CAMBIO DI VARIABILE NEGLI INT. DOPPI

14-5-2020

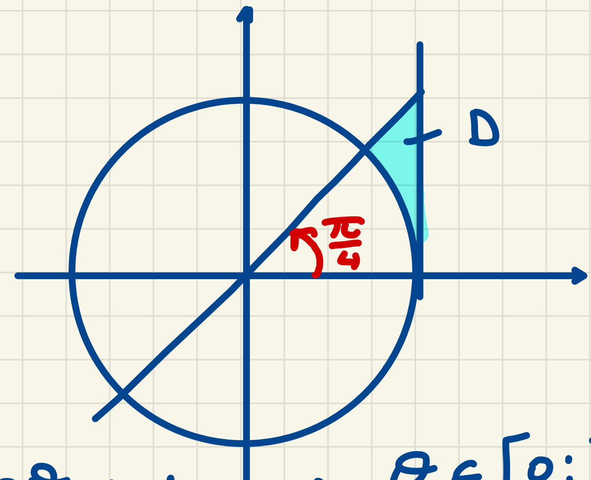
ESERCIZIO 1. Calcolare $\iint_D \frac{x}{x^2+y^2}\,dx\,dy$ con

$D = \{ (x;y) \in \mathbb{R}^2 / \ x^2+y^2 \geq 1, \ 0 \leq y \leq x, \ 0 \leq x \leq 1 \}.$

Sol.

$$D: \begin{cases} x^2+y^2 \geq 1 \\ 0 \leq y \leq x \\ 0 \leq x \leq 1 \end{cases} \qquad \begin{cases} x = \rho\cos\theta \\ y = \rho\operatorname{sen}\theta \end{cases}$$

$$dx\,dy = \rho\,d\rho\,d\theta$$



$$D': \begin{cases} \rho^2 \geq 1 \to \rho \geq 1 \\ 0 \leq \rho\operatorname{sen}\theta \leq \rho\cos\theta \to 0 \leq \operatorname{tg}\theta \leq 1 \to \theta \in [0; \frac{\pi}{4}] \\ 0 \leq \rho\cos\theta \leq 1 \to 0 \leq \rho \leq \frac{1}{\cos\theta} \end{cases}$$

$$D' = \{ (\rho,\theta) \ / \ 0 \leq \theta \leq \frac{\pi}{4}, \ 1 \leq \rho \leq \frac{1}{\cos\theta} \}$$

$\Longrightarrow$ D' è ρ-semplice.

$$\iint_D \frac{x}{x^2+y^2}\,dx\,dy = \iint_{D'} \frac{\cancel{\rho}\cos\theta}{\cancel{\rho^2}}\,\cancel{\rho}\,d\rho\,d\theta = \int_0^{\pi/4}\left(\int_1^{1/\cos\theta}\cos\theta\,d\rho\right)d\theta$$

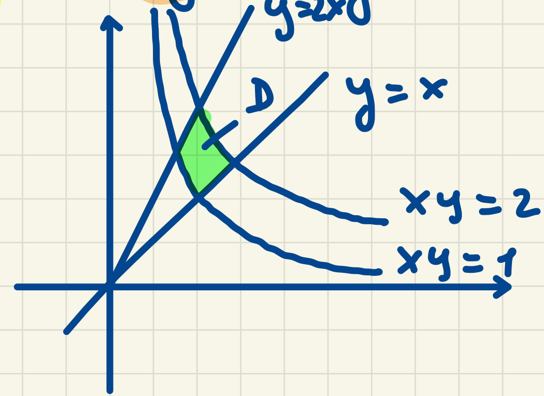
$$= \int_0^{\pi/4}\cos\theta\,\Big[\rho\Big]_1^{1/\cos\theta}\,d\theta = \int_0^{\pi/4}\cos\theta\left(\frac{1}{\cos\theta}-1\right)d\theta =$$

$$= \int_0^{\pi/4}(1-\cos\theta)\,d\theta = \Big[\theta-\operatorname{sen}\theta\Big]_0^{\pi/4} = \frac{\pi}{4}-\frac{\sqrt2}{2}$$

A CASA: $\displaystyle\iint_D \frac{x-1}{(x-1)^2+y^2}\,dx\,dy$, $D=\{(x,y)\in\mathbb{R}^2 \,/\, (x-1)^2+y^2\geq 1,$
$0\leq y\leq\sqrt3\,(x-1),\ 1\leq x\leq 2\}$ $\left[R;\ \frac{\pi}{3}-\frac{\sqrt3}{2}\right]$

ESERCIZIO 2. Calcolare $\iint_D \frac{x}{y} \operatorname{sen} xy \, dx\,dy$ con

D regione in figura



SOL.

- $y = x \rightarrow \frac{x}{y} = 1$
- $y = 2x \rightarrow \frac{x}{y} = \frac{1}{2}$

- $xy = 1$
- $xy = 2$

$$F^{-1}: \begin{cases} u = xy \\ v = \frac{x}{y} \end{cases} \qquad F: \begin{cases} x = g(u,v) \\ y = h(u,v) \end{cases}$$

NON SERVE CALCOLARLA

$u \in [1;2]$, $v \in [\frac{1}{2};1]$

$$J_{F^{-1}} = \begin{bmatrix} y & x \\ \frac{1}{y} & -\frac{x}{y^2} \end{bmatrix}$$

$$\det J_{F^{-1}} = -\frac{x}{y} - \frac{x}{y} = -2\frac{x}{y}$$

$$\det J_F(u,v) = \frac{1}{\det J_{F^{-1}}} = -\frac{y}{2x} = -\frac{1}{2}\frac{y}{x} = -\frac{1}{2v}$$

$$dx\,dy = \left|\frac{-1}{2v}\right| du\,dv \underset{v>0}{=} \frac{1}{2v}\,du\,dv$$

$$\iint_D \frac{x}{y}\,\operatorname{sen} xy\,dx\,dy = \iint_{D'} \cancel{v}\,\operatorname{sen} u \cdot \frac{1}{2\cancel{v}}\,du\,dv =$$

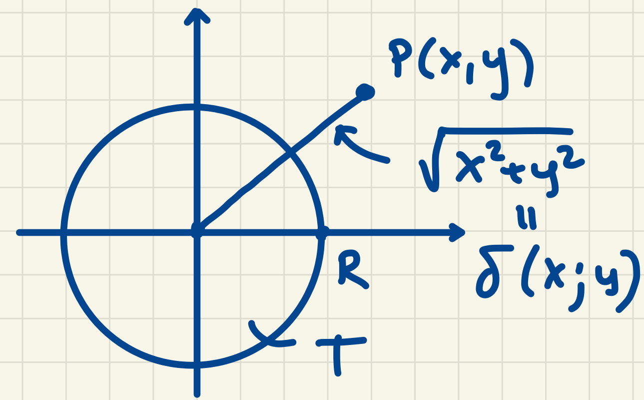
$$D' = [1;2]\times\left[\tfrac{1}{2};1\right]$$

$$= \frac{1}{2}\int_{1/2}^{1}\left(\int_1^2 \operatorname{sen} u\,du\right)dv = \frac{1}{2}\int_{1/2}^{1} dv \cdot \int_1^2 \operatorname{sen} u\,du =$$

$$= \frac{1}{2}\cdot\left(1-\tfrac{1}{2}\right)\cdot\left[-\cos u\right]_1^2 = \frac{1}{4}\left(\cos 1-\cos 2\right).$$

ESERCIZIO 3. Dimostrare che il momento di inerzia di una lamina omogenea di massa M e raggio R è $\frac{1}{2}MR^2$ in rotazione intorno all'asse z.

Sol.



$$I = \frac{M}{\text{Area}(T)} \iint_T \delta^2(x,y)\,dx\,dy =$$

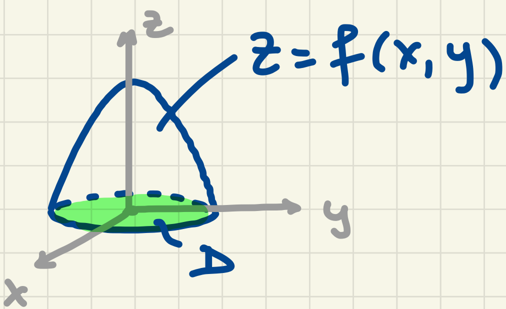
$$= \frac{M}{\pi R^2} \iint_T (x^2+y^2)\,dx\,dy =$$

$$= \frac{M}{\pi R^2} \int_0^{2\pi} \left(\int_0^R \rho^2 \rho \, d\rho \right) d\theta = \frac{M}{\cancel{\pi} R^2} \cdot 2\cancel{\pi} \cdot \left[\frac{\rho^4}{4} \right]_0^R =$$

$$= \frac{\cancel{2}M}{\cancel{R^2}} \left(\frac{R^{\cancel{4}\,2}}{\cancel{4}\,2} - \cancel{0} \right) = \frac{1}{2} M R^2.$$

ESERCIZIO 4. Calcolare il volume della porzione di spazio racchiusa tra il paraboloide $z = \underbrace{1-x^2-y^2}_{f(x,y)}$ e il piano xy $(z=0)$.

Sol.



$$\partial D : \begin{cases} z = 1-x^2-y^2 \\ z = 0 \end{cases} \Rightarrow x^2+y^2 = 1$$

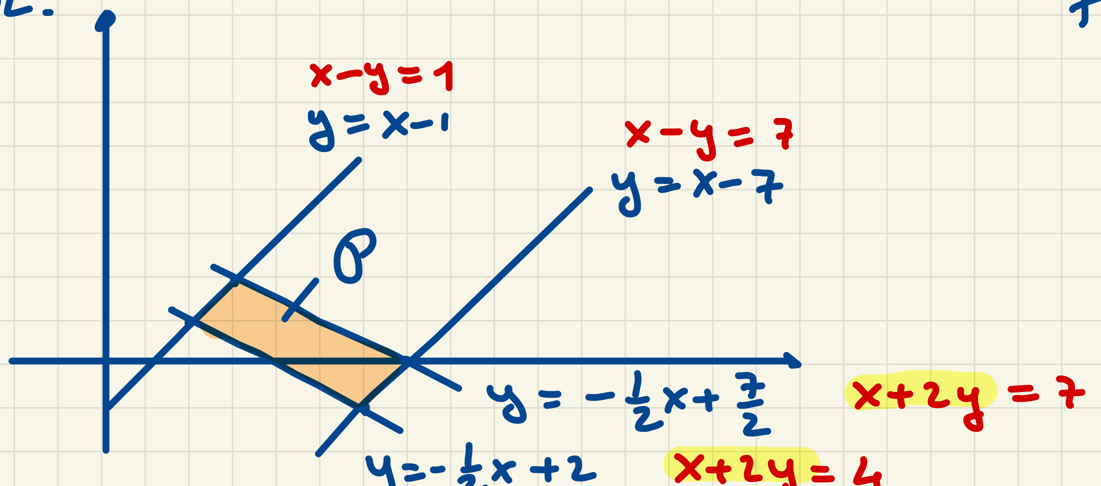
$$D : \{ x^2+y^2 \leq 1 \}$$

$$Vol = \iint_D \underbrace{(1-x^2-y^2)}_{[1-(x^2+y^2)]} \, dx\,dy = \int_0^{2\pi} \left(\int_0^1 (1-\rho^2)\rho \, d\rho \right) d\theta =$$

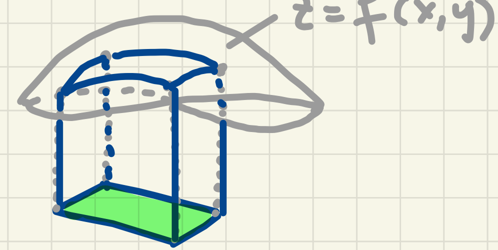
$$= 2\pi \left[\frac{\rho^2}{2} - \frac{\rho^4}{4} \right]_0^1 = 2\pi \left(\frac{1}{2} - \frac{1}{4} \right) = \frac{2\pi}{4} = \frac{\pi}{2}.$$

ESERCIZIO 5. Calcolare il volume del cilindroide con generatrici parallele all'asse z compreso tra il parallelogramma P di vertici $(2;1)$, $(6;-1)$, $(7,0)$, $(3,2)$ e la porzione di superficie di eq. $z = e^{x+2y}$ che si proietta su P.

SOL.



$$f(x,y)=e^{x+2y}$$



$$\text{Vol} = \iint_P f\,dx\,dy$$

CAMBIO DI VARIABILE

$$\begin{cases} x+2y = u \\ x-y = v \end{cases} \qquad u \in [4;7] \qquad (u,v) \in [4,7]\times[1,7] = R$$

$$v \in [1;7]$$

$$F^{-1}: \begin{cases} u = x+2y \\ v = x-y \end{cases} \qquad \det J_{F^{-1}} = \det \begin{bmatrix} 1 & 2 \\ 1 & -1 \end{bmatrix} = -3$$

$$\det J_F = -\frac{1}{3} \qquad dx\,dy = \frac{1}{3}\,du\,dv$$

$$Vol = \iint_P e^{x+2y}\,dx\,dy = \iint_R e^u \cdot \frac{1}{3}\,du\,dv = \frac{1}{3}\int_1^7 \left(\int_4^7 e^u\,du\right) dv$$

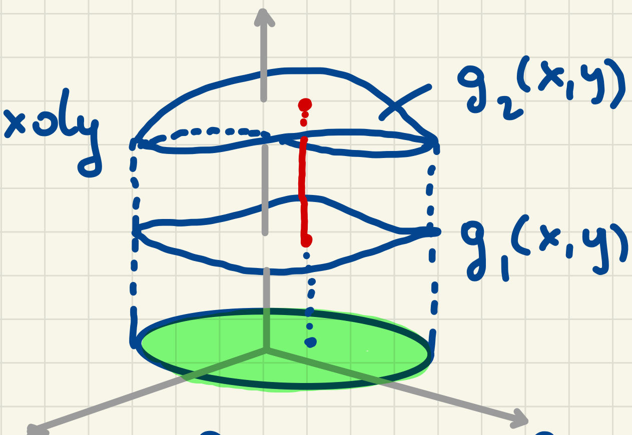
$$= \frac{1}{3} \cdot 6 \cdot \left[e^u\right]_4^7 = 2\left(e^7 - e^4\right).$$

INTEGRALI TRIPLI

1) "PER FILI"

$$\Omega = \{(x,y,z)\in\mathbb{R}^3 \,/\, g_1(x,y)\le z\le g_2(x,y)\ ,\ (x,y)\in D\}$$

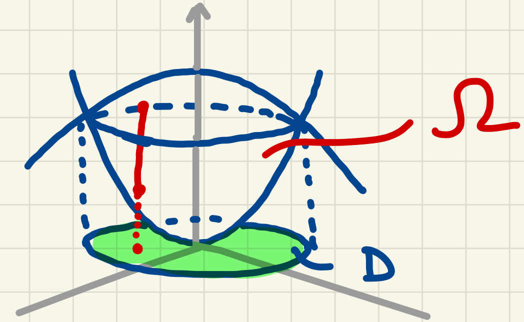
$$\iiint_\Omega f\,dx\,dy\,dz = \iint_D \left(\int_{g_1}^{g_2} f\,dz\right)dx\,dy$$



ESERCIZIO 6. Calcolare il volume della porzione di spazio compresa tra i paraboloidi $z = \underbrace{x^2+y^2}_{=g_1}$ e $z = \underbrace{\frac{4}{3} - \frac{x^2+y^2}{3}}_{=g_2}$

SOL.

$$\partial D: \begin{cases} z = x^2+y^2 \\ z = \frac{4}{3} - \frac{x^2+y^2}{3} \end{cases}$$



$$x^2+y^2 = \frac{4}{3} - \frac{x^2+y^2}{3}$$

$$\frac{4}{3}(x^2+y^2) = \frac{4}{3} \implies x^2+y^2 = 1 \qquad D: \{x^2+y^2 \leq 1\}$$

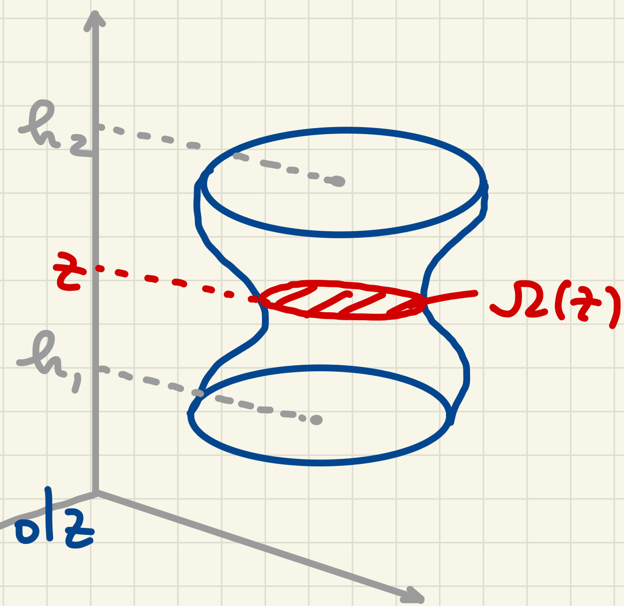
$$\text{Vol} = \iiint_\Omega dx\,dy\,dz = \iint_D \left(\int_{x^2+y^2}^{\frac{4}{3}-\frac{x^2+y^2}{3}} dz \right) dx\,dy =$$

$$= \iint_D \left[\frac{4}{3} - \frac{x^2+y^2}{3} - (x^2+y^2)\right] dx\,dy = \iint_D \left[\frac{4}{3} - \frac{4}{3}(x^2+y^2)\right] dx\,dy$$

$$= \frac{4}{3}\int_0^{2\pi}\left(\int_0^1 (1-\rho^2)\rho\, d\rho\right) d\theta = \frac{4}{3}\cdot 2\pi\left[\frac{\rho^2}{2}-\frac{\rho^4}{4}\right]_0^1$$

$$= \frac{8}{3}\pi\left(\frac{1}{2}-\frac{1}{4}\right) = \frac{2}{3}\pi .$$

2) "PER STRATI"



$$\Omega = \left\{(x,y,z)\in\mathbb{R}^3 \,/\, h_1\le z\le h_2,\ (x,y)\in\Omega(z)\right\}$$

$$\iiint_{\Omega} f\, dx\,dy\,dz = \int_{h_1}^{h_2}\left(\iint_{\Omega(z)} f(x,y)\,dx\,dy\right)dz$$

ESERCIZIO 7. Calcolare $\iiint_{\Omega} z \, dx\,dy\,dz$ con

$$\Omega = \{ (x,y,z) \in \mathbb{R}^3 \,/\, \underbrace{x^2 + (y-z)^2 \le 1}_{\text{STRATO } \Omega(z)}, \; 0 \le z \le 1 \}$$

SOL.

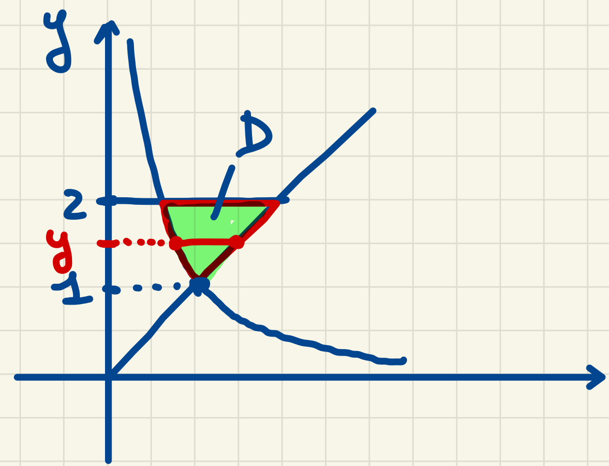
$\Omega(z)$ è il cerchio di centro $(0;z)$ e raggio 1

$$\iiint_{\Omega} z \, dx\,dy\,dz = \int_0^1 \left(\iint_{\Omega(z)} z \, dx\,dy \right) dz =$$

$$= \int_0^1 z \Big(\underbrace{\iint_{\Omega(z)} dx\,dy}_{\text{Area}(\Omega(z)) = \pi \cdot 1^2 = \pi.} \Big) dz = \int_0^1 z \cdot (\pi \cdot 1) \, dz = \pi \left[\frac{z^2}{2} \right]_0^1$$

$$= \frac{\pi}{2}.$$

ESERCIZIO 8. Sia D la regione di piano racchiusa tra l'iperbole $y=\frac{1}{x}$, la retta $y=2$ e la retta $y=x$. Calcolare il volume del solido S ottenuto ruotando D intorno all'asse y.



SOL.

Lo strato è una corona circolare

$$\Omega(y)=\left\{\frac{1}{y^2}\le x^2+z^2\le y^2\right\}$$

$$\mathrm{Vol}(S)=\iiint_S dx\,dy\,dz=\int_1^2\Big(\underbrace{\iint_{\Omega(y)}dx\,dz}_{\text{Area di }\Omega(y)}\Big)dy=$$

$$\text{Area}(\Omega(y)) = \pi y^2 - \pi \frac{1}{y^2} = \pi\left(y^2 - \frac{1}{y^2}\right)$$

$$= \int_1^2 \pi\left(y^2 - \frac{1}{y^2}\right)dy = \pi\left[\frac{y^3}{3} + \frac{1}{y}\right]_1^2 = \pi\left(\frac{8}{3} + \frac{1}{2} - \frac{1}{3} - 1\right)$$

$$= \frac{11}{6}\pi.$$

ESERCIZIO A CASA

Calcolare il volume ottenuto ruotando intorno all'asse y il triangolo di vertici $(0,0,2)$, $(0,-1,1)$, $(0,1,1)$.

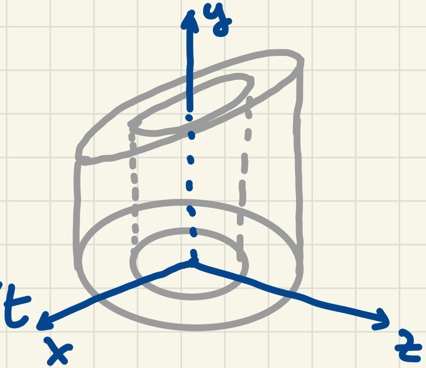
ESERCIZIO 9 (CAMBI DI COORDINATE)

Calcolare il volume del solido

$$S = \{(x,y,z) \in \mathbb{R}^3 / 1 \leq x^2 + z^2 \leq 9, \quad 0 \leq y \leq z+3\}$$

SOL.

COORDINATE CILINDRICHE



$$\begin{cases} x = \rho\cos\theta \\ y = t \\ z = \rho\,\text{sen}\,\theta \end{cases} \qquad dx\,dy\,dz = \rho\,d\rho\,d\theta\,dt$$

$$\theta \in [0, 2\pi]$$

$$1 \le x^2 + z^2 \le 9 \longrightarrow 1 \le \rho \le 3$$

$$0 \le y \le z + 3 \longrightarrow 0 \le t \le 3 + \rho\,\text{sen}\,\theta$$

$$Vol(S) = \iiint_S dx\,dy\,dz = \int_0^{2\pi} \left(\int_1^3 \left(\int_0^{3+\rho\,\text{sen}\,\theta} \rho\,dt \right) d\rho \right) d\theta$$

$$= \int_0^{2\pi} \left(\int_1^3 \rho(3 + \rho\,\text{sen}\,\theta)\,d\rho \right) d\theta =$$

$$= \int_0^{2\pi} \left(\int_1^3 (3\rho + \rho^2 \operatorname{sen}\theta)\, d\rho \right) d\theta =$$

$$= \int_0^{2\pi} \left[\frac{3\rho^2}{2} + \frac{\rho^3}{3} \operatorname{sen}\theta \right]_1^3 d\theta = \int_0^{2\pi} \left(\frac{27}{2} + 9 \operatorname{sen}\theta - \frac{3}{2} - \frac{1}{3} \operatorname{sen}\theta \right) d\theta$$

$$= \int_0^{2\pi} \left(12 + \frac{26}{3} \operatorname{sen}\theta \right) d\theta = \left[12\theta - \frac{26}{3} \cos\theta \right]_0^{2\pi} =$$

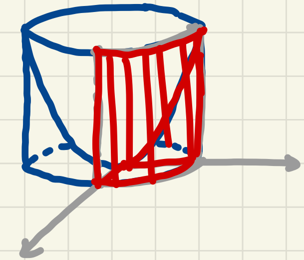
$$= 24\pi - \frac{26}{3} + \frac{26}{3} = 24\pi.$$

ESERCIZIO 10. Sia $T = \{ (x,y,z) \in \mathbb{R}^3 \,/\, 0 \le z \le x^2+y^2,\ x^2+y^2 \le 1,\ x \ge 0,\ y \ge 0 \}$. Calcolare

$$\iiint_T y\sqrt{x}\, dx\,dy\,dz$$

SOL. In coord. cilindriche:

$$\begin{cases} x = \rho\cos\theta \\ y = \rho \operatorname{sen}\theta \\ z = t \end{cases} \qquad \begin{array}{l} \theta \in \left[0; \frac{\pi}{2}\right] \\ \rho \in [0,1] \\ 0 \le t \le \rho^2 \end{array}$$



$$\iiint_T y\sqrt{x}\,dx\,dy\,dz = \int_0^1 \left(\int_0^{\pi/2} \left(\int_0^{\rho^2} \rho \operatorname{sen}\theta \sqrt{\rho\cos\theta}\,\rho\,dt \right) d\theta \right) d\rho$$

$$= \int_0^1 \left(\int_0^{\pi/2} \rho^2 \operatorname{sen}\theta \sqrt{\rho\cos\theta}\,\rho^2\,d\theta \right) d\rho =$$

$$= -\int_0^1 \rho^{9/2} \left(\int_0^{\pi/2} -\operatorname{sen}\theta\,(\cos\theta)^{1/2}\,d\theta \right) d\rho =$$

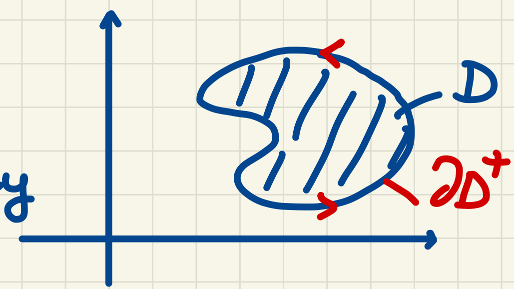
$$= \left[\rho^{11/2}\cdot\frac{2}{11}\right]_0^1 \left(-\left[(\cos\theta)^{3/2}\cdot\frac{2}{3}\right]_0^{\pi/2}\right) =$$

$$\frac{2}{11}\left(+\frac{2}{3}\right) = \frac{4}{33}$$

ESERCIZIO A CASA. Calcolare il volume della sfera di raggio R con un integrale triplo.

TEOREMA DI GAUSS-GREEN NEL PIANO E SUE APPLICAZIONI

$\underline{F} = (F_1, F_2)$ $\quad \underline{F} \in C^1, \quad D \subseteq \mathbb{R}^2$



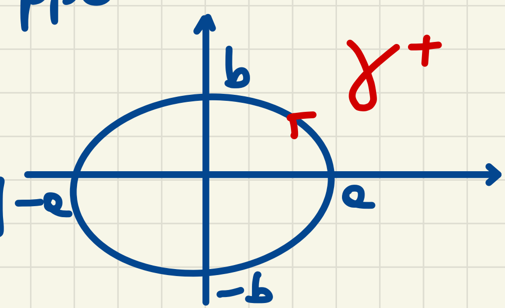
$$L = \int_{\partial^+ D} \underline{F}\cdot d\underline{r} = \iint_D \left(F_{2x} - F_{1y}\right) dx\,dy$$

• CALCOLO DI UN'AREA MEDIANTE UN OPPORTUNO INTEGRALE DI LINEA.

NOTA. $\underline{F}(x,y) = (0; x) = 0\underline{i} + x\underline{j}$ è un campo tale che: $F_{2x} - F_{1y} = 1 - 0 = 1$

$$A(D) = \iint_D 1 \, dx dy = \int_{\partial D^+} \underline{F} \cdot \underline{dr}$$

ESERCIZIO 11. Calcolare l'area dell'ellisse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ usando un opportuno integrale di linea.



SOL

$$\gamma_+ : \begin{cases} x = a\cos t \\ y = b \operatorname{sen} t \end{cases} \quad t \in [0, 2\pi]$$

$$\gamma_+' : \begin{cases} x' = -a \operatorname{sen} t \\ y' = b \cos t \end{cases}$$

$$\underline{F}(x,y) = (0,x) \qquad A(D) = \iint_D dx\,dy = \int_{\gamma^+} \underline{F} \cdot d\underline{r} =$$

$$= \int_0^{2\pi} (0; a\cos t) \cdot (-a \operatorname{sen} t; b \cos t)\,dt =$$

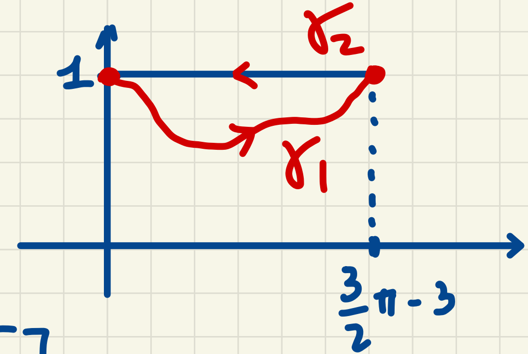
$$= \int_0^{2\pi} ab \cos^2 t\,dt = ab \left[\frac{t + \operatorname{sen} t \cos t}{2} \right]_0^{2\pi} =$$

$$= ab(\pi - 0) = \pi a b.$$

ESERCIZIO 12. Calcolare l'area della regione di piano compresa tra le curve $y=1$ e $\underline{r}(t) = 3(t - \operatorname{sen} t)\,\underline{i} + (1 - \operatorname{sen} 2t)\,\underline{j}$ $t \in [0, \pi/2]$.

SOL.

$\underline{r}(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ $\qquad$ $\underline{r}\left(\frac{\pi}{2}\right) = \left(\frac{3\pi}{2} - 3, 1\right)$



$$\gamma_1 : \begin{cases} x = 3(t - \operatorname{sen} t) \\ y = 1 - \operatorname{sen} 2t \end{cases} \qquad t \in \left[0; \frac{\pi}{2}\right]$$

$$\gamma_1' : \begin{cases} x' = 3(1 - \cos t) \\ y' = -2\cos 2t \end{cases}$$

$$\gamma_2 : \begin{cases} x = \frac{3}{2}\pi - 3 - t \\ y = 1 \end{cases} \qquad t \in \left[0; \frac{3}{2}\pi - 3\right]$$

$$\gamma_2' : \begin{cases} x' = -1 \\ y' = 0 \end{cases}$$

$$A(D) = \iint_D dx\,dy = \int_{\gamma_1} \underline{F}\cdot d\underline{r} + \int_{\gamma_2} \underline{F}\cdot d\underline{r} =$$

$$= \int_0^{\pi/2} (0;\, 3t - 3\,\mathrm{sen}\,t)\cdot(3-3\cos t,\, -2\cos 2t)\,dt \;+$$

$$\cancel{+ \int_0^{3/2\pi - 3} \left(0;\, \tfrac{3}{2}\pi - 3 - t\right)\cdot(-1,\, 0)\,dt} \;=$$

$$= \int_0^{\pi/2} (-6t\cos 2t + 6\,\mathrm{sen}\,t\cos 2t)\,dt = \ldots\; 1 .$$