ESERCIZIO 1 (CURVE PLANARI) 11-3-2021

Sia data la famiglia di curve $\Gamma_a \subseteq \mathbb{R}^3$ parametrizzata da

$$\underline{r} : \mathbb{R} \longrightarrow \Gamma_a$$

$$\underline{r}(t) = (t^2 ; t+1 ; at^3 + t^2 + 2t + 1)$$

Per quali valori di $a \in \mathbb{R}$, Γ_a è piana?
Per tali valori di a scrivere l'eq. del piano che la contiene.

SOL. (2 MODI $\longrightarrow$ 1) $\vec{B}$ costante e trovo π_{osc})
$\longrightarrow$ 2) Trovare subito il piano

MODO 1

$$\underline{r}(t) = (t^2 ; t+1 ; at^3 + t^2 + 2t + 1)$$

$$\underline{r}'(t) = (2t ; 1 ; 3at^2 + 2t + 2)$$

$$\underline{r}''(t) = (2;\ 0;\ 6at+2)$$

$$\underline{r}'(t) \times \underline{r}''(t) = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 2t & 1 & 3at^2+2t+2 \\ 2 & 0 & 6at+2 \end{vmatrix} =$$

$$= \underline{i}(6at+2) - \underline{j}(12at^2 + \cancel{4t} - 6at^2 - \cancel{4t} - 4) + \underline{k}(-2) =$$

$$= \underline{i}(6at+2) - \underline{j}(6at^2-4) - 2\underline{k}$$

Se $a = 0$:

$$\underline{r}'(t) \times \underline{r}''(t) = 2\underline{i} + 4\underline{j} - 2\underline{k} = (2;\ 4;\ -2)$$ COSTANTE

$\Longrightarrow \Gamma_0$ è una curva PLANARE

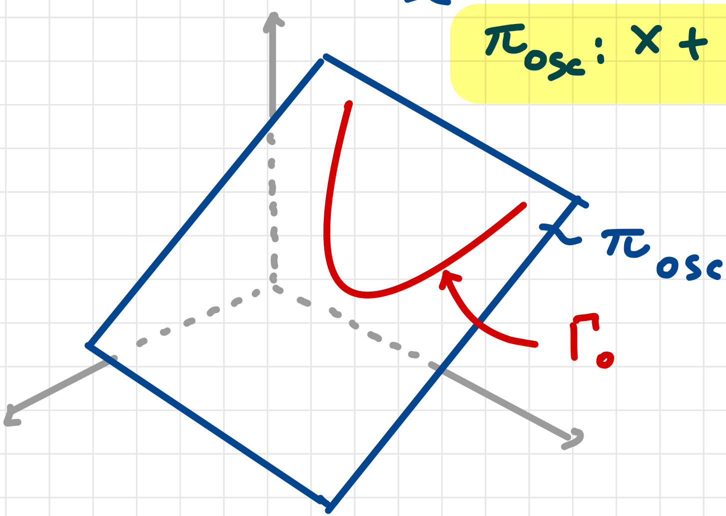
Γ_0 : $\underline{r}(t) = (t^2;\ t+1;\ t^2+2t+1)$

Il piano che la contiene è il piano $\perp$ a $\underline{r}'(t) \times \underline{r}''(t)$ e passante per un punto $P \in \Gamma_0$, per es. $P(0;1;1) = \underline{r}(0)$

$\underline{r}' \times \underline{r}'' = (2;4;-2)$ $\qquad$ $P(0;1;1)$

$$\pi_{osc}: \cancel{2}(x-0) + \cancel{4}^{\,2}(y-1) - \cancel{2}(z-1) = 0$$

$$\boxed{\pi_{osc}: x + 2y - z = 1}$$



MODO 2.

$$\underline{r}(t) = \left(t^2 ;\ t+1 ;\ at^3 + t^2 + 2t + 1 \right)$$

$$\pi : \ \alpha x + \beta y + \gamma z = \delta$$

Voglio che $\underline{r}(t) \subset \pi$:

$$\alpha t^2 + \beta t + \beta + \gamma a t^3 + \gamma t^2 + 2\gamma t + \gamma = \delta$$

$$\gamma a t^3 + (\alpha+\gamma) t^2 + (\beta + 2\gamma) t + \beta + \gamma - \delta = 0 t^3 + 0 t^2 + 0 t + 0$$

$$\begin{cases} \gamma a = 0 \\ \alpha + \gamma = 0 \\ \beta + 2\gamma = 0 \\ \beta + \gamma - \delta = 0 \end{cases} \qquad \gamma a = 0 \ \begin{cases} \gamma = 0 \to \text{N.A.} \\ a = 0 \end{cases} \qquad \longrightarrow \qquad \begin{cases} a = 0 \\ \alpha = -\gamma \\ \beta = -2\gamma \\ \delta = -2\gamma + \gamma = -\gamma \end{cases}$$

per es. $\gamma = -1 \implies \begin{cases} a = 0 \\ \alpha = 1 \\ \beta = 2 \\ \gamma = -1 \\ \delta = 1 \end{cases}$

$$\pi : \ x + 2y - z = 1$$

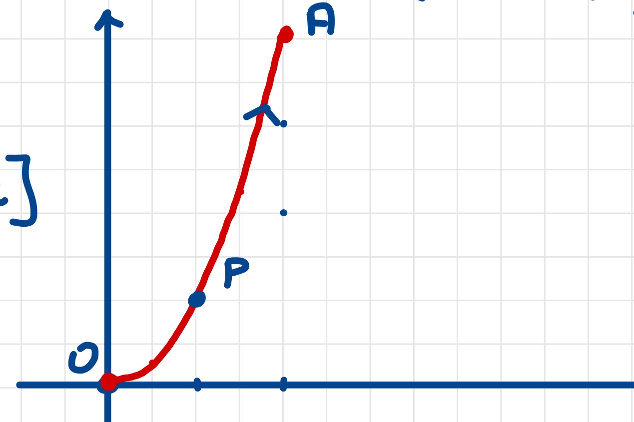
ESERCIZIO 2.

Sia data la parabola $y = x^2$. Scrivere una sua parametrizzazione da $O(0;0)$ ad $A(2;4)$ e:

1) scrivere i versori $\vec{T}$ e $\vec{N}$ in $(1;1)$

2) scrivere l'eq. del cerchio osculatore in $(1;1)$

SOL.



$$y = x^2 \Rightarrow \underline{r}(t): \begin{cases} x = t \\ y = t^2 \end{cases} \quad t \in [0;2]$$

1) $P(1;1) \qquad P = \underline{r}(1)$

$$\underline{r}'(t) = (1;\ 2t) \qquad |\underline{r}'(t)| = \sqrt{1+4t^2}$$

$$\vec{T}(1) = \frac{(1;2)}{\sqrt{5}} = \left(\frac{1}{\sqrt{5}};\frac{2}{\sqrt{5}}\right)$$

$$\vec{N}(t) = \frac{\vec{T}'(t)}{|\vec{T}'(t)|} \qquad \text{con} \qquad \vec{T}(t) = \frac{(1;2t)}{\sqrt{1+4t^2}} =$$

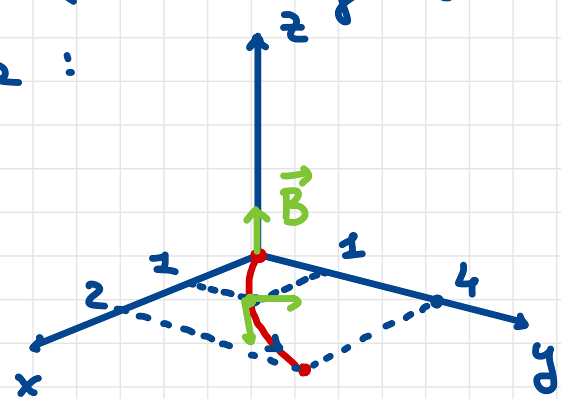
$$= \left(\frac{1}{\sqrt{1+4t^2}};\frac{2t}{\sqrt{1+4t^2}}\right)$$

I calcoli potrebbero essere lunghi e quindi cambio strategia:

$$\vec{T}(1) = \left(\frac{1}{\sqrt{5}};\frac{2}{\sqrt{5}};0\right)$$

$$\underline{r}(t) = (t;t^2;0)$$

$$\underline{r}'(t) = (1;2t;0)$$

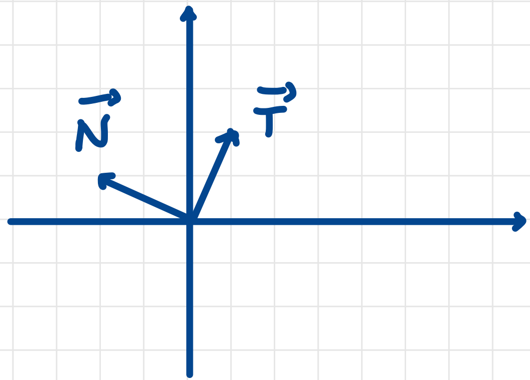


$$\underline{r}''(t) = (0;\ 2;\ 0)$$

$$r'(t) \times r''(t) = \begin{vmatrix} i & j & k \\ 1 & 2t & 0 \\ 0 & 2 & 0 \end{vmatrix} = 2\underline{k} = (0;0;2)$$

$$\vec{B}(t) = (0;0;1)$$

$$\vec{N}(1) = \vec{B}(1) \times \vec{T}(1) = \begin{vmatrix} i & j & k \\ 0 & 0 & 1 \\ 1/\sqrt{5} & 2/\sqrt{5} & 0 \end{vmatrix} = \left(-\frac{2}{\sqrt{5}};\frac{1}{\sqrt{5}};0\right)$$



2) CENTRO CERCHIO OSCULATORE

$$C = P + \rho \vec{N}$$

$$\rho = \frac{1}{k} \quad , \quad k = \frac{|\underline{r}' \times \underline{r}''|}{|\underline{r}'|^3} \quad , \quad t = 1$$

$$\underline{r}(t) = (t; t^2; 0) \longrightarrow \underline{r}(1) = (1; 1; 0)$$

$$\underline{r}'(t) = (1; 2t; 0) \longrightarrow \underline{r}'(1) = (1; 2; 0)$$

$$\underline{r}''(t) = (0; 2; 0) \longrightarrow \underline{r}''(1) = (0; 2; 0)$$

$$\underline{r}'(1) \times \underline{r}''(1) = (0; 0; 2) \qquad |\underline{r}'(1)| = \sqrt{5}$$

$$k(1) = \frac{|\underline{r}' \times \underline{r}''|}{|\underline{r}'|^3} = \frac{2}{5\sqrt{5}} \Rightarrow \rho(1) = \frac{5\sqrt{5}}{2}$$

$$C = P + \rho \vec{N} = (1; 1) + \frac{5\cancel{\sqrt{5}}}{2}\left(-\frac{2}{\cancel{\sqrt{5}}}; \frac{1}{\cancel{\sqrt{5}}}\right) =$$

$$= (1;1) + \left(-5; \frac{5}{2}\right) = \left(-4; \frac{7}{2}\right)$$

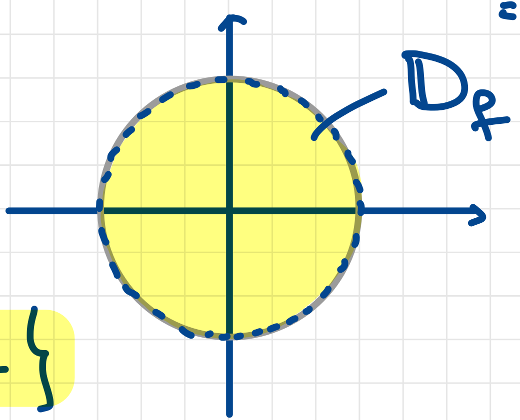
CERCHIO OSC.: circonf. di centro $\left(-4; \frac{7}{2}\right)$ e raggio $\frac{5\sqrt{5}}{2}$.

$$(x+4)^2 + \left(y - \frac{7}{2}\right)^2 = \frac{125}{4}$$

FUNZIONI IN PIÙ VARIABILI

ESERCIZIO 3. Determinare il dominio delle seguenti funzioni in due variabili:

1) $f(x;y) = \log(1-x^2-y^2)$



D_f: $1-x^2-y^2>0 \rightarrow x^2+y^2<1$

$$D_f = \{(x;y) \in \mathbb{R}^2 \mid x^2+y^2<1\}$$

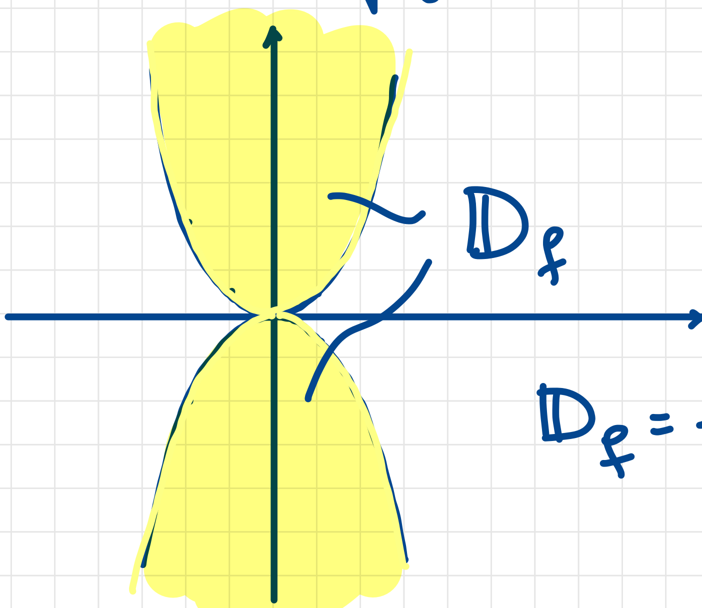
2) $f(x;y) = \log(x^2+y^2)$

$D_f : x^2+y^2 > 0$ vero $\forall (x;y) \in \mathbb{R}^2 - \{(0;0)\}$

$$\boxed{D_f = \mathbb{R}^2 - \{(0;0)\}}$$

3) $f(x;y) = \sqrt{y^2 - x^4}$ $\quad y^2 - x^4 \geq 0 \rightarrow y^2 \geq x^4$

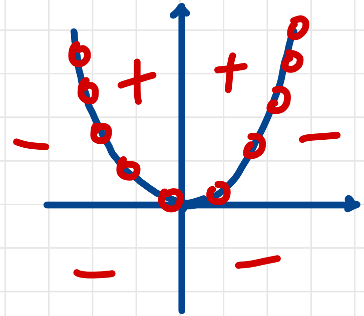
$\rightarrow \quad y \leq -x^2 \vee y \geq x^2$



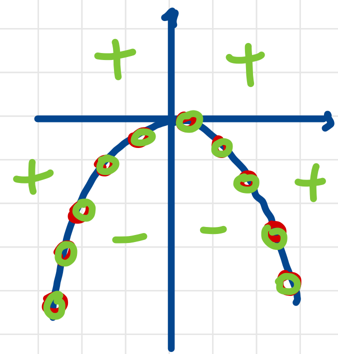
$$D_f = \{(x;y) \in \mathbb{R}^2 / y \leq -x^2 \vee y \geq x^2\}$$

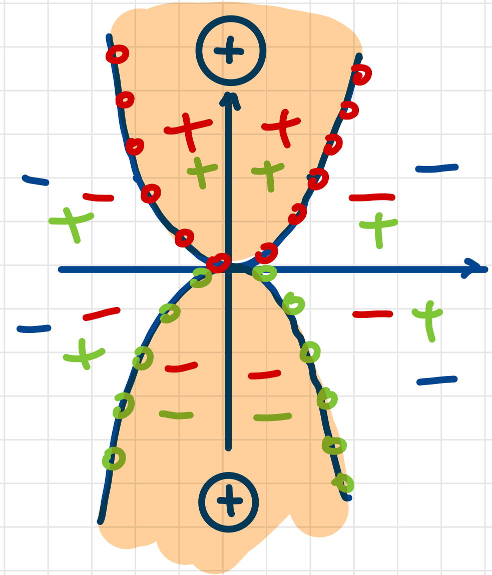
OPPURE: $y^2 - x^4 \geq 0 \longrightarrow (y-x^2)(y+x^2) \geq 0$

- I F. $> 0 \longrightarrow y > x^2$



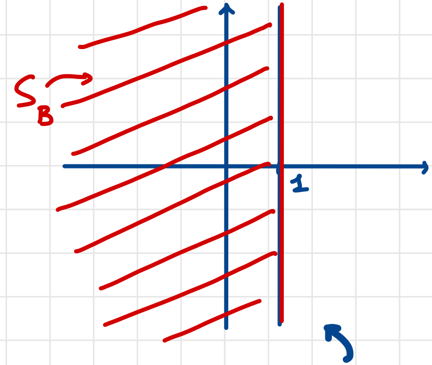
- II F. $> 0 \longrightarrow y > -x^2$





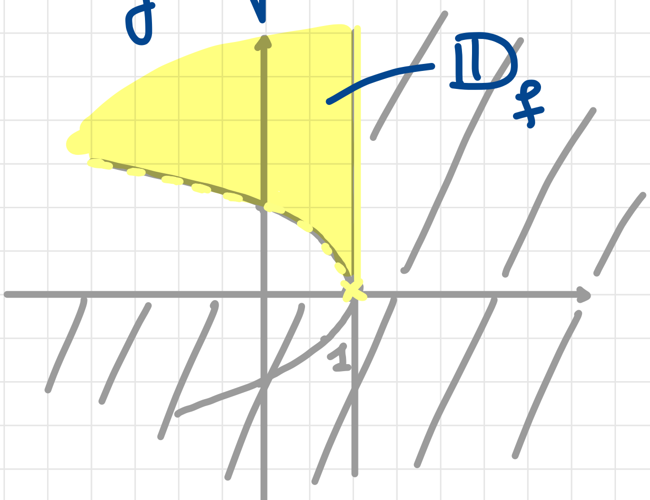
4) $f(x;y) = \log\left(y - \sqrt{1-x}\right)$ $\quad D_f: \begin{cases} y - \sqrt{1-x} > 0 & A \\ 1-x \geq 0 & B \end{cases}$

B: $1-x \geq 0 \to x \leq 1$



A: $y-\sqrt{1-x} > 0$

$y > \sqrt{1-x}$



$$\begin{cases} 1-x \geq 0 \to x \leq 1 \\ y > 0 \\ y^2 > 1-x \end{cases}$$

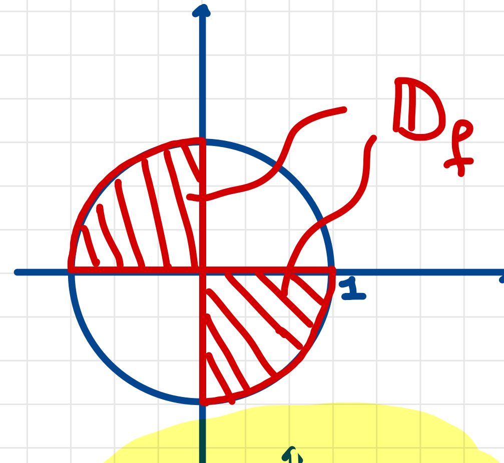
(COND. DI CONCORDANZA DEI SEGNI)

$y^2 > 1-x \to x > 1-y^2$

5) $f(x,y) = \sqrt{-xy} + \arcsin(x^2+y^2)$

$$\begin{cases} -xy \geq 0 \rightarrow xy \leq 0 & (x, y \text{ discordi o nulli}) \\ \underbrace{-1 \leq x^2+y^2 \leq 1} \end{cases}$$

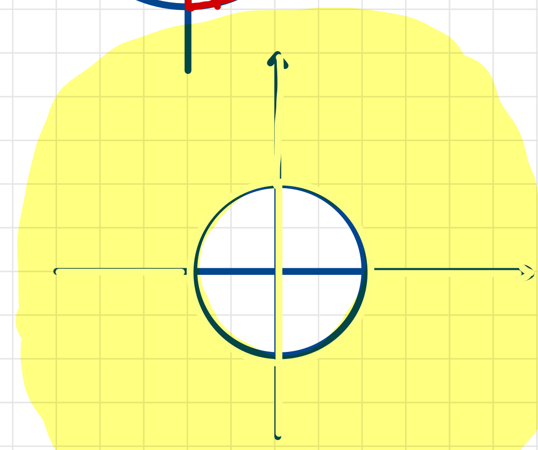
→ interno e bordo della crf.



6) $f(x,y) = \sqrt{|x|(x^2+y^2-4)}$

$|x|(x^2+y^2-4) \geq 0$

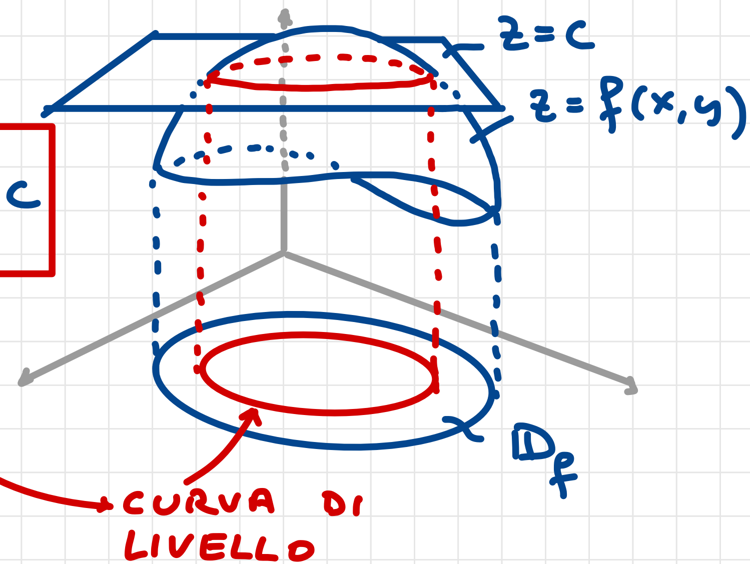
$$\begin{cases} |x| \geq 0 \\ x^2+y^2 \geq 4 \end{cases}$$



CURVE DI LIVELLO

EQ. CURVA DI LIVELLO

$$\begin{cases} z = f(x,y) \\ z = c \end{cases} \Longrightarrow \boxed{f(x,y) = c}$$



ESERCIZIO 4.

Determinare le curve di livello della funzione $f(x;y) = e^{-x^2-y^2}$

SOL.

$$\begin{cases} z = c \\ z = e^{-x^2-y^2} \end{cases} \Longrightarrow \underbrace{e^{-x^2-y^2} = c}_{c>0} \Longrightarrow -x^2-y^2 = \ln c$$

$$x^2+y^2 = -\ln c$$

$$\Rightarrow \quad x^2 + y^2 = \ln \frac{1}{c}$$

Le curve di livello sono circonferenze di centro l'origine e raggio $\sqrt{\ln \frac{1}{c}}$

$$\begin{cases} c > 0 \\ \ln \frac{1}{c} \geq 0 \rightarrow \frac{1}{c} \geq 1 \rightarrow c \leq 1 \end{cases}$$

Ci sono curve di livello per $0 < c \leq 1$

oss. Studiando le curve di livello si è riusciti a determinare $\operatorname{Im} f = (0; 1]$.

$$\max_{\mathbb{R}^2} f = 1 \qquad \inf_{\mathbb{R}^2} f = 0$$

ESERCIZIO 5. Mediante le curve di livello e mediante opportune restrizioni individuare la natura delle seguenti superfici:

1) $f(x,y) = \frac{x^2}{4} + \frac{y^2}{4}$

- CURVE DI LIVELLO

$$\frac{x^2}{4} + \frac{y^2}{4} = c \quad , \quad c \geq 0$$

$$x^2 + y^2 = 4c$$

$\Rightarrow$ circonf. di centro $(0;0)$ e raggio $2\sqrt{c}$.

- Restringo $z = f(x,y)$ all'asse x $(y=0)$

$$z = f(x;0) = \frac{x^2}{4}$$ PARABOLA

- Restringo $z = f(x,y)$ all'asse y $(x=0)$

$$z = f(0;y) = \frac{y^2}{4}$$ PARABOLA

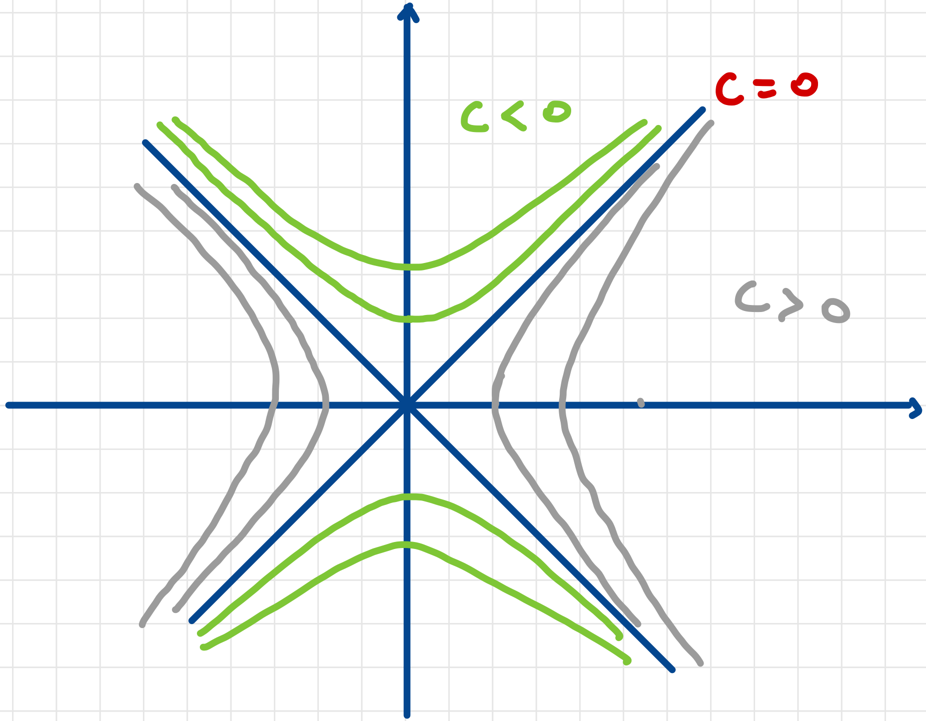
La superficie descritta da $z = f(x,y)$ è un PARABOLOIDE ELLITTICO

$$z = \frac{x^2}{a^2} + \frac{y^2}{b^2}$$

2) $z = f(x;y) = x^2 - y^2$

CURVE DI LIVELLO: $x^2 - y^2 = c \gtrless 0$

- $c > 0 \to x^2 - y^2 = c$ iperbole equilatera con fuochi su asse x
- $c < 0 \to x^2 - y^2 = c$ iperbole equilatera con fuochi su asse y
- $c = 0 \to x^2 - y^2 = 0 \to (x-y)(x+y) = 0$ coppia di rette incidenti!



Restrizione $x = 0$: $z = f(0; y) = -y^2$ PARABOLA

Restrizione $y = 0$: $z = f(x; 0) = x^2$ PARABOLA

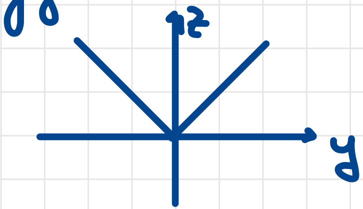
$z = x^2 - y^2$ è un PARABOLOIDE IPERBOLICO

3) $z = f(x,y) = \sqrt{x^2+y^2}$

• **CURVE LIVELLO**: $\sqrt{x^2+y^2} = c \geq 0$

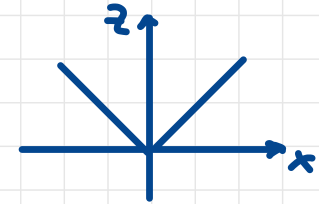
$$x^2+y^2 = c^2$$

circonf. di centro $(0;0)$ e raggio c



• $x=0 \rightarrow z = f(0;y) = \sqrt{y^2} = |y|$

• $y=0 \rightarrow z = f(x;0) = \sqrt{x^2} = |x|$



$z = \sqrt{x^2+y^2}$ è un CONO

ESERCIZIO 6.

Sia $f(x;y) = \sqrt{9-2x^2-6y^2}$

$$\left(\begin{array}{c} z^2 = 9-2x^2-6y^2 \\ 2x^2+6y^2+z^2 = 9 \end{array} \right)$$

ELLISSOIDE

1) Determinare D_f

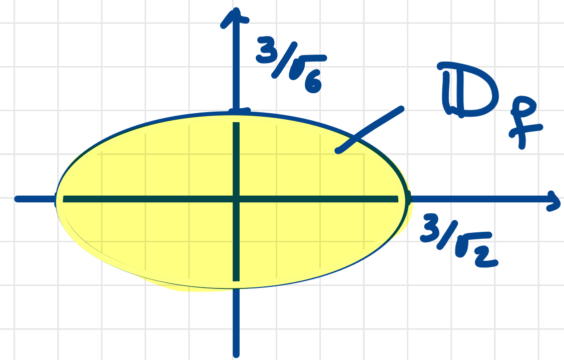
2) Disegnare le curve di livello $0;\ 1;\ 3$.

3) Si determini la curva di livello passante per $P(1;1)$.

SOL.

1) $9-2x^2-6y^2 \geq 0 \qquad 2x^2+6y^2 \leq 9 \qquad \frac{2x^2}{9}+\frac{6y^2}{9} \leq 1$

$$\frac{x^2}{9/2}+\frac{y^2}{9/6} \leq 1$$



2) CURVE DI LIVELLO:

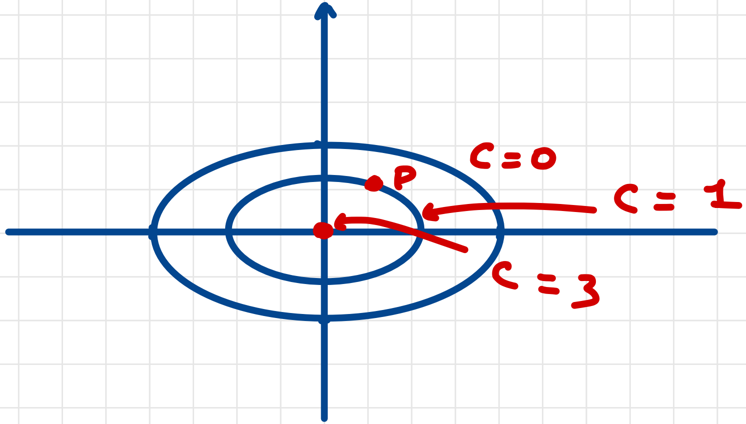
$$\sqrt{9-2x^2-6y^2} = c$$

$$9-2x^2-6y^2 = c^2$$

$c=0 \longrightarrow 2x^2+6y^2=9 \longrightarrow \frac{x^2}{9/2}+\frac{y^2}{9/6}=1$

$c=1 \longrightarrow 9-2x^2-6y^2=1 \longrightarrow 2x^2+6y^2=8 \rightarrow \frac{x^2}{4}+\frac{y^2}{4/3}=1$

$\boxed{c=3}$ $\rightarrow$ $\cancel{9}-2x^2-6y^2=\cancel{9}$ $\rightarrow$ $2x^2+6y^2=0$ $\rightarrow$ $(0;0)$



3) $P(1;1)$

$\sqrt{9-2x^2-6y^2}=c \rightarrow \sqrt{9-2-6}=c \rightarrow c=1$

FUNZIONI IN TRE VARIABILI

ESERCIZIO 7. Determinare il dominio della funzione $w=f(x,y,z)=\log(x^2+y^2)+z$

SOL. $\mathbb{D}_f$: $x^2+y^2>0$ $\quad (x;y)\neq(0;0)$

$$D_f = \{ (x,y,z) \in \mathbb{R}^3 \ / \ (x;y) \neq (0;0), \ \forall z \}$$

D_f è tutto $\mathbb{R}^3$ tranne l'asse z.

ESERCIZIO 8. Determinare le superfici di livello delle seguenti funzioni:

1) $w = f(x,y,z) = x + 3y + 5z$

SUP. DI LIVELLO:

$$\begin{cases} w = x + 3y + 5z \\ w = c \end{cases} \Longrightarrow \quad x + 3y + 5z = c$$

PIANI PARALLELI

2) $w = f(x,y,z) = x^2 + 3y^2 + 5z^2$

SUP. LIVELLO: $x^2 + 3y^2 + 5z^2 = c \quad , \quad c \geq 0$

Se $c > 0$: $\dfrac{x^2}{c} + \dfrac{y^2}{c/3} + \dfrac{z^2}{c/5} = 1 \longrightarrow$ ELLISSOIDE di sezioni $\sqrt{c}$, $\sqrt{\frac{c}{3}}$, $\sqrt{\frac{c}{5}}$.

$\mathcal{E}: \frac{x^2}{A^2}+\frac{y^2}{B^2}+\frac{z^2}{C^2}=1$ sezioni: A, B, C.

3) $w = f(x;y;z) = \frac{1}{\sqrt{x^2+y^2+z^2}}$ $\quad D_f = \mathbb{R}^3 - \{(0;0;0)\}$

$\frac{1}{\sqrt{x^2+y^2+z^2}} = c$, $\quad c > 0$

$\sqrt{x^2+y^2+z^2} = \frac{1}{c} \longrightarrow x^2+y^2+z^2 = \frac{1}{c^2}$

SFERA DI CENTRO $(0;0;0)$ e RAGGIO $\frac{1}{c}$.