# On GCD and LCM Matrices

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ABSTRACT

Let $S = \{x_1, x_2, \dots, x_n\}$ be a set of distinct positive integers. The matrix $(S)$ having the greatest common divisor $(x_i, x_j)$ of $x_i$ and $x_j$ as its $i, j$ entry is called the greatest common divisor (GCD) matrix on $S$. The matrix $[S]$ having the least common multiple of $x_i$ and $x_j$ as its $i, j$ entry is called the least common multiple (LCM) matrix on $S$. The set $S$ is factor-closed if it contains every divisor of each of its elements. If $S$ is factor-closed, we calculate the inverses of the GCD and LCM matrices on $S$ and show that $[S](S)^{-1}$ is an integral matrix. We also extend a result of H. J. S. Smith by calculating the determinant of $[S]$ when $(x_i, x_j) \in S$ for $1 \leqslant i, j \leqslant n$.

## 1. INTRODUCTION

Let $S = \{x_1, x_2, \dots, x_n\}$ be a set of distinct positive integers. The matrix $(S)$ having the greatest common divisor $(x_i, x_j)$ of $x_i$ and $x_j$ as its $i, j$ entry is called the greatest common divisor (GCD) matrix on $S$. The study of GCD matrices was initiated by Beslin and Ligh [3]. They have shown that every GCD matrix is positive definite, and, in fact, is the product of a specified matrix and its transpose. An immediate consequence of this factorization for the GCD matrix is a result of H. J. S. Smith. Smith showed [10] that the determinant of the GCD matrix $(S)$ on a factor-closed set is the product $\phi(x_1)\phi(x_2)\cdots\phi(x_n)$, where $\phi$ is Euler's totient function. The set $S$ is factor-closed if it contains every divisor of $x$ for any $x \in S$.

The matrix $[S]$ having the least common multiple $[x_i, x_j]$ of $x_i$ and $x_j$ as its $i, j$ entry is called the least common multiple (LCM) matrix on $S$. Smith




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[10] also considered the determinant of the LCM matrix on a factor-closed set. It was shown to be $\phi(x_1)\phi(x_2)\cdots\ \phi(x_n)\pi(x_1)\pi(x_2)\cdots\ \pi(x_n)$, where $\pi$ is the multiplicative function which is defined for the prime power $p^r$ by $\pi(p^r) = -p$.

From these results one can see that the GCD matrix $(S)$ and the LCM matrix $[S]$ are invertible when $S$ is a factor-closed set. In this paper we shall obtain a formula for the inverses of these matrices on factor-closed sets.

It can also be seen from these results that $\det(S)$ divides $\det[S]$ whenever $S$ is a factor-closed set. We will show that the product of the LCM matrix $[S]$ and the inverse of the GCD matrix $(S)$ is an integral matrix (i.e., has integer entries) when $S$ is factor-closed. Hence the GCD matrix divides the LCM matrix on a factor-closed set.

Finally, Beslin and Ligh have generalized Smith's result on GCD matrices by calculating the determinant of the GCD matrix on a gcd-closed set [4]. A set $S$ is gcd-closed if $(x_i, x_j) \in S$ for $1 \leqslant i, j \leqslant n$. Clearly, a factor-closed set is gcd-closed but not conversely. We will extend Smith's result on LCM matrices by calculating the determinant of the LCM matrix on a gcd-closed set.

## 2. INVERSES OF GCD MATRICES

Our study of GCD matrices was motivated by the work of Smith, Beslin, Ligh, and Li. Smith [10] showed that the determinant of the GCD matrix $(S)$ defined on $S = \{1, 2, \dots, n\}$ is $\phi(1)\phi(2)\cdots\ \phi(n)$, where $\phi$ is Euler's totient function. He also commented in that paper that this result remains valid if $S$ is any factor-closed set. Recently, Beslin and Ligh have shown that every GCD matrix is positive definite [3]. Thus if $S = \{x_1, x_2, \dots, x_n\}$ is any set of distinct positive integers, $\det(S) \leqslant x_1 x_2 \cdots x_n$. This upper bound for the determinant of a GCD matrix was sharpened by Li [7], who showed that $\phi(x_1)\phi(x_2)\cdots\ \phi(x_n) \leqslant \det(S) \leqslant x_1 x_2 \cdots x_n - n!/2$, where the equality $\det(S) = \phi(x_1)\phi(x_2)\cdots\ \phi(x_n)$ holds if and only if $S$ is factor-closed.

These results show that the GCD matrix $(S)$ defined on any set $S$ of distinct positive integers is invertible. In this section we calculate the inverse of the GCD matrix on a factor-closed set. In Section 5 we will extend this result to a gcd-closed set.

Throughout this paper $\phi$ and $\mu$ will denote Euler's totient function and the Möbius function, respectively. The structure of a GCD matrix $(S)$ depends on the values $\phi(d)$, where $d \in T$ and $T$ is a factor-closed set containing $S$. Therefore, it is not surprising to see that these values of $\phi$ can be used to calculate the inverse of a GCD matrix.

THEOREM 1. *Let $S = \{x_1, x_2, \dots, x_n\}$ be a set of distinct positive integers. If $S$ is factor-closed, then the inverse of the GCD matrix $(S)$ defined on $S$ is the matrix $A = (a_{ij})$, where*

$$a_{ij} = \sum_{\substack{x_i | x_k \\ x_j | x_k}} \frac{1}{\phi(x_k)} \mu(x_k/x_i)\mu(x_k/x_j).$$

*Proof.* Let the $n \times n$ matrices $E = (e_{ij})$ and $U = (u_{ij})$ be defined as follows:

$$e_{ij} = \begin{cases} 1 & \text{if} \quad x_j | x_i, \\ 0 & \text{otherwise}, \end{cases} \qquad u_{ij} = \begin{cases} \mu(x_i/x_j) & \text{if} \quad x_j | x_i, \\ 0 & \text{otherwise}. \end{cases}$$

Calculating the $i, j$ entry of the product $EU$ gives

$$(EU)_{ij} = \sum_{k=1}^{n} e_{ik} u_{kj} = \sum_{\substack{x_j | x_k \\ x_k | x_i}} \mu(x_k/x_j) = \sum_{x_k | x_i/x_j} \mu(x_k)$$

$$= \begin{cases} 1 & \text{if} \quad x_j = x_i, \\ 0 & \text{otherwise}. \end{cases}$$

Hence $U = E^{-1}$. If $\Lambda = \operatorname{diag}(\phi(x_1), \phi(x_2), \dots, \phi(x_n))$, then by Remark 1 of [3], $(S) = E\Lambda E^{\mathrm{T}}$. Therefore, $(S)^{-1} = U^{\mathrm{T}}\Lambda^{-1}U = (a_{ij})$, where

$$a_{ij} = (U^{\mathrm{T}}\Lambda^{-1}U)_{ij} = \sum_{k=1}^{n} \frac{1}{\phi(x_k)} u_{ki} u_{kj} = \sum_{\substack{x_i | x_k \\ x_j | x_k}} \frac{1}{\phi(x_k)} \mu(x_k/x_i)\mu(x_k/x_j). \blacksquare$$

EXAMPLE 1. Let $S = \{1, 2, 3, 6\}$. Calculating the GCD matrix on $S$ gives

$$(S) = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 1 & 2 \\ 1 & 1 & 3 & 3 \\ 1 & 2 & 3 & 6 \end{bmatrix}.$$

By Theorem 1, $(S)^{-1} = (a_{ij})$, where

$$a_{11} = \frac{1}{\phi(1)} + \frac{1}{\phi(2)} + \frac{1}{\phi(3)} + \frac{1}{\phi(6)} = 3$$

and

$$a_{21} = \frac{-1}{\phi(2)} + \frac{-1}{\phi(6)} = -\frac{3}{2}, \qquad a_{22} = \frac{1}{\phi(2)} + \frac{1}{\phi(6)} = \frac{3}{2},$$

$$a_{31} = \frac{-1}{\phi(3)} + \frac{-1}{\phi(6)} = -1, \qquad a_{32} = \frac{1}{\phi(6)} = \frac{1}{2},$$

$$a_{33} = \frac{1}{\phi(3)} + \frac{1}{\phi(6)} = 1,$$

$$a_{41} = \frac{1}{\phi(6)} = \frac{1}{2}, \qquad a_{42} = \frac{-1}{\phi(6)} = -\frac{1}{2}, \qquad a_{43} = \frac{-1}{\phi(6)} = -\frac{1}{2},$$

$$a_{44} = \frac{1}{\phi(6)} = \frac{1}{2}.$$

Therefore, since $(S)^{-1}$ is symmetric,

$$(S)^{-1} = \begin{bmatrix} 3 & -\frac{3}{2} & -1 & \frac{1}{2} \\ -\frac{3}{2} & \frac{3}{2} & \frac{1}{2} & -\frac{1}{2} \\ -1 & \frac{1}{2} & 1 & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{bmatrix}.$$

## 3. INVERSES OF LCM MATRICES

It follows from Smith's work [10] that the determinant of the LCM matrix $[S]$ on a factor-closed set $S = \{x_1, x_2, \ldots, x_n\}$ is $\phi(x_1)\phi(x_2) \cdots \phi(x_n)\pi(x_1)\pi(x_2) \cdots \pi(x_n)$, where $\pi$ is the multiplicative function which is defined for the prime power $p^r$ by $\pi(p^r) = -p$. Thus the LCM matrix on a factor-closed set is invertible, but in general an LCM matrix is never positive definite, since its leading principal minor is always negative. Moreover, an LCM matrix may not be invertible. For example, if $S = \{1, 2, 15, 42\}$, then $\det[S] = 0$.

In this section we calculate the inverse of the LCM matrix on a factor-closed set. In the following let the function $g$ be defined for each positive integer $m$ by

$$g(m) = \frac{1}{m} \sum_{d|m} d\mu(d) = \frac{\pi(m)\phi(m)}{m^2}.$$

This function determines the structure of the LCM matrix $[S]$ in almost the same way that Euler's totient function determines the structure of the GCD matrix $(S)$.

THEOREM 2. *Let $S = \{x_1, x_2, \ldots, x_n\}$ be a set of distinct positive integers. If $S$ is factor-closed, then the inverse of the LCM matrix $[S]$ defined on $S$ is the matrix $B = (b_{ij})$, where*

$$b_{ij} = \frac{1}{x_i x_j} \sum_{\substack{x_i | x_k \\ x_j | x_k}} \frac{1}{g(x_k)} \mu(x_k/x_i)\mu(x_k/x_j).$$

*Proof.* Let $D = \mathrm{diag}(x_1, x_2, \ldots, x_n)$. By [2, Theorem 2], $[S] = DAA^{\mathrm{T}}D$, where $A = (a_{ij})$ is the $n \times n$ matrix with $a_{ij}$ defined as follows:

$$a_{ij} = \begin{cases} \sqrt{g(x_j)} & \text{if} \quad x_j | x_i \\ 0 & \text{otherwise.} \end{cases}$$

If $E$ is the $n \times n$ matrix of Theorem 1 and $\Delta = \mathrm{diag}\,(g(x_1), g(x_2), \ldots, g(x_n))$, then $AA^{\mathrm{T}} = (E\Delta^{1/2})(E\Delta^{1/2})^{\mathrm{T}} = E\Delta E^{\mathrm{T}}$. Therefore, if $U$ is the matrix of Theorem 1, $[S]^{-1} = D^{-1}U^{\mathrm{T}}\Delta^{-1}UD^{-1} = (b_{ij})$, where

$$b_{ij} = \frac{1}{x_i x_j} \sum_{\substack{x_i | x_k \\ x_j | x_k}} \frac{1}{g(x_k)} \mu(x_k/x_i)\mu(x_k/x_j). \qquad \blacksquare$$

EXAMPLE 2. Let $S = \{1, 2, 3, 6\}$. Calculating the LCM matrix on $S$ gives

$$[S] = \begin{bmatrix} 1 & 2 & 3 & 6 \\ 2 & 2 & 6 & 6 \\ 3 & 6 & 3 & 6 \\ 6 & 6 & 6 & 6 \end{bmatrix}.$$

Using Theorem 2 and calculations similar to those of Example 1, we have the following:

$$[S]^{-1} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{4} & \frac{1}{2} & -\frac{1}{4} \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{6} & -\frac{1}{6} \\ \frac{1}{2} & -\frac{1}{4} & -\frac{1}{6} & \frac{1}{12} \end{bmatrix}.$$

## 4. A FACTORIZATION OF THE LCM MATRIX

If $S$ is a factor-closed set, then one can see from Smith's results that the determinant of the GCD matrix on $S$ divides the determinant of the LCM matrix on $S$. In this section we apply the results of Section 2 to show that when $S$ is factor-closed the GCD matrix $(S)$ is a factor of the LCM matrix $[S]$ in the ring $M_n(\mathbb{Z})$ of $n \times n$ matrices over the integers. The key to obtaining this result is the following lemma.

LEMMA 1. *Let $m$ and $r$ be positive integers and $t = r/(m, r)$. If $m = p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k}$ is a product of distinct primes $p_i$, then we have the following*:

$$\Psi(m, r) = \sum_{d|m} [d, r]\mu(m/d) = \begin{cases} 0 & \text{if } p_i^{a_i}|r \text{ for some } 1 \leqslant i \leqslant k \\ t\phi(m) & \text{otherwise.} \end{cases}$$

*Proof.* If $f$ is the multiplicative function which is defined for each positive integer $m$ by

$$f(m) = \sum_{d|m} \frac{d}{(d, r)} \mu(m/d),$$

then $\Psi(m, r) = rf(m)$. For a prime power $p^\epsilon$ $(\epsilon \geqslant 1)$ we have

$$f(p^\epsilon) = \frac{p^\epsilon}{(p^\epsilon, r)} - \frac{p^{\epsilon-1}}{(p^{\epsilon-1}, r)} = \begin{cases} 0 & \text{if } p^\epsilon|r, \\ \dfrac{\phi(p^\epsilon)}{(p^\epsilon, r)} & \text{otherwise.} \end{cases}$$

Thus $f(m) = 0$ if $p_i^{a_i}|r$ for some $1 \leqslant i \leqslant k$, and is equal to $\phi(m)/(m, r)$ otherwise. ∎

**Theorem 3.** *Let $S = \{x_1, x_2, \dots, x_n\}$ be a set of distinct positive integers. If $S$ is factor-closed, then the LCM matrix $[S]$ is the product of an integral matrix and the GCD matrix $(S)$.*

*Proof.* Let $B = (b_{ij})$ be the $n \times n$ matrix with

$$b_{ij} = \sum_{x_j | x_k} \frac{1}{\phi(x_k)} \mu(x_k/x_j) \sum_{d | x_k} [d, x_i] \mu(x_k/d).$$

By Lemma 1 each $b_{ij}$ is an integer. We want to show that $[S] = B(S)$. Using Theorem 1 we have the following:

$$\begin{aligned}\left([S](S)^{-1}\right)_{ij} &= \sum_{m=1}^{n} [x_i, x_m] \sum_{\substack{x_m | x_k \\ x_j | x_k}} \frac{1}{\phi(x_k)} \mu(x_k/x_m) \mu(x_k/x_j) \\ &= \sum_{x_j | x_k} \frac{1}{\phi(x_k)} \mu(x_k/x_j) \sum_{d | x_k} [d, x_i] \mu(x_k/d).\end{aligned}$$

Thus $[S] = B(S)$. ∎

**Remark 1.** Since $(S)$ and $[S]$ are symmetric, we also have $[S] = (S)B^{\mathrm{T}}$.

**Remark 2.** If $S$ is not factor-closed, then the LCM matrix $[S]$ may not be the product of the GCD matrix $(S)$ and an integral matrix. For example, if $S = \{2, 3, 5\}$, then we have the following

$$(S) = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 5 \end{bmatrix}, \qquad [S] = \begin{bmatrix} 2 & 6 & 10 \\ 6 & 3 & 15 \\ 10 & 15 & 5 \end{bmatrix},$$

$$(S)^{-1} = \begin{bmatrix} \frac{7}{11} & -\frac{2}{11} & -\frac{1}{11} \\ -\frac{2}{11} & \frac{9}{22} & -\frac{1}{22} \\ -\frac{1}{11} & -\frac{1}{22} & \frac{5}{22} \end{bmatrix}.$$

Therefore $\left([S](S)^{-1}\right)_{11} = -\frac{8}{11}$.

## 5. GCD AND LCM MATRICES ON GCD-CLOSED SETS

Many generalizations of Smith's result on GCD matrices have been published [1, 3, 4, 6, 8, 9]. Beslin and Ligh [4] have extended his result by showing that the determinant of the GCD matrix defined on the gcd-closed set $S = \{x_1, x_2, \ldots, x_n\}$ is the product $\alpha_1 \alpha_2 \cdots \alpha_n$, where

$$\alpha_i = \sum_{\substack{d \mid x_i \\ d \nmid x_t \\ x_t < x_i}} \phi(d).$$

In fact, we have calculated the inverse of a GCD matrix on a gcd-closed set. The proof is similar to Theorem 1.

THEOREM 4. *Let $S = \{x_1, x_2, \ldots, x_n\}$ be a set of distinct positive integers. If $S$ is gcd-closed, then the inverse of the GCD matrix $(S)$ defined on $S$ is the matrix $A = (a_{ij})$, where*

$$a_{ij} = \sum_{\substack{x_i \mid x_k \\ x_j \mid x_k}} \frac{c_{ik} c_{jk}}{b_k}, \qquad \text{with } b_i = \sum_{\substack{d \mid x_i \\ d \nmid x_t \\ x_t < x_i}} \phi(d) \text{ and } c_{ij} = \sum_{\substack{dx_i \mid x_j \\ dx_i \leq x_t \\ x_t < x_j}} \mu(d).$$

*Proof.* Let $C = (c_{ij})$, and $E$ be the matrix defined in Theorem 1. Calculating the $i, j$ entry of the product $CE^{\mathrm{T}}$ gives

$$(CE^{\mathrm{T}})_{ij} = \sum_{k=1}^{n} c_{ik} e_{jk} = \sum_{x_k \mid x_j} \sum_{\substack{dx_i \mid x_k \\ dx_i \nmid x_t \\ x_t < x_k}} \mu(d). \tag{1}$$

Suppose $ex_i \mid x_j$, and let $\{x_t : ex_i \mid x_t \text{ and } x_t \mid x_j\} = \{t_1, t_2, \ldots, t_r\}$. Since $S$ is gcd-closed, $x_m = (t_1, t_2, \ldots, t_r) \in S$. By our choice of $x_m$, $ex_i \mid x_m$ but $ex_i x_t$ for $x_t < x_m$. Thus $\mu(e)$ is one term in (1). Furthermore, if $ex_i \mid x_t$ and $x_t \mid x_j$ for some $x_t \neq x_m$, then $ex_i \mid x_m$ and $x_m < x_t$. Hence the term $\mu(e)$ occurs exactly once in (1). Therefore we have the following:

$$(CE^{\mathrm{T}})_{ij} = \sum_{d \mid x_j / x_i} \mu(d) = \begin{cases} 1 & \text{if } x_j = x_i, \\ 0 & \text{otherwise.} \end{cases}$$

Hence $C^{\mathrm{T}} = E^{-1}$. If $B = \mathrm{diag}(b_1, b_2, \dots, b_n)$, then from [4, Theorem 1] it follows that $(S) = EBE^{\mathrm{T}}$. Therefore, $(S)^{-1} = CB^{-1}C^{\mathrm{T}}$. ∎

Our next result will generalize Smith's result on LCM matrices by calculating the determinant of an LCM matrix on a gcd-closed set.

THEOREM 5. *Let $S = \{x_1, x_2, \dots, x_n\}$ be a set of distinct positive integers. If $S$ is gcd-closed, then*

$$\det[S] = \prod_{i=1}^{n} x_i^2 \alpha_i, \qquad \text{where} \quad \alpha_i = \sum_{\substack{d \mid x_i \\ d \nmid x_t \\ x_t < x_i}} g(d).$$

*Proof.* Since the determinant of the LCM matrix defined on $S$ is independent of the order of the elements in $S$, we may assume that $x_1 < x_2 < \cdots < x_n$. Let $D = \mathrm{diag}(x_1, x_2, \dots, x_n)$, and define the $n \times n$ matrix $\Delta = (\delta_{ij})$ via

$$\delta_{ij} = \begin{cases} \alpha_i & \text{if} \quad x_i \mid x_j, \\ 0 & \text{otherwise.} \end{cases}$$

If $E$ is the matrix of Theorem 1, we have the following:

$$(DE\Delta D)_{ij} = x_i x_j \sum_{k=1}^{n} e_{ik} \delta_{kj} = x_i x_j \sum_{x_k \mid (x_i, x_j)} \alpha_k.$$

By an argument similar to the one in the proof of [3, Proposition 1],

$$\sum_{x_k \mid (x_i, x_j)} \alpha_k = \sum_{d \mid (x_i, x_j)} g(d).$$

From the Möbius inversion formula, $\sum_{d \mid m} g(d) = 1/m$. Thus $[S] = DE\Delta D$. Therefore, since $E$ and $\Delta$ are triangular matrices with $e_{ii} = 1$ and $\delta_{ii} = \alpha_i$, we have $\det[S] = \prod_{i=1}^{n} x_i^2 \alpha_i$. ∎

COROLLARY 1 (Smith [10]). *Let $S = \{x_1, x_2, \dots, x_n\}$ be a factor-closed set, and $[S]$ be the LCM matrix defined on $S$. Then* $\det[S] = \prod_{i=1}^{n} \pi(x_i)\phi(x_i)$.

In general an LCM matrix need not be invertible, as was pointed out in Section 3. Although the determinant of an LCM matrix on a gcd-closed set can be calculated using Theorem 5, it is not clear that the product $\alpha_1 \alpha_2 \cdots \alpha_n$ is always nonzero. Hence we make the following conjecture.

CONJECTURE. The LCM matrix on a gcd-closed set is invertible.

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*Received 24 May 1991; final manuscript accepted 12 August 1991*