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博 士 学 位 论 文

偏序集分拆函数的计算及其算术性质

Computations and Arithmetic Properties of P-partition Functions

论文作者 杜康　　　　指导教师 侯庆虎 教授

申请学位 理学博士　　　　培养单位 组合数学中心

学科专业 应用数学　　　　研究方向 组合数学

答辩委员会主席 王军　　　　评 阅 人 匿名评审

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论文题目	偏序集分拆函数的计算及其算术性质				
姓　名	杜康	学号	1120100006	答辩日期	2013 年 5 月 30 日
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院/系/所	组合数学中心	专业	应用数学		
联系电话	15620622586	**Email**	daniel.kang.du@gmail.com		
通信地址(邮编)：天津市南开区卫津路 94 号					
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中文摘要

偏序集分拆作为经典分拆在偏序集上的推广，是计数组合学的一个重要研究对象。著名组合数学家Stanley于1972年首先引入偏序集分拆这一概念，从而统一了许多经典的组合结构，包括分拆、有序分拆、多重分拆、平面分拆等。近四十年来，偏序集分拆的组合性质和算术性质引起了人们极大的兴趣，已得到了一系列丰富的研究成果。偏序集分拆在对称函数、表示论、丢番图方程组和不等式组等问题的研究中有非常广泛的应用。

本文主要研究偏序集分拆的计数函数，即偏序集分拆函数，主要结果包括：1. 我们引入偏序集上的两类变换和一类运算，以此为工具给出一种求偏序集分拆生成函数的递归方法。2. 利用模形式理论，我们研究了偏序集分拆生成函数的算术性质，得到一系列裂钻分拆和多重分拆满足的拉马努金型同余式。

本论文结构如下。第一章简要介绍偏序集分拆的基本概念和相关背景知识，包括几类经典的求偏序集分拆生成函数的方法，经典分拆函数及多重分拆函数的拉马努金型同余式。

第二章给出一种求偏序集分拆生成函数的递归方法。我们定义了偏序集上的两类变换：删元和部分线性扩展，并分别给出了它们作用前后的偏序集上分拆的生成函数所满足的关系。我们将证明通过删元和部分线性扩展，任意偏序集均可变换为一个空偏序集。事实上，这一变换过程给出了求偏序集分拆生成函数的递归方法。另外，我们将偏序集上的直和及序和推广为部分偏序和，很多著名的偏序集都可由简单偏序集的部分偏序和生成。作为应用，我们考虑偏序集P与其自身的n次部分偏序和P_n，文中构造性地证明了其分拆的生成函数序列$\{f_{P_n}(\mathbf{x})\}_{n\geq 1}$满足一组递推关系式。特别地，我们解决了Souza等人利用“五项准则”方法无法处理的3行偏序集的分拆生成函数问题，并得到了基于重立方体偏序集的偏序集分拆生成函数。

第三章主要研究裂钻分拆函数及多重分拆函数的算术性质。通过构造适当的η-商，我们利用Sturm定理得到了6个2裂钻分拆函数的拉马努金型同余式。关于n的r重分拆函数$p_r(n)$的算术性质，我们证明了序列$p_r((m^kn+r)/24)$的生成函数模m^k后在空间$S_{\gamma,\lambda}$中，其中$S_{\gamma,\lambda}$是由复值函数$\eta(24z)^{\gamma}\phi(24z)$张成的空

间，此处 $\eta(z)$ 是 η-函数，$\phi(z)$ 是 $M_\lambda(\mathrm{SL}_2(\mathbb{Z}))$ 中的一个整模形式。利用该结果，我们得到了一系列著名的拉马努金型同余式，如经典分拆函数 $p(n)$ 模 $5,7,11$ 的拉马努金同余式，Gandhi 关于 $p_2(n)$ 模 5 和 $p_8(n)$ 模 11 的同余式，同时我们也得到了很多新的关于 $p_r(n)$ 的拉马努金型同余式。利用空间 $S_{\gamma,\lambda}$ 是 Hecke 算子 T_{ℓ^2} 的不变子空间这一性质，我们得到了两类关于 $p_r(n)$ 模素数次幂的同余式。最后，针对一类特殊的多重分拆函数，我们给出了对应的拉马努金猜想和纽曼猜想之间的具体联系。

关键词： 分拆，偏序集分拆，多重分拆，裂钻分拆，生成函数，模形式，同余式，拉马努金同余式

Abstract

P-partition is defined as a mapping that assigns nonnegative integral entries to the elements of a poset, which is one of the most important objects in enumerative combinatorics. As a generalization of the ordinary partition function, P-partition function, the counting function of P-partitions, has many applications in symmetric functions, representation theory, linear diophantine equations and inequalities.

The main results of this thesis consist of some progress in P-partition functions, including a recursive method for computing the generating functions of P-partitions, and the arithmetic properties of two classes of P-partitions: the broken 2-diamond partitions and multipartitions.

This thesis is organized as follows. The first chapter is devoted to an introduction of P-partitions. We first give a brief review of the theory and present the important special cases of P-partitions. Then we recall several classical methods for computing generating functions of P-partitions. Meanwhile, we display some important classical Ramanujan-type congruences of P-partitions and related background.

In Chapter 2, we present an approach to compute the generating function $f_P(\mathbf{x})$ of P-partitions for a given poset P. To this end, we first introduce two kinds of transformations on posets, which are named as deletion and partially linear extension, respectively. We show that the generating function of P-partition of a poset can be expressed in terms of corresponding generating functions of its transformations. In fact, one can compute $f_P(\mathbf{x})$ by recursively applying these two transformations. As an application, we consider the partially ordinal sum P_n of n copies of a given poset, which generalizes both the direct sum and the ordinal sum. We constructively prove that the sequence $\{f_{P_n}(\mathbf{x})\}_{n\geq 1}$ satisfies a finite system of recurrence relations with respect to n. Finally, we illustrate our method by giving several examples, including the well-known multi-cube posets and a kind of 3-rowed posets which cannot be dealt with by Souza's "five guidelines".

In Chapter 3, we focus on the arithmetic properties of the broken k-diamond partitions and multipartitions. By constructing appropriate eta-quotients and applying Sturm's theorem, we discuss the Ramanujan-type congruences of the broken k-diamond partitions. Specially, we give new proofs of six Ramanujan-type congruences of the broken 2-diamond partition. Then, we study the arithmetic properties of multipartitions in the framework of modular forms. Let $p_r(n)$ denote the number of r-component multipartitions of n, and let $S_{\gamma,\lambda}$ be the space spanned by $\eta(24z)^{\gamma}\phi(24z)$, where $\eta(z)$ is the Dedekind's eta function and $\phi(z)$ is a holomorphic modular form in $M_{\lambda}(\mathrm{SL}_2(\mathbb{Z}))$. We show that the generating function of $p_r((m^k n+r)/24)$ with respect to n is congruent to a function in the space $S_{\gamma,\lambda}$ modulo m^k. As special cases, this relation leads to many well-known congruences, including the Ramanujan congruences of $p(n)$ modulo $5,7,11$, Gandhi's congruences of $p_2(n)$ modulo 5 and of $p_8(n)$ modulo 11, as well as a series of new congruences of multipartitions. Furthermore, using the invariance property of $S_{\gamma,\lambda}$ under the Hecke operator T_{ℓ^2}, we obtain two classes of congruences pertaining to the m^k-adic property of $p_r(n)$. In the end, we establish an interesting connection between Ramanujan's conjecture and Newman's conjecture for multipartitions.

Key Words: partition, P-partition, multipartition, broken k-diamond partition, generating function, modular form, congruence, Ramanujan congruence

CONTENTS

Chapter 1 Introduction

The theory of partition is an interesting and far-reaching branch of combinatorics and number theory. Certain special problems in partitions date back to the Middle Ages and many great mathematicians have contributed to the development of the theory. The applications of partitions, as said by Rota in [7], are found wherever discrete objects are to be counted or classified, whether in the molecular and the atomic studies of matter, in the theory of numbers, or in combinatorial problems from all sources.

In 1972, based on the works of MacMahon [58] and Knuth [54], Stanley [76] generalized the notion of partition into a whole new class of objects, called *P-partition*, which is one of the most interesting and fruitful combinatorial objects. For its rich arithmetic and combinatorial properties, P-partition has been extensively studied by MacMahon, Ramanujan, Andrews, Stanley, Zeilberger, Paule, Ono and so on. It has been widely applied in symmetric functions, representation theory, linear diophantine equations and inequalities.

In this thesis, we are mainly concerned with the computing of the generating functions and the arithmetic properties of P-partition functions. In the following sections of this chapter, we introduce the basic definitions and notation on P-partition functions. We also give a brief overview of the generating functions and the arithmetic properties of P-partition functions.

1.1 *P*-partitions

We first recall the terminology of partially ordered sets in [78, Chapter 3]. A *partially ordered set* P (or *poset*, for short) is a set, together with a binary relation denoted $\leq_P$ (or $\leq$ when there is no confusion), satisfying the following three axioms:

1. For all $x \in P$, $x \leq x$. (*reflexivity*)
2. If $x \leq y$ and $y \leq x$, then $x = y$. (*antisymmetry*)
3. If $x \leq y$ and $y \leq x$, then $x \leq z$. (*transitivity*)

All posets considered in this paper are finite. Here we give three examples as follows.

1. Let $n \in \mathbb{P}$. The set $[n]$ with its usual order forms an n-element poset with the special property that any two elements are comparable. This poset is denoted by $\mathbf{n}$.
2. Let $n \in \mathbb{N}$. One can make the set $2^{[n]}$ of all subsets of $[n]$ into a poset B_n by defining $S \leq T$ in B_n if $S \subset T$ as sets. We say that B_n consists of the subsets of $[n]$ "ordered by inclusion".
3. Let $n \in \mathbb{P}$. The set of all positive integral divisors of n can be made into a poset D_n in a way by defining $i \leq j$ in D_n if j is divisible by i.

Let $x, y \in P$, we say y *covers* x (or, x *is covered by* y), denoted by $x \lessdot y$, if $x < y$ and if no element $z \in P$ such that $x < z < y$. In the framework of these cover relations, one can depict any given poset by its graphical representation. The *Hasse diagram* of a finite poset P is the graph whose vertices are the elements of P, whose edges are the cover relations, and such that if $x \lessdot y$ then y is drawn "above" x. To coincide with the descriptions used by Andrews, Paule and Riese (see, [15]), we rotate the Hasse diagram by 90 degree clockwise so that smaller elements lie to the left throughout this thesis. For example, Figure 1.1 illustrates the Hasse diagrams of some of the posets considered in above examples. And the diamond poset $D = \{1, 2, 3, 4\}$ with cover

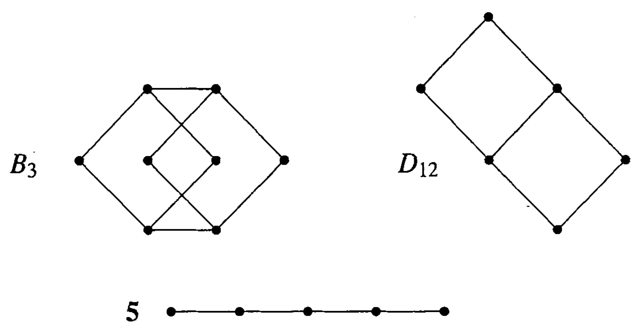


Figure 1.1 Hasse diagrams of posets.

relations $\{1 \lessdot 2 \lessdot 4, 1 \lessdot 3 \lessdot 4\}$ can be represented by Figure 1.2.

Sometimes it is useful to consider the degenerate cases of posets. A *chain* (or *totally ordered set* or *linearly ordered set*) is a poset in which any two elements are comparable. It is straightforward to see that the poset $\mathbf{n}$ is a chain. A subset C of a

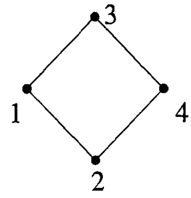


Figure 1.2 The Hasse diagram of the Diamond poset D.

poset P is called a *chain* if C is a chain when regarded as a subposet of P. The *length* $\ell(C)$ of a finite chain is defined by $\ell(C) = |C| - 1$. And the *length* (or *rank*) of a finite poset P is

$$\ell(P) := \max\{\ell(C) : C \text{ is a chain of } P\}.$$

On the other extreme, an *anti-chain* is a poset in which any two elements are not comparable.

Based on the knowledge of posets, we begin to introduce the P-partitions. As said by Zeilberger in [40], compositions and partitions are the two extremes of non-negative integer arrays, where in the former there is no order at all imposed, while in the latter there is a total order. In 1972, Stanley [76], standing on the shoulders of MacMahon [58] and Knuth [54], defined a whole new class of objects that bridges these two extremes. He coined them *P-partitions*[1].

Definition 1.1 *Let P be a finite poset, then a P-partition of n is an* order-reversing *map* $\sigma : P \to \mathbb{N}$, *i.e.,*

$$i \leq_P j \Rightarrow \sigma(i) \geq_{\mathbb{N}} \sigma(j),$$

satisfying

$$\sum_{i \in P} \sigma(i) = n,$$

denoted $|\sigma| = n$.

Denote the set of P-partitions of a poset P by $\mathrm{Par}(P)$. Then the (multivariate) generating function of P-partitions of P is given by

$$f_P(\mathbf{x}) = \sum_{\sigma \in \mathrm{Par}(P)} \prod_{a \in P} x_a^{\sigma(a)}, \tag{1.1}$$

where $\mathbf{x}$ is the variable vector $(x_a)_{a \in P}$.

[1] It would be better to refer to these as "poset partitions", but we follow tradition instead.

As special cases of P-partitions, the compositions, ordinary partitions, multipartitions and plane partitions are the most important objects in the study of combinatorics.

Compositions

A *composition* of an integer n into m parts is a sequence of positive integers $(\lambda_1, \lambda_2, \ldots, \lambda_m)$ with

$$\lambda_1 + \lambda_2 + \cdots + \lambda_m = n.$$

Let P be a disjoint union of m points, one can see that a P-partition of n is equivalent to a composition of n into m parts.

Ordinary Partitions

Let P be a ℓ-element chain, then a P-partition of n is equivalent to an ordinary partition of n into at most ℓ parts. More specifically, an ordinary *partition* λ of a nonnegative integer n into ℓ parts means a non-increasing sequence of positive integers $(\lambda_1, \lambda_2, \ldots, \lambda_\ell)$ satisfying

$$\lambda_1 + \lambda_2 + \cdots + \lambda_\ell = n,$$

where n is called the weight of λ. Let $p(n)$ denote the ordinary partition function, the number of partitions of n. For convenience, we agree that $p(0) = 1$, that is, there is only one certain partition of 0 denoted by $\emptyset$. We also set $p(n) = 0$ for $n < 0$, and $p(\alpha) = 0$, if $\alpha \notin \mathbb{Z}$. By a classical identity of Euler, we know that the generating function of $p(n)$ has an infinite product representation

$$\sum_{n=0}^{\infty} p(n)q^n = \prod_{n=1}^{\infty} \frac{1}{(1-q^n)}. \tag{1.2}$$

Another effective elementary device for studying partitions is their graphical representation. Each partition $(\lambda_1, \lambda_2, \ldots, \lambda_\ell)$ is associated with its graphical representation, named *Ferrer's diagram*, depicted as ℓ rows of dots, the i-th row containing λ_i of them. For example, Figure 1.3 shows the Ferrer's diagram for one possible partition of 32 into 6 parts.

Multipartitions

In 1882, Sylvester [80] broke new ground in the theory of partitions. Throughout most of the nineteenth century, partitions of integers were viewed primarily as an auxiliary aid in the theory of invariants. Sylvester's monumental paper [80] revealed that partitions were themselves interesting mathematical objects with a surprising rich

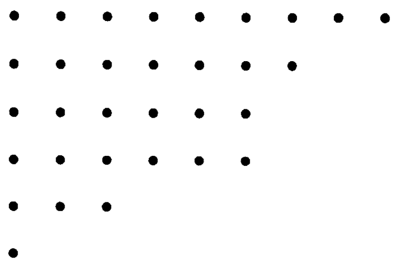


Figure 1.3 Ferrer's diagram for the partition $(9,7,6,6,3,1)$

arithmetic combinatorial properties. And there are many interesting generalizations of ordinary partitions, one of which is multipartition, also called colored partitions. Here we adopt the notation in [6] given by Andrews.

Definition 1.2 *A* multipartition *of a non-negative integer n with r components is an r-tuple*

$$\lambda = (\lambda^{(1)}, \ldots, \lambda^{(r)})$$

of partitions whose weights sum to n.

For example, the ten 2-component multipartitions of 3 are

$$\begin{array}{lllll} ((3),\emptyset), & (\emptyset,(3)), & ((2,1),\emptyset), & (\emptyset,(2,1)) & ((1),(2)), \\ ((2),(1)), & ((1,1,1),\emptyset), & (\emptyset,(1,1,1)), & ((1),(1,1)), & ((1,1),(1)). \end{array}$$

It is no hard to see that if P is a poset with r disjoint chains, then a P-partition of n is equivalent to a multipartition of n with r components.

One can read a r-component multipartition as a non-increasing sequence of positive integers in which each positive integer is assigned one of r distinct colors, the order of the colors is not considered, and the sum of all of these positive integers is n. Let $p_r(n)$ denote the number of the r-component multipartitions of n, then the generating function of $p_r(n)$ is given as follows

$$\sum_{n=0}^{\infty} p_r(n) q^n = \prod_{n=1}^{\infty} \frac{1}{(1-q^n)^r}, \tag{1.3}$$

which is indeed a power of the generating function (1.2) of ordinal partition functions. Multipartitions arise in combinatorics, representation theory and physics. As pointed out by Fayers [41], the representations of the Ariki–Koike algebra are naturally indexed by multiparititions. Bouwknegt [26] showed that the Durfee square formulas of multipartitions are useful in deriving expressions for the characters of modules of affine Lie algebras in terms of the universal chiral partition functions.

Plane Partitions

One can see that an ordinary partition can be regarded as a sequence of integers standing on a line orderly, while a r-component multipartition can been seen as a r-tuple of integral sequence standing on r lines orderly. Similarly, an array of integers standing on a plane orderly also defines a partition, which is called plane partition. More precisely, a *plane partition*, which was first considered by MacMahon [63], with n rows and l columns is an array $\pi = \left(\pi_{i,j}\right)_{n\times l}$ of nonnegative integers $\pi_{i,j}$ that is weakly decreasing in rows and columns, i.e.,

$$\pi_{i,1} \geq \pi_{i,2} \geq \cdots \geq \pi_{i,l}, \quad (i = 1,2,\ldots,n),$$

and

$$\pi_{1,j} \geq \pi_{2,j} \geq \cdots \geq \pi_{n,j}, \quad (j = 1,2,\ldots,l).$$

Furthermore, if the array π is in three-dimensional pile with non increasing in each directions, we say π is a *solid partition*, which is a natural generalization of the plane partition.

If P is a poset with regular $l\times n$ Hasse diagram, then a P-partition of n is equivalent to a $n\times l$ plane partition of n.

The Broken k-Diamond Partitions

In 2007, Andrews and Paule [9] continued their study on MacMahon's Partition Analysis by considering families of plane partitions. Instead of using squares, they introduced the k-diamond, depicted in Figure 1.4, as building blocks of the chain. Besides studying k-elongated diamond partitions of length n, Andrews and Paulse also defined the object known as *the broken k-diamond* in Figure 1.5, consisting of two separated k-elongated partition diamonds of length n where in one of them, the source is deleted. And the *broken k-diamond partitions* are the P-partitions with respect to the poset depicted in Figure 1.5.

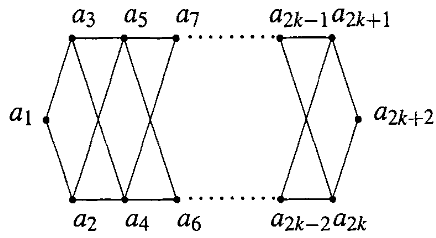


Figure 1.4 A k-elongated diamond partition of length 1.

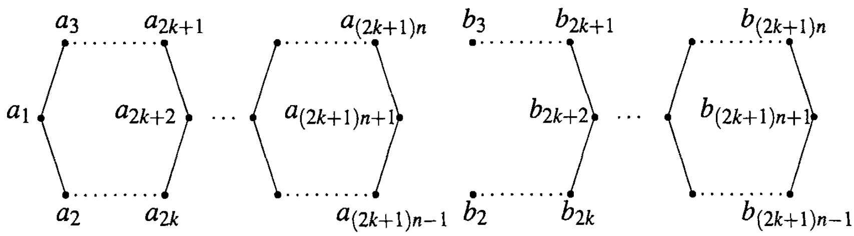


Figure 1.5 A broken k-diamond of length $2n$.

1.2 Computations of P-partition Functions

As Wilf articulated in his monograph [84], *a generating function is a clothesline on which we hang up a sequence of numbers for display.* Generating functions are a bridge between discrete mathematics, on the one hand, and continuous analysis (particularly complex variable theory) on the other hand. The full beauty of the subject of generating functions emerges only from tuning in on both channels: the discrete and the continuous. And the computation of the generating functions (1.1) has preoccupied a growing number of mathematicians in recent years.

During the late nineteenth and early twentieth centuries, MacMahon discussed plane and solid partitions using partition analysis [58, Section VIII, Ch.II]. He began by considering plane partitions on a square, and he [58, pp.183] showed the multi-variable generating function of 2×2 plane partitions as follows

$$F(x_1,x_2,x_3,x_4) = \frac{1-x_1^2x_2x_3}{(1-x_1)(1-x_1x_2)(1-x_1x_3)(1-x_1x_2x_3)(1-x_1x_2x_3x_4)}.$$

Futhermore, he observed that if $x_1 = x_2 = x_3 = x_4 = q$, the resulting generating function is

$$\frac{1}{(1-q)(1-q^2)^2(1-q^3)}.$$

More generally, the generating function of $n \times l$ plane partitions is still an open problem in enumerative combinatorics. In order to solve general systems of linear diophantine inequalities and equations, MacMahon [59–61] primarily developed the method *Partition Analysis*, which was first suggested by Cayley [30] in 1875. The key ingredient of MacMahon's partition analysis is the Omega operator $\Omega_{\geq}$ defined as follows

$$\Omega_{\geq} \sum_{s_1=-\infty}^{\infty} \cdots \sum_{s_r=-\infty}^{\infty} A_{s_1,\ldots,s_r} \lambda_1^{s_1} \cdots \lambda_r^{s_r} := \sum_{s_1=-\infty}^{\infty} \cdots \sum_{s_r=-\infty}^{\infty} A_{s_1,\ldots,s_r},$$

where the $A_{s_1,\ldots,s_r}$ are functions of several complex variables and the λ_i are restricted to a neighborhood of the unit circle. MacMahon also extended this operator to another operator $\Omega_{=}$ which is defined by

$$\Omega_{=} \sum_{s_1=-\infty}^{\infty} \cdots \sum_{s_r=-\infty}^{\infty} A_{s_1,\ldots,s_r} \lambda_1^{s_1} \cdots \lambda_r^{s_r} := \sum_{s_1=-\infty}^{\infty} \cdots \sum_{s_r=-\infty}^{\infty} A_{0,\ldots,0}.$$

The operator $\Omega_{=}$ indeed computes the constant term of the formal series, also denoted by CT (see such as [85]). However, MacMahon failed to prove the major theorems on plane partition using Partition Analysis, and he [58] said "*Our knowledge of the Omega operator is not sufficient to enable us to establish the final form of result. This will be accomplished by the aid of new ideas which will be brought forward in the following chapters.*"

This comment of Partition Analysis almost led to total neglect of this method. Until the early 1970's, Stanley [77] successfully utilized Partition Analysis in his beautiful solution to the Anand-Dumir-Gupta conjecture. In addition, Stanley showed that $f_P(\mathbf{x})$ in (1.1) can be expressed as a sum over linear extensions of P in his monograph [78, Theorem 4.5.4].

Apart from this one shining moment, Partition Analysis has lain dormant. In 1998, Andrews' second paper on Partition Ananlysis [5] sucessfully derives MacMahon's generating function for 2-rowed plane partitions. Then Andrews, Paule and

Riese [4–6, 8, 10–18] computed $f_{P_n}(\mathbf{x})$ for several sequences $\{P_n\}_{n\geq 1}$ of posets by developing MacMahon's partition analysis. For example, they treated solid partitions on a cube, a generalization of Hermite's problem and k-gon partitions in [11]. In most of their investigations, they were aided by the computer algebra package `Omega` that they developed.

In the 2000's, Corteel, Savage [35,36] et. al., presented a new approach to computing generating functions for partition enumeration problems which overcomes the limitations of the constraint matrix technique and that proves as powerful as the Partition Analysis techniques [75]. They gave "five guidelines" to build a recurrence relation of a generating function for the sequences of integers constrained by them. Indeed, these guidelines can be viewed as a simplification of Partition Analysis. They also computed the generating function of 2-rowed partition to illustrate their method.

In 2005, Souza [75] provided a Maple package `GFPartitions` which generates recurrence relations of $f_{P_n}(\mathbf{x})$ once the decomposition of the posets P_n is given manually. Ekhad and Zeilberger [40] considered the umbral operator on "grafting" of posets in 2007. The corresponding Maple package `RotaStanley` can generate the recurrence relations automatically.

In Chapter 2, we [37] present an alternative approach to computing the generating function $f_P(\mathbf{x})$ in (1.1) for a given poset P. We show that the generating function of P-partition of a poset can be expressed in terms of its two transformations: deletion and partially linear extension. As applications, we give a recurrence relations of the generating function $f_P(\mathbf{x})$, where P is a kind of 3-rowed posets which cannot be dealt with by Souza's "five guidelines" [75]. And we also show the generating functions of P-partition with respect to the n-th multi-cube poset, i.e.,

$$f_{C_n}(q) = \frac{P_n(q)}{(q;q)_{4n}}$$

where $n = 1, 2, \ldots, 6$, and $P_n(q)$ is some fixed polynomials of q. Furthermore, in Section 2.4.1 we present an algorithm to compute $f_P(\mathbf{x})$ in (1.1) automatically. Our corresponding `Maple` package is available at

`http://www.combinatorics.net.cn/homepage/hou/.`

1.3 Ramanujan-type Congruences of *P*-partitions

In the groundbreaking observations [70–73], Ramanujan proved that the partition function $p(n)$ satisfies

$$p(An+B) \equiv 0 \pmod{M} \tag{1.4}$$

for all nonnegative integers n and the triples

$$(A,B,M) \in \{(5,4,5),(7,5,7),(11,6,11)\},$$

and he conjectured further such congruences modulo arbitrary powers of 5,7 and 11. In general, congruences of the form (1.4) are called *Ramanujan-type congruences*. For $m=5$ and 7, Watson [82] proved that

$$p(m^k n+\beta_{m,k}) \equiv 0 \pmod{m^{r_k}}, \tag{1.5}$$

where $k \geq 1$, $\beta_{m,k} \equiv 1/24 \pmod{m^k}$, $r_k = k$ for $m=5$ and $r_k = \lfloor k/2 \rfloor + 1$ for $m=7$. Indeed, the case $m=5$ in (1.5) was first considered by Ramanujan, see Berndt and Ono [25]. Atkin [20] showed that (1.5) is also valid for $m=11$. Since then, the problem of finding more such Ramanujan-type congruences has attracted a great deal of attention. For example, subsequent works by Andrews, Atkin, Dyson, Garvan, Kim, Stanton, and Swinnerton-Dyer [3,7,20,23,38,45], in the spirit of Dyson, have a long way towards providing combinatorial and physical explanations for their existence. When M is not a power of 5,7 and 11, until the 1960's, Atkin and O'Brien [22] discovered the following congruence

$$p(11^3 13n+237) \equiv 0 \pmod{13}.$$

In general, congruences of the form (1.4) are exceptionally rare for M not a power of 5,7 or 11. It was until the end of the twentieth century, there are only a handful of such examples. For the cases of A be a prime, Ahlgren and Boylan [2] proved that the congruences proved or conjectured by Ramanujan are the only ones. On the other hand, for the cases of A be nonprime, in a surprising result of piece of work, Ono [66] proved that for any prime $m \geq 5$ and any positive integer k, a positive proportion of the primes ℓ has the property

$$p\left(\frac{m^k \ell^3 n+1}{24}\right) \equiv 0 \pmod{m} \tag{1.6}$$

for every nonnegative integer n coprime to ℓ. Ahlgren [1] then extended this result to composite number m, with $(m,6)=1$.

Both Ono [66] and Ahlgren [1] mainly proved the existence of the congruence of the form (1.6). Weaver [83] then presented an algorithm for generating explicit 76,065 such congruences for primes $m \in \{13,17,19,23,29,31\}$. Chua [34] gave the explicit congruences of the form

$$\sum_{\substack{n\geq 0 \\ m^k n\equiv -1 \pmod{24}}}^{\infty} p\left(\frac{m^k n+1}{24}\right) q^n \equiv \eta(24z)^{r_{m,k}} F_{m,k}(24z) \pmod{m},$$

where $F_{m,k}(z)$ is an explicitly computable level one holomorphic modular forms of small weight, and $r_{m,k}$ is an integer depending on m and k. Then Ahlgren and Boylan [2] extended this result to the cases of powers of primes. Garvan [45] and Yang [86] proved that the space

$$S_{r,s} := \{\eta(24z)^r F(24z) : F(z) \in M_s(SL_2(z))\} \tag{1.7}$$

is an invariant subspace of $S_{s+r/2}(\Gamma_0(576),\chi_{12})$ under the action of Hecke operators. Remarkably, Yang [86, Theorem 3.6] gave an explicit congruence of the form

$$p\left(\frac{m\ell^{2\mu K-1}n+1}{24}\right) q^n \equiv 0 \pmod{m} \tag{1.8}$$

for all positive integers μ and all positive integers n not divisible by ℓ, where K is a computable integer. Recently, Folsom, Kent and Ono [42] provided a very general theorem which gave new generalized partition congruences systematically. In this framework, they proved that if $5\leq m\leq 31$ is a prime and k is a positive integer, then there exists an integer $A_m(b_1,b_2,k)$ such that

$$p\left(\frac{m^{b_1}n+1}{24}\right) \equiv A_m(b_1,b_2,k)\, p\left(\frac{m^{b_2}n+1}{24}\right) \pmod{m^k} \tag{1.9}$$

for all positive integers n and $b_1 \equiv b_2 \pmod{2}$ larger than some fixed integer.

As similar to the ordinary partition, the r-component multipartition possesses a number of outstanding congruences properties. The most well-known examples of these congruences are the Ramanujan-type congruences of $p_r(n)$, which have been extensively studied, see for example [6,21,43,46,48,52,64,81]. Gandhi [43] derived the following congruences of $p_r(n)$ by applying the identities of Euler and Jacobi

$$p_2(5n+3) \equiv 0 \pmod{5}, \tag{1.10}$$

$$p_8(11n+4) \equiv 0 \pmod{11}. \tag{1.11}$$

With the aid of Sturm's theorem [79], Eichhorn and Ono [39] computed an upper bound $C(A,B,r,m^k)$ such that

$$p_r(An+B) \equiv 0 \pmod{m^k}$$

holds for all nonnegative integers n if and only if it is true for $n \leq C(A,B,r,m^k)$. For example, to prove (1.10), it suffices to check that it holds for $n \leq 3$. In the same vain, one can prove (1.11) by verifying that it holds for $n \leq 11$. Treneer [81] extended (1.6) to weakly holomorphic modular forms and showed that for any prime $m \geq 5$ and positive integers k, there is a positive proportion of primes ℓ such that

$$p_r\left(\frac{m^k\ell^{\mu_r}n+r}{24}\right) \equiv 0 \pmod{m}$$

for every nonnegative integer n coprime to ℓ, where μ_r equals 1 if r is even and 3 if r is odd. Using the methods of Folsom, Kent and Ono [42], Belmont et. al. [24, Corollary 1.2] generalized congruence (1.9) to the cases of $p_r(n)$. They proved that if the rank of the corresponding space is no more than 1, then there exists an integer $C_m(r,b_1,b_2,k)$ such that

$$p_r\left(\frac{m^{b_1}n+r}{24}\right) \equiv C_m(r,b_1,b_2,k)p_r\left(\frac{m^{b_2}n+r}{24}\right) \pmod{m^k}, \tag{1.12}$$

where n is a positive integer and $b_1 \equiv b_2 \pmod{2}$ are large enough integers.

None of [39,43,64,71,81] addressed the algorithmic aspect of finding congruences of the form (1.4) for multipartition. Motivated by the works of Weaver, Chua, Ahlgren, Boylan and Yang [2, 34, 83, 86], we get an explicitly modulo formula of the generating function of multipartition sequences in [32]. As an application, in Chapter 3, we shall show a series of Ramanujan-type congruences of $p_r(n)$ including Gandhi's congruences

(1.10) and (1.11). Especially, we find the following new congruences of multipartitions

$$\begin{aligned} p_2(5^2n+23) &\equiv 0 \pmod{5^2}, \\ p_3(11^2n+106) &\equiv 0 \pmod{11^2}, \\ p_4(7^2n+41) &\equiv 0 \pmod{7^2}, \\ p_5(11^2n+96) &\equiv 0 \pmod{11^2}, \\ p_8(11^2n+81) &\equiv 0 \pmod{11^2}. \end{aligned}$$

And we construct two classes of congruences as follows

$$p_r\left(\frac{m^k\ell^{2\mu K-1}n+r}{24}\right) \equiv 0 \pmod{m^k}, \tag{1.13}$$

$$p_r\left(\frac{m^k\ell^i n+r}{24}\right) \equiv p_r\left(\frac{m^k\ell^{2M+i}n+r}{24}\right) \pmod{m^k}, \tag{1.14}$$

where r is an odd integer, ℓ is any prime other than $2,3$ and m, and μ is an arbitrary positive integer, K and M are fixed positive integers, and n is a positive integer coprime to ℓ. For example, we have

$$p_3\left(\frac{5^2 13^{199}n+3}{24}\right) \equiv 0 \pmod{5^2}.$$

Although there are many congruences of multipartitions, numerical evidence suggests that congruences of the following special form only hold for a few small primes

$$p_r(mn+\beta_{m,r}) \equiv 0 \pmod{m}, \tag{1.15}$$

where $m \geq 5$ is prime, and $1 \leq \beta_{m,r} < 24$ is defined by the condition

$$24\beta_{m,r} \equiv r \pmod{m}.$$

A natural problem is whether there exists Ramanujan-type congruence of form (1.15) for $p_r(n)$. Parallel to the cases of ordinary partitions, we have the following Ramanujan's conjecture for multipartitions.

Conjecture 1.1 (Ramanujan's conjecture) *Let r be positive integer. If $m \geq M_r$ is prime, then there are infinitely many integers n for which*

$$p_r(mn+\beta_{m,r}) \not\equiv 0 \pmod{m}.$$

For example, as shown in Table 3.2, one can guess that $M_1 = 13$, $M_2 = 7$, $M_3 = 19$, $M_4 = 11$ and so on. The following classical conjecture, named Newman's conjecture, concerns the distribution of the multipartition function among the complete set of residue classes modulo an integer M.

Conjecture 1.2 (Newman's conjecture) *Let M and r be positive integers, there are infinitely many nonnegative integers n such that*

$$p_r(n) \equiv \gamma \pmod{M}$$

holds for every residue class $\gamma \pmod{M}$.

In 2003, using the theory of modular forms, Bruinier and Ono [28] established an interesting connection between the Newman's conjecture for ordinary partitions and the Ramanujan-type congruences. In 2009, Brown and Li [27] proved that this conjecture is true for some M and r. In Chapter 3, we establish a connection between Conjecture 1.1 and Conjecture 1.2, in the sense that, for a prime modulus $M = m \geq 5$ with $\gcd(r+2, m-1) = 1$, Conjecture 1.2 holds follows from the existence of a single n for which $p_r(mn+\beta_{m,r}) \not\equiv 0 \pmod{m}$.

Chapter 2 Generating Functions of P-partitions

2.1 Introduction

The main objective of this chapter[2] is to find an efficient method of computing $f_P(\mathbf{x})$ with respect to a given poset P. To this end, we introduce two kinds of transformations on posets in Section 2.2, which are the *deletion* and the *partially linear extension*. We show that there exist simple relations between the generating function of P-partitions of a poset and generating functions of its transformations. Applying these two transformations, one can compute the generating function $f_P(\mathbf{x})$ recursively .

In Section 2.3, we consider the poset P_n composed of n copies of a given poset P. More precisely, we introduce the *partially ordinal sum* $\oplus_R$ of posets, and denote

$$P_n := P \oplus_R P \oplus_R \cdots \oplus_R P,$$

where P occurs n times. By applying the deletions and the partially linear extensions, we find that the sequence $\{f_{P_n}(\mathbf{x})\}_{n\geq 1}$ satisfies a system of *substituted recurrence relations* with respect to n.

Finally, the corresponding algorithm is presented in Section 2.4. And we provide several examples in Section 2.4, including a kind of 3-rowed posets which cannot be dealt with by the packages `GFPartitions` and `RotaStanley`, and multi-cube which cannot be solved by the package `Omega`. For example, we have

$$f_{C_6}(q) = \frac{q^{192} + 2q^{190} + \cdots + 40660110q^{96} + \cdots + 2q^2 + 1}{(q;q)_{24}},$$

where C_6 is the 6th multi-cube poset.

2.2 Two Transformations on Posets

In this section, we introduce the deletions and the partially linear extensions on posets. On one hand, they reduce a poset to simpler ones. On the other hand, the generating

[2]The content of this chapter is largely taken from a joint paper with Hou [37].

function of P-partitions of a poset can be expressed in terms of generating functions of its transformations. This enables us to compute $f_P(\mathbf{x})$ by recursively transforming the given poset P.

2.2.1 The Deletion

The first transformation we consider is removing an element from a poset. We are only concern with the *removable* elements, which cover at most one element and are covered by at most one element.

Recall that an element of a poset is *minimal* if it covers no element and *maximal* if no element covers it. Therefore, in terms of the number of elements that cover or are covered by b, there are four cases of removable elements as follows.

1. There exist a and c such that $a \lessdot b \lessdot c$.
2. There exists a such that $a \lessdot b$ and b is maximal.
3. There exists c such that $b \lessdot c$ and b is minimal.
4. b is an isolated element.

Let P be a poset and $b \in P$ be a removable element of P. The deletion of P with respect to b is the transformation from P to the induced sub-poset $P \setminus \{b\}$ of P by deleting b from P. We shows the relation between $f_P(\mathbf{x})$ and $f_{P\setminus\{b\}}(\mathbf{x})$ in the following theorem.

Theorem 2.1 *Let P be a poset and $b \in P$ be a removable element of P. Then*

$$f_P(\mathbf{x}) = \frac{g(\mathbf{x}) - h(\mathbf{x})}{1 - x_b}, \tag{2.1}$$

where

$$g(\mathbf{x}) = \begin{cases} f_{P\setminus\{b\}}(\mathbf{x})|_{x_c = x_b x_c} & \text{if } \exists c \in P \text{ such that } b \lessdot c, \\ f_{P\setminus\{b\}}(\mathbf{x}) & \text{otherwise,} \end{cases}$$

and

$$h(\mathbf{x}) = \begin{cases} x_b f_{P\setminus\{b\}}(\mathbf{x})|_{x_a = x_a x_b} & \text{if } \exists a \in P \text{ such that } a \lessdot b, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. We only give the proof for the case when there exist a and c such that $a \lessdot b \lessdot c$. The other three cases can be proved in a similar way.

By the definition of P-partitions, we have

$$\begin{aligned}
f_P(\mathbf{x}) &= \sum_{\sigma\in Par(P)} \prod_{u\in P} x_u^{\sigma(u)} \\
&= \sum_{\sigma\in Par(P\setminus\{b\})} \left(\sum_{m=\sigma(c)}^{\sigma(a)} x_b^m \prod_{u\in P\setminus\{b\}} x_u^{\sigma(u)} \right) \\
&= \sum_{\sigma\in Par(P\setminus\{b\})} \left(\frac{x_b^{\sigma(c)} - x_b^{\sigma(a)+1}}{1-x_b} \prod_{u\in P\setminus\{b\}} x_u^{\sigma(u)} \right) \\
&= \frac{1}{1-x_b} \sum_{\sigma\in Par(P\setminus\{b\})} (x_b x_c)^{\sigma(c)} \prod_{u\in P\setminus\{b,c\}} x_u^{\sigma(u)} \\
&\quad - \frac{x_b}{1-x_b} \sum_{\sigma\in Par(P\setminus\{b\})} (x_a x_b)^{\sigma(a)} \prod_{u\in P\setminus\{a,b\}} x_u^{\sigma(u)},
\end{aligned}$$

as desired. ∎

Now we give an example to illustrate the applications of Theorem 2.1.

Example 2.1 Let us consider the Diamond poset D as shown in Figure 1.2. We see that 2 is a removable element of D and $D\setminus\{2\}$ is a chain C of length 3 *with*

$$f_C(x_1,x_2,x_3) = \frac{1}{(1-x_1)(1-x_1x_2)(1-x_1x_2x_3)}.$$

Invoking equation (2.1)*, see Figure 2.1, we derive that*

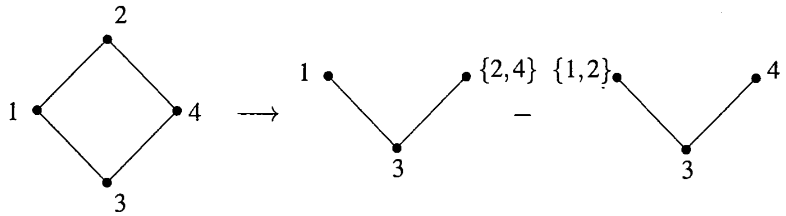


Figure 2.1 Deletion of the Diamond poset.

$$\begin{aligned}
f_D(\mathbf{x}) &= \frac{1}{1-x_2}\left(f_C(x_1,x_3,x_2x_4) - x_2 f_C(x_1x_2,x_3,x_4)\right) \\
&= \frac{1-x_1^2x_2x_3}{(1-x_1)(1-x_1x_2)(1-x_1x_3)(1-x_1x_2x_3)(1-x_1x_2x_3x_4)}.
\end{aligned}$$

2.2.2 The Partially Linear Extension

The second transformation we consider is partially ordering the elements of an anti-chain of a poset.

Let A be an anti-chain of a poset P and let M be a non-empty subset of A. *The partially linear extension* (*PLE* in short) of P with respect to the pair (M,A) is the transformation from the poset P to the poset $P(M,A)$ by gluing the elements of M together and setting the glued element cover the elements of $A\setminus M$. More precisely, $P(M,A)$ is the poset defined on $P\setminus M\cup\{M\}$, in which M is viewed as a whole point, and partially ordered by $x\le y$ if and only if one of the following conditions holds.

1. $x,y\in P\setminus M$ and $x\le_P y$.
2. $x,y\in P\setminus M$ and there exist $x'\in A$ and $y'\in M$ such that $x\le_P x', y'\le_P y$.
3. $x=M$ and there exists $y'\in M$ such that $y'\le_P y$.
4. $y=M$ and there exists $x'\in A$ such that $x\le_P x'$.
5. $x=y=M$.

Example 2.2 Let $P=\{1,2,3,4,5\}$ *be the poset as shown in Figure 2.2. Then the posets*

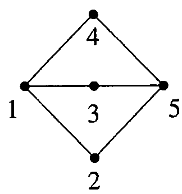
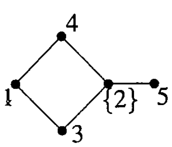


Figure 2.2 The Hasse diagram of the poset P.

$P(\{2\},\{2,3,4\})$ *and* $P(\{2,3\},\{2,3,4\})$ *are given as in Figure 2.3.*

$P(\{2\},\{2,3,4\}) =$ 

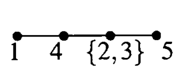
$P(\{2,3\},\{2,3,4\}) =$ 

Figure 2.3 Two partially linear extensions of P.

As a generalization of Theorem 2.3 in [44], in the following theorem, we show that the generating function of P-partitions of a poset can be expressed by the generating functions of its PLE's.

Theorem 2.2 *Let P be a poset and A be an anti-chain of P. Then*

$$f_P(\mathbf{x}) = \sum_{\emptyset \neq M \subseteq A} (-1)^{|M|-1} f_{P(M,A)}(\mathbf{x})|_{x_M = \prod_{a\in M} x_a}. \tag{2.2}$$

Proof. For each $a \in A$, we define

$$S_a = \{\sigma \in \mathrm{Par}(P) \colon \sigma(a) \leq \sigma(x), \forall x \in A\}.$$

Since $\mathrm{Par}(P) = \cup_{a\in A} S_a$, by the inclusion-exclusion principle we derive that

$$f_P(\mathbf{x}) = \sum_{\emptyset \neq M \subseteq A} (-1)^{|M|-1} \sum_{\sigma \in S_M} \prod_{u \in P} x_u^{\sigma(u)},$$

where $S_M = \bigcap_{a\in M} S_a$.

Given a P-partition $\sigma \in S_M$, we denote $m = \min\{\sigma(a) \colon a \in A\}$. Then by the definition of S_M we have $\sigma(x) = m$ for any $x \in M$. Let

$$\sigma'(u) = \begin{cases} \sigma(u) & u \in P \setminus M, \\ m & u = M. \end{cases}$$

One sees that σ' is a P-partition of $P(M,A)$. Conversely, let σ' be a P-partition of $P(M,A)$. By defining

$$\sigma(u) = \begin{cases} \sigma'(u) & u \in P \setminus M, \\ \sigma'(M) & u \in M, \end{cases}$$

we obtain a P-partition of P in S_M. We thus set up a one-to-one correspondence between the P-partitions of P in S_M and the P-partitions of $P(M,A)$. Therefore,

$$\begin{aligned} f_P(\mathbf{x}) &= \sum_{\emptyset \neq M \subseteq A} (-1)^{|M|-1} \sum_{\sigma' \in \mathrm{Par}(P(M,A))} \prod_{u\in M} x_u^{\sigma'(M)} \cdot \prod_{u \in P\setminus M} x_u^{\sigma'(u)} \\ &= \sum_{\emptyset \neq M \subseteq A} (-1)^{|M|-1} f_{P(M,A)}(\mathbf{x})|_{x_M = \prod_{a\in M} x_a}. \end{aligned}$$

This completes the proof. ∎

The Hasse diagrams provide a simple graphical representation for equation (2.2), see the following example.

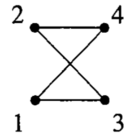


Figure 2.4 The Hasse diagram of the poset P.

Example 2.3 Let $P=\{1,2,3,4\}$ be the poset as shown in Figure 2.4. Taking the antichain $\{1,2\}$ into account, we find that the PLE's are shown as in Figure 2.5, from which we read out

$$f_P(\mathbf{x}) = f_{P(\{1\},\{1,2\})}(x_2,x_1,x_3,x_4) + f_{P(\{2\},\{1,2\})}(x_1,x_2,x_3,x_4) - f_{P(\{1,2\},\{1,2\})}(x_1x_2,x_3,x_4). \quad (2.3)$$

Since 3 is a common removable element of $P(\{1\},\{1,2\}), P(\{2\},\{1,2\})$ and $P(\{1,2\},\{1,2\})$,

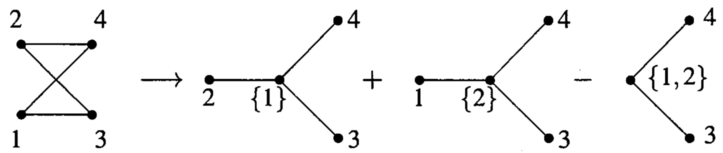


Figure 2.5 A graphical representation of Equation (2.2).

as in Example 2.1, we have the following generating functions

$$f_{P(\{1\},\{1,2\})}(\mathbf{x}) = \frac{1-x_1^2x_2^2x_3x_4}{(1-x_2)(1-x_1x_2)(1-x_1x_2x_3)(1-x_1x_2x_4)(1-x_1x_2x_3x_4)},$$

$$f_{P(\{2\},\{1,2\})}(\mathbf{x}) = \frac{1-x_1^2x_2^2x_3x_4}{(1-x_1)(1-x_1x_2)(1-x_1x_2x_3)(1-x_1x_2x_4)(1-x_1x_2x_3x_4)},$$

$$f_{P(\{1,2\},\{1,2\})}(\mathbf{x}) = \frac{1-x_1^2x_2^2x_3x_4}{(1-x_1x_2)(1-x_1x_2x_3)(1-x_1x_2x_4)(1-x_1x_2x_3x_4)}.$$

Substituting the above three generating functions into formula (2.3), *we have*

$$f_P(\mathbf{x}) = \frac{1-x_1^2x_2^2x_3x_4}{(1-x_1)(1-x_2)(1-x_1x_2x_3)(1-x_1x_2x_4)(1-x_1x_2x_3x_4)}.$$

2.2.3 Computing $f_P(\mathbf{x})$ via Two Transformations

In this part, we shall show that the deletion and the partially linear extension are powerful enough for computing $f_P(\mathbf{x})$ for any poset P. Specifically, we have

Theorem 2.3 *Any poset can be reduced to the empty poset by applying the deletion and the partially linear extension finite times.*

Proof. Let $ac(P)$ denote the number of distinct anti-chains of a poset P. Since each element of P forms an anti-chain, the deletion reduces $ac(P)$ by at least one.

Suppose that A is an anti-chain of P with cardinality at least two and M is a non-empty subset of A. If $M \neq A$, there exist $x \in M$ and $y \in A \setminus M$. Then $\{x,y\}$ is an anti-chain of P but is not an anti-chain of $P(M,A)$. If $M = A$, the cardinality of $P(M,A)$ is strictly less than that of P. Thus in either case, we have $ac(P(M,A)) \leq ac(P) - 1$.

Now iteratively apply the deletion whenever there is a removable element and apply the partially linear extension whenever there is an anti-chain with cardinality at least two. Since $ac(P)$ is a finite number, the procedure eventually stops. The final poset contains no removable element and no anti-chain with cardinality at least two. The only poset satisfying this property is the empty poset. ∎

2.3 Partially Ordinal Sums

In this section, we consider a kind of posets composed of small blocks. We generalize the direct sums and the ordinal sums of posets by giving the definition of partially ordinary sum of two posets as follows.

Definition 2.1 *Let P, Q be two posets and R be a subset of the Cartesian product $P \times Q$. The* partially ordinal sum *(or R-*plus*, for short) of P and Q with respect to R is the poset $P \oplus_R Q$ defined on the disjoint union of P and Q and partially ordered by $x \leq y$ in $P \oplus_R Q$ if and only if*

1. *$x, y \in P$ and $x \leq_P y$, or*
2. *$x, y \in Q$ and $x \leq_Q y$, or*
3. *$x \in P, y \in Q$ and there exists $(x', y') \in R$ such that $x \leq_P x'$ and $y' \leq_Q y$.*

As special cases, R-plus reduces to the direct sum if $R = \emptyset$ and to the ordinal sum if $R = P \times Q$, respectively.

Example 2.4 Figure 2.6 gives the R-plus of P and Q, where $P = \{1,2\}$ is an anti-chain, $Q = \{3,4\}$ is a chain, and

$$R = \{(1,4),(2,3),(2,4)\}.$$

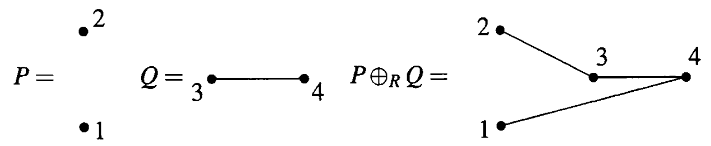


Figure 2.6 The partially ordinal sum of P and Q w.r.t. R.

It is easy to check that the partially ordinal sum is associative up to isomorphic. Therefore we can naturally extend the definition of partially ordinal sum of two posets to several posets. In particular, given a poset P and $R \subseteq P \times P$, we can define the partially ordered sum of n copies of P: $P \oplus_R P \oplus_R \cdots \oplus_R P$, where P appears n times. We denote the poset by P_R^n for brevity.

The sequence $\{f_{P_R^n}(\mathbf{x})\}_{n\geq 1}$ satisfies a kind of recurrence relation which is given as follows.

Definition 2.2 *We say a sequence* $\{f_n(\mathbf{x})\}_{n\geq 1}$ *of functions is* substituted recursive *if there are finitely many sequences*

$$\{f_n^{(0)}(\mathbf{x})\}_{n\geq 1},\ \{f_n^{(1)}(\mathbf{x})\}_{n\geq 1},\ \ldots,\ \{f_n^{(I)}(\mathbf{x})\}_{n\geq 1}$$

such that $f_n^{(0)}(\mathbf{x}) = f_n(\mathbf{x})$ *and for* $i = 0, 1, \ldots, I$,

$$f_n^{(i)}(\mathbf{x}) = \sum_{j=0}^{I} \sum_{k=0}^{K} r_{i,j,k}(\mathbf{x}) f_{n-1}^{(j)}(\mathbf{y}^{(j,k)}), \tag{2.4}$$

where $r_{i,j,k}(\mathbf{x})$ *are rational functions and each component of the variable vector* $\mathbf{y}^{(j,k)}$ *is a monomial in* $x_1, \ldots, x_n$.

For example, suppose that

$$f_n(x_1, \ldots, x_n) = \frac{1}{1-x_1} g_{n-1}(x_2, \ldots, x_n)$$

and

$$g_n(x_1, \ldots, x_n) = \frac{g_{n-1}(x_1x_2, x_3, \ldots, x_n)}{1-x_1x_2} - \frac{g_{n-1}(x_1, x_2x_3, x_4, \ldots, x_n)}{1-x_1x_3}.$$

Then both $\{f_n(\mathbf{x})\}_{n\geq 1}$ and $\{g_n(\mathbf{x})\}_{n\geq 1}$ are substituted recursive.

To compute $f_{P_R^n}(\mathbf{x})$, we consider the more general posets

$$X_n = A \oplus_{R_1} P_R^n \oplus_{R_2} B, \tag{2.5}$$

where A, B are posets and $R_1 \subseteq A \times P$, $R_2 \subseteq P \times B$. In the following theorem, we present an important property of the generating function $f_{X_n}(\mathbf{x})$.

Theorem 2.4 *Let X_n be given as in* (2.5). *Then the sequence $\{f_{X_n}(\mathbf{x})\}_{n\geq 1}$ of generating functions of P-partitions of X_n is substituted recursive.*

Proof. Let $\mathscr{S}$ denote the set of all pairs (C, R') such that C is a chain, $R' \subseteq C \times P$, and none of the elements of C in $C \oplus_{R'} P$ is removable. Since each element of C is not removable, it must be covered by a certain element of P. Moreover, two distinct elements of C can not be covered by the same element in P. This implies that the cardinality of C is less than that of P. Therefore, $\mathscr{S}$ is a finite set.

Now we apply the deletion and the partially linear extension to the elements of $A \oplus_{R_1} P$ in X_n whenever possible. As shown in the proof of Theorem 2.3, we eventually arrive at posets of the form $C \oplus_{R'} P_R^{n-1} \oplus_{R_2} B$ with $(C, R') \in \mathscr{S}$. Moreover, we have

$$f_{X_n}(\mathbf{x}) = \sum_{(C,R')\in\mathscr{S}} r(C,R',\mathbf{x}) f_{C\oplus_{R'}P_R^{n-1}\oplus_{R_2}B}(\mathbf{y}^{(C,R')}),$$

where $r(C, R', \mathbf{x})$ are rational functions of $\mathbf{x}$ depending on C and R', and each component of the variable vector $\mathbf{y}^{(C,R')}$ is a monomial in $\mathbf{x}$. By a similar discussion, for each $(C, R') \in \mathscr{S}$ we have

$$f_{C\oplus_{R'}P_R^n\oplus_{R_2}B}(\mathbf{x}) = \sum_{(C',R'')\in\mathscr{S}} r'(C,R',C',R'',\mathbf{x}) f_{C'\oplus_{R''}P_R^{n-1}\oplus_{R_2}B}(\mathbf{y}^{(C,R',C',R'')}),$$

where $r'(C, R', C', R'', \mathbf{x})$ are rational functions of $\mathbf{x}$ that depend on C, R', C' and R''. This completes the proof. ∎

2.4 Algorithms and Examples

In this section, we first give an algorithm for computing the generating functions of P-partitions, and then some examples will be presented to illustrate our approach to the computation of $f_{X_n}(\mathbf{x})$. We begin with an introductory example, i.e., the 3-rowed plane partition introduced by Souza [75]. Then we provide more examples, including the zigzag posets, the 2-rowed posets with double diagonals and the multi-cube posets.

In these examples, we consider the q-generating function $f_P(q)$ obtained from $f_P(\mathbf{x})$ by setting all variables x_i equal to the indeterminant q. For brevity, we omit some variables equalling to q and write $f(x_1,x_2,\dots,x_k)$ instead of $f(x_1,x_2,\dots,x_k,q,q,\dots,q)$. We also adopt the following standard notation

$$\begin{aligned}(a;q)_n &:= (1-a)(1-aq)\cdots(1-aq^{n-1}),\\ (a;q)_\infty &:= \prod_{n=0}^{\infty}(1-aq^n).\end{aligned}$$

2.4.1 Algorithms for Computing $f_{X_n}(\mathbf{x})$

Note that the proof of Theorem 2.4 provides an algorithm for generating substituted recurrence relations of $f_{X_n}(\mathbf{x})$, where

$$X_n = A \oplus_{R_1} P_R^n \oplus_{R_2} B,$$

as defined by (2.4), A,B are posets and $R_1 \subseteq A\times P$, $R_2 \subseteq P\times B$ are Cartesian product.

Algorithm Reduce

Input: a poset Q and a set $\mathscr{S}$ of pairs (P,R) where P is a poset and $R\subset P\times Q$.

Output: an irreducible set $\mathscr{S}'$ which can be obtained from $\mathscr{S}$ by applying the reduce transformation recursively.

Step 1. Initially set $\mathscr{S}'=\mathscr{S}$.

Step 2. Suppose there exists a pair $(P,R)\in\mathscr{S}'$ with an element $b\in P$ such that b is removable in $P\oplus_R Q$. Then we remove the pair (P,R) from $\mathscr{S}'$ and add all pairs in $\mathscr{D}(P,R,b)$. Repeat Step 2 until there is no such poset and b.

Step 3.

(a). If there is $(P,R)\in\mathscr{S}'$ such that P is not a chain, we choose an anti-chain A of P of cardinality larger than one. Then remove the pair (P,R) from $\mathscr{S}'$ and add all the pairs in $\mathscr{L}(P,R,A)$. Go to Step 2.

(b). If for each pair $(P,R)\in\mathscr{S}'$, P is a chain, return the set $\mathscr{S}'$ and the algorithm terminates.

Remark. $\mathscr{S}'$ may contain only the special element $(\emptyset,\emptyset)$. Consider, for example, $Q=\{2\}$ and $\mathscr{S}=\{(\{1\},\{(1,2)\})\}$.

Algorithm GFR

Input: posets A,P and two relations $R_1\subseteq A\times P$, $R\subseteq P\times P$.

Output: a set $\mathscr{S}$ consists of pairs (C,R'), where C is a chain and $R'\subseteq C\times P$.

Step 1. Initially set $\mathscr{S} = \{(A \oplus_{R_1} P, R)\}$.
Step 2. Choose a pair $(A', R') \in \mathscr{S}$, and apply the `Algorithm Reduce` to obtain a set $\mathscr{S}'$. Remove the pair (A', R') from $\mathscr{S}$, and add all the elements in $\mathscr{S}'$. Repeat the procedure until $\mathscr{S}$ does not change any more.
Step 3. Return $\mathscr{S}$.

One can see that Theorem 2.3 and Theorem 2.4 ensure the correctness of `Algorithm Reduce` and `Algorithm GFR`, respectively. The corresponding `Maple` package is available at

`http://www.combinatorics.net.cn/homepage/hou/.`

2.4.2 An Introductory Example

Souza [75] introduced the 3-rowed plane partition whose corresponding poset P_n is given by Figure 2.7. He failed to find out recurrence relations of the generating function $f_{P_n}(\mathbf{x})$. Our approach gives substituted recurrence relations of $f_{P_n}(\mathbf{x})$.

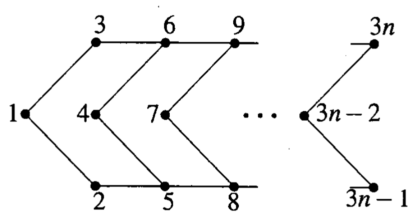


Figure 2.7 The graphical representation of the 3-rowed poset P_n.

It is easy to see that $P_n = P_R^n$, where P is the poset on $\{1,2,3\}$ with $1 < 2$ and $1 < 3$, and $R = \{(2,2),(3,3)\}$.

By deleting the removable elements 2 and 3 of P, we reduce the poset P_n to

$$Q_{n-1} = \mathbf{1} \oplus_{R'} P_R^{n-1},$$

where $\mathbf{1}$ is the poset with only one element and

$$R' = \{(1,2),(1,3)\} \subset \mathbf{1} \times P.$$

See Figure 2.8 for a demonstration. According to Theorem 2.1, we derive that

$$f_{P_n}(x_1,x_2,x_3) = \frac{1}{(1-x_2)(1-x_3)}$$
$$\times \big(f_{Q_{n-1}}(x_1,q,qx_2,qx_3) - x_2 f_{Q_{n-1}}(x_1x_2,q,q,qx_3)$$
$$- x_3 f_{Q_{n-1}}(x_1x_3,q,qx_2,q) + x_2x_3 f_{Q_{n-1}}(x_1x_2x_3,q,q,q)\big). \quad (2.6)$$

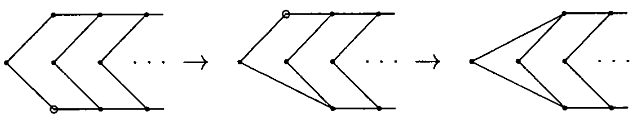


Figure 2.8 The transformation from P_n to Q_{n-1}.

Now let us consider $Q_n = \mathbf{1} \oplus_{R'} P_R^n$. It is readily to see that the unique element of $\mathbf{1}$ and the minimal element of P are not comparable. Thus we can apply PLE to the anti-chain consisting of these two elements, as shown in Figure 2.9.

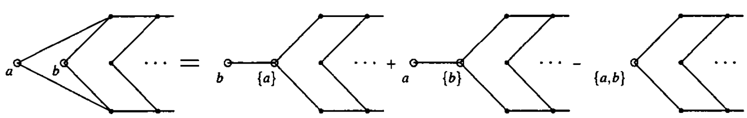


Figure 2.9 The PLE transformation of Q_n.

After further deletions, all the posets generated by the PLE transformation reduce to Q_{n-1}. We thus obtain the recurrence relation

$$f_{Q_n}(x_1,x_2,x_3,x_4) = \frac{1-x_1x_2}{(1-x_1)(1-x_2)(1-x_3)(1-x_4)}$$
$$\times \big(f_{Q_{n-1}}(x_1x_2,q,qx_3,qx_4) - x_3 f_{Q_{n-1}}(x_1x_2x_3,q,q,qx_4)$$
$$- x_4 f_{Q_{n-1}}(x_1x_2x_4,q,qx_3,q) + x_3x_4 f_{Q_{n-1}}(x_1x_2x_3x_4,q,q,q)\big). \quad (2.7)$$

By the recurrence relations (2.6) and (2.7) and the initial condition

$$f_{Q_1}(x_1,x_2,x_3,x_4) = \frac{1-x_1^2x_2^2x_3x_4}{(1-x_1)(1-x_2)(1-x_1x_2x_3)(1-x_1x_2x_4)(1-x_1x_2x_3x_4)},$$

we can compute $f_{P_n}(q)$ recursively.

2.4.3 More Examples

In this subsection, we give five more examples to illustrate our method.

Example 2.5 Let $P=\{1,2\}$ be a chain with $2 \lessdot 1$ and let $R=\{(2,1)\}$. The zigzag poset of length n is given by $Z_n = P_R^n$. We have

$$f_{Z_n}(x_1,x_2) = \frac{f_{Z_{n-1}}(qx_2,q)}{(1-x_1)(1-x_2)} - \frac{x_1 f_{Z_{n-1}}(qx_1x_2,q)}{(1-x_1)(1-x_1x_2)}.$$

The initial condition is given by

$$f_{Z_1}(x_1,x_2) = \frac{1}{(1-x_2)(1-x_1x_2)}.$$

Note that the P-partitions of Z_n is exactly the up-down sequences defined by Carlitz [29].

Example 2.6 Consider the plane partition diamonds D_n depicted in Figure 2.10, we have

$$f_{D_n}(q) = \frac{(-q^2,q^3)_n}{(q;q)_{3n+1}}. \tag{2.8}$$

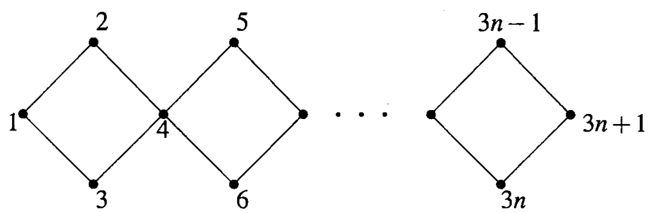


Figure 2.10 Plane partition diamonds D_n.

Andrews, Paule and Riese first introduced the plane partition diamonds D_n in [15]. Using MacMahon's Partition Analysis, they obtained the generating function of this family. The simplest case of plane partition diamonds, D_1, has exhibited in Figure 1.2. Now we consider the more general case, i.e., plane partition diamonds D_n by `Algorithm GFR`.

Let $f_n(\mathbf{x})$ be the generating function of P-partition related to diamonds D_n. Since 2 is an removable element of D_n, according to Theorem 2.1 we have

$$f_{D_n}(\mathbf{x}) = \frac{1}{1-x_2}\left(g_n(\mathbf{x})|_{x_4=x_2x_4} - x_2 g_n(\mathbf{x})|_{x_1=x_1x_2}\right), \tag{2.9}$$

where $g_n(\mathbf{x})$ is the generating function of P-partition related to poset $D_n \setminus \{2\}$, which equals to

$$g_n(\mathbf{x}) = \frac{1}{1-x_1}\frac{1}{1-x_1x_3} f_{n-1}(\mathbf{x})|_{x_4=x_1x_3x_4}.$$

Substituting the above generating function into the formula (2.11) gives a recursion for $f_n(\mathbf{x})$ as follows

$$f_{D_n}(\mathbf{x}) = \frac{1-x_1^2x_2x_3}{(1-x_1)(1-x_1x_2)(1-x_1x_3)(1-x_1x_2x_3)} f_{D_{n-1}}(\mathbf{x})|_{x_4=x_1x_2x_3x_4},$$

with initial value $f_{D_1}(x_1,x_2,x_3,x_4)$, which have obtained in Example 2.1. It is straightforward to show by induction that $f_n(\mathbf{x})$ satisfies

$$f_{D_n}(\mathbf{x}) = \frac{\prod_{i=1}^{n}\left(1-\prod_{j=1}^{3i-2} x_j^2 x_{3i-1}x_{3i}\right)}{\prod_{i=1}^{3n+1}\left(1-\prod_{j=1}^{i} x_j\right)\prod_{i=1}^{n}\left(1-\prod_{j=1}^{3i-2} x_j x_{3i}\right)}.$$

Finally, substituting all $x_i = q$ gives the generating function (2.8).

Example 2.7 Let P_n be the 2-rowed plane partition as show in Figure 2.11, then

$$f_{P_n}(q) = \frac{1}{(q;q)_n(q^2;q)_n}. \tag{2.10}$$

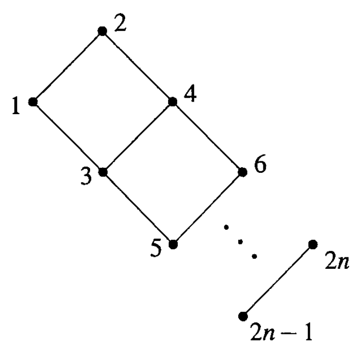


Figure 2.11 2-rowed plane partition P_n.

MacMahon [63] first considered the 2-rowed plane partition, and he gave the following generating function (2.10). In [5], Andrews proved (2.10) using MacMahon's Partition Analysis. Corteel, Souza approached the same problem in [75] using the Seven Guidelines technique. We now demonstrate the ease with `Algorithm GFR`.

Let $f_{P_n}(\mathbf{x})$ be the generating function of P-partition of poset P_n. For the removable element 2 of P_n, in terms of Theorem 2.1, we get

$$f_{P_n}(\mathbf{x}) = \frac{1}{1-x_2}\left(g_n(\mathbf{x})|_{x_4=x_2x_4} - x_2 g_n(\mathbf{x})|_{x_1=x_1x_2}\right), \tag{2.11}$$

where $g_n(\mathbf{x})$ is the generating function of P-partition of poset $P_n \setminus \{2\}$, which equals to

$$g_n(\mathbf{x}) = \frac{1}{1-x_1} f_{n-1}(\mathbf{x})|_{x_3=x_1x_3}.$$

Substituting the above generating function into the formula (2.11), we have the following recursion

$$\begin{aligned} f_{P_n}(x_1,x_2,\ldots,x_{2n}) &= \frac{1}{(1-x_1)(1-x_2)} f_{P_{n-1}}(x_1x_3,x_2x_4,\ldots,x_{2n}) \\ &\quad -\frac{x_2}{(1-x_2)(1-x_1x_2)} f_{P_{n-1}}(x_1x_2x_3,x_4,\ldots,x_{2n}). \end{aligned}$$

Set

$$f_{P_n}(x_1,x_2) = f_{P_n}(x_1,x_2,q,q,\ldots,q),$$

then the recurrence for $f_n(\mathbf{x})$ can be rewrite as

$$f_{P_n}(x_1,x_2) = \frac{f_{P_{n-1}}(x_1q,x_2q)}{(1-x_1)(1-x_2)} - \frac{x_2 f_{P_{n-1}}(x_1x_2q,q)}{(1-x_2)(1-x_1x_2)},$$

with initial condition $f_1(x_1,x_2) = 1/(1-x_1)/(1-x_1x_2)$. It is straightforward to verify by induction that

$$f_{P_n}(x_1,x_2) = \frac{g_n(x_1,x_2)}{(x_1;q)_n(x_1x_2;q)_n},$$

where $g_n(x_1,x_2)$ is a rational function of x_1 and x_2 with $g_n(q,q)=1$. Then, setting $x_1=x_2=q$, we arrives at the generating function (2.10).

Example 2.8 Let P_n be the 2-rowed poset with double diagonals depicted in Figure 2.12. Then we have

$$f_{P_n}(q) = \frac{(-q^2;q^2)_{n-1}}{(q;q)_{2n}}. \tag{2.12}$$

Davis, Souza, Lee and Savage [36] used the "digraph method" to derive formula (2.12). Using the inclusion-exclusion principle, Gao, Hou and Xin [44] obtained the same generating function.

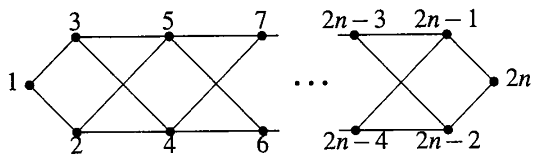


Figure 2.12 The 2-rowed poset with double diagonals.

Let $f_{P_n}(\mathbf{x})$ be the generating function of P-partition related to poset P_n. Since $\{2,3\}$ is an anti-chain of length 2 of P_n, according to Theorem 2.2 we have

$$f_{P_n}(\mathbf{x}) = f_{P_n(\{2\},\{2,3\})}(\mathbf{x}) + f_{P_n(\{3\},\{2,3\})}(\mathbf{x}) - f_{P_n(\{2,3\},\{2,3\})}(\mathbf{x}). \tag{2.13}$$

We can readily to see that

$$\begin{aligned} f_{P_n(\{2\},\{2,3\})}(\mathbf{x}) &= \frac{1}{1-x_1x_3} f_{P_n(\{2,3\},\{2,3\})}(\mathbf{x}), \\ f_{P_n(\{3\},\{2,3\})}(\mathbf{x}) &= \frac{1}{1-x_1x_2} f_{P_n(\{2,3\},\{2,3\})}(\mathbf{x}), \end{aligned}$$

and

$$f_{P_n(\{2,3\},\{2,3\})}(\mathbf{x}) = \frac{1}{1-x_1} f_{n-1}(\mathbf{x})|_{x_3=x_1x_2x_3}.$$

Substituting the above three formulas into (2.13) gives a resursion of $f_{P_n}(\mathbf{x})$, namely

$$f_{P_n}(x_1,x_2,x_3,\ldots) = \frac{1-x_1^2x_2x_3}{(1-x_1)(1-x_1x_2)(1-x_1x_3)} f_{P_{n-1}}(x_1x_2x_3,x_4,x_5,\ldots),$$

with initial value

$$f_2(x_1,x_2,x_3,x_4) = f_{D_1}(x_1,x_2,x_3,x_4),$$

which have been obtained in Example 2.1. Iterating the recursion for $f_{P_n}(\mathbf{x})$ gives

$$f_{P_n}(\mathbf{x}) = \frac{\prod_{i=1}^{n-1}(1-X_{2i-1}X_{2i}x_{2i+1})}{\prod_{i=1}^{2n}(1-X_i)\prod_{i=1}^{n-1}(1-X_{2i-1}x_{2i+1})}.$$

Finally, substituting all $x_i = q$, we arrive at the generating function (2.12).

Example 2.9 Let $D = \{1,2,3,4\}$ be the Diamond poset shown as in Figure 1.2 and

$$R = \{(1,1),(2,2),(3,3),(4,4)\}.$$

The n-th multi-cube poset C_n is defined by $C_n = D_R^n$, as shown in Figure 2.13. Using the substituted recurrence relations, we compute $f_{C_n}(q)$ for $n \le 6$. For example,

$$f_{C_6}(q) = \frac{q^{192}+2q^{190}+\cdots+40660110q^{96}+\cdots+2q^2+1}{(q;q)_{24}}.$$

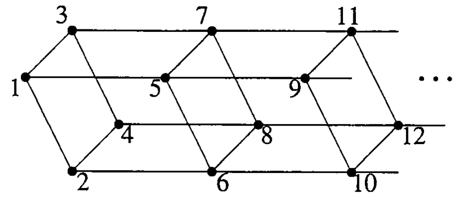


Figure 2.13 The graphical representation of multi-cube posets.

MacMahon devoted *Art.* 98 of [62, Section 7] to be consideration of the simplest "*lattice in solido*", namely, the lattice "in which the points are the summits of a cube and the branches the edges of the cube". In other words, following MacMahon, the Hasse diagram of cube as described by Figure 2.13 with $n=2$.

Under the framework of Partition Analysis, Andrews, Paule and Rises [11] find the generating function of P-partitions related to D_R^2 by the Omega package as follows

$$f_{C_2}(q)=\frac{P_2(q)}{(q;q)_8}, \tag{2.14}$$

where $P_2(q)$ is a fixed polynomial of q with order 16. For lager n, this package, unfortunately, is too slow to find out the corresponding generating functions of P-partitions related to D_R^n.

Using our method, we would obtain a system of substituted recurrence relations for the generating functions $f_{C_n}(\mathbf{x})$ as follows.

$$\begin{aligned}
&f_{C_n}(x_1,x_2,x_3,x_4,\ldots)\\
=&R_1(\mathbf{x})f_{C_{n-1}}(x_1x_3x_5,x_2x_6,x_7,x_4x_8,\ldots)-R_2(\mathbf{x})f_{C_{n-1}}(x_1x_3x_5,x_2x_4x_6,x_7,x_8,\ldots)\\
&+R_3(\mathbf{x})f_{C_{n-1}}(x_1x_2x_5,x_6,x_3x_7,x_4x_8,\ldots)-R_4(\mathbf{x})f_{C_{n-1}}(x_1x_2x_5,x_6,x_3x_4x_7,x_8,\ldots)\\
&-R_5(\mathbf{x})f_{C_{n-1}}(x_1x_2x_3x_5,x_6,x_7,x_4x_8,\ldots)-R_6(\mathbf{x})f_{C_{n-1}}(x_1x_2x_3x_4x_5,x_6,x_7,x_8,\ldots),
\end{aligned}$$

where $R_i(\mathbf{x})(i=1,2,\ldots,6)$ are rational functions on $x_1,x_2,\ldots,x_5$ as follows

$$\begin{aligned}
R_1(\mathbf{x}) &= \frac{1-x_1^2x_2x_5}{(1-x_1)(1-x_2)(1-x_4)(1-x_1x_3)(1-x_1x_5)},\\
R_2(\mathbf{x}) &= -\frac{x_4(1-x_1^2x_3x_5)}{(1-x_1)(1-x_4)(1-x_2x_4)(1-x_1x_3)(1-x_1x_5)},
\end{aligned}$$

$$\begin{aligned}
R_3(\mathbf{x}) &= \frac{1-x_1^2x_2x_5}{(1-x_1)(1-x_3)(1-x_4)(1-x_1x_2)(1-x_1x_5)},\\
R_4(\mathbf{x}) &= -\frac{x_4(1-x_1^2x_2x_5)}{(1-x_1)(1-x_4)(1-x_3x_4)(1-x_1x_2)(1-x_1x_5)},\\
R_5(\mathbf{x}) &= -\frac{(1-x_2x_3)(1-x_1^2x_2x_3x_5)}{(1-x_1)(1-x_2)(1-x_3)(1-x_4)(1-x_1x_2x_3)},\\
R_6(\mathbf{x}) &= -\frac{P(\mathbf{x})}{(1-x_1)(1-x_4)(1-x_1x_2)(1-x_1x_3)(1-x_1x_5)}\\
&\quad\times\frac{1}{(1-x_2x_4)(1-x_3x_4)(1-x_1x_2x_3x_4)},
\end{aligned}$$

where $P(\mathbf{x})$ is a polynomial on $x_1,\ldots,x_5$ containing 52 terms of order 14. Since it is too huge, we would not write it here. With the aid of this system, one can see that

$$f_{C_n}(q)=\frac{P_n(q)}{(q;q)_{4n}},$$

where $P_n(q)$ is some fixed polynomials of q. As a trivial example, $P_1(q)=1+q^2$, which leads to the generating function of Diamonds partition. And for $n=2$, we obtain the Andrews's generating function (2.14). Moreover, we have

$$P_3(q)=q^{42}+2q^{40}+\cdots+140q^{21}+\cdots+2q^2+1,$$

$$P_4(q)=q^{80}+2q^{78}+\cdots+7164q^{40}+\cdots+2q^2+1,$$

$$P_5(q)=q^{130}+2q^{128}+\cdots+479990q^{65}+\cdots+2q^2+1,$$

$$P_6(q)=q^{192}+2q^{190}+\cdots+40660110q^{96}+\cdots+2q^2+1.$$

Chapter 3 Arithmetic Properties of P-partitions

3.1 Introduction

The objective of this chapter[3] is to use the theory of modular forms to derive certain congruences of P-partitions.

Section 3.2 presents basic notation and results of the theory of modular forms, containing the elementary tools for computing congruences of P-partitions.

In Section 3.3, we first present a brief overview of the theory of congruences of the broken k-diamond partitions. Then we reprove the following congruences

$$\Delta_2(An+B) \equiv 0 \pmod 3,$$

where

$$(A,B) \in \{(15,1),(15,7),(15,10),(15,13),(27,16),(27,15)\}$$

and $\Delta_2(n)$ denotes the broken 2-diamond partitions.

Section 3.4 is devote to establishing congruences of multipartitions modulo powers of primes. For example, we shall show that

$$p_r\left(\frac{m^k\ell^{2\mu K-1}n+r}{24}\right) \equiv 0 \pmod{m^k}, \tag{3.1}$$

where r is an odd integer, ℓ is a prime other than $2,3$ and m, and μ is a positive integer, K is a fixed positive integer, and n is a positive integer coprime to ℓ.

To derive congruences of $p_r(n)$, one may consider the congruence properties of the generating functions of $p_r(n)$. For the case of ordinary partitions, i.e., $r=1$, Chua [34] showed that

$$\sum_{mn\equiv -1 \pmod{24}} p\left(\frac{mn+1}{24}\right) q^n \equiv \eta(24z)^{\gamma_m}\phi_m(24z) \pmod m, \tag{3.2}$$

[3] The content of this chapter is largely taken from a joint paper with Chen, Hou and Sun [32].

where $\eta(z)$ is Dedekind's eta function, γ_m is an integer depending on m and $\phi_m(z)$ is a holomorphic modular form. Ahlgren and Boylan [2] extended (3.2) to congruences modulo powers of primes, namely,

$$\begin{aligned} F_{m,k}(z) &= \sum_{m^k n \equiv -1 \pmod{24}} p\left(\frac{m^k n+1}{24}\right) q^n \\ &\equiv \eta(24z)^{\gamma_{m,k}} \phi_{m,k}(24z) \pmod{m^k}, \end{aligned} \tag{3.3}$$

where $\gamma_{m,k}$ is an integer and $\phi_{m,k}(z)$ is a holomorphic modular form.

In order to prove the existence of congruences of $p_r(n)$ modulo powers of primes, Brown and Li [27] introduced the generating function

$$G_{m,k,r}(z) \equiv \sum_{\left(\frac{n}{m}\right)=-\left(\frac{-r}{m}\right)} p_r\left(\frac{n+r}{24}\right) q^n \pmod{m^k}, \tag{3.4}$$

and showed that $G_{m,k,r}(z)$ is a modular form of level $576m^3$. Kilbourn [50] used the generating function

$$H_{m,k,r}(z) \equiv \sum_{mn \equiv -r \pmod{24}} p_r\left(\frac{mn+r}{24}\right) q^n \pmod{m^k}, \tag{3.5}$$

and he proved that $H_{m,k,r}(z)$ is a modular form of level $576m$. However, due to the large dimensions of the spaces $M_\lambda(\Gamma_0(576m^3))$ and $M_\lambda(\Gamma_0(576m))$, it does not seem to be a feasible task to compute explicit bases. To derive explicit congruence formulas of $p_r(n)$, it is desirable to find a generating function of $p_r(n)$ that can be expressed in terms of modular forms of a small level.

In this chapter, we find the following extension of the generating function $F_{m,k}(z)$, namely,

$$F_{m,k,r}(z) = \sum_{m^k n \equiv -r \pmod{24}} p_r\left(\frac{m^k n+r}{24}\right) q^n, \tag{3.6}$$

where $q = e^{2\pi i z}$. We show that $F_{m,k,r}(z)$ is congruent to a weakly holomorphic modulo form modulo m^k. More precisely, we find

$$F_{m,k,r}(z) \equiv \eta(24z)^{\gamma_{m,k,r}} \phi_{m,k,r}(24z) \pmod{m^k}, \tag{3.7}$$

where $\phi_{m,k,r}(z)$ is a holomorphic modular form in $M_{\lambda_{m,k,r}}(\mathrm{SL}_2(\mathbb{Z}))$ and $\gamma_{m,k,r}$ is an integer. Noting that any element of $M_{\lambda_{m,k,r}}(\mathrm{SL}_2(\mathbb{Z}))$ can be expressed as a polynomial of the Eisenstein series $E_4(z)$ and $E_6(z)$ as pointed in Proposition 3.1. This enables us to derive explicit congruences of generating functions of $p_r(n)$ modulo m^k.

If $\phi_{m,k,r}(z) = 0$, then (3.7) yields a Ramanujan-type congruence as follows

$$p_r\left(\frac{m^k n + r}{24}\right) \equiv 0 \pmod{m^k}. \tag{3.8}$$

For example, when

$$\phi_{5,1,2}(z) = \phi_{11,1,2}(z) = 0$$

in congruence (3.7), the congruence (3.8) reduces to Gandhi's congruences (1.10) and (1.11). We also find

$$p_2(5^2 n + 23) \equiv 0 \pmod{5^2}, \tag{3.9}$$

$$p_8(11^2 n + 81) \equiv 0 \pmod{11^2}, \tag{3.10}$$

since

$$\phi_{5,2,2}(z) = \phi_{11,2,8}(z) = 0$$

in (3.7). For more congruences of the form (3.8), see Table 3.2.

On the other hand, if $\phi_{m,k,r}(z) \neq 0$ in (3.7), we may use Yang's method [86] to find congruences of the form (3.1). For example, since $F_{5,2,3}(z)$ is congruent to a modular form in the invariant space $S_{21,48}$ of T_{5^2} modulo 5^2, we have

$$p_3\left(\frac{5^2 13^{199} n + 3}{24}\right) \equiv 0 \pmod{5^2}.$$

3.2 Background on Modular Forms

In this section, we briefly recall some basic notation and facts about modular forms. For more details, one can see, such as [19,51,55].

We define the *full modular group* $\mathrm{SL}_2(\mathbb{Z})$ (also known simply as *the modular group*) to be the group of 2×2 matrices with integral elements and determinant 1, that is,

$$\mathrm{SL}_2(\mathbb{Z}) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, d, c \in \mathbb{Z}, ad - bc = 1 \right\}.$$

Modular forms are meromorphic functions which transform in a suitable way with respect to groups of such transformations. For this purpose, we first introduce certain congruence subgroups of $\mathrm{SL}_2(\mathbb{Z})$, see, for example [67].

Definition 3.1 *If N is a positive integer, then define the level N* congruence subgroups *$\Gamma_0(N)$, $\Gamma_1(N)$ and $\Gamma(N)$ by*

$$\Gamma_0(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) : c \equiv 0 \pmod{N} \right\},$$

$$\Gamma_1(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) : a \equiv d \equiv 1 \pmod{N} \text{ and } c \equiv 0 \pmod{N} \right\},$$

$$\Gamma(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) : a \equiv d \equiv 1 \pmod{N} \text{ and } b \equiv c \equiv 0 \pmod{N} \right\}.$$

It is straightforward to verify that $\Gamma_0(N)$, $\Gamma_1(N)$ and $\Gamma(N)$ are indeed subgroups of $\mathrm{SL}_2(\mathbb{Z})$. Specially, we have

$$\mathrm{SL}_2(\mathbb{Z}) = \Gamma_0(1) = \Gamma_1(1) = \Gamma(1).$$

We now introduce the definition of modular forms of congruence subgroup Γ. Let

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$$

act on the complex upper half plane

$$\mathfrak{H} := \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$$

by the linear fractional transformation

$$\gamma z = \frac{az+b}{cz+d}.$$

Then the *k slash operator* $|_k$ on the function $f : \mathfrak{H} \to \mathbb{C}$ is defined by

$$(f|_k\gamma)(z) = (\det\gamma)^{k/2}(cz+d)^{-k} f(\gamma z) \tag{3.11}$$

for all $\gamma \in \Gamma$ and $z \in \mathfrak{H}$.

Definition 3.2 [51] *Suppose that k is a positive integer, that Γ is a congruence subgroup of level N, and that $f : \mathfrak{H} \to \mathbb{C}$ is a meromorphic function which satisfies the* transformation formula

$$(f|_k\gamma)(z) = f(z) \tag{3.12}$$

for all $\gamma \in \Gamma$ and all $z \in \mathfrak{H}$, then we say that f is a weakly modular of weight k *for* Γ.
In particular, substituting γ by

$$S = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \text{ and } T = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

in (3.12), respectively, we have

$$f(z+1) = f(z), \tag{3.13}$$

$$f(-1/z) = (-1)^k f(z), \tag{3.14}$$

for all $z \in \mathfrak{H}$. Since S and T generate $\mathrm{SL}_2(\mathbb{Z})/\{\pm 1\}$ (see [53, Chapter 1. Corollary 3]), we see that if $f : \mathfrak{H} \to \mathbb{C}$ is a meromorphic function which satisfies the formulas (3.13) and (3.14), then it is a weakly modular for $\mathrm{SL}_2(\mathbb{Z})$.

Definition 3.3 [51, 67] *Let k be a positive integer and Γ be a congruence subgroup of level N. We say that a meromorphic function $f : \mathfrak{H} \to \mathbb{C}$ is a* meromorphic modular form with integer weight k for Γ *if*

1. *f is a weakly modular of weight k for Γ,*
2. *f is holomorphic on $\mathfrak{H}$, and*
3. *$f|_k\gamma$ is also holomorphic at ∞ for all $\gamma \in \mathrm{SL}_2(\mathbb{Z})$.*

Furthermore, if the values of f at all cusp points of Γ equal to zero, then we say that f is a cusp from of weight k for Γ.

We now make a remark about this definition. In order to verify the third condition of Definition 3.3, it is suffice to prove that f is holomorphic at all cusps of Γ. We denote the complex vector space of modular forms (cusp forms, respectively) of weight k with respect to congruence subgroup Γ by $M_k(\Gamma)$ ($S_k(\Gamma)$, respectively). By the transformation formula (3.14), one can verify that there are no nonzero modular forms of odd weight k on $\Gamma_0(N)$. Furthermore, since $\Gamma_1(N)$ contains the element S, modular forms in $M_k(\Gamma_1(N)$ are fixed under the substitution $z \mapsto z+1$, and thus we have a *Fourier expansion at infinity* as form

$$f(z) = \sum_{n \geq n_0} a(n)q^n,$$

where $q := e^{2\pi i z}$.

We now introduce several classical modular forms on $M_k(\mathrm{SL}_2(\mathbb{Z}))$ and $M_k(\Gamma_0(N)$. Let k be a positive integer, then let $\sigma_k(n)$ be the divisor function

$$\sigma_k(n) := \sum_{1 \leq d | n} d^k,$$

and the Bernoulli numbers B_k be defined by its exponential generating function

$$\sum_{n=0}^{\infty} B_k \frac{t^k}{k!} = \frac{t}{e^t - 1} = 1 - \frac{1}{2}t + \frac{1}{12}t^2 - \cdots.$$

Definition 3.4 *Let $k \geq 2$ be even, then the* weight k Eisenstein series $E_k(z)$ *is defined by*

$$E_k(z) = 1 - \frac{2k}{B_k} \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n.$$

One can see that $E_k(z)$ is a modular form of weight k on $\mathrm{SL}_2(\mathbb{Z})$ if $k \geq 4$ is even. The reader interested in the proof can see any textbook of modular forms, such as [53,55]. Here we list the first three Eisenstein series as follows

$$\begin{aligned} E_4(q) &= 1 + 240 \sum_{n=1}^{\infty} \sigma_3(n) q^n, \\ E_6(q) &= 1 - 504 \sum_{n=1}^{\infty} \sigma_5(n) q^n, \\ E_8(q) &= 1 + 480 \sum_{n=1}^{\infty} \sigma_7(n) q^n. \end{aligned}$$

A fundamental fact is that the two Eisentein series $E_4(z)$ and $E_6(z)$ generate the whole späce $M_k(\mathrm{SL}_2(\mathbb{Z}))$ for all even positive integer $k \geq 4$, see [67, Theorem 1.23]. Specifically, we have

Proposition 3.1 *If $k \geq 4$ is even, then $M_k(\mathrm{SL}_2(\mathbb{Z}))$ is generated by monomials of $E_4(Z)$ and $E_6(z)$, that is,*

$$M_k(\mathrm{SL}_2(\mathbb{Z})) = \left\{ E_4(z)^a E_6(z)^b : a, b \geq 0, 4a + 6b = k \right\}.$$

Using Proposition 3.1, one can obtain the following dimension formulas for $M_k(\mathrm{SL}_2(\mathbb{Z}))$ and $S_k(\mathrm{SL}_2(\mathbb{Z}))$ immediately.

Proposition 3.2 [67, Proposition 1.25] *If $k \geq 4$ is even, then*

$$\dim\left(S_k(\mathrm{SL}_2(\mathbb{Z}))\right) = \dim\left(M_k(\mathrm{SL}_2(\mathbb{Z}))\right),$$

and

$$\dim\left(M_k(\mathrm{SL}_2(\mathbb{Z}))\right) = \begin{cases} \lfloor k/12 \rfloor + 1 & \text{if } k \not\equiv 2 \pmod{12}, \\ \lfloor k/12 \rfloor & \text{if } k \equiv 2 \pmod{12}. \end{cases}$$

A classical example of cusp form is the so called *Ramanujan τ function* defined by

$$\Delta(z) := \frac{E_4(z)^3 - E_6(z)^2}{1728},$$

which is an unique cusp form of weight 12 on $\mathrm{SL}_2(\mathbb{Z})$.

Of particular interests are certain modular forms in $M_k(\Gamma_1(N))$ with nice modular transformation properties with respect to $\Gamma_0(N)$ and half-integral weight modular forms, see for example [67, Definition 1.15].

Definition 3.5 *If χ is a Dirichlet character modulo N, then we say that a form $f(z) \in M_k(\Gamma_1(N))$ ($S_k(\Gamma_1(N))$, respectively) has* Nebentypus character χ *if*

$$(f|_k\gamma)(z) = \chi(d)f(z) \tag{3.15}$$

for all $z \in \mathfrak{H}$ and all

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N).$$

The space of such modular forms (cusp forms, respectively) is denoted by $M_k(\Gamma_0(N),\chi)$ ($S_k(\Gamma_0(N),\chi)$, respectively).

Definition 3.6 [67, Definition 1.36] *Suppose that $k \in \frac{1}{2}\mathbb{Z} \setminus \mathbb{Z}$ is a half integer and that N is a positive integer. Furthermore, suppose that χ is a Dirichlet character modulo $4N$. A meromorphic function $f : \mathfrak{H} \to \mathbb{C}$ is called a* half-integral weight modular form with Nebentypus χ and weight $\lambda + \frac{1}{2}$ *if*

1. f satisfies the transformation

$$f(\gamma z) = \chi(d)\left(\frac{c}{d}\right)^k \varepsilon_d^{-k}(cz+d)^k f(z),$$

2. *f is holomorphic on $\mathfrak{H}$, and*
3. *f is also holomorphic at cusps of $\gamma_0(4N)$.*

Recall that *Dedekind's eta-function*, denoted by $\eta(z)$, is defined by the infinite product

$$\eta(z) = q^{\frac{1}{24}} \prod_{n=1}^{\infty}(1-q^n). \tag{3.16}$$

One of the useful transformation formulas of eta function is as follows

$$\eta(\gamma z) = \varepsilon_{a,b,c,d}(cz+d)^{\frac{1}{2}}\eta(z), \tag{3.17}$$

where $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$, $\varepsilon_{a,b,c,d}$ is a 24th root of unity. Specially, we have

$$\eta(-1/z) = \sqrt{z/i}\,\eta(z). \tag{3.18}$$

It is also known that

$$\eta(24z) = q\prod_{n=1}^{\infty}(1-q^{24n})$$

is a weight $\frac{1}{2}$ cusp form in $S_{\frac{1}{2}}(\Gamma_0(576), \chi_{12})$.

One can see that $\eta(z)$ is holomorphic and is nonvanishing on $\mathfrak{H}$. The eta-function is useful for providing explicit descriptions of many modular forms, and is also useful for constructing combinatorial generating functions. To this end, we introduce the following notation.

Definition 3.7 [67, Definition 1.63] *Any function $f(z)$ of the form*

$$f(z) = \prod_{\delta \mid N} \eta(\delta z)^{r_\delta},$$

where $N \geq 1$ and each r_δ is an integer, is known as an eta-quotient. *If $r_\delta \geq 0$, then $f(z)$ is known as an* eta-product.

The following general result of Gordon, Hughes and Newman [47, 65] comes in handy when working with eta-quotients and eta-products.

Proposition 3.3 *Let*

$$f(z) = \prod_{\delta \mid N} \eta(\delta z)^{r_\delta}$$

be an eta-quotient, with the additional properties that

$$\sum_{\delta \mid N} \delta r_\delta \equiv 0 \pmod{24} \tag{3.19}$$

and

$$\sum_{\delta \mid N} \frac{N}{\delta} r_\delta \equiv 0 \pmod{24} \tag{3.20}$$

then $f(z)$ *satisfies the transformation formula* (3.15) *for every*

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N),$$

where $k = \frac{1}{2}\sum_{\delta \mid N} r_\delta \in \mathbb{Z}$ *and the character* χ *is defined by*

$$\chi(d) := \left(\frac{(-1)^k \prod_{\delta \mid N} \delta^{r_\delta}}{d}\right).$$

For instance, one can verify that

$$\frac{\eta(5z)^5}{\eta(z)} \in M_2\left(\Gamma_0(5), \left(\frac{\cdot}{5}\right)\right)$$

and

$$\eta(4z)^2\eta(8z)^2 \in S_2(\Gamma_0(32)).$$

We now introduce some useful operators acting on Fourier series. Let $f(z) \in M_k(\Gamma_0(N), \chi)$ with the following Fourier expansion at ∞

$$f(z) = \sum_{n \geq 0} a(n)q^n,$$

where $q = e^{2\pi i z}$. Recall that k slash operator defined by (3.11)

$$(f|_k\gamma)(z) = (\det\gamma)^{k/2}(cz+d)^{-k}f(\gamma z),$$

where $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be a 2×2 real matrix with positive determinant. In particular, let ℓ be an integer and

$$\gamma_\ell = \begin{pmatrix} 0 & -1 \\ \ell & 0 \end{pmatrix}.$$

The *Fricke involution* W_ℓ is given by

$$f|W_\ell = f|_k\gamma_\ell. \tag{3.21}$$

Furthermore, the U*-operator* U_ℓ and V*-operator* V_ℓ are defined by

$$f(z)|U_\ell = \sum_{n\geq 0} a(\ell n)q^n, \tag{3.22}$$

and

$$f(z)|V_\ell = \sum_{n\geq 0} a(n)q^{\ell n}, \tag{3.23}$$

respectively. It is known that

$$f(z)|_kU_\ell = \ell^{\frac{k}{2}-1}\sum_{\mu=0}^{\ell-1} f(z)\Big|_k\begin{pmatrix} 1 & \mu \\ 0 & \ell \end{pmatrix}. \tag{3.24}$$

Let ψ be a Dirichlet character. The ψ-twist of $f(z)$ is defined by

$$(f\otimes\psi)(z) = \sum_{n\geq 0}\psi(n)a(n)q^n.$$

Let ℓ be a prime and $f(z)\in M_{\lambda+\frac{1}{2}}(\Gamma_0(N),\chi)$ be a modular form of half-integral weight. The Hecke operator T_{ℓ^2} is defined by

$$f(z)|T_{\ell^2} = \sum_{n\geq 0}\left(a(\ell^2 n) + \chi(\ell)\left(\frac{(-1)^\lambda n}{\ell}\right)\ell^{\lambda-1}a(n) + \chi(\ell^2)\ell^{2\lambda-1}a\left(\frac{n}{\ell^2}\right)\right)q^n. \tag{3.25}$$

We will use the following level reduction properties of the operators U_ℓ and $Tr_\ell = U_\ell + \ell^{\frac{k}{2}-1}W_\ell$ (see [56, Lemma 1] and [34, Lemma 2.2]).

Proposition 3.4 *Let $k\in\mathbb{Z}$, N be a positive integer, χ be a character modulo N, and $f(z)\in M_k(\Gamma_0(N),\chi)$. Assume that ℓ is a prime factor of N and χ is also a character modulo N/ℓ.*

1. *If $\ell^2\,|\,N$, then $f|U_\ell\in M_k(\Gamma_0(N/\ell),\chi)$.*
2. *If $N=\ell$ and χ is the trivial character, then $f|Tr_\ell\in M_k(\mathrm{SL}_2(\mathbb{Z}))$.*

For proving the desired congruence, one difficulty is that the given function is not a modular form. Thus we need to construct a suitable modular form whose coefficients are closely related to the original. To this end, the following lemma (see for example [39] or [57]) plays an elementary and important role.

Proposition 3.5 *Let*

$$f(z)g(z)=\sum_{n=0}^{\infty}c(n)q^n$$

be products of two formal series

$$f(z)=\sum_{n\geq 0}a(n)q^n$$

and

$$g(z)=\sum_{n\geq 0}b(mn)q^{mn}$$

with $b(0)=1$. *Then for each residue class* γ *mod* m, *we have*

1. *If* $a(mn+\gamma)\equiv 0 \pmod{M}$ *for* $0\leq n\leq n_0$, *then* $c(mn+\gamma)\equiv 0 \pmod{M}$ *for* $0\leq n\leq n_0$.
2. *If* $c(mn+\gamma)\equiv 0 \pmod{M}$ *for all* n, *then* $a(mn+\gamma)\equiv 0 \pmod{M}$ *for all* n.

By setting $1/g=\sum_{n\geq 0}b^{'}(mn)q^{mn}$, one can see that the above conditions are also "if and only if".

To conclude this section, we present a powerful criterion of Sturm to verify whether two modular forms with integer coefficients are congruent modulo a prime. Let m be a positive integer. Define the order of $f(z)$ by

$$\mathrm{ord}_m(f(z)):=\min\{n: a(n)\not\equiv 0 \pmod{m}\},$$

if there is no such n, we say $\mathrm{ord}_m(f(z))=\infty$, which implies

$$f(z)\equiv 0 \pmod{m}.$$

Furthermore, we have Sturm's Theorem [79].

Theorem 3.1 (Sturm(1984)) *Let* m *be a prime. Let* $f(z)$ *and* $g(z)$ *are integral coefficient modular forms in* $M_k(\Gamma_0(N))$. *If*

$$\mathrm{ord}_m(f(z)-g(z))>\frac{kN}{12}\prod_{\text{primes } p\mid N}\frac{p+1}{p},$$

then $f(z)\equiv g(z) \pmod{m}$. *Moreover,* $f(z)=g(z)$ *provided that the above relation holds for all primes* m.

3.3 Congruences for the Broken k-Diamond Partitions

In [9], Andrews and Paule continued their study on MacMahon's Partition Analysis by considering families of plane partitions. And they introduced the conception of broken k-diamond partitions. Let $\Delta_k(n)$ denote the number of broken k-diamond partitions of nonnegative integer n. They noted that the generating function of $\Delta_k(n)$ is essentially a modular form. More precisely, for $k \geq 1$,

$$\begin{aligned}\sum_{n=0}^{\infty} \Delta_k(n) q^n &= \prod_{n\geq 1}^{\infty} \frac{(1+q^n)}{(1-q^n)^2(1+q^{(2k+1)n})} \\ &= q^{\frac{k+1}{12}} \frac{\eta(2z)\eta((2k+1)z)}{\eta(z)^3\eta((4k+2)z)}. \end{aligned} \tag{3.26}$$

They then proved the Ramanujan-type congruence

$$\Delta_1(2n+1) \equiv 0 \pmod 3$$

for all $n \geq 0$ and conjectured the following three congruence properties.

Conjecture 3.1 *For all $n \geq 0$,*

$$\Delta_2(10n+2) \equiv 0 \pmod{2}, \tag{3.27}$$

$$\Delta_2(25n+14) \equiv 0 \pmod{5}, \tag{3.28}$$

$$\Delta_2(625n+314) \equiv 0 \pmod{25}. \tag{3.29}$$

For these conjectures, Hirschhorn and Sellers [49] proved the first one. In fact, they obtain the following theorem including four congruence formulas modulo 2.

Theorem 3.2 (Hirschhorn and Sellers [49]) *For all $n \geq 0$, we have*

$$\Delta_1(4n+2) \equiv \Delta_1(4n+3) \equiv 0 \pmod{2}, \tag{3.30}$$

$$\Delta_2(10n+2) \equiv \Delta_2(10n+6) \equiv 0 \pmod{2}. \tag{3.31}$$

In 2008, Chan [31] reproved the congruences (3.31), and in addition, he proved the following infinite family of congruence for $\Delta_2(n)$.

Theorem 3.3 (Chan [31]) *Let $l \geq 1$ be an integer, we have*

$$\Delta_2\left(5^{l+1}n + \frac{3}{4}(5^l - 1) + 2\cdot 5^l + 1\right) \equiv 0 \pmod 5$$

and

$$\Delta_2\left(5^{l+1}n+\frac{3}{4}(5^l-1)+4\cdot 5^l+1\right)\equiv 0 \pmod 5$$

for all integers n.

Specially, the case $l=1$ in the above theorem settled the conjecture (3.28), and we obtained

$$\Delta_2(25n+14)\equiv\Delta_2(25n+24)\equiv 0 \pmod{5}.$$

Otherwise, when $l=2$ in Theorem 3.3 they established two new congruences as follows

$$\Delta_2(125n+69)\equiv\Delta(125n+119)\equiv 0 \pmod{5}.$$

Then in 2009, Paule and Radu [68] gave two "strange" infinite families of $\Delta_2(n)$'s congruences in the following two theorems.

Theorem 3.4 (Paule and Radu [68]) *If p is a prime such that $p\equiv 13 \pmod{20}$ or $p\equiv 17 \pmod{20}$, then we have*

$$\Delta_2\left((5n+4)p-\frac{p-1}{4}\right)\equiv 0 \pmod 5$$

for all nonnegative integers n such that $20n+15\not\equiv 0 \pmod{p}$.

Theorem 3.5 (Paule and Radu [68])) *For all nonnegative integers k, we have*

$$\Delta_2\left((5n+4)p-\frac{p-1}{4}\right)\equiv 0 \pmod{5}.$$

Recently, Radu and Sellers [69] proved the following congruence result related to the generating function for $\Delta_2(3n+1)$.

Theorem 3.6 (Radu and Sellers [69])

$$\sum_{n=0}^{\infty}\Delta_2(3n+1)q^n\equiv 2q\prod_{n=1}^{\infty}\frac{(1-q^{10n})^4}{(1-q^{5n})^2} \pmod{3}. \tag{3.32}$$

Using this 3-dissection formula of the generating function of $\Delta_2(n)$ modulo 3, Radu and Sellers [69] obtained:

Theorem 3.7 *For all $n\geq 0$, we have*

$$\Delta_2(An+B)\equiv 0 \pmod{3},$$

where $A=15$, $B=1,7,10$ *and* 13.

Furthermore, based on the generating function (3.32), Radu and Sellers [69] proved that

$$\Delta_2\left(3p^2n+\frac{3}{4}((4k+3)p-1)+1\right)\equiv 0 \pmod 3, \tag{3.33}$$

for prime $p\equiv 3 \pmod 4$, $0\le k\le p-1$ and $k\neq \frac{p-3}{4}$, which implies the following congruences.

Theorem 3.8 *For all $n\geq 0$, we have*

$$\Delta_2(An+B)\equiv 0 \pmod 3, \tag{3.34}$$

where $A=27$, $B=16$ and 25.

Applying the same generating function (3.32) and properties of operators on modular forms, Chen [33] et.al. proved the following theorem, which covers Theorem 3.8.

Theorem 3.9 (Chen, et. al. [33]) *Let $l\geq 1$ be an integer, we have*

$$\Delta_2\left(3^{2l+1}n+\frac{3}{4}(3^{2l}-1)+3^{2l}+1\right)\equiv 0 \pmod 3$$

and

$$\Delta_2\left(3^{2l+1}n+\frac{3}{4}(3^{2l}-1)+2\cdot 3^{2l}+1\right)\equiv 0 \pmod 3$$

for all integers n.

In this section, we mainly consider the arithmetic properties of $\Delta_2(n)$ modulo 3. By constructing appropriate eta-quotient and using Sturm's Theorem, we shall reprove Theorem 3.7 and Theorem 3.8.

Proof of Theorem 3.7. Let $f(z)$ be an eta-quotient given by

$$f(z)=\frac{\eta(2z)\eta(5z)}{\eta(z)^3\eta(10z)}\eta(15z)^{34}\eta(30z)^4.$$

Using Proposition 3.3, one can verify that $f(z)$ is a modular form in $M_{18}(\Gamma_0(90))$ with the trivial character. By setting $k=2$ in (3.26), it is straightforward to verify that

$$\begin{aligned} f(z) &= \sum_{n=0}^{\infty} a(n)q^n \\ &= \sum_{n=0}^{\infty}\Delta_2(n)q^{n+26}\prod_{n=1}^{\infty}(1-q^{15n})^{34}(1-q^{30n})^4. \end{aligned}$$

By Lemma 3.5, the desired congruences are shared by

$$a(15n+\gamma) \equiv 0 \pmod{3}, \tag{3.35}$$

where $\gamma = 3, 6, 9, 12$. Applying Proposition 3.4, we find that

$$\begin{aligned} g(z) &:= f(z)|U(3) \\ &= \sum_{n=0}^{\infty} a(3n)q^n \end{aligned}$$

is a modular form in the space $M_{18}(\Gamma_0(30))$. Let ψ be the trivial character with modulo 5, then

$$g_\psi(z) = \sum_{i=1,2,3,4} \sum_{n=0}^{\infty} a(15n+3i)q^{5n+i}$$

is a modular form falling into $M_{18}(\Gamma_0(750), \psi^2)$. In order to prove Theorem 3.7, it is suffice to prove

$$g_\psi(z) \equiv 0 \pmod{3}.$$

By Theorem 3.1, we only need to verify the first 2701 terms. In other words, we should verify congruence (3.35) holds for all integers $n \in [0, 540]$, which can be verified by machine computation. ∎

To conclude this section, we give the proof of Theorem 3.8 as follows.

Proof of Theorem 3.8. The proof is similar to the above. Let

$$f(z) = \frac{\eta(2z)\eta(5z)}{\eta(z)^3\eta(10z)} \frac{\eta(z)^{27}}{\eta(9z)^3} \eta(27z)^{10}\eta(54z)^4$$

be an eta quotient. By Proposition 3.3, one can see that $f(z)$ is a cusp form in $S_{18}(\Gamma_0(270))$. Note that

$$\frac{\eta(z)^{27}}{\eta(9z)^3} \equiv 1 \pmod{3}.$$

Combining the definition of the broken 2-diamond partition in (3.26), we have

$$f(z) \equiv \sum_{n=0}^{\infty} \Delta_2(n)q^{n+20} \prod_{n=1}^{\infty} (1-q^{27n})^{10}(1-q^{54n})^4 \pmod{3}.$$

Suppose that $f(z)$ has the following expansion

$$f(z) = \sum_{n \geq 0} a(n)q^n,$$

then by Lemma 3.5, the desired congruences are shared by

$$a(27n+\gamma) \equiv 0 \pmod 3 \tag{3.36}$$

for $\gamma = 9, 18$. Applying the level reduction Lemma 3.4, we have

$$f(z)|U(3) = \sum_{n\geq 0} a(3n)q^n \in S_{18}(\Gamma_0(90)),$$

and

$$g(z) := f(z)|U(3)|U(3) = \sum_{n\geq 0} a(9n)q^n \in S_{18}(\Gamma_0(30)).$$

Let ψ be the trivial character with modulo 3, then

$$\begin{aligned} g_\psi(z) &= \sum_{n\equiv 1,2 \pmod 3} a(9n)q^n \\ &= \sum_{n\geq 0} a(27n+9)q^{3n+1} + \sum_{n\geq 0} a(27n+18)q^{3n+2} \end{aligned}$$

is a cusp form in space $S_{18}(\Gamma_0(270), \psi^2)$. Now we need to prove

$$g_\psi(z) \equiv 0 \pmod 3.$$

By Sturm's theorem 3.1, it is suffice to verify the first 973 terms. And thus, we should verify (3.36) holds for $0 \leq n \leq 305$, which also can be verified by machine computation. ∎

3.4 Arithmetic Properties of Multipartitions

In this section, the arithmetic of multipartitions, which has been proved so useful in the study of Lie algebras, is studied for its own intrinsic interest. Following up on the work of Ono and Yang, we shall present an infinite family of congruences for $P_r(n)$, the number of r-component multipartitions of n.

3.4.1 The Generating Function of $p_r(n)$ Modulo m^k

In this part, we mainly consider the generating function $F_{m,k,r}(z)$ defined by (3.6), namely,

$$F_{m,k,r}(z) = \sum_{m^k n \equiv -r \pmod{24}} p_r\left(\frac{m^k n + r}{24}\right) q^n,$$

and we are devoted to deriving an congruence of $F_{m,k,r}(z)$ in the following theorem.

Theorem 3.10 *Let $m \geq 5$ be a prime, and let k and r be positive integers. Then there exists a modular form $\phi_{m,k,r}(z) \in M_{\lambda_{m,k,r}}(\mathrm{SL}_2(\mathbb{Z}))$, such that*

$$F_{m,k,r}(z) \equiv \eta(24z)^{\gamma_{m,k,r}} \phi_{m,k,r}(24z) \pmod{m^k}, \tag{3.37}$$

where

$$\lambda_{m,k,r} = \begin{cases} \frac{m^k - m^{k-1}}{2} r - \frac{\gamma_{m,k,r}+r}{2} & \text{if } k \text{ is odd}, \\ (m^k - m^{k-1}) r - \frac{\gamma_{m,k,r}+r}{2} & \text{if } k \text{ is even}, \end{cases} \tag{3.38}$$

$$\gamma_{m,k,r} = \frac{24\beta_{m,k,r} - r}{m^k}, \tag{3.39}$$

and $\beta_{m,k,r}$ is the unique integer in the range $0 \leq \beta_{m,k,r} < m^k$ congruent to $r/24$ modulo m^k.

The first step of the proof of Theorem 3.10 is to express $F_{m,k,r}(z)$ in terms of a modular form. Consider the η-quotient

$$f_{m,k,r}(z) = \left(\frac{\eta(m^k z)^{m^k}}{\eta(z)} \right)^r, \tag{3.40}$$

which is a modular form in $M_{\frac{(m^k-1)r}{2}}\left(\Gamma_0(m^k), \left(\frac{\cdot}{m}\right)^{kr}\right)$. The following lemma shows that $F_{m,k,r}(z)$ can be obtained from $f_{m,k,r}(z)$ by applying the U-operator and the V-operator.

Lemma 3.1 *Let $m \geq 5$ be a prime, and let k and r be positive integers. Then we have*

$$F_{m,k,r}(z) = \frac{\left(f_{m,k,r}(z) | U_{m^k}\right) | V_{24}}{\eta(24z)^{m^k r}}. \tag{3.41}$$

Proof. Since

$$\sum_{n=0}^{\infty} p_r(n) q^n = \prod_{n=1}^{\infty} \frac{1}{(1-q^n)^r},$$

we find that

$$\begin{aligned} f_{m,k,r}(z) &= q^{\frac{m^{2k}-1}{24} r} \prod_{n=1}^{\infty} \frac{1}{(1-q^n)^r} \prod_{n=1}^{\infty} (1 - q^{m^k n})^{m^k r} \\ &= q^{\frac{m^{2k}-1}{24} r} \sum_{n=0}^{\infty} p_r(n) q^n \prod_{n=1}^{\infty} (1 - q^{m^k n})^{m^k r}. \end{aligned}$$

One can see that $(m^{2k}-1)/24$ is a positive integer for primes $m \geq 5$. Applying the operator U_{m^k} to the above relation, we obtain

$$f_{m,k,r}(z)|U_{m^k} = q^{\frac{m^{2k}-1}{24m^k}r} \sum_{n=0}^{\infty} p_r(m^k n) q^n \prod_{n=1}^{\infty} (1-q^n)^{m^k r}. \tag{3.42}$$

Let $0 \leq \beta_{m,k,r} \leq m^k - 1$ be the integer uniquely determined by the congruence $24\beta_{m,k,r} \equiv r \pmod{m^k}$. Substituting n by $n + \frac{\beta_{m,k,r}}{m^k}$ in the summation in (3.42), we find

$$f_{m,k,r}(z)|U_{m^k} = \sum_{n=0}^{\infty} p_r(m^k n + \beta_{m,k,r}) q^{n+\frac{r(m^{2k}-1)+24\beta_{m,k,r}}{24m^k}} \prod_{n=1}^{\infty} (1-q^n)^{m^k r},$$

which belongs to $\mathbb{Z}[[q]]$. So we deduce that

$$\sum_{n=0}^{\infty} p_r(m^k n + \beta_{m,k,r}) q^{n+\frac{r(m^{2k}-1)+24\beta_{m,k,r}}{24m^k}} = \frac{f_{m,k,r}(z)|U_{m^k}}{\prod_{n=1}^{\infty}(1-q^n)^{m^k r}}.$$

Applying the operator V_{24}, we get

$$\sum_{n=0}^{\infty} p_r(m^k n + \beta_{m,k,r}) q^{24n+\frac{24\beta_{m,k,r}-r}{m^k}} = \frac{\left(f_{m,k,r}(z)|U_{m^k}\right) | V_{24}}{\eta(24z)^{m^k r}}. \tag{3.43}$$

Substituting $24n + \frac{24\beta_{m,k,r}-r}{m^k}$ by n in (3.43), the sum on the left-hand side becomes $F_{m,k,r}(z)$. This completes the proof. ∎

The second step of the proof of Theorem 3.10 is to derive a congruence relation for $f_{m,k,r}(z)|U_{m^k}$ modulo m^k. We find it can be embedded in a space of modular forms on the complete modular group $\mathrm{SL}_2(\mathbb{Z})$, and we show the following theorem.

Theorem 3.11 *Let $m \geq 5$ be a prime, and let k and r be positive integers. Then there exists a modular form $G_{m,k,r}(z) \in M_{w_{m,k,r}}(\mathrm{SL}_2(\mathbb{Z}))$ such that*

$$f_{m,k,r}(z)|U_{m^k} \equiv G_{m,k,r}(z) \pmod{m^k},$$

where

$$w_{m,k,r} = \begin{cases} \frac{2m^k - m^{k-1} - 1}{2} r & \text{if } k \text{ is odd,} \\ \frac{3m^k - 2m^{k-1} - 1}{2} r & \text{if } k \text{ is even.} \end{cases}$$

Proof. Let

$$g_{m,k,r}(z) = \left(\frac{\eta(z)^m}{\eta(mz)}\right)^{c_k m^{k-1} r},$$

where

$$c_k = \begin{cases} 1 & \text{if } k \text{ is odd,} \\ 2 & \text{if } k \text{ is even.} \end{cases}$$

Since $g_{m,k,r}(z)$ is an η-quotient, using Proposition 3.3, we deduce that

$$g_{m,k,r}(z) \in M_{\frac{c_k(m^k-m^{k-1})r}{2}}\left(\Gamma_0(m), \left(\frac{\cdot}{m}\right)^{kr}\right).$$

Moreover, since

$$1-q^{mn} \equiv (1-q^n)^m \pmod{m},$$

we see that

$$\eta(z)^m \equiv \eta(mz) \pmod{m},$$

which implies

$$g_{m,k,r}(z) \equiv 1 \pmod{m^k}. \tag{3.44}$$

Since $f_{m,k,r}(z) \in S_{\frac{(m^k-1)r}{2}}\left(\Gamma_0(m^k), \left(\frac{\cdot}{m}\right)^{kr}\right)$, using Proposition 3.4 (1) repeatedly, we obtain that

$$f_{m,k,r}(z)|U_{m^{k-1}} \in S_{\frac{(m^k-1)r}{2}}\left(\Gamma_0(m), \left(\frac{\cdot}{m}\right)^{kr}\right).$$

Thus, $f_{m,k,r}(z)|U_{m^{k-1}} \cdot g_{m,k,r}(z)$ is a modular form on $\Gamma_0(m)$ of the trivial character and of weight

$$w_{m,k,r} = \frac{c_k(m^k-m^{k-1})r}{2} + \frac{(m^k-1)r}{2}.$$

Invoking Proposition 3.4 (2), we find that

$$G_{m,k,r}(z) = \left(f_{m,k,r}(z)|U_{m^{k-1}} \cdot g_{m,k,r}(z)\right)|F_m \tag{3.45}$$

is a modular form in $M_{w_{m,k,r}}(\mathrm{SL}_2(\mathbb{Z}))$.

To complete the proof of Theorem 3.11, it remains to show that

$$\left(f_{m,k,r}(z)|U_{m^{k-1}} \cdot g_{m,k,r}(z)\right)|F_m \equiv f_{m,k,r}(z)|U_{m^k} \pmod{m^k}, \tag{3.46}$$

where

$$F_m = U_m + m^{\frac{w_{m,k,r}}{2}-1}W_m,$$

and the operator W_m is given by (3.21). By congruence (3.44), we see that the left-hand side of (3.46) equals

$$f_{m,k,r}(z)|U_{m^k}+m^{\frac{w_{m,k,r}}{2}-1}\left(f_{m,k,r}(z)|U_{m^{k-1}}\cdot g_{m,k,r}(z)\right)|W_m \pmod{m^k}.$$

To prove (3.46), it suffices to show that

$$m^{\frac{w_{m,k,r}}{2}-1}\left(f_{m,k,r}(z)|U_{m^{k-1}}\cdot g_{m,k,r}(z)\right)|W_m \equiv 0 \pmod{m^k}. \tag{3.47}$$

We only consider the case when k is odd. The case when k is even can be dealt with in the same manner. In light of the transformation formula (3.18) of the eta function, we find that

$$\begin{aligned} g_{m,k,r}(z)|W_m &= m^{\frac{(m^k-m^{k-1})r}{4}}(mz)^{-\frac{(m^k-m^{k-1})r}{2}}g_{m,k,r}\left(-\frac{1}{mz}\right) \\ &= m^{-\frac{(m^k-m^{k-1})r}{4}}z^{-\frac{(m^k-m^{k-1})r}{2}}\left(\frac{(\sqrt{mz/i}\,\eta(mz))^m}{\sqrt{z/i}\,\eta(z)}\right)^{m^{k-1}r} \\ &= m^{\frac{(m+1)m^{k-1}r}{4}}(-i)^{\frac{(m-1)m^{k-1}r}{2}}\left(\frac{\eta(mz)^m}{\eta(z)}\right)^{m^{k-1}r}. \end{aligned}$$

Therefore, (3.47) can be deduced from the following congruence

$$m^{\frac{(3m^k-1)r}{4}-1}\left(f_{m,k,r}(z)\big|\,U_{m^{k-1}}\right)\big|\,W_m \equiv 0 \pmod{m^k}. \tag{3.48}$$

By the property of the U-operator as in (3.24), we have

$$\begin{aligned} & m^{\frac{(3m^k-1)r}{4}-1}\, f_{m,k,r}(z)\big|\,U_{m^{k-1}}\big|\,W_m \\ = \;& m^{\frac{(k+2)m^kr-(r+4)k}{4}}\sum_{\mu=0}^{m^{k-1}-1} f_{m,k,r}(z)\Big|_{\frac{(m^k-1)r}{2}}\begin{pmatrix} 1 & \mu \\ 0 & m^{k-1}\end{pmatrix}\Big|W_m \\ = \;& m^{\frac{(k+2)m^kr-(r+4)k}{4}}\sum_{\mu=0}^{m^{k-1}-1} f_{m,k,r}(z)\Big|_{\frac{(m^k-1)r}{2}}\begin{pmatrix} \mu m & -1 \\ m^k & 0\end{pmatrix}. \end{aligned} \tag{3.49}$$

Using the transformation formula (3.18) of the eta function, (3.49) can be written as

$$\begin{aligned} & m^{\frac{m^kr}{2}-k}z^{-\frac{(m^k-1)r}{2}}\sum_{\mu=0}^{m^{k-1}-1}\left(\frac{\eta(m\mu-\frac{1}{z})^{m^k}}{\eta(\frac{m\mu z-1}{m^kz})}\right)^r \\ = \;& m^{\frac{m^kr}{2}-k}z^{\frac{r}{2}}\eta(z)^{m^kr}\sum_{\mu=0}^{m^{k-1}-1}\frac{\alpha_\mu}{\eta(\frac{m\mu z-1}{m^kz})^r}, \end{aligned} \tag{3.50}$$

where α_μ is a 24th root of unity.

For $\mu \neq 0$, we write $\mu = m^s t$ where $m \nmid t$. For $\mu = 0$, we set $s = k-1$ and $t = 0$. In either case, there exist integers b and d such that

$$bt + dm^{k-s-1} = -1.$$

It follows that

$$\begin{pmatrix} m\mu & -1 \\ m^k & 0 \end{pmatrix} = \begin{pmatrix} t & d \\ m^{k-s-1} & -b \end{pmatrix} \begin{pmatrix} m^{s+1} & b \\ 0 & m^{k-s-1} \end{pmatrix}.$$

Applying the corresponding slash operator to $\eta(z)$, we obtain that

$$\eta\left(\frac{m\mu z - 1}{m^k z}\right) = \varepsilon_\mu m^{\frac{s+1}{2}} z^{\frac{1}{2}} \eta\left(\frac{m^{s+1}z + b}{m^{k-s-1}}\right),$$

where ε_μ is a 24th root of unity. Since the coefficients of the Fourier expansion of $\eta(z)$ at ∞ are integers and the coefficient of the term with the lowest degree is 1, the Fourier coefficients of each term in (3.50) are divisible by $m^{\frac{m^k-s-1}{2}r-k}$ in the ring $\mathbb{Z}[\zeta_{24}]$. Clearly, $0 \leq s \leq k-1$. Thus we have

$$\frac{m^k - s - 1}{2} r - k \geq \frac{m^k - k}{2} r - k \geq \frac{m^k - k}{2} - k \geq k$$

for $m \geq 5$ and $k \geq 1$. Hence the Fourier coefficients of each term in (3.50) are divisible by m^k. So we arrive at (3.48). This completes the proof. ∎

We are now in a position to finish the proof of Theorem 3.10.

Proof of Theorem 3.10. By Theorem 3.11, there exists a modular form $G_{m,k,r}(z) \in M_{w_{m,k,r}}(\mathrm{SL}_2(\mathbb{Z}))$ such that

$$f_{m,k,r}(z) | U_{m^k} \equiv G_{m,k,r}(z) \pmod{m^k}. \tag{3.51}$$

Let

$$\phi_{m,k,r}(z) = \frac{G_{m,k,r}(z)}{\Delta(z)^{\frac{m^k r + \gamma_{m,k,r}}{24}}},$$

where $\Delta(z) = \eta(z)^{24}$ is Ramanujan's Δ-function. In the proof of Lemma 3.1, we have shown that

$$f_{m,k,r}(z) | U_{m^k} = \sum_{n=0}^{\infty} p_r(m^k n + \beta_{m,k,r}) q^{n + \frac{r(m^{2k}-1) + 24\beta_{m,k,r}}{24 m^k}} \cdot \prod_{n=1}^{\infty} (1-q^n)^{m^k r},$$

which implies that the order of the Fourier expansion of $f_{m,k,r}(z)|U_{m^k}$ at ∞ is at least

$$\frac{r(m^{2k}-1)+24\beta_{m,k,r}}{24m^k}=\frac{m^kr+\gamma_{m,k,r}}{24}.$$

Thus $\phi_{m,k,r}(z)$ is a modular form in $M_{\lambda_{m,k,r}}(\mathrm{SL}_2(\mathbb{Z}))$. Combining (3.51) and Lemma 3.1, we conclude that

$$F_{m,k,r}(z)\equiv\frac{\left(\Delta(z)^{\frac{m^kr+\gamma_{m,k,r}}{24}}\phi_{m,k,r}(z)\right)\Big|V_{24}}{\eta(24z)^{m^kr}}$$
$$=\eta(24z)^{\gamma_{m,k,r}}\phi_{m,k,r}(24z)\pmod{m^k},$$

as required. ∎

One can see that Theorem 3.10 implies that the generating function of multipartitions $F_{m,k,r}(z)$ falls into the invariant Subspace $S_{r,s}$ defined as (1.7). Using the pigeonhole principle, it is readily to see that the following periodicity result of multipartitions.

Corollary 3.1 *Let $m\geq 5$ be a prime. Then there exist integers $N(m)$ and $P(m)$ with $0\leq(N_{m,k,r}-1)/2\leq m^{d_{m,k,r}}$ and $0\leq P_{m,k,r}\leq m^{d_{m,k,r}}$ such that*

$$p_r\left(\frac{m^kn+r}{24}\right)\equiv p_r\left(\frac{m^{2P_{m,k,r}+k}n+r}{24}\right)\pmod{m^k}$$

for all nonnegative integers n and all $k\geq N_{m,k,r}$, where $d_{m,k,r}$ be the dimension of $S_{\gamma_{m,k,r},\lambda_{m,k,r}}$.

3.4.2 Congruences of $p_r(n)$ Modulo m^k

In this section, we apply Theorem 3.10 on the congruence relation for the generating function $F_{m,r,k}(z)$ and Yang's method [86] to derive two classes of congruences of $p_r(n)$ modulo m^k.

Let

$$S_{\gamma,\lambda}=\{\eta(24z)^{\gamma}\phi(24z)\colon\phi(z)\in M_{\lambda}(\mathrm{SL}_2(\mathbb{Z}))\}.$$

Yang [86] showed that when γ is an odd integer such that $0<\gamma<24$ and λ is a nonnegative even integer, $S_{\gamma,\lambda}$ is an invariant subspace of $S_{\lambda+\gamma/2}(\Gamma_0(576),\chi_{12})$ under the action of Hecke operators. More precisely, for all primes $\ell\neq 2,3$ and all $f\in S_{\gamma,\lambda}$, we have

$f|T_{\ell^2} \in S_{\gamma,\lambda}$. By the invariant property of $S_{\gamma,\lambda}$, we obtain two classes of congruences of $p_r(n)$ modulo m^k as follows.

Theorem 3.12 *Let $m \geq 5$ be a prime, k be a positive integer, r be an odd positive integer less than m^k, and ℓ be a prime different from $2,3$ and m. Then there exists an explicitly computable positive integer K such that*

$$p_r\left(\frac{m^k\ell^{2\mu K-1}n+r}{24}\right) \equiv 0 \pmod{m^k} \tag{3.52}$$

for all positive integers μ and all positive integers n relatively prime to ℓ. There is also a positive integer M such that

$$p_r\left(\frac{m^k\ell^i n+r}{24}\right) \equiv p_r\left(\frac{m^k\ell^{2M+i}n+r}{24}\right) \pmod{m^k} \tag{3.53}$$

for all nonnegative integers i and n.

Proof. According to congruence relation (3.37), the generating function $F_{m,k,r}(z)$ is congruent to a modular form in $S_{\gamma_{m,k,r},\lambda_{m,k,r}} \cap \mathbb{Z}[[q]]$, where $\lambda_{m,k,r}$ and $\gamma_{m,k,r}$ are integers as given in (3.38) and (3.39). It is known that $S_{\gamma_{m,k,r},\lambda_{m,k,r}} \cap \mathbb{Z}[[q]]$ has a basis $\{f_1(z),\ldots,f_d(z)\}$ of the form

$$f_i(z) = E_4(z)^{u_i}E_6(z)^{v_i}\Delta(z)^{w_i},$$

where u_i, v_i and w_i are nonnegative integers satisfying

$$4u_i + 6v_i + 12w_i = \lambda_{m,k,r} + \gamma_{m,k,r}/2.$$

For more details, see [67]. Suppose that

$$f_i(z) = \sum_{n\geq 0} a_i(n)q^n,$$

where $i = 1,2,\ldots,d$.

To prove (3.52), it suffices to show that there exists a positive integer K such that for any $1 \leq i \leq d$,

$$a_i(\ell^{2\mu K-1}n) \equiv 0 \pmod{m^k} \tag{3.54}$$

for all n coprime to ℓ.

From the relation $\gamma_{m,k,r}m^k = 24\beta_{m,k,r} - r$, one sees that $\gamma_{m,k,r}$ and r have the same parity. Since $r < m^k$ is odd, we have $0 < \gamma_{m,k,r} < 24$, and hence $S_{\gamma_{m,k,r},\lambda_{m,k,r}}$ is invariant under the Hecke operator T_{ℓ^2}. So there exists a $d \times d$ matrix A such that

$$\left.\begin{pmatrix} f_1 \\ \vdots \\ f_d \end{pmatrix}\right| T_{\ell^2} = A \begin{pmatrix} f_1 \\ \vdots \\ f_d \end{pmatrix}. \tag{3.55}$$

Let

$$X = \begin{pmatrix} A & I_d \\ -\ell^{\gamma_{m,k,r}+2\lambda_{m,k,r}-2} I_d & 0 \end{pmatrix}.$$

Using the property of the basis $\{f_1(z), \dots, f_d(z)\}$ under the action of the U-operator as given by Yang [86, Corollary 3.4], we obtain

$$\left.\begin{pmatrix} f_1 \\ \vdots \\ f_d \end{pmatrix}\right| U_{\ell^2}^s = A_s \begin{pmatrix} f_1 \\ \vdots \\ f_d \end{pmatrix} + B_s \begin{pmatrix} g_1 \\ \vdots \\ g_d \end{pmatrix} + C_s \left.\begin{pmatrix} f_1 \\ \vdots \\ f_d \end{pmatrix}\right| V_{\ell^2}, \tag{3.56}$$

where s is a positive integer, $g_i = f_i \otimes \left(\frac{\cdot}{\ell}\right)$, and A_s, B_s and C_s are $d \times d$ matrices given by

$$\begin{pmatrix} A_s & A_{s-1} \end{pmatrix} = \begin{pmatrix} I_d & 0 \end{pmatrix} X^s, \tag{3.57}$$

$$B_s = -\ell^{\lambda_{m,k,r}+(\gamma_{m,k,r}-3)/2} \left(\frac{(-1)^{(\gamma_{m,k,r}-1)/2}12}{\ell} \right) A_{s-1},$$

$$C_s = -\ell^{\gamma_{m,k,r}+2\lambda_{m,k,r}-2} A_{s-1}.$$

Since $\gcd(m, \ell) = 1$, the matrix $X \pmod{m^k}$ is invertible in the ring $\mathscr{M}$ consisting of $2d \times 2d$ matrices over $\mathbb{Z}_{m^k}$. By the finiteness of $\mathscr{M}$, we see that there exist integers $a > b$ such that X^a and X^b are linearly dependent over $\mathbb{Z}_{m^k}$, i.e., there exists a constant $c \in \mathbb{Z}_{m^k}$ such that $X^a \equiv cX^b \pmod{m^k}$. Thus $X^K \equiv cI_{2d} \pmod{m^k}$, where $K = a - b$. In view of the relation

$$\begin{pmatrix} A_{\mu K} & A_{\mu K-1} \end{pmatrix} \equiv c^{\mu} \begin{pmatrix} I_d & 0 \end{pmatrix} \pmod{m^k},$$

we find that $A_{\mu K-1} \equiv 0 \pmod{m^k}$. Hence, from (3.56) it follows that

$$\begin{pmatrix} f_1 \\ \vdots \\ f_d \end{pmatrix} \Bigg| U_{\ell^2}^{\mu K-1} \equiv B_{\mu K-1} \begin{pmatrix} g_1 \\ \vdots \\ g_d \end{pmatrix} + C_{\mu K-1} \begin{pmatrix} f_1 \\ \vdots \\ f_d \end{pmatrix} \Bigg| V_{\ell^2} \pmod{m^k}.$$

Applying the U-operator U_ℓ and observing that

$$g_i | U_\ell = f_i \otimes \left(\frac{\cdot}{\ell}\right) \Big| U_\ell = 0,$$

relation (3.56) leads to the following congruence

$$\begin{pmatrix} f_1 \\ \vdots \\ f_d \end{pmatrix} \Bigg| U_{\ell^2}^{\mu K-1} U_\ell \equiv C_{\mu K-1} \begin{pmatrix} f_1 \\ \vdots \\ f_d \end{pmatrix} \Bigg| V_\ell \pmod{m^k},$$

namely,

$$\sum_{n\geq 0} a_i(\ell^{2\mu K-1} n) q^n \equiv \sum_{n\geq 0} a_i(n) q^{\ell n} \pmod{m^k},$$

which implies (3.54).

We now turn to the proof of congruence (3.53). By the finiteness of $\mathcal{M}$, we see that there exists a positive integer M such that $X^M \equiv I_{2d} \pmod{m^k}$. Thus matrix equation (3.57) reduces to the following congruence

$$\begin{pmatrix} A_M & A_{M-1} \end{pmatrix} \equiv \begin{pmatrix} I_d & 0 \end{pmatrix} \pmod{m^k}.$$

It follows that $A_M \equiv I_d \pmod{m^k}$ and $B_M \equiv C_M \equiv 0 \pmod{m^k}$. Thus, relation (3.56) implies

$$\begin{pmatrix} f_1 \\ \vdots \\ f_d \end{pmatrix} \Bigg| U_{\ell^2}^{M} \equiv \begin{pmatrix} f_1 \\ \vdots \\ f_d \end{pmatrix} \pmod{m^k}.$$

So the coefficient of q^n is congruent to the coefficient of $q^{\ell^{2M} n}$ in $f_i(z)$ modulo m^k for all i and n. Since $F_{m,k,r}(z)$ is a linear combination of $f_i(z)$ with integer coefficients, we obtain congruence (3.53). This completes the proof. ∎

3.4.3 Newman's Conjecture and Ramnujan's Conjecture

In this part, we consider the Newman's conjecture for multiparitions, which states that if M and r are positive integers, then for every residue class $\gamma \pmod{M}$ there are infinitely many nonnegative integers n such that

$$p_r(n) \equiv \gamma \pmod{M}.$$

Brown and Li [27] proved that this conjecture is true for some M and r. Bruinier and Ono [28] established an interesting connection between the Newman's conjecture for ordinary partitions and Ramanujan-type congruences. We find that there exists a similar connection for multipartitions. More explicitly, we have the following theorem.

Theorem 3.13 *Let r be a positive integer, $m \geq 5$ be a prime with*

$$\gcd(r+2, m-1) = 1.$$

Then at least one of the following claims is true:

1. *Newman's conjecture is true for $M = m$, and for $0 \leq \gamma < m$ we have*

$$\#\{0 \leq n \leq X : p_r(n) \equiv \gamma \pmod{m}\} > \begin{cases} c_{m,r}\sqrt{X}/\log X & \text{if } 1 \leq \gamma < m, \\ c_{m,r}X & \text{if } \gamma = 0, \end{cases}$$

   where $c_{m,r} > 0$ is a constant dependent on m and r.

2. *There is a Ramanujan-type congruence modulo m, namely,*

$$p_r(mn + \beta_{m,r}) \equiv 0 \pmod{m}$$

   holds for every positive integers n, where $0 \leq \beta_{m,r} < m$ is determined by $24\beta_{m,r} \equiv r \pmod{m}$.

Bruinier and Ono [28] proved that for coefficients of half-integral weight modular form-s either Newman's conjecture holds or there exists the Ramanujan-type congruences. To prove the corresponding result for multipartition functions, we would need the following theorem.

Theorem 3.14 ([28, Theorem 1]) *Let*

$$f(z) = \sum_{n=1}^{\infty} a(n)q^n \in S_{\lambda+\frac{1}{2}}(N, \chi) \cap \mathbb{Z}[[q]]$$

be a half-integral weight cusp form, and let χ be a real Dirichlet character. If m is an odd integer and there is a positive integer n for which $\gcd(a(n),m)=1$, *then at least one of the following is true:*

1. *If $0 \le \gamma < m$, then*

$$\#\{0 \le n \le X : a(n) \equiv \gamma \pmod{m}\} \gg_{\gamma,m} \begin{cases} \sqrt{X}/\log X & \text{if } 1 \le \gamma < m, \\ X & \text{if } \gamma = 0. \end{cases}$$

2. *There are finitely many square-free integers, say $n_1, n_2, \ldots, n_t$, for which*

$$f(z) \equiv \sum_{i=1}^{t} \sum_{k=1}^{\infty} a(n_i k^2) q^{n_i k^2} \pmod{m}.$$

Moreover if $\gcd(m,N)=1, \varepsilon \in \{\pm 1\}$ *and $p \nmid Nm$ is a prime with* $\left(\frac{n_i}{p}\right) \in \{0, \varepsilon\}$ *for each $1 \le i \le t$, then $(p-1)f(z)$ is an eigenform modulo m of the half-integral weight Hecke operator T_{p^2}. In particular, we have*

$$(p-1)f(z) \mid T_{p^2} \equiv \varepsilon\chi(p)\left(\frac{(-1)^{\lambda}}{p}\right)(p^{\lambda}+p^{\lambda-1})(p-1)f(z) \pmod{m}.$$

Based on this fundamental property, we would prove the multipartition case.

Proof of Theorem 3.13. Let

$$F_{m,r}(z) = \sum a_{m,r}(n) q^n := \eta(24z)^{\gamma_{m,1,r}} \phi_{m,1,r}(24z)$$

is a half-integral weight modular form in $S_{\frac{(m-2)r}{2}}(\Gamma_0(576), \chi_{12})$, where $\phi_{m,k,r}(z)$ defined as Theorem 3.10, then we have

$$F_{m,r}(z) \equiv \sum_{\substack{n \ge 0 \\ mn \equiv -r \pmod{24}}} p_r\left(\frac{mn+r}{24}\right) q^n \pmod{m}.$$

It is readily to see that the coefficients of $F_{m,r}(z) \pmod{m}$ are precisely the values $p_r(mn+\beta_{m,r}) \pmod{m}$.

Suppose that Theorem 3.13 (2) is false, which implies that there exists an integer n_0 such that

$$a_{m,r}(n_0) \not\equiv 0 \pmod{m}.$$

In order to complete the proof, we just need to prove Theorem 3.13 (1) is true.

First, if r is an even positive integer, then $F_{m,r}(z)$ is an integral-weight cusp form. Serre [74, 6.4] observed that there is a set of primes p with positive density with the property that

$$a_{m,r}(n_0 p^{\gamma}) \equiv (\gamma+1) a_{m,r}(n_0) \pmod{m} \tag{3.58}$$

for every positive integer γ. Obviously, (3.58) implies that each residue class $\gamma \pmod{m}$ contains infinitely many coefficients $a_{m,r}(n)$.

Otherwise, if r is an odd positive integer, then $F_{m,r}(z) \in S_{\lambda+\frac{1}{2}}$ is a half-integral weighted cusp form, where $\lambda = \frac{(m-2)r-1}{2}$. Once Theorem 3.14 (1) is true, one can deduce Theorem 3.13 (1) directly. Otherwise, in term of Theorem 3.14 (2) there are finitely many square-free integers $n_1, n_2, \ldots, n_t$, such that

$$F_{m,r}(z) \equiv \sum_{i=1}^{t} \sum_{k=1}^{\infty} a_{m,r}(n_i k^2) q^{n_i k^2} \pmod{m},$$

Without loss of generality, we assume that

$$0 \not\equiv F_{m,r}(z) \equiv \sum_{k=1}^{\infty} a_{m,r}(n_1 k^2) q^{n_1 k^2} \pmod{m}.$$

Now we apply Theorem 3.14 (2) with $\varepsilon = \left(\frac{n_0}{p}\right)$, where n_0 be the integer such that

$$a_{m,r}(n_0) \not\equiv 0 \pmod{m}$$

as before. If $p_0 \nmid n_0$ is such a prime, then we have

$$\begin{aligned} a_{m,r}(p_0^2 n_0) &\equiv \left(\frac{(-1)^{\lambda} n_0}{p_0}\right) \chi_{12}(p_0) p_0^{-\frac{r+1}{2}} a_{m,r}(n_0) \pmod{m}, \\ a_{m,r}(p_0^4 n_0) &\equiv -p_0^{-r-2} a_{m,r}(n_0) \pmod{m}. \end{aligned}$$

This follows from the definition of Hecke operators, Theorem 3.14, the fact p_0 is a quadratic non-residue modulo m and the fact that

$$\lambda = (m^2 - m - 1)r/2 - 1/2.$$

More generally, we could select such a prime p_0 with the additional property that

$$\left(\frac{(-1)^{\lambda} n_0}{p_0}\right) \chi_{12}(p_0) = 1.$$

And thus for every positive integer k we have

$$a_{m,r}(p_0^{2k}n_0) \equiv \begin{cases} (-1)^{\frac{k-1}{2}} p_0^{-\frac{(r+2)k-1}{2}} a_{m,r}(n_0) \pmod{m} & \text{if } k \text{ is odd}, \\ (-1)^{\frac{k}{2}} p_0^{-\frac{(r+2)k}{2}} a_{m,r}(n_0) \pmod{m} & \text{if } k \text{ is even}, \end{cases}$$

which can be rewriten as

$$a_{m,r}(p_0^{2k}n_0) \equiv (-1)^{\lfloor \frac{k}{2} \rfloor} p_0^{-\lfloor \frac{(r+2)k}{2} \rfloor} a_{m,r}(n_0) \pmod{m}. \tag{3.59}$$

If $\gcd(r+2, m-1) = 1$, then $p_0^{-(r+2)}$ is a primitive root modulo m, so does $-p_0^{-(r+2)}$. Then (3.59) implies that each nonzero residue class $\gamma \pmod{\ell}$ contains infinitely many $a_{m,r}(n)$. Otherwise, if $\gcd(r+2, m-1) \geq 3$, then the order of $p_0^{-(r+2)}$ modulo m must be equal to or smaller than $\frac{m-1}{3}$, which implies that the coefficients in (3.59) cannot cover nonzero residue class $\gamma \pmod{m}$, and thus we cannot determine whether the same result is correct or not in this case.

For the estimates in Theorem 3.13 (1), one could obtain by simple arguing as Bruinier and Ono [28, Theorem 1], and thereby we omit it here. ∎

3.4.4 Examples

In this section, we present some consequences of Theorem 3.10 and Theorem 3.12. We first give some examples for the congruences of the generating function $F_{m,k,r}(z)$ of $p_r(n)$.

Example 3.1 By Theorem 3.10, *we find*

$$F_{m,k,r}(z) \equiv \eta(24z)^{\gamma_{m,k,r}} \phi_{m,k,r}(24z) \pmod{m^k},$$

where $\gamma_{m,k,r}$ *is an integer,* $\phi_{m,k,r}(z)$ *is a polynomial of* $\Delta(z)$ *and the Eisenstein series* $E_4(z)$ *and* $E_6(z)$. Table 3.1 *gives the list of explicit expressions of* $\eta(z)^{\gamma_{m,1,r}} \phi_{m,1,r}(z)$ *for* $m \leq 19$ *and* $2 \leq r \leq 7$.

r	m	$\eta(z)^{\gamma_{m,1,r}} \phi_{m,1,r}(z)$
2	5	0
	7	$3\eta(z)^{10}$
	11	$2\eta(z)^2 E_4(z)^2$
	13	$8\eta(z)^{22}$

	17	$5\eta(z)^{14}E_4(z)^2$
	19	$\eta(z)^{10}(14E_4(z)^3+12\Delta(z))$
3	5	$4\eta(z)^9$
	7	$3\eta(z)^3E_6(z)$
	11	0
	13	$\eta(z)^9(4E_4(z)^3+6\Delta(z))$
	17	0
	19	$\eta(z)^{15}(2E_6(z)^3+3E_6(z)\Delta(z))$
4	5	$4\eta(z)^4E_4(z)$
	7	0
	11	$\eta(z)^4(3E_4(z)^4+8E_4(z)\Delta(z))$
	13	$\eta(z)^{20}(7E_4(z)^3+4\Delta(z))$
	17	$\eta(z)^4(6E_4(z)^7+11E_4(z)^4\Delta(z)+4E_4(z)\Delta(z)^2)$
	19	$\eta(z)^{20}(16E_4(z)^6+18E_4(z)^3\Delta(z)+2\Delta(z)^2)$
5	5	$\eta(z)^{-1}E_4(z)^2$
	7	$\eta(z)^{13}E_6(z)$
	11	0
	13	$\eta(z)^7(8E_4(z)^6+11E_4(z)^3\Delta(z)+5\Delta(z)^2)$
	17	$\eta(z)^{11}(16E_4(z)^8+16E_4(z)^5\Delta(z)+4E_4(z)^2\Delta(z)^2)$
	19	$\eta(z)(5E_6(z)^7+15E_6(z)^5\Delta(z)+16E_6(z)^3\Delta(z)^2)$
6	5	0
	7	$\eta(z)^6(6E_4(z)^3+6\Delta(z))$
	11	$\eta(z)^6(10E_4(z)^6+E_4(z)^3\Delta(z))$
	13	$\eta(z)^{18}(7E_4(z)^6+8E_4(z)^3\Delta(z)+6\Delta(z)^2)$
	17	$\eta(z)^{18}(3E_4(z)^9+3E_4(z)^6\Delta(z)+5E_4(z)^3\Delta(z)^2)$
	19	$\eta(z)^6(6E_4(z)^{12}+E_4(z)^9\Delta(z)+14\Delta(z)^4)$
7	5	0
	7	$\eta(z)^{-1}E_6(z)^3$
	11	0
	13	$\eta(z)^5(10E_4(z)^9+6E_4(z)^6\Delta(z)+9E_4(z)^3\Delta(z)^2+11\Delta(z)^3)$
	17	$\eta(z)(7E_4(z)^{13}+2E_4(z)^{10}\Delta(z)+E_4(z)^7\Delta(z)^2+3E_4(z)^4\Delta(z)^3)$

19	0

Table 3.1: Explicit congruences derived from Theorem 3.10.

Example 3.2 Let $0 \leq \beta < m^k$ *be an integer with* $\beta \equiv r/24 \pmod{m^k}$*. If*

$$\phi_{m,k,r}(z) \equiv 0 \pmod{m^k},$$

using Theorem 3.10 *we obtain the following Ramanujan-type congruences of multipartition functions*

$$p_r(m^k n + \beta) \equiv 0 \pmod{m^k}. \tag{3.60}$$

The values of m *and* β *for* $r \leq 9$ *and* $k = 1, 2$ *are given in Table 3.2.*

r	(m, β)	(m^2, β)
1	$(5,4),(7,5),(11,6)$	$(25,24),(49,47),(121,116)$
2	$(5,3)$	$(25,23)^*$
3	$(11,7),(17,15)$	$(121,106)^*$
4	$(7,6)$	$(49,41)^*$
5	$(11,8),(23,5)$	$(121,96)^*$
6	$(5,4)$	$(25,19)$
7	$(5,3),(11,9),(19,9)$	$(25,18),(121,86)$
8	$(7,5),(11,4)$	$(121,81)^*$
9	$(17,11),(19,17),(23,9)$	—

Table 3.2 Ramanujan-type congruences of multipartitions.

It can be seen that Table 3.2 contains the Ramanujan congruences (1.4) of $p(n)$ modulo 5,7 and 11, as well as Gandhi's congruences (1.10) for $p_2(n)$ and (1.11) for $p_8(n)$. The congruences marked by $*$ in the table seem to be new.

Moreover, we conjecture that all Ramanujan-type congruences with a further restriction $24\gamma \equiv r \pmod{m}$ for $r \leq 9$ are the only ones listed in the table. In fact, our computation strongly support this conjecture.

The following examples demonstrate how to derive certain congruences of $p_r(n)$ with the aid of Theorem 3.12.

Example 3.3 For the values of ℓ and K_ℓ as given in Table 3.3, we have

$$p_3\left(\frac{7\ell^{2\mu K_\ell-1}n+3}{24}\right)\equiv 0 \pmod 7 \tag{3.61}$$

for all positive integers μ and all positive integers n not divisible by ℓ.

ℓ	5	11	13	17	19	23	29	31	37	41	43	47	53	59
a_ℓ	6	4	0	4	3	6	2	5	3	0	0	3	5	5
K_ℓ	6	7	2	6	8	7	7	8	3	2	2	8	3	8

Table 3.3 Eigenvalues a_ℓ of $F_{7,1,3}(z)$ acted by T_{ℓ^2} and the corresponding K_ℓ.

Proof. By Theorem 3.10, we find

$$F_{7,1,3}(z)\equiv 3\eta(24z)^3E_6(24z) \pmod 7.$$

Since $\eta(24z)^3E_6(24z)$ belongs to the 1-dimensional space $S_{3,6}$, for any prime $\ell\neq 2,3,7$, there exists an integer a_ℓ such that

$$F_{7,1,3}(z)|T_{\ell^2}\equiv a_\ell F_{7,1,3}(z) \pmod 7.$$

Inspecting the proof of Theorem 3.12, we obtain the corresponding orders K_ℓ for which congruence (3.61) holds. ∎

Example 3.4 We have

$$p_3\left(\frac{5^2 13^{199}n+3}{24}\right)\equiv 0 \pmod{5^2}$$

for all integers n coprime to 13 and

$$p_3\left(\frac{5^2 13^{i}n+3}{24}\right)\equiv p_3\left(\frac{5^2 13^{200+i}n+3}{24}\right) \pmod{5^2}$$

for all nonnegative integers n and i.

Proof. By Theorem 3.10, $F_{5,2,3}(z)$ is congruent to a modular form in the space $S_{21,48}$ of dimension 5. Setting

$$f_i=\eta(24z)^{21}E_4(24z)^{3(5-i)}\Delta(24z)^{i-1}$$

for $1 \leq i \leq 5$. Clearly, $f_1, f_2, \ldots, f_5$ form a $\mathbb{Z}$-basis of $S_{21,48} \cap \mathbb{Z}[[q]]$. Let A be the matrix of T_{ℓ^2} with respect to this basis. By computing the first five Fourier coefficients of f_i and $f_i|T_{13^2}$ and equating the Fourier coefficients of both sides of (3.55), we find

$$A \equiv \begin{pmatrix} 17 & 21 & 18 & 3 & 3 \\ 0 & 19 & 5 & 5 & 5 \\ 0 & 0 & 22 & 4 & 19 \\ 0 & 0 & 0 & 22 & 10 \\ 0 & 0 & 0 & 0 & 12 \end{pmatrix} \pmod{5^2},$$

with the corresponding orders $K = M = 100$. Setting $\mu = 1$ in Theorem 3.12, we complete the proof. ∎

Below are two more examples for $p_3(n)$ and $p_5(n)$ modulo 7^2. The proofs are analogous to the proof of the above example, and hence are omitted.

Example 3.5 We have

$$p_3\left(\frac{7^2 11^{2351} n + 3}{24}\right) \equiv 0 \pmod{7^2}$$

for all positive integers n coprime to 7 and

$$p_3\left(\frac{7^2 11^{i} n + 3}{24}\right) \equiv p_3\left(\frac{7^2 11^{1176+i} n + 3}{24}\right) \pmod{7^2}$$

for all nonnegative integers n and i.

Example 3.6 We have

$$p_5\left(\frac{7^2 17^{195} n + 5}{24}\right) \equiv 0 \pmod{7^2}$$

for all positive integers n coprime to 17 and

$$p_5\left(\frac{7^2 17^{i} n + 5}{24}\right) \equiv p_5\left(\frac{7^2 17^{588+i} n + 5}{24}\right) \pmod{7^2}$$

for all nonnegative integers n and i.

At last, we give an example for $p_3(n)$ modulo 11^2 without proof.

Example 3.7 We have

$$p_3\left(\frac{11^2 7^{2420-1} n + 3}{24}\right) \equiv 0 \pmod{11^2}$$

for all positive integers n coprime to 11 *and*

$$p_3\left(\frac{11^2 7^i n+3}{24}\right) \equiv p_3\left(\frac{11^2 7^{2420+i} n+3}{24}\right) \pmod{11^2}$$

for all nonnegative integers n and i, where the corresponding K and M are both equal to 1210.

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个人简历

个人情况：

1986 年 10 月 7 日出生于甘肃省定西市安定区香泉镇云山村，2003 年 9 月考入陕西师范大学数学与信息科学学院学习，于 2007 年 7 月本科毕业并获得理学学士学位，2008 年考入南开大学组合数学中心应用数学专业学习，2010 年提前攻读博士至今。

在学期间完成：

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