Trasformate di Fourier di distribuzioni

Problema: Data $T \in \mathcal{D}'(\mathbb{R}^n)$, come definire $\hat{T}$?

Idea: scaricare $\mathcal{F}$ sulle funzioni test

Sappiamo: $\displaystyle\int_{\mathbb{R}^n} \hat{u}\, v = \int_{\mathbb{R}^n} u\, \hat{v} \quad \forall u, v \in \mathcal{S}$

Data $\varphi \in \mathcal{D}(\mathbb{R}^n)$, potremmo definire $\hat{T}$

$$\langle \hat{T}, \varphi \rangle := \langle T, \hat{\varphi} \rangle$$

"<u>Inconveniente</u>": la def sopra è mal posta, perché

$$\varphi \in \mathcal{D}(\mathbb{R}^n) \not\Longrightarrow \hat{\varphi} \in \mathcal{D}(\mathbb{R}^n)$$

cioè $\mathcal{F}$ NON manda $\mathcal{D}(\mathbb{R}^n)$ in sé stesso

(la trasformata di una $\varphi \in \mathcal{D}(\mathbb{R}^n)$ è <u>analitica</u>, quindi non può avere supporto compatto).

<u>Per rimediare</u>: usare funzioni test in $\mathcal{S}(\mathbb{R}^n)$.

A tale scopo, introduco una convergenza in $\mathcal{S}(\mathbb{R}^n$

Def. Data $\{\varphi_n\} \subseteq \mathcal{S}(\mathbb{R}^n)$, e $\varphi \in \mathcal{S}(\mathbb{R}^n)$ diciamo che $\varphi_n \longrightarrow \varphi$ in $\mathcal{S}(\mathbb{R}^n)$ se $\forall \alpha, \beta$, $x^\alpha D^\beta \varphi_n \longrightarrow x^\alpha D^\beta \varphi$ unif. su $\mathbb{R}^n$

Oss. 1) Data $\{\varphi_n\} \subseteq \mathcal{D}(\mathbb{R}^n) \subseteq \mathcal{S}(\mathbb{R}^n)$

$$\varphi_n \longrightarrow \varphi \text{ in } \mathcal{D}(\mathbb{R}^n) \Rightarrow \varphi_n \longrightarrow \varphi \text{ in } \mathcal{S}(\mathbb{R}^n)$$

↑

(Recall: $\varphi_n \to \varphi$ in $\mathcal{D}(\mathbb{R}^n)$ se $\exists$ K compatto t.c. $\operatorname{supp} \varphi_n \subseteq K$ $\forall n$ e $\varphi_n \to \varphi$ unif. su K con tutte le derivate).

2). $\mathcal{F}: \mathcal{S}(\mathbb{R}^n) \longrightarrow \mathcal{S}(\mathbb{R}^n)$ lineare è CONTINUO:

ovvero $\varphi_n \to \varphi$ in $\mathcal{S}(\mathbb{R}^n) \Rightarrow \hat{\varphi}_n \to \hat{\varphi}$ in $\mathcal{S}(\mathbb{R}^n)$

$$\Big(\underbrace{\xi^\alpha D^\beta \hat{\varphi}_n}_{\simeq\ \widehat{D^\alpha(x^\beta \varphi_n)}} \longrightarrow \underbrace{\xi^\alpha D^\beta \hat{\varphi}}_{\simeq\ \widehat{D^\alpha(x^\beta \varphi)}} \text{ unif. su } \mathbb{R}^n$$

perché $D^\alpha(x^\beta \varphi_n) \longrightarrow D^\alpha(x^\beta \varphi)$ in $L^1(\mathbb{R}^n)$

es. $n=1$

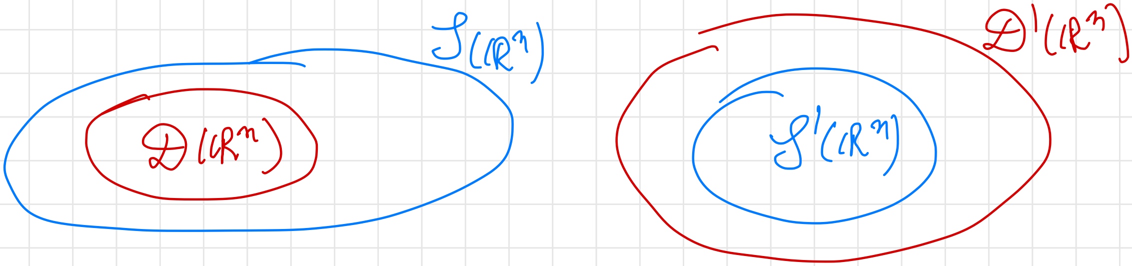
$$\int_{\mathbb{R}} \frac{|D^\alpha(x^\beta \varphi_n) - D^\alpha(x^\beta \varphi)|\,(1+x^2)}{1+x^2} \longrightarrow 0$$

$$\le \sup_{\mathbb{R}^n} \Big|\, |D^\alpha(x^\beta \varphi_n) - D^\alpha(x^\beta \varphi)|\,(1+x^2) \Big| \cdot \int_{\mathbb{R}} \frac{1}{1+x^2}\,dx$$

Def. Una distribuzione $T \in \mathcal{D}'(\mathbb{R}^n)$ si dice **DISTRIBUZIONE TEMPERATA** se

$$\varphi_h \subseteq \mathcal{D}(\mathbb{R}^n) : \quad \varphi_h \to 0 \text{ in } \mathcal{S}(\mathbb{R}^n) \implies \langle T, \varphi_h \rangle \longrightarrow 0$$

Def. $\mathcal{S}'(\mathbb{R}^n) := \{\text{distribuzioni temperate}\}$.



Esempi $(n=1)$

1) $u(x) = p(x)$ polinomio $\in L^1_{loc}(\mathbb{R}) \subseteq \mathcal{D}'(\mathbb{R})$

Dico che $u(x) \in \mathcal{S}'(\mathbb{R})$. Per ogni $\varphi \in \mathcal{D}(\mathbb{R})$

$$\left| \int_{\mathbb{R}} u \cdot \varphi_h \right| \leq \int_{\mathbb{R}} \left| \frac{u}{p} \cdot p\, \varphi_h \right| dx \leq \left\| \frac{u}{p} \right\|_{L^1} \| p\, \varphi_h \|_{L^\infty}$$

dove p polinomio tale che $\frac{u}{p} \in L^1(\mathbb{R})$.

Quindi: se $\varphi_h \to 0$ in $\mathcal{S}(\mathbb{R})$, $\| p\, \varphi_h \|_{L^\infty} \longrightarrow 0$

e quindi $\int_{\mathbb{R}} u\, \varphi_h \longrightarrow 0$.

2) $u \in L^1_{loc}(\mathbb{R})$ si dice A CRESCITA LENTA se $u = q\, w$, con q polinomio e $w \in L^1(\mathbb{R})$

Tutte le funzioni a crescita lenta stanno in $\mathcal{S}'(\mathbb{R})$. Per ogni $\varphi \in \mathcal{D}(\mathbb{R})$

$$\left| \int_{\mathbb{R}} u \varphi_h \right| \leq \int_{\mathbb{R}} |w q \varphi_h| \leq \|w\|_{L^1} \|q \varphi_h\|_{L^\infty}$$

Quindi: se $\varphi_h \to 0$ in $\mathcal{S}(\mathbb{R}^n)$, $\|q \varphi_h\|_{L^\infty} \to 0$

e quindi $\int_{\mathbb{R}} u \varphi_h \longrightarrow 0$

3) Come caso particolare: $u \in L^p(\mathbb{R}) \Longrightarrow$ u è a crescita lenta, e quindi sta in $\mathcal{S}'(\mathbb{R})$

Data $u \in L^p(\mathbb{R})$, $u = q\, w$ con q polinomio e $w \in L^1(\mathbb{R})$.

- $u \in L^1(\mathbb{R}) \Rightarrow$ prendo $q = 1$, $w \in u$
- $u \in L^\infty(\mathbb{R}) \Rightarrow$ prendo q: $\frac{1}{q} \in L^1(\mathbb{R})$

$$u = q \cdot \frac{u}{q} \qquad w = \frac{u}{q} \in L^1(\mathbb{R})$$

Hölder

- $u \in L^p(\mathbb{R}) \Rightarrow$ prendo q: $\frac{1}{q} \in L^{p'}(\mathbb{R})$

$$u = q \cdot \frac{u}{q} \qquad w = \frac{u}{q} \in L^1(\mathbb{R})$$

$\uparrow$ Hölder

4) $\delta_0 \in \mathcal{S}'(\mathbb{R})$, $D^{(k)}\delta_0 \in \mathcal{S}'(\mathbb{R})$

$\{\varphi_h\} \subseteq \mathcal{D}(\mathbb{R})$: $\varphi_h \to 0$ in $\mathcal{S}(\mathbb{R})$.

$$\Rightarrow \underset{\varphi_h(0)}{\overset{}{\langle \delta_0, \varphi_h \rangle}} \longrightarrow 0$$

$\langle \delta_0, \varphi_h \rangle = \varphi_h(0)$

(perché ho cv. unif. $\Rightarrow$ puntuale)

Altre osservazioni su $\mathcal{S}'(\mathbb{R}^n)$

1) $T \in \mathcal{S}'(\mathbb{R}^n) \Rightarrow T$ può agire più in generale su funzioni test di $\mathcal{S}(\mathbb{R}^n)$

$$\langle T, \varphi \rangle_{\mathcal{D}', \mathcal{D}} \qquad \langle T, \varphi \rangle_{\mathcal{S}', \mathcal{S}}$$

Infatti, se $T \in \mathcal{S}'(\mathbb{R}^n)$, posso definire $\langle T, \varphi \rangle$ $\forall \varphi \in \mathcal{S}(\mathbb{R}^n)$ come

$\lim_h \langle T, \varphi_h \rangle$ dove $\varphi_h \subseteq \mathcal{D}(\mathbb{R}^n)$:

$\varphi_h \longrightarrow \varphi$ in $\mathcal{S}(\mathbb{R}^n)$

2) Se $T \in \mathcal{S}'(\mathbb{R}^n)$, allora vale:
$\{\varphi_n\} \subseteq \mathcal{S}(\mathbb{R}^n)$ tale che $\varphi_n \to 0$ in $\mathcal{S}(\mathbb{R}^n)$
$\Rightarrow \langle T, \varphi_n \rangle \longrightarrow 0$.

Trasformata di Fourier in $\mathcal{S}'(\mathbb{R}^n)$

Def. Data $T \in \mathcal{S}'(\mathbb{R}^n)$, definisco $\hat{T} \in \mathcal{S}'(\mathbb{R}^n)$ come

$$\boxed{\langle \hat{T}, \varphi \rangle := \langle T, \hat{\varphi} \rangle \quad \forall \varphi \in \mathcal{D}(\mathbb{R}^n)}$$

Oss.

1) $\varphi \in \mathcal{D}(\mathbb{R}^n) \Rightarrow \varphi \in \mathcal{S}(\mathbb{R}^n) \Rightarrow \hat{\varphi} \in \mathcal{S}(\mathbb{R}^n)$
Quindi $\langle T, \hat{\varphi} \rangle$ ha senso perché $T \in \mathcal{S}'(\mathbb{R}^n)$.

2) Verifichiamo che $\hat{T} \in \mathcal{D}'(\mathbb{R}^n)$
$\{\varphi_n\} \longrightarrow 0$ in $\mathcal{D}(\mathbb{R}^n) \Rightarrow \langle \hat{T}, \varphi_n \rangle \to 0$
Infatti $\{\varphi_n\} \longrightarrow 0$ in $\mathcal{D}(\mathbb{R}^n) \Rightarrow$

$\varphi_h \to 0$ in $\mathcal{S}(\mathbb{R}^n) \Rightarrow \hat{\varphi}_h \to 0$ in $\mathcal{S}(\mathbb{R}^n)$

$\Rightarrow \langle T, \hat{\varphi}_h \rangle \longrightarrow 0$ perché $T \in \mathcal{S}'(\mathbb{R}^n)$.

3) Si ha anche $\hat{T} \in \mathcal{S}'(\mathbb{R}^n)$.
(cf. sopra).

<u>Proprietà</u>: valgono in $\mathcal{S}'(\mathbb{R}^n)$
tutte le proprietà di $\mathcal{F}$ in $\mathcal{S}(\mathbb{R}^n)$.

$$\check{T} = (2\pi)^{-n} \hat{\hat{T}}$$ F. INVERSIONE.

(dove $\langle \check{T}, \varphi \rangle := \langle T, \check{\varphi} \rangle$)

<u>Dim.</u>

$$\langle \check{T}, \varphi \rangle = \langle T, \check{\varphi} \rangle = (2\pi)^{-n} \langle T, \hat{\hat{\varphi}} \rangle =$$
$$= (2\pi)^{-n} \langle \hat{\hat{T}}, \varphi \rangle$$

$\blacksquare$

Esempi $(n=1)$

1) $T = \delta_0 \qquad \widehat{\delta_0} \qquad \hat{\varphi}(\xi) = \int_{\mathbb{R}} \varphi(x) e^{-i\xi x} dx$

$$\langle \widehat{\delta_0}, \varphi \rangle = \langle \delta_0, \hat{\varphi} \rangle \overset{= \hat{\varphi}(0) =}{\underset{\uparrow}{=}} \int_{\mathbb{R}} \varphi(x) dx =$$

metto $\xi = 0$ nella def di $\hat{\varphi}(\xi)$

$$= \int_{\mathbb{R}} \underbrace{1}_{u} \cdot \varphi(x) dx = \langle 1, \varphi \rangle$$

$$\Rightarrow \widehat{\delta_0} = 1$$

2) $\hat{1} = \hat{\hat{\delta_0}} = (2\pi) \check{\delta_0} = (2\pi) \delta_0$

$$\langle \check{\delta_0}, \varphi \rangle = \langle \delta_0, \underbrace{\check{\varphi}}_{\varphi(-x)} \rangle = \langle \delta_0, \varphi \rangle$$

3) $\hat{x} = \mathcal{F}(x \cdot 1) = i (\hat{1})' = 2\pi i (\delta_0)'$

$$(\langle \delta_0', \varphi \rangle = \langle \delta_0, \varphi' \rangle = \varphi'(0)).$$

Recap:

$$\mathcal{F}: L^1 \longrightarrow L^\infty$$

$$\mathcal{F}: \mathcal{S} \longrightarrow \mathcal{S}$$

$$\mathcal{F}: L^2 \longrightarrow L^2$$

$$\mathcal{F}: \mathcal{S}' \longrightarrow \mathcal{S}'$$