ESERCIZIO 1. 7-5-2020

Calcolare il lavoro compiuto dal campo $\underline{F}(x,y,z) = x^2\underline{i} - yz\,\underline{k}$ lungo l'elica cilindrica $\underline{r}(t) = 2\cos t\,\underline{i} + 2\operatorname{sen} t\,\underline{j} + t\,\underline{k}$ $t \in [0; 2\pi]$.

SOL.

$\mathbb{D}_{\underline{F}} = \mathbb{R}^3$ sempl. connesso.

$$\operatorname{rot}\underline{F} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \partial_x & \partial_y & \partial_z \\ x^2 & 0 & -yz \end{vmatrix} = \underline{i}(-z) - \underline{j}(0) + \underline{k}(0) =$$

$$= -z\,\underline{i} \neq \underline{0} \Rightarrow \underline{F} \text{ non}$$

è irrotazionale $\Rightarrow$ $\underline{F}$ non è conservativo.

$$\underline{r}(t) = \begin{cases} x = 2\cos t \\ y = 2\,\text{sen}\, t \\ z = t \end{cases} \qquad \underline{r}'(t) = \begin{cases} x' = -2\,\text{sen}\, t \\ y' = 2\cos t \\ z' = 1 \end{cases}$$

$$\mathcal{L} = \int_\gamma \underline{F} \cdot d\underline{r} = \int_0^{2\pi} \underline{F}(\underline{r}(t)) \cdot \underline{r}'(t)\, dt =$$

$$= \int_0^{2\pi} \left(4\cos^2 t;\ 0;\ -2t\,\text{sen}\, t\right) \cdot \left(-2\,\text{sen}\, t, 2\cos t, 1\right) dt =$$

$$= \int_0^{2\pi} \left(-8\,\text{sen}\, t\cos^2 t - 2t\,\text{sen}\, t\right) dt = +8\int_0^{2\pi} (-\text{sen}\, t)\cos^2 t\, dt$$

$$-2\int_0^{2\pi} t\,\text{sen}\, t\, dt = \cancel{8\left[\frac{\cos^3 t}{3}\right]_0^{2\pi}} - 2\int_0^{2\pi} t\,\text{sen}\, t\, dt =$$

$$= -2 \left\{ \left[-t\cos t \right]_0^{2\pi} + \int_0^{2\pi} \cos t \, dt \right\} =$$

$$= -2 \left[-t\cos t + \operatorname{sen} t \right]_0^{2\pi} = -2(-2\pi) = 4\pi.$$

ESERCIZIO 2. Si consideri la famiglia di campi in $\mathbb{R}^2 - \{(0;0)\}$

$$\underline{F}(x,y) = \left(\frac{x+\alpha y}{2x^2+2y^2} ; \frac{\beta x + y}{2x^2+2y^2} \right) \qquad \alpha, \beta \in \mathbb{R}.$$

1) Per quali valori di α, β $\underline{F}(x,y)$ è irrotazionale in $\mathbb{R}^2 - \{(0,0)\}$?

2) Per gli α, β trovati calcolare $\int_\Gamma \underline{F} \cdot d\underline{r}$ dove Γ è la circonferenza di centro

l'origine e raggio 1 orientata positivamente.

3) Esistono valori di α e β per cui $\underline{F}$ possa essere conservativo in $\mathbb{R}^2 - \{(0,0)\}$? Per tali valori determinarne, se esiste, un potenziale.

SOL. $\underline{F}(x,y) = \left(F_1(x,y) \, ; \, F_2(x,y) \right)$

1) $$\partial_x F_2(x,y) = \partial_x \left(\frac{\beta x + y}{2x^2 + 2y^2} \right) = \frac{-2\beta x^2 + 2\beta y^2 - 4xy}{(2x^2+2y^2)^2}$$

$$\partial_y F_1(x,y) = \partial_y \left(\frac{x + \alpha y}{2x^2 + 2y^2} \right) = \frac{2\alpha x^2 - 2\alpha y^2 - 4xy}{(2x^2+2y^2)^2}$$

$\underline{F}$ è irrotazionale $\Longleftrightarrow \begin{cases} -2\beta = 2\alpha \\ 2\beta = -2\alpha \end{cases} \Longleftrightarrow \boxed{\alpha = -\beta}$

2) $\alpha = -\beta$, $\int_\Gamma \underline{F} \cdot d\underline{r}$ $\quad r: \begin{cases} x = \cos t \\ y = \operatorname{sen} t \end{cases} \quad t \in [0; 2\pi]$

$r': \begin{cases} x' = -\operatorname{sen} t \\ y' = \cos t \end{cases}$

$$\int_\Gamma \underline{F} \cdot d\underline{r} = \int_0^{2\pi} \underline{F}(\underline{r}(t)) \cdot \underline{r}'(t)\, dt = \quad (\beta = -\alpha)$$

$$= \int_0^{2\pi} \left(\frac{\cos t + \alpha \operatorname{sen} t}{2} ; \frac{-\alpha \cos t + \operatorname{sen} t}{2} \right) \cdot (-\operatorname{sen} t, \cos t)\, dt$$

$$= \frac{1}{2} \int_0^{2\pi} \left(\cancel{-\operatorname{sen} t \cos t} - \alpha \operatorname{sen}^2 t - \alpha \cos^2 t + \cancel{\operatorname{sen} t \cos t} \right) dt =$$

$$= \frac{1}{2}\int_0^{2\pi} -\alpha\, dt = -\frac{1}{2}\alpha(2\pi) = -\pi\alpha.$$

3) $$\int_\Gamma \underline{F}\cdot d\underline{r} = -\alpha\pi = 0 \iff \alpha = 0.$$

$\underline{F}$ PUÒ essere conservativo se e solo se $\alpha = \beta = 0$.

Cerco un potenziale:

$$\underline{F}(x,y) = \left(\frac{x}{2x^2+2y^2} ; \frac{y}{2x^2+2y^2} \right)$$

$$\begin{cases} U_x = F_1 \\ U_y = F_2 \end{cases}$$

$$U_x = \frac{x}{2x^2+2y^2} \implies U(x,y) = \frac{1}{4}\int \frac{4x}{2x^2+2y^2}\,dx =$$

$$= \frac{1}{4}\ln(2x^2+2y^2) + c(y)$$

$$F_2 = U_y = \partial_y\left(\frac{1}{4}\ln(2x^2+2y^2) + c(y)\right) =$$

$$= \frac{1}{\cancel{4}} \cdot \frac{1\cdot\cancel{4}y}{2x^2+2y^2} + c_y(y)$$

$$\cancel{\frac{y}{2x^2+2y^2}} + c_y(y) = \cancel{\frac{y}{2x^2+2y^2}}$$

$$c_y = 0 \implies c(y) = c$$

$$\implies U(x,y) = \frac{1}{4}\ln(2x^2+2y^2) + c$$

$$= \frac{1}{4}\ln 2 + \frac{1}{4}\ln(x^2+y^2) + c$$

$$= \frac{1}{4}\ln(x^2+y^2) + C_1 \, , \quad C_1 \in \mathbb{R}.$$

Se $\alpha = \beta = 0$:

$U(x,y) = \frac{1}{4}\ln(x^2+y^2) + C$, $\underline{F}$ è conservativo

ESERCIZIO 3. Si consideri il campo vettoriale

$$\underline{F}(x,y) = (\cos y + 3)\,\underline{i} + (-x\sin y + g(x))\,\underline{j}$$

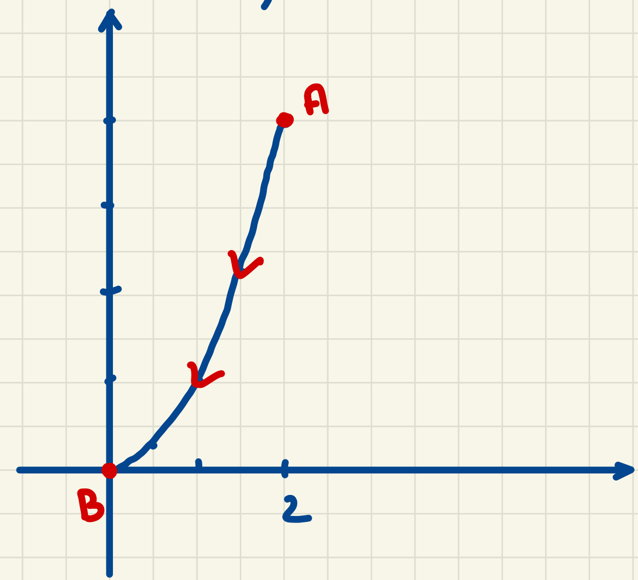
dove $g(x)$ è definita e derivabile su $\mathbb{R}$ tale che $g(1) = 1$.

Per quali $g(x)$ $\underline{F}$ è conservativo?

Per tali funzioni si determini un potenziale e si calcoli $\int_\gamma \underline{F} \cdot d\underline{r}$ essendo γ l'arco di parabola $y = x^2$ $x \in [0; 2]$ dall'alto verso il basso.

SOL.

$D_{\underline{F}} = \mathbb{R}^2$ sempl. connesso



$\partial_x F_2(x,y) = \partial_x \left(-x \operatorname{sen} y + g(x)\right) =$

$= -\operatorname{sen} y + g'(x)$

$\partial_y F_1(x,y) = \partial_y (\cos y + 3) = -\operatorname{sen} y.$

$\underline{F}$ è irrotazionale $\Longleftrightarrow$ $\partial_x F_2 = \partial_y F_1$ $\Longleftrightarrow$

$$-\operatorname{sen} y + g'(x) = -\operatorname{sen} y$$

$$g'(x) = 0$$

$$g(x) = c$$

$$g(1) = 1 \implies c = 1$$

$$\boxed{g(x) = 1}$$

$$\underline{F}(x,y) = (\cos y + 3;\ -x \operatorname{sen} y + 1)$$

Cerco $U(x,y)$ t.c. $\nabla U(x,y) = \underline{F}(x,y)$

$$\begin{cases} U_x = F_1 \\ U_y = F_2 \end{cases} \longrightarrow \begin{cases} U_x(x,y) = \cos y + 3 \\ U_y(x,y) = -x \operatorname{sen} y + 1 \end{cases}$$

$$U(x,y) = \int (\cos y + 3)\,dx = x\cos y + 3x + c(y)$$

$$F_2 = U_y = \partial_y\left(x\cos y + 3x + c(y)\right) =$$

$$= -x\,\text{sen}\,y + c_y(y)$$

$$-x\,\cancel{\text{sen}\,y} + 1 = -\cancel{x\,\text{sen}\,y} + c_y(y)$$

$$c_y(y) = 1$$

$$c(y) = \int 1\,dy = y + c.$$

$$U(x,y) = x\cos y + 3x + y + c \qquad c \in \mathbb{R}.$$

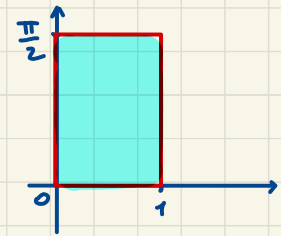
$$\int_\gamma \underline{F}\cdot \underline{dr} = U(0,0) - U(2;4) = \cancel{c} - 2\cos 4 - 6 - 4 - \cancel{c}$$

$$= -2\cos 4 - 10$$

INTEGRALI DOPPI IN COORDINATE CARTESIANE

ESERCIZIO 4. Sia $D=[0;1]\times\left[0;\frac{\pi}{2}\right]$. Calcolare

$$\iint_D y\,\mathrm{sen}\,xy\,dx\,dy.$$



SOL.

$$\iint_D y\,\mathrm{sen}\,xy\,dx\,dy = \begin{cases} \displaystyle\int_0^1\left(\int_0^{\pi/2} y\,\mathrm{sen}\,xy\,dy\right)dx & \text{NON CONVIENE}\\[2ex] \displaystyle\int_0^{\pi/2}\left(\int_0^1 y\,\mathrm{sen}\,xy\,dx\right)dy \end{cases} =$$

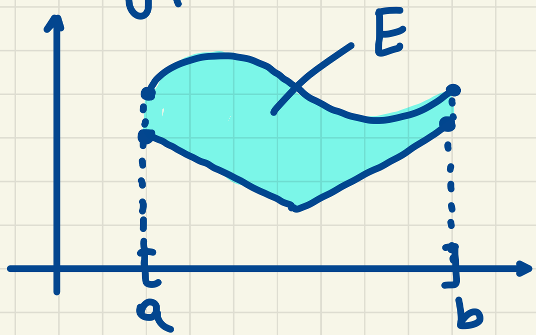
$$= \int_0^{\pi/2}\left[-\cos xy\right]_0^1 dy = \int_0^{\frac{\pi}{2}}\left(-\cos y+1\right)dy =$$

$$= \left[-\operatorname{sen} y + y\right]_0^{\pi/2} = -1 + \frac{\pi}{2} - 0 = \frac{\pi}{2} - 1.$$

NOTA. Sia $E \subset \mathbb{R}^2$. E è y-semplice

$$E = \{(x,y) \in \mathbb{R}^2 \mid x \in [a,b],\ g_1(x) \le y \le g_2(x)\}$$

$$g_1, g_2 : [a,b] \to \mathbb{R} \text{ continue}$$

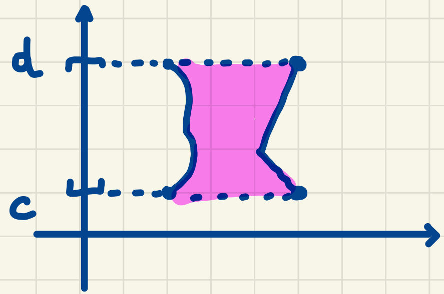


$$\int_E f(x,y)\,dx\,dy = \int_a^b \left(\int_{g_1(x)}^{g_2(x)} f(x,y)\,dy \right) dx$$

E è x-semplice se

$$E = \{(x,y) \in \mathbb{R}^2 / \ y \in [c,d],\ h_1(y) \le x \le h_2(y)\}$$

$h_1, h_2 : [c, d] \longrightarrow \mathbb{R}$ continue.

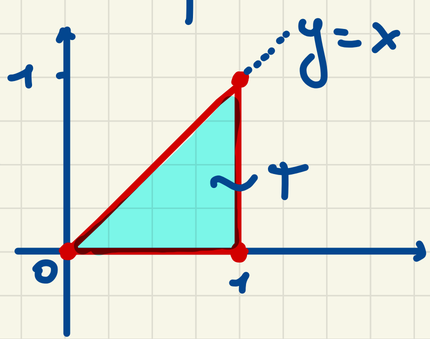


$$\iint_E f(x,y)\,dx\,dy = \int_c^d \left(\int_{h_1(y)}^{h_2(y)} f \, dx \right) dy$$

ESERCIZIO 5. Sia T il triangolo di vertici $(0;0)$ $(1,1)$, $(1,0)$. Calcolare $\iint_T xy\,dx\,dy$.

SOL.

$$T = \{ (x,y) \in \mathbb{R}^2 / \ x \in [0,1] \quad 0 \le y \le x \}$$



T è y-semplice

$$\iint_T xy\,dx\,dy = \int_0^1 \left(\int_0^x xy\,dy \right) dx =$$

$$= \int_0^1 \left[x \frac{y^2}{2} \right]_0^x dx = \int_0^1 \left(\frac{x^3}{2} - 0 \right) dx = \left[\frac{x^4}{8} \right]_0^1 = \frac{1}{8}.$$

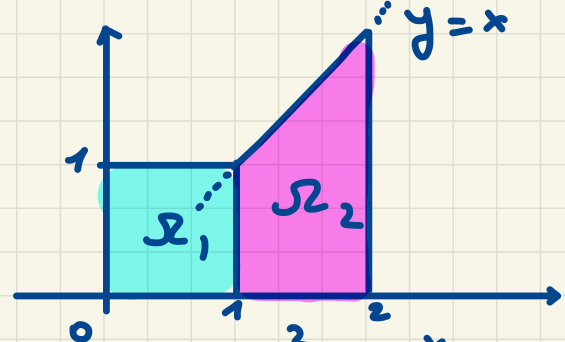
NOTA. Sia Ω un dominio regolare tale che $\Omega = \Omega_1 \cup \Omega_2 \cup \dots \cup \Omega_k$, $|\Omega_i \cap \Omega_j| = 0 \ \forall i \neq j$. Allora

$$\iint_\Omega f \, dx dy = \sum_{i=1}^k \iint_{\Omega_i} f \, dx dy$$

ESERCIZIO 6. Calcolare $\iint_\Omega xy \, dx dy$ con

$$\Omega = \{ (x,y) \in \mathbb{R}^2 / \ x \in [0,1], \ y \in [0,1] \} \cup \{ (x,y) \in \mathbb{R}^2 / \ 1 \leq x \leq 2, \ 0 \leq y \leq x \} = \Omega_1 \cup \Omega_2 .$$

SOL.



$$\iint_{\Omega} xy\,dx\,dy = \iint_{\Omega_1} xy\,dx\,dy +$$

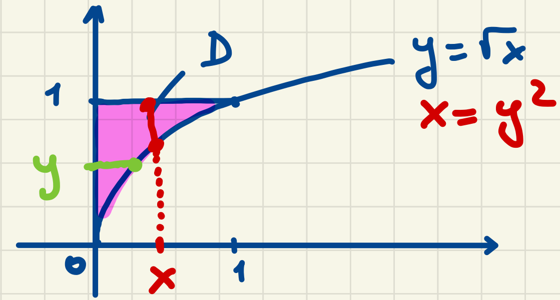
$$+ \iint_{\Omega_2} xy\,dx\,dy = \int_0^1 \left(\int_0^1 xy\,dy \right) dx + \int_1^2 \left(\int_0^x xy\,dy \right) dx$$

$$= \int_0^1 \left[x\frac{y^2}{2} \right]_0^1 dx + \int_1^2 \left[x\frac{y^2}{2} \right]_0^x dx =$$

$$= \int_0^1 \frac{x}{2}\,dx + \int_1^2 \frac{x^3}{2}\,dx = \left[\frac{x^2}{4} \right]_0^1 + \left[\frac{x^4}{8} \right]_1^2 = \frac{17}{8}.$$

ESERCIZIO 7. Sia $D = \{(x,y) \in \mathbb{R}^2 / 0 \le x \le 1, \sqrt{x} \le y \le 1\}$. Calcolare $\iint_D \operatorname{sen} y^3 \, dx\,dy$.

SOL.



D è y-semplice:

$$\iint_D \operatorname{sen} y^3 \, dx\,dy = \int_0^1 \left(\int_{\sqrt{x}}^1 \operatorname{sen} y^3 \, dy \right) dx$$

NON LO SO CALCOLARE!

Trasformo il dominio da y-semplice a x-semplice e scambio l'ordine di integrazione:

$$D' = \{(x,y) \in \mathbb{R}^2 / 0 \le y \le 1, \quad 0 \le x \le y^2\}:$$

$$\iint_D \operatorname{sen} y^3\,dx\,dy = \iint_{D'} \operatorname{sen} y^3\,dx\,dy = \int_0^1\left(\int_0^{y^2}\operatorname{sen} y^3\,dx\right)dy$$

$$= \int_0^1 \operatorname{sen} y^3\,[x]_0^{y^2}\,dy = \frac{1}{3}\int_0^1 3y^2 \operatorname{sen} y^3\,dy =$$

$$= \frac{1}{3}\left[-\cos y^3\right]_0^1 = -\frac{1}{3}(\cos 1 - 1) = \frac{1-\cos 1}{3}$$

ESERCIZIO 8. Sia

$$\iint_D f\,dx\,dy = \underbrace{\int_0^1\left(\int_{-3x}^{x^4} f\,dy\right)dx}_{1} + \underbrace{\int_1^2\left(\int_{x^2-4}^{(2-x)^3} f\,dy\right)dx}_{2}$$

Rappresentare D e scambiare l'ordine di

integrazione.

$$D_1 = \{(x,y) \in \mathbb{R}^2 / \ x \in [0;1], \ -3x \leq y \leq x^4\}$$

$$D_2 = \{(x,y) \in \mathbb{R}^2 / \ x \in [1;2], \ x^2-4 \leq y \leq (2-x)^3\}$$

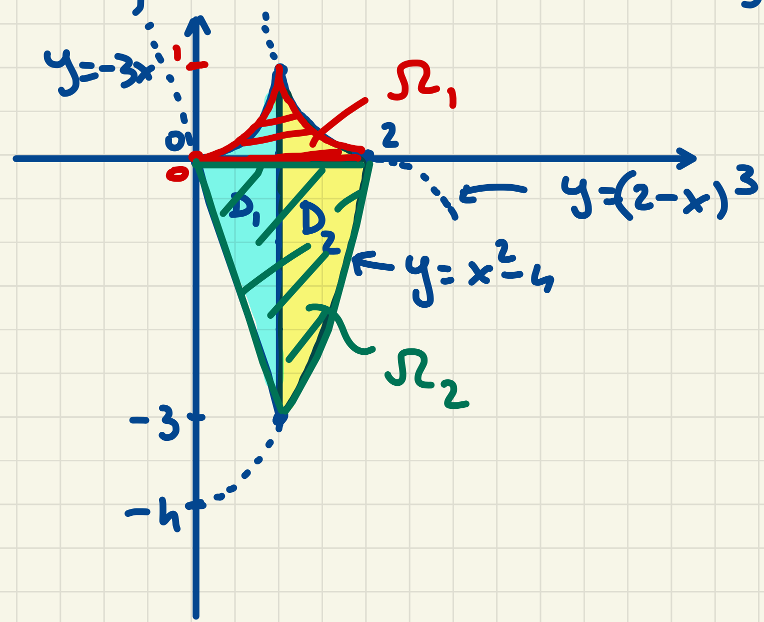
$$D = D_1 \cup D_2$$

$$y = -3x \longrightarrow x = -\frac{1}{3}y$$

$$y = x^4 \longrightarrow x = \sqrt[4]{y}$$

$$y = (2-x)^3 \longrightarrow x = 2-\sqrt[3]{y}$$

$$y = x^2-4 \longrightarrow x = \sqrt{y+4}$$



$$\Omega_1 = \left\{ (x,y) \in \mathbb{R}^2 / \ y \in [0;1] \ ; \ \sqrt[4]{y} \le x \le 2 - \sqrt[3]{y} \right\}$$

$$\Omega_2 = \left\{ (x,y) \in \mathbb{R}^2 / \ y \in [-3;0] \ ; \ -\frac{1}{3}y \le x \le \sqrt{y+4} \right\}.$$

$$\iint_D f \, dx dy = \iint_\Omega f \, dx dy = \int_0^1 \left(\int_{\sqrt[4]{y}}^{2-\sqrt[3]{y}} f \, dx \right) dy + \int_{-3}^{0} \left(\int_{-\frac{1}{3}y}^{\sqrt{y+4}} f \, dx \right) dy.$$

INTEGRALI DOPPI E PARTICOLARI SIMMETRIE

- f è dispari rispetto all'asse y se $f(-x,y) = -f(x,y)$
- f " " " " x " $f(x,-y) = -f(x,y)$
- f è simmetrica rispetto ad $O(0,0)$ se $f(-x,-y) = -f(x,y)$

Se D è un dominio simmetrico rispetto all'asse y (x) e se f è dispari rispetto all'asse y (x) allora:

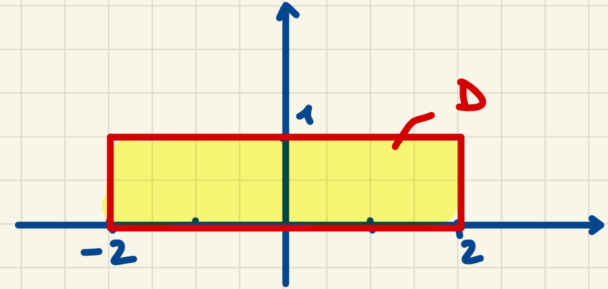
$$\iint_D f \, dx\,dy = 0$$

Se D è simmetrico rispetto all'origine ed f è simmetrica rispetto all'origine, allora

$$\iint_D f \, dx\,dy = 0.$$

ESERCIZIO 9. Calcolare $\iint_D \left(2 + 3x \log(x^4 + y^3)\right) dx\,dy$ essendo $D = [-2;2] \times [0;1]$.

SOL.



$$\iint_D \left(2+3x\log(x^4+y^3)\right)dx\,dy =$$

$$= \iint_D 2\,dx\,dy + \iint_D \underbrace{\cancel{3x\log(x^4+y^3)}}_{g(x,y)}\,dx\,dy = (*)$$

D è simm. risp. all'asse y.

- D è simm. risp. asse y
- $g(-x,y) = -3x\log(x^4+y^3) = -g(x,y)$

$$\left.\begin{array}{l} \bullet\ D \text{ è simm. risp. asse } y \\ \bullet\ g(-x,y) = -3x\log(x^4+y^3) = -g(x,y) \end{array}\right\} \Rightarrow \iint_D g = 0$$

$$(*) = 2\iint_D dx\,dy = 2\,\text{Area}(D) = 2\cdot 4 = 8.$$

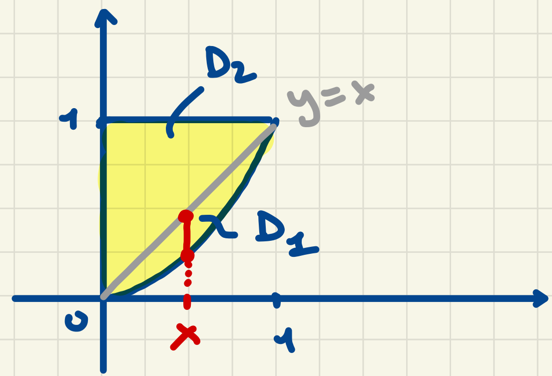
INTEGRALI DOPPI E VALORI ASSOLUTI

ESERCIZIO 10. Calcolare $\iint_D |x-y|\,dx\,dy$ con

$$D = \{ (x,y) \in \mathbb{R}^2 / \ 0 \le x \le 1 \ \wedge \ x^2 \le y \le 1 \}$$

SOL.

$$|x-y| = \begin{cases} x-y & \text{se } y \le x \\ y-x & \text{se } y > x \end{cases}$$



Su D_1 $\quad f(x,y) = x-y$

Su D_2 $\quad f(x,y) = y-x$

$$\iint_D |x-y|\,dx\,dy = \iint_{D_1} (x-y)\,dx\,dy + \iint_{D_2} (y-x)\,dx\,dy =$$

$$= \int_0^1 \left(\int_{x^2}^{x} (x-y)\,dy \right) dx + \int_0^1 \left(\int_x^1 (y-x)\,dy \right) dx =$$

$$= \int_0^1 \left[xy - \frac{y^2}{2} \right]_{x^2}^{x} dx + \int_0^1 \left[\frac{y^2}{2} - xy \right]_x^1 dx =$$

$$= \int_0^1 \left(x^2 - \cancel{\frac{x^2}{2}} - x^3 + \frac{x^4}{2} \right) dx + \int_0^1 \left(\frac{1}{2} - x - \cancel{\frac{x^2}{2}} + \cancel{x^2} \right) dx$$

$$= \int_0^1 \left(\frac{1}{2}x^4 - x^3 + x^2 - x + \frac{1}{2} \right) dx = \left[\frac{x^5}{10} - \frac{x^4}{4} + \frac{x^3}{3} - \frac{x^2}{2} + \frac{1}{2}x \right]_0^1$$

$$= \frac{1}{10} - \frac{1}{4} + \frac{1}{3} - \cancel{\frac{1}{2}} + \cancel{\frac{1}{2}} = \boxed{\frac{11}{60}}$$