Graph Polynomials and Their Applications I: The Tutte Polynomial

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1 Introduction

We begin our exploration of graph polynomials and their applications with the Tutte polynomial, a renown tool for analyzing properties of graphs and networks. This two-variable graph polynomial, due to W. T. Tutte [Tut47, Tut54, Tut67], has the important universal property that essentially any multiplicative graph invariant with a deletion/contraction reduction must be an evaluation of it. These deletion/contraction operations are natural reductions for many network models arising from a wide range of problems at the hearts of computer science, engineering, optimization, physics, and biology.

In addition to surveying a selection of the Tutte polynomial's many properties and applications, we use the Tutte polynomial to showcase a variety of principles and techniques for graph polynomials in general. These include several ways in which a graph polynomial may be defined and methods for extracting combinatorial information and algebraic properties from a graph polynomial. We also use the Tutte polynomial to demonstrate how graph polynomials may be both specialized and generalized, and how they can encode information relevant to physical applications.

We begin with the Tutte polynomial because it has a rich and well-developed theory, and thus it serves as an ideal model for exploring other graph polynomials in the next chapter, Graph Polynomials and Their Applications II: Interrelations and Interpretations. Furthermore, because of the Tutte polynomial's long history, extensive study, and its universality property, it is often a 'point of contact' for research into other graph polynomials in that their study frequently includes exploring their relations to the Tutte polynomial. These interrelationships will be a central theme of the following chapter.

In this chapter we give both recursive and generating function formulations of the Tutte polynomial, and state its universality in the form of a recipe theorem.

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We give a number of properties and combinatorial interpretations for various evaluations of the Tutte polynomial. We recover colorings, flows, orientations, network reliability, etc., and related polynomials as specializations of the Tutte polynomial. We discuss the coefficients, zeros, and derivatives of the Tutte polynomial, and conclude with a brief discussion of computational complexity.

2 Preliminary Notions

The graph terminology that we use is standard and generally follows Diestel [Die00]. Graphs may have loops and multiple edges. For a graph G we denote by V(G) its set of vertices and by E(G) its set of edges. An oriented graph, \vec{G} , also called a digraph, has a direction assigned to each edge.

2.0.1 Basic Concepts

We first recall some of the notions of graph theory most used in this chapter. Two graphs G_1 and G_2 are isomorphic, denoted $G_1 \simeq G_2$, if there exists a bijection $\phi: V(G_1) \to V(G_2)$ with $xy \in E(G_1)$ if and only if $\phi(x)\phi(y) \in E(G_2)$. We denote by $\kappa(G)$ the number of connected components of a graph G, and by c(G) the number of non-trivial connected components, that is the number of connected components not counting isolated vertices. A graph is k-connected if at least k vertices must be removed to disconnect the graph.

A cycle in a graph G is a set of edges e_1, \ldots, e_k such that, if $e_i = (v_i, w_i)$ for $1 \le i \le k$, then $w_i = v_{i+1}$ for $1 \le i \le k-1$; also $w_k = v_1$ and $v_i \ne v_j$ for $i \ne j$. A trail is a path that may revisit a vertex, but not retrace an edge. A circuit is a closed trail, and thus a cycle is just a circuit that does not revisit any vertices. In the case of a digraph, the edges of a trail or circuit must be consistently oriented.

The dual notion of a cycle is that of cut or cocycle. If $\{V_1, V_2\}$ is a partition of the vertex set, and the set C, consisting of those edges with one end in V_1 and one end in V_2 , is not empty, then C is called a cut. A cycle with one edge is called a cut-edge or bridge. We refer to an edge that is neither a loop nor a bridge as ordinary.

A tree is a connected graph without cycles. A forest is a graph whose connected components are all trees. A subgraph H of a graph G is spanning if V(H) = V(G). Spanning trees in connected graphs will play a fundamental role in the theory of the Tutte polynomial. Observe that a loop in a connected graph can be characterized as an edge that is in no spanning tree, while a bridge is an edge that is in every spanning tree.

If $V' \subseteq V(G)$, then the *induced subgraph* on V' has vertex set V' and edge set those edges of G with both endpoints in V'. If $E' \subseteq E(G)$, then the *spanning subgraph* induced by E' has vertex set V(G) and edge set E'.

2.0.2 Deletion and Contraction

The operations of deletion and contraction of an edge are essential to the study of the Tutte polynomial. The graph obtained by deleting an edge $e \in E(G)$ is just $G \setminus e = (V, E \setminus e)$. The graph obtained by contracting an edge e in G results from identifying the endpoints of e followed by removing e, and is denoted G/e. When e is a loop, G/e is the same as $G \setminus e$. It is not difficult to check that both deletion and contraction are commutative, and thus, for a subset of edges A, both $G \setminus A$ and G/A are well defined. Also, if $e \neq f$, then $G \setminus e/f$ and $G/f \setminus e$ are isomorphic; thus for disjoint subsets $A, A' \subseteq E(G)$, the graph $G \setminus A/A'$ is well-defined. A graph $G \setminus A/A'$ is called a minor of $G \setminus A/A'$ for some choice of disjoint edge sets A and A' is called a minor of $G \setminus A$ class $G \setminus A$ of graphs is minor closed if whenever $G \setminus A$ is $G \setminus A$ then any minor of $G \setminus A$ is also in the class.

A graph invariant is a function f on the class of all graphs such that

$$f(G_1) = f(G_2)$$
 whenever $G_1 \simeq G_2$.

A graph polynomial is a graph invariant where the image lies in some polynomial ring.

2.0.3 The Rank and Nullity Functions for Graphs

To simplify notation, we typically identify a subset of edges A of a graph G with the spanning subgraph of G that A induces. Thus, for a fixed graph G we have the following rank and nullity functions on the lattice of subsets of E(G).

Definition 1 For $A \subseteq E(G)$, the rank and nullity of A, denoted r(A) and n(A) respectively, are defined as

$$r(A) = |V(G)| - \kappa(A)$$
 and $n(A) = |A| - r(A)$.

Three special graphs are important. One is the rank 0 graph L consisting of a single vertex with one loop edge, another is the rank 1 graph B consisting of two vertices with one bridge edge between them, and the third one is the edgeless graph E_1 on 1 vertex.

2.0.4 Planar Graphs and Duality

A graph is planar if it can be drawn in the plane without edges crossing, and it is a plane graph if it is so drawn in the plane. A drawing of a graph in the plane separates the plane into regions called faces. Every plane graph G has a dual graph, G^* , formed by assigning a vertex of G^* to each face of G and joining two vertices of G^* by k edges if and only if the corresponding faces of G share k edges in their boundaries. Notice that G^* is always connected. If G is connected, then $(G^*)^* = G$. If G is planar, in principle it may have many plane duals, but when G is 3-connected, all its plane duals are isomorphic. This is not the case when G is only 2-connected.

There is a natural bijection between the edge set of a planar graph G and the edge set of G^* , any one of its plane duals, so we can assume that G and G^* have the same edge set E. It is easy to check that $A \subseteq E$ is a spanning tree of G if and only if $E \setminus A$ is a spanning tree of G^* . Thus, a planar graph and any of its plane duals have the same number of spanning trees. Furthermore, if G is a planar graph with rank function r, and G^* is any of its plane duals, then the rank function of G^* , denoted r^* , can be expressed as

$$r^*(A) = |A| - r(E) + r(E \setminus A). \tag{2.1}$$

These observations reflect a deeper relation between G and G^* that we will see captured by the Tutte polynomial at the end of Subsection 3.2.

3 Defining the Tutte Polynomial

Here we present several very different, but nevertheless equivalent, definitions of the Tutte polynomial. The interplay among these different formulations is a source for many powerful tools developed to analyze the Tutte polynomial. Furthermore, each formulation lends itself to different proof techniques, for example induction with the linear recursion form and Möbius inversion with the generating function form. These different formulations also are representative of some of the most common ways of defining any graph polynomial, although we will also see other methods in the next chapter.

While space prohibits including full proofs of the equivalence of these various expressions for the Tutte polynomial, we note that there are several approaches. One direct way is to specify the linear recursion form as the definition of the Tutte polynomial and then use induction on the number of edges to show that it is equivalent to either the rank-nullity generating function or the spanning trees expansion. Showing that the linear recursion form is equivalent to rank generating function form also establishes the essential fact that it is well-defined, that is, independent of the order in which the edges are deleted and contracted. Another common approach is to establish some definition of the Tutte polynomial, then prove from it that the Tutte polynomial has the universality property discussed in Section 4. This universality property may then be applied to show that some other function is equivalent to, or an evaluation of, the Tutte polynomial.

The spanning trees expansion formulation in Subsection 3.3 was the approach originally used by Tutte to develop versions of this and similar polynomials. See Tutte [Tut47, Tut54, Tut67]. A particularly lucid proof of the equivalence between rank-nullity generating function definition of Definition 3 and the spanning trees expansion definition of Definition 4 can be found in [Bol98]. That the Tutte polynomial has a deletion-contraction reduction was shown by Tutte [Tut47, Tut48, Tut67] (also see Brylawski [Bry72], and Oxley and Welsh [OW79]).

3.1 Linear Recursion Definition

Broadly speaking, a linear recursion relation is a set of reduction rules together with an evaluation for the terminal forms. The reduction rules rewrite a graph as a weighted (formal) sum of graphs that are in some way "smaller" or "simpler" than the original graph. Furthermore, the reduction rules again apply to the newly generated simpler graphs, hence the recursion. This recursion process eventually terminates in a well-defined set of "most simple" graphs, which are no longer reducible by the reduction rules. These are then each identified with a monomial of independent variables to yield a polynomial. It is essential to show that the reduction rules are independent of the order in which they are applied and that they do in fact terminate. See Yetter [Yet90] for a more formal treatment.

The Tutte polynomial may be defined by a linear recursion relation given by deleting and contracting ordinary edges. The "most simple" terminal graphs are then just forests with loops.

Definition 2 If G = (V, E) is a graph, and e is an ordinary edge, then

$$T(G; x, y) = T(G \setminus e; x, y) + T(G/e; x, y). \tag{3.1}$$

Otherwise, G consists of i bridges and j loops and

$$T(G; x, y) = x^i y^j. (3.2)$$

In other words, T may be calculated recursively by specifying an ordering of the edges and repeatedly applying (3.1). Remarkably, the Tutte polynomial is well defined in that the polynomial resulting from this recursive process is independent of the order in which the edges are chosen. One way to prove this is by showing that this definition is equivalent to the rank generating form we will see in Definition 3. A proof can be found in [BO92] for example.

Figure 1 gives a small example of computing T using (3.1) and (3.2) for K_4 minus one edge. By adding the monomials at the bottom of Figure 1 we find that $T(G; x, y) = x^3 + 2x^2 + x + 2xy + y + y^2$.

Recall that a *one-point join* G * H of two graphs G and H is formed by identifying a vertex u of G and a vertex v of H into a single vertex w (necessarily a cut vertex) of G * H. Also, $G \cup H$ is the disjoint union of G and H.

Proposition 1 If G and H are graphs then

$$T(G \cup H) = T(G)T(H)$$
 and $T(G * H) = T(G)T(H)$.

This follows readily from Definition 2 by induction on the number of ordinary edges in G*H or $G\cup H$.

3.2 Rank-Nullity Generating Function Definition

A generating function can often be thought of as a (possible infinite) polynomial whose coefficients count structures that are encoded by the exponents of

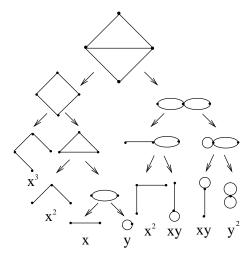


Figure 1: An example of computing the Tutte polynomial recursively

the variables. Because generating functions count, which is at the very heart of enumerative combinatorics, there is extensive literature on them. The two volumes [Sta96a] and [Sta99] are an excellent resource. In the case of the Tutte polynomial, there are several different generating function formulations, each of which has its advantages. We give one here and another in Subsection 3.3, with a variation in Subsection 7.2, and refer the reader to [BO92] for additional forms.

Definition 3 If G = (V, E) is a graph, then the Tutte polynomial of G, T(G; x, y), has the following expansion

$$T(G; x, y) = \sum_{A \subseteq E} (x - 1)^{r(E) - r(A)} (y - 1)^{n(A)}.$$
 (3.3)

The advantage of a generating function formulation is that it facilitates counting. For example, interpretations for several evaluations of the Tutte polynomial given in Section 5 follow immediately from Definition 3.

We can also deduce the following pleasing property of the Tutte polynomial.

Proposition 2 If G is a planar graph with dual G^* then

$$T(G; x, y) = T(G^*; y, x).$$
 (3.4)

This result follows from routine checking using Definition 3 and (2.1).

3.3 Spanning Trees Expansion Definition

We need to develop a little terminology before presenting the spanning trees definition of the Tutte polynomial. First, given a spanning tree S and an edge

 $e \notin S$, there is a cycle defined by e, namely the unique cycle in $S \cup e$. Similarly, for an edge $f \in S$, there is a cut defined by f, namely the set of edges C such that if $f' \in C$, then $(S - f) \cup f'$ is a spanning tree.

Assume there is a fixed ordering \prec on the edges of G, say $E = \{e_1, \ldots, e_m\}$, where $e_i \prec e_j$ if i < j. Given a fixed tree S, an edge f is called *internally active* if $f \in S$ and it is the smallest edge in the cut defined by f. Dually, an edge e is *externally active* if $e \not\in S$ and it is the smallest edge in the cycle defined by e. The internal activity of S is the number of its internally active edges and its external activity is the number of externally active edges. With this, we have the following definition of the Tutte polynomial.

Definition 4 If G is a graph with a total order on its edge set, then

$$T(G; x, y) = \sum_{i,j} t_{ij} x^i y^j, \tag{3.5}$$

where t_{ij} is the number of spanning trees with internal activity i and external activity j.

Two important observations follow immediately from the equivalence of Definitions 3 and 4. One is that the terms t_{ij} in Definition 4 are independent of the total order used in the edge set, since there is no ordering of the edges in Definition 3. The other is that the coefficients in Definition 3 must be non-negative since the coefficients in Definition 4 clearly are.

4 Universality of the Tutte Polynomial

The universality property discussed here is one of the most powerful aspects of the Tutte polynomial. It says that essentially any graph invariant that is multiplicative on disjoint unions and one-point joins of graphs and that has a deletion/contraction reduction must be an evaluation of the Tutte polynomial. We will see several applications of this theorem throughout the rest of this chapter and in the next chapter as well. Various generalizations of the Tutte polynomial are careful to retain this essential property, and analogous universality properties are sought in the context of other graph polynomials.

Definition 5 Let \mathcal{G} be a minor closed class of graphs. A graph invariant f from \mathcal{G} to a commutative ring \mathcal{R} with unity is called a generalized Tutte-Gröthendieck invariant, or T-G invariant, if $f(E_1)$ is the unity of R, if there exist fixed elements $a, b \in R$ such that for every graph $G \in \mathcal{G}$ and every ordinary edge $e \in G$, then

$$f(G) = af(G \setminus e) + bf(G/e); \tag{4.1}$$

and if for every $G, H \in \mathcal{G}$, whenever $G \cup H$ or G * H is in \mathcal{G} , then

$$f(G \cup H) = f(G)f(H) \text{ and } f(G * H) = f(G)f(H).$$
 (4.2)

Thus, the Tutte polynomial is a T-G invariant, and in fact, since the following two results give both universal and unique extension properties, it is essentially the *only* T-G invariant, in that any other must be an evaluation of it. Theorem 1 is known as a recipe theorem since it specifies how to recover a T-G invariant as an evaluation of the Tutte polynomial.

Theorem 1 Let \mathcal{G} be a minor closed class of graphs, let R be a commutative ring with unity, and let $f: \mathcal{G} \to R$. If there exists $a, b \in R$ such that f is a T-G invariant, then

$$f(G) = a^{|E(G)| - r(E(G))} b^{r(E(G))} T\left(G; \frac{x_0}{b}, \frac{y_0}{a}\right), \tag{4.3}$$

where $f(B) = x_0$ and $f(L) = y_0$.

Furthermore, we have the following unique extension property, which says that if we specify any four elements $a, b, x_0, y_0 \in R$, then there is a unique well-defined T-G invariant on these four elements.

Theorem 2 Let \mathcal{G} be a minor closed class of graphs, let R be a commutative ring with unity, and let $a, b, x_0, y_0 \in R$. Then there is a unique T-G invariant $f: \mathcal{G} \to R$ satisfying Definition 5 with $f(B) = x_0$ and $f(L) = y_0$. Furthermore, this function f is given by

$$f(G) = a^{|E(G)| - r(E(G))} b^{r(E(G))} T\left(G; \frac{x_0}{b}, \frac{y_0}{a}\right). \tag{4.4}$$

If a or b are not units of R, then (4.3) and (4.4) are interpreted to mean using expansion (3.3) of Definition 3, and cancelling before evaluating.

These results can be proved by induction on the number of ordinary edges from the deletion/contraction definition of the Tutte polynomial. See, for example, Brylawski [Bry72], Oxley and Welsh [OW79], Brylawski and Oxley [BO92], Welsh [Wel93], and Bollobás [Bol98] for detailed discussions of these theorems and their consequences.

Examples applying this important theorem may found throughout Section 6, where it may be used to show that all of the graph polynomials surveyed there are evaluations of the Tutte polynomial.

5 Combinatorial Interpretations of Some Evaluations

A graph polynomial encodes information about a graph. The challenge is in extracting combinatorially useful information from this algebraic object. A number of successful techniques have evolved for meeting this challenge, and we use the Tutte polynomial to showcase some of them while simultaneously demonstrating the richness of the information encoded by the Tutte polynomial.

5.1 Spanning Subgraphs

Spanning subgraphs, and in particular spanning trees, play a fundamental role in the theory of Tutte polynomials as we have already seen in Definition 4. This is also reflected in the most readily attainable interpretations for evaluations of the Tutte polynomial, which enumerate various spanning subgraphs. We begin with these here, writing $\tau(G)$ for the number of spanning trees of a connected graph G.

Theorem 3 If G = (V, E) is a connected graph then:

- 1. T(G; 1, 1) equals $\tau(G)$.
- 2. T(G; 2, 1) equals the number of spanning forests of G.
- 3. T(G; 1, 2) equals the number of spanning connected subgraphs of G.
- 4. T(G; 2, 2) equals $2^{|E|}$.

Proof. To illustrate common proof techniques, we give two short proofs of the first statement. The remaining statements may be proved similarly. When x = y = 1, the non-vanishing terms in the rank-nullity expansion (3.3) are $A \subseteq E$ such that r(E) = r(A) and |A| = r(A). That r(E) = r(A) implies that A has the same number of connected components as G, namely one, so (V, A) is connected. Then |A| = r(A) implies that |A| = |V| - 1, so A must be a tree, and hence a spanning tree.

Alternatively, we can use Theorem 1. Let $\tau'(G)$ be the number of maximal spanning forests in a general (not necessarily connected) graph G. We prove that $T(G;1,1)=\tau'(G)$. If G is connected, we have that $T(G;1,1)=\tau'(G)=\tau(G)$. First note that the number of maximal spanning forests has a deletion-contraction reduction for ordinary edges, that is, if G is a graph and e is an ordinary edge of G, then $\tau'(G)=\tau'(G\setminus e)+\tau'(G/e)$. This follows because the maximal spanning forests of G can be partitioned into the maximal spanning forests that do not contain e and those that do contain e. The former are the maximal spanning forests of $G\setminus e$ and the latter are in one-to-one correspondence with the maximal spanning forests of G/e.

The result then follows immediately from Theorem 1 with $a=b=x_0=y_0=1$.

Computing the number of spanning trees of a graph is easy in that there are polynomial time algorithms to do it. One of these involves a determinant. Recall that the *Laplacian matrix* L of an graph G with vertices v_1, \ldots, v_n is the $n \times n$ -matrix defined by

$$L_{ij} = \begin{cases} \deg(i) & \text{if } i = j \\ -r & \text{if } r \text{ is the number of edges between vertices } i \text{ and } j. \end{cases}$$
 (5.1)

Theorem 4 If G is a connected graph with Laplacian L, then

$$T(G; 1, 1) = \tau(G) = \text{Det}(L'),$$
 (5.2)

where L' is any cofactor of L.

A proof of this can be found in [AZ01] using the Binet-Cauchy formula and that $L = DD^t$, where D is the incidence matrix of (an orientation of) G.

This result not only provides an interpretation of the Tutte polynomial at (1,1) in terms of the incidence matrix of a graph, but also proves that (1,1) is one of the (very few, as we will see in Section 8) points where the Tutte polynomial can be computed in polynomial time.

5.2 The Tutte Polynomial at y = x

Combinatorial interpretations are known for the Tutte polynomial at all integer values along the line y = x. In addition to those for T(G, 1, 1) and T(G, 2, 2) previously given, we have the following interpretation for T(G; -1, -1) due to Read and Rosenstiehl [RR78]. We will also see alternative interpretations for T(G; -1, -1) and T(G; 3, 3) in Subsection 5.3.

The incidence matrix D of a graph G defines a vector space over \mathbb{Z}_2 , called the cycle space \mathcal{C} . The bicycle space \mathcal{B} is then just $\mathcal{C} \cap \mathcal{C}^{\perp}$.

Theorem 5
$$T(G; -1, -1) = (-1)^{|E|} (-2)^{\dim(\mathcal{B})}$$
.

One method of extracting information from a graph polynomial is via its relation to some other graph invariant. The following interpretations for the Tutte polynomial of a planar graph along the line x=y derive from its relation to the Martin polynomial [Mar77], a one variable graph polynomial that we will discuss in further in the next chapter.

We first recall that the *medial graph* of a connected planar graph G is constructed by placing a vertex on each edge of G and drawing edges around the faces of G. The faces of this medial graph are colored black or white, depending on whether they contain or do not contain, respectively, a vertex of the original graph G. This face two-colors the medial graph. The edges of the medial graph are then directed so that the black face is on the left. We refer to this as the directed medial graph of G and denote it by \overline{G}_m . An example is given in Fig. 2.

Martin [Mar77] showed that, for a planar graph G, the relation between the Martin polynomial and Tutte polynomial is $m(\vec{G}_m; x) = T(G; x, x)$. Evaluations for the Martin polynomial in [E-M04a] then give the following interpretations of the Tutte polynomial.

Let $D_n(\vec{G}_m) = \{(D_1, \dots, D_n)\}$, where (D_1, \dots, D_n) is an ordered partition of $E(\vec{G}_m)$ into n subsets such that G restricted to D_i is 2-regular and consistently oriented for all i.

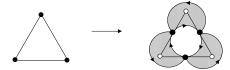


Figure 2: On the left hand side we have a planar graph G. On the right hand side we have \vec{G}_m with the vertex faces colored black, oriented so that black faces are to the left of each edge

Theorem 6 Let G be a planar graph with oriented medial graph \vec{G}_m . Then, for n a positive integer,

$$(-n)^{c(G)}T(G;1-n,1-n) = \sum_{D_n(\vec{G}_m)} (-1)^{\sum_{i=1}^n c(D_i)}.$$

Theorem 7 Let G be a planar graph with oriented medial graph \vec{G}_m . Then, for n a positive integer,

$$n^{c(G)}T(G;1+n,1+n) = \sum 2^{\mu(\phi)},$$

where the sum is over all edge colorings ϕ of \vec{G}_m with n colors so that each (possibly empty) set of monochromatic edges forms an Eulerian digraph, and where $\mu(\phi)$ is the number of monochromatic vertices in the coloring ϕ .

5.3 Orientations and Score Vectors

The combinatorial interpretations of the Tutte polynomial in Theorem 3 are given in terms of the number of certain subgraphs of the graph G. However, they can also be given in terms of orientations of the graph and its score vectors. Given a graph G = (V, E), an orientation of G may be obtained by directing every edge from either of its ends to the other. From this follows that T(G; 2, 2) equals the number of possible orientations of G.

The score vector of an orientation \vec{G} is the vector (s_1, s_2, \ldots, s_n) such that vertex i has outdegree s_i in the orientation. In the following theorem we gather several similar results about the Tutte polynomial and orientations of a graph.

Theorem 8 Let G = (V, E) be a connected graph with Tutte polynomial T(G; x, y). Then

- 1. T(G; 2, 0) equals the number of acyclic orientations of G, that is orientations without oriented cycles.
- 2. T(G; 0, 2) equals the number of totally cyclic orientations of G, that is orientations in which every arc is in a directed cycle.

- 3. T(G; 1, 0) equals the number of acyclic orientations with exactly one predefined source v.
- 4. T(G; 2, 1) equals the number of score vectors of orientations of G.

Item 1 was first proved by Stanley in [Sta73] by using the Ehrhart polynomial; a proof using the universality of the Tutte polynomial is given by Brylawski and Oxley in [BO92]. For Items 2 and 3, see Green and Zaslavsky [GZ83] and Las Vergnas [Las77]. The former also gives the number of strongly connected orientations of G, and note that the latter is independent of the choice of source vertex v. Item 4 was first proved by Stanley in [Sta80], with a bijective proof given by Kleitman and Winston in [KW81], and a proof using Theorem 1 by Brylawski and Oxley in [BO92]. Comparing Item 4 with Theorem 3, Item 2, shows that the number of score vectors equals the number of spanning forests of G.

Some other evaluations of the Tutte polynomial can also be interpreted in terms of orientations. Recall that an anticircuit in a digraph is a closed trail so that the directions of the edges alternate as the trail passes through any vertex of degree greater than 2. Note that in a 4-regular Eulerian digraph such as \vec{G}_m , the set of anticircuits can be found by pairing the two incoming edges and the two outgoing edges at each vertex.

Two surprising results from Las Vergnas [Las88] and Martin [Mar78] are the following:

Theorem 9 Let G be a connected planar graph. Then

$$T(G; -1, -1) = (-1)^{|E(G)|} (-2)^{a(\vec{G}_m) - 1}$$
(5.3)

and

$$T(G;3,3) = K2^{a(\vec{G}_m)-1},$$
 (5.4)

where $a(\vec{G}_m)$ is the number of anticircuits in the directed medial graph of G and K is some odd integer.

Comparing (5.3) to Theorem 5 gives the following corollary.

Corollary 1 If G be a connected planar graph, then the dimension of the bicycle space is $a(\vec{G}_m) - 1$.

6 Some Specializations

Here we illustrate the wide range of applicability of the Tutte polynomial while demonstrating some proof techniques for showing that a graph invariant is related to the Tutte polynomial. The advantage of recognizing an application-driven function as a specialization of the Tutte polynomial is that the large body of knowledge about the Tutte polynomial is then available to inform the desired application. We say a graph polynomial is a specialization of the Tutte

polynomial if it may be recovered from the Tutte polynomial by some substitution for x and y, with possibly some prefactor.

For various substitutions along different algebraic curves in x and y, the Tutte polynomial has interpretations as the generating function of combinatorial quantities or numerical invariants associated with a graph. Some of these were considered long before the development of the Tutte polynomial, and others were discovered to be unexpectedly related to the Tutte polynomial. We survey six of the more well known of these application-driven generating functions.

6.1 The Chromatic Polynomial

The chromatic polynomial, because of its theoretical and applied importance, has generated a large body of work. Chia [Chi97] provides an extensive bibliography on the chromatic polynomial, and Dong, Koh, and Teo [DKT05] give a comprehensive treatment.

For positive integer λ , a λ -coloring of a graph G is a mapping of V(G) into the set $[\lambda] = \{1, 2, ..., \lambda\}$. Thus there are exactly λ^n colorings for a graph on n vertices. If ϕ is a λ -coloring such that $\phi(i) \neq \phi(j)$ for all $ij \in E$, then ϕ is called a *proper* (or *admissible*) coloring.

We wish to find the number, $\chi(G;\lambda)$, of admissible λ -colorings of a graph G. As noted by Whitney [Whi32], the 4-color theorem can be formulated in this general setting as follows: If G is planar graph, then $\chi(G;4)>0$.

The following theorem is due to G. D. Birkhoff in [Bir12] and independently by H. Whitney in [Whi32]. We sketch the second proof.

Theorem 10 If G = (V, E) is a graph, then

$$\chi(G;\lambda) = \sum_{A \subseteq E} (-1)^{|A|} \lambda^{\kappa(A)}. \tag{6.1}$$

Proof. Let P_{ij} be the set of λ -colorings such that vertices i and j receive the same color. Let \bar{P}_{ij} be the complement of P_{ij} in the set of λ -colorings. Then, the value $\chi(G;\lambda)$ can be computed using the inclusion-exclusion principle.

$$\chi(G;\lambda) = \left| \bigcap_{ij \in E} \bar{P}_{ij} \right|
= \lambda^n - \sum_{ij \in E} |P_{ij}| + \sum_{\substack{ij,kl \in E \\ ij \neq kl}} |P_{ij} \cap P_{kl}| - \cdots + (-1)^{|E|} \left| \bigcap_{ij \in E} P_{ij} \right|.$$
(6.2)

Every term is of the form $|\cap_{ij\in A} P_{ij}|$ for some $A\subseteq E$, and hence corresponds to the subgraph (V,A), where A is the set of edges given by the indices of the P_{ij} 's. Thus, the cardinality of this set is the number of λ -colorings that have

a constant value on each of the connected components of (V, A), that is, $\lambda^{\kappa(A)}$. The sum on the right-hand side of (6.1) is then precisely (6.2).

Thus, $\chi(G; \lambda)$ is a polynomial on λ and it is called the *chromatic polynomial* of G. Some easily seen properties of $\chi(G; \lambda)$, which can be found in Read's seminal work [Rea68] on the chromatic polynomial, are the following:

Proposition 3 If G is a graph with chromatic polynomial $\chi(G; \lambda)$, then:

- 1. If G has no edges, then $\chi(G; \lambda) = \lambda^n$.
- 2. If G has a loop, then $\chi(G; \lambda) = 0$, for all λ .
- 3. $\chi(K_n; \lambda) = \lambda(\lambda 1) \cdots (\lambda n + 1)$.
- 4. If e is any edge of G, then

$$\chi(G; \lambda) = \chi(G \setminus e; \lambda) - \chi(G/e; \lambda).$$

Note that Items 1 and 4 give a recursive alternative definition of the chromatic polynomial.

Also in Read [Rea68] is the following not so trivial, but not difficult to prove, property of the chromatic polynomial.

Theorem 11 If G is the union of two vertex set induced subgraphs H_1 and H_2 such that the intersection $H_1 \cap H_2$ is a vertex set induced subgraph isomorphic to K_p , then

$$\chi(G;\lambda) = \frac{\chi(H_1;\lambda)\chi(H_2;\lambda)}{\chi(K_p;\lambda)}.$$

Thus, although Proposition 3 Item 4 suggests that the chromatic polynomial might be a T-G invariant, by Theorem 11, it is *not* multiplicative on the one point join of two graphs. However, as is frequently the case, this can be addressed by a simple multiplier; it is easy to check that $\lambda^{-\kappa(G)}\chi(G;\lambda)$ is a T-G invariant. The relation between the Tutte and chromatic polynomials may then be found by applying Theorem 1 with the help of Proposition 3, Item 4. We give an alternative proof of this relationship deriving from Theorem 10.

Theorem 12 If G = (V, E) is a graph, then

$$\chi(G;\lambda) = (-1)^{r(E)} \lambda^{\kappa(G)} T(G; 1 - \lambda, 0).$$

Proof. Since $r(E) - r(A) = \kappa(A) - \kappa(G)$ we have that

$$\begin{split} \chi(G;\lambda) &= \sum_{A\subseteq E} (-1)^{|A|} \lambda^{\kappa(A)} \\ &= (-1)^{r(E)} \lambda^{\kappa(G)} \sum_{A\subseteq E} (-1)^{|A|-r(A)} (-\lambda)^{r(E)-r(A)} \\ &= (-1)^{r(E)} \lambda^{\kappa(G)} T(G; 1-\lambda, 0), \end{split}$$

with the last equality following from Definition 3.

6.2 The Bad Coloring Polynomial

One way to generalize the chromatic polynomial is to count all possible colorings of the graph G, not just proper colorings. In order to differentiate between proper and improper colorings, we keep track of the edges between vertices of the same color, calling them $bad\ edges$. This leads to the $bad\ coloring\ polynomial$.

Definition 6 The bad coloring polynomial is the generating function

$$B(G; \lambda, t) = \sum_{j} b_{j}(G; \lambda) t^{j},$$

where $b_j(G; \lambda)$ is the number of λ -colorings of G with exactly j bad edges.

Now consider $B(G; \lambda, t+1)$, which can be written as

$$B(G; \lambda, t+1) = \sum_{\phi: V \to [\lambda]} (1+t)^{|b(\phi)|}, \tag{6.3}$$

where $b(\phi)$ is the set of bad edges in the λ -coloring ϕ . With this last expression is again easy to get the relation to the Tutte polynomial using the following derivation of S. D. Noble (private communication).

Theorem 13 For a graph G = (V, E) we have that

$$B(G; \lambda, t+1) = t^{r(E)} \lambda^{\kappa(G)} T\left(G; \frac{\lambda+t}{t}, 1+t\right).$$

Proof.

$$\begin{split} B(G;\lambda,t+1) &= \sum_{\phi:V\to[\lambda]} (1+t)^{|b(\phi)|} \\ &= \sum_{\phi:V\to[\lambda]} \sum_{A\subseteq b(\phi)} t^{|A|} \\ &= \sum_{A\subseteq E} \sum_{\phi:V\to[\lambda]} t^{|A|} \\ &= \sum_{A\subseteq E} t^{|A|} \lambda^{\kappa(A)}. \end{split}$$

Thus,

$$\begin{split} B(G;\lambda,t+1) &= \sum_{A\subseteq E} t^{|A|} \lambda^{\kappa(A)} \\ &= t^{r(E)} \lambda^{\kappa(G)} \sum_{A\subseteq E} t^{|A|-r(A)} \left(\frac{\lambda}{t}\right)^{r(E)-r(A)} \\ &= t^{r(E)} \lambda^{\kappa(G)} T\left(G;1+\frac{\lambda}{t},t+1\right). \end{split}$$

Again, the above result could also be obtained from the universal property of the Tutte polynomial given in Theorem 1 by applying it to $\bar{B}(G;\lambda,t)=\lambda^{-\kappa(G)}B(G;\lambda,t)$ and verifying that

- 1. $\bar{B}(G; \lambda, t) = \bar{B}(G \setminus e; \lambda, t) + (t 1)\bar{B}(G/e; \lambda, t)$, if e is an ordinary edge.
- 2. $\bar{B}(G; \lambda, t) = t\bar{B}(G \setminus e; \lambda, t)$, if e is a loop.
- 3. $\bar{B}(G; \lambda, t) = (t + \lambda 1)\bar{B}(G/e; \lambda, t)$, if e is a bridge.

6.3 The Flow Polynomial

The dual notion to a proper λ -coloring is a nowhere zero λ -flow. A standard resource for the material in this subsection is Zhang [Zha97], while Jaeger [Jae88] gives a good survey.

Let G be a graph with an arbitrary but fixed orientation, and let H be an Abelian group with 0 as its identity element. An H-flow is a mapping ϕ of the oriented edges $\vec{E}(G)$ into the elements of the group H such that Kirchhoff's law is satisfied at each vertex of G, that is

$$\sum_{\vec{e}=u\to v} \phi(\vec{e}) + \sum_{\vec{e}=u\leftarrow v} \phi(\vec{e}) = 0,$$

for every vertex v, and where the first sum is taken over all arcs towards v and the second sum is over all arcs leaving v. An H-flow is nowhere zero if ϕ never takes the value 0.

By replacing the group element on an edge e by its inverse, it is clear that two orientations that differ only in the direction of exactly one arc \vec{e} have the same number of nowhere zero H-flows for any H. Thus, this number does not depend on the choice of orientation of G. In fact, when H is finite, it does not depend on the structure of the group, but rather only on its cardinality. The following, due to Tutte [Tut54], relates the number of nowhere zero flows of G over a finite group and Tutte polynomial of G.

Theorem 14 Let G = (V, E) be a graph and H a finite Abelian group. If $\chi^*(G; H)$ denotes the number of nowhere zero H-flows then

$$\chi^*(G;H) = (-1)^{|E|-r(E)} T(G;0,1-|H|).$$

Proof. [sketch] Here we use the universality of the Tutte polynomial. If e is an ordinary edge of G, then the number of nowhere zero H-flows in G/e can be partitioned into two sets P_1 and P_2 . We let P_1 consist of those that are also nowhere zero H-flows in $G \setminus e$, and P_2 be the complement of P_1 . Clearly then $|P_1| = \chi^*(G \setminus e; H)$. Furthermore, there is a bijection between the elements in P_2 and the nowhere zero H-flows in G, and thus $|P_2| = \chi^*(G; H)$. It follows that

$$\chi^*(G; H) = \chi^*(G \setminus e; H) - \chi^*(G/e; H),$$

and hence $\chi^*(G; H)$ satisfies (4.1). It is also easy to check that $\chi^*(G; H)$ satisfies (4.2). Since $\chi^*(L; H) = 0$ and $\chi^*(B; H) = |H| - 1$, the result follows from Theorem 1.

Consequently, $\chi^*(G; \lambda)$ is a polynomial called the *flow polynomial* which for λ an integer at least 1 gives the number of nowhere zero flows of G in a group of order λ . We call any nowhere zero H-flow simply a λ -flow if $|H| = \lambda$.

If the Abelian group is \mathbb{Z}_3 , and the graph is 4-regular, then the Tutte polynomial at (0, -2) counts the number of nowhere zero \mathbb{Z}_3 -flows on G. But these flows are in one-to-one correspondence with orientations such that at each vertex exactly two edges are directed in and two out. Such an orientation is called an *ice configuration* of G (see Lieb [Lie67] and Pauling [Pau35] for this important model of ice and its physical properties). Thus, we have the following corollary.

Corollary 2 If G is a 4-regular graph, then T(G; 0, -2) equals the number of ice configurations of G.

We mentioned that proper colorings are the dual concept of nowhere zero flows, and now with Theorem 6.3 and (3.4) we observe that

$$\chi(G;\lambda) = \lambda \chi^*(G^*;\lambda),$$

for G a connected planar graph and G^* , any of its plane duals. Thus, to each λ -proper coloring in G corresponds λ nowhere zero \mathbb{Z}_{λ} -flows of G^* . A bijective proof can be found in Diestel [Die00].

Thus, by the 4-color theorem and the duality relation between colorings and nowhere zero H-flows, every bridgeless planar graph has a 4-flow. For cubic graph, having a nowhere zero $\mathbb{Z}_2 \times \mathbb{Z}_2$ -flow is equivalent to be 3-edge-colorable. Therefore, as the Petersen graph is not 3-edge-colorable, it has no 4-flow. However, the Petersen graph does have a 5-flow. In fact, the famous 5-flow conjecture of Tutte [Tut54] postulates that every bridgeless graph has a 5-flow.

The 5-flow conjecture is clearly difficult as it is not even apparent that every graph will have a λ -flow for some λ . However, Jaeger [Jae76] proved that every bridgeless graph has an $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ -flow, thus every bridgeless graph has a 8-flow. Subsequently Seymour [Sey81] proved that every bridgeless graph has a $\mathbb{Z}_2 \times \mathbb{Z}_3$ -flow, thus every graph has a 6-flow.

Not much is currently known about properties of the flow polynomial apart from those that can be deduced from its duality with the chromatic polynomial and efforts to solve the 5-flow conjecture. However, for some recent work in this direction, see Dong and Koh [DK07] and Jackson [Jac07].

6.4 Abelian Sandpile Models

Self-organized criticality is a concept widely considered in various domains since Bak, Tang and Wiesenfeld [BTW88] introduced it. One of the paradigms in this framework is the Abelian sandpile model, introduced by Dhar [Dha90].

We begin by recalling the definition of the general Abelian sandpile model on a set of N sites labeled 1, 2, ..., N, that we refer to as the system. A sandpile at each site i has height given by an integer h_i . The set $\vec{h} = \{h_i\}$ is called a *configuration* of the system. For every site i, a threshold H_i is defined; configurations with $h_i < H_i$ for all i are called *stable*. For every stable configuration, the height h_i increases in time at a constant rate; this is called the *loading* of the system. This loading continues until $h_i \geq H_i$ for some i. The site i then 'topples' and all the values h_j , for $1 \leq j \leq N$, are updated according to the rule:

$$h_j = h_j - M_{ij}, \quad \text{for all } j, \tag{6.4}$$

where M is a given fixed integer matrix satisfying

$$M_{ii} > 0$$
, $M_{ij} \le 0$ and $s_i = \sum_j M_{ij} \ge 0$.

If, after this redistribution, the height at some vertex exceeds its threshold, we again apply the toppling rule (6.4), and so on, until we arrive at a stable configuration and the loading resumes. The sequence of topplings is called an *avalanche*. We assume that an avalanche is "instantaneous", so that no loading occurs during an avalanche.

The value s_i is called the dissipation at site i. We say that s_i is dissipative if $s_i > 0$ and non-dissipative if $s_i = 0$. It may happen that an avalanche continues without end. We can avoid this possibility by requiring that from every non-dissipative site i, there exists a path to a dissipative site j. In other words, there is a sequence i_0, \ldots, i_n , with $i_0 = i$, $i_n = j$ and $M_{i_{k-1}, i_k} < 0$, for $k = 1, \ldots, n$. In this case, following Gabrielov [Gab93], we say that the system is weakly-dissipative, and we assume that a system is always weakly-dissipative. In a weakly-dissipative system, any configuration \vec{h} will eventually arrive at a stable configuration. But the process is infinite, and the stable configurations are clearly finite. Thus, some stable configurations recur, and these are called critical configurations.

The sandpile process has an Abelian property, in that if at some stage, two sites can topple, the resulting stable configurations after the avalanche is independent of the order in which the sites toppled. Thus, for any configuration \vec{h} , the process eventually arrives at a unique critical configuration \vec{c} .

Let G be a graph, $q \in V(G)$, and L' be the minor of the Laplacian of G resulting from deleting the row and column corresponding to q. When the matrix M is L' for some vertex q, the Abelian sandpile model coincides with the chip-firing game or dollar game on a graph that was defined by Biggs [Big96b]. For the rest of this Subsection we assume M is given in this way.

For the rest of this Subsection we assume M is given in this way. For a configuration \vec{h} , we define its weight to be $w(\vec{h}) = \sum_{i=0}^{N} h_i$. If \vec{c} is a critical configuration, we define its level as

$$\operatorname{level}(\vec{c}) = w(\vec{c}) - |E(G)| + \deg(q).$$

This definition may seem a little unnatural, but it is justified by the following theorem of Biggs [Big96b], which tell us that it is actually the right quantity to consider if we want to grade the critical configurations.

Theorem 15 If G = (V, E) is a graph and \vec{h} a critical configuration of G, then

$$0 \le \operatorname{level}(\vec{h}) \le |E| - |V| + 1.$$

The right-most quantity is called the *cyclomatic number* of G. We now consider the generating function of the of these critical configurations.

Definition 7 Let G = (V, E) be a graph and for nonnegative integers i let c_i be the number of critical configurations with level i. Then the critical configuration polynomial is

$$P_q(G; y) = \sum_{i=0}^{|E|-|V|+1} c_i y^i.$$

Theorem 16 For a graph G and any vertex q, the generating function of the critical configurations equals the Tutte polynomial of G along the line x = 1, that is,

$$P_q(G; y) = T(G; 1, y),$$

and thus $P_q(G; y)$ is independent of the choice of q.

A proof using deletion and contraction of an edge incident with the special vertex q can be found in [Mer97].

New combinatorial identities frequently arise when a new generating function can be shown to be related to the Tutte polynomial, as in the following corollary.

Corollary 3 If G is a connected graph, then the number of critical configurations of G is equal to the number of spanning trees, and the number of critical configurations with level 0 is equal to the number of acyclic orientations with a unique source.

This follows from comparing Theorem 16 with Theorems 3 and 8.

6.5 The Reliability Polynomial

Many of the invariants reviewed thus far have applications in the sciences, engineering and computer science, but the reliability polynomial we discuss next is among the most directly applicable.

Definition 8 Let G be a connected graph or network with n vertices and m edges, and suppose that each edge is independently chosen to be active with probability p. Then the (all terminal) reliability polynomial is

$$R(G; p) = \sum_{\substack{A \text{ spanning} \\ connected}} p^{|A|} (1-p)^{|E-A|}$$

$$= \sum_{k=0}^{m-n+1} g_k p^{k+n-1} (1-p)^{m-k-n+1},$$
(6.5)

where g_k is the number of spanning connected subgraphs with k + n - 1 edges.

Thus the reliability polynomial, R(G; p), is the probability that in this random model there is a path of active edges between each pair of vertices of G.

Theorem 17 If G is a connected graph with m edges and n vertices, then

$$R(G; p) = p^{n-1}(1-p)^{m-n+1}T\left(G; 1, \frac{1}{1-p}\right).$$

Proof. We first note from the rank generating expansion of Definition 3 that

$$T(G; 1, y + 1) = \sum_{k=0}^{m-n+1} g_k y^k,$$

since the only non-vanishing terms are those corresponding to $A \subseteq E$ with r(E) = r(A), that is spanning connected subgraphs.

We then observe that

$$R(G;p) = \sum_{k=0}^{m-n+1} g_k p^{k+n-1} (1-p)^{m-k-n+1}$$

$$= p^{n-1} (1-p)^{m-n+1} \sum_{k=0}^{m-n+1} g_k \left(\frac{p}{1-p}\right)^k$$

$$= p^{n-1} (1-p)^{m-n+1} T\left(G; 1, 1 + \frac{p}{1-p}\right)$$

$$= p^{n-1} (1-p)^{m-n+1} T\left(G; 1, \frac{1}{1-p}\right).$$

If we extend the reliability polynomial to graphs with more than one component by defining $R(G \cup H; p) = R(G, p)R(H, p)$, then this result may also be proved using the universality property of the Tutte polynomial. Observe that if an ordinary edge is not active (this happens with probability 1-p), then the reliability of the network is the same as if the edge were deleted. Similarly, if an edge is active (which happens with probability p), then the reliability is the same as it would be if the edge were contracted. Thus, the reliability polynomial has the following deletion/contraction reduction:

$$R(G; p) = (1 - p)R(G \setminus e) + pR(G/e).$$

With this, and noting that R(G * H; p) = R(G; p)R(H; p) with R(L, p) = 1 and R(B; p) = p, Theorem 17 also follows immediately from Theorem 1.

There is a vast literature about reliability and the reliability polynomial; for a good survey, including a wealth of open problems, we refer the reader to Chari and Colbourn [CC97].

6.6 The Shelling Polynomial

A simplicial complex Δ is a collection of subsets of a set of vertices V such that if $v \in V$, then $\{v\} \in \Delta$ and also if $F \in \Delta$ and $H \subseteq F$, then $H \in \Delta$. The elements of Δ are called *faces*. Maximal faces are called *facets*, and if all the facets have the same cardinality, Δ is called *pure*. The dimension of a face is its size minus one and the dimension of a pure simplicial complex is the dimension of any of its facets.

If f_k is the number of faces of size k in a simplicial complex Δ , then the vector (f_0, f_1, \ldots, f_d) is called the *face vector* or f-vector of Δ , and

$$f_{\Delta}(x) = \sum_{k=0}^{d} f_k x^{d-k},$$
 (6.6)

is the generating function of the faces of Δ , or face enumerator.

The collection of spanning forests of a connected graph G forms a pure (d-1) dimensional simplicial complex $\Delta(G)$. The points of $\Delta(G)$ are the non-loop edges of G and its facets are the spanning trees, so d=r(E). The collection of complements of spanning connected subgraphs of G also forms a pure (d^*-1) dimensional simplicial complex $\Delta^*(G)$. Here the elements are the non-bridge edges, while the facets are complements of spanning trees, when viewed as subsets of E; in general, if A is the edge-set of a spanning connected subgraph of G of cardinality k+n-1, then $E\setminus A$ is a face of size m-n+1-k in $\Delta^*(G)$. Thus, $d^*=m-n+1$ and if, as before, g_k is the number of spanning connected subgraphs with k+n-1 edges, the f-vector of $\Delta^*(G)$ is $(f_0^*,\ldots,f_{d^*}^*)$, where $f_i^*=g_{d^*-i}$.

Theorem 18 The Tutte polynomial gives the face enumerators for both $\Delta(G)$ and $\Delta^*(G)$:

$$T(G; x+1, 1) = \sum_{k=0}^{d} f_k x^{d-k} = f_{\Delta(G)}(x),$$

and

$$T(G; 1, y + 1) = \sum_{i=0}^{d^*} f_i^* y^{d^* - i} = f_{\Delta^*(G)}(x).$$

Proof. This follows readily by comparing (6.6) with Definition 3.

For a pure simplicial complex Δ , a *shelling* is a linear order of the facets F_1 , F_2, \ldots, F_t such that, if $1 \leq k \leq t$, then F_k meets the complex generated by its predecessors, denoted Δ_{k-1} , in a non-empty union of maximal proper faces. A complex is said to be *shellable* if it is pure and admits a shelling. A good exposition of the following results can be found in Björner [Bjo92].

For $1 \le k \le t$, define $\mathcal{R}(F_k) = \{x \in F_k | F_k \setminus x \in \Delta_{k-1}\}$, where here $\Delta_0 = \emptyset$. The number of facets such that $|F_k - \mathcal{R}(F_k)| = i$ is denoted by h_i and it does not

depend on the particular shelling (this follows for example from (6.7) below). The vector (h_0, h_1, \ldots, h_d) is called the h-vector of Δ . The shelling polynomial is the generating function of the h-vector, and is given by

$$h_{\Delta}(x) = \sum_{i=0}^{d} h_i x^{d-i}.$$

The face enumerator and shelling polynomial are related in a somewhat surprising way, namely

$$h_{\Delta}(x+1) = f_{\Delta}(x). \tag{6.7}$$

Both $\Delta(G)$ and $\Delta^*(G)$ are known to be shellable, see for example Provan and Billera [PB80]), and thus (6.7) gives the following corollary to Theorem 18, relating the two shelling polynomials to the Tutte polynomial (see Björner [Bjo92]).

Corollary 4 Let G be a graph. Then

$$T(G; x, 1) = h_{\Delta(M)}(x) = \sum_{i=0}^{d} h_i x^{d-i}$$

and

$$T(G; 1, y) = h_{\Delta * (G)}(y) = \sum_{i=0}^{d^*} h_i^* y^{d^* - i}.$$

The reader may have noticed that the reliability polynomial as well as the face enumerator and shelling polynomial of $\Delta^*(G)$ are all specializations of the Tutte polynomial along the line x=1. There is an important open conjecture in algebraic combinatorics about the h-vectors (and hence the shelling polynomials), of the two complexes coming from a graph (or, more generally, a matroid), namely that they are 'pure O-sequences'. For more details see Stanley [Sta96b] or [Mer01]. The latter also relates the shelling polynomial and the chip firing game. Let G be a graph with n vertices and m edges. From Corollary 4 and Theorem 16, we get that $c_i = h_{m-n+1-i}^*$, where c_i is the number of critical configurations of level i of G and $(h_0^*, \ldots, h_{m-n+1}^*)$ is the h-vector of $\Delta^*(G)$. In [Mer01] it is proved that (c_{m-n+1}, \ldots, c_0) is a pure O-sequence. Thus, the conjecture is true for the simplicial complex $\Delta^*(G)$ but is still open for $\Delta(G)$.

It is also clear from Theorem 17 and Corollary 4 that the reliability and shelling polynomials are related. This connection is explored, and open questions related to it presented, by Chari and Colbourn [CC97].

7 Some Properties of the Tutte Polynomial

There is a large and ever-growing body of information about properties of the Tutte polynomial. Here, we present some of them, again with an emphasis on illustrating general techniques for extracting information from a graph polynomial.

7.1 The Beta Invariant

Even a single coefficient of a graph polynomial can encode a remarkable amount of information. It may characterize entire classes of graphs and have a number of combinatorial interpretations. A noteworthy example is the β invariant, introduced (in the context of matroids) by Crapo in [Cra67].

Definition 9 Let G = (V, E) be a graph with at least two edges. The β invariant of G is

$$\beta\left(G\right) = (-1)^{r(G)} \sum_{A \subset E} (-1)^{|A|} r\left(A\right).$$

The beta invariant is a deletion/contraction invariant, that is, it satisfies (4.1). However, the β invariant is zero if and only if G either has loops or is not two-connected. Thus, the β invariant is not a Tutte-Gröthendieck invariant in the sense of Section 4. While the β invariant may be defined to be 1 for a single edge or a single loop, it still will not satisfy (4.2), and it is not multiplicative with respect to disjoint unions and one-point joins. Nevertheless, the β invariant derives from the Tutte polynomial.

Theorem 19 If G has at least two edges, and we write T(G; x, y) in the form $\sum t_{ij}x^iy^j$, then $t_{0,1} = t_{1,0}$, and this common value is equal to the β invariant.

Proof. This can easily be proved by induction, using deletion/contraction for an ordinary edge, and otherwise noting that the β invariant is zero if the graph has loops or is not two-connected.

The β invariant does not change with the insertion of parallel edges or edges in series. Thus, homeomorphic graphs have the same β invariant. The β invariant is also occasionally called the chromatic invariant, because $\chi'(G;1) = (-1)^{r(G)} \beta(G)$, where $\chi(G;x)$ is the chromatic polynomial.

Definition 10 A series-parallel graph is a graph constructed from a digon (two vertices joined by two edges in parallel) by repeatedly adding an edge in parallel to an existing edge, or adding an edge in series with an existing edge by subdividing the edge. Series-parallel graphs are loopless multigraphs, and are planar.

Brylawski, [Bry71] and also [Bry82], in the context of matroids, showed that the β invariant completely characterizes series-parallel graphs.

Theorem 20 *G* is a series-parallel graph if and only if $\beta(G) = 1$.

Using the deletion/contraction definition of the Tutte polynomial, it is quite easy to show that the β invariant is unchanged by adding an edge in series or in parallel to another edge in the graph. This, combined with the β invariant of a digon being one, suffices for one direction of the proof. The difficulty is in the reverse direction, and the proof is provided in [Bry72] by a set of equivalent characterizations for series-parallel graphs, one by excluded minors and another

that the β invariant is 1 for series-parallel graphs. For graphs, the excluded minor is K_4 (cf. Duffin [Duf65] and Oxley [Oxl82]). Succinct proofs may also be found in Zaslavsky [Zas87]. The fundamental observation, which may be applied to other situations, is that there is a graphical element, here an edge which is in series or parallel with another edge, which behaves in a tractable way with respect to the computation methods of the polynomial.

The β invariant has been explored further, for example by Oxley in [Oxl82] and by Benashski, Martin, Moore and Traldi in [BMMT95]. Oxley characterized 3-connected matroids with $\beta \leq 4$, and a complete list of all simple 3-connected graphs with $\beta \leq 9$ is given in [BMMT95].

A wide variety of combinatorial interpretations have also been found for the β invariant. Most interpretations involve objects other than graphs, but we give two graphical interpretations below. The first is due to Las Vergnas [Las84].

Theorem 21 Let G be a connected graph. Then $2\beta(G)$ gives the number of orientations of G that have a unique source and sink, independent of their relative locations.

This result is actually a consequence of a more general theorem giving an alternative formulation of the Tutte polynomial, which will be discussed further in Subsection 7.2. We also have the following result from [E-M04a].

Theorem 22 Let G = (V, E) be a connected planar graph with at least two edges. Then

$$\beta = \frac{1}{2} \sum \left(-1\right)^{c(E \setminus P) + 1},$$

where the sum is over all closed trails P in \vec{G}_m which visit all its vertices at least once.

Like the interpretations for T(G; x, x) given in the Subsection 5.2, this result follows from the Tutte polynomials relation to the Martin polynomial.

Graphs in a given class may have β invariants of a particular form. Mc-Kee [McK01] provides an example of this in dual-chordal graphs. A dual-chordal graph is 2-connected, 3-edge-connected, such that every cut of size at least four creates a bridge. A θ graph has two vertices with three edges in parallel between them. A dual-chordal graph has the property that it may be reduced to a θ graph by repeatedly contracting induced subgraphs of the following forms: digons, triangles, and $K_{2,3}$'s, where in all cases each vertex has degree 3 in G.

Theorem 23 If G is a dual-chordal graph, then $\beta(G) = 2^a 5^b$. Here, a is the number of triangles in G, where each vertex has degree 3 in G, that are contracted in reducing G to a θ graph. Similarly, b is the number of induced $K_{2,3}$ in G, again where each vertex has degree 3 in G, that are contracted in reducing G to a θ graph.

The proof follows from considering the acyclic orientations of G with unique source and sink and applying the results of Green and Zaslavsky [GZ83].

7.2 Coefficient Relations

After observing that $t_{1,0} = t_{0,1}$ in the development of the β invariant, it is natural to ask if there are similar relations among the coefficients t_{ij} of the Tutte polynomial $T(G; x, y) = \sum t_{ij} x^i y^j$ and whether there are combinatorial interpretations for these coefficients as well. The answer is yes, although less is known. The most basic fact, and one which is not obvious from the rank-nullity formulation of Definition 3, is that all the coefficients of the Tutte polynomial are non-negative.

That $t_{1,0} = t_{0,1}$ is one of an infinite family of relations among the coefficients of the Tutte polynomial. Brylawski [Bry82] has shown the following:

Theorem 24 If G is a graph with at least m edges, then

$$\sum_{i=0}^{k} \sum_{j=0}^{k-i} (-1)^j \binom{k-i}{j} t_{ij} = 0,$$

for $k = 0, 1, \dots, m - 1$.

Additionally, Las Vergnas in [Las84] found combinatorial interpretations in the context of oriented matroids for these coefficients by determining yet another generating function formulation for the Tutte polynomial. Gioan and Las Vergnas [GLV05] give the following specialization to orientations of graphs.

Theorem 25 Let G be a graph with a linear ordering of its edges. Let $o_{i,j}$ be the number of orientations of G such that the number of edges that are smallest on some consistently directed cocycle is i and the number of edges that are smallest on a consistently directed cycle is j. Then

$$T(G; x, y) = \sum_{i,j} o_{i,j} 2^{-(i+j)} x^i y^j,$$

and thus $t_{ij} = o_{i,j}/(2^{i+j})$.

The proof is modeled on Tutte's proof that the t_{ij} 's are independent of the ordering of the edges by using deletion/contraction on the greatest edge in the ordering.

Another natural question is to ask if these coefficients are unimodular or perhaps log concave, for example in either x or y. While this was originally conjectured to be true (see Seymour and Welsh [SW75], Tutte [Tut84]), then Schwärzler [Sch93] found a contradiction in the graph in Fig. 3. This counterexample can be extended to an infinite family of counterexamples by increasing the number of edges parallel to e or f.

The unimodularity question for the chromatic polynomial, raised by Read in [Rea68], is still unresolved.

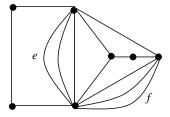


Figure 3: A counterexample to the conjecture

7.3 Zeros of the Tutte Polynomial

Because the Tutte polynomial is after all a polynomial, it is very natural to ask about its zeros and factorizations. The importance of its zeros is magnified by their interpretations. For example, since T(G;0,y) is essentially the flow polynomial, a root of the form $(0,1-\lambda)$, for λ a positive integer, means that G does not have a nowhere zero flow for any Abelian group of order λ . Similarly, since T(G;x,0) is essentially the chromatic polynomial, a root of the form $(1-\lambda,0)$ with λ a positive integer, means that G cannot be properly colored with λ colors. In particular, a direct proof the four-color theorem would follow if it could be shown that the Tutte polynomial has no zero of the form (-3,0) on the class of planar graphs. Of course, because of the duality between the flow and chromatic polynomials, results for the zeros of the one informs the other, and vice versa. Jackson [Jac03] surveys zeros of both chromatic and the flow polynomials.

As we will see in detailed in the next chapter, the chromatic polynomial has an additional interpretation as the zero-temperature antiferromagnetic Potts model of statistical mechanics. In this context, its zeros correspond to numbers of spins for which the ground state degeneracy function may be nonanalytic. This has led to research into its zeros by theoretical physicists as well as mathematicians. Traditionally, the focus from a graph theory perspective was on positive integer roots of the chromatic polynomial, corresponding to graph not being properly colorable with q colors. In statistical mechanics however, the relevant quantity involves the limit of an increasing family of graphs as the number, n, of vertices goes to infinity. This shifted the focus to the complex roots of the chromatic polynomial, since the sequence of complex roots as $n \to \infty$ may have an accumulation point on the real axis.

Because of this, a significant body of work has emerged in recent years devoted to clearing regions of the complex plane (in particular regions containing intervals of the real axis) of roots of the chromatic polynomial. Results showing that certain intervals of the real axis and certain complex regions are free of zeros of chromatic polynomials include those of Woodall [Woo92], Jackson [Jac93], Shrock and Tsai [ST97a, ST97b], Thomassen [Tho97], Sokal [Sok01b], Procacci, Scoppola, and Gerasimov [PSG03], Choe, Oxley, Sokal, and

Wagner [COSW04], Borgs [Bor06], and Fernandez and Procacci [FP]. One particular question concerns the maximum magnitude of a zero of a chromatic polynomial and of zeros comprising region boundaries in the complex plane as the number of vertices $n \to \infty$. An upper bound is given in [Sok01b], depending on the maximal vertex degree. There are, however, families of graphs where both of these magnitudes are unbounded (see Read and Royle [RR91], Shrock and Tsai [ST97a,ST98], Brown, Hickman, Sokal and Wagner [BHSW01], and Sokal [Sok04]). For recent discussions of some relevant research directions concerning zeros of chromatic polynomials and properties of their accumulation sets in the complex plane, as well as approximation methods, see, e.g., Shrock and Tsai [ST97b], Shrock [Shr01], Sokal [Sok01a, Sok01b], Chang and Shrock [CS01b], Chang, Jacobsen, Salas, and Shrock [CJSS04], Choe, Oxley, Sokal, and Wagner [COSW04], Dong and Koh [DK04], and more recently Royle [Roya, Royb].

If G is a graph with chromatic number k+1, then $\chi(G;x)$ has integer roots at $0,1,\ldots,k$. Thus, the chromatic polynomial of G can be written as

$$\chi(G; x) = x^{a_0} (x - 1)^{a_1} \cdots (x - k)^{a_k} q(x),$$

where a_0, \ldots, a_k are integers and q(x) is a polynomial with no integer roots in the interval [0, k]. In contrast to this we have the following result of Merino, de Mier and Noy [MMN01].

Theorem 26 If G is a 2-connected graph, then T(G; x, y) is irreducible in $\mathbb{Z}[x, y]$.

The proof is quite technical and it heavily relies on Theorem 24 and that $\beta(G) \neq 0$ if and only if G has no loops and it is 2-connected.

If G is not 2-connected, then T(G; x, y) can be factored. From Proposition 1 we get that if G is a disconnected graph with connected components G_1, \ldots, G_{κ} , then $T(G; x, y) = \prod_{i=1}^{\kappa} T(G_i; x, y)$. So let us consider when G is connected but not 2-connected.

One of the basic properties mentioned in [BO92] is that $y^s|T(G;x,y)$ if and only if G has s loops. Thus, let us focus on loopless connected graphs that are not 2-connected. It is well-known that such graphs have a decomposition into its blocks, see for example [Bol98]. A block of a graph G is either a bridge or a maximal 2-connected subgraph. If two blocks of G intersect, they do so in a cut vertex. By Theorem 26 and Proposition 1 we get the following.

Corollary 5 If G is a loopless connected graph that is not 2-connected with blocks H_1, \ldots, H_p , then the factorization of T(G; x, y) in $\mathbb{Z}[x, y]$ is exactly

$$T(G; x, y) = T(H_1; x, y) \cdots T(H_p; x, y).$$

7.4 Derivatives of the Tutte Polynomial

It is also most natural to differentiate the Tutte polynomial and to ask for combinatorial interpretations of its derivatives. For example, Las Vergnas [Las] has

found the following combinatorial interpretation of the derivatives of the Tutte polynomial. It first requires a slight generalization of the notions of internal and external activities given in Subsection 3.3.

Definition 11 Let G = (V, E) be a graph with a linear order on its edges, and let $A \subseteq E$. An edge $e \in A$ and a cut C are internally active with respect to A if $e \in C \subseteq (E \setminus A) \cup \{e\}$ and e is the smallest element in C. Similarly, an edge $e \in E \setminus A$ and a cycle C are externally active with respect to A if $e \in C \subseteq A \cup \{e\}$.

In the case that A is a spanning tree, this reduces to the previous definitions of internally and externally active.

Theorem 27 Let G be a graph with a linear ordering on its edges. Then

$$\frac{\partial^{p+q}}{\partial x^p \partial y^q} T(G; x, y) = p! \, q! \, \sum x^{\operatorname{in}(A)} y^{\operatorname{ex}(A)},$$

where the sum is over all subsets A of the edge set of G such that r(G)-r(A)=p and |A|-r(A)=q, and where in(A) is the number of internally active edges with respect to A, and ex(A) is the number of externally active edges with respect to A.

The proof begins by differentiating the spanning tree definition of the Tutte polynomial, Definition 4, which gives a sum over i and j restricted by p and q. This is followed by showing that the coefficients of $x^{i-p}y^{j-q}$ enumerate the edge sets described in the theorem statement. The enumeration comes from examining, for each subset A of E, the set of $e \in E \setminus A$ such that there is a cut-set of G contained in $E \setminus A$ with e as the smallest element (and dually for cycles).

The Tutte polynomial along the line x=y is a polynomial in one variable that, for planar graphs, is related to the Martin polynomial via a medial graph construction. From this relationship, [E-M04b] derives an interpretation for the n-th derivative of this one variable polynomial evaluated at (2,2) in terms of edge disjoint closed trails in the oriented medial graph.

Definition 12 For an oriented graph \vec{G} , let P_n be the set of ordered n-tuples $\bar{p} := (p_1, \ldots, p_n)$, where the p_i 's are consistently oriented edge-disjoint closed trails in \vec{G} .

Theorem 28 If G is a connected planar graph with oriented medial graph $\overrightarrow{G_m}$, then, for all non-negative integers n,

$$\left. \frac{\partial^n}{\partial x} T(G; x, x) \right|_{x=2} = \sum_{k=0}^n (-1)^{n-k} \frac{n!}{k!} \sum_{\bar{p} \in P_k(\vec{G}_m)} 2^{m(\bar{p})},$$

where $m(\bar{p})$ is the number of vertices of \vec{G} not belonging to any of the trails in \bar{p} .

7.5 Convolution and the Tutte Polynomial

Since the Tutte polynomial can also be formulated as a generating function, the tools of generating functions, such as Möbius inversion and convolution, are available to analyze it. A comprehensive treatment of convolution and Möbius inversion can be found in Stanley [Sta96a]. Convolution identities are valuable because they write a graph polynomial in terms of the polynomials of its substructures, thus facilitating induction techniques. We have the following result from Kook, Reiner, and Stanton [KRS99] using this approach.

Theorem 29 The Tutte polynomial can be expressed as

$$T(G;x,y) = \sum T(G/A;x,0) T\left(\left.G\right|_A;0,y\right),$$

where the sum is over all subsets A of the edge set of G, and where $G|_A$ is the restriction of G to the edges of A, i.e. $G|_A = G \setminus (E \setminus A)$.

This result is particularly interesting in that it essentially writes the Tutte polynomial of a graph in terms of the chromatic and flow polynomials of its minors. It may be proved in several ways, for example by induction using the deletion/contraction relation, or from the spanning trees expansion of the Tutte polynomial. However, we present the first proof from [KRS99] to illustrate the technique, which is dependent on results of Crapo [Cra69].

Proof. [sketch] We begin with a convolution product of two functions on graphs into the ring $\mathbb{Z}[x,y]$ given by $f*g = \sum_{A\subseteq E(G)} f(G|_A)g(G/A)$. The identity for convolution is $\delta(G)$ which is 1 if and only if G is edgeless and 0 otherwise. From Crapo [Cra69], we have that

$$T(G; x + 1, y + 1) = (\zeta(1, y) * \zeta(x, 1))(G),$$

where $\zeta(x,y)(G) = x^{r(G)}y^{r(G^*)}$. Kook, Reiner, and Stanton [KRS99] then show that $\zeta(x,y)^{-1} = \zeta(-x,-y)$. From this it follows that $T(G;x+1,0) = (\zeta(1,-1)*\zeta(x,1))(G)$ and $T(G;0,y+1) = (\zeta(1,y)*\zeta(-1,1))(G)$. Thus, $\sum T(G|_A;0,y+1) = T(G/A;x+1,0) = T(G,y)*\zeta(-1,1)*\zeta(1,-1)*\zeta(1,1)(G)$. By associativity, the last expression is the same as T(G;x+1,y)*T(G,y)*T(

A formula, known as Tutte's identity for the chromatic polynomial, with a similar flavor, exists for the chromatic polynomial.

Theorem 30 The chromatic polynomial can be expressed as

$$\chi(G;x+y) = \sum \chi(\left.G\right|_A;x)\,\chi(\left.G\right|_{A^c};y),$$

where the sum is over all subsets A of the set of vertices of G, and where $G|_A$ is the restriction of G to the vertices of A.

Proof. Consider an (m+n)-coloring of G, and let A be the vertices colored by the first m colors. Then an (m+n)-coloring of G decomposes into an m coloring of $G|_A$ using the first m colors and an n coloring of $G|_{A^c}$ using the remaining colors. Thus, for any two non-negative integers m and n, it follows that $\chi(G; m+n) = \sum \chi(G|_A; m) \chi(G|_{A^c}; n)$. Since the expressions involve finite polynomials, this establishes the result for indeterminates x and y.

8 The Complexity of the Tutte Polynomial

We assume the reader is familiar with the basic notions of computational complexity, but for formal definitions in the present context, see, for example, Welsh [Wel93].

We have seen that along different algebraic curves in the XY plane, the Tutte polynomial evaluates to many diverse quantities. Some of these, such as $T(G;2,2)=2^{|E|}$ are very easy to compute, and others such as T(G;1,1) may also be computed efficiently, as in Subsection 5.1. In general though, the Tutte polynomial is intractable, as shown in the following theorem of Jaeger, Vertigan and Welsh [JVW90].

Theorem 31 The problem of evaluating the Tutte polynomial of a graph at a point (a,b) is #P-hard except when (a,b) is on the special hyperbola

$$H_1 \equiv (x-1)(y-1) = 1$$

or when (a,b) is one of the special points (1,1), (-1,-1), (0,-1), (-1,0), (i,-i), (-i,i), (j,j^2) and (j^2,j) , where $j=e^{2\pi i/3}$. In each of these exceptional cases the evaluation can be done in polynomial time.

For planar graphs there is a significant difference. The technique developed using the Pfaffian to solve the Ising problem for the plane square lattice by Kasteleyn [Kas61] can be extended to give a polynomial time algorithm for the evaluation of the Tutte polynomial of any planar graph along the special hyperbola

$$H_2 \equiv (x-1)(y-1) = 2.$$

However, even restricting a class of graphs to its planar members, or further restricting colouring enumeration on the square lattice, does not necessarily yield any additional tractability, as shown by the following results, the first due to Vertigan and Welsh [VW92], and the second to Farr [Far06].

Theorem 32 The evaluation of the Tutte polynomial of bipartite planar graphs at a point (a, b) is #P-hard except when

$$(a,b) \in H_1 \cup H_2 \cup \{(1,1), (-1,-1), (j,j^2), (j^2,j)\},\$$

at which points it is computable in polynomial time.

Theorem 33 For $\lambda \geq 3$, computing the number of λ -colorings of induced subgraphs of the square lattice is #P-complete.

A natural question then arises as to how well an evaluation of the Tutte polynomial might be approximated. That is, if there is a fully polynomial randomized approximation scheme, or FPRAS, for T at a point (x, y) for a well-defined family of graphs. Here, FPRAS refers to a probabilistic algorithm that takes the input s and the degree of accuracy ϵ to produce, in polynomial time on |s| and ϵ^{-1} , a random variable which approximates T(G; x, y) within a ratio of $1+\epsilon$ with probability greater than or equal to 3/4. For example, Jerrum and Sinclair [JS93] show that there exits an FPRAS for T along the positive branch of the hyperbola H_2 .

However, in general approximating is provably difficult as well. Recently, Goldberg and Jerrum [GJ07] have extended the region of the x-y plane for which the Tutte polynomial does not have an FPRAS, to essentially all but the first quadrant (under the assumption that $RP \neq NP$). A consequence of this is that there is no FPRAS for counting nowhere zero λ -flows for $\lambda > 2$. They also provide a good overview of prior results. For a somewhat more optimistic prognosis in the case of dense graphs, we refer the reader to [WM00], and to Alon, Frieze, and Welsh [AFW95].

There has been an increasing body of work since the seminal results of Robertson and Seymour [RS83,RS84,RS86] impacting computational complexity questions for graphs with bounded tree-width (see Bodlaender's accessible introduction in [Bod93]). A powerful aspect of this work is that many NP-Hard problems become tractable for graphs of bounded tree-width. For example, Noble [Nob98] has shown that the Tutte polynomial may be computed in polynomial time (in fact requires only a linear number of multiplications and additions) for rational points on graphs with bounded tree width. Makowsky, Rotics, Averbouch and Godlin [MRAG06] provide similar results for bounded clique-width (a notion with significant computation complexity consequences analogous to those for bounded tree-width – see Oum and Seymour [OS06]). Noble [Nob07] gives a recent survey of complexity results for this area, including new monadic second order logic methods and extensions to the multivariable generalizations of the Tutte polynomial discussed in the next chapter.

Although the Tutte polynomial is not in general computationally tractable, there are some resources for reasonably sized graphs (about 100 edges). These include Sekine, Imai, and Tani [SIT95], which provides an algorithm to implement the recursive definition. Common computer algebra systems such as Maple and Mathematica will compute the Tutte polynomial for smallish graphs, and there are also some implementations freely available in the Web, such as http://ada.fciencias.unam.mx/~rconde/tulic/ by R. Conde or http://homepages.mcs.vuw.ac.nz/~djp/tutte/ by G. Haggard and D. Pearce.

9 Conclusion

For further exploration of the Tutte polynomial and its properties, we refer the reader to the relevant chapters of Welsh [Wel93] and Bollobás [Bol98] for excellent introductions, and to Brylawki [Bry82], Brylawski and Oxley [BO92], and Welsh [Wel99] for an in-depth treatment of the Tutte polynomial, including generalizations to matroids. Although we focused on graphs here to broaden accessibility, matroids, rather than graphs, are the natural domain of the Tutte polynomial, and Crapo [Cra69] gives a compelling justification for this viewpoint. Farr [Far07] gives a recent treatment and engaging history of the Tutte polynomial. Finally, we especially recommend Tutte's own account of how he "...became acquainted with the Tutte polynomial..." in [Tut04].

References

- [AFW95] Alon, N., Frieze, A. M., Welsh, D. J. A.: Polynomial time randomized approximation schemes for Tutte-Gröthendieck invariants: the dense case. Random Structures Algorithms, **6**, 459–478 (1995)
- [AZ01] Aigner, M., Ziegler, G.M.: Proofs from the Book. Springer-Verlag, Berlin Heidelberg New York (2001)
- [BHSW01] Brown, J. I., Hickman, C. A., Sokal, A. D., Wagner, D. G.: On the chromatic roots of generalized theta graphs. J. Combin. Theory Ser. B, 83, 272–297 (2001)
- [Big96a] Biggs, N.: Algebraic Graph Theory. Cambridge University Press, Cambridge, second edition (1996)
- [Big96b] Biggs, N.: Chip firing and the critical group of a graph. Research Report, London School of Economics, London (1996).
- [Bir12] Birkhoff, G.D.: A determinant formula for the number of ways of coloring a map. Annals of Mathematics, 14, 42–46 (1912)
- [Bjo92] Björner, A.: Homology and shellability of matroids and geometric lattices. In: White, N. (ed) Matroid Applications, Encyclopedia of Mathematics and its Applications. Cambridge University Press, Cambridge (1992)
- [BMMT95] Benashski, J., Martin, R., Moore, J., Traldi, L.: On the β -invariant for graphs. Congr. Numer., **109**, 211-221 (1995)
- [BO92] Brylawski, T., Oxley, J.: The Tutte Polynomial and its Applications. In: White, N. (ed) Matroid Applications, Encyclopedia of Mathematics and its Applications. Cambridge University Press, Cambridge (1992)

- [Bod93] Bodlaender, H. L.: A tourist guide through treewidth. Acta Cybernet, 11, 1–21 (1993)
- [Bol98] Bollobás, B.: Modern Graph Theory, Graduate Texts in Mathematics. Springer, New York (1998)
- [Bor06] Borgs, C.: Absence of zeros for the chromatic polynomial on bounded degree graphs. Combin. Probab. Comput., **15**, 63–74 (2006)
- [BR99] Bollobás, B., Riordan, O.: A Tutte Polynomial for Colored Graphs. Combin. Probab. Comput., **8**, 45–93 (1999)
- [Bry71] Brylawski, T.: A combinatorial model for series-parallel networks. Trans. Amer. Math. Soc., **154**, 1–22 (1971)
- [Bry72] Brylawski, T.: A Decomposition for Combinatorial Geometries. Trans. Amer. Math. Soc., **171**, 235–282 (1972)
- [Bry82] Brylawski, T.: The Tutte polynomial, Part 1: General Theory. In: Barlotti, A. (ed) Matroid Theory and Its Applications. Proceedings of the Third International Mathematical Summer Center (C.I.M.E. 1980) (1982)
- [BTW88] Bak, P., Tang, C., Wiesenfeld K.: Self-organized criticality. Phys. Rev. A, 38, 364–374 (1988)
- [CC97] Chari, M. K., Colbourn, C. J.: Reliability polynomials: A survey. Journal of Combinatorics, Information and System Sciences, 22, 177–193 (1997)
- [Chi97] Chia, G. L.: A Bibliography on Chromatic Polynomials. Discrete Math., **172**, 175–191 (1997)
- [CJSS04] Chang, S. C., Jacobsen, J., Salas, J., Shrock, R.: Exact Potts model partition functions for strips of the triangular lattice. J. Stat. Phys., 114, 768–823 (2004)
- [COSW04] Choe, Y. B., Oxley, J. G., Sokal, A. D., Wagner, D. G.: Homogeneous multivariate polynomials with the half-plane property. Adv. in Appl. Math., 32, 88–187 (2004)
- [Cra67] Crapo, H. H.: A higher invariant for matroids. J. Combin. Theory, **2**, 406–417 (1967)
- [Cra69] Crapo, H. H.: The Tutte polynomial. Aeq. Math., **3**, 211–229 (1969)
- [CS01b] Chang, S. C., Shrock, R.: Exact Potts model partition functions on wider arbitrary-length strips of the square lattice. Physica A, **296**, 234-288 (2001)

- [Dha90] Dhar, D.: Self-organized critical state of sandpile automaton models. Phys. Rev. lett., **64**, 1613–1616 (1990)
- [Die00] Diestel, R.: Graph Theory, Graduate Texts in Mathematics. Springer, New York (2000)
- [DKT05] Dong, F. M., Koh, K. M., Teo, K. L.: Chromatic polynomials and chromaticity of graphs. World Scientific, Hackensack, NJ, (2005)
- [DK04] Dong, F. M.; Koh, K. M.: On upper bounds for real roots of chromatic polynomials. Disc. Math. **282**, 95–101 (2004)
- [DK07] Dong, F. M.; Koh, K. M.: Bounds for the coefficients of flow polynomials. J. Combin. Theory Ser. B, **97**, 413–420 (2007)
- [Don04] Dong, F. M. The largest non-integer zero of chromatic polynomials of graphs with fixed order. Disc. Math. **282**, 103-112, (2004)
- [Duf65] Duffin, R.J.: Topology of series-parallel networks. J. Math. Anal. Appl. 10, 303–318 (1965)
- [E-M04a] Ellis-Monaghan, J.: Identities for the circuit partition polynomials, with applications to the diagonal Tutte polynomial. Advances in Applied Mathematics, **32**, 188–197 (2004)
- [E-M04b] Ellis-Monaghan, J.: Exploring the Tutte-Martin connection. Discrete Mathematics, **281**, 173–187 (2004)
- [Far06] Farr, G. E.: The complexity of counting colourings of subgraphs of the grid. Comb. Probab. Comput., 15, 377–383 (2006)
- [Far07] Farr, G. E.: Tutte-Whitney polynomials: some history and generalizations. In: Grimmett, G. R., McDiarmid C. J. H.(eds) Combinatorics, Complexity, and Chance: A Tribute to Dominic Welsh. Oxford University Press, Oxford (2007)
- [FP] Fernandez, R., Procacci, A.: Regions without complex zeros for chromatic polynomials on graphs with bounded degree. To appear in Comb. Probab. Comput.
- [Gab93] Gabrielov, A.: Abelian avalanches and the Tutte polynomials. Physica A, 195, 253–274 (1993)
- [GJ79] Garey, M. R., Johnson, D. S.: Computers and Intractability— A guide to the theory of NP-completeness. W. H. Freeman, San Francisco (1979)
- [GJ07] Goldberg, L. A., Jerrum M. R.: Inapproximability of the Tutte polynomial. In STOC '07: Proceedings of the 39th Annual ACM Symposium on Theory of Computing. ACM Press, New York (2007)

- [GLV05] Gioan, E., Las Vergnas, M.: Activity preserving bijections between spanning trees and orientations in graphs. Discrete Math., **298**, 169–188 (2005)
- [GZ83] Green, C., Zaslavsky, T.: On the interpretation of Whitney numbers through arrangements of hyperplanes, zonotopes, non-Radon partitions and orientations of graphs. Trans. Amer. Math. Soc., **280**, 97–126 (1983)
- [Jac93] Jackson, B.: A zero-free interval for chromatic polynomials of graphs. Combin. Probab. Comput., 2, 325–336 (1993)
- [Jac03] Jackson, B.: Zeros of chromatic and flow polynomials of graphs. J. Geom., **76**, 95–109 (2003)
- [Jac07] Jackson, B.: Zero-free intervals for flow polynomials of near-cubic graphs. Comb. Probab. Comput., **16**, 85–108 (2007)
- [Jae76] Jaeger, F.: On nowhere-zero flows in multigraphs. In: Nash-Williams, C. St. J. A., Sheehan, J. (eds) Proceedings of the Fifth British Combinatorial Conference. Utilitas Math., Winnipeg (1976)
- [Jae88] Jaeger, F.: Nowhere-zero flow problems. In: Beineke, L. W., Wilson, R. J. (eds) Selected Topics in Graph Theory 3. Academic Press, New York (1988)
- [Jer87] Jerrum, M. R.: 2-dimensional monomer-dimer systems are computationally intractable. J. Statist. Phys. 48, 121–134 (1987)
- [JS93] Jerrum, M. R., Sinclair, A.: Polynomial time approximation algorithms for the Ising model. SIAM J. Comput., **22**, 1087–1116 (1993)
- [JVW90] Jaeger, F., Vertigan, D. L., Welsh, D. J. A.: On the computational complexity of the Jones and Tutte polynomials. Math. Proc. Camb. Phil. Soc., 108, 35–53 (1990)
- [Kas61] Kasteleyn, P.W.: The statistics of dimers on a lattice. Physica, **27**, 1209–1225 (1961)
- [KRS99] Kook, W., Reiner, V., Stanton, D.: A Convolution Formula for the Tutte Polynomial. J. Combin. Theory Ser. B, **76**, 297–300 (1999)
- [KW81] Kleitman, D. J., Winston, K. J.: Forests and score vectors. Combinatorica, 1, 49–54 (1981)
- [Las04] Lass, B.: Orientations acycliques et le polynome chromatique. European J. Combin., **22**, 1101–1123 (2001)

- [Las77] Las Vergnas, M.: Acyclic and totally cyclic orientations of combinatorial geometries. Discrete Mathematics, **20**, 51–61 (1977)
- [Las84] Las Vergnas, M.: The Tutte polynomial of a morphism of matroids II. Activities of orientations. In: Bondy, J. A., Murty, U. S. R. (eds) Progress in Graph Theory, Proceedings of Waterloo Silver Jubilee Combinatorial Conference 1982. Academic Press, Toronto (1984)
- [Las88] Las Vergnas, M.: On the evaluation at (3,3) of the Tutte polynomial of a graph. J. Combin. Theory Ser. B, 44, 367–372 (1988)
- [Las] Las Vergnas, M.: The Tutte polynomial of a morphism of matroids V. Derivatives as generating functions. Preprint (2007)
- [Lie67] Lieb, E. H.: Residual entropy of square ice. Phys. Rev., **162**, 162–172 (1967)
- [Mak05] Makowsky, J. A.: Colored Tutte polynomials and Kauffman brackets for graphs of bounded tree width. Discrete Appl. Math., 145, 276–290 (2005)
- [Mar77] Martin, P.: Enumérations eulériennes dans le multigraphs et invariants de Tutte-Gröthendieck. PhD Thesis, Grenoble (1977)
- [Mar78] Martin, P.: Remarkable valuation of the dichromatic polynomial of planar multigraphs. J. Combin. Theory Ser. B, **24**, 318–324 (1978)
- [McK01] McKee, T. A.: Recognizing dual-chordal graphs. Congr. Numer., **150**, 97–103 (2001)
- [Mer97] Merino, C.: Chip firing and the Tutte polynomial. Annals of Combinatorics, 1, 253–259 (1997)
- [Mer01] Merino, C.: The chip firing game and matroid complex. Discrete Mathematics and Theoretical Computer Science, Proceedings vol. **AA**, 245–256 (2001)
- [MMN01] Merino, C., de Mier, A., Noy, M.: Irreducibility of the Tutte polynomial of a connected matroid. J. Combin. Theory Ser. B, 83, 298–304 (2001)
- [MRAG06] Makowsky, J. A., Rotics, U., Averbouch, I., Godlin, B.: Computing graph polynomials on graphs of bounded clique-width. In: Lecture Notes in Computer Science 4271. Springer-Verlag, New York (2006)
- [Nob98] Noble, S. D.: Evaluating the Tutte polynomial for graphs of bounded tree-width. Comb. Probab. Comput., 7, 307–321 (1998)

- [Nob07] Noble, S. D.: The complexity of graph polynomials. In: Grimmett, G. R., McDiarmid C. J. H.(eds) Combinatorics, Complexity, and Chance: A Tribute to Dominic Welsh. Oxford University Press, Oxford (2007)
- [OS06] Oum, S., Seymour, P. D.: Approximating clique-width and branch-width. J. Combin. Theory Ser. B, **96**, 514-528 (2006)
- [Oxl82] Oxley, J.: On Crapos beta invariant for matroids. Stud. Appl. Math. **66**, 267–277 (1982)
- [OW79] Oxley, J., Welsh D. J. A.: The Tutte Polynomial and Percolation. In: Bondy, J. A., Murty U. S. R. (eds) Graph Theory and Related Topics. Academic Press, London (1979)
- [Pau35] Pauling, L.: The structure and entropy of ice and of other crystals with some randomness of atomic arrangement. J. Am. Chem. Soc., **57**, 2680–2684 (1935)
- [PB80] Provan, J. S., Billera, L. J.: Decompositions of simplicial complexes related to diameters of convex polyhedra. Math. Oper. Res., 5, 576–594 (1980)
- [PSG03] Procacci, A., Scoppola, B., Gerasimov, V.: Potts model on infinite graphs and the limit of chromatic polynomials. Communications in Mathematical Physics, **235**, 215–231 (2003)
- [Rea68] Read, R. C.: An introduction to chromatic polynomials. J. Combin. Theory Ser. B, 4, 52–71 (1968)
- [Roya] Royle, G.: Planar triangulations with real chromatic roots arbitrarily close to four. Preprint math.CO.0511205
- [Royb] Royle, G.: Graphs with chromatic roots in the interval (1,2). Preprint arXiv:0704.2264
- [RR78] Read, R. C., Rosenstiehl, P.: On the principal edge tripartition of a graph. Ann. Discrete Math., 3, 195–226 (1978)
- [RR91] Read, R. C., Royle, G.: Chromatic roots of families of graphs. In: Alavi, Y. et al. (eds) Graph Theory, Combinatorics, and Applications. Wiley, New York (1991)
- [RS83] Robertson, N., Seymour, P. D.: Graph minors. I. Excluding a forest. J. Combin. Theory Ser. B, **35**, 39–61 (1984)
- [RS84] Robertson, N., Seymour, P. D.: Graph minors. III. Planar treewidth. J. Combin. Theory Ser. B, 36, 49–64 (1984)
- [RS86] Robertson, N., Seymour, P.D.: Graph minors. II. Algorithmic aspects of tree-width. J Algorithms, 7, 309–322 (1986)

- [Sch93] Schwärzler, W.: The coefficients of the Tutte polynomial are not unimodal. J. Combin. Theory Ser. B, **58**, 240–242 (1993)
- [Sey81] Seymour, P. D.: Nowhere-zero 6-flows. J. Combin. Theory Ser. B, 30, 130–135 (1981)
- [Shr01] Shrock, R.: Chromatic polynomials and their zeros and asymptotic limits for families of graphs. Discrete Math., **231**, 421–446 (2001)
- [SIT95] Sekine, K., Imai, H., Tani, S.: Computing the Tutte polynomial of a graph of moderate size. Lecture Notes in Computer Science. Springer, Berlin (1995)
- [Sok01a] Sokal, A. D.: A personal list of unsolved problems concerning lattice gases and antiferromagnetic Potts models. Markov Process and Related Fields, **7**, 21–38 (2001)
- [Sok01b] Sokal, A. D.: Bounds on the complex zeros of (di)chromatic polynomials and Potts-model partition functions. Combin. Probab. Comput., **10**, 41–77 (2001)
- [Sok04] Sokal, A. D.: Chromatic roots are dense in the whole complex plane. Combin. Probab. Comput., 13, 221–261 (2004)
- [ST97a] Shrock, R., Tsai, S. H.: Asymptotic limits and zeros of chromatic polynomials and ground state entropy of Potts antiferromagnets. Phys. Rev. E, **55**, 5165–5179 (1997)
- [ST97b] Shrock, R., Tsai, S. H.: Families of graphs with $W_r(G,q)$ functions that are nonanalytic at 1/q = 0. Phys. Rev. E **56**, 3935–3943 (1997)
- [ST98] Shrock, R., Tsai, S.H.: Ground state entropy of Potts antiferromagnets: cases with noncompact W boundaries having multiple points at 1/q = 0. Physica A, **259**, 315–348 (1998)
- [Sta73] Stanley, R.: Acyclic orientations of graphs. Discrete Mathematics, 5, 171–178 (1973)
- [Sta80] Stanley, R.: Decomposition of rational polytopes. Annals of Discrete Mathematics, **6**, 333–342 (1980)
- [Sta96a] Stanley, R.: Enumerative Combinatorics, vol. 1. Cambridge University Press, Cambridge (1996)
- [Sta96b] Stanley, R.: Combinatorics and Commutative Algebra, Progress in Mathematics. Birkhäuser, Boston Besel Stuttgart, 2nd Edition (1996)
- [Sta99] Stanley, R.: Enumerative Combinatorics, vol. 2. Cambridge University Press, New York Cambridge (1999)

- [SW75] Seymour, P. D., Welsh, D. J. A.: Combinatorial applications of an inequality of statistical mechanics. Math. Proc. Cambridge Philos. Soc., 77, 485-495 (1975)
- [Thi87] Thistlethwaite, M. B.: A spanning tree expansion of the Jones polynomial. Topology, **26**, 297–309 (1987)
- [Tho97] Thomassen, C.: The zero-free intervals for chromatic polynomials of graphs. Combin. Probab. Comput., 6, 497–506 (1997)
- [Tra00] Traldi, L.: Series and parallel reductions for the Tutte polynomial. Discrete Mathematics, **220**, 291–297 (2000)
- [Tra06] Traldi, L.: On the colored Tutte polynomial of a graph of bounded treewidth. Discrete Applied Mathematics, **154**, 1032–1036 (2006)
- [Tut47] Tutte, W. T.: A ring in graph theory. Proc. Cambridge Phil. Soc., 43, 26–40 (1947)
- [Tut48] Tutte, W. T.: An Algebraic Theory of Graphs, PhD thesis, University of Cambridge (1948)
- [Tut54] Tutte, W. T.: A contribution to the theory of chromatic polynomials. Can. J. Math., **6**, 80–91 (1954)
- [Tut67] Tutte, W. T.: On dichromatic polynomials. J. Combin. Theory, $\mathbf{2}$, 301-320 (1967)
- [Tut84] Tutte, W. T.: Graph Theory. Cambridge University Press, Cambridge (1984)
- [Tut79] Tutte, W. T.: All the kings horses. In: Bondy, J. A., Murty U. S. R. (eds) Graph Theory and Related Topics. Academic Press, London (1979)
- [Tut04] Tutte, W. T.: Graph-polynomials. Special issue on the Tutte polynomial, Adv. in Appl. Math., **32**, 5–9, (2004)
- [VW92] Vertigan, D. L., Welsh, D. J. A.: The computational complexity of the Tutte plane: the bipartite case. Comb. Probab. Comput., 1, 181–187 (1992)
- [Wel93] Welsh, D. J. A.: Complexity: Knots, Colorings and Counting. Cambridge University Press, Cambridge (1993)
- [Wel99] Welsh, D. J. A.: The Tutte Polynomial, in Statistical physics methods in discrete probability, combinatorics, and theoretical computer science. Random Structures Algorithms, **15**, 210–228 (1999)

- [Whi32] Whitney, H.: A Logical Expansion in Mathematics. Bull. Amer. Math. Soc., **38**, 572-579 (1932)
- [WM00] Welsh, D. J. A., Merino, C.: The Potts model and the Tutte polynomial. Journal of Mathematical Physics, 41, 1127-1152 (2000)
- [Woo92] Woodall, D.: A zero-free interval for chromatic polynomials. Discrete Math., **101**, 333–341 (1992)
- [Yet90] Yetter, D.: On graph invariants given by linear recurrence relations. J. Combin. Theory Ser. B, 48, 6–18 (1990)
- [Zas75] Zaslavsky, T.: Facing up to arrangements: Face-count formulas for partitions of spaces by hyperplanes. Mem. Amer. Math. Soc., 154, (1975)
- [Zas87] Zaslavsky, T.: The Möbius function and the characteristic polynomial.In: White, N. (ed) Combinatorial Geometries, Encyclopedia of Mathematics and its Applications. Cambridge University Press, Cambridge (1987)
- [Zha97] Zhang, C. Q.: Integer flows and cycle covers of graphs. Marcel Dekker Inc., New York (1997)

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