Trasformate

TRASFORMATA DI FOURIER

FUNZIONE $u(x)$

TRASFORMATA $\mathcal{F}(u(x)) = \hat{u}(\omega) = \int_{\mathbb{R}} e^{-i\omega x} u(x)$

LA DEFINIZIONE NON E' UNIVOCA

PROPRIETA'

- REGOLARITA' E CONTINUITA'
- $\mathcal{F}$(PARI) = REALE PARI
- $\mathcal{F}$(DISPARI) = IMMAGINARIA PURA DISPARI

TEORIA

$$u \in AC \rightarrow u \text{ INTEGRABILE} \rightarrow u \in L^1 \rightarrow \begin{cases} \hat u \in L^\infty \\ \hat u \in C^0 \quad \text{(RL LEMMA)} \\ \lim_{\omega \to \infty} \hat u = 0 \end{cases}$$

$\mathcal{F} \quad L^1 \rightarrow L^\infty$

$\mathcal{F} \quad S \rightarrow S$

$\mathcal{F} \quad L^2 \rightarrow L^2$

$\mathcal{F} \quad S' \rightarrow S'$

$$\begin{tikzcd} u \in S \arrow[r] \arrow[d] & u \in L^2 \arrow[d] \\ \hat u \in S' \arrow[r] & \hat u \in L^2 \end{tikzcd}$$

$x^N f \in L^1(\mathbb{R}) \rightarrow \hat f^{(N)} \in C^\infty$ E NULLA AD ∞ $(\forall N \geq 1)$

$f, xf \in L^2 \rightarrow \hat f, \hat f' \in L_2(\mathbb{R})$ $\quad f \in AC_{loc}(\mathbb{R})$

$u x^K \in L^1(\mathbb{R}) \rightarrow \hat u \in C^K$

TRASFORMA $u(x) = e^{-|x|}$

$$\mathcal{F}(e^{-|x|}) = \int_{-\infty}^{+\infty} e^{-i\omega x} \cdot e^{-|x|} dx = \int_{-\infty}^{0} e^{(1-i\omega)x} dx + \int_0^\infty e^{-(1+i\omega)x} dx = \left| \frac{1}{1-i\omega} e^{(1-i\omega)x} \right|_{-\infty}^{0} + \left| \frac{-1}{1+i\omega} e^{-(1+i\omega)x} \right|_0^\infty = \frac{1}{1-i\omega} + \frac{1}{1+i\omega} = \frac{2}{1+\omega^2}$$

TRASFORMA $\chi_{(-1,+1)}$

$$\mathcal{F}(\chi_{(-1,+1)}) = \int_{-\infty}^{+\infty} e^{-i\omega x} \chi_{(-1,+1)}(x)\, dx = \int_{-1}^{+1} e^{-i\omega x} dx = \left| \frac{-1}{i\omega} e^{-i\omega x} \right|_{-1}^{+1} = -\frac{1}{i\omega} e^{-i\omega} + \frac{1}{i\omega} e^{+i\omega} = \frac{1}{i\omega}(e^{i\omega} - e^{-i\omega}) = \frac{2 \sin(i\omega)}{\omega}$$

TRASFORMA $u(x) = \frac{1}{1+x^2}$

$\mathcal{F}\left(\frac{1}{1+x^2}\right) = \int_{-\infty}^{+\infty} e^{i\omega x} \frac{1}{1+x^2} dx$ $\quad \frac{1}{1+x^2}$

ANALISI COMPLESSA

$f(z) = e^{-i\omega z} \frac{1}{1+z^2}$

LEMMA DI JORDAN

DIPENDE DAL SEGNO DI ω

$\omega > 0$ SEMICERCHIO SOTTO $(-)$

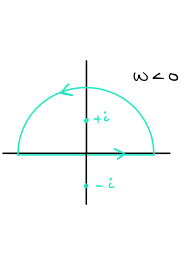
$\omega < 0$ SEMICERCHIO SOPRA $(+)$

$\rightarrow$ f PARI, CI BASTA FARLO 1 SOLA VOLTA

SINGOLARITA' $\quad z \neq \pm i \quad$ POLO ORDINE 1

RESIDUI $\quad RES(f, z=i) = \frac{NUM}{DEN'}$

TEOREMA DEI RESIDUI



$$\lim_{R \to \infty} \int_R^R f(z) dz = 2\pi i \sum RES(f) = 2\pi i\, RES(f, z=i) = 2\pi i \frac{e^{-i\omega i}}{2i} = \pi e^{\omega}$$

PER $\omega < 0$ $\quad \hat u(\omega) = \pi e^{\omega}$ DEVE ESSERE PARI

TRASFORMATA $\quad \hat u(\omega) = \pi e^{-|\omega|}$

RISOLVI $u' + 2xu = 0$

$u = e^{-x^2} \quad u' = -2x e^{-x^2}$

PROBLEMA DI CAUCHY $\quad u(0) = 1$

TRASFORMATE NOTEVOLI

$\mathcal{F}(u) = \hat u$

$\mathcal{F}(u') = i\omega \hat u$

$\mathcal{F}(xu) = i \frac{d}{d\omega} \hat u = i \hat u'$

EQUAZIONE LINEARE I ORDINE

$y'(t) + a\, y(t) = 0$

$y(t) = K e^{-\int a(t) dt}$

SOSTITUISCO

$\cancel{i}\omega \hat u + 2\cancel{i} \frac{d}{d\omega} \hat u = 0$

$\omega \hat u + 2 \hat u' = 0$

$\hat u' + \frac{\omega}{2} \hat u = 0$

$\hat u(\omega) = K e^{-\omega^2/4}$

DETERMINO K

$\hat u(\omega) = \int_{\mathbb{R}} e^{-i\omega x} u(x) dx$

$\hat u(0) = \int_{\mathbb{R}} u(x) dx$

$K = \hat u(0) = \int_{\mathbb{R}} e^{-x^2} dx = \sqrt{\pi}$

SOLUZIONE

$\mathcal{F}(e^{-x^2}) = \sqrt{\pi} e^{-\omega^2/4}$

$u = \frac{1}{\cosh(x)}$

$\mathcal{F}\left(\frac{1}{\cosh x}\right) = \int_{\mathbb{R}} e^{-i\omega x} \frac{1}{\cosh x} dx$

ANALISI COMPLESSA

$f(z) = \frac{e^{-i\omega z}}{\cosh z} = \frac{2 e^{-i\omega z}}{e^z + e^{-z}}$ $\quad$ ES DECRESCENZA RAPIDA

SINGOLARITA'

$\cosh z = 0 \quad \frac{e^z + e^{-z}}{2} = 0 \qquad e^{x+iy} + e^{-x-iy} = 0$

$e^x(\cos y + i \sin y) + e^{-x}(\cos(-y) + i \sin(-y))$

$e^x \cos y + e^x i \sin y + e^{-x} \cos y - e^{-x} i \sin y = 0$

$$\begin{array}{l|l} \text{Re} & \cos y\,(e^x + e^{-x}) = 0 \\ \text{Im} & \sin y\,(e^x - e^{-x}) = 0 \end{array}$$

$$\left| \begin{array}{l} \cos y = 0 \rightarrow y = \frac{\pi}{2} + k\pi \\ (-1)^k (e^x - e^{-x}) = 0 \end{array} \right| \begin{array}{l} y = \frac{\pi}{2} + k\pi \\ x = 0 \end{array}$$

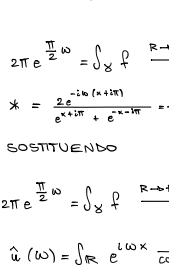
$z = \frac{\pi}{2} + k\pi$ $\quad$ POLI DEL I ORDINE

RESIDUO

$RES(f, POLO) = \frac{NUM}{DEN'}$

$$RES\left(f, z = i\frac{\pi}{2}\right) = \left. \frac{2 e^{-i\omega z}}{e^z - e^{-z}} \right|_{z = i\frac{\pi}{2}} = \frac{2 e^{\frac{\pi}{2}\omega}}{e^{i\frac{\pi}{2}} - e^{-i\frac{\pi}{2}}} = \frac{2 e^{\frac{\pi}{2}\omega}}{2i}$$

PERCORSO RETTANGOLARE



$\int_\gamma f(z) dz = 2\pi i\, RES(z = i\frac{\pi}{2})$

$RES(f, z = i\frac{\pi}{2}) = \frac{2 e^{\frac{\pi}{2}\omega}}{2i}$

$2\pi e^{\frac{\pi}{2}\omega} = \int_\gamma f \xrightarrow{R \to +\infty} \int_{-\infty}^{+\infty} e^{-i\omega x} \frac{1}{\cosh x} dx - \int *$ $\quad$ (PARTE SUPERIORE)

$* = \frac{2 e^{-i\omega(x + i\pi)}}{e^{x+i\pi} + e^{-x-i\pi}} = -\frac{2 e^{-i\omega x}}{e^x + e^{-x}} e^{\omega\pi}$

SOSTITUENDO

$2\pi e^{\frac{\pi}{2}\omega} = \int_\gamma f \xrightarrow{R \to +\infty} \int_{-\infty}^{+\infty} e^{-i\omega x} \frac{1}{\cosh x} dx + \int_{-\infty}^{+\infty} \frac{e^{-i\omega x}}{\cosh x} e^{\omega\pi} dx$

$\hat u(\omega) = \int_{\mathbb{R}} e^{i\omega x} \frac{dx}{\cosh x} = \frac{2\pi e^{\omega\pi/2}}{e^{\omega\pi} + 1} = \frac{2\pi}{e^{\omega\pi/2} + e^{-\omega\pi/2}} = \frac{\pi}{\cosh(\frac{\pi}{2}\omega)}$

RISOLVI IN $L^1(\mathbb{R})$

$u''(x) + u(x) = \chi_{[-1,+1]}(x) \qquad x \in \mathbb{R}$

TRASFORMATA

$\mathcal{F}(u(x)) = \hat u(\omega)$

$\mathcal{F}(u'(x)) = i\omega \hat u(\omega)$

$\mathcal{F}(u''(x)) = -\omega^2 \hat u(\omega)$

$\mathcal{F}(\chi_{[-1,+1]}) = \frac{\sin(\omega) - \sin(-\omega)}{\omega} = \frac{2\sin(\omega)}{\omega}$

$-\omega^2 \hat u + \hat u = \frac{2 \sin(\omega)}{\omega}$

$\hat u(1 - \omega^2) = \frac{2 \sin(\omega)}{\omega}$

$\hat u(\omega) = \frac{2 \sin(\omega)}{\omega(1-\omega^2)} \notin L^\infty(\mathbb{R})$

CONTRADDIZIONE CON RIEMANN-LEBESGUE

NON CI SONO SOLUZIONI IN L^1

RISOLVI $u' + u = \chi(-\frac{1}{2}, +\frac{1}{2})$

TRASFORMATE

$\mathcal{F}(u) = \hat u$

$\mathcal{F}(u') = i\omega \hat u$

$\mathcal{F}(\chi_{(a,b)}) = \frac{\sin(b\omega) - \sin(a\omega)}{\omega}$

$\mathcal{F}(\chi(-\frac{1}{2}, +\frac{1}{2})) = \frac{\sin(\omega/2) - \sin(-\omega/2)}{\omega} = \frac{2\sin(\omega/2)}{\omega}$

SOSTITUISCO

$i\omega \hat u + \hat u = \frac{2 \sin(\omega/2)}{\omega}$

$\hat u = \frac{1}{1+i\omega} \cdot \frac{2\sin(\omega/2)}{\omega}$

$= \mathcal{F}(e^{-x} H(x))\, \mathcal{F}(\chi(-\frac{1}{2}, +\frac{1}{2}))$

$= \mathcal{F}(e^{-x} H(x) * \chi(-\frac{1}{2}, +\frac{1}{2}))$

$u(x) = \int e^{-(x-y)} H(x-y)\, \chi(-\frac{1}{2}, +\frac{1}{2})(y)\, dy$

CASO 1 $\quad x < -\frac{1}{2} \qquad u(x) = 0$

CASO 2 $\quad x \in (-\frac{1}{2}, +\frac{1}{2})$

$u(x) = \int_{-\frac{1}{2}}^{x} e^{y-x} dy = \left| e^{y-x} \right|_{-\frac{1}{2}}^{x} = 1 - e^{-\frac{1}{2} - x}$

CASO 3 $\quad x > +\frac{1}{2}$

$u(x) = \int_{-\frac{1}{2}}^{+\frac{1}{2}} e^{y-x} = e^{\frac{1}{2} - x} - e^{-\frac{1}{2} - x}$

SOLUZIONE

$$u(x) = \begin{cases} 0 & x \leq -\frac{1}{2} \\ 1 - e^{-\frac{1}{2} - x} & x \in (-\frac{1}{2}, +\frac{1}{2}) \\ e^{\frac{1}{2} - x} - e^{-\frac{1}{2} - x} & x \geq +\frac{1}{2} \end{cases}$$

PER QUALI $K \in \mathbb{N}$ $\quad u(x) \in C^K$?

CONTINUITA' DI U

$\lim_{x \to -\frac{1}{2}^+} u(x) = \lim_{x \to -\frac{1}{2}^+} 1 - e^{0} = 0$

$\lim_{x \to +\frac{1}{2}^+} u(x) = e^0 - e^{-1} = 1 - e^{-1}$ $\quad \rightarrow u \in C^0$

$\lim_{x \to +\frac{1}{2}^-} u(x) = 1 - e^{-1}$

$$u'(x) = \begin{cases} 0 & x \leq -\frac{1}{2} \\ +e^{-\frac{1}{2} - x} & x \in (-\frac{1}{2}, +\frac{1}{2}) \\ -e^{\frac{1}{2} - x} + e^{-\frac{1}{2} - x} & x \geq +\frac{1}{2} \end{cases}$$

$\lim_{x \to -\frac{1}{2}^+} u'(x) = e^{-1} \neq 0$ $\qquad \nexists$ LIMITE $\rightarrow u(x) \notin C^1$

RISPOSTA $\quad u \in C^0 \setminus C^1$

CALCOLA LA TRASFORMATA DI $f(x)$

$f(x) = \frac{2(3x^2 - 1)}{(x^2+1)^3}$

CONSIDERANDO

$g(x) = \frac{1}{1+x^2}$

$g'(x) = \frac{(1)'(1+x^2) - (1)(1+x^2)'}{(1+x^2)^2} = \frac{-2x}{(1+x^2)^2}$

$g''(x) = \frac{(-2x)'(1+x^2)^2 - (-2x) \cdot 2(1+x^2)(1+x^2)'}{(1+x^2)^4} = \frac{2(3x^2-1)}{(x^2+1)^3}$

QUINDI $\quad f(x) = g''(x)$

TRASFORMATE

$\mathcal{F}(u) = \hat u$

$\mathcal{F}(u') = i\omega \hat u$

$\mathcal{F}(u'') = -\omega^2 \hat u$

$\mathcal{F}\left(\frac{1}{1+x^2}\right) = \pi e^{-|\omega|}$ $\quad$ NOTEVOLE

$\mathcal{F}(f(x)) = -\pi \omega^2 e^{-|\omega|}$

RISOLVI $\quad f(x) * u(x) = x f(x)$

$\mathcal{F}(f * u) = \mathcal{F}(f)\mathcal{F}(u) = -\pi\omega^2 e^{-|\omega|} \hat u$

$\mathcal{F}(xf) = i \frac{d}{d\omega}(-\pi\omega^2 e^{-|\omega|}) = -\pi i (2\omega e^{-|\omega|} - \omega^2 \operatorname{sign}(\omega) e^{-|\omega|})$

$+\pi\omega^2 e^{-|\omega|} \hat u = +\pi i (2\omega e^{-|\omega|} - \omega^2 \operatorname{sign}(\omega) e^{-|\omega|})$

$\hat u = i\left(\frac{2}{\omega} - \operatorname{sign}(\omega)\right)$

$\lim_{\omega \to \infty} \hat u \neq 0 \rightarrow$ NO SOLUZIONI IN $L^1(\mathbb{R})$

TRASFORMATA $e^{-2|x|}$

$\int_{-\infty}^{+\infty} e^{-i\omega x} e^{-2|x|} dx$

$= \int_{-\infty}^{0} e^{-i\omega x} e^{+2x} dx + \int_0^\infty e^{-i\omega x} e^{-2x} dx$

$= \int_{-\infty}^{0} e^{x(2-i\omega)} dx + \int_0^\infty e^{x(-2-i\omega)} dx$

$= \frac{1}{2-i\omega} \left| e^{x(2-i\omega)} \right|_{-\infty}^{0} - \frac{1}{2+i\omega} \left| e^{-x(2+i\omega)} \right|_0^\infty$

$= \frac{1}{2-i\omega}(1 - e^{-\infty}) - \frac{1}{2+i\omega}(e^{-\infty} - e^0)$

$= \frac{1}{2-i\omega} + \frac{1}{2+i\omega} = \frac{4}{4+\omega^2}$

TRASFORMATA PARAMETRICA IN a

$\mathcal{F}(e^{-a|x|}) = \frac{2a}{a^2+\omega^2}$

$\int_{-\infty}^{+\infty} e^{-i\omega x - a|x|} dx$

$\int_{-\infty}^{0} e^{x(a-i\omega)} + \int_0^\infty e^{x(-a-i\omega)}$

$= \dots = \frac{1}{a+i\omega} + \frac{1}{a-i\omega} = \frac{2a}{a^2+\omega^2}$

SOLUZIONI IN $L^1(\mathbb{R})$

$u^{(6)} + u = \chi_{(-2,+2)} * e^{-2|x|} \qquad x \in \mathbb{R}$

$\mathcal{F}(u) = \hat u$

$\mathcal{F}(u^{(6)}) = (i\omega)^6 \hat u = -\omega^6 \hat u$

$\mathcal{F}(\chi_{(-2,+2)}) = \frac{2\sin(2\omega)}{\omega}$

$\mathcal{F}(e^{-2|x|}) = \frac{4}{4+\omega^2}$

$\mathcal{F}(A * B) = \mathcal{F}(A) \cdot \mathcal{F}(B)$

$\hat u(1 - \omega^6) = \frac{2\sin(2\omega)}{\omega} \cdot \frac{4}{4+\omega^2}$

$\hat u(\omega) = \frac{8 \sin(2\omega)}{\omega(4+\omega^2)(1-\omega^6)}$

$\hat u(\omega)$ NON CONTINUA $\rightarrow \nexists$ SOLUZIONI IN $L^1(\mathbb{R})$

CONSIDERA L'EQUAZIONE

$u''(x) - u(x) = f$

SOLUZIONI IN $L^1(\mathbb{R})$

SUPPONIAMO $\quad u, u' \in L^1(\mathbb{R}) \cap AC(\mathbb{R})$

$(i\omega)^2 \hat u - \hat u = \hat f$

$-\omega^2 \hat u - \hat u = \hat f$

$\hat u = -\frac{\hat f}{1+\omega^2}$

$u(x) = -f(x) * e^{-|x|}$

$u'(x) = -f'(x) * e^{-|x|}$

$u''(x) = -f''(x) * e^{-|x|}$

$f^{(n)}(x), e^{-|x|} \in L^1(\mathbb{R}) \rightarrow \exists u(x) \in L^1(\mathbb{R})$

$f^{(n)}(x), e^{-|x|} \in L^2(\mathbb{R}) \rightarrow \exists u(x) \in L^2(\mathbb{R})$

$f(x), e^{-|x|} \in C^2(\mathbb{R}) \rightarrow \exists u(x) \in C^2(\mathbb{R})$

DERIVATA CONVOLUZIONE

$\frac{\partial^n}{\partial x^n}(A * B) = A^{(n)} * B$

TRASFORMA $u(x) = |x| e^{-|x|}$

$\hat u(\omega) = \int_{\mathbb{R}} e^{-i\omega x} |x| e^{-|x|} dx =$

$= \int_{-\infty}^{0} e^{-i\omega x} (-x) e^{+x} dx + \int_0^\infty e^{-i\omega x} x e^{-x} dx$

$= -\underbrace{\int_{-\infty}^{0} x e^{x(1-i\omega)} dx}_{A} + \underbrace{\int_0^\infty x e^{x(-1-i\omega)} dx}_{B}$

$= -A + B$

INTEGRO A PER PARTI

$f = x \qquad f' = 1$

$g' = e^{x(1-i\omega)} \qquad g = \frac{e^{x(1-i\omega)}}{1-i\omega}$

$A = fg - \int f'g = \cancel{\left| \frac{x e^{x(1-i\omega)}}{1-i\omega} \right|_{-\infty}^{0}} - \int \frac{e^{x(1-i\omega)}}{1-i\omega} dx$

$= -\frac{1}{(1-i\omega)^2} \left| e^{x(1-i\omega)} \right|_{-\infty}^{0} =$

$= -\frac{1}{(1-i\omega)^2}$

INTEGRO B PER PARTI

$f = x \qquad f' = 1$

$g' = e^{x(-1-i\omega)} \qquad g = -\frac{1}{1+i\omega} e^{x(-1-i\omega)}$

$B = fg - \int f'g = \cancel{\left| \frac{x e^{x(-1-i\omega)}}{-(1+i\omega)} \right|_0^\infty} + \int \frac{e^{x(-1-i\omega)}}{1+i\omega}$

$= -\frac{1}{(1+i\omega)^2} \left| e^{x(-1-i\omega)} \right|_0^\infty =$

$= +\frac{1}{(1+i\omega)^2}$

SOLUZIONE

$\hat u(\omega) = -A + B = \frac{1}{(1-i\omega)^2} + \frac{1}{(1+i\omega)^2}$

$= \frac{1 - 2i\omega - \omega^2 + 1 + 2i\omega - \omega^2}{(1-i\omega)^2 (1+i\omega)^2}$

$= \frac{2(1-\omega^2)}{(1+\omega^2)^2}$

STUDIA E TRASFORMA $f(x)$

$f(x) = \cos^2(x)\, \chi_{(-1,+1)}(x) \qquad x \in \mathbb{R}$

STUDIO

f REALE PARI $\rightarrow \hat f$ REALE PARI

$f \in L^1(\mathbb{R}) \rightarrow \hat f \in C^0$ E NULLA AD ∞

$x^N f \in L^1(\mathbb{R}) \rightarrow \hat f^{(N)} \in C^\infty$ E NULLA AD ∞ $(\forall N \geq 1)$

$f, xf \in L^2 \rightarrow \hat f, \hat f' \in L_2(\mathbb{R}) \qquad f \in AC_{loc}(\mathbb{R})$

TRASFORMATA

$f(x) = \cos^2(x)\, \chi_{(-1,+1)}(x)$

$f(x) = \frac{e^{2ix} + e^{-2ix} + 2}{4}$

$\hat f(\omega) = \frac{1}{4}\int_{-1}^{+1} e^{2ix} e^{-i\omega x} dx + \frac{1}{4}\int_{-1}^{+1} e^{-2ix} e^{-i\omega x} + \frac{1}{2}\int_{-1}^{+1} e^{-i\omega x}$

$= \frac{1}{4}\int_{-1}^{+1} e^{x(2i - \omega i)} dx + \frac{1}{4}\int_{-1}^{+1} e^{x(-2i-\omega i)} - \frac{1}{2(i\omega)} \left| e^{-i\omega x} \right|_{-1}^{+1}$

$= \frac{1}{4} \frac{1}{(2-\omega)i} \left| e^{x(2-\omega)i} \right|_{-1}^{+1} - \frac{1}{4} \frac{1}{(2+\omega)i} \left| e^{-x(2+\omega)i} \right|_{-1}^{+1} - \frac{1}{2i\omega}(e^{-i\omega} - e^{+i\omega})$

$= \frac{1}{4i(2-\omega)}(e^{(2-\omega)i} - e^{-(2-\omega)i}) - \frac{1}{4i(2+\omega)}(e^{-(2+\omega)i} - e^{+(2+\omega)i}) + \frac{1}{2i\omega}(e^{i\omega} - e^{-i\omega})$

$= \frac{\sin(2-\omega)}{2(2-\omega)} + \frac{\sin(2+\omega)}{2(2+\omega)} + \frac{\sin(\omega)}{\omega}$

METODO ALTERNATIVO

$f(x) = \frac{1}{4}(e^{2ix}\chi + e^{-2ix}\chi + 2\chi)$

$\hat f(\omega) = \frac{1}{4}(\mathcal{F}(e^{2ix}\chi) + \mathcal{F}(e^{-2ix}\chi) + \mathcal{F}(2\chi))$

$\mathcal{F}(\chi_{(-a,a)}) = \frac{2\sin(a\omega)}{\omega} = \left. \frac{2 \sin \omega}{\omega} \right|_{a=1}$

$\mathcal{F}(u\, e^{iax}) = \hat u(\omega - a)$

$\hat f(\omega) = \frac{1}{4}\left(\frac{2\sin(\omega-2)}{(\omega-2)} + \frac{2\sin(\omega+2)}{(\omega+2)} + \frac{4 \sin \omega}{\omega}\right)$

$= \frac{\sin(2-\omega)}{2(2-\omega)} + \frac{\sin(2+\omega)}{2(2+\omega)} + \frac{\sin(\omega)}{\omega}$

RISOLVI

$u + iu' * e^{-|x|} = f$

$\mathcal{F}(u) = \hat u$

$\mathcal{F}(iu' * e^{-|x|}) = \mathcal{F}(iu')\mathcal{F}(e^{-|x|})$

$\mathcal{F}(iu') = i(i\omega)\hat u = -\omega \hat u$

$\mathcal{F}(e^{-|x|}) = \frac{2}{1+\omega^2}$

$\hat u - \frac{2\omega \hat u}{1+\omega^2} = \left(1 - \frac{2\omega}{1+\omega^2}\right)\hat u = \frac{\omega^2 - 2\omega + 1}{1+\omega^2} \hat u = \hat f$

$\hat u = \frac{1+\omega^2}{(1-\omega)^2}\left(\frac{\sin(2-\omega)}{2(2-\omega)} + \frac{\sin(2+\omega)}{2(2+\omega)} + \frac{\sin(\omega)}{\omega}\right)$

$\hat u \notin L^\infty(\mathbb{R}) \rightarrow$ NO SOLUZIONI IN $L^1(\mathbb{R})$

CALCOLA $I(\omega)$ $\forall \omega \in \mathbb{R}$

$I(\omega) = \int_{\mathbb{R}} \frac{x}{(x^2+2)^2} e^{i(2-\omega^2)x} dx$

NOTEVOLI

$\mathcal{F}(u') = i\omega \mathcal{F}(u)$

$\mathcal{F}\left(\frac{1}{1+(\frac{x}{a})^2}\right) = a\pi e^{-a|\omega|}$

RISOLVIAMO $\quad I(\omega) = \mathcal{F}(f(x)) = \hat f(2 - \omega^2)$

$f(x) = \frac{x}{(x^2+2)^2} = -\frac{1}{2} \frac{\partial}{\partial x}\left(\frac{1}{x^2+2}\right)$

$\hat f(\omega) = \mathcal{F}\left(-\frac{1}{2} \frac{\partial}{\partial x}\left(\frac{1}{x^2+2}\right)\right)$

$= -\frac{1}{2}(i\omega)\mathcal{F}\left(\frac{1}{x^2+2}\right)$

$= -\frac{\omega i}{4} \mathcal{F}\left(\frac{1}{(\frac{x}{\sqrt{2}})^2 + 1}\right)$

$= -\frac{\omega i}{4} \sqrt{2}\pi e^{-\sqrt{2}|\omega|}$

$I(\omega) = \hat f(2 - \omega^2) = -\frac{i\pi(2-\omega^2)}{4} \sqrt{2} e^{-\sqrt{2}|2-\omega^2|}$

$\varphi \in L^\infty(\mathbb{R}) \qquad u(x) = \chi_{(-1,+1)} \varphi(x)$

DETERMINA $\quad K_0 = \sup\{k > 0 : \hat u(x) \in C^k(\mathbb{R})\}$

CRITERIO $\quad x^k u \in L^1(\mathbb{R}) \rightarrow \hat u \in C^k(\mathbb{R})$

APPARTENENZA AD $L^1(\mathbb{R})$

$\int_{-1}^{+1} \varphi(x) x^k$ $\quad \forall K$ POICHE $\varphi(x) \in L^\infty \rightarrow \varphi(x)$ LIMITATA $\qquad K_0 = +\infty$

TRASFORMA

$u(x) = \cos x\, \chi_{(-1,+1)}(x)$

$\cos x = \frac{e^{ix} + e^{-ix}}{2}$

$\mathcal{F}(\chi_{(-a,a)}) = \left. \frac{2\sin(a\omega)}{\omega} \right|_{a=1} = \frac{2\sin(\omega)}{\omega}$

$\mathcal{F}(u \cdot e^{iax}) = \hat u(\omega - a)$

$\hat u(x) = \frac{1}{2}(\mathcal{F}(\chi e^{+ix}) + \mathcal{F}(\chi e^{-ix}))$

$= \frac{1}{2}(\hat\chi(\omega - 1) + \hat\chi(\omega+1))$

$= \frac{1}{2}\left(\frac{2\sin(\omega-1)}{\omega-1} + \frac{2\sin(\omega+1)}{\omega+1}\right)$

$= \frac{\sin(\omega-1)}{\omega-1} + \frac{\sin(\omega+1)}{\omega+1}$

EQUAZIONE $\quad u - 4u'' = \varphi \qquad x \in \mathbb{R} \qquad \varphi \in L^2(\mathbb{R})$

ESISTENZA ED UNICITA' SOLUZIONI

$\mathcal{F}(u) = \hat u$

$\mathcal{F}(u'') = (i\omega)^2 \hat u = -\omega^2 \hat u$

$\mathcal{F}(\varphi) = \hat\varphi$

$\mathcal{F}^{-1}\left(\frac{1}{1+(2\omega)^2}\right) = \frac{1}{2} e^{-\frac{|x|}{2}}$

$(1 + 4\omega^2)\hat u = \hat\varphi \qquad \hat u = \frac{\hat\varphi}{1+(2\omega)^2}$

$\hat u = \mathcal{F}(\varphi)\, \mathcal{F}(\frac{1}{2} e^{-\frac{|x|}{2}})$

$u = \varphi * \frac{1}{2} e^{-|x|/2}$

$\varphi \in L^2, \ \frac{1}{2} e^{-|x|/2} \in L^2 \rightarrow u \in L^2$

$u'(x) + 3u(x) = H(x)\, e^{-x} \qquad x \in \mathbb{R}$

TROVA FUNZIONI $u \in L^1(\mathbb{R})$

$\mathcal{F}(u) = \hat u$

$\mathcal{F}(u') = i\omega \hat u$

$\mathcal{F}(H(x) e^{-x}) = \frac{1}{1+i\omega}$

$\hat u(i\omega + 3) = \frac{1}{1+i\omega}$

$\hat u = \frac{1}{(3+i\omega)(1+i\omega)}$

$\lim_{\omega \to \infty} \hat u(\omega) = 0 \qquad \hat u \in L^1(\mathbb{R})$

$u'(x) + 3u(x) = H(x)\, e^{-|x|} \qquad x \in \mathbb{R}$

TROVA FUNZIONI $u \in L^1(\mathbb{R})$

$\hat u(i\omega + 3) = \hat f$

$\hat u = \frac{1}{i\omega + 3} \cdot \mathcal{F}(H(x) e^{-|x|})$

$\frac{1}{i\omega + 3} = \mathcal{F}(H(x) e^{-3|x|})$

$\hat u(x) = \mathcal{F}(H(x) e^{-3|x|})\, \mathcal{F}(H(x) e^{-|x|})$

$u(x) = H(x) e^{-3|x|} * H(x) e^{-|x|}$

CONVOLUZIONE

$u(x) = \int_0^x H(y) e^{-3|y|} H(x-y) e^{-|x-y|} dy$

$\left| \begin{array}{l} y > 0 \\ x > y \end{array} \right. \rightarrow x > y > 0 \qquad \underline{x > 0}$

$u(x) = \int_0^x e^{-3y} e^{-x+y} dy \qquad x > 0$

$= \int_0^x e^{-x-2y} dy$

$= -\frac{1}{2} \left| e^{-x-2y} \right|_0^x$

$= -\frac{1}{2}(e^{-3x} - e^{-x})$

$$u(x) = \begin{cases} 0 & x < 0 \\ \frac{1}{2}(e^{-x} - e^{-3x}) & x > 0 \end{cases} \in L^1(\mathbb{R})$$

$2u''(x) - u(x) * e^{-|x|} = \chi_{(-1,1)} \qquad x \in \mathbb{R}$

- $\exists!\ u(x) \in L^2(\mathbb{R})$?

$\mathcal{F}(u) = \hat u$

$\mathcal{F}(u'') = -\omega^2 \hat u$

$\mathcal{F}(e^{-|x|}) = \frac{2}{1+\omega^2}$

$\mathcal{F}(\chi_{(-1,+1)}) = \frac{2\sin(\omega)}{\omega}$

$\mathcal{F}(2u'') - \mathcal{F}(u * e^{-|x|}) = \mathcal{F}(\chi)$

$-2\omega^2 \hat u - \mathcal{F}(u)\mathcal{F}(e^{-|x|}) = \frac{2\sin\omega}{\omega}$

$-2\omega^2 \hat u - \frac{2\hat u}{1+\omega^2} = \frac{2 \sin\omega}{\omega}$

$\hat u\left(-\omega^2 - \frac{1}{1+\omega^2}\right) = \frac{\sin\omega}{\omega}$

$\hat u = -\frac{\sin\omega}{\omega} \frac{1+\omega^2}{\omega^4 + \omega^2 + 1}$

$\lim_{\omega \to \infty} \frac{\omega^2}{\omega^5} \sim \omega^{-3} \quad \rightarrow \hat u \in L^2(\mathbb{R})$

$\hat u$ TRASFORMATA DI u UNICA IN $L^2(\mathbb{R})$

STUDIO

$\hat u$ PARI E REALE $\rightarrow u$ PARI E REALE

$u \in L^2(\mathbb{R}) \qquad \forall \omega \in \mathbb{R}$

$(\mathcal{F}u)(\omega) = (\sin\omega) \cdot \chi_{(-\pi,\pi)}(\omega)$

HO TROVATO UN MODO BELLISSIMO DI
RISOLVERLO MA PURTROPPO MI E' MORTO
IL GATTO FELIX, POI LO SCRIVO GIURO

$\varphi \in L^1(\mathbb{R}) \qquad e^{-x^2} = \int_{\mathbb{R}} e^{-|x-y|} \varphi(y)\, dy$

DETERMINA $\hat\varphi(\omega)$

$e^{-x^2} = e^{-|x|} * \varphi(x)$

$\mathcal{F}(e^{-x^2}) = \mathcal{F}(e^{-|x|})\, \mathcal{F}(\varphi(x))$

$\sqrt{\pi} e^{-\omega^2/4} = \frac{2}{1+\omega^2} \hat\varphi$

$\hat\varphi(\omega) = \frac{\sqrt{\pi}}{2}(1+\omega^2) e^{-\omega^2/4}$

MOSTRA CHE $\varphi \in S$

$$\left. \begin{array}{ll} \hat\varphi \in L^1(\mathbb{R}) & \omega^N \hat\varphi \in L^1(\mathbb{R}) \ \forall N \\ \hat\varphi \in C^\infty(\mathbb{R}) & \hat\varphi^{(N)} = P(\omega) e^{-\omega^2/4} \end{array} \right\} \quad x^N \varphi(x) \rightarrow 0 \qquad \varphi \in S$$

CALCOLO $\varphi(x)$

C'E' UNA FORMULA ELEGANTISSIMA
PER CALCOLARE φ MA NON CI STA'
NEL MARGINE DI QUESTA PAGINA

SOLUZIONI IN $L^2(\mathbb{R})$

$u'' + u = (4x^2 - 1) e^{-x^2}$

$\mathcal{F}(u'') + \mathcal{F}(u) = 4\mathcal{F}(x^2 e^{-x^2}) - \mathcal{F}(e^{-x^2})$

$\mathcal{F}(u) = \hat u$

$\mathcal{F}(u'') = (i\omega)^2 \hat u = -\omega^2 \hat u$

$\mathcal{F}(e^{-x^2}) = \sqrt{\pi} e^{-\omega^2/4}$

$\mathcal{F}(xu) = i \frac{\partial}{\partial\omega} \hat u$

$\mathcal{F}(x^2 u) = -\frac{\partial^2}{\partial\omega^2} \hat u$

TRASFORMATA

$\hat u(1 - \omega^2) = -4\frac{\partial^2}{\partial\omega^2}(\sqrt{\pi} e^{-\omega^2/4}) - \sqrt{\pi} e^{-\omega^2/4}$

$\hat u(1 - \omega^2) = -4\sqrt{\pi}(e^{-\omega^2/4})'' - \sqrt{\pi} e^{-\omega^2/4}$

$\frac{\partial}{\partial\omega} e^{-\omega^2/4} = -\frac{\omega}{2} e^{-\omega^2/4}$

$\frac{\partial^2}{\partial\omega^2} e^{-\omega^2/4} = -\frac{1}{2} e^{-\omega^2/4} + \frac{\omega^2}{4} e^{-\omega^2/4}$

$\hat u(1 - \omega^2) = -4\sqrt{\pi}\left(-\frac{1}{2} e^{-\omega^2/4}\right) - 4\sqrt{\pi}\left(\frac{\omega^2}{4} e^{-\omega^2/4}\right) - \sqrt{\pi} e^{-\omega^2/4}$

$= +2\sqrt{\pi} e^{-\omega^2/4} - \sqrt{\pi}\omega^2 e^{-\omega^2/4} - \sqrt{\pi} e^{-\omega^2/4}$

$= \sqrt{\pi} e^{-\omega^2/4}(1 - \omega^2)$

$\hat u = \sqrt{\pi} e^{-\omega^2/4} \in L^2(\mathbb{R}) \quad \rightarrow \exists!\ u(x) \in L^2(\mathbb{R})$

TROVA SOLUZIONE $u(x) \in L^1(\mathbb{R})$

$(u * e^{-x^2})' + 2x(u * e^{-x^2}) = 0$

ORDINARIA $\quad f = u * e^{-x^2} \qquad f' + 2xf = 0$

TRASFORMO

$\mathcal{F}(f) = \hat f$ $\qquad$ NON E' LA VIA GIUSTA

~~$\mathcal{F}(f') = i\omega \hat f$~~

~~$\mathcal{F}(2xf) = 2\mathcal{F}(xf) = 2i\frac{\partial}{\partial\omega}(\hat f)$~~

~~$i\omega \hat f + 2i\frac{\partial}{\partial\omega}\hat f = 0$~~

RISOLVO L'ORDINARIA

$f'(x) + a(x) f(x) = 0 \qquad a(x) = 2x \qquad \int a(x) = x^2$

$f(x) = K e^{-\int a(x)} = K e^{-x^2}$

$\hat f(\omega) = K\sqrt{\pi} e^{-\omega^2/4}$

RISOLVO PER $u(x)$

f RISOLVE LA DIFFERENZIALE

$f(x) = u(x) * e^{-x^2}$

$\mathcal{F}(f(x)) = K\cancel{\mathcal{F}(e^{-x^2})} = \mathcal{F}(u) \cdot \cancel{\mathcal{F}(e^{-x^2})}$

$\hat u = K$ $\quad$ NON TENDE A ZERO AD ∞ PER $K \neq 0$

$\exists!\ u(x) = 0 \in L^1(\mathbb{R})$

$u \in L^1(\mathbb{R}) \qquad$ DIMOSTRA

$\mathcal{F}(u(a^{-1}x))(\omega) = a\hat u(a\omega)$

$\mathcal{F}(u(a^{-1}x))(\omega) = \int_{-\infty}^{+\infty} u(a^{-1}x) e^{-i\omega x}$

SOSTITUISCO $\quad y = a^{-1}x \qquad dx = a\, dy$

$= a\int_{-\infty}^{+\infty} u(y) e^{-i\omega a y} dy = a\hat u(a\omega)$

CALCOLA $\mathcal{F}(2^{-x^2})$

$2^{-x^2} = e^{\ln 2^{-x^2}} = e^{-x^2 \ln 2} = e^{-(\sqrt{\ln 2}\, x)^2}$

$a^{-1} = \sqrt{\ln 2}$

$\mathcal{F}(e^{-x^2}) = \sqrt{\pi} e^{-\omega^2/4}$

$\mathcal{F}(2^{-x^2}) = \frac{1}{\sqrt{\ln 2}} \mathcal{F}(e^{-x^2})\left(\frac{\omega}{\sqrt{\ln 2}}\right)$

$= \frac{1}{\sqrt{\ln 2}} \sqrt{\pi} e^{-\frac{\omega^2}{4 \ln 2}}$

SOLUZIONI IN $L^2(\mathbb{R})$

$u - \frac{1}{4} e^{-|x|} * u = x e^{-|x|}$

$\mathcal{F}(u) = \hat u$

$\mathcal{F}(-\frac{1}{4} e^{-|x|} * u) = -\frac{1}{4}\mathcal{F}(e^{-|x|})\mathcal{F}(u) = -\frac{1}{4}\frac{2}{1+\omega^2}\hat u$

$\mathcal{F}(x e^{-|x|}) = i\frac{\partial}{\partial\omega}\mathcal{F}(e^{-|x|}) = i\frac{\partial}{\partial\omega}\left(\frac{2}{1+\omega^2}\right)$

$\hat u - \frac{1}{2(1+\omega^2)}\hat u = i\frac{\partial}{\partial\omega}\left(\frac{2}{1+\omega^2}\right)$

$\hat u\left(1 - \frac{1}{2(1+\omega^2)}\right) = -2i(2\omega)(1+\omega^2)^{-2}$

$\hat u \frac{2\omega^2 + 1}{2(1+\omega^2)} = \frac{-4i\omega}{(1+\omega^2)^2}$

$\hat u = \frac{-4i\omega}{(1+\omega^2)^2} \cdot \frac{2}{2\omega^2+1} = \frac{-8i\omega}{(1+\omega^2)(2\omega^2+1)} \in L^2(\mathbb{R})$

FRATTI SEMPLICI

$\hat u = -8i\omega\left(\frac{A}{1+\omega^2} + \frac{B}{2\omega^2+1}\right)$

$2\omega^2 A + A + B + \omega^2 B = 1$

$$\begin{array}{l|l} \omega^0 & A + B = 1 \\ \omega^1 & 0 \ \checkmark \\ \omega^2 & 2A + B = 0 \end{array} \quad \left| \begin{array}{l} -A = 1 \\ - \\ B = -2A \end{array} \right. \quad \left| \begin{array}{l} A = -1 \\ B = +2 \end{array} \right.$$

$\hat u = -8i\omega\left(\frac{-1}{1+\omega^2} + \frac{+2}{2\omega^2+1}\right)$

$\hat u = +4\omega i\underbrace{\left(\frac{2}{1+\omega^2}\right)}_{\hat g_1} - 4\omega i\underbrace{\left(\frac{4}{1+(\sqrt{2}\omega)^2}\right)}_{\hat g_2}$

SAPENDO CHE $\quad \mathcal{F}(g') = \omega i \hat g$

$\hat g_1 = \frac{2}{1+\omega^2} \rightarrow g_1 = e^{-|x|}$

$\hat g_2 = \frac{2\sqrt{2}}{1+(\sqrt{2}\omega)^2} \rightarrow g_2 = e^{-\frac{|x|}{\sqrt{2}}}$

$u(x) = 4\frac{\partial}{\partial x}(e^{-|x|}) - 4\frac{\partial}{\partial x}(e^{-\frac{|x|}{\sqrt{2}}})$

$= \dots = 4\operatorname{sign}(x) \cdot (e^{-\frac{|x|}{\sqrt{2}}} - e^{-|x|})$

FOURIER PARAMETRICO

1) CALCOLA $\mathcal{F}(e^{-a|x|}) \qquad a \in \mathbb{R}^+$

PER DEFINIZIONE

$\mathcal{F}(e^{-a|x|}) = \int_{-\infty}^{+\infty} e^{-a|x|} e^{-i\omega x} dx$

$= \int_{-\infty}^{0} e^{x(a-i\omega)} dx + \int_0^\infty e^{x(-a-i\omega)}$

$= \left| \frac{1}{a-i\omega} e^{x(a-i\omega)} \right|_{-\infty}^{0} + \left| \frac{-1}{a+i\omega} e^{x(-a-i\omega)} \right|_0^\infty$

$= \frac{1}{a-i\omega} + \frac{1}{a+i\omega}$

$= \frac{2a}{a^2+\omega^2} = \frac{2}{a}\frac{1}{1+(\frac{\omega}{a})^2}$

$\rightarrow \mathcal{F}(e^{-a|x|}) = \frac{2a}{a^2+\omega^2}$

FOURIER PARAMETRICO

2) CALCOLA $\mathcal{F}\left(\frac{1}{1+(ax)^2}\right)$

PER DEFINIZIONE

$\mathcal{F}(u) = \int_{-\infty}^{+\infty} \frac{1}{1+(ax)^2} e^{-i\omega x} dx$

INTEGRALE IN PIANO COMPLESSO

$\mathcal{F}(u) = \int_\gamma \frac{1}{1+(az)^2} e^{-i\omega z} dz$

SINGOLARITA'

$1 + a^2z^2 = 0 \qquad a^2z^2 = -1 \qquad z = \pm\frac{1}{a}i$

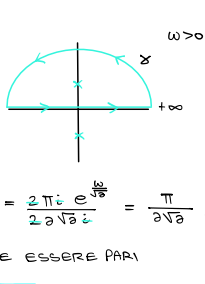
POLI SEMPLICI (GRADO 1)

RESIDUI

$RES(z_k) = \frac{e^{-i\omega z}}{2a^2 z} = \frac{e^{-i\omega(\pm\frac{i}{a})}}{2a^2(\pm\frac{i}{a})} = \pm\frac{e^{\pm\frac{\omega}{a}}}{2ai}$

LEMMA DI JORDAN

APPLICO IL TDR
NEL SEMIPIANO Im⁺



INTEGRALE

$\mathcal{F}(u) = 2\pi i\, RES(+\frac{i}{a}) = \frac{2\pi i\, e^{\frac{\omega}{a}}}{2ai} = \frac{\pi}{a} e^{\frac{\omega}{a}}$

POICHE LA FUNZIONE DEVE ESSERE PARI

$\rightarrow \mathcal{F}(u) = \frac{\pi}{a} e^{-\frac{1}{a}|\omega|}$

FOURIER PARAMETRICO

3) CALCOLA $\mathcal{F}\left(\frac{1}{a^2+x^2}\right)$

PER DEFINIZIONE

$\mathcal{F}(u) = \int_{-\infty}^{+\infty} \frac{1}{a^2+x^2} e^{-i\omega x} dx$

INTEGRALE IN PIANO COMPLESSO

$\mathcal{F}(u) = \int_\gamma \frac{1}{a^2+z^2} e^{-i\omega z} dz$

SINGOLARITA'

$a^2 + z^2 = 0 \qquad z = \pm ai$

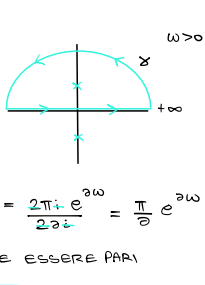
POLI SEMPLICI (GRADO 1)

RESIDUI

$RES(z_k) = \frac{e^{-i\omega z}}{2z} = \frac{e^{-i\omega(\pm ai)}}{2(\pm ai)} = \pm\frac{e^{\pm a\omega}}{2ai}$

LEMMA DI JORDAN

APPLICO IL TDR
NEL SEMIPIANO Im⁺



INTEGRALE

$\mathcal{F}(u) = 2\pi i\, RES(+a) = \frac{2\pi i\, e^{a\omega}}{2ai} = \frac{\pi}{a} e^{a\omega}$

POICHE LA FUNZIONE DEVE ESSERE PARI

$\rightarrow \mathcal{F}(u) = \frac{\pi}{a} e^{-a|\omega|}$

FOURIER PARAMETRICO

4) CALCOLA $\mathcal{F}(H(x) \cdot e^{-ax}) \qquad a \in \mathbb{R}^+$

PER DEFINIZIONE

$\mathcal{F}(u) = \int_{-\infty}^{+\infty} H(x) e^{-ax} e^{-i\omega x} dx$

$= \int_0^\infty e^{x(-a-i\omega)} dx$

$= -\frac{1}{a+i\omega} \left| e^{x(-a-i\omega)} \right|_0^\infty$

$= +\frac{1}{a+i\omega}$

$\rightarrow \mathcal{F}(u) = \frac{1}{a+i\omega}$

TRUCCHETTO ALLA "DIO E' MORTO"

CALCOLA $\mathcal{F}(e^{-x^2})$

$u = e^{-x^2} \qquad u' = -2x e^{-x^2}$

$\mathcal{F}(u') = \omega i \mathcal{F}(e^{-x^2}) = \omega i \hat u$

$\mathcal{F}(-2x e^{-x^2}) = -2\mathcal{F}(x e^{-x^2}) = -2i\frac{\partial}{\partial\omega}\mathcal{F}(e^{-x^2}) = -2i\hat u'$

EQUAZIONE DIFFERENZIALE

$\omega i \hat u = -2i \hat u'$

$+2i\hat u' + \omega i \hat u = 0$

$\hat u' + \frac{\omega}{2}\hat u = 0$

$\hat u(\omega) = K e^{-\int \frac{\omega}{2} d\omega} = K e^{-\omega^2/4}$

CONDIZIONE INIZIALE

$K = \hat u(0) = \int_{-\infty}^{+\infty} e^{-t^2} dt = \sqrt{\pi}$

$\rightarrow \mathcal{F}(e^{-x^2}) = \sqrt{\pi} e^{-\omega^2/4}$

CALCOLA TRASFORMATA

$u_N(x) = N\, \chi_{(0,\frac{1}{N})} \cos x$

PER DEFINIZIONE

$\hat u_N(x) = \int_{-\infty}^{+\infty} N \chi_{(0,\frac{1}{N})} \cos x\, e^{-i\omega x} dx$

DOVE $\quad \cos x = \frac{e^{ix} + e^{-ix}}{2}$

$\hat u_N(x) = \frac{N}{2}\int_0^{1/N} e^{xi(1-\omega)} dx + \frac{N}{2}\int_0^{1/N} e^{xi(-1-\omega)} dx$

$= \frac{N}{2}\left(\frac{1}{i(1-\omega)} \left| e^{xi(1-\omega)} \right|_0^{1/N} - \frac{1}{i(1+\omega)} \left| e^{xi(-1-\omega)} \right|_0^{1/N}\right)$

$= \frac{N}{2}\left(\frac{1}{i(1-\omega)}\left(e^{\frac{i(1-\omega)}{N}} - 1\right) - \frac{1}{i(1+\omega)}\left(e^{-\frac{i(1+\omega)}{N}} - 1\right)\right)$

$= \frac{N}{2}\left(\frac{e^{\frac{i(1-\omega)}{N}} - 1}{i(1-\omega)} - \frac{e^{-\frac{i(1+\omega)}{N}} - 1}{i(1+\omega)}\right)$

TRASFORMA $\quad f(x) = \frac{x}{x^2+4}(1 - 4\sin(2x))$

RISCRIVO LA FUNZIONE

$f(x) = x \cdot \frac{1}{x^2+4} - 4x \cdot \frac{1}{x^2+4} \cdot \frac{e^{2ix} - e^{-2ix}}{2i}$

$= \underbrace{x \cdot \frac{1}{x^2+4}}_{A} - \frac{2}{i}\underbrace{x \cdot \frac{1}{x^2+4} e^{2ix}}_{B} + \frac{2}{i}\underbrace{x \cdot \frac{1}{x^2+4} e^{-2ix}}_{C}$

TRASFORMO A

$\mathcal{F}(xu) = i\frac{\partial}{\partial\omega}\hat u$

$\mathcal{F}\left(\frac{1}{x^2+a^2}\right) = \frac{\pi}{a} e^{-a|\omega|}$

$A = i\frac{\partial}{\partial\omega}\left(\frac{\pi}{2} e^{-2|\omega|}\right) = -i\pi \operatorname{sign}(\omega) e^{-2|\omega|}$

TRASFORMO B

$\mathcal{F}(xu) = i\frac{\partial}{\partial\omega}\hat u$

$\mathcal{F}(u \cdot e^{ibx}) = \hat u(\omega - b)$

$\mathcal{F}\left(\frac{1}{x^2+a^2}\right) = \frac{\pi}{a} e^{-a|\omega|}$

$B = i\frac{\partial}{\partial\omega}\hat u(\omega - b) = i\frac{\partial}{\partial\omega}\left(\frac{\pi}{a} e^{-a|\omega-b|}\right)$

$= -i\frac{\pi}{a} a \operatorname{sign}(\omega - b) e^{-a|\omega-b|}$

$= -i\pi \operatorname{sign}(\omega - 2) e^{-2|\omega-2|}$

TRASFORMO C

$\mathcal{F}(xu) = i\frac{\partial}{\partial\omega}\hat u$

$\mathcal{F}(u \cdot e^{ibx}) = \hat u(\omega - b)$

$\mathcal{F}\left(\frac{1}{x^2+a^2}\right) = \frac{\pi}{a} e^{-a|\omega|}$

$B = i\frac{\partial}{\partial\omega}\hat u(\omega - b) = i\frac{\partial}{\partial\omega}\left(\frac{\pi}{a} e^{-a|\omega-b|}\right)$

$= -i\frac{\pi}{a} a \operatorname{sign}(\omega - b) e^{-a|\omega-b|}$

$= -i\pi \operatorname{sign}(\omega + 2) e^{-2|\omega+2|}$

TRASFORMATA DI $f(x)$

$\hat f(\omega) = -\pi i \operatorname{sign}(\omega) e^{-2|\omega|} + 2\pi \operatorname{sign}(\omega - 2) e^{-2|\omega-2|} - 2\pi \operatorname{sign}(\omega+2) e^{-2|\omega+2|}$

$f(x) = \chi_{(-\pi,\pi)} \sin x$

SENZA CALCOLARE $\hat f$

1) $\hat f \in L^\infty(\mathbb{R})$?

$f \in L^1 \rightarrow \hat f \in L^\infty \quad \checkmark$

DIMOSTRIAMO CHE $f \in L^1$

$\|f\|_1 = \int_{-\pi}^{+\pi} \sin x = -\cos(\pi) + \cos(-\pi) = 0$

2) $\hat f \in L^2(\mathbb{R})$?

$f \in L^2 \rightarrow \hat f \in L^2 \quad \checkmark$

DIMOSTRIAMO CHE $f \in L^2$

$\|f\|_2^2 = \int_{-\pi}^{+\pi} \sin^2 x < \infty$

3) PER QUALI $K \in \mathbb{N}$ $\quad \hat f \in C^K(\mathbb{R})$?

$x^k f \in L^1 \rightarrow \hat f \in C^k$

SI HA CHE $\hat f \in C^k$ $\forall K$ POICHE

$\int_{-\pi}^{+\pi} \sin x\, x^k < \infty \rightarrow x^k f \in L^1$

4) TRASFORMA $f(x)$

$f(x) = \chi_{(-\pi,+\pi)} \sin x$

$\hat f(\omega) = \int_{-\pi}^{+\pi} \frac{e^{ix} - e^{-ix}}{2i} e^{-i\omega x} dx$

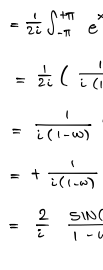
$= \frac{1}{2i}\int_{-\pi}^{+\pi} e^{xi(1-\omega)} dx - \frac{1}{2i}\int_{-\pi}^{+\pi} e^{xi(-1-\omega)} dx$

$= \frac{1}{2i}\left(\frac{1}{i(1-\omega)}\left(e^{\pi i(1-\omega)} - e^{-\pi i(1-\omega)}\right) - \frac{1}{i(1+\omega)}\left(e^{-\pi i(1+\omega)} - e^{\pi i(1+\omega)}\right)\right)$

$= \frac{1}{i(1-\omega)} \sin(\pi - \pi\omega) - \frac{1}{i(1+\omega)} \sin(\pi + \pi\omega)$

$= +\frac{1}{i(1-\omega)} \sin(\omega\pi) + \frac{1}{i(1+\omega)} \sin(\omega\pi)$

$= \frac{2}{i}\frac{\sin(\omega\pi)}{1-\omega^2} = \frac{2i \sin(\omega\pi)}{\omega^2 - 1}$



$\sin(\pi - \alpha) = \sin\alpha$

$\sin(\pi + \alpha) = -\sin\alpha$

SOLUZIONI IN $L^1(\mathbb{R})$

$u'' + u = \chi_{(-1,+1)} \qquad x \in \mathbb{R}$

$\mathcal{F}(u) = \hat u$

$\mathcal{F}(u'') = -\omega^2 \hat u$

$\mathcal{F}(\chi_{(-1,+1)}) = \frac{2\sin(\omega)}{\omega}$

$\hat u(1 - \omega^2) = \frac{2\sin\omega}{\omega}$

$\hat u = \frac{2\sin\omega}{\omega(1-\omega^2)} \notin L^\infty \rightarrow u \notin L^1$

CALCOLA $\quad I = \int_{\mathbb{R}} \frac{e^{i3x}}{4+x^2} dx$

$f(x) = \frac{1}{4+x^2}$

$\mathcal{F}\left(\frac{1}{x^2+a^2}\right) = \left. \frac{\pi}{a} e^{-a|\omega|} \right|_{a=2} = \frac{\pi}{2} e^{-2|\omega|}$

$I = \hat f(-3) = \frac{\pi}{2} e^{-2|-3|} = \frac{\pi}{2} e^{-6}$

TRASFORMA $\quad f(x) = -\frac{x}{(x^2+4)^2}$

$\frac{\partial}{\partial x}\left(\frac{1}{x^2+4}\right) = -\frac{2x}{(x^2+4)^2}$

$f(x) = +\frac{1}{2}\left(\frac{1}{x^2+4}\right)'$

$\mathcal{F}(u') = \omega i \hat u$

$\mathcal{F}\left(\frac{1}{a^2+x^2}\right) = \frac{\pi}{a} e^{-a|\omega|} \qquad (a = 2)$

$\hat f = \omega i\left(\frac{1}{2}\right)\left(\frac{\pi}{2} e^{-2|\omega|}\right) = \frac{\omega\pi i}{4} e^{-2|\omega|}$

SOLUZIONE $u \in L^2(\mathbb{R})$ DI:

$u(x) - \frac{1}{4}\int_{-\infty}^{+\infty} e^{-|x-y|} u(y)\, dy = x e^{-|x|}$

CONVOLUZIONE

$-\frac{1}{4}\int_{\mathbb{R}} e^{-|x-y|} u(y)\, dy = e^{-|x|} * u(x)$

$u - \frac{1}{4} e^{-|x|} * u = x e^{-|x|}$

TRASFORMATA

$\mathcal{F}(u) = \hat u$

$\mathcal{F}(e^{-|x|}) = \frac{2}{1+\omega^2}$

$\mathcal{F}(xu) = i\frac{\partial}{\partial\omega}\hat u$

$\mathcal{F}(x e^{-|x|}) = i\frac{\partial}{\partial\omega}\mathcal{F}(e^{-|x|})$

$\mathcal{F}(-\frac{1}{4} e^{-|x|} * u) = -\frac{1}{4}\mathcal{F}(e^{-|x|})\mathcal{F}(u)$

$\hat u - \frac{1}{\cancel{4}} \frac{\cancel{2}}{1+\omega^2}\hat u = i\frac{\partial}{\partial\omega}\left(\frac{2}{1+\omega^2}\right)$

$\hat u\left(1 - \frac{1}{2(1+\omega^2)}\right) = -2i\frac{2\omega}{(1+\omega^2)^2}$

$\hat u \cdot \frac{2 + 2\omega^2 - 1}{2\cancel{(1+\omega^2)}} = -2i\frac{2\omega}{(1+\omega^2)^{\cancel{2}}}$

$\hat u = \frac{-8i\omega}{(1+\omega^2)(2\omega^2+1)} \in L^2(\mathbb{R}) \qquad \sim \frac{1}{\omega^3} \qquad \underline{3 > 1}$

SVILUPPO HEAVYSIDE

$\hat u = -8i\omega\left(\frac{A}{1+\omega^2} + \frac{B}{2\omega^2+1}\right)$

$A(2\omega^2 + 1) + B(1 + \omega^2) = 1$

$2\omega^2 A + A + B + \omega^2 B = 1$

$$\begin{array}{l|l} \omega^0 & A + B = 1 \\ \omega & 0 = 0 \ \checkmark \\ \omega^2 & 2A + B = 0 \end{array} \quad \left| \begin{array}{l} -A = 1 \\ B = -2A \end{array} \right. \quad \left| \begin{array}{l} A = -1 \\ B = +2 \end{array} \right.$$

$\hat u = -8i\omega\left(\frac{-1}{1+\omega^2} + \frac{+2}{2\omega^2+1}\right)$

ANTITRASFORMATA

$\mathcal{F}^{-1}(\omega i \hat u) = \frac{\partial}{\partial x} u$

$u(x) = +4\frac{\partial}{\partial x}\mathcal{F}^{-1}\left(\frac{2}{1+\omega^2}\right) - 8\frac{\partial}{\partial x}\mathcal{F}^{-1}\left(\frac{2}{2\omega^2+1}\right)$

$\mathcal{F}^{-1}\left(\frac{2a}{a^2+\omega^2}\right) = e^{-a|x|}$

$\mathcal{F}^{-1}\left(\frac{2}{1+\omega^2}\right) = e^{-|x|}$

IN PARTICOLARE

$\mathcal{F}^{-1}\left(\frac{2}{2\omega^2+1}\right) = \mathcal{F}^{-1}\left(\frac{1}{\omega^2 + 1/2}\right) = \frac{\sqrt{2}}{2} e^{-\frac{|x|}{\sqrt{2}}}$

$u(x) = +4\frac{\partial}{\partial x} e^{-|x|} - 8\left(\frac{\sqrt{2}}{2}\right)\frac{\partial}{\partial x} e^{-\frac{|x|}{\sqrt{2}}}$

$= -4\operatorname{sign}(x) e^{-|x|} + 4\operatorname{sign}(x) e^{-\frac{|x|}{\sqrt{2}}}$

$u(x) = 4\operatorname{sign}(x)\left(e^{-\frac{|x|}{\sqrt{2}}} - e^{-|x|}\right) \in L^2(\mathbb{R})$

$a > 0 \qquad$ TRASFORMA $\quad f_a(x)$

$f_a(x) = \frac{2\sin x}{x^2+a^2}$

$f_a(x) = 2 \cdot \frac{1}{x^2+a^2} \cdot \frac{e^{ix} - e^{-ix}}{2i}$

$= \frac{1}{i}\left(\frac{1}{x^2+a^2} e^{xi} - \frac{1}{x^2+a^2} e^{-ix}\right)$

$\mathcal{F}\left(\frac{1}{x^2+a^2}\right) = \frac{\pi}{a} e^{-a|\omega|}$

$\mathcal{F}(u(x) \cdot e^{i\alpha x}) = \hat u(\omega - \alpha)$

$\mathcal{F}(f_a) = \left(\frac{\pi}{a} e^{-a|\omega-1|} - \frac{\pi}{a} e^{-a|\omega+1|}\right)(-i)$

$= \frac{\pi i}{a}\left(e^{-a|\omega+1|} - e^{-a|\omega-1|}\right)$

$a > 0 \qquad$ TRASFORMA $\quad f_a(x)$

$f_a(x) = \frac{x \sin x}{x^2+a^2}$

$f_a(x) = \frac{1}{2i}\left(\frac{x}{x^2+a^2} e^{xi} - \frac{x}{x^2+a^2} e^{-ix}\right)$

$u(x) = \frac{1}{x^2+a^2}$

$\mathcal{F}(u) = \frac{\pi}{a} e^{-a|\omega|}$

$\mathcal{F}(u(x) \cdot e^{i\alpha x}) = \hat u(\omega - \alpha)$

$\mathcal{F}(x \cdot u(x)) = i\frac{\partial}{\partial\omega}\hat u$

$\mathcal{F}(f_a) = \left(i\frac{\partial}{\partial\omega}\hat u(\omega - 1) - i\frac{\partial}{\partial\omega}\hat u(\omega+1)\right)\frac{1}{2i}$

$= \frac{1}{2}\frac{\partial}{\partial\omega}\left(\frac{\pi}{a} e^{-a|\omega-1|}\right) - \frac{1}{2}\frac{\partial}{\partial\omega}\left(\frac{\pi}{a} e^{-a|\omega+1|}\right)$

$= -\frac{a}{2}\frac{\pi}{a}\operatorname{sign}(\omega - 1) e^{-a|\omega-1|} + \frac{a}{2}\frac{\pi}{a}\operatorname{sign}(\omega+1) e^{-a|\omega+1|}$

$= \frac{\pi}{2}\left(\operatorname{sign}(\omega+1) e^{-a|\omega+1|} - \operatorname{sign}(\omega-1) e^{-a|\omega-1|}\right)$