Measure Theory

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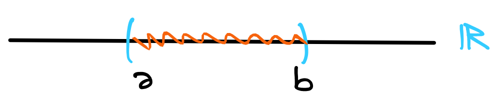


LENGTH = $b-a$

WHAT IS THE MEASURE OF A SUBSET ?
HOW WE CAN GENERALIZE N-VOLUMES IN $\mathbb{R}^N$?

POWER SET

$\mathcal{X}$ SET , $P(\mathcal{X})$ IS THE SET OF ALL SUBSETS

EXAMPLE $\mathcal{X} = \{a, b\}$ $P(\mathcal{X}) = (\phi, \mathcal{X}, \{a\}, \{b\})$

σ - ALGEBRA

COLLECTIONS OF SETS THAT ARE MEASURABLE

$\Sigma \subset P(\mathcal{X})$ IS A σ - ALGEBRA IF

1) $\phi, \mathcal{X} \in \Sigma$ — IT CONTEINS ITSELF
2) $A \in \Sigma \rightarrow A^C \in \Sigma$ — CLOSED UNDER COMPLEMENT
3) $A_i \in \Sigma \ \forall i \in \mathbb{N} \rightarrow \bigcup_{i=1}^{\infty} A_i \in \Sigma$ — CLOSED UNDER COUNTABLE UNIONS
4) $A_i \in \Sigma \ \forall i \in \mathbb{N} \rightarrow \bigcap_{i=1}^{\infty} A_i \in \Sigma$ — CLOSED UNDER COUNTABLE INTERSECTIONS

EXAMPLES

$\Sigma = \{\phi, \mathcal{X}\}$ SMALLEST POSSIBLE σ-ALGEBRA

$\Sigma = P(\mathcal{X})$ BIGGEST POSSIBLE σ - ALGEBRA
BEST CASE SCENARIO, ALL SUBSETS ARE MEASURABLE

MEASURABLE SET

AN ELEMENT IN A σ ALGEBRA $A \in \Sigma$ A IS Σ-MEASURABLE

MESURABILITY IS GIVEN WITH RESPECT TO A GIVEN σ ALGEBRA

GENERATING σ - ALGEBRA

FOR $\mathcal{M} \subseteq P(\mathcal{X})$ ∃ THE SMALLEST σ-ALGEBRA THAT CONTAINS $\mathcal{M}$

$\sigma(\mathcal{M}) = \bigcap_{\Sigma \supseteq \mathcal{M}} \Sigma$ SMALLEST σ-ALGEBRA GENERATED BY $\mathcal{M}$ (THE MOST EFFICIENT)

EXAMPLE

$\mathcal{X} = \{a, b, c, d\}$ $M = (\{a\}, \{b\})$

$$\Sigma = \sigma(M) = (\underbrace{\phi, \mathcal{X}}_{\text{ALWAYS}}, \underbrace{\{a\}, \{b\}}_{M}, \underbrace{\{a,b\}}_{\text{UNIONS}}, \underbrace{\{b,c,d\}, \{a,c,d\}, \{c,d\}}_{\text{COMPLEMENTS}})$$

THIS IS MUCH HARDER WITH AN INFINITE STEP

BOREL σ - ALGEBRA

LET $\mathcal{X}$ BE "TOPOLOGICAL" SPACE OR "METRIC" SPACE OR $\mathbb{R}^N$

σ- ALGEBRA GENERATED BY OPEN SETS

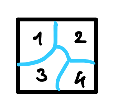
CONTAIN ALL THE SETS WE WANT TO MEASURE

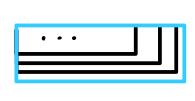
MEASURE

$(\mathcal{X}, \Sigma)$ MEASURABLE SPACE — ALL N-VOLUMES ARE POSITIVE

THE MEASURE μ IS A SPECIAL MAP $\mu : \Sigma \rightarrow [0, +\infty]$ (∞ AS A NEW SYMBOL) THAT SATISFIES:

1) $\mu(\phi) = 0$

2) $\mu(\bigcup_{i=1}^{N} A_i) = \sum_{i=1}^{N} \mu(A_i)$ $A_i \cap A_j = \phi$ FOR $i \neq j$ $\forall A_i \in \Sigma$ — DISSOINTED UNION 

2) $\mu(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} \mu(A_i)$ $A_i \cap A_j = \phi$ FOR $i \neq j$ $\forall A_i \in \Sigma$ — APPROXIMATION 

} σ - ADDITIVITY

A MEASURE HAS TO LIVE ON A σ - ALGEBRA

MEASURING SUBSET MEANS GIVING THEM A GENERALIZED VOLUME $\mathbb{R}^+$

2 PROPERTIES OF VOLUMES : NOTHING SHOULD HAVE ZERO VOLUME
AND YOU CAN'T GET RID OF A VOLUME BY SPLITTING IT UP IN
COUNTABLE MEASURABLE SUBSETS

(X, Σ, μ) MEASURE SPACE

EXAMPLE

COUNTING MEASURE $\mu(A) = \begin{cases} \#A & \text{NUMBER OF ELEMENTS} \\ \infty \end{cases}$

DIRAC MEASURE $\delta_p(A) = \begin{cases} 1 & p \in A \\ 0 & p \notin A \end{cases}$

MEASURE IN $\mathbb{R}^N$

1) $\mu([0,1]^N) = 1$ — NORMAL VOLUME
2) $\mu(\vec{v} + A) = \mu(A)$ — INDEPENDENT FROM TRANSLATIONS

THE MEASURE PROBLEM

WE CANNOT USE $P(X)$ AS THE LEBESGUE MEASURE

WE SEARCH MEASURE μ ON $P(\mathbb{R})$ WITH:

1) $\mu([a,b]) = b - a$ $b > a$
2) $\mu(x + A) = A$ $A \in P(\mathbb{R})$ $x \in \mathbb{R}$

} $\nexists \mu$

THIS IS NOT SOLVABLE BUT WE CAN SET A SUITABLE σ-ALGEBRA

MEASURABLE MAPS

(Ω_1, Σ_1) (Ω_2, Σ_2) MEASURABLE SPACES

$f : \Omega_1 \rightarrow \Omega_2$ MEASURABLE IF $f^{-1}(A_2) \in \Sigma_1$ $\forall A_2 \in \Sigma_2$

WE WANT TO GUARANTEE THAT BOTH ELEMENTS ON X , Y AXIS ARE MEASURABLE

EXAMPLE

CHARACTERISTIC FUNCTION

(Ω, Σ) $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$

$\mathcal{X}_A : \Omega \rightarrow \mathbb{R}$ $\mathcal{X}_A = \begin{cases} 1 & w \in A \\ 0 & w \notin A \end{cases}$

FOR ALL MEASURABLE SETS $(A \in \Sigma)$ $\mathcal{X}_A$ IS A MEASURABLE MAP

$\mathcal{X}_A^{-1}(\phi) = \phi$ $\mathcal{X}_A^{-1}(\mathbb{R}) = \Omega$

$\mathcal{X}_A^{-1}(\{1\}) = A$ $\mathcal{X}_A^{-1}(\{0\}) = A^c$

$\phi, A, \Omega, A^c \in \Sigma$

COMPOSITION OF MEASURABLE MAPS

$f : \Omega_1 \rightarrow \Omega_2 \rightarrow \Omega_3$
COMPOSITION IS MEASURABLE AS WELL

FUNCTIONS ARE MAPS TO THE $\mathbb{R}$ REAL NUMBER LINE

$f+g$ $f-g$ $f \times g$ $f : g$ $f \circ g$ $|f|$ MEASURABLE

LEBESGUE INTEGRAL

$(\mathcal{X}, \Sigma, \mu)$ ABSTRACT MEASURE SPACE

$\mu : \Sigma \rightarrow [0, +\infty]$ MEASURE
σ- ALGEBRA COLLECTION OF SUBSETS OF $\mathcal{X}$
SET

WITH RESPECT TO THIS MEASURE SPACE WE WANT TO INTEGRATE SOME SPECIAL FUNCTIONS

MEASURABLE MAPS $f : \underset{(\Sigma)}{\mathcal{X}} \rightarrow \underset{(\mathcal{B})}{\mathbb{R}}$ WHERE $f^{-1}(E) \in A$ $\forall$ BOREL SETS $E \subseteq \mathcal{B}(\mathbb{R})$

$$I(f) = \int_{\mathcal{X}} f(x)\, d\mu(x) = \int_{\mathcal{X}} f\, d\mu = \sum_{i=1}^{N} c_i \cdot \mu(A_i) \in [0, \infty]$$