Lavagne Virtuali di Analisi III

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Ing. Fisica

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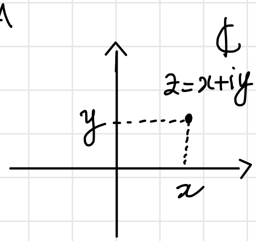
I. FRAGALÀ

ANALISI COMPLESSA

Def. Una FUNZIONE DI VARIABILE COMPLESSA è una funzione $f: \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$



$z = x + iy \qquad x, y \in \mathbb{R} \qquad (i)^2 = -1$

$f(z) = u(z) + i\, v(z) = u(x,y) + i\, v(x,y)$

$f \rightsquigarrow u, v: \Omega \subseteq \mathbb{R}^2 \longrightarrow \mathbb{R}$

Esempi

- $f(z) = z_0 \in \mathbb{C} \qquad z_0 = x_0 + iy_0$

  $u(x,y) = x_0 \quad , \quad v(x,y) = y_0$

- $f(z) = z \qquad z = x + iy$

  $u(x,y) = x \quad , \quad v(x,y) = y$

- $f(z) = \operatorname{Re} z \qquad z = x + iy \Rightarrow f(z) = x$

  $u(x,y) = x \quad , \quad v(x,y) = 0$

- $f(z) = \operatorname{Im} z \qquad z = x + iy \Rightarrow f(z) = y$

  $u(x,y) = y \quad , \quad v(x,y) = 0$

- $f(z) = |z| \qquad z = x + iy \Rightarrow f(z) = \sqrt{x^2 + y^2}$

  $u(x,y) = \sqrt{x^2 + y^2} \quad , \quad v(x,y) = 0$

- $f(z) = P(z) = \underset{\uparrow}{a_n} z^n + \underset{\uparrow}{a_{n-1}} z^{n-1} + \dots + \underset{\uparrow}{a_1} z + \underset{\uparrow}{a_0}$ $\in \mathbb{C}$

<u>Es.</u> $z^2 + iz + 1 =$

$= (x+iy)^2 + i(x+iy) + 1 = x^2 - y^2 + 2ixy + ix - y + 1$

$= (\underbrace{x^2 - y^2 - y + 1}_{u(x,y)}) + i(\underbrace{2xy + x}_{v(x,y)})$

- $f(z) = \dfrac{P(z)}{Q(z)}$, con P, Q funzioni polinomiali.

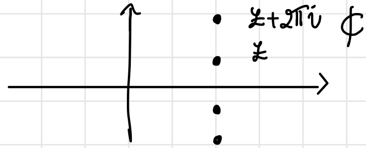
$\Omega = \{ z \in \mathbb{C} : Q(z) \neq 0 \}$

<u>Es.</u> $f(z) = \dfrac{z^2+1}{z^2-1}$ $\qquad \Omega = \{ z \in \mathbb{C} : z \neq \pm 1 \}$

$f(z) = \dfrac{z^2-1}{z^2+1}$ $\qquad \Omega = \{ z \in \mathbb{C} : z \neq \pm i \}$

- $f(z) = e^z, \ \sin z, \ \cos z, \ \sinh z, \ \cosh z.$

Funzione e^z



$$z = x+iy \Rightarrow e^z := e^x \cdot e^{iy} := e^x \cdot (\cos y + i \sin y)$$

- $z = x \in \mathbb{R} \Rightarrow e^z = e^x$ (la funzione exp in campo reale)
- $u(x,y) = e^x \cos y \,, \quad v(x,y) = e^x \sin y$
- $e^{z+2\pi i} = e^{x+iy+2\pi i} = e^{x+i(y+2\pi)} =$

$$= e^x \cdot (\cos(y+\cancel{2\pi}) + i \sin(y+\cancel{2\pi})) = e^x (\cos y + i \sin y) = e^z$$

$\Rightarrow$ la funzione exp in $\mathbb{C}$ è PERIODICA di periodo $T = 2\pi i$

- Es. $e^{z_1+z_2} = e^{z_1} \cdot e^{z_2}$

$(z_1 = x_1 + iy_1 , \quad z_2 = x_2 + iy_2 \,, \quad z_1 + z_2 = (x_1+x_2) + i(y_1+y_2))$

- $e^z = 0$?? $\iff |e^z| = 0$

$\overset{\shortparallel}{e^x (\cos y + i \sin y)}$

$$|e^z| = |e^x| \, |\cos y + i \sin y| = e^x \underset{\neq}{>} 0$$

$$|e^z| = e^{\mathrm{Re}\, z} \neq 0$$

Funzioni $\cos z$, $\sin z$

$$\cos z := \frac{1}{2}\left(e^{iz} + e^{-iz}\right), \qquad \sin z := \frac{1}{2i}\left(e^{iz} - e^{-iz}\right)$$

- $z = x \in \mathbb{R} \Rightarrow$

$$\frac{1}{2}\left(e^{ix} + e^{-ix}\right) =$$

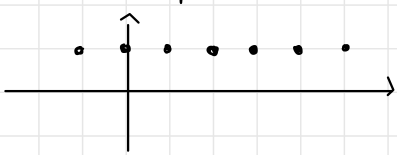
$$= \frac{1}{2}\left[\cos x + i\cancel{\sin x} + \cos(-x) + i\cancel{\sin(-x)}\right] = \cos x$$

- $\cos(z + 2\pi) = \cos z$

$$\cos(z+2\pi) = \frac{1}{2}\left(e^{i(z+2\pi)} + e^{-i(z+2\pi)}\right) =$$

$$= \frac{1}{2}\left(e^{iz+\cancel{2\pi i}} + e^{-iz\cancel{-2\pi i}}\right) = \frac{1}{2}\left(e^{iz} + e^{-iz}\right) = \cos z$$

$\Rightarrow$ $\cos z$ è periodica di periodo $T = 2\pi$.



- $\cos z = 0 \iff z = \frac{\pi}{2} + k\pi, \quad k \in \mathbb{Z}$

- $\begin{cases} u(x,y) = \cos x \cosh y \\ v(x,y) = -\sin x \sinh y \end{cases}$

- $x = 0 \Rightarrow \cos(iy) = \cosh y.$

$\rightarrow$ Esercitazione

$$\cos z = \cos x \cosh y - i \sin x \sinh y$$

Per $\sin z$, $\cosh z$, $\sinh z$ valgono proprietà analoghe.

In particolare

$\sin z = \sin x \cosh y + i \cos x \sinh y$.

- $\cosh z = \dfrac{e^{z} + e^{-z}}{2}$, $\sinh z = \dfrac{e^{z} - e^{-z}}{2}$

$$\begin{cases} \cosh z = \cosh x \cos y + i \sinh x \sin y \\ \sinh z = \sinh x \cos y + i \cosh x \sin y . \end{cases}$$

periodiche di periodo $2\pi i$