INTEGRALI DI SUPERFICIE

21-5-2020

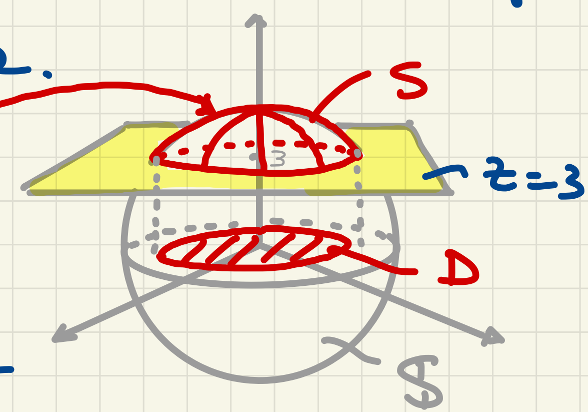
ESERCIZIO 1.

In $\mathbb{R}^3$ sia S la calotta sferica ottenuta intersecando la sfera di centro $(0;0;0)$ e raggio $\sqrt{10}$ con il semispazio $z \geq 3$. Si disegni S e se ne calcoli l'area.

SOL.

$$\mathcal{S}: \; x^2+y^2+z^2=10$$

$$z = f(x,y) = \sqrt{10-x^2-y^2}$$



$$\partial D = \begin{cases} x^2+y^2+z^2=10 \\ z=3 \end{cases} \Rightarrow x^2+y^2=1$$

$$D = \{(x,y) \in \mathbb{R}^2 / \; x^2+y^2 \leq 1\}$$

$$A(S) = \iint_S d\sigma \quad \text{con} \quad d\sigma = \sqrt{1+|\nabla f|^2}\, dx\,dy$$

$$A(S) = \iint_S d\sigma = \iint_D \sqrt{1+|\nabla f|^2}\, dx\, dy = (*)$$

$$f(x,y) = \sqrt{10-x^2-y^2} \;; \quad \nabla f(x,y) = \begin{bmatrix} \dfrac{-x}{\sqrt{10-x^2-y^2}} \\ \dfrac{-y}{\sqrt{10-x^2-y^2}} \end{bmatrix}$$

$$|\nabla f(x,y)| = \sqrt{\frac{x^2}{10-x^2-y^2} + \frac{y^2}{10-x^2-y^2}} =$$

$$= \sqrt{\frac{x^2+y^2}{10-x^2-y^2}}$$

$$(*) = \iint_D \sqrt{1+\frac{x^2+y^2}{10-(x^2+y^2)}}\, dx\, dy = \int_0^{2\pi} \left(\int_0^1 \sqrt{1+\frac{\rho^2}{10-\rho^2}}\, \rho\, d\rho \right) d\theta$$

$$= 2\pi \int_0^1 \rho \sqrt{\frac{10}{10-\rho^2}}\, d\rho = \frac{\cancel{2}\pi\sqrt{10}}{-\cancel{2}} \int_0^1 -2\rho\,(10-\rho^2)^{-1/2}\, d\rho =$$

$$= -\pi\sqrt{10}\left[2(10-\rho^2)^{1/2}\right]_0^1 = -\pi\sqrt{10}\left(2\cdot 3 - 2\sqrt{10}\right) =$$

$$= 2\pi\sqrt{10}\left(\sqrt{10}-3\right).$$

$$= 2\pi\left(10-3\sqrt{10}\right).$$

ESERCIZIO 2.

Calcolare l'area della superficie

$$\Sigma:\ \underline{r}(u,v) = (u\cos v,\ u\operatorname{sen} v,\ u^2)$$

con $(u,v) \in [0;1]\times[0;\pi) = T$.

SOL. Ricordo $d\sigma = |\underline{r}_u \times \underline{r}_v|\,du\,dv.$

$$\underline{r}_u(u,v) = (\cos v, \ \text{sen}\, v, \ 2u)$$

$$\underline{r}_v(u,v) = (-u\,\text{sen}\, v, \ u\cos v, \ 0)$$

$$\underline{r}_u \times \underline{r}_v = \left(-2u^2\cos v; \ -2u^2\,\text{sen}\, v; \ u\right)$$

$$|\underline{r}_u \times \underline{r}_v| = \sqrt{4u^4 + u^2} = |u|\sqrt{4u^2+1} = u\sqrt{4u^2+1}$$

$$A(\Sigma) = \iint_{\Sigma} d\sigma = \iint_{T} u\sqrt{4u^2+1}\, du\, dv =$$

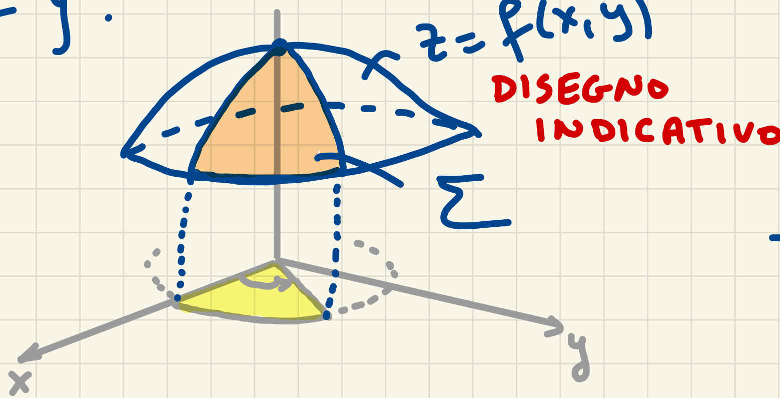
$$= \int_0^{\pi} \left(\int_0^1 u\,(4u^2+1)^{1/2}\, du \right) dv =$$

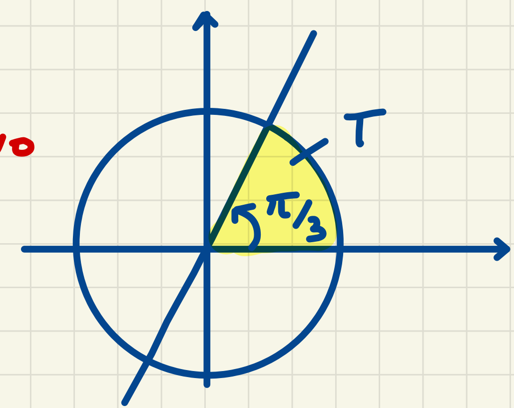
$$= \frac{\pi}{8} \int_0^1 8u\,(4u^2+1)^{1/2}\, du = \frac{\pi}{\cancel{8}_{4}} \left[(4u^2+1)^{3/2} \cdot \frac{\cancel{2}}{3} \right]_0^1 =$$

$$= \frac{\pi}{12}\left(5^{3/2} - 1\right) = \frac{\pi}{12}\left(5\sqrt{5} - 1\right).$$

ESERCIZIO 3. Calcolare $\iint_\Sigma z \, d\sigma$ dove Σ è la porzione di superficie di eq. $z = xy$ che si proietta in $T = \{(x,y) \in \mathbb{R}^2 \mid 0 \le y \le \sqrt{3}x, \ x^2+y^2 \le 1\}$.

SOL.





$$z = f(x,y) = xy \qquad \nabla f(x,y) = \begin{bmatrix} y \\ x \end{bmatrix} \qquad |\nabla f| = \sqrt{x^2+y^2}$$

$$d\sigma = \sqrt{1+|\nabla f|^2}\,dx\,dy = \sqrt{1+x^2+y^2}\,dx\,dy$$

$$\iint_{\Sigma} z\,d\sigma = \iint_{T} xy\sqrt{1+(x^2+y^2)}\,dx\,dy =$$

$$= \int_0^{\pi/3}\left(\int_0^1 \rho^2 \operatorname{sen}\theta\cos\theta\sqrt{1+\rho^2}\,\rho\,d\rho\right)d\theta =$$

$$= \int_0^{\pi/3}\operatorname{sen}\theta\cos\theta\,d\theta \cdot \int_0^1 \rho^3\sqrt{1+\rho^2}\,d\rho =$$

$$= \left[\frac{\operatorname{sen}^2\theta}{2}\right]_0^{\pi/3} \cdot \int_0^1 \rho^2 \cdot \rho\sqrt{1+\rho^2}\,d\rho =$$

$\sqrt{1+\rho^2} = t \qquad 1+\rho^2 = t^2 \qquad \rho^2 = t^2-1$

$$\cancel{2}\rho\, d\rho = \cancel{2}t\, dt$$

$$= \frac{1}{2}\left(\frac{\sqrt{3}}{2}\right)^2 \cdot \int_1^{\sqrt{2}} (t^2-1)t^2\, dt = \frac{3}{8}\left[\frac{t^5}{5} - \frac{t^3}{3}\right]_1^{\sqrt{2}} =$$

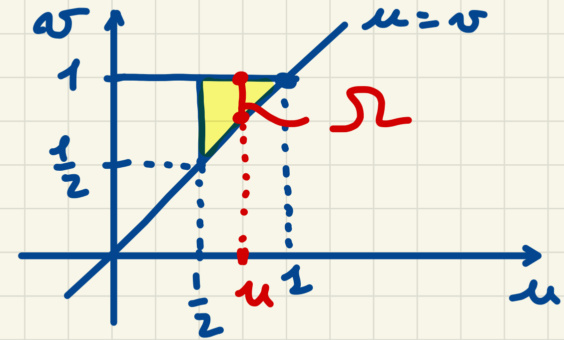
$$= \frac{3}{8}\left(\frac{4}{5}\sqrt{2} - \frac{2}{3}\sqrt{2} - \frac{1}{5} + \frac{1}{3}\right) = \frac{\cancel{3}}{8}\,\frac{12\sqrt{2}-10\sqrt{2}-3+5}{\cancel{15}\ 5} =$$

$$= \frac{2\sqrt{2}+2}{40} = \frac{\sqrt{2}+1}{20}.$$

ESERCIZIO 4. Sia data la superficie $\Sigma = \{(x,y,z) \in \mathbb{R}^3 / (x,y,z) = (\operatorname{sen} uv, \cos uv, u),\ (u,v) \in \Omega\}$ essendo $\Omega = \{(u,v) \in \mathbb{R}^2 / \frac{1}{2} \leq u \leq v,\ v \leq 1\}$.

1) Calcolare Area (Σ)

2) Calcolare $\displaystyle\iint_{\Sigma} \frac{x^2+y^2}{z^3}\, d\sigma$



SOL.

$$\underline{r}(u,v) = (\operatorname{sen} uv,\ \cos uv,\ u)$$

$$\underline{r}_u(u,v) = (v\cos uv,\ -v\operatorname{sen} uv,\ 1)$$

$$\underline{r}_v(u,v) = (u\cos uv,\ -u\operatorname{sen} uv,\ 0)$$

$$\underline{r}_u \times \underline{r}_v = (u\operatorname{sen} uv;\ u\cos uv,\ 0)$$

$$|\underline{r}_u \times \underline{r}_v| = \sqrt{u^2\operatorname{sen}^2 uv + u^2\cos^2 uv} = \sqrt{u^2} = u$$

$$d\sigma = |\underline{r}_u \times \underline{r}_v|\, du\, dv = u\, du\, dv.$$

1) $A(\Sigma) = \iint_{\Sigma} d\sigma = \iint_{\Omega} u \, du \, dv = \int_{1/2}^{1} \left(\int_{u}^{1} u \, dv \right) du =$

$= \int_{1/2}^{1} u(1-u) \, du = \int_{1/2}^{1} (u - u^2) \, du = \left[\frac{u^2}{2} - \frac{u^3}{3} \right]_{1/2}^{1} =$

$= \frac{1}{2} - \frac{1}{3} - \frac{1}{8} + \frac{1}{24} = \frac{12 - 8 - 3 + 1}{24} = \frac{1}{12}.$

2) $\iint_{\Sigma} \frac{x^2 + y^2}{z^3} \, d\sigma = \iint_{\Omega} \frac{\operatorname{sen}^2 uv + \cos^2 uv}{u^{\cancel{3}\,2}} \cdot \cancel{u} \, du \, dv =$

$\underline{r}(u, v) = (\operatorname{sen} uv; \cos uv; u)$

$= \iint_{\Omega} u^{-2} \, du \, dv = \int_{1/2}^{1} \left(\int_{u}^{1} u^{-2} \, dv \right) du =$

$$= \int_{1/2}^{1} u^{-2}(1-u)\,du = \int_{1/2}^{1} (u^{-2} - u^{-1})\,du =$$

$$= \left[-\frac{1}{u} - \ln u \right]_{1/2}^{1} = -1 - \cancel{\ln 1} + 2 + \ln\frac{1}{2} =$$

$$= 1 - \ln 2 .$$

FLUSSI

ESERCIZIO 5. Calcolare il flusso del campo $\underline{F}(x,y,z) = xy\,\underline{i} + xy\,\underline{j} + z\,\underline{k}$ attraverso la superficie

$$\Sigma = \{ (x,y,z) \in \mathbb{R}^3 / z = 1 - x^2 - y^2,\ z \geq 0 \}$$

scegliendo il versore $\underline{n}$ con la terza componente non negativa.

SOL.

$$\phi_{\Sigma}(F) = \iint_{\Sigma} F \cdot \underline{n} \, d\sigma = \iint_{T} F \cdot (r_u \times r_v) \, du \, dv.$$

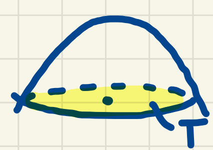
$$\underline{r}(x,y) = (x; y; f(x,y)) = (x, y, 1-x^2-y^2)$$

$$\underline{r}_x(x,y) = (1, 0, -2x)$$

$$\underline{r}_y(x,y) = (0, 1, -2y)$$

$$\underline{r}_x \times \underline{r}_y = (2x, 2y, 1) = (-f_x, -f_y, 1)$$

$$\partial T : \begin{cases} z = 1-x^2-y^2 \\ z = 0 \end{cases} \Rightarrow \begin{aligned} 1-x^2-y^2 &= 0 \\ x^2+y^2 &= 1 \end{aligned}$$



$$T = \{ (x,y) \in \mathbb{R}^2 \ / \ x^2+y^2 \leq 1 \}$$

$$\iint_{\Sigma} F \cdot u \, d\sigma =$$

$$\underline{F}(x,y,z) = (xy, \ xy, \ z)$$

$$\underline{\sigma}(x,y) = (x, \ y, \ 1-x^2-y^2)$$

$$= \iint_T (xy, xy, 1-x^2-y^2) \cdot (2x, 2y, 1) \, dx \, dy =$$

$$= \iint_T (2x^2y + 2xy^2 + 1 - x^2 - y^2) \, dx \, dy =$$

$$= \iint_T \underbrace{\cancel{(2x^2y + 2xy^2)}}_{g(x,y)} \, dx \, dy + \iint_T [1 - (x^2+y^2)] \, dx \, dy =$$

$$g(-x,-y) = 2x^2(-y) + 2y^2(-x) = -g(x,y)$$

$$\left.\begin{array}{l} g \text{ è simm. risp. ad } O \\ T \text{ è simm. risp. ad } O \end{array}\right\} \Rightarrow \iint_T g = 0$$

$$= \int_0^{2\pi} \left(\int_0^1 (1-\rho^2)\rho \, d\rho \right) d\theta = 2\pi \left[\frac{\rho^2}{2} - \frac{\rho^4}{4} \right]_0^1 =$$

$$= 2\pi \cdot \frac{1}{4} = \frac{\pi}{2}.$$

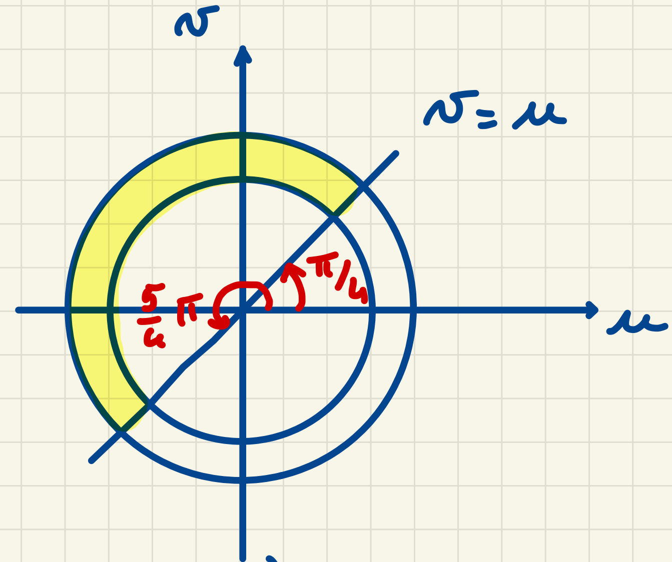
ESERCIZIO 6. Calcolare il flusso del campo $\underline{F}(x,y,z) = \underline{i} + \underline{k}$ attraverso la superficie Σ di equazione

$$\underline{r}(u,v) = u^2 \underline{i} + \sqrt{2}\, uv \, \underline{j} + v^2 \underline{k}$$

con $(u,v) \in T = \{ (u,v) \in \mathbb{R}^2 \,/\, 1 \le u^2 + v^2 \le 2,$

$u < v \}$.

SOL.



$$\underline{r}(u,v) = \left(u^2, \sqrt{2}\,uv, v^2\right)$$

$$\underline{r}_u(u,v) = \left(2u, \sqrt{2}\,v, 0\right)$$

$$\underline{r}_v(u,v) = \left(0; \sqrt{2}\,u, 2v\right)$$

$$\underline{r}_u \times \underline{r}_v = \left(2\sqrt{2}\,v^2, -4uv, 2\sqrt{2}\,u^2\right)$$

$$\underline{F}(x,y,z) = (1, 0, 1)$$

$$\iint_{\Sigma} \underline{F} \cdot \underline{n}\, d\sigma = \iint_{T} (1,0,1) \cdot \left(2\sqrt{2}\,v^2, -4uv, 2\sqrt{2}\,u^2\right) du\, dv$$

$$= \iint_T \left(2\sqrt{2}\,v^2 + 2\sqrt{2}\,u^2\right) du\,dv =$$

$$= 2\sqrt{2} \iint_T (u^2 + v^2)\, du\, dv = 2\sqrt{2}\int_{\pi/4}^{5/4\pi} \left(\int_1^{\sqrt{2}} \rho^2 \rho\, d\rho \right) d\theta =$$

$$= 2\sqrt{2}\pi \left[\frac{\rho^4}{4} \right]_1^{\sqrt{2}} = 2\sqrt{2}\pi \left(1 - \frac{1}{4}\right) = 2\sqrt{2}\,\pi \cdot \frac{3}{4} = \frac{3}{2}\sqrt{2}\,\pi.$$

ESERCIZIO 7 Sia C la regione di spazio tra la superficie conica $z = 2\sqrt{x^2+y^2}$ e il piano $z = 4$. Supponiamo che C sia occupata da un solido di densità $\rho(x,y,z) = 4 - z$.

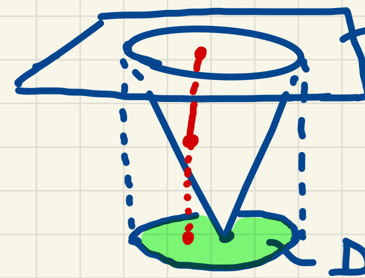
1) Calcolare la massa del solido

2) Calcolare il flusso di $\underline{F}(x,y,z) = (x,y,z)$ uscente da ∂C.

SOL.

$$\partial D: \begin{cases} z = 2\sqrt{x^2+y^2} \\ z = 4 \end{cases}$$

$$\sqrt{x^2+y^2} = 2$$
$$x^2+y^2 = 4$$



$$D = \{(x,y) : x^2+y^2 \leq 4\}$$

$$M = \iiint_C \rho(x,y,z)\,dx\,dy\,dz \underset{\text{FILI}}{=} \iint_D \left(\int_{2\sqrt{x^2+y^2}}^{4} (4-z)\,dz \right) dx\,dy =$$

$$= \iint_D \left[4z - \frac{z^2}{2} \right]_{2\sqrt{x^2+y^2}}^{4} dx\,dy =$$

$$= \iint_D \Big(\underbrace{16-8}_{8} - 8\sqrt{x^2+y^2} + 2(x^2+y^2) \Big)\, dx\, dy =$$

$$= \int_0^{2\pi} \left(\int_0^2 (8 - 8\rho + 2\rho^2)\rho \, d\rho \right) d\theta =$$

$$= 2\pi \left[\overset{4}{\cancel{8}} \frac{\rho^2}{\cancel{2}} - 8\frac{\rho^3}{3} + \cancel{2}\frac{\rho^4}{\underset{2}{\cancel{4}}} \right]_0^2 = 2\pi \left(16 - \frac{64}{3} + 8 \right) =$$

$$= 2\pi \left(24 - \frac{64}{3} \right) = \frac{16}{3}\pi .$$

2) $$\underset{\partial C}{\phi}(\underline{F}) = \iiint_C \operatorname{div} \underline{F} \, dx\, dy\, dz = \iiint_C 3 \, dx\, dy\, dz =$$

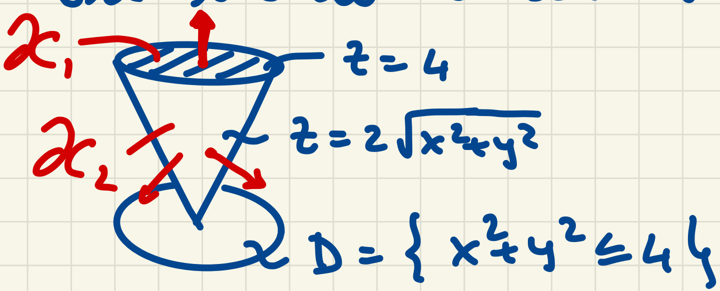
NOTA: $\underline{F}(x,y,z) = (\underset{F_1}{x}, \underset{F_2}{y}, \underset{F_3}{z})$ $\quad \operatorname{div} \underline{F} = F_{1x} + F_{2y} + F_{3z}$

$$= 1 + 1 + 1 = 3.$$

$$= 3 \iiint_C dx\,dy\,dz = 3 \; \mathrm{Vol}(C) = \cancel{3} \cdot \frac{\pi \cdot 4 \cdot 4}{\cancel{3}} =$$

$$= 16\pi .$$

In modo diretto:



$$\underset{\partial C}{\phi}(\underline{F}) = \underset{\partial C_1}{\phi}(\underline{F}) + \underset{\partial C_2}{\phi}(\underline{F})$$

$$\underset{\partial C_1}{\phi}(\underline{F}) = \iint_{\partial C_1} \underline{F} \cdot \underline{n} \, d\sigma = \iint_D (x, y, 4) \cdot (0, 0, 1)\, dx\,dy =$$

$$= \iint_D 4 \, dx\,dy = 4 \; \mathrm{Area}(D) = 16\pi$$

$$\underset{\partial C_2}{\phi(\underline{F})} = \iint_{\partial C_2} \underline{F} \cdot \underline{n} \, d\sigma =$$

$$= \iint_D \left(x ; y ; 2\sqrt{x^2+y^2} \right) \cdot \left(\frac{2x}{\sqrt{x^2+y^2}} ; \frac{2y}{\sqrt{x^2+y^2}} ; -1 \right) dx\,dy =$$

$$= \iint_D \left(\frac{2x^2}{\sqrt{x^2+y^2}} + \frac{2y^2}{\sqrt{x^2+y^2}} - 2\sqrt{x^2+y^2} \right) dx\,dy = 0 .$$

$$\phi_{TOT} = \phi_1 + \phi_2 = 16\pi + 0 = 16\pi$$

come visto con il teorema della div.

ESERCIZIO 8. Sia B la porzione di sfera di centro $(0,0,0)$ e raggio R contenuta nel primo ottante.

$$B = \{ (x,y,z) \in \mathbb{R}^3 \,/\, x^2+y^2+z^2 \le R^2, x \ge 0, y \ge 0, z \ge 0 \}$$

Dato il campo $\underline{F} = (xy, xz, yz)$:

1) Calcolare $\operatorname{div} \underline{F}$, $\operatorname{rot} \underline{F}$, $\operatorname{div}(\operatorname{rot} \underline{F})$

2) si calcoli il flusso di $\underline{F}$ uscente dalla superficie che delimita B. Quanto vale il flusso su ciascuna faccia della superficie?

SOL.

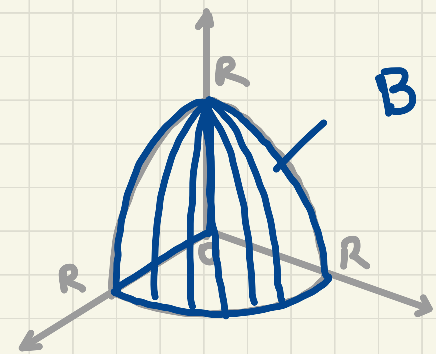
1) $\operatorname{rot} \underline{F} = (z-x;\ 0;\ z-x)$

$\operatorname{div} \underline{F} = 2y$

$\operatorname{div}(\operatorname{rot} \underline{F}) = 0$

FARE I CONTI

2) $\Phi_{\partial B}(\underline{F}) = \iiint_B \operatorname{div} F \, dx\,dy\,dz =$



$= \iiint_B 2y \, dx\,dy\,dz =$

COORD. SFERICHE

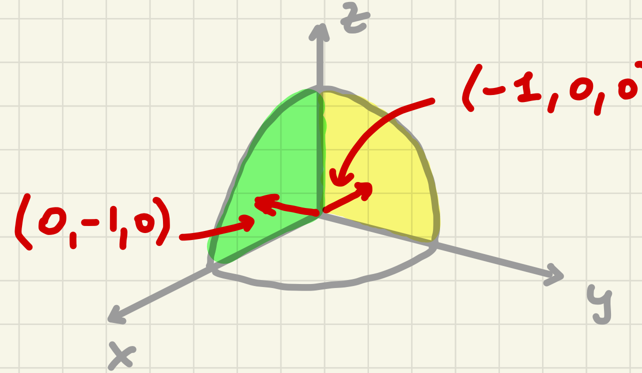
$= 2 \int_0^{\pi/2} \left(\int_0^{\pi/2} \left(\int_0^R \rho \operatorname{sen}\varphi \operatorname{sen}\theta \, \rho^2 \operatorname{sen}\varphi \, d\rho \right) d\theta \right) d\varphi$

$$= 2\int_0^{\pi/2}\left(\int_0^{\pi/2} \operatorname{sen}^2\varphi \operatorname{sen}\theta \left[\frac{\rho^4}{4}\right]_0^R\right) d\theta \bigg) d\varphi$$

$$= 2\frac{R^4}{4}\int_0^{\pi/2} \operatorname{sen}^2\varphi \left[-\cos\theta\right]_0^{\pi/2} d\varphi = \frac{R^4}{2}(1)\int_0^{\pi/2} \operatorname{sen}^2\varphi \, d\varphi$$

$$= \frac{R^4}{2}\left[\frac{\varphi - \operatorname{sen}\varphi\cos\varphi}{2}\right]_0^{\pi/2} = \frac{R^4\pi}{8}$$

$$F(x,y,z) = (xy,\ xz,\ yz)$$



$\partial B \cap \{x=0\}$

$$F(x,y,z) = (0,\ 0,\ yz)$$

↑ nulla sup.

$F \cdot \underline{n} = (0, 0, yz) \cdot (-1, 0, 0) = 0.$ $\phi_1 = 0$

$\partial B \cap \{y = 0\}$

$F(x, y, z) \underset{\uparrow}{=} (0; xz; 0)$ $n = (0, -1, 0)$

nella sup.

$$\phi_2 = \iint_{\partial B \cap \{y=0\}} \underline{F} \cdot \underline{n} \, d\sigma = \iint_{\{x^2+z^2 \leq R, \, x \geq 0, \, z \geq 0\}} -xz \, dx dz = \cdots = -\frac{R^4}{8}$$

coord. polari

Analogamente a ϕ_1 anche $\phi_3 = 0$ $(n = (0, 0, -1))$

$$\phi_4 = \phi_{TOT} - \cancel{\phi_1} - \phi_2 - \cancel{\phi_3} = \frac{R^4}{8}\pi - \left(-\frac{R^4}{8}\right)$$

$$= \frac{R^4}{8}(1 + \pi).$$