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Enumeration of skew Ferrers diagrams*

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Abstract

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In this paper, we show that the generating function for skew Ferrers diagrams according to their width and area is the quotient of new basic Bessel functions.

Résumé

Delest, M.P. and J.M. Fedou, Enumeration of skew Ferrers diagrams, Discrete Mathematics 112 (1993) 65–79.

Nous montrons dans cet article que la fonction génératrice des diagrammes de Ferrers gauches selon les paramètres périmètre et aire s'exprime en fonction du quotient des q analogues de deux fonctions de Bessel.

Introduction

Ferrers diagrams, related to the well-known partitions of an integer, have been extensively studied. See for instance Andrews' book [3]. A partition of an integer n is a decreasing sequence of integers, $n_1, n_2, \ldots, n_k$, such that $n_1+n_2+\cdots+n_k=n$. The geometric figure formed by the k rows having respectively $n_1, n_2, \ldots, n_k$ cells (see Fig. 1) is called the Ferrers diagram associated to the partition $(n_1, n_2, \ldots n_k)$ of n. Filling Ferrers diagrams with numbers gives plane partitions, which are related to representations of the symmetric group [13]. Young tableaux are examples of plane partitions and are of great interest in the computation of Schur functions. The literature on these subjects is plentiful.

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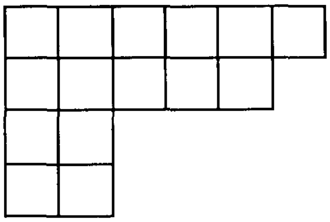

Fig. 1. Ferrers diagram corresponding to the sequence (6, 5, 2, 2).

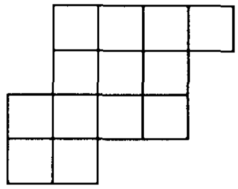

Fig. 2. A parallelogram polyomino having area 13 and perimeter 20.

The difference between two Ferrers diagrams is called a skew Ferrers diagram. Thus a skew Ferrers diagram is defined by two increasing sequences of integers, $n_1, n_2, \ldots, n_k$ and $p_1, p_2, \ldots, p_k$ such that, for every $1 \leqslant i \leqslant k$, $n_i \leqslant p_i$ (see Fig. 2). If the skew Ferrers diagrams have no cut point and are connected then they are a particular case of polyominoes, the so-called parallelogram polyominoes.

Unit squares with vertices at integer points in the cartesian plane are called *cells*. A polyomino is a finite connected union of cells such that the interior is also connected. Polyominoes are defined up to translation. The perimeter of a polyomino is the length of its border and its area is the number of cells which comprise it. For example, the skew Ferrers diagram shown in Fig. 2 is defined by the two sequences (5, 4, 4, 2) and (1, 1). It is also a parallelogram polyomino having perimeter 18 and area 13.

Counting polyominoes according to the area or perimeter is a major unsolved problem in combinatorics. See for review [19, 21]. The problem is also well-known in statistical physics. Usually, physicists consider animals instead of polyominoes, equivalent objects obtained by taking the center of each elementary cell. They attempt to find some relations for the number of animals having an area or a perimeter n. For results on this subjects the reader should see [25].

A column (resp. a row) is the intersection of the polyomino with an infinite vertical (resp. horizontal) unit strip. A polyomino is said to be convex when all its columns and rows are connected. Recently, convex polyominoes have been enumerated according to the perimeter [8]. The enumeration according to the area is still an open problem.

A parallelogram polyomino is a convex polyomino bordered by two non-intersecting paths having only North and East steps (see Fig. 2). Parallelogram polyominoes are well-known in combinatorics (see Polya [19], Gessel [14]). The number of such polyominoes having perimeter $2n+2$ is the Catalan number C_n,

$$C_n = \frac{1}{n+1}\binom{2n}{n}.$$

The enumeration of parallelogram polyominoes according to area has been studied by Gessel in [14], as an application of a q-analog of the Lagrange inversion formula, but no explicit formula is given.

As shown in Section 2, it is easy to get from what we call a q-analog of an algebraic grammar, a functional equation for the generating function of skew Ferrers diagrams. When A.M. Garsia was visiting in Bordeaux during September 1989, we did not know how to proceed further. Computing the first terms using Macsyma led us to a known sequence of integers, related to the zeroes of Bessel functions [18] and classified in the *Handbook of Integer Sequences* [23] of Sloane. Here, Garsia initiated us into the q-calculus and the Ehrhart theory and his help was valuable.

The main result of this paper is to show that the generating function for the skew Ferrers diagrams according to their area and number of columns is,

$$F(t;q)=\sum_{n=0}^{+\infty}\frac{(-1)^n q^{\binom{n+1}{2}}}{(q;q)_n(q;q)_{n+1}}q^{n+1}t^{n+1}\Bigg/\sum_{n=0}^{+\infty}\frac{(-1)^n q^{\binom{n}{2}}}{(q;q)_n^2}q^n t^n,$$

where $(a;q)_n=(1-a)(1-aq)(1-aq^2)\cdots(1-aq^{n-1})$.

We remark that the basic Bessel functions appearing here are different from those defined by Ismail [15] and Jackson [16]. The subject is so rich that it leads us to several combinatorial interpretations for these functions. These can be made in terms of weight enumerators of trees, multichains in Dyck paths, and multiwalks in trees [12]. This not withstanding, further work still needs to be done. Some open questions are given in the conclusion.

1. Definitions and notations

A *path* is a sequence of points in $\mathbb{N}\times\mathbb{N}$. A *step* of a path is a pair of two consecutive points in the path. A *Dyck path* is a path $w=(s_0, s_1,\ldots, s_{2n})$ such that $s_0=(0,0)$, $s_{2n}=(2n,0)$, having only steps North-East ($s_i=(x,y)$, $s_{i+1}=(x+1,y+1)$) or South-East ($s_i=(x,y)$, $s_{i+1}=(x+1,y-1)$). A *peak* (resp. *trough*) is a point s_i such that the step (s_{i-1}, s_i) is North-East (resp. South-East) and the step (s_i, s_{i+1}) is South-East (resp. North-East). The *height* $h(s_i)$ of a point s_i is its ordinate.

A *Dyck word* is a word $w\in\{x,\bar{x}\}^*$ satisfying both conditions:

(i) $|w|_x=|w|_{\bar{x}}$,

(ii) for every factorization $w=uv$, $|u|_x\geqslant|u|_{\bar{x}}$.

Classically, a Dyck path having length $2n$ is coded by a Dyck word of length $2n$, $w=x_1\cdots x_{2n}$: each North-East (resp. South-East) step (s_{i-1}, s_i) corresponds to the letter $x_i=x$ (resp. $x_i=\bar{x}$). The peaks (resp. troughs) of a Dyck path correspond with the factors $x\bar{x}$ (resp. $\bar{x}x$) of the associated Dyck word. We denote by D_n the set of Dyck words having length $2n$.

Example. The Dyck path shown in Fig. 3 is coded by the following Dyck word

$$w=x\,x\,\bar{x}\,x\,x\,x\,\bar{x}\,\bar{x}\,x\,\bar{x}\,x\,\bar{x}\,\bar{x}\,\bar{x}\,x\,\bar{x}.$$

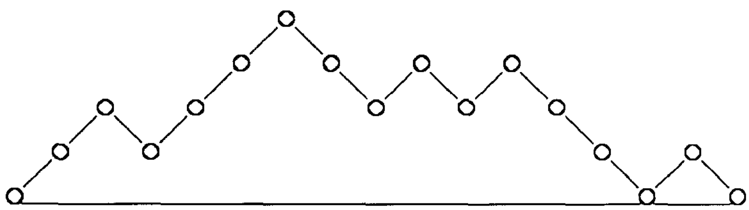

Fig. 3. A Dyck path of D_8.

Delest and Viennot give in [8] a bijection μ between the parallelogram polyominoes having perimeter $2n+2$ and the Dyck words having length $2n$. A parallelogram polyomino P can be defined by the two sequences of integers $(a_1,\ldots,a_n)$ and $(b_1,\ldots,b_{n-1})$, where a_i is the number of cells belonging to the ith column and (b_i+1) is the number of cells adjacent to columns i and $i+1$. The Dyck word $\mu(P)$ is the Dyck word having n peaks, whose heights (resp. troughs) are $a_1,\ldots,a_n$ (resp. $b_1,\ldots,b_{n-1}$). They deduce the following.

Proposition 1. *The map μ transforms a parallelogram polyomino having perimeter $2p+2$, n columns and area k into a Dyck word having length $2p$ and n peaks and such that the sum of the height of the peaks is k.*

Example. The parallelogram polyomino shown in Fig. 2 is defined by the two sequences (2, 4, 3, 3, 1) and (1, 2, 2, 0) and corresponds to the Dyck path showed Fig. 3.

Bessel functions occur in analysis where they are particularly useful for the solution of differential equations. There are a lot of works on these functions. See for instance [9, 26,]. We recall here their classical definition and also a result by Lehmer [18] about the quotient of such functions.

Bessel functions are defined for $\nu>-1$, by

$$J_\nu(x)=\sum_{n=0}^{\infty}\frac{(-1)^n(\frac{1}{2}x)^{2n+\nu}}{n!\,\Gamma(\nu+n+1)}.$$

All the zeros of $J_\nu(x)$ are real. Let $j_{\nu,k}$ be the kth positive zero of $J_\nu(x)$. The symmetric function,

$$\sigma_{2n}(\nu)=\sum_{k=1}^{\infty}(j_{\nu,k})^{-2n},$$

is rational in ν for any positive integer ν. Some arithmetic properties of $\sigma_{2n}(\nu)$ are studied by Carlitz in [6].

Before, Rayleigh [20] and others have used this result for the computation of the first zeros of the Bessel functions. The functions σ_{2n} were known as coefficients of the meromorphic functions,

$$\frac{J_{\nu+1}(x)}{2J_\nu(x)}=\sum_{n=1}^{\infty}\sigma_{2n}(\nu)x^{2n-1}.$$

k \ n	1	2	3	4	5	6	7	8
0	1	1	2	11	38	946	4580	202738
1				5	14	1026	4324	311387
2						362	1316	185430
3						42	132	53752
4								7640
5								429

Fig. 4. Coefficients $a_k^{(n)}$ for $n \leqslant 8$.

The first values of

$$\sigma_{2n}(v) = 2^{-2n} \frac{\Phi_n(v)}{\pi_n(v)},$$

are given by Lehmer in [18]. Let $\lfloor x \rfloor$ be the integer part of x. Then,

$$\pi_n(v) = \prod_{k=1}^{n} (k+v)^{\lfloor n/k \rfloor},$$

and $\Phi_n(v) = a_0^{(n)} + a_1^{(n)} v + \cdots + a_d^{(n)} v^d$ is a polynomial of degree

$$d = 1 - n + \sum_{k=2}^{n} \left\lfloor \frac{n}{k} \right\rfloor.$$

The values of the first coefficients are given in Fig. 4.

Remark. In this array the Catalan numbers appear. An explanation of this fact is given in [12].

2. Enumeration of parallelogram polyominoes

In this paragraph, we use the bijection μ between parallelogram polyominoes and Dyck words described in Section 1. We apply a method due to Schützenberger [22] in order to get first the generating function of Dyck words according to the parameters length and number of peaks. A particular 'reading' of the derivation rules of the Dyck grammar allows us to get the third parameter, sum of the height of the peaks. This method will be described in [7]. We deduce an explicit formula for the generating function,

$${}^{q}f(t) = \sum_{n,k \geqslant 1} a_{k,n} q^k t^n,$$

where $a_{k,n}$ is the number of parallelogram polyominoes having n columns and area k, and we show some recurrence on $a_{k,n}$.

Proposition 2. *The number of Dyck words having length* $2n$ *and* k *peaks is*

$$\frac{1}{n}\binom{n}{k}\binom{n}{k-1}.$$

This is a classical property related to the Narayana numbers (see for instance [17]).

Proof. Let D' be the set of words written over the alphabet $\{x, \bar{x}, t\}$ obtained by substituting $xt\bar{x}$ for each factor $x\bar{x}$ in the non-empty Dyck words. We say that we 'mark' each peak with the letter t. This language is the solution of the following equation,

$$D' = xt\bar{x} + xt\bar{x}D' + xD'\bar{x} + xD'\bar{x}D'.$$

Let

$$d(t, x) = \sum_{n,k \geqslant 0} a_{n,k} x^n t^k,$$

where $a_{n,k}$ is the number of Dyck words having length $2n$ and k peaks. Commuting the variables in the equation of D' gives the following equation,

$$d(t, x) = xt + xt\, d(t, x) + xd(t, x) + x\, d(t, x)^2.$$

Finally, the Lagrange inversion formula proves Proposition 2. □

Proposition 3. *Let* ${}^{q}f(t)$ *be the generating function for Dyck words according to the number of peaks and the sum of the height of the peaks. Then* ${}^{q}f(t)$ *satisfies the following functional equation*:

$${}^{q}f(t) = qt + qt\,{}^{q}f(t) + {}^{q}f(qt) + {}^{q}f(t)\,{}^{q}f(qt).$$

Proof. The method used is described in [7]. It deals with the more general problem of getting the generating function of some combinatorial objects according to two parameters, for instance perimeter and area. More details can be found in [7, 11]. We just recall here the principle of the method which is divided into four steps.

(1) We code the studied objects by the words of an algebraic language L so that the perimeter can be directly read from the length of the words. This is the classical methodology of Schützenberger [22]. Commuting the variables in the algebraic system, one obtains from a grammar G of L the generating function according to the perimeter.

(2) For each word w of L, we consider the monomial $\varphi(w) = q^k$ where k is the area of the object coded by w. The idea is to define recursively the function φ from the derivation rules of the grammar G in order to construct the q-analog ${}^{q}L$ of the language L. It is the set of words $(w; q)$ obtained by applying the recursive definition of φ to w.

(3) We consider the formal series

$${}^{q}S = \sum_{w \in L} (w; q),$$

which satisfies a q-analog of the system of algebraic equation satisfied by

$$S=\sum_{w\in L} w.$$

(4) Commuting the variables, we get a functional equation satisfied by the generating function,

$${}^q l(t)=\sum_{n,k=0}^{\infty} a_{n,k}t^n q^k,$$

where $a_{n,k}$ is the number of studied objects having perimeter n and area q. □

Let g be the map which associates to each word from D' the monomial q^k where k is the sum of the height of the peaks of w. The following recursive relations allow us to construct the q-analog ${}^qD'$ of the language D' which is the set of the words $(w;q)$ when w describes D'.

$$(1;q)=1,$$

$$(xt\bar{x};q)=xqt\bar{x},$$

$$(xt\bar{x}\,u;q)=xqt\bar{x}(u;q), \quad \text{for every word } u \text{ in } D',$$

$$(xu\bar{x};q)=xq^{|u|_t}(u;q)\bar{x}, \quad \text{for every word } u \text{ in } D',$$

$$(xu\bar{x}v;q)=xq^{|u|_t}(u;q)\bar{x}(v;q)), \quad \text{for } u \text{ and } v \text{ words in } D' \text{ (see Fig. 5).}$$

Let us consider the formal series ${}^qS$,

$${}^qS=\sum_{u\in D'}(u;q).$$

The image of ${}^qS$ by the morphism χ sending t on t, and x, $\bar{x}$ on 1, is the function ${}^qf(t)$,

$${}^qf(t)=\sum_{u\in D'} g(u)\chi(u).$$

So the generating function for Dyck words according to the parameters number of peaks and sum of the height of these peaks, which is also the generating function of the skew Ferrers diagrams or parallelogram polyominoes according to the parameters

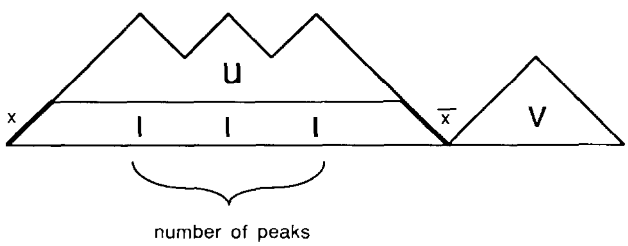

Fig. 5. The equality $(xu\bar{x}v;q)=xq^{|u|_t}(u;q)\bar{x}(v;q)$.

number of columns and area, is exactly the function ${}^{q}f(t)$. Applying the recursive definition of the function g gives

$$ {}^{q}f(t)=qt+qt\sum_{u\in D'} g(u)\chi(u)+\sum_{u,v\in D'} q^{|u|_t} g(u)(g(v)+1)\chi(u)\chi(v), $$

$$ {}^{q}f(t)=qt+qt\,{}^{q}f(t)+{}^{q}f(t)+{}^{q}f(t)\,{}^{q}f(qt). $$

Let

$$ {}^{q}f(t)=\sum_{n=1}^{\infty} a_n(q)\,t^n. $$

We denote $a_n(q)$ by a_n for short. The functional equation gives $a_1=q+qa_1$, and if $n>1$,

$$ a_n=q^n a_n+qa_{n-1}+\sum_{k=1}^{n-1} a_k q^k a_{n-k}. $$

Thus, setting

$$ a_n=\frac{\alpha_n}{(1-q)^{2n-1}}, $$

we have $\alpha_1=1$ and for every n, $n>1$,

$$ (1-q^n)\alpha_n=(1-q)^2\alpha_{n-1}+\sum_{k=1}^{n-1}(1-q)q^k\alpha_k\alpha_{n-k}, $$

or, denoting $[n]=1+q+\cdots+q^{n-1}$,

$$ [n]\alpha_n=(1-q)\alpha_{n-1}+q\alpha_1\alpha_{n-1}+q^{n-1}\alpha_1\alpha_{n-1}+\sum_{\kappa=2}^{n-2} q^k\alpha_k\alpha_{n-k}, $$

which is

$$ \alpha_2=\frac{1}{[2]}, $$

and for every n, $n\geqslant 3$,

$$ [n]\alpha_n=(1+q^{n-1})\alpha_{n-1}+\sum_{k=2}^{n-2} q^k\alpha_k\alpha_{n-k}. $$

Let us denote by $f_0(t)$ the formal power series

$$ f_0(t)=\sum_{n=1}^{\infty}\alpha_n t^n. $$

Then we get

$$ f_0\left(\frac{qt}{(1-q)^2}\right)=(1-q)\,{}^{q}f(t), $$

which gives the following.

Theorem 4. *The generating function* ${}^{q}f(t)$ *of Dyck words according to the parameters sum of the height of the peaks and number of peaks is* $(1-q)f_0(qt/(1-q)^2)$, *where the coefficients of* f_0 *satisfy*

$$[n]\alpha_n=(1+q^{n-1})\alpha_{n-1}+\sum_{k=2}^{n-2} q^k\alpha_k\alpha_{n-k}.$$

We show in the next section that the function f_0 can be expressed using Bessel functions. For explaining the introduction of the function f_0, we must say that first, we computed (using Macsyma) the first values of the sequence (a_n), which is classified in Sloane [23].

3. New basic Bessel functions

Let us first recall some techniques of q-calculus. The q-analog of an integer n is the polynomial

$$[n]=1+q+q^2+\cdots+q^{n-1},$$

and the q-analog of n factorial is

$$[n]!=\prod_{i=1}^{n}[i].$$

The q-derivative of a function $f(x)$ is defined by

$$D_q(f(x))=\frac{f(qx)-f(x)}{qx-x}.$$

This q-derivative coincides with the usual one when $q\to 1$.

Example. $D_q(x^n)=[n]x^{n-1}$.

Classical formulas for derivatives are easily extended to the q-derivative. For instance, if u and v are two functions,

$$D_q(u+v)=D_q(u)+D_q(v),$$

$$D_q(uv)(x)=D_q(u)(x)\cdot v(x)+u(qx)\cdot D_q(v)(x),$$

$$D_q\left(\frac{1}{u}\right)(x)=-\frac{D_q(u)(x)}{u(x)u(qx)}.$$

The reader will find in [1, 2, 4, 10] the q-analogs of classical functions and their properties.

Here, we will use a slightly different form of the Bessel functions. This form is close to the one used by some combinatorists (see for instance [5]).

Definition 5. For any integer v, let $T_v(x)$ be the function defined by

$$T_v(x)=\sum_{n=0}^{+\infty}\frac{(-1)^n x^{n+v}}{n!(n+v)!}$$

Remark. One gets $T_v(x)$ from $J_v(x)$ by changing the variable x,

$$J_v(x)=\left(\frac{x}{2}\right)^{-v} T_v\left(\frac{x^2}{2}\right).$$

The functions $T_v(x)$ satisfy a property similar to Lehmer's for $J_v(x)$.

Property 6. $T_{v+1}(x)/T_v(x)=\sum_{n=1}^{+\infty}(\Phi_n(v)/\pi_n(v))\,x^n$

The usual q-analog of the Bessel function would be

$${}^qT_v(x)=\sum_{n=0}^{+\infty}\frac{(-1)^n x^{n+v}}{[n]!\,[n+v]!},$$

where each occurrence of a factorial has been replaced by its q-analog. Here, we need a slightly different definition.

Definition 7. Let ${}_qT_v(x)$ be the q-analog of the Bessel function T_v,

$${}_qT_v(x)=\sum_{n=0}^{+\infty}\frac{(-1)^n q^{\binom{n+v}{2}} x^{n+v}}{[n]!\,[n+v]!}.$$

We define $\varphi_v(x)$ by,

$$\varphi_v(x)=\frac{{}_qT_{v+1}(x)}{{}_qT_v(x)}.$$

4. Properties of the functions $\varphi_v(x)$

In this paragraph, we first give formulas for q-derivatives of the functions ${}_qT_v(x)$ in order to get a q-differential equation satisfied by $\varphi_0(x)$. Then we show the q-analog of Property 6 in the particular case when $v=0$.

Theorem 8. *The function $\varphi_0(x)$ satisfies the following q-differential equation*:

$$D_q(\varphi_0(x))=1+(1-q)\,\varphi_0(x)+\frac{1}{x}\varphi_0(x)\,\varphi_0(qx).$$

Proof. This theorem comes from the formulas for the q-derivative of the functions ${}_qT_\nu(x)$, combined with the formulas of q-derivation. Indeed,

$$D_q({}_qT_0(x))=\sum_{n=0}^{+\infty}\frac{(-1)^n q^{\binom{n}{2}}}{[n]![n]!}D_q(x^n)=\sum_{n=0}^{+\infty}\frac{(-1)^n q^{\binom{n}{2}}}{[n-1]![n]!}x^{n-1}=-\frac{1}{x}\,{}_qT_1(x).$$

When $\nu>0$, we get similarly,

$$D_q({}_qT_\nu(x))=\sum_{n=0}^{+\infty}\frac{(-1)^n q^{\binom{n+\nu-1}{2}}(qx)^{n+\nu-1}}{[n]![n+\nu-1]!}.$$

So we have,

$$D_q({}_qT_\nu(x))={}_qT_{\nu-1}(qx)={}_qT_{\nu-1}(x)+(qx-x)D_q({}_qT_{\nu-1}(x)).$$

In particular,

$$D_q({}_qT_1(x))={}_qT_0(x)-(q-1)\,{}_qT_1(x)).$$

Finally, using q-derivative formulas we get

$$D_q\left(\frac{{}_qT_1(x)}{{}_qT_0(x)}\right)=\frac{{}_qT_0(x)-(q-1)\,{}_qT_1(x)}{{}_qT_0(x)}+\frac{1}{x}\frac{{}_qT_1(qx)\,{}_qT_1(x)}{{}_qT_0(qx)\,{}_qT_0(x)},$$

and Theorem 8 follows. □

We conjecture the following property.

Conjecture 9. The functions $\varphi_\nu(x)$ are given by

$$\varphi_\nu(x)=\sum_{n=1}^{\infty}\frac{[\Phi_n](\nu)}{[\pi_n](\nu)}x^n,$$

where $[\pi_n](\nu)$ is the natural q-analog of π_n,

$$[\pi_n](\nu)=\prod_{k=1}^{n}[k+\nu]^{\lfloor n/k\rfloor},$$

and $[\Phi_n](\nu)$ is a polynomial in the variables q and ν and with positive coefficients.

Definition 10. We denote by λ_n the natural q-analog of $\pi_n(0)$ which is the polynomial $[\pi_n](0)$, that is,

$$\lambda_n=\prod_{i=1}^{n}[i]^{\lfloor n/i\rfloor}.$$

Remark. The polynomials λ_n satisfy the following equalities,

$$\lambda_n = \prod_{j=1}^{n} ([n/j]!)^j,$$

$$\text{if } n>1, \quad \lambda_n = \lambda_{n-1} \prod_{d/n} [d].$$

The first values of λ_n are 1, [2], [2][3], $[2]^2[3][4], \ldots$.

Definition 11. For all integers $n \geqslant 1$ and $i \leqslant n$, define the q-binomial of shape λ as

$$\begin{bmatrix} n \\ i \end{bmatrix}_\lambda = \frac{\lambda_n}{\lambda_i \lambda_{n-i}}.$$

It is possible to construct some posets (binomial in Stanley's sense [24]) such that the number of maximal chains of length n is λ_n. We obtain posets which are too complicated to be analyzed.

Lemma 12. *For all integers $n \geqslant 2$ and $1 \leqslant i \leqslant n-1$.*

$$\frac{1}{[n]} \begin{bmatrix} n \\ i \end{bmatrix}_\lambda,$$

is a polynomial with integer coefficients.

This lemma is a direct consequence of a basic property of the integer part. The definition of λ_n gives,

$$\frac{1}{[n]} \begin{bmatrix} n \\ i \end{bmatrix}_\lambda = \frac{1}{[n]} \prod_{1 \leqslant j \leqslant n} [j]^{\lfloor n/j \rfloor - \lfloor i/j \rfloor - \lfloor (n-i)/j \rfloor}.$$

For every pair (x, y) of reals, we have $\lfloor x+y \rfloor \leqslant \lfloor x \rfloor + \lfloor y \rfloor$, so that each factor of the above product is a polynomial. Moreover, for $j=n$ the nth term in the product is exactly $[n]$ so the equality is trivial.

Then we can prove the following.

Proposition 13. $\not{f}_0(x) = \varphi_0(x)$.

Proof. Let

$$\varphi_0(x) = \sum_{n=1}^{\infty} \alpha_n x^n,$$

where α_n depends on q. The equality of Theorem 4 can be written as

$$\sum_{n=1}^{\infty} [n] \alpha_n x^{n-1} = 1 - \sum_{n=1}^{\infty} (q-1) \alpha_n x^n + \sum_{i,j=1}^{\infty} \alpha_i \alpha_j q^j x^{i+j-1}.$$

We have $\alpha_1 = 1$ and for every $n \geqslant 1$,

$$[n+1]\alpha_{n+1} = (1-q)\alpha_n + \sum_{k=1}^{n} \alpha_k \alpha_{n-k+1} q^k.$$

Expanding gives $\alpha_2 = 1/[2]$ and for every n, $n \geqslant 2$,

$$[n+1]\alpha_{n+1} = (1+q^n)\alpha_n + \sum_{k=2}^{n-1} \alpha_k \alpha_{n-k+1} q^k.$$

Using this last equality, we easily finish the proof using Theorem 4. □

Remark. We have $\alpha_1 = 1$, $\alpha_2 = 1/[2]$ and the above equality is enough to define the function φ_0 by recursion.

Theorem 14. *Property* 9 *holds for* $v = 0$.

Proof. The following proof is made using calculus. A more elegant combinatorial proof using valued trees is given in [12]. Let

$$\alpha_n = \frac{\beta_n}{\lambda_n}.$$

we have $\beta_1 = 1$, $\beta_2 = 1$, and for every integer $n \geqslant 2$,

$$\beta_{n+1} = (1+q^n)\frac{1}{[n+1]}\frac{\lambda_{n+1}}{\lambda_n} + \sum_{k=2}^{n-1} \frac{1}{[n+1]} \begin{bmatrix} n+1 \\ k \end{bmatrix}_\lambda \beta_k \beta_{n-k+1} q^k.$$

This gives

$$\beta_3 = 1 + q^2,$$

$$\beta_4 = 1 + q + 2q^2 + 3q^3 + 2q^4 + q^5 + q^6,$$

$$\beta_5 = 1 + q + 3q^2 + 5q^3 + 6q^4 + 6q^5 + 6q^6 + 5q^7 + 3q^8 + q^9 + q^{10}.$$

Using Definition 10 and Lemma 12, an induction gives the proof of Theorem 14. □

Conclusions

(1) The method we used here seems to be a powerful generalization of the Schützenberger methodology. In particular, it can be used even when the generating function is not algebraic.

(2) We showed that the generating functions $a_n(q)$ of skew Ferrers diagrams having a fixed number n of rows according to the area are rational. These functions have other interesting combinatorial interpretations. In [12], it is shown that on the one hand they are related to Ehrhart's theory of the enumeration of points with integer coordinates in a convex polytope. This allows us to describe these functions by means

of valued binary trees. On the other hand, these functions appear also in the enumeration of some multichains in the cartesian plane.

(3) The main open problem about this work is to find a combinatorial interpretation of numerators and denominators of the functions a_n.

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