Osservazioni sull'esempio

- $u \rightsquigarrow T_u$ è iniettiva su $L^1(\Omega)$.

$\rightarrow u_1 = u_2$ q.o. su $\Omega \Rightarrow T_{u_1} = T_{u_2}$ in $\mathcal{D}'(\Omega)$

$$T_{u_1}(\varphi) = \int_\Omega u_1 \varphi \underset{\substack{\uparrow \\ u_1 = u_2 \text{ q.o. su } \Omega}}{=} \int_\Omega u_2 \varphi = T_{u_2}(\varphi).$$

(*) $\left[T_{u_1} = T_{u_2} \text{ in } \mathcal{D}'(\Omega) \Rightarrow u_1 = u_2 \text{ q.o. su } \Omega \right]$

$$\int_\Omega u_1 \varphi = \int_\Omega u_2 \varphi \quad \forall \varphi \in \mathcal{D}(\Omega) \Rightarrow u_1 = u_2 \text{ q.o. su } \Omega$$

$$\int_\Omega (u_1 - u_2)\varphi = 0 \quad \forall \varphi \in \mathcal{D}(\Omega) \Rightarrow u_1 - u_2 = 0 \text{ q.o. su } \Omega$$

- Notazione: $T_u(\varphi)$ equivale a $\langle u, \varphi \rangle_{(\mathcal{D}'(\Omega), \mathcal{D}(\Omega))}$

$$\langle u, \varphi \rangle = T_u(\varphi) = \int_\Omega u \varphi$$

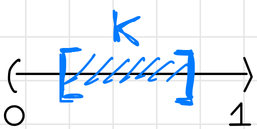
- Per definire T_u, basta una condiz. più debole:

$u \in L^1_{loc}(\Omega) := \left\{ u : \Omega \to \mathbb{R} : \int_K |u| < +\infty \ \forall K \text{ compatto} \subseteq \Omega \right\}$.

$L^1(\Omega) \underset{\not\supseteq}{\subseteq} L^1_{loc}(\Omega)$ $\left(\text{se } \int_\Omega |u| < +\infty \Rightarrow \forall K \subset\subset \Omega \int_K |u| < +\infty \right)$

Es. $\Omega = (0,1)$ $\quad u(x) = \frac{1}{x} \notin L^1(\Omega)$ ma $u \in L^1_{loc}(\Omega)$

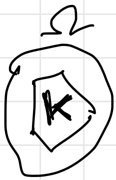
infatti, se $K \subset\subset \Omega$ (compatto $\subseteq \Omega$)



$u \in L^1(K)$.

- In particolare, possiamo associare una distribuzione a qualsiasi $u \in L^p(\Omega)$ con $p \in [1,+\infty]$.

Infatti $L^p(\Omega) \not\subseteq L^1(\Omega)$, MA:



$$L^p(\Omega) \subseteq L^1_{loc}(\Omega) \quad \forall p \in [1,+\infty] :$$

$$u \in L^p(\Omega) \Rightarrow u \in L^p(K) \ \forall K \subset\subset \Omega \longrightarrow \int_K |u|^p < +\infty$$

$$\int_\Omega |u|^p < +\infty$$

$$\Rightarrow u \in L^1(K) \ \forall K \subset\subset \Omega \Rightarrow u \in L^1_{loc}(\Omega)$$

perché $|K| < +\infty$.

Tutte le funzioni $u \in L^p(\Omega)$ possono essere viste come distribuzioni

$$u \in L^p(\Omega) \rightsquigarrow T_u$$

$$\langle u, \varphi \rangle_{(\mathcal{D}'(\Omega), \mathcal{D}(\Omega))} := \int_\Omega u \varphi \, dx .$$

$$\boxed{\mathcal{D}'(\Omega) = \{ T : \mathcal{D}(\Omega) \to \mathbb{R} \ \text{distribuzioni} \}.}$$

- è uno spazio vettoriale

$$\begin{cases} (T_1 + T_2)(\varphi) := T_1(\varphi) + T_2(\varphi) & \forall \varphi \in \mathcal{D}(\Omega) \\ (\lambda T)(\varphi) := \lambda \cdot T(\varphi) & \forall \varphi \in \mathcal{D}(\Omega). \end{cases}$$

$$T_1 = T_{u_1}, \quad T_2 = T_{u_2} \Rightarrow \left((T_1+T_2)(\varphi) = \int_\Omega (u_1\varphi + u_2\varphi) = \int_\Omega (u_1+u_2)\varphi = T_{u_1+u_2}(\varphi) \right)$$

$$\lambda T_u = T_{\lambda u} \qquad \left(\lambda T_u(\varphi) = \lambda \int_\Omega u\varphi = \int_\Omega \lambda u \varphi = T_{\lambda u}(\varphi) \right).$$

- Introduciamo su $\mathcal{D}'(\Omega)$ una CONVERGENZA:

$\{T_h\} \subseteq \mathcal{D}'(\Omega)$, $T_h \xrightarrow{\text{in } \mathcal{D}'(\Omega)} 0$ se

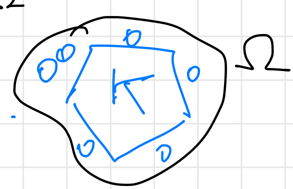
$$T_h(\varphi) \longrightarrow 0 \quad \forall \varphi \in \mathcal{D}(\Omega).$$

$$\left(T_h \to T \quad \text{se} \quad T_h(\varphi) \longrightarrow T(\varphi) \quad \forall \varphi \in \mathcal{D}(\Omega) \right):$$

<u>Esempio</u> $T_h = T_{u_h}$, con $u_h \subseteq L^1(\Omega)$

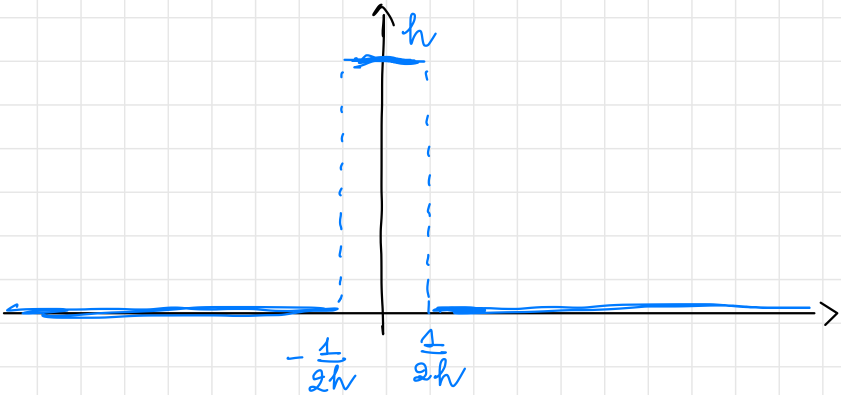
$$u_h \to 0 \text{ in } L^1(\Omega) \Rightarrow T_{u_h} \longrightarrow 0 \text{ in } \mathcal{D}'(\Omega)$$

$$|T_{u_h}(\varphi)| = \left| \int_\Omega u_h \varphi \right| \leq \int_\Omega |u_h|\,|\varphi| \leq \max_{\cancel{\Omega}\,K} |\varphi| \cdot \underbrace{\int_\Omega |u_h|}_{\downarrow \atop 0}$$



2) La delta di Dirac.

$\{u_h\} \subseteq L^1(\mathbb{R})$



- non converge in $L^1(\mathbb{R})$:

$u_h \longrightarrow 0$ q.o. su $\mathbb{R}$. $\left(\forall x \in \mathbb{R}\setminus\{0\},\ u_h(x) \to 0\right)$

$\Rightarrow$ se $\exists \lim\limits_{h\to+\infty} u_h$ in $L^1(\mathbb{R})$,

$\lim\limits_{h\to+\infty} u_h = 0$, , T/A. $\lim\limits_{h\to+\infty} u_h \neq 0$ in $L^1(\mathbb{R})$

perché $\|u_h\|_{L^1(\mathbb{R})} = \int_{-\frac{1}{2h}}^{\frac{1}{2h}} h = 1$.

- converge in $\mathcal{D}'(\mathbb{R})$

$$\langle u_h, \varphi\rangle = T_{u_h}(\varphi) = \int_{\mathbb{R}} u_h \varphi = \overbrace{h\int_{-\frac{1}{2h}}^{\frac{1}{2h}} \varphi}^{\text{media di } \varphi \text{ su } \left[-\frac{1}{2h}, \frac{1}{2h}\right]} \longrightarrow \varphi(0)$$

(MASSA)

Def. δ_0 DELTA DI DIRAC IN 0

$$\langle \delta_0, \varphi \rangle_{(\mathcal{D}'(\mathbb{R}),\mathcal{D}(\mathbb{R}))} := \varphi(0)$$

$$\parallel$$

$$\delta_0(\varphi)$$

Osservazioni:

- Se $u_h = h \cdot \chi_{\left[-\frac{1}{2h}, \frac{1}{2h}\right]}$, allora $u_h \longrightarrow \delta_0$ in $\mathcal{D}'(\mathbb{R})$.

- Verifica che $\delta_0 \in \mathcal{D}'(\mathbb{R})$

(i) lineare: $\underbrace{\delta_0(\alpha\varphi+\beta\psi)}_{\substack{\parallel \\ (\alpha\varphi+\beta\psi)(0) \\ \parallel \\ \alpha\varphi(0)+\beta\psi(0)}} \stackrel{?}{=} \underbrace{\alpha\,\delta_0(\varphi)}_{\alpha\varphi(0)} + \underbrace{\beta\,\delta_0(\psi)}_{\beta\psi(0)}$

(ii) continuo: $\varphi_h \longrightarrow 0$ in $\mathcal{D}(\mathbb{R}) \Rightarrow \underbrace{\delta_0(\varphi_h)}_{\varphi_h(0)} \longrightarrow 0$.

vero per def. di convergenza in $\mathcal{D}(\mathbb{R})$

$\mathrm{Supp}(\varphi_h) \subseteq K$ compatto

$\varphi_h \longrightarrow 0$ uniformemente.

$$\Downarrow$$

$\varphi_h(0) \longrightarrow 0$.

• Generalizzazioni ovvie

→ $0 \rightsquigarrow x^0 \in \mathbb{R}$ $\qquad \delta_{x^0}(\varphi) = \varphi(x^0)$.

→ caso n-dimensionale:

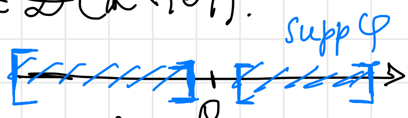
$$x^0 \in \mathbb{R}^n \qquad \delta_{x^0}(\varphi) := \varphi(x^0) \qquad \delta_{x^0} \in \mathcal{D}'(\mathbb{R}^n)$$

• $\underset{\nexists\, T_u}{\delta_0}$ non è associata ad alcuna funzione di $u \in L^1_{loc}(\Omega)$

Dim. Supponiamo per assurdo $\delta_0 \overset{\text{in } \mathcal{D}'(\mathbb{R})}{=} T_u$, con $u \in L^1_{loc}(\mathbb{R})$

$$\int_{\mathbb{R}} u\,\varphi\,dx = \varphi(0) \qquad \forall \varphi \in \mathcal{D}(\mathbb{R})$$

In particolare, posso prendere $\varphi \in \mathcal{D}(\mathbb{R}\setminus\{0\})$.



$$\int_{\mathbb{R}\setminus\{0\}} u\varphi = \varphi(0) = \underset{\int_{\mathbb{R}\setminus\{0\}} 0\cdot\varphi}{\underset{\parallel}{0}}. \qquad \forall \varphi \in \mathcal{D}(\mathbb{R}\setminus\{0\})$$

Applico la proprietà $(*)$ con $\begin{cases} \Omega = \mathbb{R}\setminus\{0\} \\ u_1 = u \\ u_2 = 0 \end{cases}$

$\Rightarrow u = 0$ q.o. su $\mathbb{R}\setminus\{0\}$. $\Rightarrow u = 0$ q.o. su $\mathbb{R}$.

$$\Rightarrow \int_{\mathbb{R}} \underbrace{u\,\varphi}_{=0 \text{ q.o. su } \mathbb{R}.}\,dx = 0 \qquad \forall \varphi \in \mathcal{D}(\mathbb{R}) \qquad \text{ASSURDO.}$$

prendendo φ: $\varphi(0) = 1$ ■

Derivazione di distribuzioni

Def. $(n=1)$ $\Omega \subseteq \mathbb{R}$

Data $T \in \mathcal{D}'(\Omega)$, definisco $T' \in \mathcal{D}'(\Omega)$ come:

$$\boxed{\langle T', \varphi \rangle := -\langle T, \varphi' \rangle} \qquad \forall \varphi \in \mathcal{D}(\Omega)$$

- è una distribuzione

(i) lineare

$$\langle T', \alpha\varphi + \beta\psi \rangle \overset{?}{=} \alpha \underbrace{\langle T', \varphi \rangle}_{\substack{\parallel \\ -\langle T, \varphi' \rangle}} + \beta \underbrace{\langle T', \psi \rangle}_{\substack{\parallel \\ -\langle T, \psi' \rangle}}$$

$$\parallel$$

$$-\langle T, (\alpha\varphi + \beta\psi)' \rangle$$

$$\parallel$$

$$-\langle T, \alpha\varphi' + \beta\psi' \rangle$$

$$\parallel$$

$$-\alpha\langle T, \varphi' \rangle - \beta\langle T, \psi' \rangle$$

(ii) $\varphi_h \to 0$ in $\mathcal{D}(\Omega) \Rightarrow \overbrace{\langle T', \varphi_h \rangle}^{\substack{-\langle T, \varphi_h' \rangle \\ \parallel}} \longrightarrow 0$

Infatti $\varphi_h \to 0$ in $\mathcal{D}(\Omega) \Rightarrow \varphi_h' \to 0$ in $\mathcal{D}(\Omega)$

($\exists K$ tale che $\operatorname{supp} \varphi_h' \subset K$ $\forall h$ e

$\varphi_h' \longrightarrow 0$ unif. su K con tutte le derivate)

Quindi $\langle T, \varphi_h' \rangle \longrightarrow 0$ perché $T \in \mathcal{D}'(\Omega)$.

• perché definirla in qs. modo?

Si consideriamo il caso $T = T_u$ con $u \in C^1(\Omega) \cap L^1_{loc}(\Omega)$

$$(T_u)' \overset{\text{in } \mathcal{D}'(\Omega)}{=} T_{u'}$$

$$\langle (T_u)', \varphi \rangle = -\langle T_u, \varphi' \rangle = \boxed{-\int_\Omega u\,\varphi'}$$

$$\| $$

$$\langle T_{u'}, \varphi \rangle = \boxed{\int_\Omega u'\,\varphi}$$

= int. per parti.

se $\Omega = (-1, 1)$

il termine

$$u\varphi \Big|_{-1}^{1} = 0$$

perché $\varphi \in \mathcal{D}(-1, 1)$