Proceedings of the Edinburgh Mathematical Society (2011) **54**, 431–441
DOI:10.1017/S0013091509000431

THE LEAST COMMON MULTIPLE OF CONSECUTIVE ARITHMETIC PROGRESSION TERMS

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(Received 28 March 2009)

Abstract Let $k \geqslant 0$, $a \geqslant 1$ and $b \geqslant 0$ be integers. We define the arithmetic function $g_{k,a,b}$ for any positive integer n by

$$g_{k,a,b}(n) := \frac{(b+na)(b+(n+1)a)\cdots(b+(n+k)a)}{\operatorname{lcm}(b+na, b+(n+1)a, \dots, b+(n+k)a)}.$$

If we let $a = 1$ and $b = 0$, then $g_{k,a,b}$ becomes the arithmetic function that was previously introduced by Farhi. Farhi proved that $g_{k,1,0}$ is periodic and that $k!$ is a period. Hong and Yang improved Farhi's period $k!$ to $\operatorname{lcm}(1, 2, \dots, k)$ and conjectured that $(\operatorname{lcm}(1, 2, \dots, k, k+1))/(k+1)$ divides the smallest period of $g_{k,1,0}$. Recently, Farhi and Kane proved this conjecture and determined the smallest period of $g_{k,1,0}$. For the general integers $a \geqslant 1$ and $b \geqslant 0$, it is natural to ask the following interesting question: is $g_{k,a,b}$ periodic? If so, what is the smallest period of $g_{k,a,b}$? We first show that the arithmetic function $g_{k,a,b}$ is periodic. Subsequently, we provide detailed p-adic analysis of the periodic function $g_{k,a,b}$. Finally, we determine the smallest period of $g_{k,a,b}$. Our result extends the Farhi–Kane Theorem from the set of positive integers to general arithmetic progressions.

Keywords: arithmetic progression; least common multiple; p-adic valuation; arithmetic function; smallest period

2010 *Mathematics subject classification:* Primary 11B25; 11N13; 11A05

1. Introduction

Many beautiful and important theorems about the arithmetic progression in number theory are known: Dirichlet's Theorem [**1**, **11**] and the Green–Tao Theorem [**9**] being the two most famous examples. For some other results, see, for example, [**4**, **12**, **15**, **21**, **22**]. Meanwhile, the topic of the least common multiple of any given sequence of positive integers has received a lot of attention from many authors: see, for example, [**2**, **3**, **5**–**7**, **10**, **11**, **13**, **14**, **16**, **19**, **20**]. For detailed background information about the least common multiple of finite arithmetic progressions, we refer readers to [**17**].

Farhi [**6**, **7**] investigated the least common multiple of a finite number of consecutive integers. Let $k \geqslant 0$ be an integer. It was proved in [**6**] and [**7**] that $\operatorname{lcm}(n, n+1, \dots, n+k)$ is divisible by $n\binom{n+k}{k}$ and also divides

$$n\binom{n+k}{k} \operatorname{lcm}\left(\binom{k}{0}, \binom{k}{1}, \dots, \binom{k}{k}\right).$$

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Farhi [**6**,**7**] showed that the last equality holds if $k!|(n+k+1)$. Farhi also introduced the arithmetic function g_k, which is defined for any positive integer n by

$$g_k(n) := \frac{n(n+1)\cdots(n+k)}{\operatorname{lcm}(n, n+1, \ldots, n+k)}.$$

Farhi then proved that the sequence $\{g_k\}_{k=0}^{\infty}$ satisfies the recursive relation $g_k(n) = \gcd(k!, (n+k)g_{k-1}(n))$ for all positive integers n, where $\gcd(a,b)$ means the greatest common divisor of integers a and b. Using this relation, we can easily show (by induction on k) that for any non-negative integer k, the function g_k is periodic of period $k!$. This is a result due to Farhi [**7**]. Define P_k to be the smallest period of the function g_k. Farhi's result then says that $P_k|k!$. Define $L_0 := 1$ and, for any integer $k \geqslant 1$, define $L_k := \operatorname{lcm}(1, 2, \ldots, k)$. Hong and Yang [**17**] showed that $g_k(1)|g_k(n)$ for any non-negative integer k and any positive integer n. Consequently, using this result, they showed that $P_k|L_k$ for all positive integers k. This improves Farhi's period. In [**17**], Hong and Yang raised a conjecture stating that $L_{k+1}/(k+1)$ divides P_k for all non-negative integers k. From this conjecture, one can read that $k|P_k$ and $P_k = L_k$ if $k+1$ is a prime. Very recently, Farhi and Kane [**8**] found a proof of the Hong–Yang conjecture. Furthermore, Farhi and Kane determined the exact value of P_k, which solved the open problem posed by Farhi in [**7**].

Throughout this paper, let $\mathbb{Q}$ and $\mathbb{N}$ denote the field of rational numbers and the set of positive integers, respectively. Define $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. Let $k, b \in \mathbb{N}_0$ and let $a \in \mathbb{N}$. We define the arithmetic function $g_{k,a,b} : \mathbb{N} \to \mathbb{N}$ by

$$g_{k,a,b}(n) = \frac{(b+na)(b+(n+1)a)\cdots(b+(n+k)a)}{\operatorname{lcm}(b+na, b+(n+1)a, \ldots, b+(n+k)a)}.$$

Note that $g_{k,1,0} = g_k$. It is natural to ask the following interesting question.

Problem 1.1. *Let $k \geqslant 0$, $a \geqslant 1$ and $b \geqslant 0$ be integers. Is $g_{k,a,b}$ periodic and, if so, what is the smallest period of $g_{k,a,b}$?*

Assume that $g_{k,a,b}$ is periodic and that $P_{k,a,b}$ is the smallest period of $g_{k,a,b}$. We can then use $P_{k,a,b}$ to give a formula for $\operatorname{lcm}(b+na, b+(n+1)a, \ldots, b+(n+k)a)$ as follows: for any positive integer n, we have

$$\operatorname{lcm}(b+na, b+(n+1)a, \ldots, b+(n+k)a) = \frac{(b+na)(b+(n+1)a)\cdots(b+(n+k)a)}{g_{k,a,b}(\langle n\rangle_{P_{k,a,b}})},$$

where $\langle n\rangle_{P_{k,a,b}}$ denotes the least non-negative residue of n modulo $P_{k,a,b}$. Therefore, it is important to determine the exact value of $P_{k,a,b}$.

In this paper, we investigate the least common multiple of consecutive terms in arithmetic progressions. As usual, for any prime number p, we let v_p be the normalized p-adic valuation of $\mathbb{Q}$, i.e. $v_p(a) = s$ if $p^s \parallel a$. For any real number x, by $\lfloor x\rfloor$ we denote the largest integer no more than x. Let $e_{p,k} := \lfloor \log_p k\rfloor = \max_{1\leqslant i\leqslant k}\{v_p(i)\}$ be the largest exponent of a power of p that is at most k. We can now give the main result of this paper.

Theorem 1.2. *Let $k \geqslant 0$, $a \geqslant 1$ and $b \geqslant 0$ be integers. The arithmetic function $g_{k,a,b}$ is then periodic, and if $\gcd(a,b)=1$, then its smallest period equals $Q_{k,a}$, where*

$$Q_{k,a} := \frac{L_k}{\delta_{k,a} \prod_{\text{prime } q \mid \gcd(a,L_k)} q^{e_{q,k}}}, \tag{1.1}$$

and

$$\delta_{k,a} := \begin{cases} p^{e_{p,k}} & \text{if } p \nmid a \text{ and } v_p(k+1) \geqslant e_{p,k} \text{ for some prime } p \leqslant k, \\ 1 & \text{otherwise.} \end{cases}$$

For $\gcd(a,b)>1$, the smallest period of $g_{k,a,b}$ is equal to $Q_{k,a'}$ with $a' = a/(\gcd(a,b))$.

Thus we answer Problem 1.1 completely. Our result extends the Farhi–Kane Theorem from the set of positive integers to general arithmetic progressions.

The paper is organized as follows. In § 2, by using a well-known result of Hua [**18**] we show that the arithmetic function $g_{k,a,b}$ is periodic (see Theorem 2.5). Then, in § 3, we provide detailed p-adic analysis of the periodic function $g_{k,a,b}$ and determine the smallest period of $g_{k,a,b}$. In the last section, we prove Theorem 1.2 and give an example to illustrate its validity.

2. The periodicity of $g_{k,a,b}$

Hong and Yang [**17**] proved that L_k is a period of g_k. In this section, we introduce a new method to show that for any integers $k \geqslant 0$, $a \geqslant 1$ and $b \geqslant 0$, the arithmetic function $g_{k,a,b}$ is periodic, and in particular L_k is also a period of $g_{k,a,b}$. First we need a well-known result of Hua. One can easily deduce this result by using the principle of inclusion–exclusion (see, for instance, [**18**, p. 11]).

Lemma 2.1 (Hua [18]). *Let $a_1, a_2, \dots, a_n$ be any n positive integers. We then have*

$$\operatorname{lcm}(a_1,a_2,\dots,a_n) = a_1 a_2 \cdots a_n \prod_{r=2}^{n} \prod_{1 \leqslant i_1 < \cdots < i_r \leqslant n} (\gcd(a_{i_1},\dots,a_{i_r}))^{(-1)^{r-1}}.$$

Lemma 2.2. *Let $a_1, a_2, \dots, a_n$ and $b_1, b_2, \dots, b_n$ be any $2n$ positive integers. Let $3 \leqslant t \leqslant n$ be a given integer. If $\gcd(a_{i_1},\dots,a_{i_t}) = \gcd(b_{i_1},\dots,b_{i_t})$ for any $1 \leqslant i_1 < \cdots < i_t \leqslant n$, we then have*

$$\begin{aligned} &\frac{a_1 a_2 \cdots a_n}{\operatorname{lcm}(a_1,a_2,\dots,a_n)} \prod_{r=2}^{t-1} \prod_{1 \leqslant i_1 < \cdots < i_r \leqslant n} (\gcd(a_{i_1},\dots,a_{i_r}))^{(-1)^{r-1}} \\ &\qquad = \frac{b_1 b_2 \cdots b_n}{\operatorname{lcm}(b_1,b_2,\dots,b_n)} \prod_{r=2}^{t-1} \prod_{1 \leqslant i_1 < \cdots < i_r \leqslant n} (\gcd(b_{i_1},\dots,b_{i_r}))^{(-1)^{r-1}}. \end{aligned}$$

Proof. If $\gcd(a_{i_1},\dots,a_{i_t}) = \gcd(b_{i_1},\dots,b_{i_t})$ for any $1 \leqslant i_1 < \cdots < i_t \leqslant n$, then we have $\gcd(a_{i_1},\dots,a_{i_k}) = \gcd(b_{i_1},\dots,b_{i_k})$ for any $1 \leqslant i_1 < \cdots < i_k \leqslant n$ and any $n \geqslant k \geqslant t$. Thus, by using Lemma 2.1, we get the result of Lemma 2.2. □

In particular, we have the following result.

Lemma 2.3. *Let $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ be any $2n$ positive integers. If, for any $1 \leqslant i_1 < i_2 < i_3 \leqslant n$, we have $\gcd(a_{i_1}, a_{i_2}, a_{i_3}) = \gcd(b_{i_1}, b_{i_2}, b_{i_3})$, then*

$$\frac{1}{\prod_{1\leqslant i<j\leqslant n}\gcd(a_i,a_j)}\frac{a_1a_2\cdots a_n}{\operatorname{lcm}(a_1,a_2,\ldots,a_n)} = \frac{1}{\prod_{1\leqslant i<j\leqslant n}\gcd(b_i,b_j)}\frac{b_1b_2\cdots b_n}{\operatorname{lcm}(b_1,b_2,\ldots,b_n)}.$$

Proof. Since $\gcd(a_{i_1}, a_{i_2}, a_{i_3}) = \gcd(b_{i_1}, b_{i_2}, b_{i_3})$ for any $1 \leqslant i_1 < i_2 < i_3 \leqslant n$, we have $\gcd(a_{i_1}, \ldots, a_{i_k}) = \gcd(a_{i_1}, \ldots, a_{i_k})$ for any $1 \leqslant i_1 < \cdots < i_k \leqslant n$ and $k \geqslant 3$. By using Lemma 2.1, we get the conclusion of Lemma 2.3. □

Notice that if $\gcd(a_i, a_j) = \gcd(b_i, b_j)$ for any $1 \leqslant i < j \leqslant n$, then $\gcd(a_{i_1}, a_{i_2}, a_{i_3}) = \gcd(b_{i_1}, b_{i_2}, b_{i_3})$ for any $1 \leqslant i_1 < i_2 < i_3 \leqslant n$. It follows immediately from Lemma 2.3 that the following is true.

Corollary 2.4. *Let $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ be any $2n$ positive integers. If $\gcd(a_i, a_j) = \gcd(b_i, b_j)$ for any $1 \leqslant i < j \leqslant n$, we then have*

$$\frac{a_1a_2\cdots a_n}{\operatorname{lcm}(a_1,a_2,\ldots,a_n)} = \frac{b_1b_2\cdots b_n}{\operatorname{lcm}(b_1,b_2,\ldots,b_n)}.$$

We can now give the main result of this section. This also gives an alternative proof of the Hong–Yang period of the periodic function g_k [**17**].

Theorem 2.5. *Let $k \geqslant 0$, $a \geqslant 1$ and $b \geqslant 0$ be integers. The arithmetic function $g_{k,a,b}$ is then periodic, and L_k is a period of $g_{k,a,b}$.*

Proof. Let n be any positive integer. For any $0 \leqslant i < j \leqslant k$, we have

$$\begin{aligned}\gcd(b+(n+i+L_k)a, b+(n+j+L_k)a) &= \gcd(b+(n+i+L_k)a, (j-i)a)\\ &= \gcd(b+(n+i)a, (j-i)a)\\ &= \gcd(b+(n+i)a, b+(n+j)a).\end{aligned}$$

Thus, by Corollary 2.4, we obtain

$$\begin{aligned}&\frac{(b+(n+L_k)a)(b+(n+1+L_k)a)\cdots(b+(n+k+L_k)a)}{\operatorname{lcm}(b+(n+L_k)a, b+(n+1+L_k)a, \ldots, b+(n+k+L_k)a)}\\ &\qquad = \frac{(b+na)(b+(n+1)a)\cdots(b+(n+k)a)}{\operatorname{lcm}(b+na, b+(n+1)a, \ldots, b+(n+k)a)}.\end{aligned}$$

In other words, for any positive integer n, we have $g_{k,a,b}(n + L_k) = g_{k,a,b}(n)$, as desired. □

Evidently, Theorem 2.5 gives an affirmative answer to the first part of Problem 1.1.

3. p-adic analysis of $g_{k,a,b}$

Throughout this section we always let $k \geqslant 0$, $a \geqslant 1$ and $b \geqslant 0$ be integers such that $\gcd(a, b) = 1$. From the main result of the previous section (Theorem 2.5), we know that the arithmetic function $g_{k,a,b}$ is periodic. Let $P_{k,a,b}$ denote the smallest period of $g_{k,a,b}$. By Theorem 2.5 we then know that $P_{k,a,b}$ is a divisor of L_k. But the exact value of $P_{k,a,b}$ is still unknown. In this section, we will determine the exact value of $P_{k,a,b}$. We need some more notation. Let

$$S_{k,a,b}(n) := \{b + na, b + (n + 1)a, \ldots, b + (n + k)a\}$$

be any $k+1$ consecutive terms in the arithmetic progression $\{b + ma\}_{m\in\mathbb{N}_0}$. For a given prime number p, define $g_{p,k,a,b}(n) := v_p(g_{k,a,b}(n))$. Since $g_{k,a,b}$ is a periodic function, $g_{p,k,a,b}$ is also a periodic function for each prime p and $P_{k,a,b}$ is a period of $g_{p,k,a,b}$. Let $P_{p,k,a,b}$ be the smallest period of $g_{p,k,a,b}$. We have the following result.

Lemma 3.1. *We have* $P_{k,a,b} = \mathrm{lcm}_{p\,\mathrm{prime}}(P_{p,k,a,b})$.

Proof. Since, for any $n \in \mathbb{N}$, we have that $v_p(g_{k,a,b}(n + P_{k,a,b})) = v_p(g_{k,a,b}(n))$, i.e. $P_{p,k,a,b}|P_{k,a,b}$ for each prime p. Hence we have $\mathrm{lcm}_{p\,\mathrm{prime}}(P_{p,k,a,b})|P_{k,a,b}$.

Conversely, for any $n \in \mathbb{N}$, we have that

$$v_p(g_{k,a,b}(n + \mathrm{lcm}_{p\,\mathrm{prime}}(P_{p,k,a,b}))) = v_p(g_{k,a,b}(n))$$

for each prime p. Thus, we have

$$g_{k,a,b}(n + \mathrm{lcm}_{p\,\mathrm{prime}}(P_{p,k,a,b})) = g_{k,a,b}(n)$$

for any $n \in \mathbb{N}$: that is, we have $P_{k,a,b}|\,\mathrm{lcm}_{p\,\mathrm{prime}}(P_{p,k,a,b})$. Therefore, we have $P_{k,a,b} = \mathrm{lcm}_{p\,\mathrm{prime}}(P_{p,k,a,b})$, as required. □

Hence we only need to compute $P_{p,k,a,b}$ for each prime p to get the exact value of $P_{k,a,b}$. The following result is due to Farhi [**6**]. An alternative proof of it was given by Hong and Feng [**13**].

Lemma 3.2. *Let* $\{u_i\}_{i\in\mathbb{N}_0}$ *be a strictly increasing arithmetic progression of non-zero integers and let* k *be any given non-negative integer. The integer* $\mathrm{lcm}(u_0, u_1, \ldots, u_k)$ *is then a multiple of*

$$\frac{u_0u_1\cdots u_k}{k!(\gcd(u_0, u_1))^k}.$$

Lemma 3.3. *For any positive integer* n, *we have* $g_{k,a,b}(n)|k!$.

Proof. Let $u_i = b + a(n + i)$ for $0 \leqslant i \leqslant k$. Then $\gcd(u_0, u_1) = 1$, since a and b are coprime. So by Lemma 3.2 we know that there is an integer A such that

$$\mathrm{lcm}(b + na, b + (n + 1)a, \ldots, b + (n + k)a) = A\frac{(b + an)(b + a(n + 1))\cdots(b + a(n + k))}{k!}.$$

It then follows that $k! = Ag_{k,a,b}(n)$. □

It follows from Lemma 3.3 that $g_{p,k,a,b}(n) = v_p(g_{k,a,b}(n)) = 0$ for each prime $p > k$ and any positive integer n. Hence we have $P_{p,k,a,b} = 1$ for each prime $p > k$. So, by Lemma 3.1, in order to determine the exact value of $P_{k,a,b}$, it suffices to compute the exact value of $P_{p,k,a,b}$ for all the primes p such that $1 < p \leqslant k$. First we consider the case in which $p|a$ and $1 < p \leqslant k$. Since $\gcd(a, b) = 1$, we have $\gcd(p, b) = 1$, and thus $\gcd(p, b+(n+i)a) = 1$ for any integer $n \in \mathbb{N}$ and if $0 \leqslant i \leqslant k$. Hence $\gcd(p, g_{k,a,b}(n)) = 1$ for any integer $n \geqslant 1$, i.e. we have $g_{p,k,a,b}(n) = 0$ for any integer $n \geqslant 1$ if $p|a$. Thus $P_{p,k,a,b} = 1$ if $p|a$. We put these facts into the following lemma.

Lemma 3.4. *Let p be a prime such that either $p > k$ or $p|a$. We then have $P_{p,k,a,b} = 1$.*

In what follows we treat the remaining case in which $p \nmid a$ and $1 < p \leqslant k$. Clearly, we have

$$
\begin{aligned}
g_{p,k,a,b}(n) &= \sum_{m \in S_{k,a,b}(n)} v_p(m) - \max_{m \in S_{k,a,b}(n)} v_p(m) \\
&= \sum_{e \geqslant 1} \sum_{m \in S_{k,a,b}(n)} (1 \text{ if } p^e|m) - \sum_{e \geqslant 1} (1 \text{ if } p^e \text{ divides some } m \in S_{k,a,b}(n)) \\
&= \sum_{e \geqslant 1} \#\{m \in S_{k,a,b}(n) : p^e|m\} - \sum_{e \geqslant 1} (1 \text{ if } p^e \text{ divides some } m \in S_{k,a,b}(n)) \\
&= \sum_{e \geqslant 1} \max(0, \#\{m \in S_{k,a,b}(n) : p^e|m\} - 1). \qquad (3.1)
\end{aligned}
$$

We then have the following lemmas.

Lemma 3.5. *If $p \nmid a$ and $e > e_{p,k}$, then there is at most one element of $S_{k,a,b}(n)$ which is divisible by p^e.*

Proof. Suppose that there exist two integers i and j such that $p^e|b + (n + i)a$ and $p^e|b + (n + j)a$, where $0 \leqslant i < j \leqslant k$. We then have $p^e|(j - i)a$. Since $\gcd(p, a) = 1$, we get $p^e|(j - i)$. From it we deduce that $v_p(j - i) \geqslant e > e_{p,k}$. This is a contradiction. □

Lemma 3.6. *Let e be a positive integer. If $p \nmid a$, then any p^e consecutive terms in the arithmetic progression $\{b + ma\}_{m \in \mathbb{N}_0}$ are pairwise incongruent modulo p^e. Furthermore, if $e \leqslant e_{p,k}$, then there is at least one element of $S_{k,a,b}(n)$ that is divisible by p^e.*

Proof. Suppose that there exist two integers i and j such that $b+(m+i)a \equiv b+(m+j)a \pmod{p^e}$, where $m \geqslant 0$ and $0 \leqslant i < j \leqslant p^e - 1$. Then $p^e|(j-i)a$. Since $\gcd(p, a) = 1$, we have $p^e|(j - i)$. This is impossible. Thus the first part is true.

Now let $e \leqslant e_{p,k}$. Then $1 \leqslant p^e \leqslant k$. Hence $S_{k,a,b}(n)$ holds p^e consecutive terms and one of these is divisible by p^e by the above discussion. Therefore the second part holds. □

By Lemma 3.5, we know that all the terms on the right-hand side of (3.1) are 0 if $e > e_{p,k}$. By Lemma 3.6, there is at least one element divisible by p^e in the set $S_{k,a,b}(n)$ if $e \leqslant e_{p,k}$. Therefore, by (3.1) we obtain

$$
g_{p,k,a,b}(n) = \sum_{e=1}^{e_{p,k}} f_e(n), \qquad (3.2)
$$

where $f_e(n) := \#\{m \in S_{k,a,b}(n) : p^e | m\} - 1$. Since $b + (n + i + p^e)a \equiv b + (n + i)a \pmod{p^e}$ for any $i \in \{0, 1, \ldots, k\}$, we have $f_e(n + p^e) = f_e(n)$. Therefore, p^e is a period of $f_e(n)$. Hence $f_e(n + p^{e_{p,k}}) = f_e(n)$ is true for each $e \in \{1, \ldots, e_{p,k}\}$. This implies that $g_{p,k,a,b}(n + p^{e_{p,k}}) = g_{p,k,a,b}(n)$. Consequently, $p^{e_{p,k}}$ is a period of $g_{p,k,a,b}(n)$. Thus $P_{p,k,a,b} | p^{e_{p,k}}$. It follows immediately that the $P_{p,k,a,b}$ are relatively prime for different prime numbers p. But Lemmas 3.1 and 3.4 tell us that $P_{k,a,b} = \mathrm{lcm}_{p \text{ prime},\, p \leqslant k,\, p \nmid a}(P_{p,k,a,b})$. Therefore, we get the following result.

Lemma 3.7. *We have*

$$P_{k,a,b} = \prod_{p \text{ prime},\, p \nmid a,\, p \leqslant k} P_{p,k,a,b},$$

where $P_{p,k,a,b}$ satisfies that $P_{p,k,a,b} | p^{e_{p,k}}$.

According to Lemma 3.7, it suffices to compute the p-adic valuation of $P_{p,k,a,b}$ for the prime numbers p satisfying $p \nmid a$ and $p \in (1, k]$. Now let us determine the p-adic valuation of $P_{k,a,b}$ for these prime numbers p.

Proposition 3.8. *Let $a \geqslant 1$ and $b \geqslant 0$ be integers such that $\gcd(a, b) = 1$. Let $k \geqslant 2$ be an integer and let $p \in (1, k]$ be a prime number such that $p \nmid a$.*

(i) *If $v_p(k + 1) < e_{p,k}$, then $v_p(P_{k,a,b}) = e_{p,k}$.*

(ii) *If $v_p(k + 1) \geqslant e_{p,k}$, then $v_p(P_{k,a,b}) = 0$.*

Proof. (i) Since $p^{e_{p,k}}$ is a period of $g_{p,k,a,b}$, it suffices to prove that $p^{e_{p,k}-1}$ is not the period of $g_{p,k,a,b}$, from which it follows that $p^{e_{p,k}}$ is the smallest period of $g_{p,k,a,b}$. By (3.2), we have

$$g_{p,k,a,b}(n) = \sum_{e=1}^{e_{p,k}} f_e(n) = \sum_{e=1}^{e_{p,k}-1} f_e(n) + f_{e_{p,k}}(n).$$

Since $p^{e_{p,k}-1}$ is a period of $\sum_{e=1}^{e_{p,k}-1} f_e(n)$, it is sufficient to prove that $p^{e_{p,k}-1}$ is not the period of $f_{e_{p,k}}(n)$. We claim that there exists a positive integer n_0 such that $f_{e_{p,k}}(n_0 + p^{e_{p,k}-1}) = f_{e_{p,k}}(n_0) - 1$.

By $v_p(k+1) < e_{p,k}$, we deduce that $p^{e_{p,k}} \nmid (k + 1)$ and $p^{e_{p,k}} \leqslant k$. We can suppose that $k + 1 \equiv l \pmod{p^{e_{p,k}}}$ for some $1 \leqslant l \leqslant p^{e_{p,k}} - 1$. We divide the proof of part (i) into the following two cases.

Case 1. $1 \leqslant l \leqslant p^{e_{p,k}} - p^{e_{p,k}-1}$. Since $p \nmid a$, we can always find a suitable n_0 such that $b + n_0 a \equiv 0 \pmod{p^{e_{p,k}}}$. Consider the following two sets:

$$S_{k,a,b}(n_0) = \{b + n_0 a, \ldots, b + (n_0 + p^{e_{p,k}-1} - 1)a, b + (n_0 + p^{e_{p,k}-1})a, \ldots, b + (n_0 + k)a\}$$

and

$$\begin{aligned} S_{k,a,b}(n_0 + p^{e_{p,k}-1}) = \{&b + (n_0 + p^{e_{p,k}-1})a, \ldots, b + (n_0 + k)a, \\ &b + (n_0 + k + 1)a, \ldots, b + (n_0 + k + p^{e_{p,k}-1})a\}. \end{aligned}$$

We now have that $\{b+(n_0+p^{e_{p,k}-1})a,\dots,b+(n_0+k)a\}$ is the intersection of $S_{k,a,b}(n_0)$ and $S_{k,a,b}(n_0+p^{e_{p,k}-1})$. So to compare the number of terms divisible by $p^{e_{p,k}}$ in the set $S_{k,a,b}(n_0)$ with the number of terms divisible by $p^{e_{p,k}}$ in the set $S_{k,a,b}(n_0+p^{e_{p,k}-1})$, it suffices to compare the number of terms divisible by $p^{e_{p,k}}$ in the set $\{b+n_0a,\dots,b+(n_0+p^{e_{p,k}-1}-1)a\}$ with the number of terms divisible by $p^{e_{p,k}}$ in the set $\{b+(n_0+k+1)a,\dots,b+(n_0+k+p^{e_{p,k}-1})a\}$. By Lemma 3.6, any $p^{e_{p,k}}$ consecutive terms in the arithmetic progression $\{b+ma\}_{m\in\mathbb{N}_0}$ are pairwise incongruent modulo $p^{e_{p,k}}$. Thus the terms divisible by $p^{e_{p,k}}$ in the arithmetic progression $\{b+ma\}_{m\in\mathbb{N}_0}$ must be of the form $b+(n_0+tp^{e_{p,k}})a$, $t\in\mathbb{Z}$. Since $k+1\equiv l\pmod{p^{e_{p,k}}}$ and $1\leqslant l\leqslant p^{e_{p,k}}-p^{e_{p,k}-1}$, we have $k+j\equiv l+j-1\not\equiv 0\pmod{p^{e_{p,k}}}$ for all $1\leqslant j\leqslant p^{e_{p,k}-1}$. Hence $p^{e_{p,k}}\nmid(b+(n_0+k+j)a)$ for all $1\leqslant j\leqslant p^{e_{p,k}-1}$. Thus none of the elements in the set $\{b+(n_0+k+1)a,\dots,b+(n_0+k+p^{e_{p,k}-1})a\}$ are divisible by $p^{e_{p,k}}$. On the other hand, since $b+an_0\equiv 0\pmod{p^{e_{p,k}}}$, it follows from Lemma 3.6 that there is exactly one term in the set $\{b+n_0a, b+(n_0+1)a,\dots,b+(n_0+p^{e_{p,k}-1}-1)a\}$ that is divisible by $p^{e_{p,k}}$. Therefore, the number of terms divisible by $p^{e_{p,k}}$ in the set $S_{k,a,b}(n_0+p^{e_{p,k}-1})$ is equal to the number of terms divisible by $p^{e_{p,k}}$ in the set $S_{k,a,b}(n_0)$ minus 1. Namely, $f_{e_{p,k}}(n_0+p^{e_{p,k}-1})=f_{e_{p,k}}(n_0)-1$ as required. The claim is proved in this case.

Case 2. $p^{e_{p,k}}-p^{e_{p,k}-1}<l\leqslant p^{e_{p,k}}-1$. Since $p\nmid a$, it is easy to see that there is a positive integer n_0 such that $b+(n_0+p^{e_{p,k}-1}-1)a\equiv 0\pmod{p^{e_{p,k}}}$. As in the discussion of Case 1, to compare the number of terms divisible by $p^{e_{p,k}}$ in the set $S_{k,a,b}(n_0)$ with the number of terms divisible by $p^{e_{p,k}}$ in the set $S_{k,a,b}(n_0+p^{e_{p,k}-1})$, it suffices to compare the number of terms divisible by $p^{e_{p,k}}$ in the set $\{b+n_0a,\dots,b+(n_0+p^{e_{p,k}-1}-1)a\}$ with the number of terms divisible by $p^{e_{p,k}}$ in the set $\{b+(n_0+k+1)a,\dots,b+(n_0+k+p^{e_{p,k}-1})a\}$. From $b+(n_0+p^{e_{p,k}-1}-1)a\equiv 0\pmod{p^{e_{p,k}}}$ one can deduce that the terms divisible by $p^{e_{p,k}}$ in the arithmetic progression $\{b+ma\}_{m\in\mathbb{N}_0}$ must be of the form $b+(n_0+p^{e_{p,k}-1}-1+tp^{e_{p,k}})a$ with $t\in\mathbb{Z}$. Since $k+1\equiv l\pmod{p^{e_{p,k}}}$ for some $p^{e_{p,k}}-p^{e_{p,k}-1}<l\leqslant p^{e_{p,k}}-1$, we have $p^{e_{p,k}}-p^{e_{p,k}-1}+1\leqslant l+j-1\leqslant p^{e_{p,k}}+p^{e_{p,k}-1}-2$ and so $k+j\equiv l+j-1\not\equiv p^{e_{p,k}-1}-1\pmod{p^{e_{p,k}}}$ for all $1\leqslant j\leqslant p^{e_{p,k}-1}$. It follows that for all $1\leqslant j\leqslant p^{e_{p,k}-1}$, we have $p^{e_{p,k}}\nmid(b+(n_0+k+j)a)$. That is, there does not exist an integer divisible by $p^{e_{p,k}}$ in the set $\{b+(n_0+k+1)a,\dots,b+(n_0+k+p^{e_{p,k}-1})a\}$. But the term $b+(n_0+p^{e_{p,k}-1}-1)a$ is the only term divisible by $p^{e_{p,k}}$ in the set $\{b+n_0a, b+(n_0+1)a,\dots,b+(n_0+p^{e_{p,k}-1}-1)a\}$. Thus the number of terms divisible by $p^{e_{p,k}}$ in the set $S_{k,a,b}(n_0+p^{e_{p,k}-1})$ equals the number of terms divisible by $p^{e_{p,k}}$ in the set $S_{k,a,b}(n_0)$ minus 1. Hence the desired result $f_{e_{p,k}}(n_0+p^{e_{p,k}-1})=f_{e_{p,k}}(n_0)-1$ follows immediately. The proof of the claim is complete.

From the claim we deduce immediately that $p^{e_{p,k}-1}$ is not a period of $g_{p,k,a,b}$. Thus $p^{e_{p,k}}$ is the smallest period of $g_{p,k,a,b}$. It follows that $v_p(P_{k,a,b})=e_{p,k}$ as desired.

(ii) By Lemma 3.7, we know that to prove part (ii) it is sufficient to prove that $v_p(P_{q,k,a,b})=0$ for each prime q with $q\leqslant k$ and $q\nmid a$. For any prime q different from p, since $P_{q,k,a,b}|q^{e_{q,k}}$, we then have $v_p(P_{q,k,a,b})=0$. In what follows we deal with the remaining case $q=p$.

From $v_p(k+1)\geqslant e_{p,k}$, we deduce that $p^{e_{p,k}}|(k+1)$ and $p^e|(k+1)$ for each $e\in\{1,\dots,e_{p,k}\}$. By Lemma 3.6, any p^e consecutive terms in the arithmetic progression $\{b+ma\}_{m\in\mathbb{N}_0}$ are pairwise incongruent modulo p^e since $p \nmid a$. Hence for each $e \in \{1,\dots,e_{p,k}\}$, there are exactly $(k+1)/p^e$ terms divisible by p^e in any $k+1$ consecutive terms of the arithmetic progression $\{b+ma\}_{m\in\mathbb{N}_0}$. So we have that $f_e(n) = ((k+1)/p^e) - 1$ for each $e \in \{1,\dots,e_{p,k}\}$. In other words, for every $n \in \mathbb{N}$, we have $f_e(n+1) = f_e(n)$. It then follows from (3.2) that for every $n \in \mathbb{N}$, we have $g_{p,k,a,b}(n+1) = g_{p,k,a,b}(n)$. Thus $P_{p,k,a,b} = 1$ and $v_p(P_{k,a,b}) = 0$. Therefore, part (ii) is proved. $\square$

4. Proof of Theorem 1.2

In this section, we first prove Theorem 1.2.

Proof of Theorem 1.2. By Theorem 2.5, we know that $g_{k,a,b}$ is periodic. Denote by $P_{k,a,b}$ its smallest period. First, let $\gcd(a,b) = 1$. Then, by Lemma 3.7, for any prime p such that $p|a$, we have $v_p(P_{k,a,b}) = 0$. For any prime p satisfying $p \nmid a$ and $p \leqslant k$, we have, by Lemma 3.7, $P_{p,k,a,b} = p^{v_p(P_{p,k,a,b})} = p^{v_p(P_{k,a,b})}$. So, by Proposition 3.8 we infer that

$$P_{k,a,b} = \prod_{p \text{ prime},\, p\leqslant k} p^{e_p(k,a)},$$

where

$$e_p(k,a) := \begin{cases} 0 & \text{if } v_p(k+1) \geqslant e_{p,k} \text{ or } p|a, \\ e_{p,k} & \text{otherwise.} \end{cases}$$

Using the integer L_k, we obtain immediately that $P_{k,a,b} = Q_{k,a}$ as required, where $Q_{k,a}$ is defined as in (1.1).

Now let $\gcd(a,b) > 1$. If $\gcd(a,b) = d$ and $a = da'$ and $b = db'$, then $\gcd(a',b') = 1$ and we can easily check that $g_{k,a,b}(n) = d^k g_{k,a',b'}(n)$ for any $n \in \mathbb{N}$. From this one can easily derive that the periodic functions $g_{k,a,b}$ and $g_{k,a',b'}$ have the same smallest period, i.e. $P_{k,a,b} = P_{k,a',b'}$. But the result for the case $\gcd(a,b) = 1$ applied to the function $g_{k,a',b'}$ gives us that $P_{k,a',b'} = Q_{k,a'}$, with $Q_{k,a'}$ defined as in (1.1). The desired result $P_{k,a,b} = Q_{k,a'}$ therefore follows immediately. This completes the proof of Theorem 1.2. $\square$

It was proved by Farhi and Kane [**8**] that there is at most one prime $p \leqslant k$ such that $v_p(k+1) \geqslant e_{p,k}$. We noticed that such a prime p was given in Proposition 3.3 of [**8**] without the condition $p \leqslant k$, but such a restriction condition is clearly necessary because otherwise Proposition 3.3 of [**8**] would not be true. For example, letting p be any prime number greater than $k+1$ gives us $v_p(k+1) = 0 = e_{p,k}$. Comparing the smallest period $P_{k,a,b}$ of the function $g_{k,a,b}$ with the smallest period P_k of the function $g_k = g_{k,1,0}$, we arrive at the relation between $P_{k,a,b}$ and P_k as follows:

$$P_{k,a,b} = \frac{P_k}{\prod_{\text{prime}\, p|\gcd(a',P_k)} p^{e_{p,k}}},$$

where $a' = a/(\gcd(a,b))$. From this one can read that $P_{k,a,b} = P_k$ if $a|b$.

Finally, we give an application of Theorem 1.2 as the conclusion of this paper.

Example 4.1. Let us consider the least common multiple of any $k+1$ consecutive positive odd numbers. To study this problem, we define an arithmetic function h_k by

$$h_k(n) := \frac{(2n+1)(2n+3)\cdots(2n+2k+1)}{\operatorname{lcm}(2n+1, 2n+3, \ldots, 2n+2k+1)} \quad (n \in \mathbb{N}).$$

By Theorem 1.2, we know that h_k is periodic and, for any integer $k \geqslant 2$, the exact period R_k of h_k is given by

$$R_k = \frac{L_k}{2^{e_{2,k}} D_k},$$

where

$$D_k = \begin{cases} p^{e_{p,k}} & \text{if } v_p(k+1) \geqslant e_{p,k} \text{ for some odd prime } p \leqslant k, \\ 1 & \text{otherwise.} \end{cases}$$

Acknowledgements. This research was partially supported by National Science Foundation of China Grant 10971145, by PhD Programs Foundation of Ministry of Education of China Grant 2010018111007 and by Program for New Century Excellent Talents in University Grant NCET-06-0785.

The authors thank Professor Smyth and the referees for their careful reading of the manuscript and for helpful suggestions that improved its presentation.

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