$e(x) = [x=1]$

$id(x) = x$

$\varphi(x)$, $\mu(x)$

$I(x) = 1$

$\sigma_0(x) = \sum_{d|n} 1$

$\sigma_1(x) = \sum_{d|n} d$

$\Omega(x)$，质数指数和，完全积性。

$\mu_p(x)$，x中质数p指数

$\omega(x)$，x中不同质数个数。

$d = I = \sigma_0$

$\mu^{-1} = I$

$I^{-1} = \mu$

1°: $\sum_{d|n} \varphi(d) = n$

$\varphi \circ I = id = \sigma_1 \circ \mu$

2°: $\sum_{d|n} \mu(d) = [n=1]$

$\mu \circ I = e$

3°: $\sum_{d|n} 1 = I \circ I = \sigma_0(x)$

4°: $id \circ I = \sigma_1(x)$

5°: $id \circ \mu = \varphi$

6°: $id \circ id = n \cdot \sigma_0$

7°: $f = d^k \cdot \mu(d)$. $f \circ id^k = e$.

8°: $\sum_{i=1}^{n} [(n,i)=1] \cdot i = \frac{n \cdot \varphi(n) + [n=1]}{2}$

9°: $\sum_{i=1}^{n} (n,i) = \sum_{d|n} \varphi(n) \cdot [\frac{n}{d}] = \varphi \circ id$

10°: $\sum_{i=1}^{n} \sum_{j=1}^{n} [(i,j)=1] = \sum_{d} \mu(d) \cdot [\frac{n}{d}]^2$

$= 2\sum_{i=1}^{n} \varphi(i) - 1$

11°: $\sum_{i=1}^{n} \sum_{j=1}^{n} (i,j) = \sum_{d} \varphi(d) \cdot [\frac{n}{d}]^2$

$= 2\sum_{d=1}^{n} d \cdot \varphi([\frac{n}{d}]) - \frac{1}{2} n \cdot (n+1)$

$= 3\sum_{i=1}^{n} i \cdot \varphi(i) - 1$

12°: $\sum_{i=1}^{n} \sum_{j=1}^{n} [(i,j)=1] \cdot (i+j)$

$= \sum_{d} \mu(d) \cdot d \cdot [\frac{n}{d}]^2 \cdot ([\frac{n}{d}]+1)$

13°: $\sum_{i=1}^{n} \sum_{j=1}^{n} [(i,j)=1] \cdot i \cdot j$

$= \frac{1}{4} \sum_{d} \mu(d) \cdot d^2 \cdot [\frac{n}{d}]^2 \cdot ([\frac{n}{d}]+1)^2$

$= \sum_{i=1}^{n} i^2 \cdot \varphi(i)$

14°: $\sum_{d|n} |\mu(d)| = 2^p$

15°: $\sum_{i=1}^{n} [n,i] = \frac{n}{2} (\sum_{d|n} d \cdot \varphi(d) + 1)$

16°: $\sum_{i=1}^{n} \sum_{j=1}^{m} [i,j]$

$= \frac{1}{4} \sum_{d} [\frac{n}{d}][\frac{m}{d}]$

$([\frac{n}{d}]+1) \cdot ([\frac{m}{d}]+1) \cdot \cancel{d} \cdot g(d)$

$g(n) = n \sum_{d|n} d \cdot \mu(d)$

$= \sum_{i=1}^{n} i \cdot \sum_{d|i} d \cdot \varphi(d)$

17. $\sum_{i=1}^{n}\sum_{j=1}^{i}\frac{1}{i}[i,j] = \frac{1}{2}(\sum_{d} d\cdot\varphi(d)\cdot[\frac{n}{d}] + n)$

18. $\sum_{i=1}^{n}\frac{1}{i}\sum_{j=1}^{i}(i,j) = \sum_{d}\frac{\varphi(d)}{d}[\frac{n}{d}]$

19. $\sum_{i}\sum_{j}[(i,j)=1]\cdot[i\cdot j\le n] = \sum_{d=1}^{\sqrt{n}}\mu(d)\cdot\sum_{i=1}^{n/d^2}[\frac{n}{d^2\cdot i}]$

20. $\sum_{i}\sum_{j}[(i,j)=1]\cdot[i+j\le n] = \frac{1}{2}\cdot\sum_{d=1}^{n}\mu(d)\cdot[\frac{n}{d}]\cdot([\frac{n}{d}]-1)$

21. $\sum_{i=1}^{n} g(i)\cdot\sum_{d|i} f(d) = \sum_{d} f(d)\sum_{k=1}^{n/d} g(kd)$

22. $h = f\circ g \quad\Rightarrow H(n) = \sum_{k=1}^{n} g(k)\cdot\sum_{d=1}^{n/k} F([\frac{n}{k}])$

$\Rightarrow F(n) = [H(n) - \sum_{k=2}^{n} g(k)\cdot F([\frac{n}{k}])]\cdot g^{-1}(1)$

23. $\sum_{i=1}^{n}\sum_{j=1}^{n}[(i,j)\le n] = 2\sum_{d=1}^{n}\phi(n/d) - n = n^2$

24. $\sum_{d=1}^{n}\varphi(d)\cdot[\frac{n}{d}] = \sum_{d=1}^{n}\phi[\frac{n}{d}] = \frac{1}{2}n\cdot(n+1)$

25. $\sum_{d=1}^{n}\mu(d)\cdot[\frac{n}{d}] = \sum_{d=1}^{n} M([\frac{n}{d}]) = 1$

26. $\sum_{d=1}^{n} f(d)\cdot[\frac{n}{d}] = \sum_{d=1}^{n} F([\frac{n}{d}])$

$\sum_{d=1}^{n} f(d)\cdot[\frac{n}{d}]^k = \sum_{p_1,p_2\cdots p_k} F([\frac{n}{p_1\cdot p_2\cdots p_k}])$

$\pi(x) = \sum_{p\le x} 1$ $\quad\theta(x) = \sum_{p\le x}\log p$ $\quad\Lambda(n) = \log p\cdot[n=p,\ \exists p\in prime]$

$p(n)$ 划分数

$\Pi(x) = \sum_{p^k\le x}\frac{1}{k}$ $\quad\psi(x) = \sum_{p^k\le x}\log p$

27. $\sum_{i=1}^{n}\sum_{j=1}^{m}[i,j] = \sum_{k} k\cdot\sum_{i=1}^{n/k}\sum_{j=1}^{m/k}[(i,j)=1]\cdot i\cdot j$

$= \frac{1}{4}\sum_{k} k \cdot S(n/k, m/k) \rightarrow$ 见 13°

28. $\varphi\circ\sigma_0 = \varphi\circ I\circ I = id\circ I = \sigma_1(x)$

29. $\sigma_k = id_k\circ I$, $id_k = \sigma_k\circ\mu$

30. $(\varphi\cdot id^k)\circ id_k = id_2$

31. Dirichlet series :

①: $\sum_d \frac{\mu(d)}{d^s} = \frac{1}{\zeta(s)}$ ③: $\sum_d \frac{\sigma_0(d)^2}{d^s} = \frac{\zeta(s)^4}{\zeta(2s)}$

②: $\sum_d \frac{\varphi(d)}{d^s} = \frac{\zeta(s-1)}{\zeta(s)}$ ④: $\sum_d \frac{2^{\omega(d)}}{d^s} = \frac{\zeta(s)^2}{\zeta(2s)}$

32. $\sigma_0^{3}\circ I = (\sigma_0\circ I)^2$

33. $DG(f;s) = \sum_{n=1}^{\infty}\frac{f(n)}{n^s}$; $DG(f;s)\cdot DG(g;s) = DG(f\circ g;s)$

34. ~~(Carmichael)~~ $\lambda(n) = (-1)^{\Omega(n)}$, $\lambda\circ I = \sum_{d|n}\lambda(d) = [\exists x,\ n=x^2]$

$\sum_{n=1}^{\infty} q^{n^2} = \sum_{n=1}^{\infty}\frac{\lambda(n)\cdot q^n}{1-q^n}$

35. 几对 dirichlet inverson.

① $n^\alpha, \alpha\ge 0$; $\mu(n)\cdot n^\alpha$

②: $\lambda(n)$; $|\mu(n)|$

③: φ ; $\sum_{d|n} d\cdot\mu(d)$

④: σ_α ; $\sum_{d|n} d^\alpha\cdot\mu(d)\cdot\mu(\frac{n}{d})$

36. dirichlet 性质.

①: $(f\circ g)^{-1} = f^{-1}\circ g^{-1}$

②: $f^{-1}(n) = \mu(n)\cdot f(n) \iff f$ 完积

③: f 完积 $\Rightarrow (f\cdot g)^{-1} = f\cdot g^{-1}$

④: 交换. 结合. 分配. 对数乘结合.

LOVE FOREVER...

37. $\lambda(n)$ - Carmichael function.

$(a, n)=1$, $a^{\lambda(n)} \equiv 1 \ (\% \ n)$

$$\lambda(n) = \begin{cases} \varphi(n), & n=2,4,\ p^k (k\geq 1, p\text{为奇质数}) \\ \frac{1}{2}\cdot\varphi(n), & n=2^k (k\geq 3) \end{cases}$$

$n=\prod_{i=1}^{w(n)} p_i^{a_i} = lcm[\lambda(p_i^{a_i})], i\in[1, w(n)]$

38. $\zeta(s)\cdot\sum_{n=1}^{\infty}\frac{\mu(n)}{n^s}=1$, $s>0$ zeta与μ互为乘法逆元