Funzione composta, derivate successive, formula di Taylor

1. Verificare che la funzione $u(x,t) = \dfrac{1}{\sqrt{t}} e^{-\frac{x^2}{4t}}$ soddisfa l'equazione del calore: $\dfrac{\partial u}{\partial t} - \dfrac{\partial^2 u}{\partial x^2} = 0$, per ogni $t > 0, x \in \mathbb{R}$.

2. Sia $h(x,y) = f(x) + g(y) + (x-y)g'(y)$, con $f, g \in \mathcal{C}^2(\mathbb{R})$. Verificare che $(x-y)\dfrac{\partial^2 h}{\partial x \partial y} \equiv \dfrac{\partial h}{\partial y}$.

3. Sia $f(x,y) = \dfrac{x}{\sqrt{1+y}} - y\sqrt{1+x}$. Scrivere il differenziale primo e secondo di f in $(0,0)$.

4. Sia $f(t) = g(a(t), b(t))$, con $a(t), b(t) : \mathbb{R} \to \mathbb{R}$ derivabili, e $g(x,y) = x^2 e^y$. Calcolare $f'(t)$ e $f'(2)$.

5. Sia $f(t) = g(t^2, e^t)$, con $g : \mathbb{R}^2 \to \mathbb{R}$ derivabile. Calcolare $f'(t)$ e $f'(2)$.

6. Sia $f(x,y) = g(x^2 + y^2)$, con $g : \mathbb{R} \to \mathbb{R}$ derivabile. Calcolare $\nabla f(x,y)$ e $\nabla f(2,1)$.

7. Sia $f(x,y) = \log(g(x,y))$, con $g : \mathbb{R}^2 \to \mathbb{R}$ derivabile e positiva. Calcolare $\nabla f(x,y)$ e $\nabla f(1,0)$.

8. Sia $f(x,y) = yg(x^2 - y^2)$, dove $g : \mathbb{R} \to \mathbb{R},\ g \in \mathcal{C}^1(\mathbb{R})$. Dimostrare che $\dfrac{1}{x}\dfrac{\partial f}{\partial x} + \dfrac{1}{y}\dfrac{\partial f}{\partial y} \equiv \dfrac{f}{y^2}$.

9. Sia $f(t) = g(a(t), t)$, con $g \in \mathcal{C}^2(\mathbb{R}^2)$, e $a \in \mathcal{C}^2(\mathbb{R})$. Calcolare $f''(t)$.

10. Sia $f \in \mathcal{C}(\mathbb{R}^2)$, e sia $F(u,v) = f(u+v, u-v)$. Verificare che: $\dfrac{\partial^2 F}{\partial u \partial v} = \dfrac{\partial^2 f}{\partial x^2} - \dfrac{\partial^2 f}{\partial y^2}$.

11. Sia $f(x,y) \in \mathcal{C}^2(\mathbb{R})$, e siano $x = \rho\cos\theta, y = \rho\sin\theta$. Calcolare le derivate parziali di $F(\rho,\theta) = f(\rho\cos\theta, \rho\sin\theta)$; calcolare $\dfrac{\partial^2 F}{\partial \rho^2}$.

12. Scrivere lo sviluppo di Taylor per la funzione $f(x,y) = x^y$, centrato in $(1,1)$, arrestato al secondo ordine.

13. Scrivere lo sviluppo di MacLaurin per la funzione $f(x,y) = \sin x \sin y$, arrestato al secondo ordine.

14. Scrivere lo sviluppo di Taylor per la funzione $f(x,y) = (x+y)\sin y$, centrato in $\left(0, \frac{\pi}{2}\right)$, arrestato al secondo ordine.

15. Sia

$$f(x,y) = \begin{cases} xy\dfrac{x^2-y^2}{x^2+y^2} & \text{se } (x,y) \neq (0,0) \\ 0 & \text{se } (x,y) = (0,0) \end{cases}$$

Verificare che $\dfrac{\partial^2 f}{\partial x \partial y}(0,0) \neq \dfrac{\partial^2 f}{\partial y \partial x}(0,0)$.

16. Sia $F(x,y,z) = f(r(x,y,z))$, dove $r(x,y,z) = \sqrt{x^2+y^2+z^2}$ e f è una funzione derivabile di una variabile. Si dimostri che

$$\Delta F = \frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial y^2} + \frac{\partial^2 F}{\partial z^2} = f''(r) + \frac{2}{r}f'(r).$$

Soluzioni.

1. $\dfrac{\partial u}{\partial t} = -\dfrac{1}{2}\dfrac{1}{\sqrt{t^3}}e^{-\frac{x^2}{4t}} + \dfrac{1}{\sqrt{t}}e^{-\frac{x^2}{4t}}\dfrac{x^2}{4t^2}$; $\dfrac{\partial u}{\partial x} = \dfrac{1}{\sqrt{t}}e^{-\frac{x^2}{4t}}\left(-\dfrac{x}{2t}\right)$; $\dfrac{\partial^2 u}{\partial x^2} = -\dfrac{1}{\sqrt{t}}\dfrac{1}{2t}e^{-\frac{x^2}{4t}} + \dfrac{1}{\sqrt{t}}e^{-\frac{x^2}{4t}}\left(-\dfrac{x}{2t}\right)^2$.

2. $\dfrac{\partial h}{\partial y} = g'(y) - g'(y) + (x-y)g''(y) = (x-y)g''(y)$; $\dfrac{\partial h}{\partial x} = f'(x) + g'(y)$; $\dfrac{\partial^2 h}{\partial x \partial y} = g''(y)$.

3. $\dfrac{\partial f}{\partial x} = \dfrac{1}{\sqrt{1+y}} - \dfrac{1}{2}\dfrac{y}{\sqrt{1+x}}$; $\dfrac{\partial f}{\partial y} = -\dfrac{1}{2}\dfrac{x}{\sqrt{(1+y)^3}} - \sqrt{1+x}$; $\dfrac{\partial^2 f}{\partial x^2} = \dfrac{1}{4}\dfrac{y}{\sqrt{(1+x)^3}}$; $\dfrac{\partial^2 f}{\partial x \partial y} = -\dfrac{1}{2}\dfrac{1}{\sqrt{(1+y)^3}} - \dfrac{1}{2}\dfrac{1}{\sqrt{1+x}}$; $\dfrac{\partial^2 f}{\partial y^2} = \dfrac{3}{4}x\dfrac{1}{\sqrt{(1+y)^5}}$. Dunque $df(0,0) = \dfrac{\partial f}{\partial x}(0,0)dx + \dfrac{\partial f}{\partial y}(0,0)dy = dx - dy$; $d^2 f(0,0) = \dfrac{\partial^2 f}{\partial x^2}(0,0)dx^2 + 2\dfrac{\partial^2 f}{\partial x \partial y}(0,0)dxdy + \dfrac{\partial^2 f}{\partial y^2}(0,0)dy^2 = -2dxdy$.

4. $f'(t) = \dfrac{\partial g}{\partial x}(a(t), b(t))a'(t) + \dfrac{\partial g}{\partial y}(a(t), b(t))b'(t) = 2a(t)e^{b(t)}a'(t) + a^2(t)e^{b(t)}b'(t)$. $f'(2) = 2a(2)e^{b(2)}a'(2) + a^2(2)e^{b(2)}b'(2)$.

5. $f'(t) = \dfrac{\partial g}{\partial x}(t^2, e^t)2t + \dfrac{\partial g}{\partial y}(t^2, e^t)e^t$. $f'(2) = 4\dfrac{\partial g}{\partial x}(4, e^2) + e^2\dfrac{\partial g}{\partial y}(4, e^2)$.

6. $\dfrac{\partial f}{\partial x} = g'(x^2+y^2)2x$, $\dfrac{\partial f}{\partial y} = g'(x^2+y^2)2y$. $\nabla f(x,y) = (g'(x^2+y^2)2x, g'(x^2+y^2)2y)$. $\nabla f(2,1) = (4g'(5), 2g'(5))$.

7. $\dfrac{\partial f}{\partial x} = \dfrac{\frac{\partial g}{\partial x}(x,y)}{g(x,y)}$, $\dfrac{\partial f}{\partial y} = \dfrac{\frac{\partial g}{\partial y}(x,y)}{g(x,y)}$. $\nabla f(x,y) = \left(\dfrac{\frac{\partial g}{\partial x}(x,y)}{g(x,y)}, \dfrac{\frac{\partial g}{\partial y}(x,y)}{g(x,y)}\right)$. $\nabla f(1,0) = \left(\dfrac{\frac{\partial g}{\partial x}(1,0)}{g(1,0)}, \dfrac{\frac{\partial g}{\partial y}(1,0)}{g(1,0)}\right)$.

8.Si ha che: $\dfrac{\partial g}{\partial x}(x,y) = 2xg'(x^2-y^2)$, $\dfrac{\partial g}{\partial y}(x,y) = -2yg'(x^2-y^2)$, $\dfrac{\partial f}{\partial x}(x,y) = y\dfrac{\partial g}{\partial x}(x^2-y^2)$, $\dfrac{\partial f}{\partial y}(x,y) = g(x^2-y^2) + y\dfrac{\partial g}{\partial y}(x^2-y^2)$. Quindi $\dfrac{1}{x}\dfrac{\partial f}{\partial x}(x,y) + \dfrac{1}{y}\dfrac{\partial f}{\partial y}(x,y) = \dfrac{1}{x}yg'(x^2-y^2)2x + \dfrac{1}{y}(g(x^2-y^2) + yg'(x^2-y^2)(-2y) = \dfrac{g(x^2-y^2)}{y} = \dfrac{f(x,y)}{y^2}$.

9. $f'(t) = \dfrac{\partial g}{\partial x}(a(t),t)a'(t) + \dfrac{\partial g}{\partial y}(a(t),t)$, $f''(t) = \left[\dfrac{\partial^2 g}{\partial x^2}(a(t),t)a'(t) + \dfrac{\partial^2 g}{\partial x\partial y}(a(t),t)\right] a'(t) + \dfrac{\partial g}{\partial x}(a(t),t)a''(t) + \dfrac{\partial^2 g}{\partial x\partial y}(a(t),t)a'(t) + \dfrac{\partial^2 g}{\partial y^2}(a(t),t) = \dfrac{\partial^2 g}{\partial x^2}a'^2 + 2\dfrac{\partial^2 g}{\partial x\partial y}a' + \dfrac{\partial g}{\partial x}a'' + \dfrac{\partial^2 g}{\partial y^2}$.

10. $\dfrac{\partial F}{\partial u}(u,v) = \dfrac{\partial f}{\partial x}(u+v, u-v) + \dfrac{\partial f}{\partial y}(u+v, u-v)$; $\dfrac{\partial^2 F}{\partial u\partial v}(u,v) = \dfrac{\partial^2 f}{\partial x^2}(u+v, u-v) - \dfrac{\partial^2 f}{\partial x\partial y}(u+v, u-v) + \dfrac{\partial^2 f}{\partial y\partial x}(u+v, u-v) - \dfrac{\partial^2 f}{\partial y^2}(u+v, u-v) = \dfrac{\partial^2 f}{\partial x^2}(u+v, u-v) - \dfrac{\partial^2 f}{\partial y^2}(u+v, u-v)$.

11. $\dfrac{\partial F}{\partial \rho}(\rho,\theta) = \dfrac{\partial f}{\partial x}(\rho\cos\theta, \rho\sin\theta)\cos\theta + \dfrac{\partial f}{\partial y}(\rho\cos\theta, \rho\sin\theta)\sin\theta;$
$\dfrac{\partial F}{\partial \theta} = \dfrac{\partial f}{\partial x}(\rho\cos\theta, \rho\sin\theta)(-\rho\sin\theta) + \dfrac{\partial f}{\partial y}(\rho\cos\theta, \rho\sin\theta)\rho\cos\theta.$
$\dfrac{\partial^2 F}{\partial \rho^2} = \left(\dfrac{\partial^2 f}{\partial x^2}\cos\theta + \dfrac{\partial^2 f}{\partial x \partial y}\sin\theta\right)\cos\theta + \left(\dfrac{\partial^2 f}{\partial x \partial y}\cos\theta + \dfrac{\partial^2 f}{\partial y^2}\sin\theta\right)\sin\theta = \dfrac{\partial^2 f}{\partial x^2}\cos^2\theta +$
$\dfrac{\partial^2 f}{\partial y^2}\sin^2\theta + 2\dfrac{\partial^2 f}{\partial x \partial y}\cos\theta\sin\theta.$

12. $f(x,y) = 1 + (x-1) + \dfrac{1}{2}\{2(x-1)(y-1)\} + o((x-1)^2 + (y-1)^2) = 1 - y + xy + o((x-1)^2 + (y-1)^2\}.$

13. $f(x,y) = xy + o(x^2+y^2).$

14. $f(x,y) = \dfrac{\pi}{2} + x + (y - \dfrac{\pi}{2}) + \dfrac{1}{2}\left\{-\dfrac{\pi}{2}\left(y-\dfrac{\pi}{2}\right)^2\right\} + o\left(x^2 + \left(y - \dfrac{\pi}{2}\right)^2\right\} =$
$x + y - \dfrac{\pi}{4}\left(y - \dfrac{\pi}{2}\right)^2 o\left(x^2 + \left(y-\dfrac{\pi}{2}\right)^2\right).$

15. Si ha che:

$$\frac{\partial f}{\partial x} = \begin{cases} \dfrac{(3x^2y - y^3)(x^2+y^2) - 2x(x^3y - xy^3)}{(x^2+y^2)^2} & \text{se } (x,y) \neq (0,0) \\ 0 & \text{se } (x,y) = (0,0) \end{cases}$$

$$\frac{\partial f}{\partial y} = \begin{cases} \dfrac{(x^3 - 3xy^2)(x^2+y^2) - 2y(x^3y - xy^3)}{(x^2+y^2)^2} & \text{se } (x,y) \neq (0,0) \\ 0 & \text{se } (x,y) = (0,0) \end{cases}$$

Siccome $\dfrac{\partial f}{\partial x}(0,y) = -y$, per ogni y (anche per $y=0$!!), si ha che: $\dfrac{\partial}{\partial y}\dfrac{\partial f}{\partial x}(0,0) = -1$. Siccome $\dfrac{\partial f}{\partial y}(x,0) = x$, per ogni x (anche per $x = 0$!!), si ha che: $\dfrac{\partial}{\partial x}\dfrac{\partial f}{\partial y}(0,0) = 1$.

16. Applicando la formula di derivazione della funzione composta si trova che:

$\dfrac{\partial F}{\partial x} = f'(r)\dfrac{\partial r}{\partial x}, \quad \dfrac{\partial F}{\partial y} = f'(r)\dfrac{\partial r}{\partial y}, \quad \dfrac{\partial F}{\partial z} = f'(r)\dfrac{\partial r}{\partial z},$

$\frac{\partial^2 F}{\partial x^2} = f''(r)\left(\frac{\partial r}{\partial x}\right)^2 + f'(r)\frac{\partial^2 r}{\partial x^2}$, $\frac{\partial^2 F}{\partial y^2} = f''(r)\left(\frac{\partial r}{\partial y}\right)^2 + f'(r)\frac{\partial^2 r}{\partial y^2}$, $\frac{\partial^2 F}{\partial z^2} = f''(r)\left(\frac{\partial r}{\partial z}\right)^2 + f'(r)\frac{\partial^2 r}{\partial z^2}$.

Le derivate parziali della funzione $r(x, y, z)$ valgono:

$$\frac{\partial r}{\partial x} = \frac{x}{\sqrt{x^2+y^2+z^2}}, \quad \frac{\partial r}{\partial y} = \frac{y}{\sqrt{x^2+y^2+z^2}}, \quad \frac{\partial r}{\partial z} = \frac{z}{\sqrt{x^2+y^2+z^2}},$$

$$\frac{\partial^2 r}{\partial x^2} = \frac{y^2+z^2}{(x^2+y^2+z^2)^{\frac{3}{2}}}, \quad \frac{\partial^2 r}{\partial y^2} = \frac{x^2+z^2}{(x^2+y^2+z^2)^{\frac{3}{2}}}, \quad \frac{\partial^2 r}{\partial z^2} = \frac{x^2+y^2}{(x^2+y^2+z^2)^{\frac{3}{2}}}$$

È immediato verificare che

$$\left(\frac{\partial r}{\partial x}\right)^2 + \left(\frac{\partial r}{\partial y}\right)^2 + \left(\frac{\partial r}{\partial z}\right)^2 = 1,$$

$$\frac{\partial^2 r}{\partial x^2} + \frac{\partial^2 r}{\partial y^2} + \frac{\partial^2 r}{\partial z^2} = \frac{2}{\sqrt{x^2+y^2+z^2}} = \frac{2}{r}$$

Dunque $\Delta F = f''(r)\left[\left(\frac{\partial r}{\partial x}\right)^2 + \left(\frac{\partial r}{\partial y}\right)^2 + \left(\frac{\partial r}{\partial z}\right)^2\right] + f'(r)\left(\frac{\partial^2 r}{\partial x^2} + \frac{\partial^2 r}{\partial y^2} + \frac{\partial^2 r}{\partial z^2}\right) = f''(r) + \frac{2}{r}f'(r)$