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Hex and combinatorics☆

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Dedicated to Claude Berge, whose love of Hex, and of combinatorics, was infectious.

Abstract

Inspired by Claude Berge's interest in and writings on Hex, we discuss some results on the game.

Keywords: Game of Hex

1. Introduction

Claude Berge was a huge Hex fan. He played the game frequently and even wrote two articles on it: "*L'Art Subtil du Hex*" [7], an introduction intended for a general audience,[1] and "*Some Remarks about a Hex Problem*" [8], a puzzle commentary included in a book published in tribute to Martin Gardner. Being both an avid Hex player and a mathematician, Berge was interested in the mathematical aspects of the game. As he stated in [8],

> It would be nice to solve some Hex problem by using nontrivial theorems about combinatorial properties of sets (the sets considered are groups of critical [locations on the board]). It is not possible to forget that a famous chess problem of Sam Loyd (the "comet"), involving parity, is easy to solve for a mathematician aware of the König theorem about bipartite graphs; also, in chess, the theory of conjugate squares of Marcel Duchamp and Alberstadt is a beautiful application of the algebraic theory of graph isomorphism (the two graphs are defined by the moves of the kings).
>
> The use of a mathematical tool may be unexpected and therefore adds some new interest to a game; but Hex exists as a most enjoyable game in its own right, for mathematician and layman alike.

In this paper we survey some results on Hex, keeping the preceding comments in mind. Our paper, modelled after Berge's introduction [7], includes sections on Hex rules, origins, fundamental properties, virtual connections, openings, and puzzles; we have also added sections on computation and "dead-cell" analysis.

☆ Originally titled "Notes on Hex". The authors gratefully acknowledge the support of NSERC.

[1] Berge wrote the article so that it could be included in a commercial 14×14 version of the game to be produced in France. Unfortunately, the venture did not come to fruition and the article was not widely distributed. A translated version will appear in [21].

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Fig. 1. Hex on a 6 × 6 board: empty to start (left), and after a game which White has won (right).

2. Rules

The two-player game Hex takes its name from the playing board, which consists of a parallelogramn-shaped $n \times m$ array of hexagonal cells, where n and m can be any positive integers. Fig. 1 shows the start and end of a game.

The players, say Black and White, are each assigned two opposite sides of the four boundary sides of the board, as indicated in the diagrams by the four stones placed off the board. Each player is also assigned a collection of coloured stones or playing pieces. Each player in turn places a stone in a vacant cell. The goal is to form a single-coloured chain joining the two correspondingly coloured sides. Each of the four corner cells is considered adjacent to both sides with which it is incident. The right diagram in Fig. 1 shows the final state of a game won by White.

An unrestricted opening move is a considerable advantage for whomever plays first, so an additional rule that limits the opening is usually adopted. One such variation uses the *swap rule*: colours are assigned to the sides of the board; one player places a stone of either colour on any cell; the other player then chooses which colour to own. The game then continues as normal, with the next move played by whomever owns the opposite colour of the stone already on the board.

The swap rule effectively forces the first player to play a neutral opening, as the second player is likely to make the swap if the opening move seems advantageous. For example, on 14 × 14 boards the cell in the second row from each of two sides that meet at an acute corner is a common opening and was often played by Berge.

3. Origins and history

In [7], Berge uses both "invention" and "discovery" to refer to the origins of Hex, raising the question of whether the game is more an artefact of man or of mathematics. An argument in support of the latter is that the game was introduced on two separate occasions, and that on each occasion its genesis had a mathematical connection.

Hex was first invented in 1942 by Piet Hein (1905–1996), the well-known Danish mathematician, engineer, architect, and poet; according to Gardner [17], Hein thought of the game while pondering the then-unsolved four colour problem at the Institute for Theoretical Physics in Copenhagen.[2] Hein wrote a series of articles on the game, initially called "Polygon", that appeared in the Danish newspaper *Politiken* in December 1942 [24]. He soon afterwards formed the company Skjøde Skern, who marketed the game under the name of "Con-Tac-Tix".

Hex was reinvented by Nobel laureate John Nash sometime around the fall of 1948, when he arrived at Princeton as a mathematics student [35,34]. Although not recalling the details surrounding the invention, he does remember that it was "a matter of connecting topology and game theory" [36]. Nash's original version of the game was abstract; he was apparently interested in it more for its game-theoretic properties than because he thought it might be interesting to play. In the spring of 1949 Nash described the game to David Gale, a Princeton instructor who briefly served as Nash's stand-in supervisor. Nash's description on that occasion was of a game that could be played on a checkerboard, with squares considered adjacent if they shared an edge or an upper-right/lower-left corner. Gale realized that a rhombus-shaped array of hexagons would make a more suitable board, and built such a board that very day. Gale subsequently donated the board to the Princeton math department common room, where the game, called "Nash", became immediately popular [16,34].

From these two starting points the game spread, especially via the world's mathematics communities. Parker Brothers produced a version of the game under the name of "Hex" in 1952. The same year Nash wrote a technical report on Hex while at the Rand Corporation [35]; the following year Hex was mentioned in a published scientific paper when Claude Shannon described an electrical network that he and E.F. Moore built to play the game [43]. Hex reached a

[2] For more on the invention of Hex, see Maarup's webpage [30] or thesis [29].

wider audience in 1957 when Martin Garder introduced it to the readers of his *Scientific American* columns [17]. Berge most likely learned of the game in the fall of 1957, when he arrived at Princeton for the start of a yearlong academic visit.

Hex continues to be popular to this day. A 9×9 board version has been commercially available since 1991 [41]. Hex is frequently mentioned in books ranging from general mathematics [5,6] to advanced graph theory [25], a book devoted to a generalization of the game (discussed in Section 5) appeared in 1975 [42], while the first book dedicated solely to Hex appeared in 2000 [10]. There is also a thriving internet community; for example, games can be played online at `http://www.kurnik.org` and `http://www.boardspace.net`, or by e-mail server at `http://www.gamerz.net/pbmserv/` and `http://www.littlegolem.net`. Hex has also been played at recent Computer Games Olympiads [3,31,47].

4. Definitions

We introduce here the game-theoretic terminology we need to discuss Hex. A *board-state* B is defined by specifying for each board cell whether it is vacant, occupied by a black stone, or occupied by a white stone. Once stone sets have been assigned to players, we refer to a π-*stone* as a stone belonging to player π. A *game-state* $G = (B, \pi)$ is defined by specifying a board-state B and the player π to move next. $\overline{\pi}$ denotes the opponent of π. The *value function* takes as input a game-state and returns the value $+1$ if the player to move can win the game against any counterplay, and -1 otherwise. In a completely filled board position this value is determined by the outcome of the game; otherwise the value is recursively determined by

$$\mathrm{val}(G) = -\min_{m} \mathrm{val}(G \oplus m),$$

where $G \oplus m$ is the game-state obtained from G by a move at m, and m ranges over all vacant board cells. With respect to G, the move at m is *winning* if $\mathrm{val}(G \oplus m) = -1$; otherwise it is *losing*. By *solving* a game-state we mean determining its value; notice that a winning strategy is implicitly specified by solving every state that arises in the course of a game. These definitions are consistent with standard texts on game theory [37].

5. Fundamental properties

As Berge remarked in [7], the appeal of Hex to mathematicians seems paradoxical:

> Pieces never move once played; the rules are simple and the game can be played at any age. One might therefore wonder why Hex seems so interesting to mathematicians.

Hex is played on a graph and so is of interest to graph theorists [25]. A reason for its broader appeal is reflected in the following well-known properties of the game, which suggest that for Hex, simplicity is a reflection of elegance and complexity rather than triviality.

For the game of Hex played on an $n \times m$ board,

(1) (Hein [24]; Nash, see [17]; see also [5,15]) *the game cannot end in a draw;*
(2) (Nash [35]) *adding an extra stone to a game-state is never disadvantageous for the player owning the stone;*
(3) (Nash, Hein, see [17][3]) *for $m = n$, there exists a winning strategy for the first player*;
(4) (Reisch [40]) *determining the value of an arbitrary game-state is PSPACE-complete.*

A proof of (1) follows from the fact that

(5) *a Hex board with all cells occupied has a winning chain for at least one of the two colours.*

As with many topological results, (5) requires some care to prove. The earliest rigorous proof we are aware of, given by Beck et al. in 1969 [5], uses induction and the Jordan Curve theorem. Gale later gave an algorithmic proof [15]

[3] Gardner comments in [17]: “The proof seems to have been discovered independently by Hein and several mathematicians both here and in Europe. The following is a condensed version of the proof … as it was worked out by John F. Nash in 1949.”

Fig. 2. The Game of Y: a game won by Black (left) and Hex as a special case of Y (right).

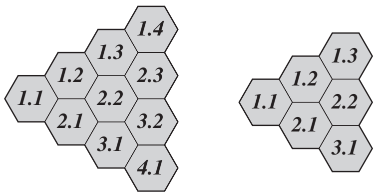

Fig. 3. The triangle-cell correspondence used in Schensted's Y-reduction: triangle $\{(x, y), (x+1, y), (x, y+1)\}$ of the larger board corresponds to cell (x, y) of the smaller board.

using only elementary graph theory and the fact that the pairwise shared edges of mutually adjacent hexagons form the complete bipartite graph $K_{1,3}$. Gale further showed that (5) is equivalent to the Brouwer fixed point theorem.[4] We present here a proof using a method due to Craige Schensted, also known as Ea Ea, that involves a game closely related to Hex, namely the game of Y.[5] This proof implies the strengthening of (5) obtained by replacing "at least one" with "exactly one".

Like Hex, Y was discovered on more than one occasion, including by John Milnor and Claude Shannon [35,17] and independently Schensted and Titus [42] in the early 1950s. A Y board consists of a triangular configuration of hexagons. As in Hex, each of the corner cells belongs to both of its bordering sides. The goal is to build a chain (here, a connected set of stones) that joins all three sides, as shown in Fig. 2.

Y is a generalization of Hex, as illustrated in Fig. 2: playing Y from the state shown in the right diagram is equivalent to playing Hex in the vacant cells of this state, since a chain wins for Y if and only if the chain restricted to the originally vacant cells wins for Hex. Thus to prove (5), it suffices to prove that

(6) *a Y board with all cells occupied has a winning chain for exactly one of the two colours.*

Schensted's method, illustrated in Fig. 4, is called *Y-reduction*. A *k-board* is a board on which each side has *k*-cells. In one Y-reduction step, a completely filled *k*-board is reduced to a completely filled $(k-1)$-board by replacing each "triangle" consisting of three pairwise adjacent hexagons oriented the same way as the board with a single hexagon whose colour is that of the majority of the colours of the triangle. For example, for the coordinate cell labelling shown in Fig. 3, triangle $\{(x, y), (x+1, y), (x, y+1)\}$ of the *k*-board corresponds to cell (x, y) of the $(k-1)$-board.

The Y-reduction process is repeated until the original *k*-board has been reduced to a 1-board as shown in Fig. 4. Now (6) follows by showing that, among the sequence of boards in the reduction, one board has a winning chain with a particular colour if and only if the next board has a winning chain with that colour. This last step is straightforward to prove and is left as an exercise for the reader.

Nash's proof of (3) uses a "strategy stealing" argument. Berge wrote in [7]:

> Nash's proof is roughly as follows. First, the game cannot end in a draw, so it follows by the Zermelo–von Neumann theorem that either the first or the second player has a winning strategy. Assume that there exists a

[4] This theorem says that any continuous mapping of an *n*-dimensional ball onto itself must have a point that maps to itself. The Nash equilibrium theorem in game theory uses this fact. See [33,26].

[5] We are grateful to Steven Meyers for communicating this result to us. Meyers independently observed that a Y *k*-board has a winning chain for colour *c* if and only if at least two of the three $k-1$ boards obtained by removing the *k* cells along a side have a winning chain for *c* [32].

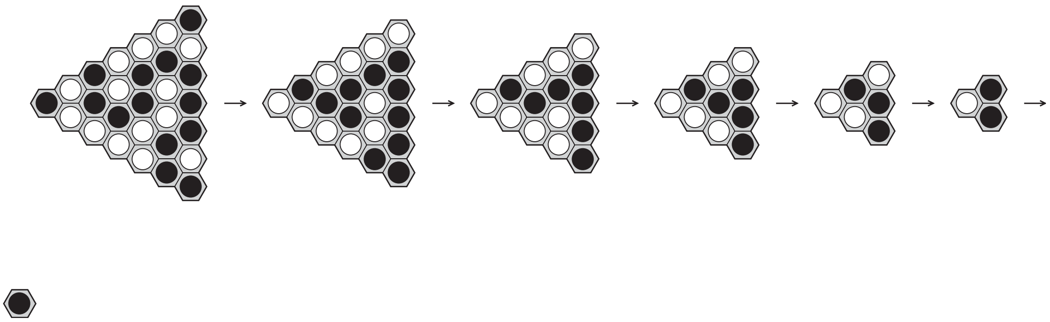

Fig. 4. Schensted's Y reduction. Each board has a winning black chain.

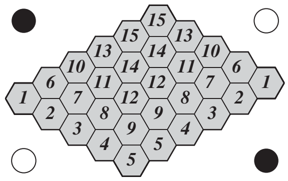

Fig. 5. A "mirror image" cell pairing which yields a winning strategy for Black.

winning strategy for the second player (say Black). Under this assumption, the first player (White) can play as follows: for the first move, play anywhere; for subsequent moves, ignore the initial move (and pretend that Black moved first) and follow Black's (winning second-player) strategy. White's required moves will always be either to a vacant location or to a location that already has a white stone, in which case White can play in any vacant location. It follows that White can follow Black's strategy, and so White (as well as Black) has a winning strategy. But only one player wins a Hex game; this contradiction completes the proof.

Nash's proof holds only for boards that "look the same" to each player, namely with a board automorphism that preserves cell adjacency and maps one player's sides to the other player's sides; for example, the proof holds for a hexagon-shaped board whose edges in cyclic order belong, respectively, to Black, Black, White, White, Black, White. The proof does not hold for asymmetric boards, and (3) does not necessarily hold either.

For example, for $n \times (n-1)$ boards, the player whose sides are closer together has a winning pairing strategy regardless of which player starts. To see why, notice that any such board can be partitioned as in Fig. 5 into two $(n-1)$-sided labelled triangles that are slightly offset mirror images of each other. By playing the mirror image of each move made by the opponent, the short-sided player manages to intersect every opponent side-to-side chain, so the opponent never wins. Shannon actually built a device that follows this strategy; see [18].

Nash's proof is tantalizing: other than showing existence, the proof reveals nothing about any first-player-win strategy. From the time that Nash's proof was first known, people have wondered: how difficult is it to find such a strategy?

In order to explain results that shed some light on this question, we describe two more games closely related to Hex. The *Shannon-switching game* is played on a graph. Two vertices are initially coloured black; players then alternately colour edges. Black wins by completing a black path between the original two vertices, while White wins by blocking all such paths. The *Shannon-vertex-colouring game* is defined similarly, except that players colour vertices instead of edges; notice that this game is a generalization of Hex.

In 1961 Oliver Gross discovered an explicit winning strategy for Bridge-It (see [19]), a game invented by Gale that is a special case of the Shannon-switching game. In 1964 Lehman generalized the Shannon-switching game to a game on matroids, and gave a criterion for winning game-states [28] that can be computed in polynomial-time.

By contrast, in 1976 Even and Tarjan [14] showed that the Shannon-vertex-colouring game is PSPACE-complete, and in 1981 Reisch [40] showed that the same conclusion holds for Hex itself, namely that (4) holds.

Reisch's result suggests that the task of discovering which of the two players has a winning strategy in an arbitrary game-state can be "difficult" computationally, but it does not exclude that this task might be "easy" for some game-states. Particularly, it might be easier to find a winning strategy for opening Hex game-states, namely for those in

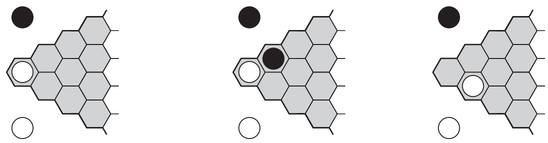

Fig. 6. For $n \times n$ boards with n at least three: a losing opening for White (left), a losing reply for Black (middle), and another losing opening for White (right).

which there are only a few stones on the board, than for arbitrary Hex game-states. Easily solved games are usually not interesting to play, so Reisch's result was greeted with relief by many Hex players.

6. Opening moves

The fact that Hex is PSPACE-complete suggests that there is likely to be no polynomial-time algorithm to solve arbitrary Hex game-states. Although the value for the empty $n \times n$ board is given by (3), even for this initial game-state there is no known polynomially computable winning strategy. In light of the swap rule, there is considerable interest in knowing which opening moves lead to wins.

A few results are known that hold for $n \times n$ boards for all n at least three. Using a strategy-stealing argument, Beck showed [5] that opening in the acute corner, as shown in the left diagram in Fig. 6, is a losing move: if White had a winning strategy for this opening, then Black could steal the same strategy to win after replying as in the middle diagram,[6] proving that White has no such winning strategy. Notice that it does not follow from Beck's argument that Black's reply leads to a win; in fact, Beck and Charles Holland later showed that this is a losing response for Black. They also showed that opening as in the right diagram of Fig. 6 loses for White [6].

Results such as these can also be proved for Rex, also known as Reverse Hex or Misère Hex, where connecting the two sides *loses* the game. Gardner mentions that Robert Winder showed that Rex is a first player win if and only if the board size is even [17]; a proof is given by Lagarias and Sleator [27], who show that the eventual loser can force the entire board to be filled. Evans proved that the opening in the acute corner is a Rex win on even boards [13].

Many Hex results are known for small boards. Solving game-states on boards up to size 5×5 is within human capacity. Enderton announced values for all 6×6 single-move openings [12]. Using a computer, Jack van Rijswijck independently verified Enderton's results and extended them by computing values for many 6×6 multiple-move openings [45]; an interactive library of these values is available online [44]. Jing Yang et al. gave values and corresponding winning strategies for several 7×7 single-move openings [49–51] and the centermost 8×8 and 9×9 single-move openings [48]. Recently, Ryan Hayward et al. computed values for all 7×7 single-move openings [22,23]. These results are summarized in Fig. 7, which shows all winning and losing opening moves on all board sizes up to 7×7.

7. Strategy: virtual connections

As Berge remarked in [7], Hex is a game "*that best lends itself to subtle analysis, to surprising combinations, to unexpected reversals of fortune*." He commented further on general Hex strategy [7]:

> In Hex, as in other games such as chess, there are different styles of play. Aggressive play consists of placing a stone where it threatens to form one or more connections. Defensive play consists of placing a stone where it can most easily cut the most threatening of the opponent's connections as soon as they appear. These two strategies eventually come together, since the only way to block all opponent threats is to form one's own connections.

Our discussion in this section, though, will be limited to one fundamental aspect of Hex strategy, namely the notion of *virtual connection*. For a further discussion of strategy, see *Mudcrack Y and Poly-Y* [42] or Cameron Browne's book *Hex Strategy: Making the Right Connections* [10].

[6] Black can steal White's strategy since the black stone nullifies the usefulness of the white stone to the point that changing its colour to black does not change the value of the game-state; we shall have more to say about such "dead cells" in Section 8.

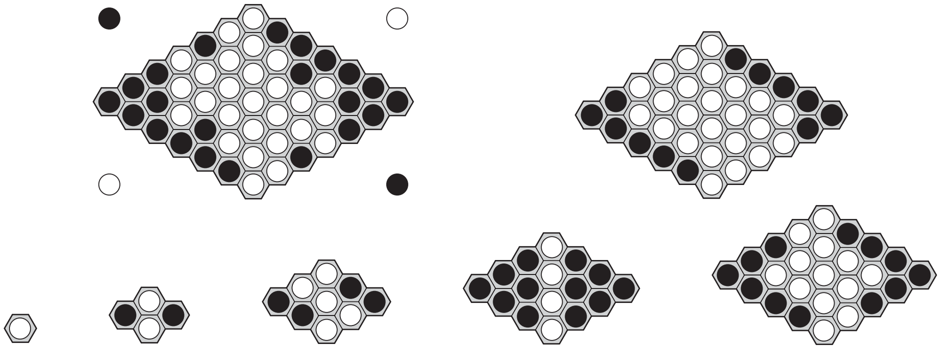

Fig. 7. Winning/losing opening moves. Stone colour indicates the winner if White opens there.

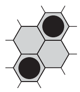

Fig. 8. A bridge virtual connection.

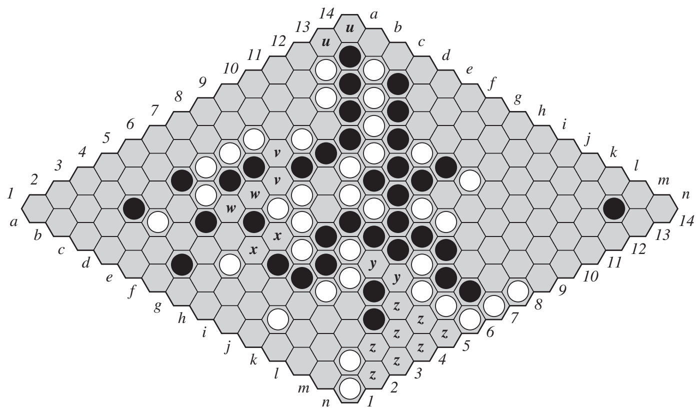

Fig. 9. From [7]: Berge's illustration of a virtual connection. The Black virtual connections with respective carriers $u, \ldots, z$ combine to form a side-to-side, and so winning, virtual connection.

At some point during a Hex game, a pair of cells might become "virtually connected": regardless of how the opponent plays, the player can connect the pair. For example, in Fig. 8 the two black stones are virtually connected via the two vacant cells: if at any time White plays in one of the vacant cells, Black can play in the other and so form the connection. A virtual connection of this form, consisting of two vacant cells mutually adjacent to the two destination cells, is called a *bridge*.

Roughly speaking, a *virtual connection* is a connection that one player can force between two cells or groups of stones even if the opponent moves first; if the connection can be forced only if the player moves first, we call it a *semi-connection*. A set of vacant cells sufficient to allow such a connection is its *carrier*. In light of the composition rules described below, it is useful to keep carriers as small as possible.

As far as we are aware, the first printed reference to the notion of virtual connection was made by Berge, who used the term *connexion virtuelle* in [7]. He did not give a definition,[7] but did present Fig. 9 as an illustrative example.

[7] This is not surprising, given that this document was written for a general audience.

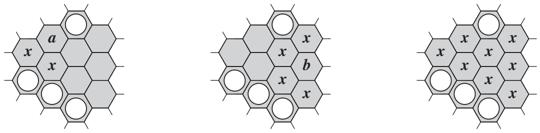

Fig. 10. Two semi-connections (left, middle) which form a virtual connection (right).

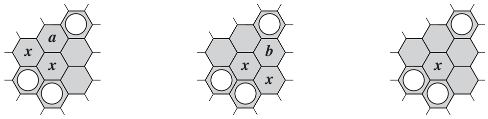

Fig. 11. Two semi-connections (left, middle) which do not form a virtual connection. Black can separate the white pieces by playing at the intersection of the two semi-connections' carriers (right).

Notice in this figure that the Black virtual connections whose carriers are marked u, v, w, x, y, z, respectively, combine to form a Black virtual connection that links the two black sides, and so wins the game for Black.

Anshelevich gave formal definitions for virtual and semi-connections together with a set of rules for constructing them [1,4]. These rules build up larger virtual connections from smaller ones, starting with "atomic" virtual connections, namely those which connect adjacent cells with an empty carrier. If we specify virtual and semi-connections by tuples $\pi(X, S, Y)$, where π is the player who can force a connection between X and Y using S as a carrier, then Anshelevich's rules can be described as follows:

AND-rule: *Let* $\pi(X,S,Y)$ *and* $\pi(Y,T,Z)$ *be virtual connections with X and S each disjoint from T and Z. Then* $\pi(X,S \cup T,Z)$ *is a virtual connection if Y is a group of* π*-stones, and* $\pi(X,S \cup T \cup Y,Z)$ *is a semi-connection if Y is a vacant cell.*

OR-rule: *Let* $\pi(X,S_j,Y)$ *be semi-connections for* $j=1,\ldots,t$ *such that* $\bigcap_j S_j$ *is empty. Then* $\pi(X,\bigcup_j S_j,Y)$ *is a virtual connection.*

Berge's virtual connection example in Fig. 9 illustrates the first part of the AND-rule.

Fig. 10 illustrates the OR-rule. The marked cells in the leftmost diagram form the carrier of a semi-connection connecting the top and bottom white stone groups, since a white stone played at cell *a* virtually connects the white groups. The marked cells in the middle diagram form a similar semi-connection. Since the two corresponding carriers do not intersect, their union forms the carrier of a virtual connection between the two white groups, as shown in the rightmost diagram.

Fig. 11 illustrates why the carriers in the OR-rule must not intersect. Since the semi-connection carriers in the first two diagrams intersect, a black stone played at their intersection, as shown in the third diagram, can prevent the two white groups from being joined.

The Anshelevich rules are incomplete in that there are connections that they cannot compose [4]. Anshelevich described without proof a complete generalization of these rules [2]; using must-play regions (described below), Rasmussen and Maire gave a generalization that can be made complete [39]; using the threat of winning the game, Noshita gave an incomplete generalization [38].

To play Hex well it is crucial to be able to recognize virtual connections. This is especially true near the end of a game, when each player is likely to have several side-to-side, and so win-threatening, semi-connections. A semi-connection can be blocked only by playing in its carrier; thus, as Berge observed both implicitly in his puzzle solution commentaries [7] and explicitly in conversation with the first author during many games of Hex, a player must play in the intersection of the carriers of all opponent side-to-side semi-connections. We refer to this intersection[8] as the *must-play* region.

Must-play analysis can reduce the number of moves that need be considered in the search for a winning move. For example, suppose the diagrams in Fig. 11 are subgames in which White's goal is to connect the two sets of white stones;

[8] More generally, since being aware of even one opponent threat can allow losing moves to be detected, it is useful to define the must-play region with respect to any set of opponent side-to-side semi-connections (instead of all such sets).

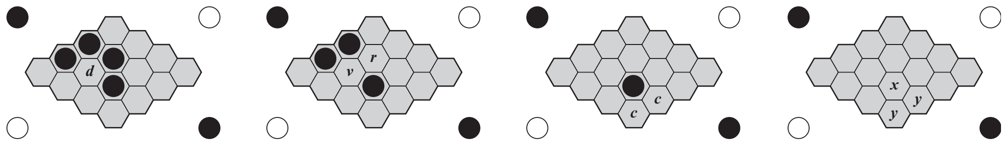

Fig. 12. From the left: a dead cell; a White-vulnerable cell; two Black-captured cells; a cell Black-dominating two other cells.

the first two diagrams show White side-to-side semi-connections and the third diagram shows the corresponding Black must-play, so Black need only consider one move.

8. Strategy: dead-cell analysis

Another form of analysis that can reduce the number of moves needing to be considered is based on the notion of a cell at which the colour of the stone is irrelevant to the outcome of the game; we call such a cell "dead". This concept has been informally discussed before, for example by Schensted and Titus [42] and Beck et al. [5,6]. The analysis we give here evolved from those works and also [45,20,22,23]; a more detailed discussion will appear in [9].

A board-state B' *completes* a board-state B if B' can be obtained from B by adding stones to all vacant cells. A cell c of B is *live* if, for some B' that completes B, changing the colour of the stone at c changes the colour of the winner on B'; otherwise, c is *dead*.[9]

For example, let B be the leftmost board-state in Fig. 12. It is easy to verify that any board B' that completes B and has a winning chain P using d still has a winning chain when d is removed from P. Thus, d is a dead cell of B.

It follows from the definition that changing the colour of a stone on a dead cell leaves the game-state's value unchanged. From this and (2) it is easy to show that a player who has a winning move but who does not yet not have a winning chain has a winning move at a live cell. Thus, the search for a winning move can ignore dead cells.

Cells that are not dead but that might be made dead, or "killed", can also be disregarded. In particular, it is pointless for π to play at a π-*vulnerable* cell, namely a cell that $\overline{\pi}$ can kill with a single stone. For example, on the second board in Fig. 12 the cell v can be killed by a black stone at r, so v is White-vulnerable.

Notice that v ceases to be White-vulnerable if a white stone is placed at r. However, on the next board in Fig. 12 Black can ensure that the two cells marked c remain White-vulnerable until either they are killed or the game ends, since a black stone placed at either of them kills the other. We call such a set of cells *Black-captured*: if White should ever play at any stone in the set, Black has a counter-strategy that kills all white stones played in the set. It is not hard to show that adding π-stones to a π-captured set does not change the value of the associated game-state.

In the rightmost board-state in Fig. 12, we say that x *Black-dominates* the two y cells, since a black stone at x Black-captures the y cells. In searching for a winning move, Black can ignore the two y cells as long as the x cell is considered.

The reader may have noticed that captured and dominating sets can be defined by mutual recursion, starting with dead cells (captured by both players) and vacant cells (dominated by both players); for details, see [9]. For further examples of dead-cell analysis in this paper, see the annotated puzzle figures in the Appendix.

9. Computation

To our mind, the approach that comes closest to fulfilling Berge's wish, expressed in [8], to solve Hex problems "using nontrivial theorems about combinatorial properties of sets" involves the mathematics and computation of winning virtual connections and captured and dominated sets. Computers facilitate the use of such methods.

Fig. 13 gives an algorithm due to van Rijswijck [46] that uses must-play sets to solve arbitrary Hex game-states.

[9] These definitions hold for vacant or occupied cells. Notice that if c is vacant, then c is *live* in B if B can be extended (by adding zero or more stones) to a board-state in which only c is vacant and neither player has a winning chain.

```
function WINVALUE(G) {
    if (B has a winning chain for π) then return (+1, ∅);
    if (B has a winning chain for π̄) then return (−1, ∅);
    W ← ∅;      [W is the carrier of a winning virtual connection]
    M ← vacant cells of B;      [M is the must-play]
    while (M ≠ ∅) {
        pick some m in M;
        (v, S) ← WINVALUE(G ⊕ m);
        if (v = −1) then return (+1, S ∪ {m});
        W ← W ∪ S;   M ← M ∩ S;
    }
    return (−1, W);
}
```

Fig. 13. Algorithm to solve an arbitrary Hex game-state G. The algorithm returns $(\mathrm{val}(G), X)$, where X is the carrier of a winning virtual connection.

The algorithm returns the value of a given game-state G, along with the carrier of a winning virtual connection in G. The variable M, corresponding to the current must-play region, contains the set of moves to investigate. The algorithm recursively solves the state arising after a move at m. If this move leads to a loss for the opponent of the player making the move then the algorithm returns a win value for G together with a corresponding carrier. If the move leads to an opponent win, say with carrier S, then all moves to cells outside of S are also losses and can be discarded; hence the update $M \leftarrow M \cap S$. When a set $\{m_i\}$ of losing moves is found whose carriers $\{S_i\}$ have an empty intersection, then G is a loss with carrier $\cup\{S_i\}$; the variable W keeps track of this union.[10]

Notice that the $M \leftarrow M \cap S$ update is guaranteed to remove at least m from M, since carriers contain only empty cells and m is not empty in $G \oplus m$. Thus the worst-case behaviour of the algorithm is identical to that of regular game tree search, which simply iterates over all available moves. The fundamental improvement is that losses can often be confirmed by investigating only a subset of the moves, whereas standard game tree search needs to try all available moves to achieve this.

By integrating some elementary dead-cell analysis [20] (essentially corresponding to the last two diagrams of Fig. 12) and the fixed-game-state virtual connection composition rules of Anshelevich [4] into van Rijswijck's dynamic-game-state algorithm, Hayward et al. recently solved all 7×7 single-move openings [22,23], as shown in Fig. 7. As most of these openings were previously unsolved, it appears as though "combinatorial properties of sets" are indeed being used to solve new Hex problems, as Berge had hoped.

10. Puzzles

> We recall Sam Loyd, the greatest chess puzzle composer of all time, who claimed that his main goal was to compose a problem where the first move is the opposite of what 999 players out of 1000 would propose. This criterion also applies in Hex: in many puzzles the solution seems paradoxical.

With these words Berge introduced in [7] a series of five puzzles, which we reproduce here. Solutions to the puzzles are given in the appendix. The Hex boards are labelled with coordinates to denote moves analogously to algebraic chess notation. Berge starts with two warm-up puzzles on a 5×5 board. The italicized text paraphrases Berge's commentary.

Puzzles 1 *and* 2: Fig. 14. *Both puzzles have a unique winning move.* Puzzle 2 is from Hein's original newspaper articles [24] and also appears in [17].

Puzzle 3: Fig. 15. *Black to play and win. It looks as though White has an impenetrable* "*wall*" *from F1 to D14. But Black can break through. How*?

Puzzle 4: Fig. 16. *Black to play and win. The two black chains seem completely cut off*, *one by a white wall from A11 to N14*, *the other by a white wall from B8 to N1. The gaps in the former wall are disjoint from the gaps in the latter. It thus seems as though Black's two chains are completely useless. And yet …*

[10] The *M*/*W* update rules are in some senses analogous to the AND/OR rules. Notice however that the AND/OR rules apply to a fixed board-state and are incomplete, whereas the *M*/*W* rules apply to dynamic game-states and are complete.

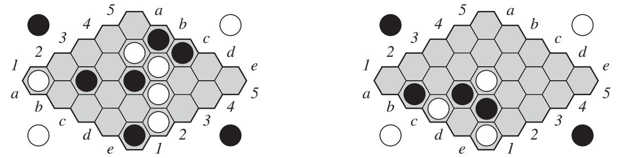

Fig. 14. Puzzles 1 (Berge) and 2 (Hein) from [7]. White to play and win.

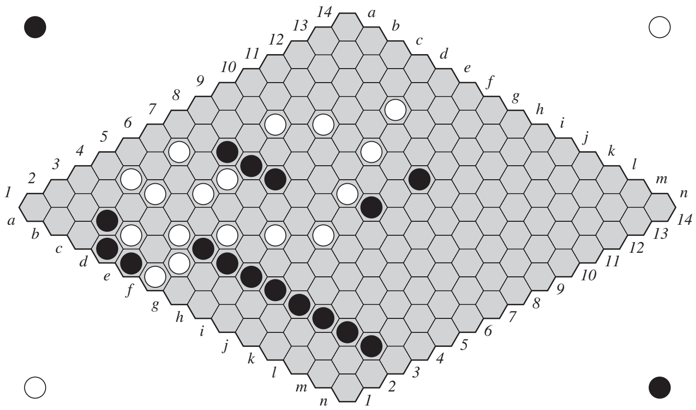

Fig. 15. Berge's Puzzle 3: Black to play and win.

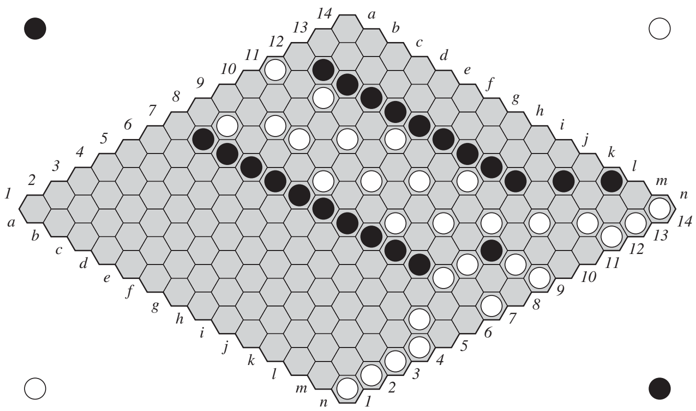

Fig. 16. Berge's Puzzle 4: Black to play and win.

Puzzle 5: Fig. 17. *White to play and win. This is a study more than a puzzle.* One question is whether White can run from C12 to the bottom-left; while this looks straightforward, Black can complicate matters considerably by well-timed moves to E4 and E3. A more *interesting question is whether White can connect G11 to the top-right*; *Black ends up trying to connect G12 to the bottom-right*, *and there are many variations to consider.*

The puzzle that is the subject of [8] was intended to be only a minor variation of Puzzle 4, but inadvertently appeared in [8] with the black stone at K7 missing. Berge's puzzle was constructed with great care, and the omission of this one stone was enough to spoil the solution: without it, there is no longer a winning move for Black. The analysis in [8] is thus invalid for the accompanying diagram; this was pointed out by Browne in [10], who was unaware of the aforementioned omission. Of course, Berge's analysis is correct for the diagram as it was supposed to appear in the article, as Browne himself was quick to point out as soon as the matter was brought to his attention [11]. A revised version of Berge's analysis in [7] of these puzzles, including the notorious Puzzle 4, appears in the Appendix.

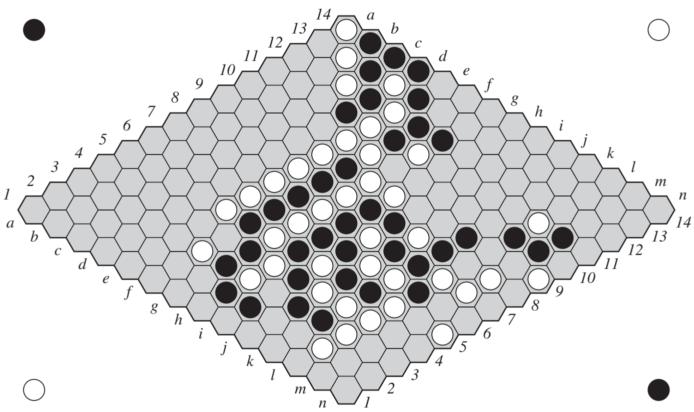

Fig. 17. Berge's Puzzle 5: White to play and win.

Acknowledgements

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Appendix: Puzzle solutions and dead-cell analysis

The annotated puzzle figures shown in Figs. A1 and A2 reflect our dead-cell analysis. Dead cells are marked with a cross, captured cells are marked with a dotted stone, and dominated or vulnerable cells are indicated, respectively, by dots of the same or opposite colour of the player to move. For example, Puzzle 1 has White to play, so in the corresponding annotated figure of Fig. A1, a vacant cell with a white (black) dot indicates a White-dominated (White-vulnerable) move.

Notice that filling in dead or captured cells can create new dead and/or captured cells. For example, in Puzzle 1 B1 is dead, as is A4 once black stones have been placed at the Black-captured A2,A3. B3,C1,D1,E3,E5 are White-vulnerable (also, C1,D1 are White-dominated by C2), while D4,D5 (also E3,E5) are White-dominated by E4. Thus, the search for a winning White move need only consider the remaining four vacant cells, namely A5,C2,D2,E4.

The following is a revised version of Berge's solutions.

Puzzle 1: The sets $\{A2, A3, B3, C2, D2\}$ and $\{A2, A3, C2, D1, D2\}$ are Black win-threatening semi-connections, as are $\{A2, A4, B3, C1, C2, D1, D2\}$ and $\{A3, A4, B3, C1, C2, D1, D2\}$. The resulting White must-play is the intersection of these four sets, namely $\{C2\}$, and it is easy to confirm that [1. C2!] is a winning move. The preceding win-threatening semi-connections yield Black winning responses to any other White move, for example [1. A2 (or A3) B3!]; [1. D1 D2!]; and [1. D2 D1!].

Puzzle 2: The winning move is [1. B3!], and Black is powerless. The most difficult Black reply is [1. – C4], to which the only correct reply is [2. D3!]. All other White opening moves lose. For example, [1. E2? D3!] [2. B3 C4!]; [1. D3? E2!]; and [1. B2? B3!].

Puzzle 3: Black can play [1. E4] threatening [2. F5]. The intermediate move B7 serves only to delay matters, and eventually White must reply with one of the moves {E5, E6, F5, F6, G4}. Black continues with D4 and then connects at either D3 or C5.

Puzzle 4: Black can start with [1. M6 L6], [2. M11 L12]; these two moves are interchangeable. Black then continues [3. L10 K11], [4. K9 K8], [5. J10 J11], [6. I10 H11], [7. I9 J8], and so on. Eventually Black reaches B9 and connects to the top-left via A9 or C8.

Puzzle 5: White can indeed connect the A14 group of stones to the bottom, and can also connect the F10 group to the top.[11] In the latter case, the key is to maintain as long as possible the double threat of joining the L11 stone with either the G11 or M8 stones, since the latter stone can be easily be joined to the bottom-left.

[11] An early version of [7] omits the white stones at N5 and M7; in that variation White can no longer connect the F10 group to the top.

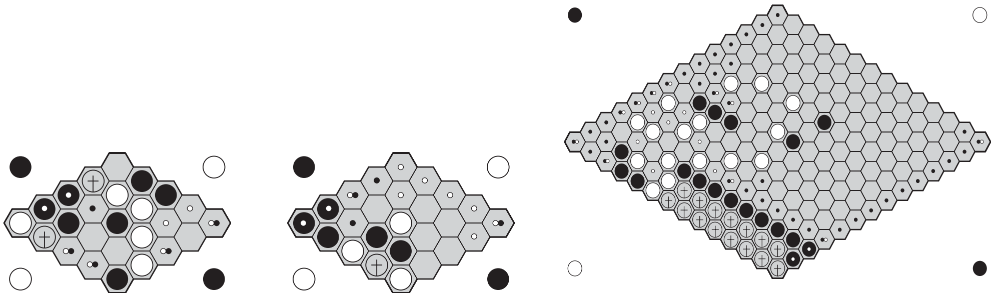

Fig. A1. Dead cell analysis of Berge Puzzles 1–3.

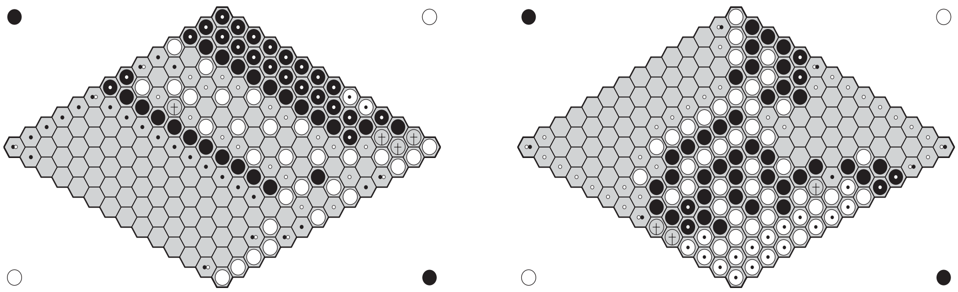

Fig. A2. Dead cell analysis of Berge Puzzles 4–5.

For example, one winning move is [1. H11], with a typical continuation being [1. – H12] [2. I11 I12] [3. K11!]. The white stones at K11,L11 are now virtually joined to the bottom-left: White can respond to [4. – J11] with [5. K10!], threatening both J10 and L9. They are also joined to the top-right; White can reply at J13 to any Black move to a location not in {J12, K12, J13, I14, J14}, and at L12 to any Black move to a location not in {L12, M12, K13, L13, M13, J14, K14, L14, M14}. The intersection of these two must-play regions is J14, and if Black moves there then [5. – J14] [6. K13 K14] [7. M13!] wins.

The move [1. K10?] is also tempting, since if Black decides to eliminate the White double threat with [1. K10 L9?] then White can reply with [2. H11!], leading eventually to a White win. But Black has a better reply, namely [1. – I11!], and now White loses.

References

[1] V. Anshelevich, The game of Hex: an automatic theorem proving approach to game programming, Proceedings of the 17th National Conference on AI (AAAI-2000), 2000, pp. 189–194.

[2] V. Anshelevich. The game of Hex: the hierarchical approach, July 2000. ⟨http://www.msri.org/publications/ln/msri/2000/gametheory/anshelevich/1/⟩.

[3] V. Anshelevich, Hexy wins Hex tournament, Internat. Comput. Games Assoc. 23 (3) (2000) 181–184.

[4] V. Anshelevich, A hierarchical approach to computer Hex, Artificial Intelligence 134 (2002) 101–120 (Also in: J. Schaeffer and J. Van den Herik, (Eds). Chips, Computers and Artificial Intelligence, Elsevier, Amsterdam, 2002, pp. 141–160.).

[5] A. Beck, M.N. Bleicher, D.W. Crowe, Games, Excursions into Mathematics, Worth, New York, 1969, pp. 317–387.

[6] A. Beck, M.N. Bleicher, D.W. Crowe, Appendix 2000. Excursions into Mathematics: the Millennium Edition, AK Peters, Natick, 2000.

[7] C. Berge, L'art subtil du Hex, Manuscript, 1977.

[8] C. Berge, Some remarks about a Hex problem, in: D. Klarner (Ed.), The Mathematical Gardner, Wadsworth International, Belmont, 1981, pp. 25–27.

[9] Y. Björnsson, R. Hayward, M. Johanson, J. van Rijswijck, Dead cell analysis in Hex and the Shannon Game, Proceedings of Graph Theory 2004, ⟨http://www.cs.ualberta.ca/~hayward/publications.html⟩, to appear.

[10] C. Browne, Hex Strategy: Making the Right Connections, AK Peters, Natick, MA, 2000.

[11] C. Browne, Puzzle J, Abstract Games Magazine, 8:21, 2001. Solution in C. Browne, Puzzle J, Abstract Games Magazine, 10:28, 2002.

[12] B. Enderton, Answers to infrequently asked questions about the game of Hex. 1995. ⟨http://www.cs.cmu.edu/~hde/hex/hexfaq/⟩.

[13] R.J. Evans, A winning opening in reverse Hex, J. Recreational Math. 7 (3) (1974) 189–192.

[14] S. Even, R. Endre Tarjan, A combinatorial problem which is complete in polynomial space, J. Comput. Mach. 23 (1976) 710–719.
[15] D. Gale, The game of Hex and the Brouwer fixed point theorem, Amer. Math. Monthly 86 (10) (1979) 818–827.
[16] D. Gale, February 2000, private communication.
[17] M. Gardner, Mathematical games, Sci. Amer. 197 (1957) 145–150 August 120–127; October 130–138.
[18] M. Gardner, The game of Hex, The Scientific American Book of Mathematical Puzzles and Diversions, Simon and Schuster, New York, 1959, pp. 73–83.
[19] M. Gardner, Mathematical games, Sci. Amer. 205 (1) (1961) 148–168.
[20] R.B. Hayward, A note on domination in Hex. January 2003. Manuscript ⟨http://www.cs.ualberta.ca/~hayward/publications.html⟩.
[21] R.B. Hayward, Berge and the art of Hex, Manuscript, 2003, in: J.A. Bondy/V. Chvátal (Eds.), A biography of Claude Berge, Princeton University Press, Princeton, NJ, ⟨http://www.cs.ualberta.ca/~hayward/publications.html⟩, to appear.
[22] R.B. Hayward, Y. Björnsson, M. Johanson, M. Kan, N. Po, J. van Rijswijck, Solving 7 × 7 Hex: virtual connections and game-state reduction, in: H.J. van den Herik, H. Iida, E.A. Heinz (Eds.), Advances in Computer Games, vol. 263 of IFIP International Federation for Information Processing, Kluwer Academic Publishers, Boston, 2003, pages 261–278.
[23] R.B. Hayward, Y. Björnsson, M. Johanson, M. Kan, N. Po, J. van Rijswijck, Solving 7 × 7 Hex with domination, fill-in, and virtual connections, Theoret. Comput. Sci. 349 (2005) 123–139.
[24] P. Hein, Vil de laere polygon? Series of articles in *Politiken* newspaper, starting 26 December 1942.
[25] T.R. Jensen, B. Toft, Graph Colouring Problems, Winning Hex, Wiley Interscience, New York, 1995, pp. 271–275. (Chapter 17.14)
[26] H.W. Kuhn, S. Nasar, (Eds.), The Essential John Nash, Princeton University Press, Princeton, NJ, 2001.
[27] J. Lagarias, D. Sleator, Who wins misère Hex?, in: E. Berlekamp, T. Rogers (Eds.), The Mathemagician and Pied Puzzler, AK Peters, Natick, MA, 1999, pp. 237–240.
[28] A. Lehman, A solution of the Shannon switching game, J. Soc. Indust. Appl. Math. 12 (1964) 687–725.
[29] T. Maarup, Hex – everything you always wanted to know about Hex but were afraid to ask. Master's Thesis, Department of Mathematics and Computer Science, University of Southern Denmark, Odense, Denmark, 2005.
[30] T. Maarup, Hex webpage. 2005 ⟨http://www.maarup.net/thomas/hex/⟩.
[31] G. Melis, R. Hayward, Six wins Hex tournament, Internat. Comput. Games Assoc. 26 (4) (2003) 277–280.
[32] S. Meyers, 2002. private communication.
[33] J. Milnor, A Nobel prize for John Nash, Math. Intell. 17 (3) (1995) 11–17.
[34] S. Nasar, A Beautiful Mind, Touchstone, New York, 1998.
[35] J. Nash, Some games and machines for playing them. Technical Report D-1164, Rand Corp., 1952.
[36] J. Nash, December 2 1999. Telephone conversation with R. Hayward and J. van Rijswijck.
[37] J. Von Neumann, O. Morgenstern, Theory of Games and Economic Behaviour, Princeton University Press, Princeton, NJ, 1944.
[38] K. Noshita, Union-connections and a simple readable winning way in 7 × 7 Hex, Proceedings of the Ninth Game Programming Japan Workshop, November 12–14, 2004, pp. 72–79.
[39] R. Rasmussen, F. Maire, An extension of the H-search algorithm for artificial Hex players, Proceedings of the 17th Australian Joint Conference on AI, 2004, pp. 646–657.
[40] S. Reisch, Hex ist PSPACE-vollständig, Acta Inform. 15 (1981) 167–191.
[41] S. Sackson and Editors of Klutz Press, The Book of Classic Board Games, Klutz Press, Palo Alto, CA, 1991.
[42] C. Schensted, C. Titus, Mudcrack Y and Poly-Y, Neo Press, Peaks Island, Maine, 1975.
[43] C. Shannon, Computers and automata, Proc. Inst. Radio Eng. 41 (10) (1953) 1234–1241.
[44] J. van Rijswijck, Queenbee's home page, 2000, ⟨http://www.cs.ualberta.ca/~queenbee⟩.
[45] J. van Rijswijck, Computer Hex: Are Bees better than Fruitflies? Master's thesis, Department of Computing Science, University of Alberta, Edmonton, Canada, 2000.
[46] J. van Rijswijck, Search and evaluation in Hex, Technical Report, University of Alberta, 2002, ⟨http://www.cs.ualberta.ca/~javhar/research.html⟩.
[47] J. Willemson, Y. Björnsson, Six wins Hex tournament, Internat. Comput. Games Assoc. 27 (3) (2004).
[48] J. Yang, Jing yang's web site, 2003–2006. ⟨http://www.ee.umanitoba.ca/~jingyang⟩.
[49] J. Yang, S. Liao, M. Pawlak, A decomposition method for finding solution in game Hex 7 × 7, in: International Conference on Application and Development of Computer Games in the 21st Century, November 2001, pp. 96–111.
[50] J. Yang, S. Liao, M. Pawlak, Another solution for Hex 7 × 7, Technical Report, University of Manitoba, Winnipeg, Canada, 2002, ⟨http://www.ee.umanitoba.ca/~jingyang/TR.pdf⟩.
[51] J. Yang, S. Liao, M. Pawlak, New winning and losing positions for 7 × 7 Hex, in: Computers and Games, Lecture Notes in Computer Science, vol. 2883, Springer, Berlin, 2002, pp. 230–248.