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Divisibility properties of power GCD matrices and power LCM matrices ☆

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Abstract

Let a, b and n be positive integers and the set $S = \{x_1, \ldots, x_n\}$ of n distinct positive integers be a divisor chain (i.e. there exists a permutation σ on $\{1, \ldots, n\}$ such that $x_{\sigma(1)}|\ldots|x_{\sigma(n)}$). In this paper, we show that if $a|b$, then the ath power GCD matrix (S^a) having the ath power $(x_i, x_j)^a$ of the greatest common divisor of x_i and x_j as its i, j-entry divides the bth power GCD matrix (S^b) in the ring $M_n(\mathbf{Z})$ of $n \times n$ matrices over integers. We show also that if $a \nmid b$ and $n \geqslant 2$, then the ath power GCD matrix (S^a) does not divide the bth power GCD matrix (S^b) in the ring $M_n(\mathbf{Z})$. Similar results are also established for the power LCM matrices.

AMS classification: 11C20; 11A05; 15A36

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1. Introduction

Smith [26] published his famous and beautiful theorem stating that for any integer $n \geqslant 1$, the determinant of the $n \times n$ matrix $[(i, j)]$ having the the greatest common divisor (i, j) of i and j as its i, j-entry is the product $\prod_{k=1}^{n} \varphi(k)$, where φ is Euler's totient function. Since then many generalizations of Smith's determinant have been published (see, for example, [1–25,27,28]).

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Let $n \geqslant 1$ be an integer and $S = \{x_1, \ldots, x_n\}$ be a set of n distinct positive integers. Let $a \geqslant 1$ be an integer. The matrix having the ath power $(x_i, x_j)^a$ of the greatest common divisor of x_i and x_j as its i, j-entry is called *ath power greatest common divisor* (*GCD*) *matrix* defined on S, denoted by (S^a). The eigen structure of power GCD matrices were received attentions by Wintner [27] as well as Lindqvist and Seip [24], and recently by Hong and Loewy [21,22] and by Hong and Knoch Lee [20]. If $a = 1$, then the power GCD matrix defined on S is called the *GCD matrix* defined on S, denoted by (S). GCD matrices have been investigated since 1875 and especially actively in the recent decades. The matrix having the ath power $[x_i, x_j]^a$ of the least common multiple of x_i and x_j as its i, j-entry is called *ath power least common multiple* (*LCM*) *matrix* defined on S, denoted by $[S^a]$. If $a = 1$, then the power LCM matrix defined on S is called the *LCM matrix* defined on S. Nonsingularity of power LCM matrices has been extensively studied by some authors [3,7,11,16–20,23]. In the field of power GCD matrices and power LCM matrices, questions of divisibility are central. The set S is said to be *factor closed* (*FC*) if it contains every divisor of x for any $x \in S$. Bourque and Ligh [3] showed that if S is an FC set, then the GCD matrix (S) divides the LCM matrix $[S]$ in the ring $M_n(\mathbf{Z})$ of $n \times n$ matrices over the integers. That is: There exists an $A \in M_n(\mathbf{Z})$ such that $[S] = (S)A$ or $[S] = A(S)$. The set S is said to be *gcd closed* if $(x_i, x_j) \in S$ for all $1 \leqslant i, j \leqslant n$. It is clear that a factor-closed set is gcd closed but not conversely. Hong [13] showed that such factorization is no longer true in general if S is gcd closed. Bourque and Ligh [6] extended their result showing that if S is factor closed, then for any positive integer a, the power GCD matrix (S^a) divides the power LCM matrix $[S^a]$ in the ring $M_n(\mathbf{Z})$.

The set S is called a *divisor chain* if there exists a permutation σ on $\{1, \ldots, n\}$ such that $x_{\sigma(1)}|\cdots|x_{\sigma(n)}$. Obviously a divisor chain is gcd closed but the converse is not true. The set S is called *multiple closed* if $y \in S$ whenever $x|y|\mathrm{lcm}(S)$ for any $x \in S$, where $\mathrm{lcm}(S)$ means the least common multiple of all elements in S. Hong [14] showed that for any divisor chain S with $|S| = n$ and for any multiple-closed set S with $|S| = n$, if a is a positive integer, then the power GCD matrix (S^a) divides the power LCM matrix $[S^a]$ in the ring $M_n(\mathbf{Z})$. It should be noted that Zhao et al. [28] showed that for any given $n \geqslant 4$, there exists an *odd-lcm-closed set* $S = \{x_1, \ldots, x_n\}$ (namely, each element in S is an odd number and $[x_i, x_j] \in S$ for all $1 \leqslant i, j \leqslant n$) such that the power GCD matrix $((x_i, x_j)^a)$ on S does not divide the power LCM matrix $([x_i, x_j]^a)$ on S in the ring $M_n(\mathbf{Z})$. We remark that Hong [15], He [8] and He–Zhao [9] obtained some results about the divisibilities of determinants of power LCM matrices.

In this paper we will concentrate on the questions of divisibility. We provide a new and interesting idea by considering the divisibility among power GCD matrices and among power LCM matrices. Let $a, b \geqslant 1$ be integers. We show that if $a|b$, then for any divisor chain S, the power GCD matrix (S^a) divides the power GCD matrix (S^b) in the ring $M_n(\mathbf{Z})$. But such factorization should not hold if $a \nmid b$. We also show that if $a|b$, then for any divisor chain S, the power LCM matrix $[S^a]$ divides the power LCM matrix $[S^b]$ in the ring $M_n(\mathbf{Z})$. But such result fails to be true if $a \nmid b$.

For any permutation σ on $\{1, \ldots, n\}$, define $S_\sigma := \{x_{\sigma(1)}, \ldots, x_{\sigma(n)}\}$. Then one can easily check that $(S^a)^{-1}(S^b) = P^t(S_\sigma^a)^{-1}(S_\sigma^b)P$, where P is the $n \times n$ permutation matrix whose ith row equals $(0, \ldots, 0, \underbrace{1}_{\sigma(i)}, 0, \ldots, 0)(1 \leqslant i \leqslant n)$. It follows that $(S^a)^{-1}(S^b) \in M_n(\mathbf{Z}) \Leftrightarrow (S_\sigma^a)^{-1}(S_\sigma^b) \in M_n(\mathbf{Z})$. Similarly, we have $[S^a]^{-1}[S^b] \in M_n(\mathbf{Z}) \Leftrightarrow [S_\sigma^a]^{-1}[S_\sigma^b] \in M_n(\mathbf{Z})$. So for our purpose of divisibility, without loss of generality, we assume throughout this paper that $x_i|x_{i+1}$ for $1 \leqslant i \leqslant n-1$ and $x_1 = 1$.

2. Divisibility among power GCD matrices

In this section we discuss the divisibility among power GCD matrices. First we give a formula for the inverse of the GCD matrix on a divisor chain.

Lemma 2.1. *Let S be a divisor chain such that $1 = x_1|x_2|\ldots|x_n$. Then the inverse of the GCD matrix (S) is tridiagonal. Furthermore, we have*

$$(S)^{-1} = \begin{pmatrix} x_2r_2 & -r_2 & 0 & \ldots & 0 & 0 \\ -r_2 & r_2+r_3 & -r_3 & \ldots & 0 & 0 \\ 0 & -r_3 & r_3+r_4 & \ldots & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \ldots & r_{n-1}+r_n & -r_n \\ 0 & 0 & 0 & \ldots & -r_n & r_n \end{pmatrix},$$

where $r_i = \frac{1}{x_i - x_{i-1}}$ for $2 \leqslant i \leqslant n$.

Proof. By direct computation, the result follows immediately. □

We are now in a position to give the first main result of this paper.

Theorem 2.2. *Let $a, b \geqslant 1$ be integers and S be a divisor chain.*

(i) *If $a|b$, then the power GCD matrix (S^a) divides the power GCD matrix (S^b) in the ring $M_n(\mathbf{Z})$;*
(ii) *If $a \nmid b$ and $n \geqslant 2$, then the power GCD matrix (S^a) does not divide the power GCD matrix (S^b) in the ring $M_n(\mathbf{Z})$.*

Proof. (i) First we consider the case $a = 1$. By Lemma 2.1 we get

$$(S)^{-1}(S^a) = \begin{pmatrix} 1 & t_1 & t_1 & t_1 & \ldots & t_1 & t_1 \\ 0 & t_2 & t_2-t_3 & t_2-t_3 & \ldots & t_2-t_3 & t_2-t_3 \\ 0 & 0 & t_3 & t_3-t_4 & \ldots & t_3-t_4 & t_3-t_4 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & 0 & \ldots & t_{n-1} & t_{n-1}-t_n \\ 0 & 0 & 0 & 0 & \ldots & 0 & t_n \end{pmatrix},$$

where

$$t_1 = \frac{x_2 - x_2^b}{x_2 - 1} \quad \text{and} \quad t_i = \frac{x_i^b - x_{i-1}^b}{x_i - x_{i-1}} \quad \text{for } 2 \leqslant i \leqslant n.$$

Clearly $t_i \in \mathbf{Z}$ for $1 \leqslant i \leqslant n$. So we have $(S)^{-1}(S^b) \in M_n(\mathbf{Z})$. This concludes part (i) for the case $a = 1$.

Now consider the general case: $a > 1$. Let $T = \{y_1, \ldots, y_n\}$ with $y_i = x_i^a$ for $1 \leqslant i \leqslant n$. Since S is a divisor chain, T is also a divisor chain. Note that for any $1 \leqslant i, j \leqslant n$

$$(y_i, y_j) = (x_i^a, x_j^a) = (x_i, x_j)^a.$$

Hence the GCD matrix (T) on T is equal to the ath power GCD matrix (S^a) on S, namely $(T) = (S^a)$. Let $c = \frac{b}{a}$. Then $c \in \mathbf{Z}$ since $a|b$. Since $(y_i, y_j)^c = (x_i, x_j)^b$ for all $1 \leqslant i, j \leqslant n$, we have $(T^c) = (S^b)$.

On the other hand, the result for the case $a = 1$ tells us that in the ring $M_n(\mathbf{Z})$, we have $(T)|(T^c)$. So the desired result $(S^a)|(S^b)$ follows immediately. Part (i) is proved.

(ii) Let $n \geqslant 2$ be an integer and $a \not| b$. Then $a \neq b$. Since the set $\{x_1^a, x_2^a, \ldots, x_n^a\}$ is a divisor chain, we get by Lemma 2.1

$$(S^a)^{-1} = \begin{pmatrix} x_2^a\bar{r}_2 & -\bar{r}_2 & 0 & \ldots & 0 & 0 \\ -\bar{r}_2 & \bar{r}_2+\bar{r}_3 & -\bar{r}_3 & \ldots & 0 & 0 \\ 0 & -\bar{r}_3 & \bar{r}_3+\bar{r}_4 & \ldots & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \ldots & \bar{r}_{n-1}+\bar{r}_n & -\bar{r}_n \\ 0 & 0 & 0 & \ldots & -\bar{r}_n & \bar{r}_n \end{pmatrix}, \tag{1}$$

where $\bar{r}_i = \frac{1}{x_i^a - x_{i-1}^a}$ for $2 \leqslant i \leqslant n$. Using (1) we can compute the $(2, 2)$ entry of the product $(S^a)^{-1}(S^b)$ and get that

$$((S^a)^{-1}(S^b))_{22} = \frac{x_2^b - 1}{x_2^a - 1}.$$

We claim that $((S^a)^{-1}(S^b))_{22} \notin \mathbf{Z}$. By the claim we have immediately that $(S^a)^{-1}(S^b) \notin \mathbf{Z}$ which concludes part (ii). In what follows we show the claim. If $a > b$, then $0 < x_2^a - 1 < x_2^b - 1$ since $x_2 > 1$. It follows that $0 < \frac{x_2^b - 1}{x_2^a - 1} < 1$ which means that $((S^a)^{-1}(S^b))_{22} \notin \mathbf{Z}$ as claimed. If $a < b$ and $a \not| b$, then $b > a \geqslant 2$. So there are unique integers $q \geqslant 1$ and $1 \leqslant r \leqslant a - 1$ such that $b = qa + r$. From this we then deduce that

$$\frac{x_2^b - 1}{x_2^a - 1} = x_2^r(1 + x_2^a + \cdots + x_2^{a(q-1)}) + \frac{x_2^r - 1}{x_2^a - 1} \notin \mathbf{Z}$$

since $0 < r < a$ together with $x_2 > 1$ implying that $0 < \frac{x_2^r - 1}{x_2^a - 1} < 1$. Therefore the claim is proved and the proof of part (ii) of Theorem 2.2 is complete. □

Remark. By Theorem 2.4 (i), we know immediately that for any integer $a \geqslant 1$ and any divisor chain S, the GCD matrix (S) divides the ath power GCD matrix (S^a).

3. Divisibility among power LCM matrices

In the present section, we consider the divisibility among power LCM matrices. We need to compute the inverse of the LCM matrix on a divisor chain.

Lemma 3.1. *Let S be a divisor chain such that $1 = x_1|x_2|\ldots|x_n$. Then the inverse of the LCM matrix $[S]$ is tridiagonal. Furthermore, we have*

$$[S]^{-1} = \begin{pmatrix} u_1 & -u_1 & 0 & \ldots & 0 & 0 \\ -u_1 & u_1+u_2 & -u_2 & \ldots & 0 & 0 \\ 0 & -u_2 & u_2+u_3 & \ldots & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \ldots & u_{n-2}+u_{n-1} & -u_{n-1} \\ 0 & 0 & 0 & \ldots & -u_{n-1} & u_{n-1}+u_n \end{pmatrix},$$

where $u_i = \frac{1}{x_i - x_{i+1}}$ for $1 \leqslant i \leqslant n$ and $x_{n+1} := 0$.

Proof. By direct computation, the result follows immediately. $\square$

We can now show the second main result of this paper.

Theorem 3.2. *Let $a, b \geqslant 1$ be integers and S be a divisor chain.*

(i) *If $a|b$, then the power LCM matrix $[S^a]$ divides the power LCM matrix $[S^b]$ in the ring $M_n(\mathbf{Z})$;*
(ii) *If $a \nmid b$ and $n \geqslant 2$, then the power LCM matrix $[S^a]$ does not divide the power LCM matrix $[S^b]$ in the ring $M_n(\mathbf{Z})$.*

Proof. (i) First we consider the case $a = 1$. By Lemma 3.1, we obtain

$$[S]^{-1}[S^b] = \begin{pmatrix} v_1 & 0 & 0 & \dots & 0 & 0 \\ v_2 - v_1 & v_2 & 0 & \dots & 0 & 0 \\ v_3 - v_2 & v_3 - v_2 & v_3 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ v_{n-1} - v_{n-2} & v_{n-1} - v_{n-2} & v_{n-1} - v_{n-2} & \dots & v_{n-1} & 0 \\ v_n - v_{n-1} & v_n - v_{n-1} & v_n - v_{n-1} & \dots & v_n - v_{n-1} & v_n \end{pmatrix},$$

where

$$v_i = \frac{x_{i+1}^b - x_i^b}{x_{i+1} - x_i} \quad \text{for } 1 \leqslant i \leqslant n$$

and

$$x_{n+1} := 0.$$

Clearly $v_i \in \mathbf{Z}$ for $1 \leqslant i \leqslant n$. So we have

$$[S]^{-1}[S^b] \in M_n(\mathbf{Z}).$$

This concludes part (i) for the case $a = 1$.

Now consider the general case: $a > 1$. Let $T = \{y_1, \dots, y_n\}$ with $y_i = x_i^a$ for $1 \leqslant i \leqslant n$. Then T is a divisor chain since S is a divisor chain. Note that for any $1 \leqslant i, j \leqslant n$, we have

$$[y_i, y_j] = [x_i^a, x_j^a] = [x_i, x_j]^a.$$

So the LCM matrix $[T]$ on T is equal to the power LCM matrix $[S^a]$ on S, namely $[T] = [S^a]$. Let $c = \frac{b}{a}$. Then $c \in \mathbf{Z}$ since $a|b$. Since for all $1 \leqslant i, j \leqslant n$, $[y_i, y_j]^c = [x_i, x_j]^b$. From this we derive that $[T^c] = [S^b]$.

On the other hand, it follows form the result for the case $a = 1$ that in the ring $M_n(\mathbf{Z})$, we have $[T]|[T^c]$. Thus the desired result $[S^a]|[S^b]$ follows immediately. Part (i) is proved.

(ii) Let $n \geqslant 2$ be an integer and $a \nmid b$. Then $a \neq b$. Since the set $\{1, x_2^a, \dots, x_n^a\}$ is a divisor chain, we get by Lemma 3.1

$$[S^a]^{-1} = \begin{pmatrix} \bar{u}_1 & -\bar{u}_1 & 0 & \dots & 0 & 0 \\ -\bar{u}_1 & \bar{u}_1 + \bar{u}_2 & -\bar{u}_2 & \dots & 0 & 0 \\ 0 & -\bar{u}_2 & \bar{u}_2 + \bar{u}_3 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & \bar{u}_{n-2} + \bar{u}_{n-1} & -\bar{u}_{n-1} \\ 0 & 0 & 0 & \dots & -\bar{u}_{n-1} & \bar{u}_{n-1} + \bar{u}_n \end{pmatrix}, \tag{2}$$

where $\bar{u}_i = \frac{1}{x_i^a - x_{i+1}^a}$ for $1 \leqslant i \leqslant n$ and $x_{n+1} := 0$. By the inverse formula (2), we can calculate the (1,1) entry of the product $[S^a]^{-1}[S^b]$ and obtain that

$$([S^a]^{-1}[S^b])_{11} = \frac{x_2^b - 1}{x_2^a - 1},$$

which is not an integer, by the proof of part (ii) of Theorem 2.2, since $a \nmid b$ and $x_2 \geqslant 2$. This implies that $[S^a]^{-1}[S^b] \notin M_n(\mathbf{Z})$. Part (ii) is proved. This completes the proof of Theorem 3.2. $\square$

Remark. Evidently for $n = 1$, we have $(S^a)|(S^b)$ and $[S^a]|[S^b]$ if $a < b$ and $a \nmid b$. By Theorem 3.2 (i) one knows immediately that for any integer $a \geqslant 1$ and any divisor chain S, the LCM matrix $[S]$ divides the ath power LCM matrix $[S^a]$.

4. Divisibility of $[S^b]$ by (S^a)

From the results presented in [14] and Sections 2 and 3 of this paper, we can derive the third main result of this paper as follows.

Theorem 4.1. *Let $a, b \geqslant 1$ be integers and S be a divisor chain.*

(i) *If $a|b$, then the power GCD matrix (S^a) divides the power LCM matrix $[S^b]$ in the ring $M_n(\mathbf{Z})$;*
(ii) *If $a \nmid b$ and $n \geqslant 2$, then the power GCD matrix (S^a) does not divide the power LCM matrix $[S^b]$ in the ring $M_n(\mathbf{Z})$.*

Proof. (i) First it follows from [14] that $(S^b)|[S^b]$. On the other hand, since $a|b$, by Theorem 2.2(i) we have $(S^a)|(S^b)$. So we have $(S^a)|[S^b]$ as desired. Part (i) also follows from [14] and Theorem 3.2(i).

(ii) By the inverse formula (1), we can calculate the (1, 1) entry of the product $(S^a)^{-1}[S^b]$ and get that

$$((S^a)^{-1}[S^b])_{11} = \frac{x_2^a - x_2^b}{x_2^a - 1} = 1 - \frac{x_2^b - 1}{x_2^a - 1},$$

which is not an integer since $a \nmid b$ together with $x_2 > 1$ implying that $\frac{x_2^b - 1}{x_2^a - 1} \notin \mathbf{Z}$ by the proof of part (ii) of Theorem 2.2. It implies that $(S^a)^{-1}[S^b] \notin M_n(\mathbf{Z})$. So part (ii) is proved. This completes the proof of Theorem 4.1. $\square$

Remark. (1) Clearly for $n = 1$, we have $(S^a)|[S^b]$ if $a < b$ and $a \nmid b$. By Theorem 4.1(i) one knows immediately that for any integer $a \geqslant 1$ and any divisor chain S, the GCD matrix (S) divides the ath power LCM matrix $[S^a]$.

(2) Let $a, b \geqslant 1$ be integers and S a divisor chain with $|S| \geqslant 2$. It follows from Theorems 2.2, 3.2 and 4.1 that $\det(S^a)|\det(S^b)$, $\det[S^a]|\det[S^b]$ and $\det(S^a)|\det[S^b]$ if $a|b$, and $(S^a) \nmid (S^b)$, $[S^a] \nmid [S^b]$ and $(S^a) \nmid [S^b]$ if $a \nmid b$. However, the non-divisibility of matrices does not imply the non-divisibility of determinants. It is still unclear whether we have $\det(S^a) \nmid \det(S^b)$, $\det[S^a] \nmid \det[S^b]$ and $\det(S^a) \nmid \det[S^b]$ if $a \nmid b$ for any divisor chain S with $|S| \geqslant 2$. We guess that the answer to this question should be affirmative. In a forthcoming paper, we will discuss this topic.

(3) Bhowmik and Hong [2] established the similar results as in the present paper when S is factor closed or multiple closed. One can show that there is a gcd-closed set S such that $(S^a) \nmid (S^b)$ (resp. $[S^a] \nmid [S^b]$ and $(S^a) \nmid [S^b]$) in the ring $M_{|S|}(\mathbf{Z})$ if $a|b$. Furthermore, we propose several conjectures about the gcd-closed case. For this purpose, we recall the concept of greatest-type divisor introduced in [11]. For any $d, x \in S$ with $d < x$, we say that d is a *greatest-type divisor* of x in S if $d|x$ and there is no other $y \in S$ such that $d|y$ and $y|x$. For $x \in S$, denote by $G_S(x)$ the set of all greatest-type divisors of x in S.

Conjecture 4.2. *Let $a, b \geqslant 1$ be integers such that $a|b$ and $S = \{x_1, \ldots, x_n\}$ be a gcd-closed set with $\max_{x\in S}\{|G_S(x)|\} = 1$. Then the a-th power GCD matrix (S^a) on S divides the b-th power GCD matrix (S^b) on S in the ring $M_n(\mathbf{Z})$.*

Conjecture 4.3. *Let $a, b \geqslant 1$ be integers such that $a|b$ and $S = \{x_1, \ldots, x_n\}$ be a gcd-closed set with $\max_{x\in S}\{|G_S(x)|\} = 1$. Then the a-th power LCM matrix $[S^a]$ on S divides the b-th power LCM matrix $[S^b]$ on S in the ring $M_n(\mathbf{Z})$.*

Conjecture 4.4. *Let $a, b \geqslant 1$ be integers such that $a|b$ and $S = \{x_1, \ldots, x_n\}$ be a gcd-closed set with $\max_{x\in S}\{|G_S(x)|\} = 1$. Then the a-th power GCD matrix (S^a) on S divides the b-th power LCM matrix $[S^b]$ on S in the ring $M_n(\mathbf{Z})$.*

Obviously, Theorems 2.2(i), 3.2(i) and 4.1(i) give evidences to Conjectures 4.2, 4.3 and 4.4, respectively. We can also prove that $(S^a)|(S^b)$ (resp. $[S^a]|[S^b]$ and $(S^a)|[S^b]$) in the ring $M_{|S|}(\mathbf{Z})$ for any gcd-closed set S with $|S| \leqslant 3$. Let now $n \geqslant 4$ and $a, b \geqslant 1$ be integers such that $a|b$. The problem of determining the necessary and sufficient conditions on the gcd-closed set S with $|S| = n$ such that $(S^a)|(S^b)$ (resp. $[S^a]|[S^b]$ and $(S^a)|[S^b]$) in the ring $M_n(\mathbf{Z})$ keeps widely open.

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