分圆多项式

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In mathematics the *n*th **cyclotomic polynomial**, for any positive integer *n*, is the unique irreducible polynomial with integer coefficients that is a divisor of $x^n - 1$ and is not a divisor of $x^k - 1$ for any $k < n$. Its roots are all *n*th primitive roots of unity $e^{2i\pi\frac{k}{n}}$, where *k* runs over the positive integers not greater than *n* and coprime to *n*. In other words, the ***n*th cyclotomic polynomial** is equal to

$$\Phi_n(x) = \prod_{\substack{1\le k\le n \\ \gcd(k,n)=1}} \left(x - e^{2i\pi\frac{k}{n}}\right).$$

It may also be defined as the monic polynomial with integer coefficients that is the minimal polynomial over the field of the rational numbers of any primitive *n*th-root of unity ($e^{2i\pi/n}$ is an example of such a root).

An important relation linking cyclotomic polynomials and primitive roots of unity is

$$\prod_{d|n} \Phi_d(x) = x^n - 1,$$

showing that x is a root of $x^n - 1$ if and only if it is a *d*th primitive root of unity for some *d* that divides *n*.

$$\begin{aligned}
\Phi_1(x) &= x - 1 \\
\Phi_2(x) &= x + 1 \\
\Phi_3(x) &= x^2 + x + 1 \\
\Phi_4(x) &= x^2 + 1 \\
\Phi_5(x) &= x^4 + x^3 + x^2 + x + 1 \\
\Phi_6(x) &= x^2 - x + 1 \\
\Phi_7(x) &= x^6 + x^5 + x^4 + x^3 + x^2 + x + 1 \\
\Phi_8(x) &= x^4 + 1 \\
\Phi_9(x) &= x^6 + x^3 + 1 \\
\Phi_{10}(x) &= x^4 - x^3 + x^2 - x + 1
\end{aligned}$$

分圆多项式的基本性质

$$x^n - 1 = \prod_{1\leqslant k\leqslant n} \left(x - e^{2i\pi\frac{k}{n}}\right) = \prod_{d|n} \prod_{\substack{1\leqslant k\leqslant n \\ \gcd(k,n)=d}} \left(x - e^{2i\pi\frac{k}{n}}\right) = \prod_{d|n} \Phi_{\frac{n}{d}}(x) = \prod_{d|n} \Phi_d(x).$$

$$\Phi_n(x) = \prod_{d|n} (x^d - 1)^{\mu\left(\frac{n}{d}\right)}$$

$$\Phi_n(x) = \prod_{k=1}^{n} \left(x^{\gcd(k,n)} - 1\right)^{\cos\left(\frac{2\pi k}{n}\right)}$$

$$\Phi_n(x) = \frac{x^n - 1}{\prod_{\substack{d|n \\ d<n}} \Phi_d(x)}$$

n为质数：

$$\Phi_n(x) = 1 + x + x^2 + \cdots + x^{n-1} = \sum_{i=0}^{n-1} x^i.$$

n为奇数，n>1：

$$\Phi_{2n}(x)=\Phi_n(-x).$$

n=2p为奇质数两倍：

$$\Phi_n(x)=1-x+x^2-\cdots+x^{p-1}=\sum_{i=0}^{p-1}(-x)^i.$$

If $n=p^m$ is a prime power (where p is prime), then

$$\Phi_n(x)=\Phi_p(x^{p^{m-1}})=\sum_{i=0}^{p-1}x^{ip^{m-1}}.$$

More generally, if $n=p^m r$ with r relatively prime to p, then

$$\Phi_n(x)=\Phi_{pr}(x^{p^{m-1}}).$$

比较一般的结论，q为n的质因子的乘积(***radical***)：

$$\Phi_n(x)=\Phi_q(x^{n/q}).$$

p为质数，且不整除n：

$$\Phi_{np}(x)=\Phi_n(x^p)/\Phi_n(x).$$

一些特殊情形：

$$\Phi_{2^h}(x)=x^{2^{h-1}}+1$$
$$\Phi_{p^k}(x)=\sum_{i=0}^{p-1}x^{ip^{k-1}}$$
$$\Phi_{2^h p^k}(x)=\sum_{i=0}^{p-1}(-1)^i x^{i2^{h-1}p^{k-1}}$$

其系数性质

1)如果n含有***最多两个不同奇质因子***，其系数绝对值不超过1

2)是本原多项式，其系数为整数，且所有系数gcd为1

3)在整数环Z内不可约

4)n>1的分圆多项式系数是对称的，即：

$$\Phi_n(x)=x^{\phi(n)}\Phi_n\left(\frac{1}{x}\right)$$

5)n>1，其一次项系数为-mu(n)

特殊值性质

$$\Phi_n(1)=\begin{cases} p & \exists p\in\mathbb{P},\alpha\in\mathbb{N}^*,\ \text{s.t. } n=p^\alpha \\ 1 & \text{otherwise}\end{cases}$$