Esempi

1) $T = T_u$ con $u \in C^1(\Omega) \Rightarrow (T_u)' = T_{u'}$

2) $T = T_u$ con $u(x) = |x|$ su $\Omega = (-1,1)$.

$$\langle (T_u)', \varphi \rangle \overset{\text{per def}}{=} -\langle T_u, \varphi' \rangle = -\int_{-1}^{1} |x| \varphi'(x)\,dx =$$

$$= -\int_0^1 x\,\varphi'(x)\,dx + \int_{-1}^{0} x\,\varphi'(x)\,dx$$

$$= \int_0^1 \varphi(x)\,dx + \cancel{x\,\varphi(x)}\Big|_0^1 - \int_{-1}^{0} \varphi(x)\,dx + \cancel{x\,\varphi(x)}\Big|_{-1}^{0}$$

$$= \int_{-1}^{1} \varphi(x)\cdot \operatorname{sign}(x)\,dx \quad \text{dove } \operatorname{sign}(x) = \begin{cases} +1 & x > 0 \\ -1 & x < 0. \end{cases}$$

$$= \langle T_{\operatorname{sign}(x)}, \varphi \rangle \qquad \forall \varphi \in \mathcal{D}(-1,1).$$

$$(T_{|x|})' = T_{\operatorname{sign}(x)} \quad \text{in } \mathcal{D}'(\Omega).$$

notazione:

$$(|x|)' = \operatorname{sign}(x) \quad \text{in } \mathcal{D}'(\Omega).$$

più in generale: se $u \in L^1_{loc}(\Omega)$, $v \in L^1_{loc}(\Omega)$

$u' = v$ in $\mathcal{D}'(\Omega)$, significa $(T_u)' = T_v$

ovvero $\forall \varphi \in \mathcal{D}(\Omega)$ $\langle (T_u)', \varphi \rangle = -\langle T_u, \varphi' \rangle = \langle T_v, \varphi \rangle$

$$\boxed{-\int_\Omega u\,\varphi' = \int_\Omega v\,\varphi \qquad \forall \varphi \in \mathcal{D}(\Omega).}$$

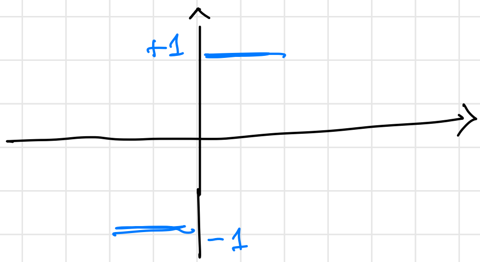
3) $u(x) = \text{sign}(x)$ $\qquad (T_u)' = ?$ $\qquad u' = ?$

$$\langle u', \varphi \rangle = -\langle u, \varphi' \rangle = -\int_{\mathbb{R}} \text{sign}(x) \cdot \varphi'(x)$$

$$= -\int_0^1 \varphi' + \int_{-1}^0 \varphi' = -\cancel{\varphi(1)} + \varphi(0) + \varphi(0) - \cancel{\varphi(-1)}$$

$$= 2\varphi(0) = 2 \langle \delta_0, \varphi \rangle.$$

$$(\text{sign}(x))' = 2\delta_0$$



4) $T = \delta_0 \qquad T' = ?$

$$\langle T', \varphi \rangle = -\langle T, \varphi' \rangle = -\langle \delta_0, \varphi' \rangle$$

$$= -\varphi'(0) \qquad \forall \varphi \in \mathcal{D}(\Omega).$$

Generalizzazioni

- $n=1$ Data $T \in \mathcal{D}'(\Omega)$, $\forall k \in \mathbb{N}$ $T^{(k)} \in \mathcal{D}'(\Omega)$

$$< T^{(k)}, \varphi > := (-1)^{(k)} < T, \varphi^{(k)} > \qquad \forall \varphi \in \mathcal{D}(\Omega)$$

Oss. $T^{(k)}$ definisce una distribuzione

(i) è lineare

(ii) $\varphi_h \to 0$ in $\mathcal{D}(\Omega) \Rightarrow < T^{(k)}, \varphi_h > \longrightarrow 0$

Infatti se $\varphi_h \to 0$ in $\mathcal{D}(\Omega)$, $\varphi_h^{(k)} \to 0$ in $\mathcal{D}(\Omega)$

$\Rightarrow < T, \varphi_h^{(k)} > \longrightarrow 0$ poiché $T \in \mathcal{D}'(\Omega)$ ☒

$$\overset{\shortparallel}{(-1)^{k} < T^{(k)}, \varphi_h >}.$$

Oss. se $T = T_u$ con $u \in C^k(\Omega) \subseteq L^1_{loc}(\Omega)$

$$\Rightarrow (T_u)^{(k)} = T_{u^{(k)}}.$$

Esempio $u(x) = |x| \Rightarrow u'' = 2\delta_0$

• $n \geq 1$ Data $T \in \mathcal{D}'(\Omega)$ $\forall \alpha$ multiindice

$$\langle D^\alpha T, \varphi \rangle = (-1)^{|\alpha|}\,^{(*)} \langle T, D^\alpha \varphi \rangle \quad \forall \varphi \in \mathcal{D}(\Omega)$$

Esempio: $n=3$ $\alpha = (3,1,2)$

$$\left\langle \frac{\partial^6 T}{\partial^3 x_1 \partial^1 x_2 \partial^2 x_3}, \varphi \right\rangle = (-1)^6 \left\langle T, \frac{\partial^6 \varphi}{\partial^3 x_1 \partial^1 x_2 \partial^2 x_3} \right\rangle$$

$\forall \varphi \in \mathcal{D}(\Omega)$

(*) $\alpha = (\alpha_1, \dots, \alpha_n) \Rightarrow |\alpha| = \alpha_1 + \dots + \alpha_n$

Oss. $D^\alpha T$ definiscono delle distribuzioni $\forall \alpha$

Oss. Si possono calcolare le derivate di tutti gli ordini, di qualsiasi $T \in \mathcal{D}'(\Omega)$.

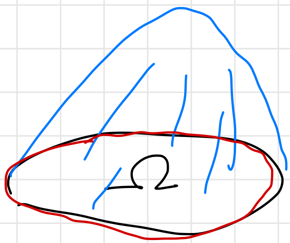
Oss. Il risultato NON dipende dall'ordine di derivazione!

• Data $T \in \mathcal{D}'(\Omega)$, si possono definire

∇T, ΔT, $\mathrm{rot}\, T$ .....

Gli spazi di Sobolev.

Motivazione: sono gli spazi dove si trovano soluz. di pb. al contorno per P.D.E.'s.



$$\begin{cases} -\Delta u = f & \text{in } \Omega \quad \text{eq. Poisson} \\ u = 0 & \text{su } \partial\Omega \end{cases}$$

condizione di Dirichlet.

(cf. ADAMS, LEONI).

<u>Def.</u> Fissato Ω aperto $\subseteq \mathbb{R}^n$, $p \in [1, +\infty]$

$$W^{1,p}(\Omega) := \Big\{ u \in L^p(\Omega) \;:\; \underbrace{\frac{\partial u}{\partial x_i}}_{\text{nel senso delle distribuzioni}} \in L^p(\Omega) \;\; \forall i = 1, .., n \Big\}$$

$\frac{\partial u}{\partial x_i} \in L^p(\Omega) \iff \exists \; v_i \in L^p(\Omega)$ tali che

$$\int_\Omega \frac{\partial u}{\partial x_i} \cdot \varphi = \int_\Omega v_i \, \varphi \qquad \forall \varphi \in \mathcal{D}(\Omega).$$

$$\int_\Omega \frac{\partial u}{\partial x_i} \cdot \varphi = -\int_\Omega u \, \frac{\partial \varphi}{\partial x_i}$$

Esempi $(n=1, \quad \Omega=(-1,1))$

- $u \in C_0^1(\Omega) \Rightarrow u \in W^{1,p}(\Omega)$ Infatti:

(1) $p<+\infty \quad \int_\Omega |u|^p < +\infty, \quad \int_\Omega |u'|^p < +\infty$

(2) $p=+\infty \quad \operatorname{ess\,sup}_\Omega |u| < +\infty, \quad \operatorname{ess\,sup}_\Omega |u'| < +\infty$

- $u(x) = |x| \Rightarrow u \in W^{1,p}(\Omega)$. Infatti, per es. $p=2$

$$\begin{cases} \int_\Omega |u|^2 = \int_{-1}^1 |x|^2 < +\infty \\ \int_\Omega |u'|^2 = \int_{-1}^1 |\operatorname{sign} x|^2 < +\infty \end{cases}$$

- $u(x) = \operatorname{sign} x \quad u \notin W^{1,2}(\Omega)$

$$\int_\Omega |u|^2 = \int_{-1}^1 |\operatorname{sign} x|^2 < +\infty \qquad (u \in L^2(\Omega))$$

MA: $\quad u'(x) = 2\delta_0 \notin L^2(\Omega)$

Def. Fissato Ω aperto $\subseteq \mathbb{R}^n$, $p \in [1, +\infty]$, $k \in \mathbb{N}$, $k \geq 1$

$$W^{k,p}(\Omega) := \{ u \in L^p(\Omega) : D^\alpha u \in L^p(\Omega) \ \forall \alpha \text{ multi-indice con } |\alpha| \leq k \}$$

Caso particolare: $p = 2$

$$W^{1,2}(\Omega) = H^1(\Omega) = \{ u \in L^2(\Omega) : \frac{\partial u}{\partial x_i} \in L^2(\Omega) \ \forall i = 1, \dots n \}$$

$$W^{k,2}(\Omega) = H^k(\Omega) = \{ u \in L^2(\Omega) : D^\alpha u \in L^2(\Omega) \ \forall \alpha \text{ multiindice con } |\alpha| \leq k \}$$

Oss.

$W^{k,p}(\Omega)$ sono spazi vettoriali

Es. $k = 1$ $\quad W^{1,p}(\Omega)$ spazio vettoriale

- $u, v \in L^p(\Omega)$, $\frac{\partial u}{\partial x_i}, \frac{\partial v}{\partial x_i} \in L^p(\Omega) \Rightarrow$

  $u + v \in L^p(\Omega)$, $\underbrace{\frac{\partial}{\partial x_i}(u+v)}_{= \frac{\partial u}{\partial x_i} + \frac{\partial v}{\partial x_i}} \in L^p(\Omega)$.

- $u \in L^p(\Omega)$, $\frac{\partial u}{\partial x_i} \in L^p(\Omega) \Rightarrow$

  $(\lambda u) \in L^p(\Omega)$, $\underbrace{\frac{\partial (\lambda u)}{\partial x_i}}_{= \lambda \frac{\partial u}{\partial x_i}} \in L^p(\Omega)$

Def. Norma su $W^{1,p}(\Omega)$: sia $u \in W^{1,p}(\Omega)$

$$\|u\|_{1,p} := \|u\|_p + \sum_{i=1}^{n} \left\| \frac{\partial u}{\partial x_i} \right\|_p$$

Def. Norma su $W^{k,p}(\Omega)$: sia $u \in W^{k,p}(\Omega)$

$$\|u\|_{k,p} := \sum_{|\alpha| \leq k} \|D^\alpha u\|_p$$

Teorema

Per ogni $p \in [1, +\infty]$, $W^{k,p}(\Omega)$ sono Banach.

Oss. $u_h \to u$ in $W^{1,p}(\Omega)$ se

$$\|u_h - u\|_{1,p} \to 0$$

$$\| \quad \|u_h - u\|_p + \left\| \frac{\partial u_h}{\partial x_i} - \frac{\partial u}{\partial x_i} \right\|_p$$

ovvero
$$\begin{cases} u_h \to u & \text{in } L^p(\Omega) \\ \dfrac{\partial u_h}{\partial x_i} \to \dfrac{\partial u}{\partial x_i} & \text{in } L^p(\Omega) \end{cases}$$

"funzioni di $W^{1,p}(\Omega)$ «nulle al bordo di Ω»"

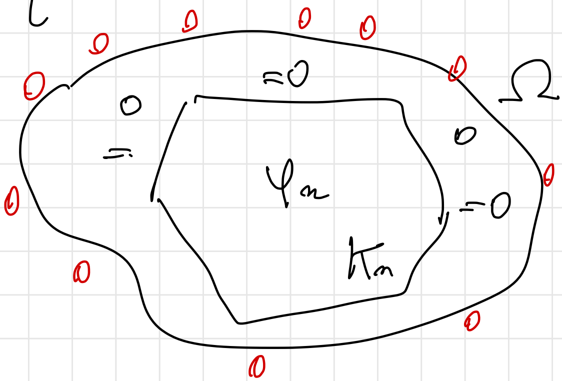
Def.

$W_0^{1,p}(\Omega)$:= chiusura di $\mathcal{D}(\Omega)$ in $W^{1,p}(\Omega)$

ovvero

$$= \left\{ u \in W^{1,p}(\Omega) : \exists \{\varphi_n\} \subseteq \mathcal{D}(\Omega) \text{ tale che } \varphi_n \xrightarrow{W^{1,p}} u \right\}.$$

$$= \left\{ u \in W^{1,p}(\Omega) : \exists \{\varphi_n\} \subseteq \mathcal{D}(\Omega) \text{ tale che } \begin{cases} \varphi_n \xrightarrow{L^p} u \\ \dfrac{\partial \varphi_n}{\partial x_i} \xrightarrow{L^p} u \end{cases} \right\}$$



Oss. Se $u \in W^{1,p}(\Omega) \cap C(\overline{\Omega})$, allora

$$u \in W_0^{1,p}(\Omega) \iff u = 0 \text{ su } \partial\Omega.$$

Teorema (di inuguaglianza di Poincaré)

Sia Ω aperto <u>limitato</u> di $\mathbb{R}^n$.

Allora esiste una costante $C_p = C_p(\Omega)$ tale che, per ogni $u \in W_0^{1,p}(\Omega)$

$$\left[\|u\|_{L^p(\Omega)} \leq C_p(\Omega) \cdot \|\nabla u\|_{L^p(\Omega)} \right]$$

<u>Quindi</u>: su $W_0^{1,p}(\Omega)$

$$\begin{cases} \|u\|_{1,p} = \|u\|_p + \|\nabla u\|_p & \text{norma su } W^{1,p}(\Omega) \\ \|\nabla u\|_p \leftarrow \text{norma equivalente.} \end{cases}$$

(Infatti $\|u\|_{1,p} \leq C_p(\Omega)\|\nabla u\|_p + \|\nabla u\|_p$
$\leq (C_p(\Omega) + 1) \|\nabla u\|_p$).

<u>Falso</u> su $W^{1,p}(\Omega)$, prendendo $u = 1$.

Oss. $\underline{n=1}$

$$u(x) = \cancel{u(0)} + \int_0^x u'(t)\, dt \Rightarrow \text{Hölder}$$

$$|u(x)| \leq \int_0^x |u'| \leq \int_0^1 |u'| \leq \left(\int_0^1 |u'|^2 \right)^{1/2}$$

integrando $\|u\|_{L^2(0,1)} \leq \|u'\|_{L^2(0,1)}$.