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Two Different Computing Methods of the Smith Arithmetic Determinant

Xing-Jian Li, Shen Qu

Abstract—The Smith arithmetic determinant is investigated in this paper. By using two different methods, we derive the explicit formula for the Smith arithmetic determinant.

Keywords—Elementary row transformation, Euler function, Matrix decomposition, Smith arithmetic determinant.

I. Introduction

SUPPOSE A is an $n \times n$ matrix, $A = [a_{i,j}] = [(i,j)]$, $(i,j = 1, 2, \ldots, n)$, and (i,j) denotes the greatest common divisor of integer i and j. We call the determinant of this kind of matrix the *Smith arithmetic determinant*([3],[4],[5],[6]), and we use the symbol S_n to denote the value of the Smith arithmetic determinant of order n.

A proof has been given in reference[3]. In this paper, we present another two different methods to derive the explicit formula of the Smith arithmetic determinant.

Firstly, we introduce some definitions and several basic results.

Definition 1 ([1]). *Suppose a, b are two integers that not all of them equal to zero. If a nonzero integer d divides both a and b, then we call d a common divisor of a and b; Moreover, if any other common divisors of a and b is the divisor of d, we call d a greatest common divisor of a and b. Using symbol (a,b) to denote the positive greatest common divisor of a and b.*

Obviously we have:

1) $(1,i) = 1$;

2) Suppose p is a prime. If n is not a multiple of p, then $(p,n) = 1$; else $(p,n) = p$.

Theorem 1 ([1, Euclid Algorithm]). *Suppose a and $b \neq 0$ are two integers, and*

$$a = q_1 b + r_1, 0 < r_1 < b,$$

$$b = q_2 r_1 + r_2, 0 < r_2 < r_1,$$

$$r_1 = q_2 r_2 + r_3, 0 < r_3 < r_2,$$

$$\vdots$$

$$r_{n-1} = q_n r_n + r_{n+1}, 0 < r_{n+1} < r_n,$$

$$r_n = q_{n+1} r_{n+1}.$$

Then $(a,b) = r_{n+1}$.

X. Li and S.Qn are with the School of Mathematical Sciences, University of Electronic Science and Technology of China, Chengdu, Sichuan, 611731, China (corresponding author: Xingjian Li. phone: 86-28-61831280; fax: 86-28-61831280; e-mail: lixingjianjan@163.com).

Theorem 2 ([1]). *Suppose integer $n \geq 2$, then there exist such primes that*

$$0 < p_1 \leq p_2 \leq \cdots \leq p_r, n = p_1 p_2 \cdots p_r.$$

Furthermore, if there are other primes

$$0 < q_1 \leq q_2 \leq \cdots \leq q_s, n = q_1 q_2 \cdots q_s,$$

then we have $r = s$ and $p_i = q_i$ for $1 \leq i \leq s$.

Definition 2 ([2]). *Suppose n is a positive integer, and let $\phi(n)$ denote the number of positive integers which are prime to n and less than n. Then we call $\phi(n)$ the Euler function of n.*

Apparently if p is a prime, then $\phi(p) = p - 1$.

In the following, we will give a classical result about Euler function, which will be used in the proof.

Theorem 3 ([2]). *Suppose m is a positive integer, then $\sum\limits_{d|m} \phi(d) = m$.*

II. Elementary Row Transformation on the Smith Arithmetic Determinant

Firstly we give two examples.

Example 1. Consider the computation of the Smith arithmetic determinant of order 4. By the definition,

$$S_4 = \begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 1 & 2 \\ 1 & 1 & 3 & 1 \\ 1 & 2 & 1 & 4 \end{vmatrix}.$$

This determinant can be computed as follows: Multiply the first row R_1 by -1 and add it to every other row. Then multiply the second row R_2 by -1 and add it to the fourth row $R_{2\times 2} = R_4$. As a result, we have a determinant of an upper triangular matrix, of which the dominating diagonal elements are $\phi(1), \phi(2), \phi(3), \phi(4)$ in order. Therefore, the value of this Smith arithmetic determinant is $\prod\limits_{i=1}^{4} \phi(i)$.

The process above can be written as follows:

$$\begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 1 & 2 \\ 1 & 1 & 3 & 1 \\ 1 & 2 & 1 & 4 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 2 & 0 \\ 0 & 1 & 0 & 3 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{vmatrix}$$

$$= \begin{vmatrix} \phi(1) & \phi(1) & \phi(1) & \phi(1) \\ 0 & \phi(2) & 0 & \phi(2) \\ 0 & 0 & \phi(3) & 0 \\ 0 & 0 & 0 & \phi(4) \end{vmatrix} = \prod_{i=1}^{4} \phi(i).$$

Example 2. Consider the computation of the Smith arithmetic determinant of order 6. According to the definition,

$$S_6 = \begin{vmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 1 & 2 & 1 & 2 \\ 1 & 1 & 3 & 1 & 1 & 3 \\ 1 & 2 & 1 & 4 & 1 & 2 \\ 1 & 1 & 1 & 1 & 5 & 1 \\ 1 & 2 & 3 & 2 & 1 & 6 \end{vmatrix}.$$

Inspired by the method above, we can compute it with the following method: multiply the first row R_1 by -1 and add it to every other row. Then multiply the second row R_2 by -1 and add it to the fourth row $R_{2\times2}$, the sixth row $R_{2\times3}$. Finally, we multiply the third row R_3 by -1 and add it to the sixth row $R_{3\times2}$. As a result, we have a determinant of an upper triangular matrix, the value of which is $\prod_{i=1}^{6}\phi(i)$.

The process above can be written as follows:

$$\begin{vmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 1 & 2 & 1 & 2 \\ 1 & 1 & 3 & 1 & 1 & 3 \\ 1 & 2 & 1 & 4 & 1 & 2 \\ 1 & 1 & 1 & 1 & 5 & 1 \\ 1 & 2 & 3 & 2 & 1 & 6 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 2 & 0 & 0 & 2 \\ 0 & 1 & 0 & 3 & 0 & 1 \\ 0 & 0 & 0 & 0 & 4 & 0 \\ 0 & 1 & 2 & 1 & 0 & 5 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 2 & 0 & 0 & 2 \\ 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 \end{vmatrix},$$

of which the diagonal elements are $\phi(1), \phi(2), \phi(3), \phi(4), \phi(5), \phi(6)$ in order. Thus, we have $S_6 = \prod_{i=1}^{6}\phi(i)$.

A general method of the computation of the Smith arithmetic determinant can be concluded from the two examples above: Multiply R_1 by -1 and add it to every other row, then multiply R_2 by -1 and add it to $R_{2\times i}(i \geq 2)$, then multiply R_3 by -1 and add it to $R_{3\times i}(i \geq 2)$, ..., until a determinant of an upper triangular matrix is yielded.

As the examples illustrated, we give out a determinant of an upper triangular matrix in which the elements on the diagonal are $\phi(1), \phi(2), \ldots, \phi(n)$ in order. Therefore, we have

$$S_n = \prod_{i=1}^{n}\phi(i).$$

Before giving a complete proof, we firstly introduce several lemmas.

Lemma 1. *1) The sequence composed by the elements of each row of the Smith arithmetic determinant is periodic, and the minimal positive period is the row index number;*

2) After the elementary row transformation used above, each row is still periodic, and the minimal positive period remains the same.

Proof. 1) It is equivalent to prove $(t, i) = (t, kt + i)$. According to the Euclid algorithm, it is obviously true. Moreover, there is a one to one correspondence between the sequence $(t, i + 1), (t, i + 2), \ldots, (t, i + t)$ and the sequence $(t, 1), (t, 2), \ldots, (t, t)$, so the minimal positive period is t.

2) For each row R_a, according to the algorithm above, only when $d|a$ that the row R_d will be timed by -1 and added to R_a. The period of R_d and R_a are respectively d and a, and $d|a$, so a is also a period of R_d. It means that R_a and R_d have the same period a.

Therefore, when R_d is timed by -1 and added to R_a, the period of R_a does not change.

If the index number of a row is a prime, then this row will be called *prime row*. Otherwise this row will be called *composite row*.

It is obvious that each element of the first row as well as the first column equals to 1 and each element of the row R_p (suppose p is a prime) equals to 1 or p (which occurs only when p divides its corresponding column index number).

Now we investigate the expression of the Smith arithmetic determinant of order n:

$$S_n = \begin{vmatrix} 1 & 1 & 1 & 1 & 1 & 1 & \cdots & 1 & \cdots & 1 \\ 1 & 2 & 1 & 2 & 1 & 2 & \cdots & 1 & \cdots & * \\ 1 & 1 & 3 & 1 & 1 & 3 & \cdots & 1 & \cdots & * \\ 1 & a_{42} & a_{43} & a_{44} & a_{45} & a_{46} & \cdots & 1 & \cdots & * \\ 1 & 1 & 1 & 1 & 5 & 1 & \cdots & 1 & \cdots & * \\ 1 & a_{62} & a_{63} & a_{64} & a_{65} & a_{66} & \cdots & 1 & \cdots & * \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & 1 & 1 & 1 & \cdots & p & \cdots & * \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 1 & a_{n2} & a_{n3} & a_{n4} & a_{n5} & a_{n6} & \cdots & a_{np} & \cdots & n \end{vmatrix},$$

where $a_{ij} = (i, j)$.

Because of the periodicity in each row R_i, we only need to consider the elements from $a_{i,1}$ to $a_{i,i}$.

Elementary row transformation on the Smith arithmetic determinant of order n:

Step 1: Multiply R_1 by -1 and add it to all the prime rows, and the result is:

$$S_n = \begin{vmatrix} 1 & 1 & 1 & 1 & 1 & 1 & \cdots & 1 & \cdots & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & \cdots & 0 & \cdots & * \\ 0 & 0 & 2 & 0 & 0 & 2 & \cdots & 0 & \cdots & * \\ 1 & a_{42} & a_{43} & a_{44} & a_{45} & a_{46} & \cdots & 1 & \cdots & * \\ 0 & 0 & 0 & 0 & 4 & 0 & \cdots & 0 & \cdots & * \\ 1 & a_{62} & a_{63} & a_{64} & a_{65} & a_{66} & \cdots & 1 & \cdots & * \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & p-1 & \cdots & * \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 1 & a_{n2} & a_{n3} & a_{n4} & a_{n5} & a_{n6} & \cdots & a_{np} & \cdots & n \end{vmatrix}.$$

Because we have $\phi(1) = 1$, $\phi(p) = p - 1$(suppose p is a prime), we replace $p - 1$ by $\phi(p)$ in R_p, and the determinant above can be written into:

$$\begin{vmatrix} \phi(1) & \phi(1) & \phi(1) & \phi(1) & \phi(1) & \phi(1) & \cdots & 1 & \cdots & * \\ 0 & \phi(2) & 0 & \phi(2) & 0 & \phi(2) & \cdots & 0 & \cdots & * \\ 0 & 0 & \phi(3) & 0 & 0 & \phi(3) & \cdots & 0 & \cdots & * \\ 1 & a_{42} & a_{43} & a_{44} & a_{45} & a_{46} & \cdots & 1 & \cdots & * \\ 0 & 0 & 0 & 0 & \phi(5) & 0 & \cdots & 0 & \cdots & * \\ 1 & a_{62} & a_{63} & a_{64} & a_{65} & a_{66} & \cdots & 1 & \cdots & * \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & \phi(p) & \cdots & * \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 1 & a_{n2} & a_{n3} & a_{n4} & a_{n5} & a_{n6} & \cdots & a_{np} & \cdots & n \end{vmatrix},$$

where every prime row has the form $(0, 0, \ldots, 0, \phi(p), 0, \ldots)$, or

$$a_{pt} = \begin{cases} \phi(p), & \text{if } p \text{ divides } t; \\ 0, & \text{otherwise.} \end{cases}$$

Step 2: We compute this determinant with the following method: if a row R_b is determined and the rows above R_b have the form

$$a_{it} = \begin{cases} \phi(i), & \text{if } i \text{ divides } t; \\ 0, & \text{otherwise.} \end{cases} \quad (i = 1, 2, \ldots, p-1),$$

we multiply R_d by -1 and add it to R_b where $d|b$ and $d < b$, and we have a new row R_b'. It is obvious to know that R_4 is the first row to be transformed, and all the rows to be transformed are composite rows. We assert that:

Property 1. *After the transformation, R_b' has the form*

$$a_{bt}' = \begin{cases} \phi(b), & \text{if } b \text{ divides } t; \\ 0, & \text{otherwise.} \end{cases}.$$

Proof. Because of the periodicity in R_b, and $a_{bb} = b$, we only need to consider the result after the transformation on $a_{bi} (i = 1, 2, \ldots, b)$. For each $a_{bi} = (b, i)$, according to the method,

$$a_{bi}' = a_{bi} - \sum_{d|b, d<b} a_{di},$$

where a_{bi}' is the result after the transformation on a_{bi}. We prove it in four different cases:

1) If b and i are prime to each other, i.e. $(b, i) = 1$. Because $d|b$ and $(b, i) = 1$, we have $(d, i) = 1$. If $d \neq 1$, d will not divide i, so $a_{di} = 0$; else if $d = 1$ then $a_{di} = 1$. Therefore,

$$a_{bi}' = a_{bi} - 1 = (b, i) - 1 = 0.$$

2) If b and i are not prime to each other and $b > i$. Suppose $(b, i) = r$, then $r|b$ and $r|i$, and for each common divisor e of b and i, we have $e|r$. Because we have

$$a_{bi}' = a_{bi} - \sum_{d|b, d<b} a_{di}, \text{ and } a_{di} = \phi(d) \text{ or } 0,$$

which depends on whether or not d divides i, so only when d is a common divisor of b and i, i.e. $d|r$, $a_{di} = \phi(d)$, otherwise $a_{di} = 0$. Then we have

$$\sum_{d|b, d<b} a_{di} = \sum_{i|r} \phi(i) = r.$$

Therefore,

$$a_{bi}' = a_{bi} - \sum_{d|b, d<b} a_{di} = (b, i) - r = 0.$$

3) If b and i are not prime to each other and $b = i$, then

$$a_{bb}' = a_{bb} - \sum_{d|b, d<b} a_{db} = b - \sum_{d|b, d<b} \phi(d).$$

Because

$$b = \sum_{d|b} \phi(d) = \sum_{d|b, d<b} \phi(d) + \phi(b),$$

then

$$a_{bb}' = b - \sum_{d|b, d<b} \phi(d) = \phi(b).$$

4) If b and i are not prime to each other and $b < i$. According to the periodicity in each row, the elements behind the diagonal element are the same as the elements before it.

In summary, after the transformation on R_b, the result R_b' has the form

$$a_{bi}' = \begin{cases} \phi(b), & \text{if } b \text{ divides } i, \\ 0, & \text{otherwise,} \end{cases}$$

which means that R_b' has the same form as the rows above, so the conclusion is true.

Step 3: Based on the method, the original determinant can be transformed into

$$\begin{vmatrix} \phi(1) & \phi(1) & \phi(1) & \phi(1) & \phi(1) & \phi(1) & \cdots & 1 & \cdots & * \\ 0 & \phi(2) & 0 & \phi(2) & 0 & \phi(2) & \cdots & 0 & \cdots & * \\ 0 & 0 & \phi(3) & 0 & 0 & \phi(3) & \cdots & 0 & \cdots & * \\ 0 & 0 & 0 & \phi(4) & 0 & 0 & \cdots & 0 & \cdots & * \\ 0 & 0 & 0 & 0 & \phi(5) & 0 & \cdots & 0 & \cdots & * \\ 0 & 0 & 0 & 0 & 0 & \phi(6) & \cdots & 0 & \cdots & * \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & \phi(p) & \cdots & * \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & \cdots & \phi(n) \end{vmatrix}.$$

The value of this determinant is $\prod_{i=1}^{n} \phi(i)$. Therefore, the value of order n Smith arithmetic determinant is $\prod_{i=1}^{n} \phi(i)$, i.e. $S_n = \prod_{i=1}^{n} \phi(i)$.

III. COMPUTING THE SMITH ARITHMETIC DETERMINANT BY MATRIX DECOMPOSITION

Firstly we consider two examples of smaller order Smith arithmetic determinant.

Example 3. Consider the computation of the Smith arithmetic determinant of order 4. According to the definition, we have

$$S_4 = \begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 1 & 2 \\ 1 & 1 & 3 & 1 \\ 1 & 2 & 1 & 4 \end{vmatrix}.$$

The matrix corresponding to this determinant is

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 1 & 2 \\ 1 & 1 & 3 & 1 \\ 1 & 2 & 1 & 4 \end{pmatrix}.$$

We find that this matrix can be regarded as the multiplicative product of two matrices

$$\begin{pmatrix} \phi(1) & 0 & 0 & 0 \\ \phi(1) & \phi(2) & 0 & 0 \\ \phi(1) & 0 & \phi(3) & 0 \\ \phi(1) & \phi(2) & 0 & \phi(4) \end{pmatrix} \text{ and } \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Therefore,

$$\begin{aligned} S_4 &= \begin{vmatrix} \phi(1) & 0 & 0 & 0 \\ \phi(1) & \phi(2) & 0 & 0 \\ \phi(1) & 0 & \phi(3) & 0 \\ \phi(1) & \phi(2) & 0 & \phi(4) \end{vmatrix} \cdot \begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} \\ &= \prod_{i=1}^{4} \phi(i) \end{aligned}$$

Example 4. Consider the computation of the Smith arithmetic determinant of order 6. The matrix corresponding to the 6 order Smith arithmetic determinant is

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 1 & 2 & 1 & 2 \\ 1 & 1 & 3 & 1 & 1 & 3 \\ 1 & 2 & 1 & 4 & 1 & 2 \\ 1 & 1 & 1 & 1 & 5 & 1 \\ 1 & 2 & 3 & 2 & 1 & 6 \end{pmatrix}.$$

This matrix can be decomposed into the multiplicative product of

$$\begin{pmatrix} \phi(1) & 0 & 0 & 0 & 0 & 0 \\ \phi(1) & \phi(2) & 0 & 0 & 0 & 0 \\ \phi(1) & 0 & \phi(3) & 0 & 0 & 0 \\ \phi(1) & \phi(2) & 0 & \phi(4) & 0 & 0 \\ \phi(1) & 0 & 0 & 0 & \phi(5) & 0 \\ \phi(1) & \phi(2) & \phi(3) & 0 & 0 & \phi(6) \end{pmatrix}$$

and

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Therefore,

$$S_6 = \begin{vmatrix} \phi(1) & 0 & 0 & 0 & 0 & 0 \\ \phi(1) & \phi(2) & 0 & 0 & 0 & 0 \\ \phi(1) & 0 & \phi(3) & 0 & 0 & 0 \\ \phi(1) & \phi(2) & 0 & \phi(4) & 0 & 0 \\ \phi(1) & 0 & 0 & 0 & \phi(5) & 0 \\ \phi(1) & \phi(2) & \phi(3) & 0 & 0 & \phi(6) \end{vmatrix}$$

$$\cdot \begin{vmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{vmatrix} = \prod_{i=1}^{6} \phi(i).$$

The examples illustrated above reveal that a Smith arithmetic determinant can be decomposed into two determinants that are simple enough to be computed.

Now we present the decomposition of a general Smith arithmetic determinant.

We define two new matrices B and C of order n in the following:

$$B = \begin{pmatrix} \phi(1) & 0 & 0 & 0 & 0 & 0 & \cdots \\ \phi(1) & \phi(2) & 0 & 0 & 0 & 0 & \cdots \\ \phi(1) & 0 & \phi(3) & 0 & 0 & 0 & \cdots \\ \phi(1) & \phi(2) & 0 & \phi(4) & 0 & 0 & \cdots \\ \phi(1) & 0 & 0 & 0 & \phi(5) & 0 & \cdots \\ \phi(1) & \phi(2) & \phi(3) & 0 & 0 & \phi(6) & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

where each element b_{ik} of row R_i satisfies

$$b_{ik} = \begin{cases} \phi(k), & \text{if } k \text{ divides } i; \\ 0, & \text{otherwise.} \end{cases}$$

$$C = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & \cdots \\ 0 & 1 & 0 & 1 & 0 & 1 & \cdots \\ 0 & 0 & 1 & 0 & 0 & 1 & \cdots \\ 0 & 0 & 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 1 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

where each element c_{kj} of column C_j satisfies

$$c_{kj} = \begin{cases} 1, & \text{if } k \text{ divides } j; \\ 0, & \text{otherwise.} \end{cases}$$

Property 2. *Suppose the matrix corresponding to the Smith arithmetic determinant of n order is A, then $A = B \cdot C$.*

Proof. Suppose each element of A is a_{ij}, and each element of $B \cdot C$ is a'_{ij}. What is needed to be proved is $a_{ij} = a'_{ij}$.

Because $a_{ij} = \sum_{k=1}^{n} b_{ik}c_{kj}$ and according to the definition of b_{ik} and c_{kj}, we can deduce that

$$\sum_{k=1}^{n} b_{ik}c_{kj} = \sum_{k|(i,j)} \phi(k) = (i,j) = a_{ij},$$

which means that $a_{ij} = a'_{ij}$, i.e. $A = B \cdot C$.

Note: For each element of the expression $\sum_{k=1}^{n} b_{ik}c_{kj}$, only when $b_{ik} \neq 0$ and $c_{kj} \neq 0$, this element does not equal to 0, and when $b_{ik} \neq 0$ and $c_{kj} \neq 0$, we have $k|i$ and $k|j$, so k is a common divisor of i and j. Moreover, the value of k ranges from 1 to n, so the value of expression $b_{ik}c_{kj}$ also ranges over every positive divisors of (i,j). Therefore,

$$\sum_{k=1}^{n} b_{ik}c_{kj} = \sum_{k|(i,j)} \phi(k) = (i,j) = a_{ij}.$$

As a consequence of Property 2, we have

$$S_n = \det(B) \cdot \det(C) = \prod_{i=1}^{n} \phi(i).$$

So

$$S_n = \begin{vmatrix} \phi(1) & 0 & 0 & 0 & 0 & 0 & \cdots \\ \phi(1) & \phi(2) & 0 & 0 & 0 & 0 & \cdots \\ \phi(1) & 0 & \phi(3) & 0 & 0 & 0 & \cdots \\ \phi(1) & \phi(2) & 0 & \phi(4) & 0 & 0 & \cdots \\ \phi(1) & 0 & 0 & 0 & \phi(5) & 0 & \cdots \\ \phi(1) & \phi(2) & \phi(3) & 0 & 0 & \phi(6) & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{vmatrix}$$

$$\cdot \begin{vmatrix} 1 & 1 & 1 & 1 & 1 & 1 & \cdots \\ 0 & 1 & 0 & 1 & 0 & 1 & \cdots \\ 0 & 0 & 1 & 0 & 0 & 1 & \cdots \\ 0 & 0 & 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 1 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{vmatrix} = \prod_{i=1}^{n} \phi(i).$$

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