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NEW RESULTS ON THE LEAST COMMON MULTIPLE OF CONSECUTIVE INTEGERS

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Abstract. When studying the least common multiple of some finite sequences of integers, the first author introduced the interesting arithmetic functions g_k ($k \in \mathbb{N}$), defined by $g_k(n) := \frac{n(n+1)\dots(n+k)}{\operatorname{lcm}(n,n+1,\dots,n+k)}$ ($\forall n \in \mathbb{N} \setminus \{0\}$). He proved that for each $k \in \mathbb{N}$, g_k is periodic and $k!$ is a period of g_k. He raised the open problem of determining the smallest positive period P_k of g_k. Very recently, S. Hong and Y. Yang improved the period $k!$ of g_k to $\operatorname{lcm}(1, 2, \dots, k)$. In addition, they conjectured that P_k is always a multiple of the positive integer $\frac{\operatorname{lcm}(1,2,\dots,k,k+1)}{k+1}$. An immediate consequence of this conjecture is that if $(k+1)$ is prime, then the exact period of g_k is precisely equal to $\operatorname{lcm}(1, 2, \dots, k)$.

In this paper, we first prove the conjecture of S. Hong and Y. Yang and then we give the exact value of P_k ($k \in \mathbb{N}$). We deduce, as a corollary, that P_k is equal to the part of $\operatorname{lcm}(1, 2, \dots, k)$ not divisible by some prime.

1. Introduction

Throughout this paper, we let $\mathbb{N}^*$ denote the set $\mathbb{N} \setminus \{0\}$ of positive integers.

Many results concerning the least common multiple of sequences of integers are known. The most famous is none other than an equivalent of the prime number theorem; it states that $\log \operatorname{lcm}(1, 2, \dots, n) \sim n$ as n tends to infinity (see, e.g., [6]). Effective bounds for $\operatorname{lcm}(1, 2, \dots, n)$ were also given by several authors (see, e.g., [5] and [10]).

Recently, the topic has undergone important developments. In [1], Bateman, Kalb and Stenger obtained a quantity equivalent to $\log \operatorname{lcm}(u_1, u_2, \dots, u_n)$ for when $(u_n)_n$ is an arithmetic progression. In [2], Cilleruelo obtained a simple analog of the least common multiple for a quadratic progression. For the effective bounds, Farhi [3], [4] found lower bounds for $\operatorname{lcm}(u_0, u_1, \dots, u_n)$ in the cases where $(u_n)_n$ is an arithmetic progression and where it is a quadratic progression. In the case of arithmetic progressions, Hong and Feng [7] and Hong and Yang [8] obtained some improvements of Farhi's lower bounds.

Among arithmetic progressions, the sequences of consecutive integers are the most well-known with regard to properties of their least common multiple. In [4],

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Farhi introduced the arithmetic function $g_k : \mathbb{N}^* \to \mathbb{N}^*$ ($k \in \mathbb{N}$), which is defined by

$$g_k(n) := \frac{n(n+1)\dots(n+k)}{\mathrm{lcm}(n, n+1, \dots, n+k)} \ (\forall n \in \mathbb{N}^*).$$

Farhi proved that the sequence $(g_k)_{k\in\mathbb{N}}$ satisfies the recursive relation

$$g_k(n) = \gcd\left(k!, (n+k)g_{k-1}(n)\right) \ (\forall k, n \in \mathbb{N}^*). \tag{1}$$

Then, using this relation, he deduced (by induction on k) that for each $k \in \mathbb{N}$, g_k is periodic and $k!$ is a period of g_k. A natural open problem, raised in [4], is to determine the exact period (i.e., the smallest positive period) of g_k.

In the following, let P_k denote the exact period of g_k. So, Farhi's result amounts to saying that P_k divides $k!$ for all $k \in \mathbb{N}$. Very recently, Hong and Yang have shown that P_k divides $\mathrm{lcm}(1, 2, \dots, k)$. This improves Farhi's result but doesn't solve the problem of determining the P_k's. In their paper [8], Hong and Yang also conjectured that P_k is a multiple of $\frac{\mathrm{lcm}(1,2,\dots,k+1)}{k+1}$ for all nonnegative integers k. According to the property that P_k divides $\mathrm{lcm}(1, 2, \dots, k)$ ($\forall k \in \mathbb{N}$), this conjecture implies that the equality $P_k = \mathrm{lcm}(1, 2, \dots, k)$ holds at least when $(k+1)$ is prime.

In this paper, we first prove the conjecture of Hong and Yang and then give the exact value of P_k ($\forall k \in \mathbb{N}$). As a corollary, we show that P_k is equal to the part of $\mathrm{lcm}(1, 2, \dots, k)$ which is not divisible by some prime and that the equality $P_k = \mathrm{lcm}(1, 2, \dots, k)$ holds for an infinite number of $k \in \mathbb{N}$ for which $(k + 1)$ is not prime.

2. Proof of the conjecture of Hong and Yang

We begin by extending the functions g_k ($k \in \mathbb{N}$) to $\mathbb{Z}$ as follows:

- We define $g_0 : \mathbb{Z} \to \mathbb{N}^*$ by $g_0(n) = 1$, $\forall n \in \mathbb{Z}$.
- If, for some $k \geq 1$, g_{k-1} is defined, then we define g_k by the relation

$$g_k(n) = \gcd\left(k!, (n+k)g_{k-1}(n)\right) \ (\forall n \in \mathbb{Z}). \tag{1'}$$

These extensions are easily seen to be periodic and to have the same period as their restrictions to $\mathbb{N}^*$. The following proposition plays a vital role in what follows.

Proposition 2.1. *For any $k \in \mathbb{N}$, we have $g_k(0) = k!$.*

Proof. This follows by induction on k upon using the relation $(1')$. □

We now arrive at the theorem implying the conjecture of Hong and Yang.

Theorem 2.2. *For all $k \in \mathbb{N}$, we have*

$$P_k = \frac{\mathrm{lcm}(1, 2, \dots, k+1)}{k+1} \cdot \gcd\left(P_k + k + 1, \mathrm{lcm}(P_k + 1, P_k + 2, \dots, P_k + k)\right).$$

The proof of this theorem needs the following lemma:

Lemma 2.3. *For all $k \in \mathbb{N}$, we have*

$$\mathrm{lcm}(P_k, P_k + 1, \dots, P_k + k) = \mathrm{lcm}(P_k + 1, P_k + 2, \dots, P_k + k).$$

Proof of the lemma. Let $k \in \mathbb{N}$ be fixed. The required equality in the lemma is clearly equivalent to saying that P_k divides $\mathrm{lcm}(P_k + 1, P_k + 2, \dots, P_k + k)$. This amounts to showing that for any prime number p,

$$v_p(P_k) \leq v_p\left(\mathrm{lcm}(P_k + 1, \dots, P_k + k)\right) = \max_{1\leq i\leq k} v_p(P_k + i). \tag{2}$$

So it remains to show (2). Let p be a prime number. As P_k divides $\mathrm{lcm}(1,2,\dots,k)$ (by the result of Hong and Yang [8]), we have $v_p(P_k) \le v_p(\mathrm{lcm}(1,2,\dots,k))$, that is, $v_p(P_k) \le \max_{1\le i\le k} v_p(i)$. So there exists $i_0 \in \{1,2,\dots,k\}$ such that $v_p(P_k) \le v_p(i_0)$. It follows, according to the elementary properties of the p-adic valuation, that we have

$$v_p(P_k) = \min\left(v_p(P_k), v_p(i_0)\right) \le v_p(P_k + i_0) \le \max_{1\le i\le k} v_p(P_k+i),$$

which confirms (2) and completes this proof. □

Proof of Theorem 2.2. Let $k \in \mathbb{N}$ be fixed. The main idea of the proof is to calculate in two different ways the quotient $\frac{g_k(P_k)}{g_k(P_k+1)}$ and then to compare the results obtained. On the one hand, we have, from the definition of the function g_k,

$$\begin{aligned}\frac{g_k(P_k)}{g_k(P_k+1)} &= \frac{P_k(P_k+1)\dots(P_k+k)}{\mathrm{lcm}(P_k,P_k+1,\dots,P_k+k)} \Big/ \frac{(P_k+1)(P_k+2)\dots(P_k+k+1)}{\mathrm{lcm}(P_k+1,P_k+2,\dots,P_k+k+1)} \\ &= P_k\frac{\mathrm{lcm}(P_k+1,P_k+2,\dots,P_k+k+1)}{(P_k+k+1)\mathrm{lcm}(P_k,P_k+1,\dots,P_k+k)}. \qquad (3)\end{aligned}$$

Next, using Lemma 2.3 and the well-known formula "$ab = \mathrm{lcm}(a,b)\gcd(a,b)$ $(\forall a,b \in \mathbb{N}^*)$", we have

$$\begin{aligned}(P_k+k+1)\mathrm{lcm}(P_k,P_k+1,\dots,P_k+k) &= (P_k+k+1)\mathrm{lcm}(P_k+1,P_k+2,\dots,P_k+k) \\ &= \mathrm{lcm}\left(P_k+k+1, \mathrm{lcm}(P_k+1,\dots,P_k+k)\right) \\ &\quad\times \gcd\left(P_k+k+1, \mathrm{lcm}(P_k+1,\dots,P_k+k)\right) \\ &= \mathrm{lcm}(P_k+1,P_k+2,\dots,P_k+k+1)\gcd\left(P_k+k+1,\mathrm{lcm}(P_k+1,\dots,P_k+k)\right).\end{aligned}$$

By substituting this into (3), we obtain

$$\frac{g_k(P_k)}{g_k(P_k+1)} = \frac{P_k}{\gcd\left(P_k+k+1,\mathrm{lcm}(P_k+1,\dots,P_k+k)\right)}. \qquad (4)$$

On the other hand, according to Proposition 2.1 and the definition of P_k, we have

$$\frac{g_k(P_k)}{g_k(P_k+1)} = \frac{k!}{g_k(1)} = \frac{\mathrm{lcm}(1,2,\dots,k+1)}{k+1}. \qquad (5)$$

Finally, by comparing (4) and (5), we get

$$P_k = \frac{\mathrm{lcm}(1,2,\dots,k+1)}{k+1}\gcd\left(P_k+k+1,\mathrm{lcm}(P_k+1,P_k+2,\dots,P_k+k)\right),$$

as required. The proof is complete. □

From Theorem 2.2, we derive the following interesting corollary, which confirms the conjecture of Hong and Yang [8].

Corollary 2.4. *For all $k \in \mathbb{N}$, the exact period P_k of g_k is a multiple of the positive integer $\frac{\mathrm{lcm}(1,2,\dots,k,k+1)}{k+1}$. In addition, for all $k \in \mathbb{N}$ such that $(k+1)$ is prime, we have precisely $P_k = \mathrm{lcm}(1,2,\dots,k)$.*

Proof. The first part of the corollary immediately follows from Theorem 2.2. Furthermore, we remark that if k is a natural number such that $(k+1)$ is prime, then we have $\frac{\mathrm{lcm}(1,2,\dots,k+1)}{k+1} = \mathrm{lcm}(1,2,\dots,k)$. So, P_k is both a multiple and a divisor of $\mathrm{lcm}(1,2,\dots,k)$. Hence $P_k = \mathrm{lcm}(1,2,\dots,k)$. This finishes the proof of the corollary. □

Now, we exploit the identity in Theorem 2.2 in order to obtain the p-adic valuation of P_k ($k \in \mathbb{N}$) for most prime numbers p.

Theorem 2.5. *Let $k \geq 2$ be an integer and let $p \in [1, k]$ be a prime number satisfying*

$$v_p(k+1) < \max_{1 \leq i \leq k} v_p(i). \tag{6}$$

Then we have

$$v_p(P_k) = \max_{1 \leq i \leq k} v_p(i).$$

Proof. The identity in Theorem 2.2 implies the following equality:

$$v_p(P_k) = \max_{1 \leq i \leq k+1}(v_p(i)) - v_p(k+1) + \min\left\{v_p(P_k + k + 1), \max_{1 \leq i \leq k}(v_p(P_k + i))\right\}. \tag{7}$$

Now, using hypothesis (6) of the theorem, we have

$$\max_{1 \leq i \leq k+1}(v_p(i)) = \max_{1 \leq i \leq k}(v_p(i)) \tag{8}$$

and

$$\max_{1 \leq i \leq k+1}(v_p(i)) - v_p(k+1) > 0.$$

According to (7), this last inequality implies that

$$\min\left\{v_p(P_k + k + 1), \max_{1 \leq i \leq k} v_p(P_k + i)\right\} < v_p(P_k). \tag{9}$$

Let $i_0 \in \{1, 2, \dots, k\}$ be such that $\max_{1 \leq i \leq k} v_p(i) = v_p(i_0)$. Since P_k divides $\mathrm{lcm}(1, 2, \dots, k)$, we have $v_p(P_k) \leq v_p(i_0)$, which in turn implies that $v_p(P_k + i_0) \geq \min(v_p(P_k), v_p(i_0)) = v_p(P_k)$. Thus $\max_{1 \leq i \leq k} v_p(P_k + i) \geq v_p(P_k)$. It follows from (9) that

$$\min\left\{v_p(P_k + k + 1), \max_{1 \leq i \leq k} v_p(P_k + i)\right\} = v_p(P_k + k + 1) < v_p(P_k). \tag{10}$$

So, we have

$$\min\left(v_p(P_k), v_p(k+1)\right) \leq v_p(P_k + k + 1) < v_p(P_k),$$

which implies that

$$v_p(k+1) < v_p(P_k)$$

and then that

$$v_p(P_k + k + 1) = \min\left(v_p(P_k), v_p(k+1)\right) = v_p(k+1).$$

According to (10), it follows that

$$\min\left\{v_p(P_k + k + 1), \max_{1 \leq i \leq k} v_p(P_k + i)\right\} = v_p(k+1). \tag{11}$$

By substituting (8) and (11) into (7), we finally get

$$v_p(P_k) = \max_{1 \leq i \leq k} v_p(i),$$

as required. The theorem is proved. $\square$

Using Theorem 2.5, we can find infinitely many natural numbers k such that $(k+1)$ is not prime and the equality $P_k = \mathrm{lcm}(1, 2, \dots, k)$ holds. The following corollary gives concrete examples of such numbers k.

Corollary 2.6. *If k is an integer having the form $k = 6^r - 1$ ($r \in \mathbb{N}, r \geq 2$), then we have*

$$P_k = \operatorname{lcm}(1, 2, \ldots, k).$$

Consequently, there are infinitely many $k \in \mathbb{N}$ for which $(k+1)$ is not prime and the equality $P_k = \operatorname{lcm}(1, 2, \ldots, k)$ holds.

Proof. Let $r \geq 2$ be an integer and let $k = 6^r - 1$. We have $v_2(k+1) = v_2(6^r) = r$, while $\max_{1 \leq i \leq k} v_2(i) \geq r+1$ (since $k \geq 2^{r+1}$). Thus $v_2(k+1) < \max_{1 \leq i \leq k} v_2(i)$.

Similarly, we have $v_3(k+1) = v_3(6^r) = r$, while $\max_{1 \leq i \leq k} v_3(i) \geq r+1$ (since $k \geq 3^{r+1}$). Thus $v_3(k+1) < \max_{1 \leq i \leq k} v_3(i)$.

Finally, for any prime $p \in [5, k]$, we clearly have $v_p(k+1) = v_p(6^r) = 0$ and $\max_{1 \leq i \leq k} v_p(i) \geq 1$. Hence $v_p(k+1) < \max_{1 \leq i \leq k} v_p(i)$.

This shows that the hypothesis of Theorem 2.5 is satisfied by any prime number p. Consequently, we have for any prime p that $v_p(P_k) = \max_{1 \leq i \leq k} v_p(i) = v_p(\operatorname{lcm}(1, 2, \ldots, k))$. Hence $P_k = \operatorname{lcm}(1, 2, \ldots, k)$, as required. □

3. Determination of the exact value of P_k

Notice that Theorem 2.5 successfully computes the value of $v_p(P_k)$ for almost all primes p (in fact we will prove in Proposition 3.3 that Theorem 2.5 fails to provide this value for at most one prime). In order to evaluate P_k, all we have left to do is to compute $v_p(P_k)$ for primes p such that $v_p(k+1) \geq \max_{1 \leq i \leq k} v_p(i)$. In particular we will prove:

Lemma 3.1. *Let $k \in \mathbb{N}$. If $v_p(k+1) \geq \max_{1 \leq i \leq k} v_p(i)$, then $v_p(P_k) = 0$.*

From this lemma the following result is immediate.

Theorem 3.2. *We have for all $k \in \mathbb{N}$:*

$$P_k = \prod_{p \text{ prime},\ p \leq k} p^{\begin{cases} 0 & \text{if } v_p(k+1) \geq \max_{1 \leq i \leq k} v_p(i) \\ \max_{1 \leq i \leq k} v_p(i) & \text{otherwise.} \end{cases}}$$

In order to prove this result, we need to look into some of the more detailed divisibility properties of $g_k(n)$. In this spirit we make the following definitions.

Let $S_{n,k} = \{n, n+1, n+2, \ldots, n+k\}$ be the set of integers in the range $[n, n+k]$.

For a prime number p, let $g_{p,k}(n) := v_p(g_k(n))$. Let $P_{p,k}$ be the exact period of $g_{p,k}$. Since a positive integer is uniquely determined by the number of times each prime divides it, $P_k = \operatorname{lcm}_{p \text{ prime}}(P_{p,k})$.

Now note that

$$\begin{aligned} g_{p,k}(n) &= \sum_{m \in S_{n,k}} v_p(m) - \max_{m \in S_{n,k}} v_p(m) \\ &= \sum_{e > 0, m \in S_{n,k}} (1 \text{ if } p^e | m) - \sum_{e > 0} (1 \text{ if } p^e \text{ divides some } m \in S_{n,k}) \\ &= \sum_{e > 0} \max(0, \#\{m \in S_{n,k} : p^e | m\} - 1). \end{aligned}$$

Let $e_{p,k} = \lfloor \log_p(k) \rfloor = \max_{1 \leq i \leq k} v_p(i)$ be the largest exponent of a power of p that is at most k. Clearly there is at most one element of $S_{n,k}$ which is divisible by p^e if

$e > e_{p,k}$; therefore terms in the above sum with $e > e_{p,k}$ are all 0. Furthermore, for each $e \le e_{p,k}$, at least one element of $S_{p,k}$ is divisible by p^e. Hence we have that

$$g_{p,k}(n) = \sum_{e=1}^{e_{p,k}} \left(\#\{m \in S_{n,k} : p^e|m\} - 1\right). \tag{12}$$

Note that each term on the right-hand side of (12) is periodic in n with period $p^{e_{p,k}}$ since the condition $p^e|(n+m)$ for fixed m is periodic with period p^e. Therefore $P_{p,k}|p^{e_{p,k}}$. Note that this implies that the $P_{p,k}$ for different p are relatively prime, and hence we have

$$P_k = \prod_{p \text{ prime},\ p \le k} P_{p,k}.$$

We are now ready to prove our main result.

Proof of Lemma 3.1. Suppose that $v_p(k+1) \ge e_{p,k}$. It clearly suffices to show that $v_p\left(P_{q,k}\right) = 0$ for each prime q. For $q \ne p$ this follows immediately from the result that $P_{q,k}|q^{e_{q,k}}$. Now we consider the case $q = p$.

For each $e \in \{1, \dots, e_{p,k}\}$, since $p^e|k+1$, it is clear that $\#\{m \in S_{n,k} : p^e|m\} = \frac{k+1}{p^e}$, which implies (according to (12)) that $g_{k,n}$ is independent of n. Consequently, we have $P_{p,k} = 1$ and hence $v_p\left(P_{p,k}\right) = 0$, which completes our proof. □

Note that a slightly more complicated argument allows one to use this technique to provide an alternate proof of Theorem 2.5.

We can also show that the result in Theorem 3.2 says that P_k is basically $\mathrm{lcm}(1, 2, \dots, k)$.

Proposition 3.3. *There is at most one prime p such that $v_p(k+1) \ge e_{p,k}$. In particular, by Theorem* 3.2*, P_k is either* $\mathrm{lcm}(1, 2, \dots, k)$ *or* $\frac{\mathrm{lcm}(1,2,\dots,k)}{p^{e_{p,k}}}$ *for some prime p.*

Proof. Suppose that for two distinct primes $p, q \le k$ we have $v_p(k+1) \ge e_{p,k}$ and $v_q(k+1) \ge e_{q,k}$. Then

$$k+1 \ge p^{v_p(k+1)} q^{v_q(k+1)} \ge p^{e_{p,k}} q^{e_{q,k}} > \min\left(p^{e_{p,k}}, q^{e_{q,k}}\right)^2 = \min\left(p^{2e_{p,k}}, q^{2e_{q,k}}\right).$$

But this would imply that either $k \ge p^{2e_{p,k}}$ or $k \ge q^{2e_{q,k}}$, thus violating the definition of either $e_{p,k}$ or $e_{q,k}$. □

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