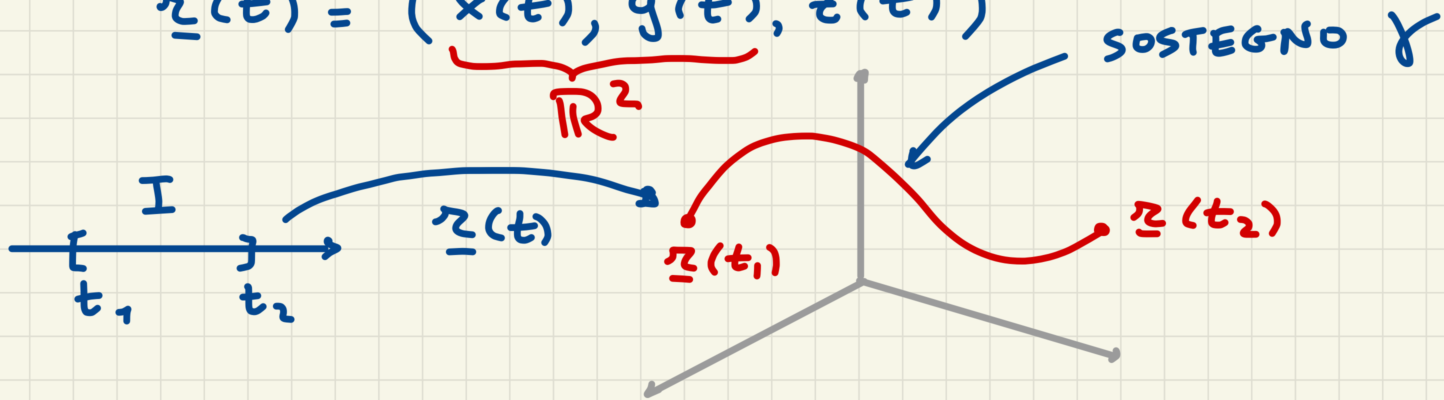
CURVE

12-3-2020

$\underline{r} : I \longrightarrow \mathbb{R}^2$ (NEL PIANO) o $\mathbb{R}^3$

$$\underline{r}(t) = (\underbrace{x(t), y(t)}_{\mathbb{R}^2}, z(t))$$



NOTA. A volte si trova scritto anche $\gamma = \gamma(t)$

- $\underline{r}(t)$ si dice ==REGOLARE== se $\underline{r}(t) \in C^1(I)$ e se $\underline{r}'(t) \neq \underline{0}$ ossia se $|\underline{r}'(t)| \neq 0$ $\forall t \in I$.

ESERCIZIO 1. Stabilire se la curva

$$\underline{r}(t) = (t; 2t; t^2) \qquad t \in [0;4]$$

è chiusa, semplice, regolare.

SOL.

- $\underline{r}(0) = (0;0;0) \neq \underline{r}(4) = (4;8;16)$

$\Rightarrow$ $\underline{r}$ non è chiusa.

- $\underline{r}(t)$ è semplice perché almeno una delle sue componenti è monotona
- $\underline{r}'(t) = (1; 2; 2t) \qquad |\underline{r}'(t)| = \sqrt{1+4+4t^2} = \sqrt{5+4t^2} \neq 0 \quad \forall t \in [0,4]$

$\Rightarrow$ $\underline{r}(t)$ è regolare.

ESERCIZIO 2. Sia γ la curva di equazioni parametriche

$$\underline{r}: \begin{cases} x = \cos t + t \operatorname{sen} t \\ y = \operatorname{sen} t - t \cos t \end{cases} \qquad t \in [-\pi, \pi]$$

1) Stabilire se γ è chiusa

2) Determinare il vettore tangente e il suo modulo. γ è regolare?

SOL.

1) γ è chiusa se $\underline{r}(-\pi) = r(\pi)$

$$\underline{r}(-\pi) = (-1; -\pi) \neq r(\pi) = (-1; \pi)$$

γ non è chiusa.

2) $\underline{r}'(t) = \begin{cases} x' = -\cancel{\operatorname{sen} t} + \cancel{\operatorname{sen} t} + t\cos t = t\cos t \\ y' = +\cancel{\cos t} - \cancel{\cos t} + t \operatorname{sen} t = t \operatorname{sen} t \end{cases}$

$\underline{r}'(t) = (t\cos t,\ t\operatorname{sen} t)$ è il vettore tangente.

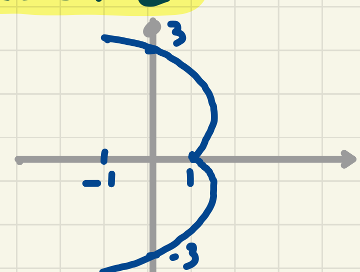
$|\underline{r}'(t)| = \sqrt{t^2\cos^2 t + t^2 \operatorname{sen}^2 t} =$

$= \sqrt{t^2} = |t|$

$|\underline{r}'(t)| = 0$ in $t = 0 \implies \gamma$ non è regolare

oss. Se una curva è regolare è sempre definito il suo VERSORE tangente.

$$\vec{T}(t) = \frac{\underline{r}'(t)}{|\underline{r}'(t)|}$$



ESERCIZIO 3. Scrivere l'equazione della retta tangente alla curva γ di equazione

$$\underline{r}(t) = \begin{cases} x = 2\sin t \\ y = -3\cos t \\ z = 4t \end{cases} \qquad t \in \mathbb{R}$$

nel suo punto P ottenuto per $t = 0$.

SOL.

$t = 0 \Rightarrow P(0; -3; 0)$

$\underline{r}(t) = (2\sin t, -3\cos t, 4t)$

$\underline{r}'(t) = (2\cos t, 3\sin t, 4)$ vettore tg $\forall t \in \mathbb{R}$

$\underline{r}'(0) = (2; 0; 4)$ direzione della retta tangente.

RETTA TG. : $\begin{cases} x = 0 + 2s \\ y = -3 + 0s \\ z = 0 + 4s \end{cases} \rightarrow \begin{cases} x = 2s \\ y = -3 \\ z = 4s \end{cases}$ $s \in \mathbb{R}$

ESERCIZIO 4. Calcolare la lunghezza della curva γ di equazioni parametriche:

$$r(t) = \begin{cases} x = t^2 \\ y = t^3 \end{cases} \qquad t \in [-1; 1]$$

SOL.

$$r : [a, b] \longrightarrow \mathbb{R}^2 \text{ o } \mathbb{R}^3$$

$$\ell(\gamma) = \int_a^b |r'(t)| \, dt$$

$r'(t) = (2t ; 3t^2)$ $\qquad |\underline{r}'(t)| = \sqrt{4t^2 + 9t^4} =$

$$= |t|\sqrt{4+9t^2}$$ (la curva non è regolare ma lo è a tratti)

$$\ell(\gamma) = \int_{-1}^{1} |\underline{r}'(t)|\,dt = \int_{-1}^{1} |t|\sqrt{4+9t^2}\,dt =$$

$$= \int_{-1}^{0} -t\sqrt{4+9t^2}\,dt + \int_{0}^{1} t\sqrt{4+9t^2}\,dt =$$

$$= -\frac{1}{18}\int_{-1}^{0} 18t\,(4+9t^2)^{1/2}\,dt + \frac{1}{18}\int_{0}^{1} 18t\,(4+9t^2)^{1/2}\,dt$$

$$= -\frac{1}{\cancel{18}\,9}\left[(4+9t^2)^{3/2}\cdot\frac{\cancel{2}}{3}\right]_{-1}^{0} + \frac{1}{\cancel{18}\,9}\left[(4+9t^2)^{3/2}\cdot\frac{\cancel{2}}{3}\right]_{0}^{1} = \dots$$

$$= \frac{2}{27}\left(13\sqrt{13}-8\right).$$

NOTA (CURVE IN FORMA POLARE)

$$\underbrace{\rho = f(\theta)}_{\text{FORMA POLARE}}, \quad \theta \in I \quad \overset{def}{\Longleftrightarrow} \quad \underline{r}(\theta) = \begin{cases} x(\theta) = f(\theta)\cos\theta \\ y(\theta) = f(\theta)\operatorname{sen}\theta \end{cases} \quad \theta \in I.$$

ESERCIZIO 5.

Calcolare la lunghezza della curva $\rho = e^{-\theta}$ con $\theta \in [0; 2\pi]$.

SOL.

$$r(\theta) = \begin{cases} x = e^{-\theta}\cos\theta \\ y = e^{-\theta}\operatorname{sen}\theta \end{cases}$$

$$r'(\theta) = \begin{cases} x' = -e^{-\theta}\cos\theta + e^{-\theta}(-\operatorname{sen}\theta) = e^{-\theta}(-\cos\theta - \operatorname{sen}\theta) \\ y' = -e^{-\theta}\operatorname{sen}\theta + e^{-\theta}\cos\theta = e^{-\theta}(\cos\theta - \operatorname{sen}\theta) \end{cases}$$

$$|r'(\theta)| = \sqrt{e^{-2\theta}(\cos\theta + \operatorname{sen}\theta)^2 + e^{-2\theta}(\cos\theta - \operatorname{sen}\theta)^2} =$$

$$= e^{-\theta}\cdot\sqrt{\cos^2\theta + \cancel{2\cos\theta\,\text{sen}\,\theta} + \text{sen}^2\theta + \cos^2\theta - \cancel{2\,\text{sen}\,\theta\cos\theta} + \text{sen}^2\theta}$$

$$= \sqrt{2}\, e^{-\theta}$$

$$\ell(\gamma) = \int_0^{2\pi} |r'(\theta)|\, d\theta = \int_0^{2\pi} \sqrt{2}\, e^{-\theta}\, d\theta = \sqrt{2}\left[-e^{-\theta}\right]_0^{2\pi} =$$

$$= \sqrt{2}\left(-e^{-2\pi} + 1\right) = \sqrt{2}\left(1 - \frac{1}{e^{2\pi}}\right).$$

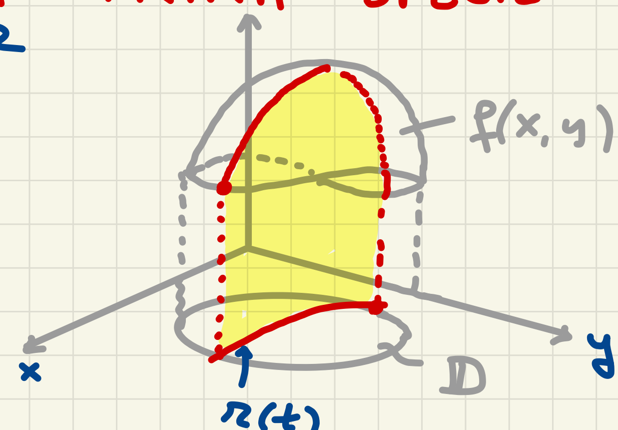
INTEGRALI CURVILINEI DI PRIMA SPECIE

Sia $f : \mathbb{D} \to \mathbb{R}$, $\mathbb{D} \subseteq \mathbb{R}^2$

$z = f(x,y)$

$\underline{r}(t) = (x(t), y(t)) \quad t \in [a,b]$

$f(\underline{r}(t)) = f(x(t), y(t))$



$$\int_\gamma f\, ds = \int_a^b f(\underline{r}(t))\,|r'(t)|\,dt$$

ESERCIZIO 6.

Calcolare $\int_\gamma f\, ds$ essendo:

1) $f(x,y) = x$ $\quad \underline{r}(t) = \begin{cases} x = t \\ y = t^2 \end{cases}$ $\quad t \in [0, a]$

- $\underline{r}(t) = (t, t^2)$ $\quad r'(t) = (1; 2t)$ $\quad |r'(t)| = \sqrt{1+4t^2}$
- $f(r(t)) = f(t, t^2) = t$

$$\int_\gamma f\, ds = \int_0^a t\sqrt{1+4t^2}\, dt = \frac{1}{8}\int_0^a 8t(1+4t^2)^{1/2}\, dt =$$

$$= \frac{1}{\cancel{8}_{4}} \left[(1+4t^2)^{3/2} \cdot \frac{\cancel{2}}{3} \right]_0^a = \frac{1}{12} \left[\sqrt{(1+4a^2)^3} - 1 \right].$$

2) $f(x,y) = \sqrt{1-y^2}$ $\quad \underline{r}(t) = (\operatorname{sen} t, \cos t) \quad t \in [0, \pi].$

- $r'(t) = (\cos t, -\operatorname{sen} t) \qquad |r'(t)| = 1$
- $f(\underline{r}(t)) = f(\operatorname{sen} t, \cos t) = \sqrt{1-\cos^2 t} = \cancel{|}\operatorname{sen} t\cancel{|}$

$$\int_\gamma f\,ds = \int_0^\pi \operatorname{sen} t \cdot 1 \cdot dt = \left[-\cos t \right]_0^\pi = 1+1 = 2.$$

3) $f(x,y) = \dfrac{x}{1+y^2}$ $\quad r(t) = (\cos t, \operatorname{sen} t) \quad t \in [0, \frac{\pi}{2}].$

$|r'(t)| = 1.$

$$f(\underline{r}(t)) = f(\cos t, \operatorname{sen} t) = \frac{\cos t}{1+\operatorname{sen}^2 t}$$

$$\int_\gamma f\,ds = \int_0^{\pi/2} \frac{\cos t}{1+\operatorname{sen}^2 t}\,dt = \left[\operatorname{arctg} \operatorname{sen} t\right]_0^{\pi/2} =$$

$$= \operatorname{arctg} 1 - \operatorname{arctg} 0 = \frac{\pi}{4}.$$

ESERCIZIO A CASA:

Calcolare $\int_\gamma (x+y^3)\,ds$ essendo γ il segmento di estremi $(0,0)$ e $(1,1)$. $\left[R.\ \frac{3}{4}\sqrt{2}\right]$

APPLICAZIONI FISICHE DEGLI INTEGRALI CURVILINEI.

1) **BARICENTRO.**

- $\underline{r}(t) : [a,b] \to \mathbb{R}^3$, $\underline{r}(t) = (x(t), y(t), z(t))$
- $\lambda(x,y,z)$ densità lineare

Allora il baricentro della curva è:

$$x_G = \frac{\int_\gamma x \lambda(x,y,z)\,ds}{\underbrace{\int_\gamma \lambda(x,y,z)\,ds}_{\text{MASSA}}} \qquad y_G = \frac{\int_\gamma y \lambda(x,y,z)\,ds}{\int_\gamma \lambda(x,y,z)\,ds}$$

Analogamente per z_G.

OSSERVAZIONE

Sia ora la densità $\lambda(x,y,z) = \lambda$ costante.

$$x_G = \frac{\int_\gamma x \cancel{\lambda}\,ds}{\int_\gamma \cancel{\lambda}\,ds} = \frac{\int_\gamma x\,ds}{\underbrace{\int_\gamma ds}_{\text{LUNGHEZZA } L}} = \frac{1}{L}\int_\gamma x\,ds$$

$$y_G = \frac{1}{L}\int_\gamma y\,ds \qquad z_G = \frac{1}{L}\int_\gamma z\,ds.$$

ESERCIZIO 7. Calcolare il baricentro della curva

$$\underline{r}(\theta) = \begin{cases} x = e^{\theta}\cos\theta \\ y = e^{\theta}\operatorname{sen}\theta \end{cases} \qquad \theta \in [0;\pi]$$

supponendo che la densità sia costante.

SOL.

$$\underline{r}(\theta) = (e^{\theta}\cos\theta ; e^{\theta}\operatorname{sen}\theta)$$

$$\underline{r}'(\theta) = (e^{\theta}\cos\theta - e^{\theta}\operatorname{sen}\theta ; e^{\theta}\operatorname{sen}\theta + e^{\theta}\cos\theta) =$$

$$= e^{\theta}(\cos\theta - \operatorname{sen}\theta ; \operatorname{sen}\theta + \cos\theta).$$

$$|\underline{r}'(\theta)| = \sqrt{2}\, e^{\theta} \qquad (\text{calcolarlo!})$$

$$L(\gamma) = \int_0^{\pi} \sqrt{2}\, e^{\theta}\, d\theta = \sqrt{2}\left[e^{\theta}\right]_0^{\pi} = \sqrt{2}(e^{\pi} - 1).$$

$$\int_\gamma x\,ds = \int_0^\pi e^\theta \cos\theta \,\sqrt{2}\, e^\theta \,d\theta = \sqrt{2}\int_0^\pi e^{2\theta}\cos\theta \,d\theta = (*)$$

A parte calcolo $\int e^{2\theta}\cos\theta\, d\theta$:

$$\int e^{2\theta}\cos\theta\, d\theta \underset{\times\,\text{PARTI}}{=} \frac{e^{2\theta}}{2}\cos\theta + \int \frac{e^{2\theta}}{2}\operatorname{sen}\theta\, d\theta =$$

$$= \frac{e^{2\theta}}{2}\cos\theta + \frac{1}{2}\left\{\frac{e^{2\theta}}{2}\operatorname{sen}\theta - \int \frac{e^{2\theta}}{2}\cos\theta\, d\theta\right\} =$$

$$= \frac{e^{2\theta}}{2}\cos\theta + \frac{e^{2\theta}}{4}\operatorname{sen}\theta - \frac{1}{4}\boxed{\int e^{2\theta}\cos\theta\, d\theta}$$

$$\frac{5}{4}\int e^{2\theta}\cos\theta\, d\theta = \left[\frac{e^{2\theta}}{\cancel{2}}\cos\theta + \frac{e^{2\theta}}{\cancel{4}_{\,2}}\operatorname{sen}\theta\right]\cdot\frac{\cancel{4}^{\,2}}{5} + C$$

$$(*) = \frac{2\sqrt{2}}{5}\left[e^{2\theta}\left(\cos\theta + \frac{1}{2}\operatorname{sen}\theta\right)\right]_0^{\pi} = \frac{2\sqrt{2}}{5}\left(-e^{2\pi} - 1\right) =$$

$$= -\frac{2}{5}\sqrt{2}\left(e^{2\pi}+1\right)$$

$$x_G = \frac{1}{L}\int_\gamma x\, ds = \frac{1}{\cancel{\sqrt{2}}(e^{\pi}-1)}\left(-\frac{2}{5}\cancel{\sqrt{2}}\left(e^{2\pi}+1\right)\right)$$

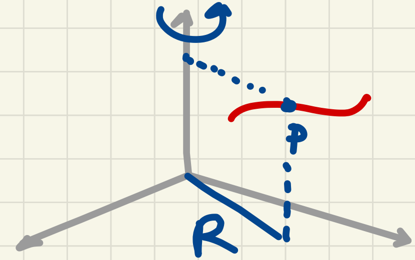
$$= -\frac{2\left(e^{2\pi}+1\right)}{5\left(e^{\pi}-1\right)}$$

A CASA CALCOLARE $\quad y_G = \frac{1}{5}\,\frac{e^{2\pi}+1}{e^{\pi}-1}$

2) MOMENTO DI INERZIA

- $\underline{r} : [a,b] \to \mathbb{R}^3$ • $\rho(x,y,z)$ densità
- $R(x,y,z)$ distanza di un punto $P(x,y,z)$ dall'asse

di rotazione.



$$I = \int_{\gamma} R^2 \rho \, ds =$$

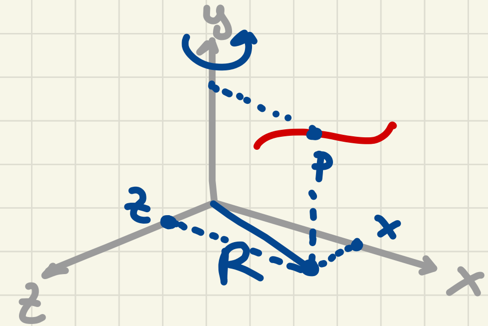
$$= \int_a^b R^2(\underline{r}(t))\, \rho(\underline{r}(t))\, |\underline{r}'(t)|\, dt$$

ESERCIZIO 8. Sia data la curva

$$\underline{r}(t) = \begin{cases} x = t-1 \\ y = t \\ z = 1 \end{cases} \qquad t \in [0;1]$$

Calcolare il suo momento di inerzia intorno all'asse y sapendo che la densità lineare è $\rho(x,y,z) = |y|$.

SOL.



$$\underline{r}(t) = (t-1\,;\, t\,;\, 1)$$

$$\underline{r}'(t) = (1\,;\, 1\,;\, 0) \qquad |\underline{r}'(t)| = \sqrt{2}$$

$$\rho(x,y,z) = |y|$$

$$R(x,y,z) = \sqrt{x^2+z^2} \implies R^2(x,y,z) = x^2+z^2$$

$$I = \int_\gamma R^2 \rho \, ds = \int_0^1 R^2(\underline{r}(t))\, \rho(\underline{r}(t))\, |\underline{r}'(t)|\, dt =$$

$$= \int_0^1 \left[(t-1)^2+1\right] \cancel{|}t\cancel{|} \sqrt{2}\, dt =$$

$t \in [0,1]$

$$= \sqrt{2} \int_0^1 (t^2-2t+2)t \, dt = \sqrt{2} \int_0^1 (t^3-2t^2+2t)\, dt =$$

$$= \sqrt{2} \left[\frac{t^4}{4} - \frac{2}{3}t^3 + \cancel{2}\frac{t^2}{\cancel{2}} \right]_0^1 = \sqrt{2}\left(\frac{1}{4} - \frac{2}{3} + 1\right) = \frac{7}{12}\sqrt{2}.$$