Convoluzione

DEFINIZIONE CONTINUA

$$f * g = \begin{cases} \int_{\mathbb{R}_y} f(x-y)\, g(y)\, dy \\ \int_{\mathbb{R}_y} f(y)\, g(x-y)\, dy \end{cases}$$

$f * g = g * f$ FORMULAZIONI EQUIVALENTI

SPAZI FUNZIONALI

$L^p(\Omega)$ $\quad \begin{cases} f \in L^1,\ g \in L^p \longrightarrow f * g \in L^p \\ \| f * g \|_p \le \| f \|_1 \| g \|_p \end{cases}$

$L^1(\Omega)$ $\quad \begin{cases} f \in L^1,\ g \in L^1 \longrightarrow f * g \in L^1 \\ \| f * g \|_1 \le \| f \|_1 \| g \|_1 \end{cases}$

$C_0^\infty(\Omega)$ $\quad f \in C_0^\infty,\ g \in L^1 \longrightarrow f * g \in C_0^\infty$

DISEQUAZIONE DI YOUNG

$\| f * g \|_R \le \| f \|_P \| g \|_Q$

CON $\frac{1}{P} + \frac{1}{Q} = \frac{1}{R} + 1$

FOURIER E LAPLACE

$\mathcal{F}(f * g) = \mathcal{F}(f) \cdot \mathcal{F}(g)$

$L(f * g) = L(f) \cdot L(g)$

PSEUDO LINEARITA' CONVOLUZIONALE

DERIVATA

$(f * g)^{(N)} = f^{(N)} * g = g^{(N)} * f$

$f^{(a)} * g^{(b)} = (f * g)^{(N)} \qquad N = a + b$

CALCOLA $U * H$

$$(U * H)(x) = \begin{cases} 1)\ \int_0^x U(y)\, H(x-y)\, dy \\ 2)\ \int_0^x U(x-y)\, H(y)\, dy \end{cases}$$

DEFINIZIONE 1

$\int_0^x U(x-y)\, H(y)$

CONDIZIONE $\longrightarrow y > 0$

$x - y = t(y) \qquad dy = -dt$

$t(0) = x$

$t(x) = 0$

$\int_x^0 - U(t)\, dt = \int_0^x U(t)\, dt$

SOLO PER $t > 0 \qquad x - y > 0 \qquad \underline{x > y > 0}$

$$(U * H)(x) = \begin{cases} 0 & x < 0 \\ \int_0^x U(t)\, dt & x \ge 0 \end{cases}$$

CALCOLA $H * H$

$(H * H)(x) = \int_0^x H(x-y)\, H(y)\, dy$

$H(y) \longrightarrow$ IMPONE $y \ge 0$

$t(y) = x - y \qquad dy = -dt$

$t(0) = x \qquad t(x) = 0$

$(H * H)(x) = -\int_x^0 H(t)\, dt = \int_0^x H(t)\, dt$

$t \ge 0 \longrightarrow x - y \ge 0 \qquad \underline{x \ge y \ge 0}$

$$(H * H)(x) = \begin{cases} 0 & x < 0 \\ x & x \ge 0 \end{cases} = x H(x)$$

CALCOLA $x^+ * H(x) \qquad x^+ = MAX(0, x)$

$$(x^+ * H)(x) = \begin{cases} \int_0^x x^+(y)\, H(x-y)\, dy \\ \int_0^x x^+(x-y)\, H(y)\, dy \end{cases}$$

SECONDA DEFINIZIONE

$H(y) \longrightarrow y \ge 0$

$x - y = t \qquad dy = -dt \qquad -\int_x^0 = +\int_0^x$

$= \int_0^x x^+(t)\, dt \qquad t > 0 \longrightarrow x > 0$

$$(x^+ * H)(x) = \begin{cases} 0 & x < 0 \\ \int_0^x t\, dt & x \ge 0 \end{cases} = \frac{x^2}{2} H(x)$$

$f(x) = (x^2 + 3x + 4)^{-1}$

$g(x) = e^{-|x|}$

CALCOLA $(f * g)(x)$

E' SEMPLICISSIMO, LASCIAMO IL LETTORE DIVERTIRSI

$f(x) = H(x) e^{-3|x|} * H(x) e^{-|x|}$

$f(x) = \int_0^x H(y)\, e^{-3|y|}\, H(x-y)\, e^{-|x-y|}\, dy$

$\begin{cases} y > 0 \\ x > y \end{cases} \longrightarrow x > y > 0 \qquad \underline{x > 0}$

$f(x) = \int_0^x e^{-3y}\, e^{-x+y}\, dy \qquad x > 0$

$= \int_0^x e^{-x-2y}\, dy$

$= -\frac{1}{2} \left| e^{-x-2y} \right|_0^x$

$= -\frac{1}{2} \left(e^{-3x} - e^{-x} \right)$

$$f(x) = \begin{cases} 0 & x < 0 \\ \frac{1}{2}(e^{-x} - e^{-3x}) & x > 0 \end{cases}$$

$f = \chi_{(0,1)} * \chi_{(0,1)} \qquad A = (0,1)$

$f = \int_{\mathbb{R}} \chi_A(y)\, \chi_A(x-y)\, dy = m([0,1] \cap [x-1, x])$

$$f(x) = \begin{cases} 0 & x \le 0 \\ x & x \in (0,1) \\ 2-x & x \in (0,2) \\ 0 & x \ge 2 \end{cases}$$

$U(x) = \chi_{[0,a]}$ CALCOLA $f = U * U$

$f(x) = U * U = \int_0^x \chi_{[0,a]}(y)\, \chi_{[0,a]}(x-y)\, dy$

$= \int_0^x \chi_{(x-a, x) \cap (0, a)}$

$$f(x) = \begin{cases} 0 & x < 0 \vee x > a \\ x & 0 < x < a \\ 2a - x & a < x < 2a \end{cases}$$

$f(x) = \chi_{(-1,+1)}(x)$

$g(x) = e^{-|x|}$

$f, g \in L^P(\mathbb{R})$ QUALSIASI

$$f * g(x) = \begin{cases} \int_{\mathbb{R}} \chi_{(-1,+1)}(t) \cdot e^{-|x-t|}\, dt = \int_{-1}^{+1} e^{-|x-t|}\, dt \\ \int_{\mathbb{R}} e^{-|t|} \cdot \chi_{(-1,+1)}(x-t)\, dt = \int_{x-1}^{x+1} e^{-|t|}\, dt \end{cases}$$

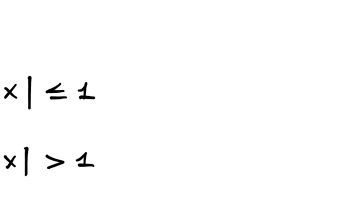
$-1 < x - t < +1$

$\underline{x-1 < t < x+1}$

χ CARATTERISTICA E' IMPORTANTE PER INTERVALLO INTEGRAZIONE

DEFINIZIONE 1

$f * g = I = \int_{-1}^{+1} e^{-|x-t|}\, dt$



$\underline{x > 1} \qquad \int_{-1}^{+1} e^{-x+t}\, dt = e^{-x} \int_{-1}^{+1} e^{t}\, dt = e^{-x}(e^{+1} - e^{-1}) = e^{-x} \sinh(1) \qquad$ A

$\underline{x < -1} \qquad \int_{-1}^{+1} e^{+x-t}\, dt = e^{x} \int_{-1}^{+1} e^{-t}\, dt = -e^{x}(e^{-1} - e^{+1}) = e^{x} \sinh(1) \qquad$ B

$\underline{-1 < x < 1} \qquad \int_{-1}^{x} e^{-x+t}\, dt + \int_x^{+1} e^{x-t}\, dt = e^{-x} \int_{-1}^{x} e^{t}\, dt + e^{+x} \int_x^{1} e^{-t}\, dt = \qquad$ C

$= e^{-x}(e^{x} - e^{-1}) + e^{x}(-e^{-1} + e^{-x})$

$= 1 - e^{-x} e^{-1} - e^{x} e^{-1} + 1$

$= 2 - e^{-1}(e^{x} + e^{-x})$

$= 2 - \frac{2}{e} \cosh(x)$

$= 2\left(1 - \frac{\cosh(x)}{e}\right)$

$$f * g(x) = \begin{cases} 2\left(1 - \frac{\cosh x}{e}\right) & |x| \le 1 \\ 2 e^{-|x|} \sinh x & |x| > 1 \end{cases}$$

DEFINIZIONE 2

$f * g = I = \int_{x-1}^{x+1} e^{-|t|}\, dt \qquad$ DIPENDE DA x

$$e^{-|t|} = \begin{cases} e^{t} & t < 0 \\ e^{-t} & t > 0 \end{cases}$$

SPEZZO INTEGRALE IN 3 CASI

$x \ge 1 \qquad \int_{x-1}^{x+1} e^{-t}\, dt = \left[-e^{-t}\right]_{x-1}^{x+1} = -e^{-x-1} + e^{-x+1} = 2 e^{-x} \sinh(1)$

$-1 < x < 1 \qquad \int_{x-1}^{0} e^{t} + \int_0^{x+1} e^{-t} = 1 - e^{x-1} + 1 - e^{-x-1} = 2\left(1 - \frac{\cosh x}{e}\right)$

$x \le -1 \qquad \int_{x-1}^{x+1} e^{+t}\, dt = \left[e^{t}\right]_{x-1}^{x+1} = e^{x+1} - e^{x-1} = 2 e^{x} \sinh(1)$

POSSIAMO POSIZIONARE GLI UGUALI ($\le, \ge$) DOVE VOGLIAMO POICHE' LA CONVOLUZIONE E' CONTINUA E I LIMITI COINCIDONO

$$f * g(x) = \begin{cases} 2\left(1 - \frac{\cosh x}{e}\right) & |x| \le 1 \\ 2 e^{-|x|} \sinh x & |x| > 1 \end{cases}$$

$f(x) = e^{-|x|} * e^{-|x|}$

BOH L' HA CHIESTO ALL' ESAME MA NON NE HO IDEA