Distribution Theory

Sunday, 21 November 2021 13:29

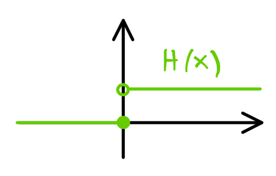
ANALYSIS FUNCTION LIMIT DERIVATE ...

NEW FIELDS DIFFERENTIAL EQUATIONS
FOURIER SERIES
FOURIER TRANSFORM

SOME SOLUTIONS WITH SHARP TURNS ARE INTERESTING AND THE DEFINITION OF A DERIVATE IS TOO STRONG TO CONSIDER FUNCTIONS WITH SHARP TURNS SINCE AT SHARP TURNS FUNCTIONS ARE NOT DIFFERENTIABLE

HISTORICAL EXAMPLE PAUL DIRAC 1927

$H:\mathbb{R}\to\mathbb{R}$ $H(x)=\begin{cases}1 & x>0\\ 0 & x\le 0\end{cases}$ HEAVYSIDE FUNCTION



DIRAC WANTED $\frac{d}{dx}H(x)$ IN $\underline{x=0}$
CLASSICAL DERIVATE HAS A PROBLEM IN $x=0$ (DISCONTINOUS)

"MORE GENERAL" DERIVATIVE SHOULD BE CALLED "DELTA FUNCTION" δ
DIRAC WANTED THAT THE FUNDAMENTAL T. OF CALCULUS HELD FOR δ

$$\begin{cases}\delta(x)=0 \quad \forall x\neq 0\\ \text{FOR } \varepsilon>0 : \int_{-\varepsilon}^{+\varepsilon}\delta(x)\,dx=\int_{-\varepsilon}^{+\varepsilon}H'(x)\,dx = H(\varepsilon)-H(-\varepsilon)=1-0=1\end{cases}$$

δ IS NOT AN ORDINARY FUNCTION

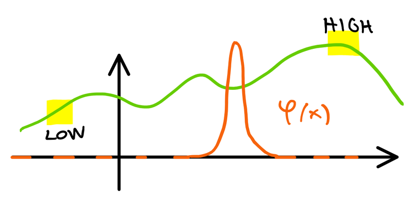
$\delta(x)=0\ \forall x\neq 0 \to \delta(x)=0$ ALMOST EVERYWHERE (WITH RESPECT TO THE LEBESGUE MEASURE)
HENCE $\int_{-\varepsilon}^{+\varepsilon}\delta(x)=0 \quad \forall\varepsilon \quad \neq 1$ CONTRADDICTION

DIRACT WANTS TO CALCULATE $\delta', \delta'', \delta'''$ ...

δ WILL BE CALLED "DISTRIBUTIONS", GENERALIZED FUNCTIONS

DISTRIBUTIONS

GENERALIZED FUNCTION, DENSITY



MASS DISTRIBUTION

IN PHYSICS : MEASUREMENT DEVICE

IN MATHS : TEST FUNCTIONS

TEST FUNCTIONS

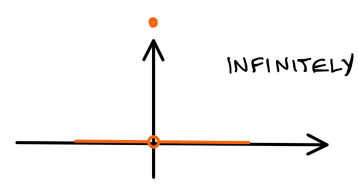
LOCALIZED & CONTINOUS $\varphi:\mathbb{R}\to\mathbb{R}$

LINEAR MAP $\varphi \mapsto \int f(x)\varphi(x)\,dx \in\mathbb{R}$

SIMILAR TO A MEASUREMENT FUNCTION,
GET BACK VALUES IN A SMALL REGIO

DELTA FUNCTION

$\varphi\mapsto\varphi(0)$



INFINITELY HIGH SUMP

TEST FUNCTIONS

$\varphi:\mathbb{R}^N\to\mathbb{R}$ (OR $\mathbb{C}$)

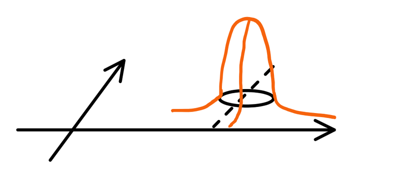
DIFFERENTIABLE ∞ TIMES
ON A COMPACT SUPPORT

SPACE OF TEST FUNCTIONS $\mathbb{D}(\mathbb{R})=C_c^\infty(\mathbb{R}^N)$ BEST SMOOTHNESS

$\mathbb{D}(\mathbb{R})$ VECTOR SPACE + SPECIFIC CONVERGENT

EXAMPLES $\varphi:\mathbb{R}^N\to\mathbb{R}$

- $\varphi=0$
- $\varphi(x)=\begin{cases}0 & \|x\|\geq 1\\ \exp\left(\frac{-1}{1-\|x\|^2}\right) & \|x\|<1\end{cases}$



NOTATIONS

SMALLEST CLOSED SUBSET OF $\mathbb{R}^N$ WHERE φ IS SUPPORTED

$\mathrm{SUPP}(\varphi)=\overline{\{x\in\mathbb{R}^N \mid \varphi(x)\neq 0\}}$ CLOUSORE IN $\mathbb{R}^N$

FOR $\alpha_1,\alpha_2,\alpha_3,\cdots,\alpha_N\in\mathbb{N}_0$ WE CALL TUPLE $\alpha(\alpha_1,\alpha_2,\cdots,\alpha_N)$ MULTI INDEX

$$\Delta^\alpha=\frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}}\frac{\partial^{\alpha_2}}{\partial x_2^{\alpha_2}}\cdots\frac{\partial^{\alpha_N}}{\partial x_N^{\alpha_N}}$$

EXAMPLE

$f:\mathbb{R}^2\to\mathbb{R}$ $f(x_1,x_2)=2x_1^2x_2^3$ $\alpha(\overset{x_1}{2},\overset{x_2}{1})\to$ HOW MANY TIME YOU HAVE TO DIFFERENTIATE A CHOSEN PARAMETER

$$(\Delta^\alpha f)(x_1,x_2)=\frac{\partial^2}{\partial x_1^2}\left(\frac{\partial}{\partial x_2}2x_1^2x_2^3\right)=\frac{\partial^2}{\partial x_1^2}(6x_1^2x_2^2)=12x_2^2$$

$f\in C^\infty(\mathbb{R}^N) \longleftrightarrow \Delta^\alpha f\in C(\mathbb{R}^N) \quad \forall\alpha$

A MULTIVARIABLE FUNCTION IS C^∞ IF IT IS DERIVABLE WITH CONTINUITY FOR ALL α MULTI INDEXES

CONVERGENCE FOR TEST FUNCTIONS

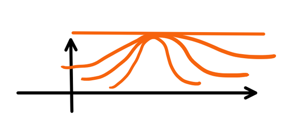
$\mathbb{D}(\mathbb{R}^N)=C_c^\infty(\mathbb{R}^N)$

$\|\cdot\|$ IF NORM $f_m\to f \longleftrightarrow \|f_m-f\|\xrightarrow{N\to\infty}0$

d IF METRIC (GENERALIZATION) $f_m\to f \longleftrightarrow d(f_m,f)=0$

WE STILL NEED A STRONGER NOTION OF CONVERGENCE
FOR ALL $(\varphi_K)_{K\in\mathbb{N}}\subset\mathbb{D}(\mathbb{R}^N)$, $\varphi\in\mathbb{D}(\mathbb{R}^N)$

ALL φ SHOULD LIVE ON THE SAME COMPACT SET



WE DON'T WANT THAT φ GET LARGER & LARGER IN THE SUPPORT SUCH THAT THE LIMIT IS NOT A TEST FUNCTION ANYMORE

$\varphi_K\to\varphi$ IF

1) $\exists M$ BOUNDED SET SUCH THAT OUTSIDE OF M $\varphi_K=0\ \forall K$

2) UNIFORM CONVERGENCE $\varphi_K\xrightarrow{\text{UNIF}}\varphi$ SO $\Delta^\alpha\varphi_K\xrightarrow{\text{UNIF}}\Delta^\alpha\varphi\ \forall\alpha$

(STRONG CONVERGENCE)

$\varphi_K\to\varphi \longleftrightarrow \exists C\subseteq\mathbb{R}^N$ COMPACT : $\mathrm{SUPP}\,\varphi_1,\mathrm{SUPP}\,\varphi_2,\cdots\subseteq C$
WHERE $\|\Delta^\alpha\varphi_K-\Delta^\alpha\varphi\|\xrightarrow{K\to\infty}0 \quad \forall\alpha$ MULTI INDEX

SPACE OF DISTRIBUTIONS

$\mathbb{D}(\mathbb{R}^N)$ VECTOR SPACE OF TEST FUNCTIONS
WITH THE NOTION OF CONVERGENCE

$T:\mathbb{D}(\mathbb{R}^N)\to\mathbb{R}$ OR $\mathbb{C}$ f AS INPUT # AS OUTPUT

T IS LINEAR
$T(\varphi_1+\varphi_2)=T(\varphi_1)+T(\varphi_2)$
$T(\lambda\varphi_1)=\lambda T(\varphi_1)$

T IS (SEQUENTIALLY) CONTINOUS $\forall\varphi\in\mathbb{D}(\mathbb{R}^N)\ \forall$ SEQ $\varphi_K\subseteq\mathbb{D}(\mathbb{R}^N)$
WITH $\varphi_K\xrightarrow{\mathbb{D}}\varphi$ SO $T(\varphi_K)\longrightarrow T(\varphi)$

NOTATION

$\mathbb{D}(\mathbb{R}^N)'$ OR $\mathbb{D}'(\mathbb{R}^N)$ ALL THE CONTINEOUS LINEAR MAPS

EXAMPLE

DELTA $\delta:\mathbb{D}(\mathbb{R}^N)\to\mathbb{R}$ $\delta(\varphi)=\varphi(0)$

ALL CONTINOUS fs ARE ALSO DISTRIBUTIONS

$\forall f\in C(\mathbb{R}^N)\ \exists T_f:\mathbb{D}(\mathbb{R}^N)\to\mathbb{R}$ WHERE $T_f(\varphi)=\int_{\mathbb{R}}f(x)\varphi(x)\,dx$

IF $f\neq g \to T_f\neq T_g$

DISTRIBUTIONS

DIST D A MAP FROM THE SPACE OF TEST FUNCTIONS TO $\mathbb{R}$

$T:\mathbb{D}(\mathbb{R}^N)\to\mathbb{R}$ OR $\mathbb{C}$

T IS A DISTRIBUTION $\longleftrightarrow \underset{\substack{K\subseteq\mathbb{R}^N\\ \text{COMPACT}}}{\forall}\ \underset{m\in\mathbb{N}_0}{\exists}\ \underset{c>0}{\exists}\ \underset{\varphi\in\mathbb{D}(\mathbb{R}^N)}{\forall}\ \mathrm{SUPP}(\varphi)\subseteq K \to |T(\varphi)|\leq c\sum_{|\alpha|\leq m}\|\Delta^\alpha\varphi\|_\infty$ BOUNDED FROM ABOVE

LOCALLY INTEGRABLE

$f:\mathbb{R}^N\to\mathbb{R}$ OR $\mathbb{C}$ L.I. IF RESTRICTED TO ALL COMPACTS SETS

$\int_K\|f(x)\|\,dx<\infty$ $f\in L^1_{loc}(\mathbb{R}^N)$

FOR $f\in L^1_{loc}(\mathbb{R}^N)$ $T_f\in\mathbb{D}'(\mathbb{R}^N)$ BY $T_f(\varphi)=\int_{\mathbb{R}}f(x)\varphi(x)\,dx$

$T\in\mathbb{D}'(\mathbb{R}^N)$ REGULAR IF $\exists f$ LOCALLY INTEGRABLE SUCH THAT $T=T_f$

DISTRIBUTIONS THAT BEHAVE LIKE NORMAL FUNCTIONS