Functional Analysis

Saturday, 20 November 2021 12:50

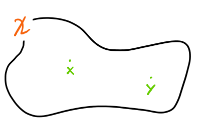
LINEAR ALGEBRA + REAL & COMPLEX ANALYSIS

STUDYING SPACES MADE OF FUNCTIONS, SEQUENCES AND LINEAR MAPS BETWEEN SPACES

TOPOLOGICAL - ALGEBRAIC STRUCTURS

SET COLLECTION OF POINTS

METRIC SPACE



$\mathcal{X}$ IS A SET

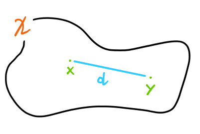
IN A SET WE CANNOT SAY ANYTHING ABOUT ITS POINTS OTHER THAN $x = y$, $x \neq y$

WE WANT TO GIVE MORE STRUCTURE TO SET $\mathcal{X}$

A SPACE W/ A DEFINITION OF DISTANCE SPACE $(\mathcal{X}, d)$

DISTANCE (MAP)

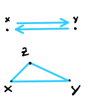
$d : \mathcal{X} \times \mathcal{X} \to [0, +\infty)$



1) $d(x,y) = 0 \longleftrightarrow x = y$ IDENTITY

2) $d(x,y) = d(y,x)$ SYMMETRY

3) $d(x,y) = d(x,z) + d(z,y) \quad \forall z$ TRIANGLE INEQUALITY



STANDARD METRICS

$\mathcal{X} = \mathbb{C} \qquad d(x,y) = \|x-y\|$ GEOMETRICAL

$\mathcal{X} = \mathbb{R}^N \qquad d(x,y) = \sqrt{(x_1-y_1)^2 + (x_2-y_2)^2 + \cdots + (x_N-y_N)^2}$ EUCLIDIAN

WE CAN DEFINE NEW METRICS

$\mathcal{X} = \mathbb{R}^N \qquad d(x,y) = \max\{|x_1-y_1|, |x_2-y_2|, \dots, |x_N-y_N|\}$ MAX

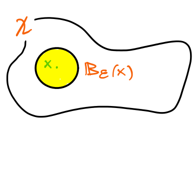
$\mathcal{X} = $ ANY SET $(\neq \emptyset) \qquad d(x,y) = \begin{cases} 0 & x=y \\ 1 & x \neq y \end{cases}$ DISCRETE

IN THE DISCRETE METRIC THERE ARE NO NEIGHBOURS AROUND A GIVEN POINTS SIMPLY BECAUSE THERE'S A FIX DISTANCE FROM ONE POINT TO ALL THE OTHER ONES ALL THE POINTS ARE ISOLATED

OPEN BALL

$(\mathcal{X}, d)$ METRIC SPACE $\mathbb{B}_\varepsilon(x)$ "OPEN ε-BALL AROUND $\mathcal{X}$"

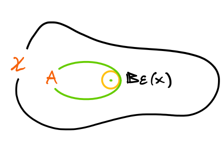
$\mathbb{B}_\varepsilon(x) = \{ y \in \mathcal{X} \mid d(x,y) < \varepsilon \}$ EVERYTHING INSIDE



NEVER EMPTY SINCE AT LEAST $\mathcal{X}$ IS INSIDE BALL CENTERED IN x WITH RADIUS ε

OPEN SETS

IF WE ARE INSIDE WE DON'T SEE THE BOUNDARY
IF YOU FIX POINT x THERE ARE ENOUGH POINTS AROUND $\mathcal{X}$ THAT ALL BELONG TO THE SET A

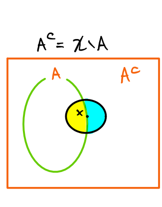


$A \subseteq \mathcal{X}$ IS OPEN IF $\forall x \in \mathcal{X} \; \exists \, \mathbb{B}_\varepsilon(x) \subseteq A$

CLOSED SETS

$A \subseteq \mathcal{X}$ IS CLOSED IF $A^c = \mathcal{X} \setminus A$ IS OPEN

BOUNDARY POINTS



$x \in \mathcal{X}$ IS A BOUNDARY POINT FOR A IF $\forall \varepsilon \; \mathbb{B}_\varepsilon(x) \cap A \neq \emptyset$, $\mathbb{B}_\varepsilon(x) \cap A^c \neq \emptyset$

IF THE BALL CENTERED IN $\mathcal{X}$ HAS BOTH POINTS OF A AND ITS COMPLEMENT

$\partial A = \{ x \in \mathcal{X} \mid x \text{ IS BOUNDARY POINT FOR } A \}$

A OPEN $\longleftrightarrow$ $A \cap \partial A = \emptyset$

A CLOSED $\longleftrightarrow$ $A \cap \partial A = A$

OPENNESS & CLOSEDNESS ARE NOT CONTRARY

CLOUSURE

SMALLEST CLOSED SET THAT CONTAINS A ADDING ALL THE MISSING BOUNDARY POINTS

$\bar{A} = A \cup \partial A$

WE CAN USE SEQUENCES TO DEFINE PROPERTIES OF A METRIC SPACE

SEQUENCES

ORDERED SET OF POINTS INSIDE THE METRIC SPACE $\mathcal{X}$

$(x_1, x_2, x_3, \dots) = (x_m)_{m \in \mathbb{N}} \qquad x : \mathbb{N} \to \mathcal{X}$

MAP FROM NATURAL NUMBERS $\mathbb{N}$ TO THE METRIC SPACE $\mathcal{X}$

CONVERGENCE

A SEQUENCE $(x_m)_{m \in \mathbb{N}}$ IN A METRIC SPACE $(\mathcal{X}, d)$ IL CALLED CONVERGENT IF THERE'S A LIMIT POINT $\tilde{x} \in \mathcal{X}$

$\forall \varepsilon > 0 \quad \exists N \in \mathbb{N} \quad \forall m \geq N \quad d(x_N, \tilde{x}) < \varepsilon$

$x_1 \quad x_2 \quad x_3 \quad \tilde{x}$

WE WRITE $x_N \to \tilde{x}$ OR $\lim_{n \to \infty} x_N = \tilde{x}$

SUCH NOTATION ARE ALWAYS GIVEN WITH RISPECT TO A METRIC d WE CAN USE SUCH NOTATION SINCE IN A METRIC $\exists! \, \tilde{x}$ THAT FULFILLS ALL THESE PROPERTIES

SEQUENCES & CLOSEDNESS

$A \subseteq \mathcal{X}$ IS CLOSED IF WE CAN'T LEAVE A FROM THE INSIDE USING JUST SEQUENCES

THE LIMIT OF A SEQUENCE MUST LIE IN A

$A \subseteq \mathcal{X}$ CLOSED $\longleftrightarrow$ $\forall$ CONVERGENT SEQ $(a_m)_{m \in \mathbb{N}} \subseteq A \qquad \lim_{N \to \infty} a_m \in A$

CHARACTERIZATION USING SEQUENCES

NON COMPLETE METRIC SPACES

$\mathcal{X} = (0,3)$

$A = (0,3)$ IS CLOSED

- $A^c = \mathcal{X} \setminus A = \emptyset$ OPEN
- EACH $(x_m)_{m \in \mathbb{N}} \subseteq A$ HAS A LIMIT $\tilde{x} \in A$

WHAT ABOUT THE SEQUENCE $\left(\frac{1}{m}\right)_{m \in \mathbb{N}}$?

SEQUENCE IN $\mathcal{X} \qquad d(x_{\underset{N}{m}}, x_{\underset{M}{m+1}}) \xrightarrow{m \to \infty} 0$

THE ONLY POSSIBLE LIMIT IS $0 \notin A$

A NON COMPLETE METRIC SPACE IS A SPACE THAT HAS A SEQUENCE THAT SHOULD CONVERGE BUT THERE IS NO POINT IN THE SPACE WHERE LE SEQUENCE LEADS TO

THE SPACE HAS A WHOLE

CAUCHY'S DEFINITION

A SEQUENCE $(x_m)_{m \in \mathbb{N}}$ IS CALLED "CAUCHY SEQUENCE" IF $\forall \varepsilon > 0 \; \exists N \in \mathbb{N} \quad \forall m \geq N : d(x_N, x_{N+1}) < \varepsilon$

GENERALIZATION OF A CONVERGENT SEQUENCE

COMPLETE SPACE

A METRIC SPACE WHERE ALL CAUCHY SEQUENCES CONVERGE

NORM

MAP $\|\cdot\| : X \to [0, +\infty) \qquad X = (\mathbb{R}, \mathbb{C})$

1) $\|x\| = 0 \longleftrightarrow x = \vec{0}$ POSITIVE DEFINITE

2) $\|\lambda x\| = \lambda \|x\|$ ABSOLUTE HOMOGENEUS

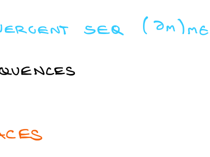
3) $\|x+y\| = \|x\| + \|y\|$ TRIANGLE INEQUALITY

$(X, \|\cdot\|)$ IS A NORM SPACE (SPECIAL CASE OF A METRIC)

IF $\|\cdot\|$ IS A NORM FOR $\mathcal{X}$ THEN $d_{\|\cdot\|}(x,y) = \|x-y\|$

BANACH SPACE

$(x, \|\cdot\|)$ NORM METRIC COMPLETE



A BANACH SPACE IS BOTH A COMPLETE METRIC SPACE AND A REAL / COMPLEX VECTOR SPACE. CONNECTION GIVEN BY THE NORM

$(\mathbb{R}, \|\cdot\|) \qquad (\{0\}, \|\cdot\|)$

$(\ell^p, \|\cdot\|_p)$ IS COMPLETE

$\|\cdot\|_p : \ell^p \to [0, +\infty)$ WITH $\|x\|_p = \left(\sum_{N=1}^{\infty} \|x_N\|^p \right)^{\frac{1}{p}}$

METRIC $\longrightarrow$ MEASURES DISTANCES

NORM $\longrightarrow$ MEASURES DISTANCES LENGTHS

INNER PRODUCT $\longrightarrow$ MEASURES DISTANCES LENGTHS ANGLES

$\langle x, y \rangle = \|x\| \|y\| \cos(\alpha)$

SPACES WITH DEFINED INNER PRODUCTS

$\mathbb{F} = (\mathbb{R}, \mathbb{C})$ LET $\mathcal{X}$ BE A $\mathbb{F}$-VECTOR SPACE

INNER PRODUCT IS A MAP $\mathcal{X} \times \mathcal{X} \to \mathbb{F}$ IF :

1) $\langle x, y \rangle \geq 0 \quad \forall x \in \mathbb{F}$ MEASURES LENGTHS
$\langle x, x \rangle = 0 \longleftrightarrow x = 0$ POSITIVE DEFINITE

2) $\langle x, y \rangle = \langle y, x \rangle$ FOR $\mathbb{F} = \mathbb{R}$ SYMMETRY
$\langle x, y \rangle = \langle y, x \rangle^{+}$ FOR $\mathbb{F} = \mathbb{C}$ CONJUCATED SYMMETRY

3) $\langle x, y_1 + y_2 \rangle = \langle x, y_1 \rangle + \langle x, y_2 \rangle \quad \forall x, y_1, y_2 \in \mathcal{X}$ LINEARITY
$\langle x, \lambda y \rangle = \lambda \langle x, y \rangle \quad \forall x, y \in \mathcal{X} \; \lambda \in \mathbb{F}$ HOMOGENEITY

$(\mathcal{X}, \langle \cdot, \cdot \rangle)$ INNER PRODUCT SPACE THEN $\|x\| = \sqrt{\langle x, x \rangle}$

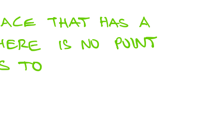
HILBERT SPACE

$(\mathcal{X}, \langle \cdot, \cdot \rangle)$ IS A HILBERT SPACE IF $(\mathcal{X}, \|\cdot\|_{\langle \cdot, \cdot \rangle})$ IS A BANACH SPACE
COMPLETE METRIC SPACE

EXAMPLES

- $\mathbb{R}^N, \mathbb{C}^N$ WITH $\langle x, y \rangle = \sum_{i=1}^{N} \overline{x_i} \, y_i$ NORMAL EUCLIDIAN GEOMETRY
- $\ell^2(\mathbb{N}, \mathbb{F})$ WITH $\langle x, y \rangle = \sum_{i=1}^{\infty} \overline{x_i} \, y_i$ ∞-DIM GENERALIZATION
- $C([0,1], \mathbb{F})$ WITH $\langle f, g \rangle = \int_0^1 \overline{f(t)} \, g(t) \, dt$ GEOMETRY FOR CONTINOUS f NOT AN HILBERT SPACE

CAUCHY-SCHWARZ INEQUALITY



WORKS FOR ALL INNER PRODUCTS

LET $(\mathcal{X}, \langle \cdot, \cdot \rangle)$ INNER PRODUCT SPACE AND $\|x\| = \sqrt{\langle x, x \rangle}$

$|\langle x, y \rangle| \leq \|x\| \cdot \|y\| \quad \forall x, y \in \mathcal{X}$

IF $|\langle x, y \rangle| = \|x\| \cdot \|y\| \longleftrightarrow x, y$ LINEARLY DEPENDENT

ORTHOGONALITY

LET $(\mathcal{X}, \langle \cdot, \cdot \rangle)$ BE AN INNER PRODUCT SPACE

1) $x, y \in \mathcal{X}$ ARE ORTHOGONAL IF $\langle x, y \rangle = 0 \qquad x \perp y$

2) FOR $U, V \subseteq \mathcal{X} \quad U \perp V$ IF $x \perp y \quad \forall \, x \in U, \, y \in V$ FOR SETS

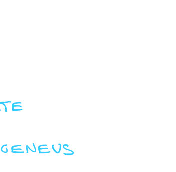
3) FOR $U \subseteq \mathcal{X}$ THE ORTHOGONAL COMPLEMENT OF U IS $U^\perp = \{ x \in \mathcal{X} \mid \langle x, u \rangle = 0 \;\; \forall u \in U \}$
ALWAYS A SUBSPACE OF $\mathcal{X}$

REMARKS

$\{0\}^\perp = \mathcal{X} \qquad \mathcal{X}^\perp = \{0\}$

$U \subseteq V \to U^\perp \supseteq V^\perp$

IF $x \perp y \to \|x+y\|^2 = \|x\|^2 + \|y\|^2$

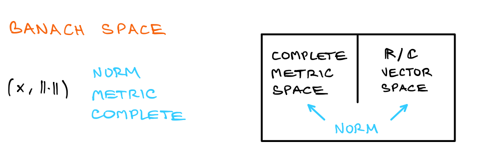


$U^\perp$ IS ALWAYS CLOSED

CONTINUITY (FOR METRIC SPACES)

$(\mathcal{X}, d_\mathcal{X}) \quad (\mathcal{Y}, d_\mathcal{Y})$ METRIC SPACES

$f : \mathcal{X} \to \mathcal{Y}$ IS CONTINOUS IF $f^{-1}[B]$ IS OPEN IN $\mathcal{X}$ $\forall$ OPEN SETS $\mathbb{B} \subseteq \mathcal{Y}$



IF NO BOUNDARY IN $\mathcal{Y}$, NO BOUNDARY POINTS IN $\mathcal{X}$

SEQUENTIALLY CONTINOUS

IF $\forall \tilde{x} \in \mathcal{X}$ AND $(x_m)_{m \in \mathbb{N}} \subseteq \mathcal{X}$ WITH $x_N \xrightarrow{m \to \infty} \tilde{x}$ HOLDS $f(x_m) \xrightarrow{m \to \infty} f(x)$

FOR METRIC SPACES C. & SEQUENTIALLY C. ARE THE SAME

FOR LINEAR OPERATOR CONTINUITY = BOUNDEDNESS

WEAK FORM PDE PROBLEM

LET V BE A HILBERT SPACE WITH NORM

FIND $u \in V$ SUCH THAT $a(u, \phi) = \ell(\phi) \quad \forall \phi \in V$

FOR ALL TEST FUNCTION ϕ IN FUNCTION SPACE V

POISSON'S PROBLEM

$V = H_0^1(\Omega)$

$a(u, \phi) = (\nabla u, \nabla \phi)$

$\ell(\phi) = (f, \phi)$

RIESZ REPRESENTATION THEOREM

LET $\ell : V \to \mathbb{R}$ BE A CONTINOUS LINEAR FUNCTIONAL OR A HILBERT SPACE V.

THEN $\exists! \; u \in V : (u, \phi)_V = \ell(\phi) \quad \forall \phi \in V$

THIS OPERATION AN ISOMETRY $\|\ell\|_{V^*} = \|u\|_V$

LAX-MILGRAM LEMMA

LET $a(\cdot,\cdot) : V \times V \to \mathbb{R}$ CONTINOUS, SYMMETRIC, COERCED

THEN $\forall \ell \in V^* \quad a(u, \phi) = \ell(\phi)$

RIESZ REPRESENTATION THEOREM

LET $(X, \langle \cdot, \cdot \rangle)$ BE A HILBERT SPACE

$\forall$ CONTINOUS LINEAR MAP $\ell : X \to \mathbb{F}$

THERE IS AN $x_\ell \in X$ SUCH THAT $\ell(x) = \langle x_\ell, x \rangle \quad \forall x \in X$

EACH LINEAR BOUNDED f HAS TO LOOK LIKE THIS

AND $\|\ell\|_{X \to \mathbb{F}} = \|x_\ell\|_X$

THIS ALWAYS WORKS IN HILBERTS (SIMPLIFYING LINEAR FUNCTIONS) $\qquad \ell = \langle \gamma |$