Applicazioni del teorema dei residui in campo reale

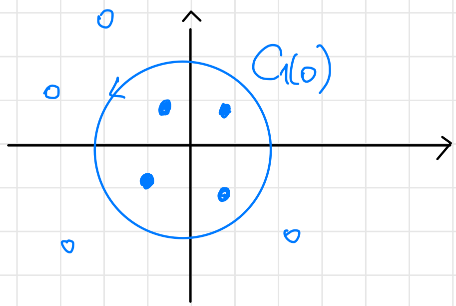
Tipo 1

$$\int_0^{2\pi} f(\cos t, \sin t)\, dt = \int_0^{2\pi} g(\underbrace{e^{it}}_{r(t)})\, \underbrace{ie^{it}}_{r'(t)}\, dt =$$

$$\begin{cases} \cos t = \dfrac{e^{it} + e^{-it}}{2} \\ \sin t = \dfrac{e^{it} - e^{-it}}{2i} \end{cases}$$

$$= \int_{C_1(0)} g(z)\, dz = 2\pi i \sum_{|z_0|<1} \operatorname{Res}(g, z_0)$$

se g soddisfa le ip. del teorema dei residui su $\Omega \supseteq C_1(0)$, con $\gamma = C_1(0)$



Esempio

$$\int_0^{2\pi} \frac{1}{2+\sin t}\,dt \underset{\substack{\uparrow \\ \sin t = \frac{e^{it}-e^{-it}}{2i}}}{=} \int_0^{2\pi} \frac{1}{2+\frac{e^{it}-e^{-it}}{2i}}\,dt$$

$$= \int_0^{2\pi} \frac{2\cancel{i}}{[4i+e^{it}-e^{-it}]}\,\frac{ie^{it}}{\cancel{i}e^{it}}\,dt$$

$$= \int_0^{2\pi} \underbrace{\frac{2}{4ie^{it}+e^{2it}-1}}_{\substack{g(r(t)) \\ \shortparallel \\ g(e^{it})}}\,\underbrace{ie^{it}}_{r'(t)}\,dt \qquad \left(\Rightarrow g(z) = \frac{2}{4iz+z^2-1}\right)$$

$$= 2\pi i \sum_{|z_0|<1} \mathrm{Res}(g, z_0) = \text{[da completare]}$$

Valor principale

$$\underbrace{V.P.}_{\text{valor principale}} \int_{\mathbb{R}} f(x)\,dx \;:=\; \lim_{\substack{R\to+\infty\\ \varepsilon\to 0}} \left[\int_{-R}^{x_0-\varepsilon} f + \int_{x_0+\varepsilon}^{R} f \right]$$

valor principale — con sing in x_0

→ Se f è integrabile (secondo Riemann), allora (in senso generalizzato)

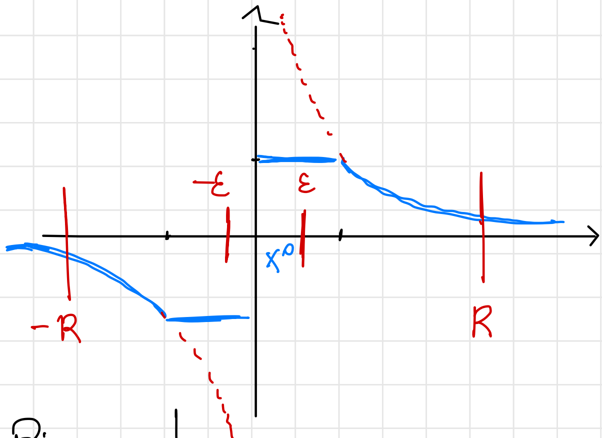
$$\int_{\mathbb{R}} f(x)\,dx = \lim_{R\to+\infty} \int_{-R}^{R} f(x)\,dx$$

→ In generale, può accadere che

$$V.P. \int_{\mathbb{R}} f(x)\,dx \in \mathbb{R}, \quad \text{ma } f \text{ non integrabile}$$

Esempio:

$$f(x) = \begin{cases} \frac{1}{x} & x \geq 1 \\ 1 & x \in [0,1] \\ -1 & x \in [-1,0] \\ \frac{1}{x} & x \leq -1 \end{cases}$$



f non integrabile secondo Riemann!

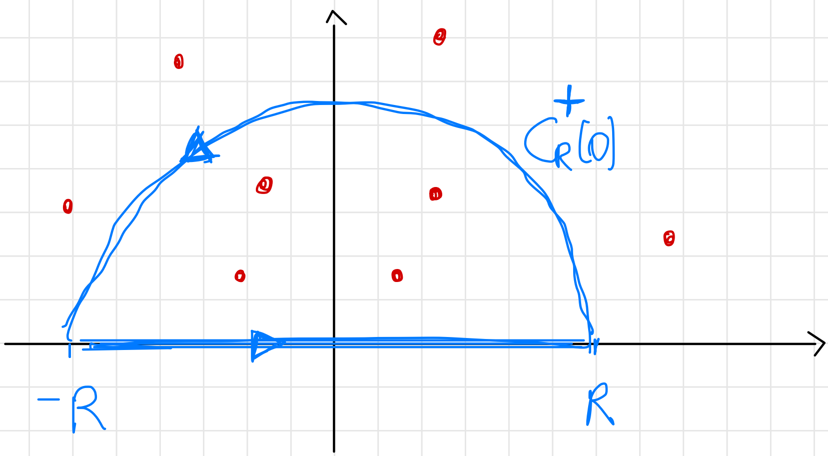
$$\int_{-R}^{R} f(x)\,dx = 0 \quad \forall R \;\Rightarrow\; V.P. \int_{\mathbb{R}} f(x)\,dx = 0.$$

Tipo 2

$$I = (V.P.)\int_{\mathbb{R}} f(x)\,dx = 2\pi i \sum_{\substack{z^0 \in S \\ \operatorname{Im} z^0 > 0}} \operatorname{Res}(f, z^0)$$

IPOTESI: $f = f(z)$ abbia un n° finito di singolarità su $\{\operatorname{Im} z > 0\}$ + ipotesi (*) (e nessuna singolarità sull'asse reale).

$$I = \lim_{R \to +\infty} \left[\underbrace{\int_{-R}^{R} f(z)\,dz + \int_{C_R^+(0)} f(z)\,dz} - \int_{C_R^+(0)} f(z)\,dz \right]$$



$$\gamma_R = [-R, R] + C_R^+(0)$$

$$I = \lim_{R\to+\infty} \left[\int_{\gamma_R} f(z)\,dz\right] - \underbrace{\cancel{\lim_{R\to+\infty}\int_{C_R^+(0)} f(z)\,dz}}_{=\,0}$$

$$= 2\pi i \sum_{\substack{z^\circ\in S\\ \operatorname{Im} z^\circ>0}} \operatorname{Res}(f, z^\circ)$$

$\uparrow$
teo residui
+
lemma di decadimento.

Lemma (di decadimento)

Se $\boxed{\exists\, \alpha>1 \text{ tale che } |f(z)| \le \dfrac{C}{|z|^\alpha}}$ $(*)$
(per $|z|$ abbastanza grande)

allora $\displaystyle\lim_{R\to+\infty}\int_{C_R^+(0)} f(t)\,dt = 0.$ $\quad \approx \dfrac{R}{R^\alpha} \xrightarrow[R\to+\infty]{} 0$

Oss. Variante analoga nel semipiano $\{\operatorname{Im} z<0\}$.

Esempio

$$I = \int_{\mathbb{R}} \frac{1}{1+x^2}\, dx$$

$$= \operatorname{arctg} x \Big|_{-\infty}^{+\infty} = \pi$$

polinomi

$\left[\int_{\mathbb{R}} \frac{P(x)}{Q(x)}\, dx \text{ con } \right.$

grado $Q \geq$ grado $P + 2$

$\left. \text{e } Q \text{ senza radici reali} \right]$

$$f(z) = \frac{1}{1+z^2}, \quad S = \{\pm i\},$$

$(*)$ vale con $\alpha = 2$ ipotesi lemma di decadimento.

$$\Rightarrow I = \begin{cases} 2\pi i \operatorname{Res}(f, i) = \pi \\ -2\pi i \operatorname{Res}(f, -i) = \pi \end{cases}$$

DA COMPLETARE

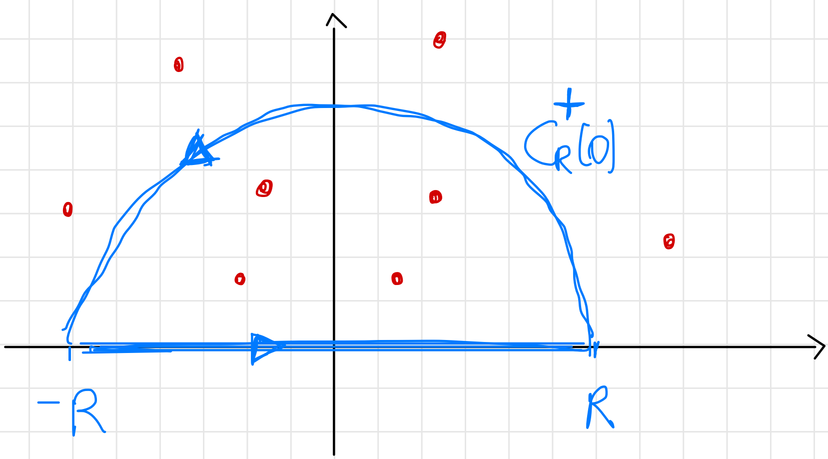
DA COMPLETARE

Tipo 3

$$I = (\text{V.P.}) \int_{\mathbb{R}} f(x)\, e^{i\omega x}\, dx = 2\pi i \sum_{\substack{z^0 \in S \\ \operatorname{Im} z^0 > 0}} \operatorname{Res}\left(f(z)\, e^{i\omega z}, z^0\right) \qquad \omega \in \mathbb{R}^+$$

IPOTESI: $f(z)e^{i\omega z}$ abbia un n° finito di singolarità su $\{\operatorname{Im} z > 0\}$ + ipotesi $(**)$ (e nessuna singolarità sull'asse reale).

$$I = \lim_{R \to +\infty} \left[\underbrace{\int_{-R}^{R} f(z)e^{i\omega z}\, dz + \int_{C_R^+(0)} f(z)e^{i\omega z}\, dz}_{} - \int_{C_R^+(0)} f(z)e^{i\omega z}\, dz \right]$$



$$\gamma_R = [-R, R] + C_R^+(0).$$

$$I = \lim_{R\to+\infty}\left[\int_{\gamma_R} f(z)e^{i\omega z}\,dz\right] - \underbrace{\cancel{\lim_{R\to+\infty}\int_{C_R^+(0)} f(z)e^{i\omega z}\,dz}}_{=0}$$

$$\underset{\substack{\uparrow\\ \text{teo residui}\\ +\\ \text{lemma di Jordan}}}{=} 2\pi i \sum_{\substack{z^0\in S\\ \operatorname{Im} z^0>0}} \operatorname{Res}\left(f(z)e^{i\omega z}, z^0\right)$$

<u>Lemma di Jordan</u>

Sotto l'ipotesi

$$\boxed{(**)\qquad \lim_{R\to+\infty}\ \sup_{z\in C_R^+(0)} |f(z)| = 0},$$

$$\lim_{R\to+\infty}\int_{C_R^+(0)} f(z)e^{i\underset{\substack{\uparrow\\ \mathbb{R}^+}}{\omega} z}\,dz = 0.$$

<u>Oss.</u> Variante analoga nel semipiano $\{\operatorname{Im} z<0\}$, <u>QUANDO</u> $\omega\in\mathbb{R}^-$

(Jordan vale anche per $\omega\in\mathbb{R}^-$ con $C_R^-(0)$).

Esempio

$$e^{ix} = \cos x + i \sin x \qquad \omega = 1$$

$$I = (V.P.)\int_{\mathbb{R}} \frac{\cos x}{1+x^2}\,dx = \int_{\mathbb{R}} \frac{e^{ix}}{1+x^2}\,dx$$

$$= \int_{\mathbb{R}} f(x)e^{i\omega x}\,dx \,, \qquad \text{con } f(z) = \frac{1}{1+z^2} \,, \ \omega = 1$$

$$\underset{\uparrow}{=} \quad 2\pi i \ \mathrm{Res}\left(f(z)e^{i\omega z}, i\right) = \text{COMPLETARE}$$

se sono OK le ip. ✓

- $S = \{\pm i\} \Rightarrow$ no singolarità su $\mathbb{R}$

- $\sup_{z \in C_R^+(0)} \left|\frac{1}{1+z^2}\right| \xrightarrow[R \to +\infty]{?} 0 \iff \inf_{z \in C_R^+(0)} |1+z^2| \to +\infty$

$$\left\| \right. \qquad \sup_{z\in C_R^+(0)} \frac{1}{|1+z^2|} = \frac{1}{\inf_{z\in C_R^+(0)} |1+z^2|}$$

$$\forall \ |1+z^2| \underset{\uparrow}{\geq} |z^2| - 1 = |z|^2 - 1 = R^2 - 1 \longrightarrow +\infty$$

TRIANGOLARE $\quad |z^2| \leq |z^2+1| + 1$

$$\hookrightarrow |(z^2+1) - 1| = |z^2+1+(-1)|$$

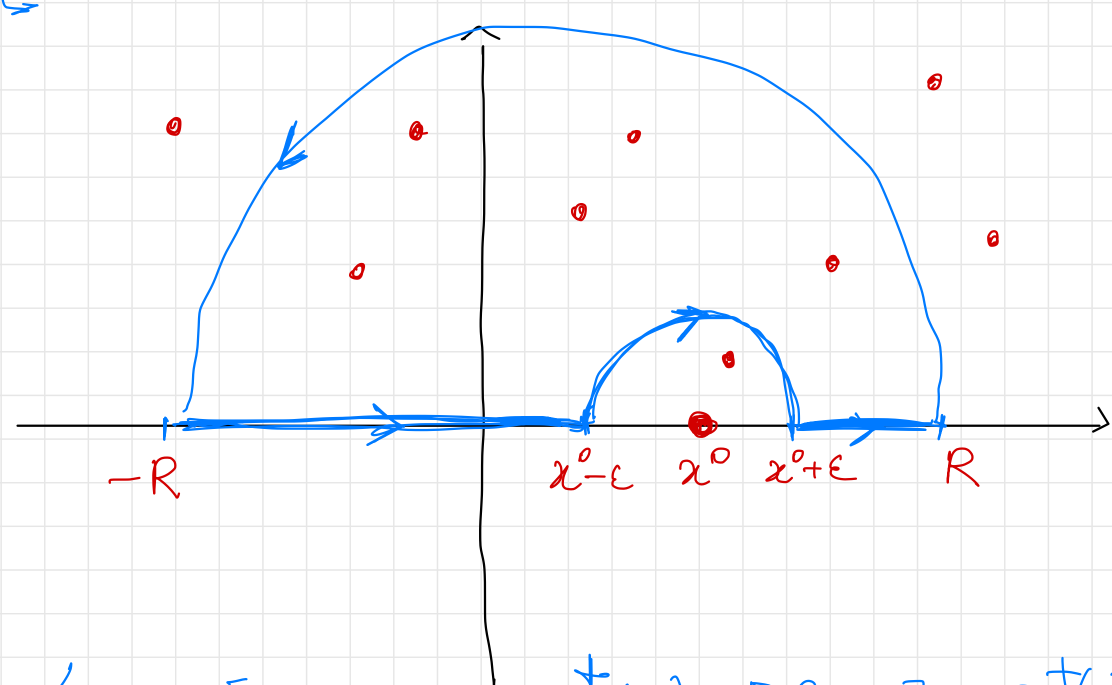
Tipo 4

$$I = (V.P.)\int_{\mathbb{R}} f(x)\,dx = 2\pi i \sum_{\substack{z^0 \in S \\ \mathrm{Im}\, z^0 > 0}} \mathrm{Res}(f, z^0) + \pi i \sum_{\substack{x^0 \in \mathbb{R} \\ x^0 \text{ poli semplici}}} \mathrm{Res}(f, x^0)$$

<u>IPOTESI</u>: $f(z)$ abbia un n° finito di singolarità su $\{\mathrm{Im}\, z > 0\}$

$+ \quad \lim_{R \to +\infty} \int_{C_R^+(0)} f(z)\,dz = 0 \quad (***)$

abbia un n° finito di POLI SEMPLICI su $\mathbb{R}$



$$\gamma_{R,\varepsilon} = [-R, x^0-\varepsilon] - C_\varepsilon^+(x^0) + [x^0+\varepsilon, R] + C_R^+(0)$$

$$I = \lim_{\substack{R \to +\infty \\ \varepsilon \to 0}} \int_{\gamma_{R,\varepsilon}} f(z)\,dz + \lim_{\varepsilon \to 0} \int_{C_\varepsilon^+(x^0)} f(z)\,dz$$

$$\cancel{- \lim_{R \to +\infty} \int_{C_R^+(0)} f(z)\,dz}$$

<u>Lemma (polo semplice)</u>

Se x^0 polo semplice,

$$\lim_{\varepsilon \to 0} \int_{C_\varepsilon^+(x^0)} f(z)\,dz = \pi i \,\mathrm{Res}(f, x^0).$$

<u>Oss.</u> Variante analoga su $\{\mathrm{Im}\, z < 0\}$.