FLUSSI

27-5-2021

ESERCIZIO 1. Calcolare il flusso del campo

$$\underline{F}(x;y;z) = xy\,\underline{i} + xy\,\underline{j} + z\,\underline{k}$$

attraverso la superficie

$$\Sigma = \{ (x;y;z) \in \mathbb{R}^3 \,/\, z = \overbrace{1-x^2-y^2}^{= f(x;y)} ,\ z \geq 0 \}$$

scegliendo il versore $\underline{n}$ con la terza componente positiva.

SOL.

$$\phi_\Sigma(\underline{F}) = \iint_\Sigma \underline{F} \cdot \underline{n}\, d\sigma = \iint_T F \cdot (\underline{r}_u \times \underline{r}_v)\, du\, dv$$

$$d\sigma = \| \underline{r}_u \times \underline{r}_v \|\, du\, dv \qquad \underline{n} = \frac{\underline{r}_u \times \underline{r}_v}{\| \underline{r}_u \times \underline{r}_v \|}$$

$$\underline{n}\, d\sigma = \frac{\underline{r}_u \times \underline{r}_v}{\cancel{\| \underline{r}_u \times \underline{r}_v \|}} \cancel{\| \underline{r}_u \times \underline{r}_v \|}\, du\, dv$$

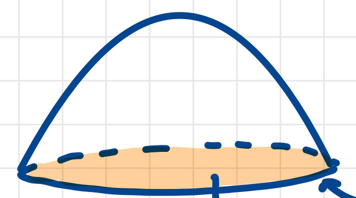
$$\underline{r}(x;y) = (x; y; f(x;y)) = (x; y; 1-x^2-y^2)$$

$$\underline{r}_x(x;y) = (1; 0; f_x(x;y)) = (1; 0; -2x)$$

$$\underline{r}_y(x;y) = (0; 1; f_y(x;y)) = (0; 1; -2y)$$

$$\underline{r}_x \times \underline{r}_y = 2x\, \underline{i} + 2y\, \underline{j} + \underline{k} = (2x; 2y; 1)$$

$$\partial T \begin{cases} z = 1-x^2-y^2 \\ z = 0 \end{cases} \longrightarrow \begin{array}{l} 1-x^2-y^2 = 0 \\ x^2+y^2 = 1 \end{array}$$



$$T = \{ (x,y) \in \mathbb{R}^2 / \ x^2+y^2 \le 1 \}$$

$$\iint_\Sigma \underline{F} \cdot \underline{n}\, d\sigma = \iint_T \begin{bmatrix} xy \\ xy \\ 1-x^2-y^2 \end{bmatrix} \cdot \begin{bmatrix} 2x \\ 2y \\ 1 \end{bmatrix} dx\, dy =$$

$$\iint_T \big(\underbrace{\cancel{2x^2y + 2xy^2}}_{=g(x;y)} + 1 - x^2 - y^2\big)\,dx\,dy =$$

$$\left.\begin{array}{l} T \text{ simmetrico risp. a } 0 \\ g \quad '' \qquad '' \quad '' \quad '' \end{array}\right\} \Rightarrow \iint_T g = 0$$

$$= \iint_T \big(1 - (x^2+y^2)\big)\,dx\,dy = \int_0^{2\pi} \left(\int_0^1 (1-\rho^2)\rho\,d\rho \right) d\theta =$$

$$= 2\pi \left[\frac{\rho^2}{2} - \frac{\rho^4}{4} \right]_0^1 = 2\pi \left(\frac{1}{2} - \frac{1}{4} \right) = \frac{2\pi}{4} = \frac{\pi}{2}.$$

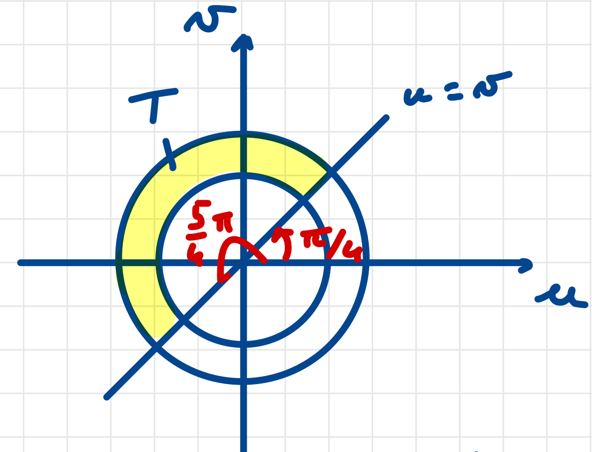
ESERCIZIO 2.

Calcolare il flusso del campo $F(x;y;z) = \underline{i} + \underline{k}$ attraverso la superficie Σ di eq.

$$\underline{r}(u,v) = u^2\,\underline{i} + \sqrt{2}\,uv\,\underline{j} + v^2\,\underline{k}$$

con $(u,v) \in T = \{ (u,v) \in \mathbb{R}^2 / \ 1 \le u^2+v^2 \le 2, u<v$

SOL.



$$\underline{r}(u,v) = \left(u^2, \ \sqrt{2}uv, \ v^2 \right)$$

$$\underline{r}_u(u,v) = \left(2u, \ \sqrt{2}v, \ 0 \right)$$

$$\underline{r}_v(u,v) = \left(0; \ \sqrt{2}u, \ 2v \right)$$

$$\underline{r}_u \times \underline{r}_v = \left(2\sqrt{2}v^2; -4uv; \ 2\sqrt{2}u^2 \right) \qquad \underline{F} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$$\phi_\Sigma(\vec{F}) = \iint_\Sigma \underline{F} \cdot \underline{n} \, d\sigma = \iint_T \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 2\sqrt{2}v^2 \\ -4uv \\ 2\sqrt{2}u^2 \end{bmatrix} du\,dv =$$

$$= \iint_T \left(2\sqrt{2}v^2 + 2\sqrt{2}u^2 \right) du\,dv = 2\sqrt{2} \int_{\pi/4}^{5/4\pi} \left(\int_1^{\sqrt{2}} \rho^3 \, d\rho \right) d\theta$$

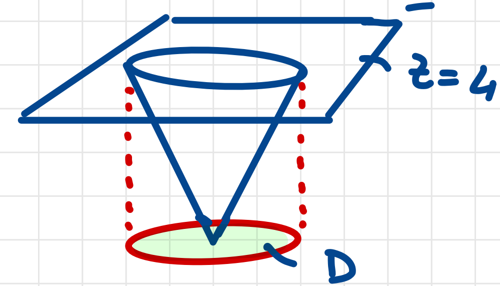
$$= 2\sqrt{2}\cdot\pi\left[\frac{\rho^4}{4}\right]_1^{\sqrt{2}} = 2\sqrt{2}\pi\left(1-\frac{1}{4}\right) = \frac{3\sqrt{2}\pi}{2}.$$

ESERCIZIO 3.

Sia C la regione di spazio tra le superf. conica $z = 2\sqrt{x^2+y^2}$ e il piano $z = 4$. Si supponga che C sia occupata da un solido di densità $\rho(x;y;z) = 4 - z$.

1) Calcolare la massa del solido

2) Calcolare il flusso di $\underline{F}(x;y;z) = x\underline{i} + y\underline{j} + z\underline{k}$ uscente da ∂C.



SOL.

1) $\partial D : \begin{cases} z = 2\sqrt{x^2+y^2} \\ z = 4 \end{cases}$

$$2\cancel{k} = \cancel{z}\sqrt{x^2+y^2} \longrightarrow x^2+y^2=4$$

$$D = \{(x;y) \in \mathbb{R}^2 \,/\, x^2+y^2 \leq 4\}.$$

$$M = \iiint_C \rho(x;y;z)\,dx\,dy\,dz = \iint_D \left(\int_{2\sqrt{x^2+y^2}}^{4} (4-z)\,dz \right) dx\,dy =$$

$$= \iint_D \left[4z - \frac{z^2}{2} \right]_{2\sqrt{x^2+y^2}}^{4} dx\,dy =$$

$$= \iint_D \left(16 - 8 - 8\sqrt{x^2+y^2} + \frac{1}{2}\cdot 4(x^2+y^2)\right) dx\,dy =$$

$$= \iint_D \left(8 - 8\sqrt{x^2+y^2} + 2(x^2+y^2)\right) dx\,dy =$$

$$= \int_0^{2\pi} \left(\int_0^2 (8 - 8\rho + 2\rho^2)\rho\,d\rho \right) d\theta =$$

$$= 2\pi \left[\frac{4\cancel{8}\,\rho^2}{\cancel{2}} - 8\frac{\rho^3}{3} + \cancel{2}\frac{\rho^4}{\cancel{4}\,2} \right]_0^2 = \frac{16}{3}\pi .$$

2)

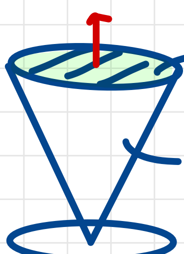
$$\oint_{\partial C} (\vec{F}) = \iiint_C \operatorname{div} F \, dx\,dy\,dz = \iiint_C 3 \, dx\,dy\,dz =$$

SUP. CHIUSA.

$$\operatorname{div} F = \partial_x F_1 + \partial_y F_2 + \partial_z F_3 = 3$$

$$= 3 \underbrace{\mathrm{Vol}(C)}_{\text{Vol. cono}} = 3 \cdot \frac{4\pi \cdot 4}{3} = 16\pi .$$

In modo alternativo:



$(\partial C)_1 : z = 4$

$(\partial C)_2 : z = 2\sqrt{x^2+y^2}$

$\Big\} \ (x,y) \in D$

$$\underset{(\partial C)_1}{\phi}(\vec{F}) = \iint_{(\partial C)_1} \underline{F} \cdot \underline{n}\, d\sigma = \iint_D \begin{bmatrix} x \\ y \\ 4 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} dx\,dy = \iint_D 4\, dx\,dy$$

$$= 4 \operatorname{Area}(D) = 4 \cdot 4\pi = 16\pi .$$

$$\underset{(\partial C)_2}{\phi}(F) = \iint_{(\partial C)_2} F \cdot n\, d\sigma = \iint_D \begin{bmatrix} x \\ y \\ 2\sqrt{x^2+y^2} \end{bmatrix} \cdot \begin{bmatrix} f_x \\ f_y \\ -1 \end{bmatrix} dx\,dy$$

$$f_x = \frac{2x}{\sqrt{x^2+y^2}} \qquad f_y = \frac{2y}{\sqrt{x^2+y^2}}$$

$$= \iint_D \left(\frac{2x^2}{\sqrt{x^2+y^2}} + \frac{2y^2}{\sqrt{x^2+y^2}} - 2\sqrt{x^2+y^2} \right) dx\,dy =$$

$$= \iint_D \frac{2x^2+2y^2-2x^2-2y^2}{\sqrt{x^2+y^2}}\,dx\,dy = 0.$$

$$\phi_{TOT} = \phi_1 + \phi_2 = 16\pi + 0 = 16\pi$$

come verificato dal teorema della div.

ESERCIZIO 4. Sia B la porzione di sfera di centro $(0;0;0)$ e raggio R contenuta nel primo ottante.

$$B = \{(x;y;z)\in\mathbb{R}^3 / x^2+y^2+z^2 \leq R^2,\ x \geq 0,\ y \geq 0,\ z \geq 0\}$$

Dato il campo $\underline{F}(x;y;z) = xy\,\underline{i} + xz\,\underline{j} + yz\,\underline{k}$

1) Calcolare $\operatorname{div}\underline{F}$, $\operatorname{rot}\underline{F}$, $\operatorname{div}(\operatorname{rot}\underline{F})$.

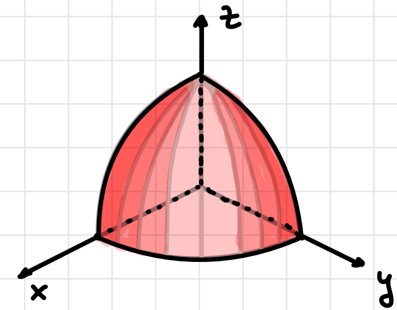
2) Si calcoli il flusso di $\underline{F}$ uscente dalla superficie che delimita B. Quanto vale il flusso attraverso ogni faccia?

SOL.

1) $$\text{rot}\,\underline{F} = \begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ xy & xz & yz \end{vmatrix} = \underline{i}(z-x) - \underline{j}(0) + \underline{k}(z-x)$$

$$\text{div}\,\underline{F} = \partial_x(xy) + \partial_y(xz) + \partial_z(yz) = y + y = 2y$$

$$\text{div}(\text{rot}\,\underline{F}) = 0$$



2)

$$\phi_{\partial B}(\underline{F}) = \iiint_B \text{div}\,F\,dx\,dy\,dz =$$

$$= \iiint_B 2y \, dx\,dy\,dz = \quad (\text{COORD. SFERICHE})$$

$$= 2 \int_0^{\pi/2} \left(\int_0^{\pi/2} \left(\int_0^R \rho \operatorname{sen}\varphi \operatorname{sen}\theta \, \rho^2 \operatorname{sen}\varphi \, d\rho \right) d\varphi \right) d\theta$$

$$= 2 \int_0^{\pi/2} \operatorname{sen}\theta \, d\theta \cdot \int_0^{\pi/2} \operatorname{sen}^2\varphi \, d\varphi \cdot \int_0^R \rho^3 \, d\rho =$$

$$= 2 \left[-\cos\theta \right]_0^{\pi/2} \cdot \left[\frac{1}{2} \left(\varphi - \operatorname{sen}\varphi \cos\varphi \right) \right]_0^{\pi/2} \cdot \left[\frac{\rho^4}{4} \right]_0^R =$$

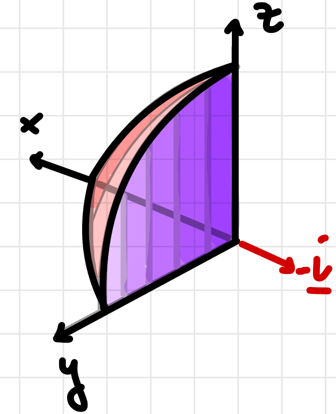
$$= \cancel{2} (1) \cdot \left(\frac{1}{\cancel{2}} \cdot \frac{\pi}{2} \right) \cdot \frac{R^4}{4} = \frac{\pi}{8} R^4 .$$

FLUSSO ATTRAVERSO LE TRE FACCE:

$$\underline{F}(x;y;z) = (xy\,;\ xz\,;\ yz)$$

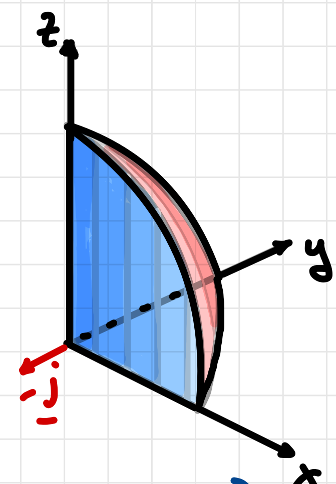
$$\boxed{\partial B \cap \{x=0\}} = S_1$$

$$F\big|_{S_1} = (0; 0; yz) \qquad \underline{n} = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}$$



$$F\big|_{S_1} \cdot n = 0 \quad \Longrightarrow \quad \phi_1 = 0.$$

$$\boxed{\partial B \cap \{y=0\}} = S_2$$



$$F\big|_{S_2} = (0;\ xz;\ 0) \qquad \underline{n} = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}$$

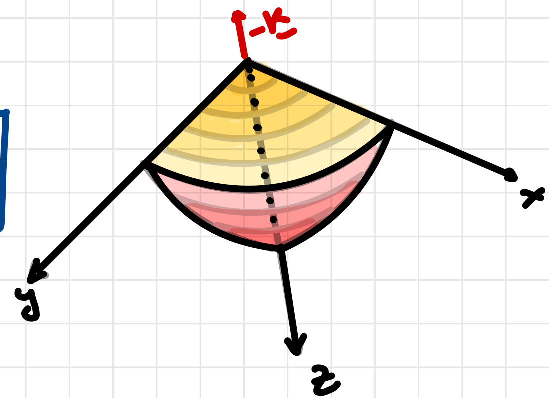
$$\phi_2 = \iint_{S_2} F \cdot n \, d\sigma = \iint_{\{x^2+z^2 \le R^2,\ x \ge 0,\ z \ge 0\}} \begin{bmatrix} 0 \\ xz \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} dx\,dz =$$

$$= \iint -xz \, dx\,dz = \int_0^{\pi/2} \left(\int_0^R -\rho^3 \operatorname{sen}\theta \cos\theta \, d\rho \right) d\theta =$$

$$= \int_0^{\pi/2} \operatorname{sen}\theta \cos\theta \, d\theta \cdot \int_0^R -\rho^3 \, d\rho = \left[\frac{\operatorname{sen}^2\theta}{2} \right]_0^{\pi/2} \left[-\frac{\rho^4}{4} \right]_0^R$$

$$= \frac{1}{2} \left(-\frac{R^4}{4} \right) = -\frac{R^4}{8}$$

$\partial B \cap \{z = 0\} = S_3$



$$F\Big|_{S_3} = (xy; 0; 0) \qquad \underline{n} = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}$$

$$\underline{F} \cdot \underline{n} = 0 \implies \phi_3 = 0$$

$$\phi_4 = \phi_{TOT} - \cancel{\phi_1} - \phi_2 - \cancel{\phi_3} = \frac{\pi R^4}{8} - \left(-\frac{R^4}{8} \right) = \frac{R^4}{8}(\pi + 1).$$

TEOREMA DI STOKES E CALCOLO DELLA CIRCUITAZIONE

ESERCIZIO 5. Si consideri la superficie

$$\Sigma = \{(x;y;z) \in \mathbb{R}^3 / z = x^2+y^2,\ x^2+y^2 \leq 4\}$$

orientata con un vettore normale avente k positivo.

1) Dato $\underline{F}(x;y;z) = (y-2z)\underline{i} + y\,\underline{j} + (2x+e^{z^2})\underline{k}$ calcolare rot $\underline{F}$ e stabilire se è conservativo.

2) Calcolare il lavoro di F lungo $\partial\Sigma$ con orientazione indotta da Σ.

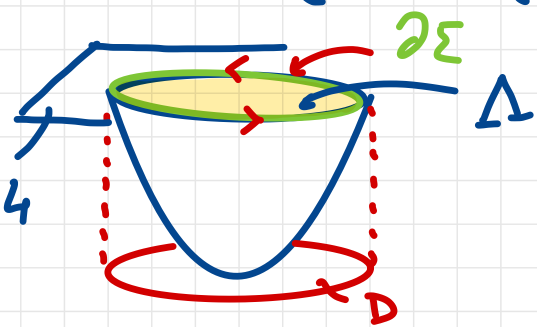
SOL.

1) $$\text{rot}\,\underline{F} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \partial_x & \partial_y & \partial_z \\ y-2z & y & 2x+e^{z^2} \end{vmatrix} =$$

$$= \underline{i}\,(0) - \underline{j}\,(4) + \underline{k}\,(-1) = \begin{bmatrix} 0 \\ -4 \\ -1 \end{bmatrix} \neq \underline{0} \Longrightarrow F$$

non è conservativo.

2)



• Calcolo della circuitazione in modo più rapido: flusso del rotore attraverso $z=4$ con $x^2+y^2 \leq 4$.

$$\oint_{\partial^+\Sigma} \underline{F} \cdot d\underline{r} = \iint_{\Lambda} \operatorname{rot} \underline{F} \cdot \underline{n}\, d\sigma = \iint_{D} \begin{bmatrix} 0 \\ -4 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} dx\, dy$$

$$= \iint_D -1 \, dx\,dy = -\text{Area}(D) = -4\pi .$$

OPPURE

- Calcolo il flusso del rotore attraverso Σ:

$$\oint_{\partial^+\Sigma} \underline{F} \cdot d\underline{r} = \iint_\Sigma \text{rot}\,\underline{F} \cdot \underline{n} \, d\sigma = \iint_D \begin{bmatrix} 0 \\ -4 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} -2x \\ -2y \\ 1 \end{bmatrix} dx\,dy$$

$$z = x^2 + y^2 \qquad \begin{bmatrix} -f_x \\ -f_y \\ 1 \end{bmatrix}$$

$$= \iint_D (8y - 1) \, dx\,dy = \cancel{8 \iint_D y \, dx\,dy} - \iint_D dx\,dy =$$

$$\underset{0}{\parallel}$$

$$= -A(D) = -4\pi .$$

OPPURE

più meccanicisticamente parametrizzo $\partial^+\Sigma$:

$$\partial^+\Sigma : \begin{cases} x = 2\cos t \\ y = 2\sin t \\ z = 4 \end{cases} \quad t \in [0; 2\pi] \quad (\partial^+\Sigma)' : \begin{cases} x' = -2\sin t \\ y' = 2\cos t \\ z' = 0 \end{cases}$$

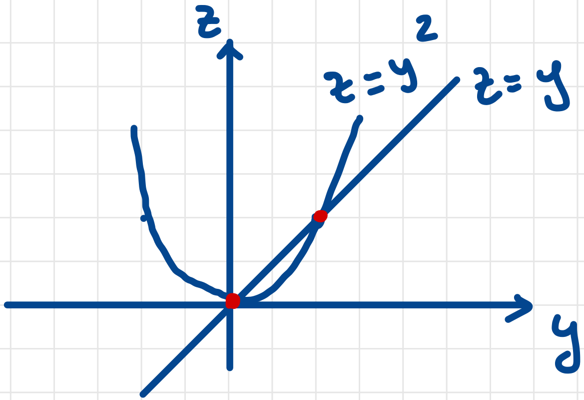
$$\oint_{\partial^+\Sigma} \underline{F} \cdot d\underline{r} = \int_0^{2\pi} \begin{bmatrix} 2\sin t - 8 \\ 2\sin t \\ 4\cos t + e^{16} \end{bmatrix} \cdot \begin{bmatrix} -2\sin t \\ 2\cos t \\ 0 \end{bmatrix} dt =$$

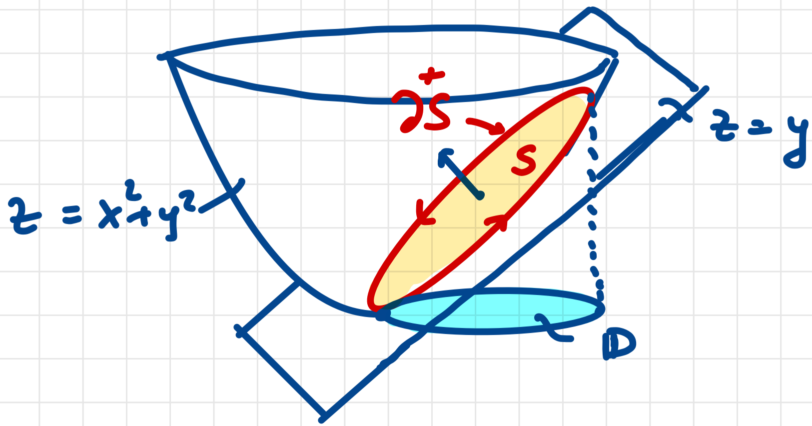
$$= \int_0^{2\pi} \left(-4\sin^2 t + 16 \cancel{\sin t} + 4 \sin t \cos t \right) dt =$$

$$= -\cancel{4}^{\,2} \left[\frac{t - \sin t \cos t}{\cancel{2}} \right]_0^{2\pi} + 4 \cancel{\left[\frac{\sin^2 t}{2} \right]_0^{2\pi}} = -4\pi .$$

ESERCIZIO 6. Utilizzare il teorema di Stokes per calcolare la circuitazione del campo $\underline{F}(x,y,z) = -z\,\underline{i} + 4x\,\underline{j} + y\,\underline{k}$ lungo la curva intersezione tra $z = x^2+y^2$ e il piano $z=y$ specificando l'orientamento scelto.

SOL.





$\partial D : \begin{cases} z = x^2+y^2 \\ z = y \end{cases}$ $\quad x^2+y^2-y=0 \longrightarrow$ circonf. di centro $(0;\frac{1}{2})$ e raggio $\frac{1}{2}$

$$D = \{(x;y) \in \mathbb{R}^2 / \ x^2+y^2-y \leq 0\}$$

$$\text{rot}\,\underline{F} = \begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ -z & 4x & y \end{vmatrix} = \underline{i}(1) - \underline{j}(1) + \underline{k}(4) = \begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix}$$

$$\oint_{\partial^+ S} \underline{F} \cdot \underline{dr} = \iint_S \text{rot}\,\underline{F} \cdot n\, d\sigma = \iint_D \begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} dx\,dy =$$

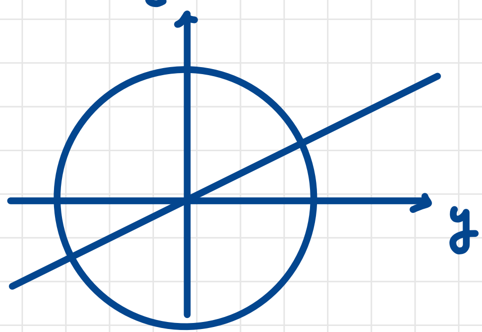
$$z = f(x;y) = y \qquad \begin{bmatrix} -f_x \\ -f_y \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

$$= \iint_D 5\, dx\,dy = 5\,\text{Area}(D) = 5 \cdot \pi \cdot \frac{1}{4} = \frac{5}{4}\pi.$$

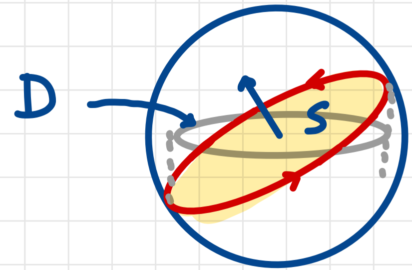
ESERCIZIO 7. Sia $\underline{F}(x;y;z) = (z-2y)\underline{i} + (z-2x)\underline{j} + (x+3y+y^2)\underline{k}$. Calcolare la circuitazione di $\underline{F}$

lungo la curva γ ottenuta intersecando la sfera $x^2+y^2+z^2=1$ con il piano $y=2z$ sia con la definizione che con il T. d. S.

SOL.



$z = \frac{1}{2} y$



$$\partial D : \begin{cases} x^2+y^2+z^2=1 \\ y=2z \end{cases} \longrightarrow x^2+5z^2=1 \rightarrow \text{ELLISSE nel piano } x\text{-}z.$$

$$\frac{x^2}{1^2}+\frac{z^2}{(1/\sqrt{5})^2}=1$$

$$\partial^+ S : \begin{cases} x = 1\cdot\cos t \\ y = \frac{2}{\sqrt{5}}\,\text{sen}\, t \\ z = \frac{1}{\sqrt{5}}\,\text{sen}\, t \end{cases} \qquad t \in [0; 2\pi]$$

$$\oint_{\partial^+ S} \underline{F} \cdot d\underline{r} = \int_0^{2\pi} \begin{bmatrix} \frac{1}{\sqrt5}\,\text{sen}\,t - \frac{4}{5}\,\text{sen}\,t \\ \frac{1}{\sqrt5}\,\text{sen}\,t - 2\cos t \\ \cos t + \frac{6}{\sqrt5}\,\text{sen}\,t + \frac{4}{5}\,\text{sen}^2 t \end{bmatrix} \cdot \begin{bmatrix} -\text{sen}\,t \\ \frac{2}{\sqrt5}\cos t \\ \frac{1}{\sqrt5}\cos t \end{bmatrix} dt$$

$$= \dots = \int_0^{2\pi} \left(\frac{3}{\sqrt5}\,\text{sen}^2 t - \frac{3}{\sqrt5}\cos^2 t + \cancel{\frac{8}{5}\,\text{sen}\,t\cos t} + \cancel{\frac{4}{5\sqrt5}\,\text{sen}^2 t\cos t} \right) dt$$

$\Downarrow$ int. nullo

$$= -\frac{3}{\sqrt5}\int_0^{2\pi} \cos 2t \, dt = 0 .$$

Con Stokes:

$$\text{rot}\, F = \begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ z-2y & z-2x & x+3y+y^2 \end{vmatrix} = \underline{i}\,(2+2y) + 0\,\underline{j} + 0\,\underline{k}$$

$$z = f(x;y) = \frac{y}{2} \qquad \begin{bmatrix} 0 \\ -1/2 \\ 1 \end{bmatrix}$$

$$\oint_{\partial^+ S} F \cdot dr = \iint_S \begin{bmatrix} 2+2y \\ 0 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ -1/2 \\ 1 \end{bmatrix} d\sigma = 0 .$$