Trasformata di Fourier

TRASFORMATA IN L^1

$u \in L^1(\mathbb{R})$

$\hat{u}(\omega) = \int_{\mathbb{R}} u(x)\, e^{-i\omega x}\, dx \qquad \omega \in \mathbb{R}$

$\hat{u}(\omega) \in L^\infty(\mathbb{R}) \longrightarrow \|\hat{u}\|_{L^\infty} \leq \|u\|_{L^1}$

- $\mathcal{F}: u \to \hat{u}$ LINEARE CONTINUO A NORMA 1
- $\hat{u}(\omega) = \int_{\mathbb{R}^N} u(\underline{x})\, e^{-i\underline{\omega}\cdot\underline{x}}\, d\underline{x} \qquad \omega \in \mathbb{R}^N$
- $\hat{u}(\omega) = \int_{\mathbb{R}} u(x)[\cos(\omega x) + i\sin(\omega x)]\, dx \qquad \hat{u} \in L^\infty(\mathbb{R}, \mathbb{C})$

TEOREMA RIEMANN LEBESGUE (RL)

$u \in L^1(\mathbb{R}) \longrightarrow \hat{u}$ LIMITATA, CONTINUA
INFINITA AD ∞

ESEMPI

$u(x) = \chi_{[a,b]}(x)$

$$\hat{u}(x) = \int_{\mathbb{R}} \chi_{[a,b]}(x)\, e^{-i\omega x}\, dx = \int_a^b e^{-i\omega x}\, dx =$$
$$= \int_a^b \cos(-\omega x) + i\sin(-\omega x)\, dx =$$
$$= \int_a^b \cos(\omega x) - i\sin(\omega x)\, dx =$$
$$= \left[\frac{\sin(\omega x)}{\omega} - i - \frac{\cos(\omega x)}{\omega} \right]_a^b =$$
$$= \frac{1}{\omega}\left(\sin(\omega a) + i\cos(\omega a) - \sin(\omega b) - i\cos(\omega b) \right)$$
$$= \frac{\sin(\omega a) - \sin(\omega b)}{\omega} + i\,\frac{\cos(\omega a) - \cos(\omega b)}{\omega}$$

PROPRIETÀ

1) $v(x) = u(x-y) \longrightarrow \hat{v}(\omega) = e^{-i\omega\cdot y}\,\hat{u}(\omega)$
2) $v(x) = e^{ix\cdot y}\, u(x) \longrightarrow \hat{v}(\omega) = \hat{u}(\omega - y)$
3) $v(x) = \bar{u}(x) \longrightarrow \hat{v}(\omega) = \overline{\hat{u}}(-\omega)$
4) $v(x) = u(A^{-1}x) \longrightarrow \hat{v}(\omega) = |\det A|\,\hat{u}(A^T\omega)$

$A \in M(N \times N)$
- $A = 2I \longrightarrow \hat{v}(\omega) = |2^N|\,\hat{u}(2\omega)$
- $A = -I \longrightarrow \hat{v}(\omega) = \hat{u}(-\omega)$

u PARI $\to$ PARI, REALE

u DISPARI $\to$ DISPARI, PURA IMMAGINARIA

ANTITRASFORMATA

$u \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ CONTINUA IN x CON $\hat{u} \in L^1(\mathbb{R}^N)$

$u(x) = (2\pi)^{-N} \int_{\mathbb{R}^N} \hat{u}(\omega)\, e^{i\omega x}\, d\omega$

CERCHIAMO UNO SPAZIO IN CUI È SEMPRE POSSIBILE

SPAZIO S

"FUNZIONI A DECRESCENZA RAPIDA"

$S(\mathbb{R}^N) = \{ u \in C_0^\infty(\mathbb{R}^N) : \forall \alpha, \beta$ MULTI INDICI $x^\alpha D^\beta u(x) \in L^\infty(\mathbb{R}^N) \}$

- $C_0^\infty(\mathbb{R}^N) \subset S(\mathbb{R}^N) \subset C^\infty(\mathbb{R}^N)$
- $S(\mathbb{R}^N) \subset L^1(\mathbb{R}^N)$
- $\mathcal{F}: S \to S$, $\exists\, \mathcal{F}^{-1}$ SEMPRE IN S

PROPRIETÀ IN $S(\mathbb{R}^N)$

$\int \hat{u} v = \int u \hat{v}$

$\int u \bar{v} = (2\pi)^{-N} \int \hat{u}\, \overline{\hat{v}}$

$\int |u|^2 = (2\pi)^{-N} \int |\hat{u}|^2$

$\mathcal{F}(u * v) = \hat{u}\hat{v}$

$\mathcal{F}(uv) = (2\pi)^{-N}\, \hat{u} * \hat{v}$

PLANCHEREL

$\|u\|_{L^2}^2 = (2\pi)^{-N}\, \|\hat{u}\|_{L^2}^2$

SI USA SPESSO PER SEMPLIFICARE INTEGRALI

DIMOSTRA CHE $\mathcal{F}, \mathcal{F}^{-1}: L^2 \to L^2$

APPLICAZIONI ALLE ODE

1) TRASFORMO EQ DIFFERENZIALE
2) RICAVO $\hat{u}(\omega)$
2) ANTITRASFORMO IN $u(x)$