LAVORO - CAMPI - INTEGRALI DI LINEA DI 2ª SPECIE

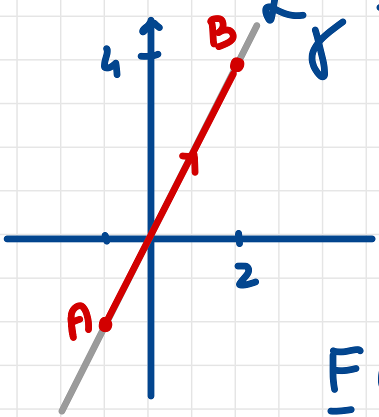
6-5-2021

ESERCIZIO 1. Calcolare il lavoro del campo piano $\underline{F}(x;y) = y\underline{i} + x\underline{j}$ lungo il segmento della retta $y = 2x$ con $x \in [-1;2]$.

SOL.

METODO 1 (DIRETTO)

$$\mathcal{L}_\gamma = \int_\gamma \underline{F} \cdot d\underline{r} = \int_a^b F(\underline{r}(t)) \cdot r'(t)\, dt$$



$A(-1;-2)$

$B(2;4)$

$$\underline{r}(t) = \begin{cases} x(t) = t \\ y(t) = 2t \end{cases} \quad t \in [-1,2]$$

$$\underline{F}(x;y) = \begin{bmatrix} y \\ x \end{bmatrix}$$

$$\underline{r}'(t) = \begin{cases} x' = 1 \\ y' = 2 \end{cases}$$

$$\mathcal{L}_\gamma = \int_\gamma \underline{F} \cdot d\underline{r} = \int_{-1}^{2} \underline{F}(\underline{r}(t)) \cdot \underline{r}'(t)\, dt = \int_{-1}^{2} \begin{bmatrix} 2t \\ t \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \end{bmatrix} dt$$

$$= \int_{-1}^{2} (2t + 2t)\, dt = \left[\cancel{4}^{\,2} \frac{t^2}{\cancel{2}} \right]_{-1}^{2} = 2(4-1) = 6.$$

METODO 2. $\underline{F}$ è conservativo ?

$\underline{F}(x;y) = \begin{bmatrix} y \\ x \end{bmatrix}$

- $\mathbb{D}_{\underline{F}} = \mathbb{R}^2 \rightarrow$ SEMPLICEMENTE CONNESSO

$$\boxed{\begin{array}{l} \underline{F}(x;y) = \begin{bmatrix} F_1(x;y) \\ F_2(x;y) \end{bmatrix} \\ F_{1y} = \partial_y(F_1) ;\ F_{2x} = \partial_x F_2 \end{array}}$$

- $F_{1y}(x;y) = 1 = F_{2x}(x;y) = 1$

$F_{1y} = F_{2x} \Rightarrow \underline{F}$ è irrotazionale.

Poiché $\underline{F}$ è irrotaz. e definito su un dominio semplicemente connesso $\Rightarrow$ $\underline{F}$ è conservativo.

$\Rightarrow \exists\, U(x;y)$ t.c. $\nabla U(x,y) = \underline{F}(x,y) \Leftrightarrow \begin{cases} U_x = F_1 \\ U_y = F_2 \end{cases}$

$\begin{cases} U_x = y \\ U_y = x \end{cases} \rightarrow U(x,y) = \int y \, dx = xy + c(y)$

$$\left. U_y(x,y) = \partial_y \left(xy + c(y) \right) = x + c_y(y) \right\} \Rightarrow$$

$\Rightarrow x + c_y(y) = x \rightarrow c_y(y) = 0$

$c(y) = c$.

Il potenziale è: $\boxed{U(x;y) = xy + c}$

$$\mathcal{L}_\gamma = U(B) - U(A) = U(2;4) - U(-1;-2) =$$
$$= 8 + c - 2 - c = 6.$$

N.B. Se $A \equiv B$, γ è chiusa $\Rightarrow$ $\mathcal{L} = 0$.

ESERCIZIO 2.

Sia $\underline{F} : \mathbb{R}^3 \to \mathbb{R}^3$ il campo

$$\underline{F}(x,y,z) = xy^2\,\underline{i} + x^2y\,\underline{j} + \frac{z^3}{3}\,\underline{k} = \begin{bmatrix} xy^2 \\ x^2y \\ z^3/3 \end{bmatrix} \begin{array}{l} \leftarrow F_1 \\ \leftarrow F_2 \\ \leftarrow F_3 \end{array}$$

F è conservativo? Se sì determinarne un potenziale.

SOL.

A) $\mathbb{D}_F = \mathbb{R}^3$ semplicemente connesso

B) $\text{rot}\,\underline{F} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \partial_x & \partial_y & \partial_z \\ F_1 & F_2 & F_3 \end{vmatrix} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \partial_x & \partial_y & \partial_z \\ xy^2 & x^2y & z^3/3 \end{vmatrix} =$

$= \underline{i}(0) - \underline{j}(0) + \underline{k}(2xy - 2xy) = \underline{0}$

$\Rightarrow \underline{F}$ è irrotazionale.

A + B $\Rightarrow$ F conservativo $\Rightarrow \exists U : \nabla U = \underline{F}$

$$\begin{cases} U_x = F_1 \\ U_y = F_2 \\ U_z = F_3 \end{cases}$$

$U(x;y;z) = \int xy^2\,dx = \frac{1}{2}x^2y^2 + c(y,z)$

$$U_y(x;y;z) = \partial_y\left(\frac{1}{2}x^2y^2 + c(y,z)\right) = \boxed{x^2y + c_y(y,z)}$$

$$\cancel{x^2y} + c_y(y,z) = \cancel{x^2y}$$

$$c_y(y,z) = 0$$

$$c(y,z) = c(z)$$

Aggiorno il potenziale:

$$U(x;y;z) = \frac{1}{2}x^2y^2 + c(z)$$

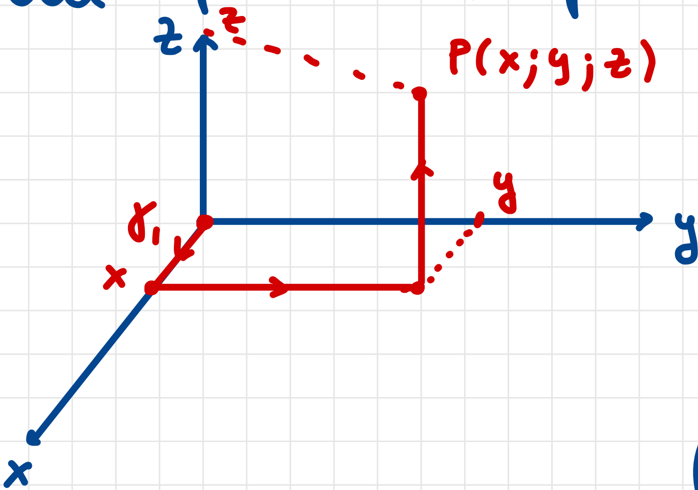
$$U_z(x;y;z) = \partial_z\left(\frac{1}{2}x^2y^2 + c(z)\right) = \boxed{c_z(z)}$$

$$c_z(z) = \frac{z^3}{3}$$

$$c(z) = \int \frac{z^3}{3}\,dz = \frac{z^4}{12} + c$$

$$\Rightarrow \boxed{U(x;y;z) = \frac{1}{2}x^2y^2 + \frac{z^4}{12} + c} \qquad c \in \mathbb{R}$$

Un potenziale poteva essere calcolato "a mano":



$$\gamma = \gamma_1 \cup \gamma_2 \cup \gamma_3$$

γ_1

$$\underline{r}_1(t): \begin{cases} x(t) = t \\ y(t) = 0 \\ z(t) = 0 \end{cases} \qquad \underline{r}_1'(t) = \begin{cases} x' = 1 \\ y' = 0 \\ z' = 0 \end{cases}$$

$$t \in [0; x]$$

γ_2

$$r_2(t) = \begin{cases} x(t) = x \\ y(t) = t \\ z(t) = 0 \end{cases} \qquad r_2'(t) = \begin{cases} x' = 0 \\ y' = 1 \\ z' = 0 \end{cases}$$

$$t \in [0; y]$$

γ_3

$$\underline{r}_3(t) = \begin{cases} x(t) = x \\ y(t) = y \\ z(t) = t \end{cases} \qquad r_3'(t) = \begin{cases} x' = 0 \\ y' = 0 \\ z' = 1 \end{cases} \qquad t \in [0, z]$$

$$U(x;y;z) = \mathcal{L}_\gamma = \int_\gamma \underline{F}\cdot d\underline{r} = \int_{\gamma_1} \underline{F}\cdot d\underline{r} + \int_{\gamma_2} \underline{F}\cdot d\underline{r} + \int_{\gamma_3} \underline{F}\cdot d\underline{r}$$

$$= \int_0^x \underline{F}(\underline{r}_1(t)) \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} dt + \int_0^y \underline{F}(\underline{r}_2(t)) \cdot \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} dt +$$

$$+ \int_0^z F(\underline{r}_3(t)) \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} dt =$$

$$= \cancel{\int_0^x t \cdot 0^2 \, dt} + \int_0^y x^2 t \, dt + \int_0^z \frac{t^3}{3} dt = x^2 \left[\frac{t^2}{2}\right]_0^y + \left[\frac{t^4}{12}\right]_0^z =$$

$$= \frac{1}{2} x^2 y^2 + \frac{z^4}{12}$$

Un potenziale è $U(x;y;z) = \frac{1}{2} x^2 y^2 + \frac{z^4}{12}$

$U(0,0,0) = 0$.

ESERCIZIO 3. (ESAME)

Calcolare $\int_\gamma \underline{F} \cdot d\underline{r}$ essendo

$$\gamma = \{ (x;y;z) \in \mathbb{R}^3 / \ x^2+y^2=4, \ z=y^2 \}$$

$$\underline{F}(x;y;z) = y\,\underline{i} - x\,\underline{j} + e^z\,\underline{k}$$

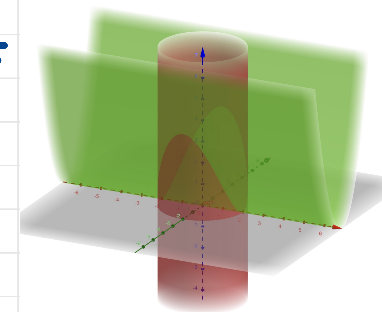
γ ha una orientazione antioraria se vista dall'alto.

SOL.

PARAMETRIZZO γ:
$$\underline{r}(t) = \begin{cases} x = 2\cos t \\ y = 2\,\text{sen}\,t \\ z = 4\,\text{sen}^2 t \end{cases}$$

$t=0: \ \underline{r}(0) = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$; $t = 2\pi : \ \underline{r}(2\pi) = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$ è chiusa.

$$\int_\gamma \underline{F}\cdot d\underline{r} = \int_0^{2\pi} \begin{bmatrix} 2\,\mathrm{sen}\,t \\ -2\cos t \\ e^{4\,\mathrm{sen}^2 t} \end{bmatrix} \cdot \begin{bmatrix} -2\,\mathrm{sen}\,t \\ 2\cos t \\ 8\,\mathrm{sen}\,t\cos t \end{bmatrix} dt$$



$$= \int_0^{2\pi} \left(-4\,\mathrm{sen}^2 t - 4\cos^2 t + 8\,\mathrm{sen}\,t\cos t\, e^{4\,\mathrm{sen}^2 t}\right) dt$$

$$= \int_0^{2\pi} \left(-4 + 8\,\mathrm{sen}\,t\cos t\, e^{4\,\mathrm{sen}^2 t}\right) dt = -4\cdot 2\pi + \left[e^{4\,\mathrm{sen}^2 t}\right]_0^{2\pi}$$

$$= -8\pi + 0 = -8\pi.$$

ESERCIZIO 4. Calcolare il lavoro compiuto dal campo $\underline{F}(x;y;z) = x^2\underline{i} - yz\,\underline{k}$ lungo l'elica cilindrica $\underline{r}(t) = 2\cos t\,\underline{i} + 2\,\mathrm{sen}\,t\,\underline{j} + t\,\underline{k}$, $t\in[0,2\pi]$.

SOL.

$D_F = \mathbb{R}^3$ s.c.

$$\operatorname{rot} \underline{F} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \partial_x & \partial_y & \partial_z \\ x^2 & 0 & -yz \end{vmatrix} = \underline{i}(-z-0) - \underline{j}(0) + \underline{k}(0) = \begin{bmatrix} -z \\ 0 \\ 0 \end{bmatrix} \neq \underline{0}$$

$\Longrightarrow$ $\underline{F}$ non è irrotazionale.

$$\mathcal{L}_\gamma = \int_\gamma \underline{F} \cdot d\underline{r} = \int_0^{2\pi} \begin{bmatrix} 4\cos^2 t \\ 0 \\ -2t \operatorname{sen} t \end{bmatrix} \cdot \begin{bmatrix} -2\operatorname{sen} t \\ 2\cos t \\ 1 \end{bmatrix} dt =$$

$$= \int_0^{2\pi} \left(-8\cos^2 t \operatorname{sen} t - 2t \operatorname{sen} t\right) dt =$$

$$= 8\int_0^{2\pi} \cos^2 t\,(-\operatorname{sen} t)\, dt - 2\int_0^{2\pi} t \operatorname{sen} t\, dt =$$

$$= 8\left[\cancel{\frac{\cos^3 t}{3}}\right]_0^{2\pi} - 2\left\{\left[-t\cos t\right]_0^{2\pi} + \cancel{\int_0^{2\pi}\cos t\,dt}\right\}$$

$$= 2\left[t\cos t\right]_0^{2\pi} = 2(2\pi - 0) = 4\pi.$$

ESERCIZIO 6.34 (ERRATO)

$$\underline{F} = -\frac{2xz}{(x^2+y^2)^2}\,\underline{i} - \frac{2yz}{(x^2+y^2)^2}\,\underline{j} + \left(\frac{1}{x^2+y^2} + \frac{1}{1+z^2}\right)\underline{k}$$

1) Verificare $\operatorname{rot}\underline{F} = \underline{0}$ in $\mathbb{D}_F$

2) Dalle risposte al punto precedente si può dedurre la conservatività di $\underline{F}$? Perché?

3) Stabilire se il campo è cons. e in caso affermativo calcolare un potenziale.

4) Calcolare il lavoro di $\underline{F}$ lungo la curva

$$\underline{r}(t) = \begin{cases} x = t \\ y = t^2 \\ z = t^3 \end{cases} \qquad t \in [0, 1].$$

$t \in [1; 2]$

ERRORE NEL TESTO.

ESERCIZIO 5.

Si consideri la famiglia di campi in $\mathbb{R}^2 - \{(0;0)\}$:

$$\underline{F}(x;y) = \frac{x + \alpha y}{2x^2+2y^2}\,\underline{i} + \frac{\beta x + y}{2x^2+2y^2}\,\underline{j} \qquad \alpha, \beta \in \mathbb{R}.$$

1) Per quali valori di α, β, $\underline{F}(x,y)$ è irrotazionale in $\mathbb{R}^2 - \{(0,0)\}$.

2) Per gli α, β trovati calcolare $\int_\Gamma \underline{F} \cdot d\underline{r}$ dove Γ è la circonf. di centro $(0,0)$

e raggio 1 orientata positivamente.

3) Esistono valori di α e β per cui $\underline{F}$ possa essere conservativo in $\mathbb{R}^2 - \{(0,0)\}$?
Per tali valori determinare, se esiste, un potenziale.

SOL.

1) $$F_{2x}(x;y) = \partial_x F_2(x;y) = \partial_x\left(\frac{\beta x + y}{2x^2+2y^2}\right) =$$

$$= \frac{\beta(2x^2+2y^2) - (\beta x + y)4x}{(2x^2+2y^2)^2} = \frac{2\beta x^2 + 2\beta y^2 - 4\beta x^2 - 4xy}{(2x^2+2y^2)^2}$$

$$= \frac{-2\beta x^2 + 2\beta y^2 - 4xy}{(2x^2+2y^2)^2}$$

$$F_{1y}(x;y) = \partial_y \left(\frac{x+\alpha y}{2x^2+2y^2} \right) =$$

$$= \frac{\alpha(2x^2+2y^2) - (x+\alpha y)4y}{(2x^2+2y^2)^2} = \frac{2\alpha x^2 + 2\alpha y^2 - 4xy - 4\alpha y^2}{(2x^2+2y^2)^2}$$

$$= \frac{2\alpha x^2 - 2\alpha y^2 - 4xy}{(2x^2+2y^2)^2}$$

$$\begin{cases} 2\alpha = -2\beta \\ 2\beta = -2\alpha \end{cases} \longrightarrow \boxed{\alpha = -\beta}$$

$F(x;y)$ è irrotazionale $\iff$ $\alpha = -\beta$.

2) Sia ora $\beta = -\alpha$, $\int_\Gamma \underline{F} \cdot d\underline{r}$, $\underline{r}(t) = \begin{cases} x = \cos t \\ y = \operatorname{sen} t \end{cases}$

$t \in [0, 2\pi]$, $\underline{r}'(t) = \begin{cases} x' = -\operatorname{sen} t \\ y' = \cos t \end{cases}$

$$\int_\Gamma \underline{F} \cdot d\underline{r} = \int_0^{2\pi} \underline{F}(\underline{r}(t)) \cdot \underline{r}'(t)\, dt =$$

$$= \int_0^{2\pi} \begin{bmatrix} \frac{\cos t + \alpha \sin t}{2} \\ \frac{-\alpha \cos t + \sin t}{2} \end{bmatrix} \cdot \begin{bmatrix} -\sin t \\ \cos t \end{bmatrix} dt =$$

$$= \frac{1}{2} \int_0^{2\pi} \left(\cancel{-\sin t \cos t} - \alpha \sin^2 t - \alpha \cos^2 t + \cancel{\sin t \cos t} \right) dt$$

$$= \frac{1}{2} \int_0^{2\pi} -\alpha\, dt = -\frac{\alpha}{2} \cdot 2\pi = -\pi\alpha .$$

3) $\int_\Gamma \underline{F} \cdot d\underline{r} = -\pi\alpha = 0 \iff \alpha = 0$

Per $\alpha = 0$ $\underline{F}$ può essere conservativo

$\alpha \neq 0 \implies$ F non è cons.

Cerco, se esiste, un potenziale di

$$\underline{F}(x;y) = \frac{x}{2x^2+2y^2}\,\underline{i} + \frac{y}{2x^2+2y^2}\,\underline{j}$$

$$U_x = F_1 \implies U(x,y) = \int F_1(x;y)\,dx =$$

$$= \frac{1}{4}\int \frac{4x}{2x^2+2y^2}\,dx = \frac{1}{4}\ln(2x^2+2y^2) + c(y)$$

$$U_y(x;y) = \partial_y\left(\frac{1}{4}\ln(2x^2+2y^2) + c(y)\right) =$$

$$= \frac{1}{4}\cdot\frac{1}{2x^2+2y^2}\cdot 4y + c_y(y) =$$

$$= \frac{y}{2x^2+2y^2} + c_y(y) = F_2 = \frac{y}{2x^2+2y^2}$$

$$\Longrightarrow \quad c_y(y) = 0 \quad \to \quad c(y) = c$$

$$\boxed{U(x;y) = \frac{1}{4}\ln\left(2x^2+2y^2\right) + c}$$

$\alpha = \beta = 0 \Longrightarrow \exists\, U : \nabla U = \underline{F} \Longrightarrow \underline{F}$ è conservativo.

ESERCIZIO 6. Si consideri il campo vettoriale

$$\underline{F}(x;y) = (\cos y + 3)\,\underline{i} + (g(x) - x \operatorname{sen} y)\,\underline{j}$$

dove $g(x)$ è definita e derivabile su $\mathbb{R}$ con $g(1) = 1$.

Per quale $g(x)$ $\underline{F}$ è conservativo?
Per tali funzioni si determini un potenziale e si calcoli $\int_\gamma \underline{F} \cdot d\underline{r}$ essendo γ l'arco della parabola $y = x^2$ con $x \in [0;2]$ dall'alto verso il basso.

SOL.

- $\mathbb{D}_{\underline{F}} = \mathbb{R}^2$ s.c.

$$F_{2x} = \partial_x \left(g(x) - x \operatorname{sen} y \right) = -\operatorname{sen} y + g'(x)$$

$$F_{1y} = \partial_y \left(\cos y + 3 \right) = -\operatorname{sen} y$$

Per essere irrotaz. si deve avere $F_{2x} = F_{1y}$

$$-\cancel{\operatorname{sen} y} + g'(x) = -\cancel{\operatorname{sen} y}$$

$$g'(x) = 0$$

$$g(x) = c$$

Poiché $g(1) = 1 \Rightarrow \underline{c = 1}$

$$\boxed{g(x) = 1}$$

$\underline{F}$ è conservativo se $g(x) = 1$ perché D_F s.c. e $\operatorname{rot} \underline{F} = \underline{0}$.

Cerco un potenziale:

$$\begin{cases} U_x = F_1 \\ U_y = F_2 \end{cases} \longrightarrow U(x,y) = \int (\cos y + 3)\, dx = x \cos y + 3x + c(y)$$

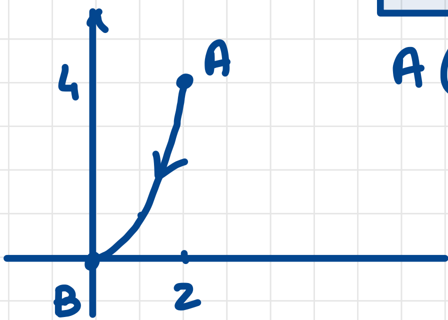
$$U_y = \partial_y \left(x \cos y + 3x + c(y) \right) = -x \operatorname{sen} y + c_y(y) = F_2$$

$$= 1 - x \operatorname{sen} y$$

$$-x\,\cancel{\sin y} + C_y(y) = 1 - x\,\cancel{\sin y}$$

$$C_y(y) = 1 \longrightarrow C(y) = y + c$$

$$\Longrightarrow \boxed{U(x;y) = x\cos y + 3x + y + c} \qquad c \in \mathbb{R}.$$



$A(2;4)$ $\quad$ $B(0;0)$

$$\mathcal{L} = U(B) - U(A) = U(0,0) - U(2;4)$$

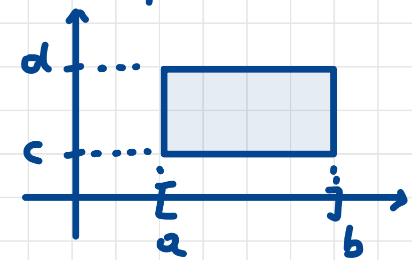
$$= \cancel{c} - 2\cos 4 - 6 - 4 - \cancel{c}$$

$$= -10 - 2\cos 4.$$

INTEGRALI DOPPI SU RETTANGOLI

$$\iint_\Omega f(x;y)\,dx\,dy = \begin{cases} \int_a^b \left(\int_c^d f(x;y)\,dy \right) dx \\ \int_c^d \left(\int_a^b f(x;y)\,dx \right) dy \end{cases}$$

$$\Omega = [a;b] \times [c,d] = \{ (x;y) \in \mathbb{R}^2 \mid a \le x \le b,\ c \le y \le d \}$$



ESERCIZIO 7. Calcolare

$$\iint_\Omega y \sin xy \,dx\,dy \qquad \Omega = [0,1] \times \left[0; \frac{\pi}{2}\right]$$

SOL.

$$\iint_{\Omega} y \operatorname{sen} xy \, dx\, dy = \begin{cases} \displaystyle\int_0^1 \left(\int_0^{\pi/2} y \operatorname{sen} xy \, dy \right) dx & \text{NON CONVIENE} \\ \displaystyle\int_0^{\pi/2} \left(\int_0^1 y \operatorname{sen} xy \, dx \right) dy = \end{cases}$$

$$= \int_0^{\pi/2} \left[-\cos xy \right]_0^1 dy = \int_0^{\pi/2} \left(-\cos y + \cos 0 \right) dy =$$

$$= \int_0^{\pi/2} (1 - \cos y)\, dy = \left[y - \operatorname{sen} y \right]_0^{\frac{\pi}{2}} = \frac{\pi}{2} - 1 .$$

ESERCIZIO 8. Calcolare i seguenti integrali su rettangoli:

1) $\iint_{\Omega} (x^2+y^2)\,dx\,dy \quad , \quad \Omega = [0;1] \times [1,2]$

$$\iint_{\Omega} (x^2+y^2)\,dx\,dy = \int_0^1 \left(\int_1^2 (x^2+y^2)\,dy \right) dx =$$

$$= \int_0^1 \left[x^2 y + \frac{y^3}{3} \right]_1^2 dx = \int_0^1 \left(2x^2 + \frac{8}{3} - x^2 - \frac{1}{3} \right) dx =$$

$$= \int_0^1 \left(x^2 + \frac{7}{3} \right) dx = \left[\frac{x^3}{3} + \frac{7}{3} x \right]_0^1 = \frac{8}{3}$$

2) $I = \iint_{\Omega} \frac{1}{(x+y)^2}\,dx\,dy \qquad \Omega = [1,2] \times [3;4]$

$$I = \int_1^2 \left(\int_3^4 \frac{1}{(x+y)^2}\, dy \right) dx$$

$$= \int_1^2 \left[-(x+y)^{-1} \right]_3^4 dx = \int_1^2 \left(-\frac{1}{x+4} + \frac{1}{x+3} \right) dx$$

$$= \left[-\ln|x+4| + \ln|x+3| \right]_1^2 = -\ln 6 + \ln 5 + \ln 5 - \ln 4$$

$$= \ln 25 - \ln 24 = \ln \frac{25}{24}.$$

3) $I = \iint_\Omega \frac{1}{(x+y)^3}\, dx\, dy \qquad \Omega = [1, +\infty) \cup [0, 1]$

$$I = \int_1^{+\infty} \left(\int_0^1 (x+y)^{-3}\, dy \right) dx =$$

$$= \int_1^{+\infty} \left[-\frac{(x+y)^{-2}}{2} \right]_0^1 dx =$$

$$= \int_1^{+\infty} \left(-\frac{1}{2}\cdot\frac{1}{(x+1)^2} + \frac{1}{2}\cdot\frac{1}{x^2} \right) dx =$$

$$= \frac{1}{2} \int_1^{+\infty} \left(\frac{1}{x^2} - \frac{1}{(x+1)^2} \right) dx = \frac{1}{2} \left[-\frac{1}{x} + \frac{1}{x+1} \right]_1^{+\infty}$$

$$= \frac{1}{2} \lim_{t \to +\infty} \left[-\frac{1}{x} + \frac{1}{x+1} \right]_1^{t} =$$

$$= \frac{1}{2} \lim_{t \to +\infty} \left(\underbrace{-\frac{1}{t} + \frac{1}{t+1}}_{\to 0} + 1 - \frac{1}{2} \right) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}.$$