SERIE DI FOURIER 3-6-2021

ESERCIZIO 1. Sia f la funzione pari definita su $\mathbb{R}$ periodica di periodo 2π e definita in $[0,\pi]$ da $f(x) = 5 - 2\operatorname{sen}\frac{x}{2}$.

La serie di Fourier di f è:

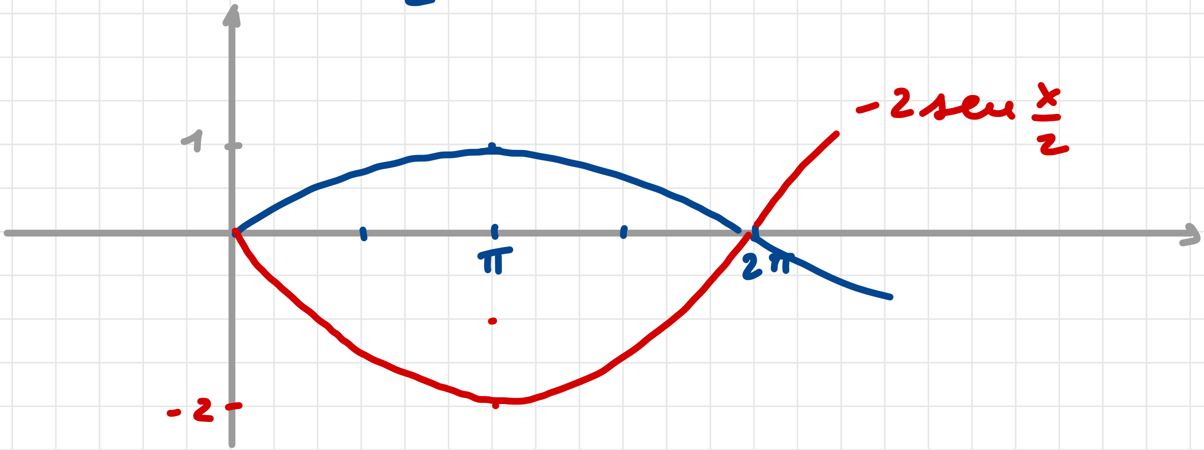
$$Sf(x) = 5 - \frac{4}{\pi} + \sum_{n=1}^{\infty} \frac{8}{(4n^2-1)\pi} \cos nx$$

1) Tracciare il grafico di f in $[-3\pi, 3\pi]$

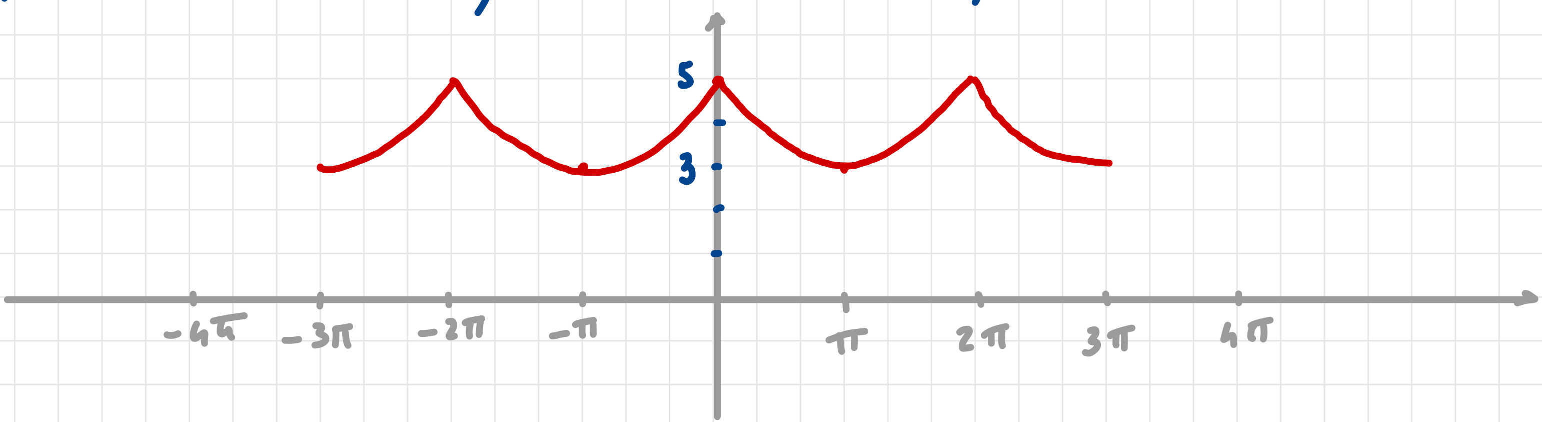
2) Dopo aver giustificato la convergenza della serie $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{4n^2-1}$, calcolarne la somma.

SOL.

1) $f(x) = 5 - 2\,\text{sen}\,\frac{x}{2} \qquad x \in [0, \pi]$



$f: \mathbb{R} \to [3;5]$, 2π- periodica , pari.



OSS.

$$Sf(x) = 5 - \frac{4}{\pi} + \sum_{n=1}^{\infty} \frac{8}{(4n^2-1)\pi} \cos nx$$

$$\frac{a_0}{2} = 5 - \frac{4}{\pi} \longrightarrow a_0 = 10 - \frac{8}{\pi}$$

$$a_n = \frac{8}{(4n^2-1)\pi} = \frac{2}{\pi} \int_0^{\pi} \left(5 - 2\sin\frac{x}{2}\right) \cos nx \, dx$$

f pari

$$b_n = 0 \qquad \forall n \geq 1 .$$

2) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{4n^2-1}$ converge ?

$$0 \leq \left| \frac{(-1)^n}{4n^2-1} \right| \leq \frac{1}{4n^2-1} \quad \text{poiché } \sum_{n=1}^{\infty} \frac{1}{4n^2-1} \text{ conv.}$$

$$\Longrightarrow \sum \left| \frac{(-1)^n}{4n^2-1} \right| \text{ conv.} \Longrightarrow \sum \frac{(-1)^n}{4n^2-1} \text{ conv.}$$

SOMMA DELLA SERIE:

$$Sf(x) = 5 - \frac{4}{\pi} + \frac{8}{\pi} \sum_{n=1}^{\infty} \frac{1}{4n^2-1} \cos nx$$

$$Sf(\pi) = 5 - \frac{4}{\pi} + \frac{8}{\pi} \sum_{n=1}^{\infty} \frac{1}{4n^2-1} \overbrace{\cos(\pi n)}^{(-1)^n}$$

$$Sf(\pi) = f(\pi) = 3$$

$$3 = 5 - \frac{4}{\pi} + \frac{8}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n}{4n^2-1}$$

$$-2 + \frac{4}{\pi} = \frac{8}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n}{4n^2-1}$$

$$\frac{\cancel{\pi}}{8}\left(\frac{4-2\pi}{\cancel{\pi}}\right) = \sum_{n=1}^{\infty} \frac{(-1)^n}{4n^2-1}$$

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{4n^2-1} = \frac{2-\pi}{4}$$

ESERCIZIO 2. Sia $f : \mathbb{R} \to \mathbb{R}$ la funzione 2π-periodica che nell'intervallo $(-\pi, \pi]$ è definita da $f(x) = 2(x + |x|)$.

1) Tracciare il grafico di f in $[-2\pi, 2\pi]$

2) Stabilire per quali $x \in \mathbb{R}$ la Sf converge puntualmente precisandone il limite.

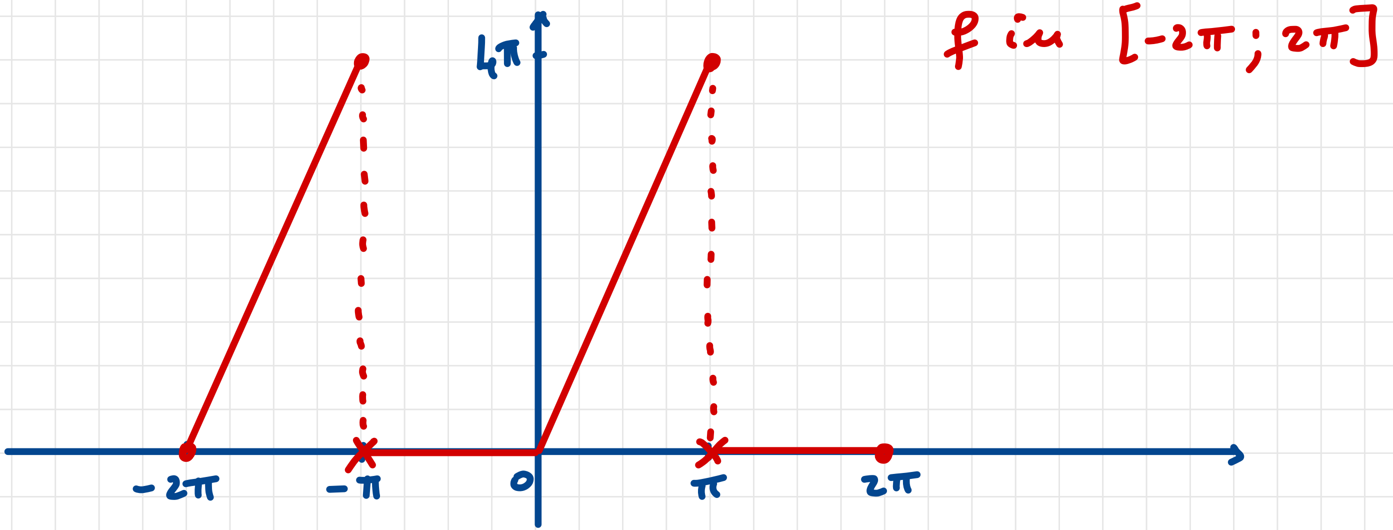
3) Scrivere il polinomio di Fourier

$$S_1 f(x) = \frac{a_0}{2} + a_1 \cos x + b_1 \operatorname{sen} x$$

SOL.

1)

$$f(x) = \begin{cases} 4x & \text{se} \quad 0 \leq x \leq \pi \\ 0 & \text{se} \quad -\pi < x < 0 \end{cases} \qquad 2\pi\text{- periodica.}$$



2) • $Sf(x)$ converge a $f(x)$ dove f è continua cioè $\forall x \in \mathbb{R} - \{(2k+1)\pi, k \in \mathbb{Z}\}$

• Se $x = x_k = (2k+1)\pi$ la serie converge alla media dei due limiti destro e sinistro di f in x_k.

$$\frac{f(x_k^+) + f(x_k^-)}{2} = \frac{0 + 4\pi}{2} = 2\pi.$$

3)
$$S_1 f(x) = \frac{a_0}{2} + a_1 \cos x + b_1 \operatorname{sen} x$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\int_0^{\pi} 4x\,dx = \frac{1}{\cancel{\pi}} \cdot \frac{4\pi^2}{2} = 2\pi$$

$$a_1 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos x\,dx = \frac{1}{\pi}\int_0^{\pi} 4x\cos x\,dx =$$

$$= \frac{4}{\pi}\int_0^{\pi} x\cos x\,dx = \frac{4}{\pi}\left\{ \cancel{\Big[x \operatorname{sen} x\Big]_0^{\pi}} - \int_0^{\pi} \operatorname{sen} x\,dx \right\}$$

$$= +\frac{4}{\pi}\Big[+\cos x\Big]_0^{\pi} = \frac{4}{\pi}(-1-1) = -\frac{8}{\pi}.$$

$$b_1 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\operatorname{sen} x\,dx = \frac{1}{\pi}\int_0^{\pi} 4x \operatorname{sen} x\,dx =$$

$$= \frac{4}{\pi}\int_0^{\pi} x \operatorname{sen} x\,dx =$$

$$= \frac{4}{\pi}\left\{\left[-x\cos x\right]_0^{\pi} + \cancel{\int_0^{\pi}\cos x\, dx}\right\} = \frac{4}{\pi}\left(+\pi\right) = 4.$$

$$\boxed{S_1 f(x) = \frac{2\pi}{2} - \frac{8}{\pi}\cos x + 4\,\mathrm{sen}\, x.}$$

ESERCIZIO 3.

Sia f la funzione dispari di periodo 2π così definita:

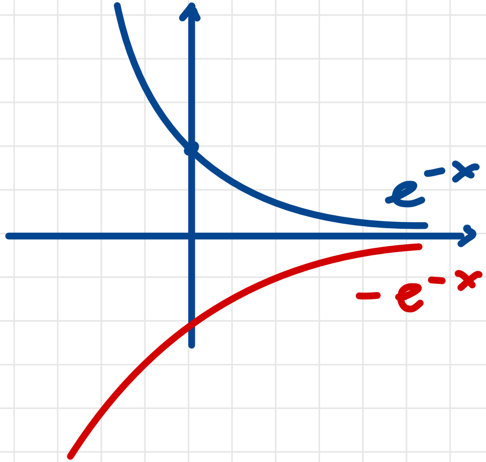
$$f(x) = \begin{cases} 1 - e^{-x} & \text{se } x \in [0; \frac{\pi}{2}] \\ 0 & \text{se } x \in (\frac{\pi}{2}; \pi] \end{cases}$$

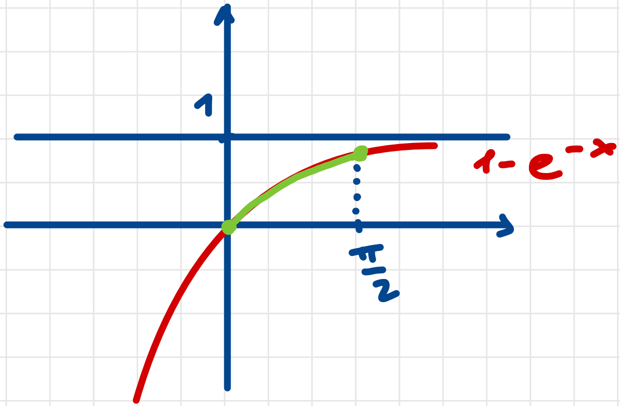
1) Tracciare il grafico di f in $[-\pi, 3\pi]$

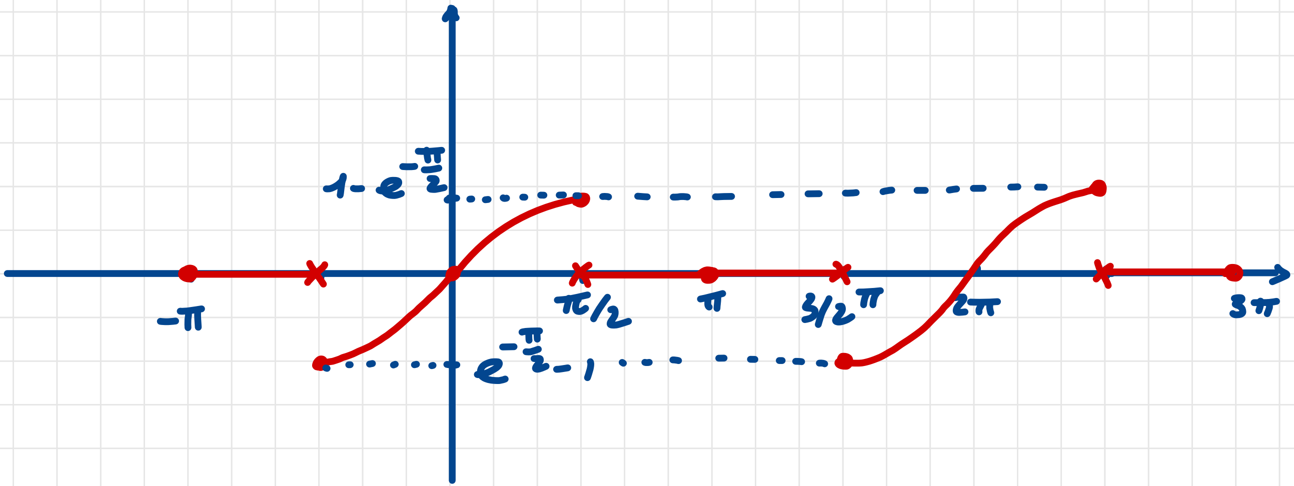
2) Si dica in quali punti $Sf(x)$ converge e e cosa converge in $[0; 2\pi]$.

SOL.

1) $f(x) = 1 - e^{-x}$







2) $Sf(x)$ converge a $f(x)$ dove f è continua cioè in $[0;\frac{\pi}{2}) \cup (\frac{\pi}{2};\frac{3}{2}\pi) \cup (\frac{3}{2}\pi; 2\pi]$.

in $x=\frac{\pi}{2}$:
$$\left.\begin{array}{l} f(\frac{\pi}{2}^-) = 1-e^{-\pi/2} \\ f(\frac{\pi}{2}^+) = 0 \end{array}\right\} \to Sf \text{ converge a } \frac{1-e^{-\pi/2}+0}{2}$$

in $x=\frac{3}{2}\pi$
$$\left.\begin{array}{l} f(\frac{3}{2}\pi^-) = 0 \\ f(\frac{3}{2}\pi^+) = e^{-\pi/2}-1 \end{array}\right\} \to Sf \text{ converge a } \frac{e^{-\pi/2}-1}{2}$$

ESERCIZIO 4.

Sia $f(x)$ la funzione 2π-periodica che vale $f(x) = x(\pi-|x|)$ se $x \in [-\pi,\pi]$. Scrivere la serie di Fourier di f e tracciare il grafico di

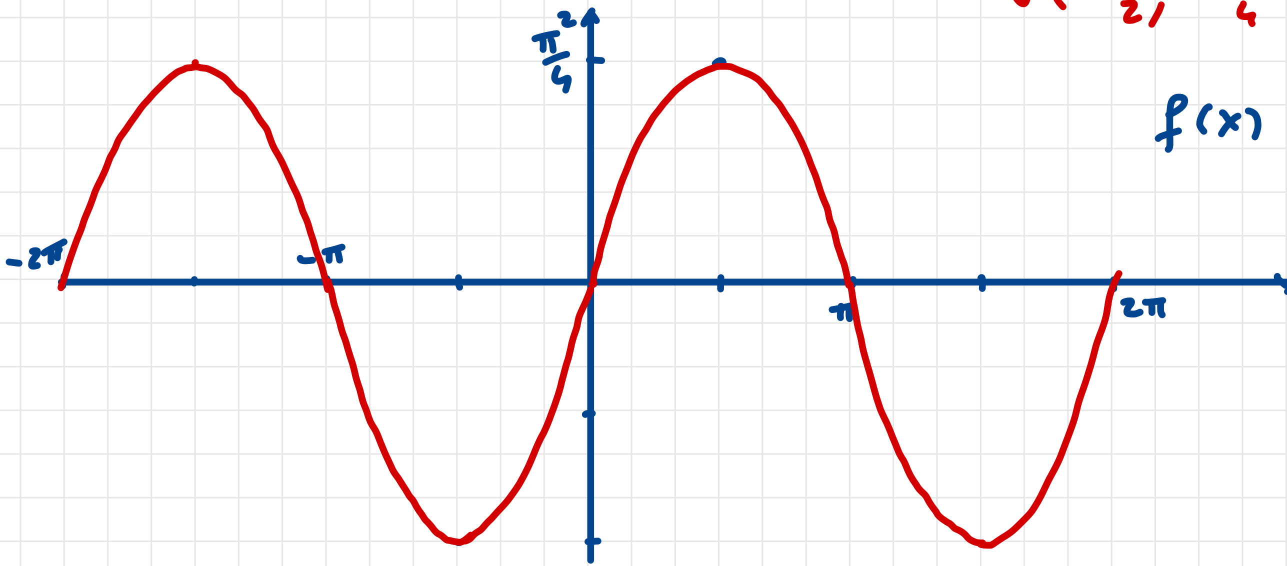
f in $[-2\pi ; 2\pi]$.

Sol.

$$f(x) = \begin{cases} x(\pi - x) & \text{se} \quad x \in [0, \pi] \\ x(\pi + x) & \text{se} \quad x \in [-\pi, 0) \end{cases}$$

PARABOLA DI VERTICE $V\left(\frac{\pi}{2}; \frac{\pi^2}{4}\right)$

PARABOLA DI VERTICE $V\left(-\frac{\pi}{2}; -\frac{\pi^2}{4}\right)$



$f(x)$ è disp.

$\Downarrow$

$a_n = 0 \ \forall n$

$$b_n = \frac{2}{\pi}\int_0^{\pi} x(\pi - x)\,\mathrm{sen}\,nx\,dx =$$

$$= \frac{2}{\pi}\left\{ \cancel{\left[-x(\pi-x)\frac{\cos nx}{n}\right]_0^{\pi}} + \int_0^{\pi}\frac{\cos nx}{n}(\pi - 2x)\,dx \right\} =$$

$$= \frac{2}{\pi n}\left\{ \cancel{\left[\frac{\mathrm{sen}\,nx}{n}(\pi - 2x)\right]_0^{\pi}} + 2\int_0^{\pi}\frac{\mathrm{sen}\,nx}{n}\,dx \right\}$$

$$= \frac{4}{\pi n^2}\cdot\left[-\frac{\cos nx}{n}\right]_0^{\pi} = -\frac{4}{\pi n^3}\left(\cos n\pi - 1\right) =$$

$$= -\frac{4}{\pi n^3}\left((-1)^n - 1\right) = \begin{cases} 0 & n \text{ pari} \\ \dfrac{8}{\pi n^3} & n \text{ dispari} \end{cases}$$

$$\boxed{Sf(x) = \sum_{n=0}^{\infty} \frac{8}{\pi(2n+1)^3}\,\mathrm{sen}\left[(2n+1)x\right]}$$

ESERCIZIO 5. Scrivere lo sviluppo di Fourier della funzione $f(x) = 1 + \operatorname{sen} x - \cos^2 x$ e calcolare

$$\int_{-\pi}^{\pi} [f(x)]^2 dx.$$

SOL.

FORMULE DI BISEZIONE:

$$\cos\frac{x}{2} = \sqrt{\frac{1+\cos x}{2}}$$

$$\operatorname{sen}\frac{x}{2} = \sqrt{\frac{1-\cos x}{2}}$$

$$\longrightarrow \quad \cos x = \sqrt{\frac{1+\cos 2x}{2}}$$

$$\operatorname{sen} x = \sqrt{\frac{1-\cos 2x}{2}}$$

$$f(x) = 1 + \operatorname{sen} x - \frac{1+\cos 2x}{2} = 1 + \operatorname{sen} x - \frac{1}{2} - \frac{1}{2}\cos 2x$$

$$= \frac{1}{2} + 0\cos x + 1\operatorname{sen} x - \frac{1}{2}\cos 2x + 0 \operatorname{sen} 2x$$

$$\frac{a_0}{2} = \frac{1}{2} \Longrightarrow a_0 = 1$$

$a_1 = 0$ $\qquad\qquad$ $b_1 = 1$

$a_2 = -\frac{1}{2}$ $\qquad\qquad$ $b_2 = 0$

$a_n = 0 \quad \forall n \geq 3$ $\qquad\qquad$ $b_n = 0 \quad \forall n \geq 2.$

PARSEVAL: $\displaystyle\int_0^{2\pi} [f(x)]^2 dx = \pi \left[\frac{a_0^2}{2} + \sum_{k=1}^{\infty} (a_k^2 + b_k^2) \right]$

$$\int_0^{2\pi} f^2(x)\, dx = \pi \left(\frac{1}{2} + \frac{1}{4} + 1 \right) = \frac{7}{4}\pi.$$

ESERCIZIO 6. Sia g la funzione 2π-periodica definite in $(-\pi, \pi]$ da

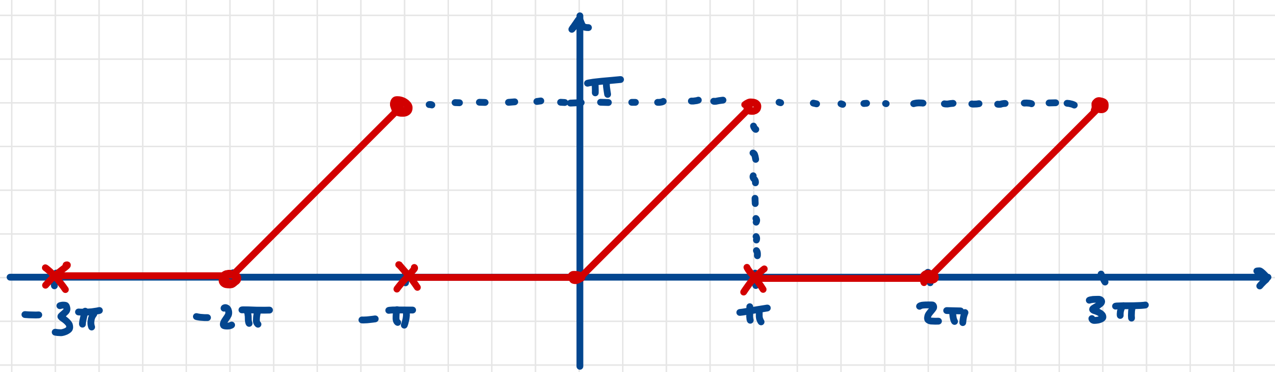
$$g(x) = \begin{cases} 0 & -\pi < x \leq 0 \\ x & 0 < x \leq \pi \end{cases}$$

1) Discutere la convergenza di $Sg(x)$

2) Calcolare i coeff. di Fourier di g fino a $n=3$. Quale proprietà di minimo è soddisfatta?

Sol.

1)



La serie di F. di g converge per ogni $x \in \mathbb{R} - \{(2k+1)\pi,\ k \in \mathbb{Z}\}$ a $g(x)$.
Se $x = (2k+1)\pi$, $S_g(x)$ converge a $\frac{\pi}{2}$

2) $$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} g(x)\,dx = \frac{1}{\pi}\int_0^{\pi} x\,dx = \frac{1}{\cancel{\pi}}\cdot\frac{\pi^2}{2} = \frac{\pi}{2}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} g(x)\cos nx\,dx = \frac{1}{\pi}\int_0^{\pi} x\cos nx\,dx =$$

$$= \frac{1}{\pi}\left\{\left[x\,\cancel{\frac{\operatorname{sen} nx}{n}}\right]_0^{\pi} - \int_0^{\pi}\frac{\operatorname{sen} nx}{n}\,dx\right\} =$$

$$= +\frac{1}{\pi n^2}\left[+\cos nx\right]_0^{\pi} = +\frac{1}{\pi n^2}\left((-1)^n - 1\right) \qquad n = 1, 2, \ldots$$

Analogamente si ottiene $b_n = \frac{(-1)^{n+1}}{n}$ $n=1,2,\dots$

(ESERCIZIO)

$$S_3 g(x) = \frac{a_0}{2} + a_1 \cos x + b_1 \sin x + a_2 \cos 2x + b_2 \sin 2x$$
$$+ a_3 \cos 3x + b_3 \sin 3x.$$

NOTA (di algebra lineare)

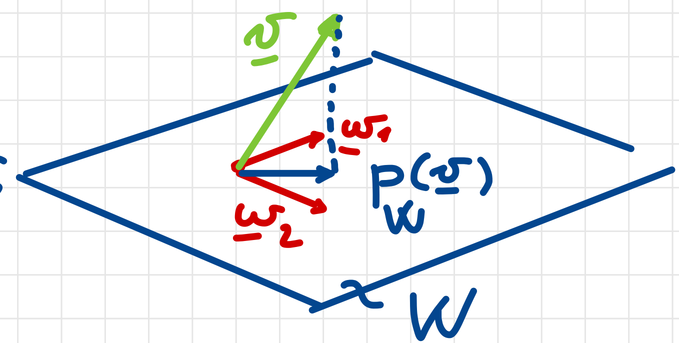
$\langle \underline{v}, \underline{w} \rangle$ prodotto scalare $\underline{v}$ per $\underline{w}$

W ssp. vett di $\mathbb{R}^3$ $\quad \mathcal{B}_W^{\perp} = \{ \underline{w}_1, \underline{w}_2 \}$ $\quad w_1 \perp w_2$

$v \notin W$

$$P_W(\underline{v}) = \underbrace{\frac{\langle \underline{v}, \underline{w}_1 \rangle}{\| \underline{w}_1 \|^2}}\, \underline{w}_1 + \underbrace{\frac{\langle \underline{v}, \underline{w}_2 \rangle}{\| \underline{w}_2 \|^2}}\, \underline{w}_2$$

COEFF. DI FOURIER.



$\| \underline{v} - p_W(\underline{v}) \|$ è la minima distanza di $\underline{v}$ da W.

Sia $V = \{ f \text{ } 2\pi\text{-periodiche} \}$ è uno sp. vett. di dimensione infinita e una sua base ortogonale è

$$\mathcal{B}_V^{\perp} = \{ 1, \operatorname{sen} nx, \cos nx \} \quad n = 1, 2, \dots$$

Sia W un s.sp. di V, $f \notin W$

$$p_W(f) = \frac{\langle f, 1 \rangle}{\| 1 \|^2} \cdot 1 + \frac{\langle f, \cos x \rangle}{\| \cos x \|^2} \cos x + \frac{\langle f, \operatorname{sen} x \rangle}{\| \operatorname{sen} x \|^2} \operatorname{sen} x$$

$$+ \dots$$

esempio $\quad < f, g > = \int_0^{2\pi} f g \, dx \quad , \quad \| f(x) \| = \sqrt{\int_0^{2\pi} f^2(x) \, dx}$

$\boxed{\| f - P_W(f) \| \text{ è la minima distanza di } f \text{ da } W.}$

Nel nostro esercizio

$S_3 g(x)$ è il polinomio trigonometrico che minimizza la distanza di g dallo spazio vettoriale generato da

$$\{ 1, \cos x, \operatorname{sen} x, \cos 2x, \operatorname{sen} 2x, \cos 3x, \operatorname{sen} 3x \}.$$

$$\| f - S_3 g(x) \|^2 = \int_0^{2\pi} | f - S_3 g(x) |^2 dx.$$

ESERCIZIO 7 (serie di pot.)

Determinare l'intervallo di convergenza delle serie di potenze

$$i) \sum_{n=1}^{\infty} \frac{2^n+1}{3^n}(x-1)^n \qquad ii) \sum_{n=0}^{\infty} \frac{1}{2n+1} x^{2n+1}$$

Calcolare poi la somma della seconda.

SOL.

i) $$\lim_{n\to\infty} \frac{2^{n+1}+1}{3^{\cancel{n}+1}} \cdot \frac{\cancel{3^n}}{2^n+1} = \lim_{n\to\infty} \frac{\cancel{2^n}\left(2+\frac{1}{2^n}\right)}{\cancel{2^n}\left(1+\frac{1}{2^n}\right)\cdot 3} = \frac{2}{3}$$

$$R = \frac{3}{2} \qquad |x-1| < \frac{3}{2} \rightarrow -\frac{3}{2} < x-1 < \frac{3}{2}$$

$$-\frac{1}{2} < x < 5/2$$

$x = -\frac{1}{2}$: $\sum_{n=1}^{\infty} \frac{2^n+1}{3^n} \left(-\frac{3}{2}\right)^n = \sum_{n=1}^{\infty} (-1)^n \frac{2^n+1}{\cancel{3^n}} \cdot \frac{\cancel{3^n}}{2^n}$

$= \sum_{n=1}^{\infty} (-1)^n \frac{2^n+1}{2^n}$ non conv.

$x = \frac{5}{2}$: $\sum_{n=1}^{\infty} \frac{2^n+1}{3^n} \cdot \left(\frac{3}{2}\right)^n = \sum_{n=1}^{\infty} \frac{2^n+1}{\cancel{3^n}} \cdot \frac{\cancel{3^n}}{2^n} = \sum \frac{2^n+1}{2^n}$

non converge perché non è soddisf.
la c. n. di conv.

La serie converge in $\left(-\frac{1}{2}; \frac{5}{2}\right)$.

ii) $\sum_{n=0}^{\infty} \frac{1}{2n+1} x^{2n+1}$ $\lim_{n\to\infty} \frac{1}{2n+3} \cdot \frac{2n+1}{1} = 1$

$-1 < x < 1$

$x = 1$: $\displaystyle\sum_{n=0}^{\infty} \frac{1}{2n+1}$ diverge come $\sum \frac{1}{n}$

$x = -1$: $\displaystyle -\sum_{n=0}^{\infty} \frac{1}{2n+1} \cancel{(-1)^{2n+1}}$ diverge come $\sum \frac{1}{n}$.

La serie di potenze converge per $x \in (-1;1)$.

Ricordiamo che $\displaystyle\sum_{n=0}^{\infty} q^n = \frac{1}{1-q}$ se $|q|<1$.

$$\sum_{n=0}^{\infty} t^{2n} = \sum_{n=0}^{\infty} (t^2)^n = \frac{1}{1-t^2} \qquad |t|<1$$

$$\underbrace{\int \sum_{n=0}^{\infty} t^{2n}\, dt}_{(A)} = \underbrace{\int \frac{1}{1-t^2}\, dt}_{(B)}$$

(A) $$\int \sum_{n=0}^{\infty} t^{2n}\,dt = \sum_{n=0}^{\infty} \frac{t^{2n+1}}{2n+1} = \sum_{n=0}^{\infty} \frac{1}{2n+1}\cdot t^{2n+1}$$

(B) $$\int \frac{1}{1-t^2}\,dt = \int \frac{1}{(1-t)(1+t)}\,dt = \int \left(\frac{A}{1-t} + \frac{B}{1+t}\right) dt \overset{(*)}{=}$$

$$\frac{1}{1-t^2} = \frac{A+At+B-Bt}{(1-t)(1+t)} = \frac{A+B+t(A-B)}{(1-t)(1+t)}$$

$$\begin{cases} A+B=1 \rightarrow A = 1/2 \\ A-B=0 \rightarrow A=B \rightarrow B = 1/2 \end{cases}$$

$$(*) = \frac{1}{2}\int \left(\frac{1}{1-t} + \frac{1}{1+t}\right) dt = \frac{1}{2}\ln|1+t| - \frac{1}{2}\ln|1-t|$$

$$= \frac{1}{2}\ln\left|\frac{1+t}{1-t}\right|$$

$$\sum_{n=0}^{\infty} \frac{1}{2n+1} x^{2n+1} = \frac{1}{2} \ln \left| \frac{1+x}{1-x} \right|$$