and Functional digraphs

If G is unicyclic and its cycle has length n, then G may be regarded as having a rooted tree , possibly the trivial one, attached to each of the n vertices of its cycle.
Thus, the counting series $U_n(x)$ for unicyclic graphs whose cycle has length n is given by

$$U_n(x) = Z(D_n, T(x)). \tag{3.16}$$

A digraph is *functional* is every vertex has outdegree 1.
The following lemma is obvious.

Lemma 3.14 *A digraph D is functional iff each of its weak components consist of exactly one directed cycle C and for each vertex u of C, the* weak component *$R(u)$ of $D - C$ which contains u is a tree to u.*

It thus follows that the series $v(x)$ for functional digraphs (with n_k cycles of length k)is given by

$$v(x) = \sum \prod_{k=2}^{\infty} Z(S_{n_k}, Z(C_k, T(x))) \tag{3.17}$$

where the sum is over each $n_k = 0$ to ∞. On interchanging the sum and product symbols, we get

$$v(x) + 1 = \prod_{k=2}^{\infty} Z(S_\infty, Z(C_k, T(x))). \tag{3.18}$$

3.4.2 Triangular cacti

A *cactus* is a connected graph in which no edge lies on more than one cycle.
A *triangular cactus* is a cactus in which every edge is in a triangle.

We follow the method of Harary and Norman.
Let the coefficient of x^n in $d(x), D(x)$ and $\bar{D}(x)$ be the number of unrooted, vertex rooted and vertex rooted triangular cacti with root degree 2, respectively, with n triangles.
Then $\bar{D}(x) = xZ(S_2, D(x))$.
And $D(x) = Z(S_\infty, \bar{D}(x)) = Z(S_\infty, xZ(S_2, D(x)))$.
The number of edge rooted cacti will be $xD(x)Z(S_2, D(x))$
and those rooted at a triangle will be $xZ(S_3, D(x))$.
To find $d(x)$ we need to find the dissimilarity characteristic theorem for triangular cacti.
Let p^*, q^* and r^* denote the number of dissimilar vertices, edges and cycles respectively, then we have

Theorem 3.15 (Dissimilarity characteristic theorem for triangular cacti)

$$1 = p^* - q^* + r^* \tag{3.19}$$

Proof: Consider a triangle, on removing its edges we obtain three cacti, each of then contributes 1 to $p^* - q^* + r^*$,and we have the following cases:
1. All edges of triangle are similar, p^* decreases by 2, q^* and r^* increase by 1.
2. Two edges were similar, then p^* decreases by 1, q^* increases by 2 and r^* increases by 1.
3. All edges dissimilar, then p^* does not change, q^* increases by 3 and r^* increases by 1.
Thus we see that $p^* - q^* + r^*$ does not change. Thus the result follows.
Thus

$$d(x) = D(x) - xD(x)Z(S_2, D(x)) + xZ(S_3, D(x)).$$

Or,

$$d(x) = D(x) - \frac{x}{3}(D^3(x) - D(x^3)) \tag{3.20}$$

□

Chapter 4

Counting Unlabeled Graphs

4.1 Introduction

Having discussed how Polya's Theorem can be applied to counting trees and tree-like structures, we are now in a position to answer probably the most basic question , How many graphs are there Γ

We shall also count the number of connected graphs, graphs rooted a given subgraph, supergraphs of a given graph and Eulerian graphs, as found by Polya and others.

4.2 Counting (p, q) graphs and (p, q) connected graphs

We first develop some notation and operations on groups.

Definition 4.1 (Line group) *Given a graph G with edge set $X(G)$. Each permutation α in $\Gamma(G)$ induces a permutation α' acting on $X(G)$ as*
If u and v are adjacent in G then $\alpha'(u, v) = (\alpha(u), \alpha(v))$. The collection of α''s form a group called the edge group of G , denoted by $\Gamma_1(G)$.

We have the following important theorem.

Theorem 4.1 *The number of dissimilar spanning subgraphs of G with q edges is the coefficient of x^q in $Z(\Gamma_1(G), 1 + x)$.*

Proof: $Z(\Gamma_1(G), 1 + x)$ enumerates the equivalence classes of sets of edges of G .These equivalence classes correspond precisely to the spanning subgraphs of G, two of which are in the same class whenever there is an automorphism of G that sends one to the other. □

Definition 4.2 (Pair Group) *Let A be a permutation group acting on the set $X = \{1, 2, \ldots, p\}$. The pair group of A , denoted by $A^{(2)}$ is the permutation group induced by A which acts on 2-subsets of X.*

Let $g_p(x)$ be the generating function for p-graphs by the number of edges.

Theorem 4.2

$$g_p(x) = Z(S_p^{(2)}, 1 + x) \tag{4.1}$$

Proof: Clear by Theorem 4.1, and noting that $\Gamma_1(K_p) = S_p^{(2)}$ □

To calculate $Z(S_p^{(2)})$, we need to determine the contribution of each $s_1^{j_1} s_2^{j_2} \ldots s_n^{j_n}$ in $Z(S_p)$

There are two separate contributions, first comes from pairs of vertices, both in a common cycle; the second, from the pair of vertices, each in different cycles. We now determine the first of these contributions. A single cycle of odd length $2n+1$ in S_p induces n cycles of length $2n+1$, and an even cycle of length $2n$ induces $n-1$ cycles of length $2n$ and one cycle of length n. Thus,

$s_{2n+1} \to s_{2n+1}^n$ and $s_{2n} \to s_n s_{2n}^{n-1}$

Thus for j_k cycles of length $2n$ in S_p we have,

$s_{2n+1}^{j_{2n+1}} \to s_{2n+1}^{nj_{2n+1}}$ and $s_{2n}^{j_{2n}} \to s_n s_{2n}^{n-1}$

For the second contribution, consider two cycles of length m and n in s_p .They induce exactly (m, n) cycles of length $[m, n]$, where (m, n) and $[m, n]$ denote respectively the *gcd* and *lcm* of m and n.

Thus , we have when $m \neq n$

$s_m^{j_m} s_n^{j_n} \to s_{[m,n]}^{j_m j_n (m,n)}$ and when $m = n = k$

$s_k^{j_k} \to s_k^{k\binom{j_k}{2}}$

Thus we have

$$Z(S_p^{(2)}) = \frac{1}{p!} \sum_{(j)} \frac{p!}{\prod k^{j_k} j_k!} \prod_k s_{2k+1}^{kj_{2k+1}} \prod_k (s_k s_{2k}^{k-1})^{j_{2k}} s_k^{k\binom{j_k}{2}} \prod_{m<n} s_{[m,n]}^{(m,n) j_m j_n} \tag{4.2}$$

Connected Graphs Let $g(x)$ and $c(x)$ be respectively the generating functions for graphs and connected graphs by the number of vertices.

And let $g(x, y)$ and $c(x, y)$ be respectively the generating functions for connected graphs by the number of vertices and edges.

It follows from Thm 3.2 that $Z(S_n, c(x))$ counts graphs with exactly n components. Thus on summing over n, $1 + g(x) = Z(S_\infty, c(x))$.

Thus,

$$1 + g(x) = \exp \sum_{k=1}^{\infty} c(x^k)/k \tag{4.3}$$

It also follows from the two variable version of Thm 3.2 that

$$1 + g(x,y) = \exp \sum_{k=1}^{\infty} c(x^k, y^k)/k \tag{4.4}$$

Let us see how c_p was computed by Cadogan .
Let $\sum_{p=1}^{\infty} a_p x^p = \log(1 + g(x))$.
It follows from Thm 2.4 that $pa_p = pg_p - \sum_{k=1}^{p-1} ka_k g_{p-k}$
Since $\sum_{p=1}^{\infty} a_p x^p = \sum_{k=1}^{\infty} c(x^k)/k$

$$pa_p = \sum_{d|p} dc_d \tag{4.5}$$

On inverting 4.5 using the *möbius function* $\mu(d)$ we get ,

$$c_p = \sum_{d|p} \frac{\mu(d)}{d} a_{p/d}. \tag{4.6}$$

4.3 Rooted Graphs and rooted supergraphs

We seek to determine the number of graphs of order p rooted at an induced graph H of order $n \leq p$.This means that no edge of G joins two nonadjacenet vertices of H.
We shall discuss the method of Harary and Palmer.
We shall denote the permutation group required by $\Gamma(H) \circ S_{p-n}$. The following operations are very useful.

Definition 4.3 (Product of groups) *Let A and B be permutation groups acting on disjoint sets X and Y respectively. Then the product AB is the group consisting of pairs of permutations (α, β), α in A and β in B which acts on $X \cup Y$, as $(\alpha, \beta)z = \alpha(z)$ if $z \in X$, or $\beta(z)$ if $z \in Y$.*

It is easy to see that,

$$Z(AB) = Z(A)Z(B)$$

Definition 4.4 (Cartesian Product of groups) *The cartesian product of two permutation groups A and B acting on sets X and Y respectively is the set of all ordered pairs (α, β) of permutations, α in A and β in B such that $(\alpha, \beta)(x, y) = (\alpha x, \beta y)$. We denote it by $A \times B$.*

Proceeding as in the dervation of 4.2, we see that

$$Z(S_m \times S_n) = \frac{1}{m!n!} \sum_{\alpha,\beta} \prod_{r,t=1}^{m,n} s_{[r,t]}^{(r,t)j_r(\alpha)j_t(\beta)} \tag{4.7}$$

We set $G = H \cup K_{p-n}$ and observe that the product $\Gamma(H)S_{p-n}$ is the subgroup of $\Gamma(G)$ which fixes H. $\Gamma(H) \circ S_{p-n}$ will be the restriction of the pair group $(\Gamma(H)S_{p-n})^{(2)}$ to pairs of vertices of G not both in H. If $\prod_k^{j_K}$ is a term of $Z(S_p)$ let the corresponding term in $Z(S_p)^{(2)}$ be denoted by $(\prod s_k^{j_k(\beta)})^{(2)}$. Now, we have the following theorem.

Theorem 4.3 *The generating function for graphs of order p rooted at an induced subgraph H of order n is $Z(\Gamma(H) \circ S_{p-n}, 1+x)$*
where

$$\Gamma(H) \circ S_{p-n} = \frac{1}{|\Gamma(H)|(p-n)!} \sum_{(\alpha,\beta)} \prod_{r,t=1}^{n,p-n} s_{[r,t]}^{(r,t)j_r(\alpha)j_t(\beta)} (\prod s_k^{j_k(\beta)})^{(2)} \qquad (4.8)$$

and sum is over all pairs with α in $\Gamma(H)$ and β in S_{p-n}.

Proof: We observe that (α, β) induces a permutation on the edges between H and K_{p-n} exactly as in the cartesian product 4.7, and $(\prod s_k^{j_k(\beta)})^{(2)}$ describes the structure of the permutation β induces on the pairs of vertices in K_{p-n}. $\square$

Counting Supergraphs If G is a supergraph of H, then H is a subgraph of G which is not necessarily induced.
The group required for counting supergraphs for H is the restriction of the pair group $(\Gamma(H)S_{p-n})^{(2)}$ to the pair of vertices which are not adjacent in H. As in [HPa] we denote it by $\Gamma_1(\bar{H}, K_p)$.
An effective procedure to compute it was given by Robinson, using the idea of a $Vertex\ edge$ group, as discussed in [HPa]

4.4 Enumerating Eulerian graphs

This method due to Robinson, illustrates an ingenious application of Burnside's lemma.
We first enumerate even graphs.
Given a permutation α in S_p we know by Thm 4.2, that $2^{v(\alpha)}$ graphs are left invariant by α, where

$$v(\alpha) = \sum_{r<t} (r,t) j_r j_t + \sum_{k=1} k\binom{j_k}{2} + \sum_{k=1} k(j_{2k} + j_{2k+1}) \qquad (4.9)$$

We will determine how many of the $2^{v(\alpha)}$ are left invariant by α are even graphs. First consider the vertices on a cycle of even length induced by α.

In a graph left invariant by α they must have the same degree. The diagonal edge cycle contributes just one adjacency at each vertex if the cycle, so that by including or excluding the diagonal cycle as need be they can all be made to have an even degree. Essentially, one degree of freedom is for each cycle of even length induced by α in passing from all labeled graphs left invariant by α to just those which are even.

Let the total number of cycles of α of odd length be m. If $m > 0$, let c be some cycle of odd length, if $c' \neq c$ is another odd cycle of α then any edge joining c and c' contributes an odd number to the degree of every vertex of c and c' .Thus the condition that the vertices of c' have even degree determines whether or not the unique edge cycle of which joins c and c' must be included in the graph.If all vertices on odd cycles other than c have even degree, then the vertices of c must have an even degree too since any graph has an even number of vertices of odd degree.

Thus we lose $m-1$ degrees of freedom if $m > 0$. When $m = 0$ there is no restriction .In general m-sg(m) degrees of freedom are lost, where $sg(m) = 1$ if $m > 0$ and $sg(0) = 0$. So,

$$\mu(\alpha) = v(\alpha) - \sum_k j_k + sg(\sum_{k=0} j_{2k+1})$$

Using 4.14 thus gives

$$\mu(\alpha) = \sum_{r<t} (r,t) j_r j_t + \sum_{k=1} k\binom{j_k}{2} + \sum_{k=1} (k-1)(j_{2k} + j_{2k+1}) + sg(\sum_{k=0} j_{2k+1}) \quad (4.10)$$

So, Burnside's lemma gives

$$w_p = \frac{1}{p!} \sum_{(j)} \frac{p!}{\prod_k k^{j_k} j_k!} 2^{e(j)} \quad (4.11)$$

For counting eulerian graphs we argue as in 4.3 .

Thus

$$1 + w(x) = Z(S_\infty, u(x)). \quad (4.12)$$

4.5 Conclusion

We see that the main technique is to define an appropriate group operation, and apply it to the appropriate edge group of some graph depending upon the problem at hand.

Another important operation called *Composition* was defined by Polya [Po]. Harary [H1] enumerated the number of bicoloured graphs with m vertices of one colour and n of the other, using cartesian product and composition.

Chapter 5

Conclusion

Though we have not discussed any application of the methods to non graphical problems many pattern and configuration problems become graphical in nature when properly formulated. Furthermore, the conceptual difficulty of the problem is more easily identified when recast in the terms of graphs.

For example, [HPa] discusses some applications to problems involving chessboard configurations, simplicial complexes etc.

A lot of graphical enumeration problems are unsolved. [HPa] gives a list of unsolved problems, some of which are closely related to the problems we have discussed.

1. *Identity graphs* Identity trees were discussed in Sec. 3.3.4

2. *Eulerian graphs* Eulerian digraphs were discussed in Sec. 4.4

3. *Eulerian trails in Eulerian graphs* The case for Eulerian digraphs was discussed in Sec. 2.4

4. *Graphs with a given diameter* The case for trees was discussed in Sec. 3.3.2

5. *k Connected Graphs* We have discussed for $k = 0, 1$, [HPa] discusses for $k = 2$. For $k \geq 3$, the problem is unsolved.

6. *Non isomorphic spanning trees of a given graph* Spanning trees were discussed in Sec. 2.3.2

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