SISTEMI DEL PRIMO ORDINE

30-4-2020

ESERCIZIO 1. Risolvere il seguente sistema del primo ordine:

$$\begin{cases} x' = -5y \\ y' = -x - 4y \end{cases}$$

SOL.

$$\begin{cases} (-y' - 4y)' = -5y \\ x = -y' - 4y \end{cases} \longrightarrow \begin{cases} -y'' - 4y' + 5y = 0 \\ x = -y' - 4y \end{cases}$$

$$y'' + 4y' - 5y = 0 \qquad \lambda^2 + 4\lambda - 5 = 0 \qquad \lambda_{1,2} = \begin{cases} 1 \\ -5 \end{cases}$$

$$y(t) = c_1 e^{t} + c_2 e^{-5t}$$

$$x(t) = -\left(c_1 e^{t} + c_2 e^{-5t}\right)' - 4\left(c_1 e^{t} + c_2 e^{-5t}\right)$$

$$= -\left(c_1 e^{t} - 5c_2 e^{-5t}\right) - 4c_1 e^{t} - 4c_2 e^{-5t}$$

$$= -5c_1 e^{t} + c_2 e^{-5t}$$

$$\begin{cases} x(t) = -5c_1 e^{t} + c_2 e^{-5t} \\ y(t) = c_1 e^{t} + c_2 e^{-5t} \end{cases}$$

$$\Rightarrow \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} -5 \\ 1 \end{bmatrix} c_1 e^{t} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} c_2 e^{-5t}$$

SOL. ALTERNATIVA

$$A = \begin{bmatrix} 0 & -5 \\ -1 & -4 \end{bmatrix} \qquad p_A(\lambda) = \det(A - \lambda I) = \det\begin{bmatrix} -\lambda & -5 \\ -1 & -4-\lambda \end{bmatrix}$$

$$= -\lambda(-4-\lambda) - 5 = \lambda^2 + 4\lambda - 5.$$

$$p_A(\lambda) = 0 \iff \lambda_{1,2} = \begin{cases} 1 \\ -5 \end{cases}$$

(DUE AUTOVALORI REALI DISTINTI)

$$V_1 : \ker[A - I] = \ker\begin{bmatrix} -1 & -5 \\ -1 & -5 \end{bmatrix}$$

$$-a - 5b = 0$$

$$a = -5b$$

$$\begin{cases} a = -5 \\ b = 1 \end{cases}$$

$$V_1 = Sp\left(\begin{bmatrix} -5 \\ 1 \end{bmatrix}\right)$$

Span

$$V_{-5} : \ker[A + 5I] = \ker\begin{bmatrix} 5 & -5 \\ -1 & 1 \end{bmatrix}$$

$$5a - 5b = 0$$

$$a = b$$

$$\begin{cases} a = 1 \\ b = 1 \end{cases}$$

$$V_{-5} = Sp\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = c_1 e^{1t} \begin{bmatrix} -5 \\ 1 \end{bmatrix} + c_2 e^{-5t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

ESERCIZIO 2. Determinare l'integrale generale del sistema

$$\begin{cases} x' = 2x - 5y \\ y' = 5x - 6y \end{cases}$$

SOL.

$$\begin{cases} y = \frac{2}{5}x - \frac{1}{5}x' \\ \frac{2}{\cancel{5}}x' - \frac{1}{\cancel{5}}x'' = 25x - \frac{12}{\cancel{5}}x + \frac{6}{\cancel{5}}x' \end{cases} \Longrightarrow x'' + 4x' + 13x = 0$$

$$p(\lambda) = \lambda^2 + 4\lambda + 13$$

$$p(\lambda) = 0 \qquad \lambda_{1,2} = -2 \pm 3i.$$

$$x(t) = e^{-2t}\left(c_1 \cos 3t + c_2 \operatorname{sen} 3t\right)$$

$$y(t) = \frac{2}{5}\,x(t) - \frac{1}{5}\,x'(t)$$

$$x'(t) = -2e^{-2t}\left(c_1 \cos 3t + c_2 \operatorname{sen} 3t\right) +$$
$$+\, e^{-2t}\left(-3c_1 \operatorname{sen} 3t + 3c_2 \cos 3t\right) =$$

$$= e^{-2t}\left(-2c_1 \cos 3t - 2c_2 \operatorname{sen} 3t - 3c_1 \operatorname{sen} 3t + 3c_2 \cos 3t\right)$$
$$= e^{-2t}\left((-2c_1 + 3c_2) \cos 3t + (-2c_2 - 3c_1) \operatorname{sen} 3t \right)$$

$$y(t) = \frac{2}{5} e^{-2t}\left(c_1 \cos 3t + c_2 \operatorname{sen} 3t\right) - \frac{1}{5} e^{2t} \cdot$$
$$\cdot \left((-2c_1 + 3c_2) \cos 3t + (-2c_2 - 3c_1) \operatorname{sen} 3t \right).$$

$$\begin{cases} x(t) = e^{-2t}\left(c_1 \cos 3t + c_2 \operatorname{sen} 3t\right) \\ y(t) = \frac{1}{5} e^{-2t}\left((3c_1+4c_2)\operatorname{sen} 3t + (4c_1-3c_2)\cos 3t\right) \end{cases}$$

In alternativa:

$$\begin{cases} x' = 2x - 5y \\ y' = 5x - 6y \end{cases} \Rightarrow \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 2 & -5 \\ 5 & -6 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

$$p_A(\lambda) = \det\begin{bmatrix} 2-\lambda & -5 \\ 5 & -6-\lambda \end{bmatrix} = (2-\lambda)(-6-\lambda)+25 =$$
$$= \lambda^2 + 4\lambda + 13$$

$$p_A(\lambda) = 0 \iff \lambda_{1,2} = -2 \pm 3i.$$

$$V_{-2+3i}: \quad \ker\left[A - (-2+3i)I\right] = \ker\begin{bmatrix} 4-3i & -5 \\ \cancel{5} & \cancel{-4-3i} \end{bmatrix}$$

$$(4-3i)a - 5b = 0$$
$$b = \frac{4-3i}{5} a \qquad \begin{cases} a = 5 \\ b = 4-3i \end{cases}$$

$$V_{-2+3i} = Sp\left(\begin{bmatrix} 5+0i \\ 4-3i \end{bmatrix}\right) = Sp\left(\begin{bmatrix} 5 \\ 4 \end{bmatrix} + i\begin{bmatrix} 0 \\ -3 \end{bmatrix}\right)$$

L'integrale generale del sistema è:

$$\underline{u}(t) = c_1\, \underline{u}_1(t) + c_2\, \underline{u}_2(t)$$

dove
$$\underline{u}_1(t) = e^{-2t}\left(\cos 3t \begin{bmatrix} 5 \\ 4 \end{bmatrix} - \operatorname{sen} 3t \begin{bmatrix} 0 \\ -3 \end{bmatrix}\right)$$
$$\underline{u}_2(t) = e^{-2t}\left(\operatorname{sen} 3t \begin{bmatrix} 5 \\ 4 \end{bmatrix} + \cos 3t \begin{bmatrix} 0 \\ -3 \end{bmatrix}\right)$$

$c_1, c_2 \in \mathbb{R}$.

ESERCIZIO 3. Determinare l'integrale generale del sistema

$$\begin{cases} x' = -x + 4y \\ y' = -x + 3y \end{cases}$$

SOL.

$$\begin{cases} 3y' - y'' = -3y + y' + 4y \\ x = 3y - y' \end{cases} \rightarrow \quad y'' - 2y' + y = 0$$

$$p(\lambda) = \lambda^2 - 2\lambda + 1 = (\lambda - 1)^2$$

$$\lambda = 1$$

$$y(t) = c_1 e^t + c_2 t e^t$$

$$y'(t) = c_1 e^t + c_2 e^t + c_2 t e^t$$

$$x(t) = 3(c_1 e^t + c_2 t e^t) - c_1 e^t - c_2 e^t - c_2 t e^t$$
$$= 2c_1 e^t - c_2 e^t + 2c_2 t e^t$$

$$\begin{cases} x(t) = 2c_1 e^t + c_2 e^t (2t - 1) \\ y(t) = c_1 e^t + c_2 t e^t \end{cases}$$

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = c_1 e^t \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 e^t \begin{bmatrix} 2t - 1 \\ t \end{bmatrix} =$$
$$= c_1 e^t \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 e^t \left(t \begin{bmatrix} 2 \\ 1 \end{bmatrix} + \begin{bmatrix} -1 \\ 0 \end{bmatrix} \right)$$

In modo alternativo:

$$\begin{cases} x' = -x + 4y \\ y' = -x + 3y \end{cases} \qquad A = \begin{bmatrix} -1 & 4 \\ -1 & 3 \end{bmatrix}$$

$$p_A(\lambda) = \det\begin{bmatrix} -1-\lambda & 4 \\ -1 & 3-\lambda \end{bmatrix} = (-1-\lambda)(3-\lambda)+4 =$$

$$= -3-2\lambda+\lambda^2+4 =$$

$$= \lambda^2-2\lambda+1 = (\lambda-1)^2.$$

$\lambda = 1$ autovalore doppio (m.a.$(1) = 2$)

$$V_1 = \ker[A-I] = \ker\begin{bmatrix} -2 & 4 \\ -1 & 2 \end{bmatrix}$$

$-a+2b=0$

$a=2b$

$\begin{cases} a=2 \\ b=1 \end{cases}$

$$V_1 = Sp\left(\underset{\underline{w}}{\begin{bmatrix} 2 \\ 1 \end{bmatrix}}\right)$$

A non è diagonalizzabile perché m.g.$(1) = 1 \neq 2$, allora cerco un vettore $\underline{v} = \begin{bmatrix} a \\ b \end{bmatrix}$ tale che $[A-\lambda I]\underline{v} = \underbrace{\underline{w}}_{\text{autovettore}}$

$$\begin{bmatrix} -2 & 4 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$\begin{cases} \cancel{-2a + 4b = 2} \\ -a + 2b = 1 \end{cases} \longrightarrow \quad a = 2b - 1 \qquad \begin{cases} b = 0 \\ a = -1 \end{cases}$$

$$\underline{v} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

L'integrale generale è :

$$\underline{u}(t) = c_1 \, \underline{u}_1(t) + c_2 \, \underline{u}_2(t)$$

con $\underline{u}_1(t) = e^{\lambda = 1 \cdot t} \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$$\underline{u}_2(t) = e^{t} \left(t \begin{bmatrix} 2 \\ 1 \end{bmatrix} + \begin{bmatrix} -1 \\ 0 \end{bmatrix} \right)$$

$$\underline{u}(t) = c_1 e^t \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 e^t \left(t \begin{bmatrix} 2 \\ 1 \end{bmatrix} + \begin{bmatrix} -1 \\ 0 \end{bmatrix} \right).$$

SISTEMI LINEARI DEL I ORDINE 2x2

$$\begin{cases} x' = ax + by \\ y' = cx + dy \end{cases} \qquad A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad \underline{u} = \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\Updownarrow \qquad \underline{u}' = \begin{bmatrix} x' \\ y' \end{bmatrix}$$

$$\underline{u}' = A\underline{u}$$

L'integrale generale del sistema è

$$\underline{u}(t) = c_1 \underline{u}_1(t) + c_2 \underline{u}_2(t)$$

$c_1, c_2 \in \mathbb{R}$ e:

1) se A ha due autovettori reali e lin. indipendenti $\underline{w}_1$ e $\underline{w}_2$ relativi agli autovalori λ_1 e λ_2

$$\underline{u}_1 = c_1 e^{\lambda_1 t} \underline{w}_1$$

$$\underline{u}_2 = c_2 e^{\lambda_2 t} \underline{w}_2$$

2) se A ha un autovalore complesso $\lambda = \alpha + i\beta$ con $\beta \neq 0$ e $\underline{w} = \underline{A} + i\,\underline{B}$ è un autovettore corrispondente, allora:

$$\underline{u}_1 = e^{\alpha t}\left[\underline{A}\cos\beta t - \underline{B}\,\mathrm{sen}\,\beta t\right],$$

$$\underline{u}_2 = e^{\alpha t}\left[\underline{A}\,\mathrm{sen}\,\beta t + \underline{B}\cos\beta t\right]$$

3) Se A ha un unico autovalore reale λ con molteplicità geometrica 1 e $\underline{w}$ è un autovettore corrispondente a λ, allora esiste un vettore $\underline{v}$ t.c.

$$[A - \lambda I]\underline{v} = \underline{w} \quad \text{e}$$

$$\underline{u}_1 = e^{\lambda t}\,\underline{w}$$

$$\underline{u}_2 = e^{\lambda t}(\underline{v} + t\,\underline{w})$$

oss.

Per ogni matrice quadrata 2×2 si è in

uno dei tre casi precedenti. Se A ha un unico autovalore reale con m.g. = 2 si scelgono come autovettori i vettori della base canonica.

ESERCIZIO 4. Determinare l'integrale generale del sistema non omogeneo

$$\begin{cases} x' = 2x+3y \\ y' = 2x+y+2e^{-2t} \end{cases}$$

SOL.

$$\begin{cases} y = \frac{1}{3}(x'-2x) \\ \cancel{\frac{1}{3}}(x''-2x') = \cancel{2}^{6}x + \cancel{\frac{1}{3}}(x'-2x) + \cancel{2}^{6}e^{-2t} \end{cases}$$

$$x'' - 2x' = 6x + x' - 2x + 6e^{-2t}$$

$$x'' - 3x' - 4x = 6e^{-2t}$$

OMOGENEA: $\lambda^2 - 3\lambda - 4 = 0 \qquad \lambda = \begin{cases} 4 \\ -1 \end{cases}$

$$x_0(t) = c_1 e^{-t} + c_2 e^{4t}$$

SOL. PART.: $x_p(t) = A e^{-2t}$

$$x_p'(t) = -2A e^{-2t}$$

$$x_p''(t) = 4A e^{-2t}$$

$$\cancel{4A e^{-2t}} + 6A e^{-2t} - \cancel{4A e^{-2t}} = 6e^{-2t}$$

$$\Longrightarrow A = 1.$$

L'int. gen. è: $x(t) = x_0(t) + x_p(t) =$

$$= c_1 e^{-t} + c_2 e^{4t} + e^{-2t}$$

$$y(t) = \frac{1}{3}\left(-c_1 e^{-t} + 4c_2 e^{4t} - 2e^{-2t} - 2c_1 e^{-t} - 2c_2 e^{4t} - 2e^{-2t}\right)$$

$$= \frac{1}{3}\left(-3c_1 e^{-t} + 2c_2 e^{4t} - 4e^{-2t}\right)$$

$$\begin{cases} x(t) = c_1 e^{-t} + c_2 e^{4t} + e^{-2t} \\ y(t) = -c_1 e^{-t} + \frac{2}{3} c_2 e^{4t} - \frac{4}{3} e^{-2t} \end{cases}$$

ESERCIZIO A CASA:

$$A = \begin{bmatrix} 1 & 5 \\ 2 & 4 \end{bmatrix} \qquad \underline{u}' = A\underline{u} \quad \text{con} \quad \underline{u} = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$

1) Scrivere l'int. gen. del sistema.

2) Risolvere il problema di Cauchy $\underline{u}(0) = \begin{bmatrix} 7 \\ 0 \end{bmatrix}$

3) Tra tutte le soluzioni infinitesime per $t \to +\infty$ determinare quelle per cui $x(0) = 10$.

[RISP. 1) $u(t) = c_1 e^{-t} \begin{bmatrix} 5 \\ -2 \end{bmatrix} + c_2 e^{6t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $c_1, c_2 \in \mathbb{R}$

2) $c_1 = 1$, $c_2 = 2$

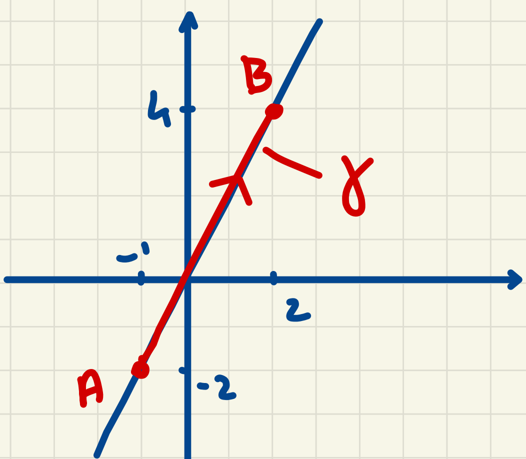
3) $u(t) = e^{-t} \begin{bmatrix} 10 \\ -4 \end{bmatrix}$]

LAVORO – CAMPI – INTEGRALI DI LINEA DI 2^a SPECIE

ESERCIZIO 5. Calcolare il lavoro del campo (piano) $\underline{F}(x,y) = y\,\underline{i} + x\,\underline{j}$ lungo il segmento della retta $y = 2x$ con $x \in [-1;2]$.

SOL.

METODO 1 (DIRETTO)



$$\mathcal{L} = \int_\gamma \underline{F} \cdot d\underline{r} = \int_a^b \underline{F}(\underline{r}(t)) \cdot \underline{r}'(t)\,dt$$

$$\underline{r}(t) = \begin{bmatrix} t \\ 2t \end{bmatrix} \qquad \underline{r}'(t) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$t \in [-1;2] \qquad \underline{F}(x,y) = \begin{bmatrix} y \\ x \end{bmatrix}$$

$$\mathcal{L} = \int_\gamma \underline{F} \cdot d\underline{r} = \int_{-1}^{2} \underline{F}(\underline{r}(t)) \cdot \underline{r}'(t)\, dt = \int_{-1}^{2} \begin{bmatrix} 2t \\ t \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \end{bmatrix} dt =$$

$$= \int_{-1}^{2} (2t + 2t)\, dt = \left[\cancel{4}^{2} \frac{t^2}{\cancel{2}} \right]_{-1}^{2} = 8 - 2 = 6.$$

METODO 2. $\underline{F}$ è conservativo ?

$\underline{F}(x,y) = \begin{bmatrix} y \\ x \end{bmatrix}$ A $D_{\underline{F}} = \mathbb{R}^2$, semplicemente connesso.

NOTA. $\underline{F}(x,y) = \begin{bmatrix} F_1(x,y) \\ F_2(x,y) \end{bmatrix}$

B $F_{1y} = 1 = F_{2x} \Rightarrow \underline{F}$ è irrotazionale

A + B $\Rightarrow$ F è conservativo.

$$\Rightarrow \exists \; U(x,y) : \quad \nabla U = \underline{F} \quad \Rightarrow \begin{cases} U_x = F_1 \\ U_y = F_2 \end{cases}$$

$$U_x = \overset{F_1}{y} \Rightarrow U(x,y) = \int y \, dx = xy + c(y)$$

$$\overset{F_2}{x} = U_y(x,y) = \frac{\partial}{\partial y}\left(xy + c(y) \right) = x + c_y$$

$$\Rightarrow \quad x = x + c_y \Rightarrow c_y = 0 \Rightarrow c(y) = c$$

Il potenziale è quindi

$$\boxed{U(x,y) = xy + c}$$

$$\mathcal{L}_{AB} = U(B) - U(A) = U(2;4) - U(-1;-2) =$$

$$= 2 \cdot 4 + \cancel{c} - (-1)(-2) - \cancel{c} = 6 .$$

N.B. Se γ è chiusa $(\gamma(A) \equiv \gamma(B)) \Rightarrow \mathcal{L} = 0$.

ESERCIZIO 2. Sia $\underline{F} : \mathbb{R}^3 \to \mathbb{R}^3$ il campo

$$\underline{F}(x,y,z) = xy^2\,\underline{i} + x^2y\,\underline{j} + \frac{z^3}{3}\,\underline{k}$$

$\underline{F}$ è conservativo? Se sì determinarne un potenziale.

SOL.

- $D = \mathbb{R}^3$ semplicemente connesso

$$\text{rot}\,\underline{F} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \partial_x & \partial_y & \partial_z \\ xy^2 & x^2y & \frac{z^3}{3} \end{vmatrix} = \underline{i}\,(0) - \underline{j}\,(0) + \underline{k}\,(2xy - 2xy)$$

$$= \underline{0} \Rightarrow \underline{F} \text{ è irrotaz.}$$

$\underline{F}$ è conservativo. Determino $U(x,y,z)$:

$$\nabla U(x,y,z) = \underline{F}(x,y,z) \iff \begin{cases} U_x = F_1 = xy^2 \\ U_y = F_2 \\ U_z = F_3 \end{cases}$$

$$U_x = xy^2 \Rightarrow U(x,y,z) = \int xy^2 \, dx = \frac{1}{2}x^2y^2 + c(y,z)$$

$$x^2y = U_y = \partial_y\left(\frac{1}{2}x^2y^2 + c(y,z)\right) = x^2y + c_y(y,z)$$

$$\cancel{x^2y} = \cancel{x^2y} + c_y(y,z) \Rightarrow$$

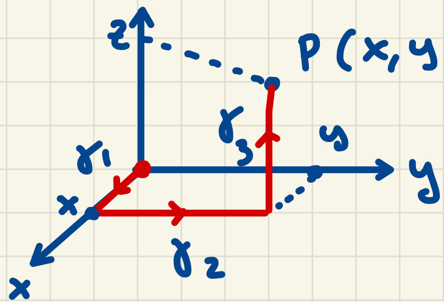
$$\Rightarrow c_y(y,z) = 0 \Rightarrow c(y,z) = c(z).$$

$$\frac{z^3}{3} = U_z = \partial_z\left(\frac{1}{2}x^2y^2 + c(z)\right) = c_z$$

$$\Rightarrow c_z = \frac{z^3}{3} \Rightarrow c(z) = \int \frac{z^3}{3} dz = \frac{z^4}{12} + c$$

$$U(x,y,z) = \frac{1}{2}x^2y^2 + \frac{z^4}{12} + c$$

In alternativa:



$$\gamma = \gamma_1 \cup \gamma_2 \cup \gamma_3$$

$$\underline{r}_1(t) = (t, 0, 0) \qquad t \in [0, x]$$

$$\underline{r}_1'(t) = (1, 0, 0) = \underline{i}$$

$$\underline{r}_2(t) = (x; t; 0) \qquad t \in [0, y]$$

$$\underline{r}_2'(t) = (0; 1; 0) = \underline{j}$$

$$\underline{r}_3(t) = (x, y, t) \quad t \in [0, z]$$

$$r_3'(t) = (0, 0, 1) = \underline{k}$$

$$U(x,y,z) = \int_\gamma \underline{F} \cdot d\underline{r} = \int_\gamma \underline{F}(\underline{r}(t)) \cdot \underline{r'(t)}\, dt =$$

$$= \sum_{i=1}^{3} \int_{\gamma_i} F(\underline{r}_i(t)) \cdot r_i'(t)\, dt =$$

$$= \cancel{\int_0^x (0;0;0) \cdot (1,0,0)\, dt} + \int_0^y (xt^2, x^2t, 0) \cdot (0,1,0)\, dt$$

$$+ \int_0^z \left(xy^2, x^2y, \frac{t^3}{3}\right) \cdot (0,0,1)\, dt =$$

$$= \int_0^y x^2 t\, dt + \int_0^z \frac{t^3}{3}\, dt = x^2 \left[\frac{t^2}{2}\right]_0^y + \left[\frac{t^4}{12}\right]_0^z =$$

$$= \frac{x^2y^2}{2} + \frac{z^4}{12}.$$