RIPASSO DI ELEMENTI DI GEOMETRIA 3D

28-2-2019

ESERCIZIO 1. Determinare l'angolo compreso tra i vettori $\underline{v} = (3,3,-6) = 3\underline{i} + 3\underline{j} - 6\underline{k}$ e $\underline{w} = (-1,-2,1) = -\underline{i} - 2\underline{j} + \underline{k}$.

SOL.

$$\underline{v} \cdot \underline{w} = |\underline{v}||\underline{w}| \cos\theta \Rightarrow \cos\theta = \frac{\underline{v} \cdot \underline{w}}{|\underline{v}||\underline{w}|}$$

$$\underline{v} \cdot \underline{w} = v_x w_x + v_y w_y + v_z w_z$$

$$\underline{v} \cdot \underline{w} = \begin{bmatrix} 3 \\ 3 \\ -6 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix} = -3 - 6 - 6 = -15$$

$$|\underline{v}| = \sqrt{9+9+36} = \sqrt{54} = 3\sqrt{6}$$

$$|\underline{w}| = \sqrt{1+4+1} = \sqrt{6}$$

$$\cos\theta = \frac{-15}{18} = -5/6$$

ESERCIZIO 2. Nello spazio $\mathbb{R}^3$ si considerino le rette

$$r: \begin{cases} x = 2t \\ y = -3t+1 \\ z = t+2 \end{cases} \quad t \in \mathbb{R} \qquad \text{ed} \qquad s: \begin{cases} x+y+3z = 5 \\ x-y+z = -5 \end{cases}$$

PARAMETRICA CARTESIANA

1) $r \cap s = \{P\}$, $P = ?$

2) calcolare l'angolo formato dalle due direzioni di r ed s.

3) scrivere un'equazione cartesiana del piano che contiene r ed s e calcolarne la distanza dall'origine.

SOL.

$$r \cap s: \begin{cases} 2t - \cancel{3t} + 1 + \cancel{3t} + 6 = 5 \implies t = -1 \\ 2t + 3t - 1 + t + 2 = -5 \implies \cancel{6}t = \cancel{-6}^{\,-1} \end{cases}$$

$r \cap s = \{P\}$ $\quad P(-2, 4, 1)$ $\quad (t = -1 \text{ in } r)$

2) $r : \begin{cases} x = 2t \\ y = -3t + 1 \\ z = t + 2 \end{cases}$ $\quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = t \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$

$$\underline{v}_r$$

$s : \begin{array}{l} \pi_1 \\ \pi_2 \end{array} \begin{cases} x + y + 3z = 5 \\ x - y + z = -5 \end{cases}$ $\quad \underline{n}_1 = \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}$ $\quad \underline{n}_2 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$

$$\underline{n}_1 \times \underline{n}_2 = \begin{vmatrix} \underline{\hat{i}} & \underline{\hat{j}} & \underline{k} \\ 1 & 1 & 3 \\ 1 & -1 & 1 \end{vmatrix} =$$

$$= \underline{\hat{i}}(1+3) - \underline{\hat{j}}(1-3) + \underline{k}(-1-1) =$$

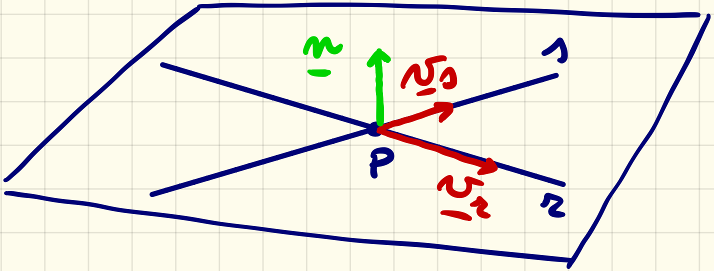
$$= 4\underline{\hat{i}} + 2\underline{\hat{j}} - 2\underline{k} = \begin{bmatrix} 4 \\ 2 \\ -2 \end{bmatrix}. \qquad \underline{v}_s = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$$

$$\widehat{rs} = \arccos \frac{\underline{v}_r \cdot \underline{v}_s}{|\underline{v}_r|\,|\underline{v}_s|} = \arccos 0 = \frac{\pi}{2}$$

$$\underline{v}_r \cdot \underline{v}_s = \begin{bmatrix}2\\-3\\1\end{bmatrix} \cdot \begin{bmatrix}2\\1\\-1\end{bmatrix} = 4 - 3 - 1 = 0$$

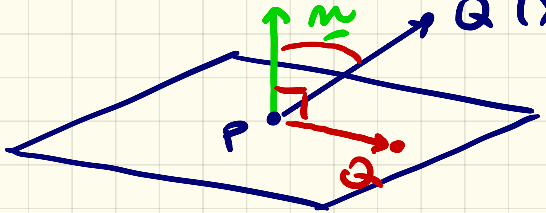
(CONDIZIONE DI PERPENDICOLARITÀ: $\underline{v} \cdot \underline{w} = 0$)

c) $r \subset \pi$ $\quad \pi : ?$
$s \subset \pi$



$$\underline{v}_r \times \underline{v}_s = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 2 & -3 & 1 \\ 2 & 1 & -1 \end{vmatrix} =$$

$$= \underline{i}\,(3-1) - \underline{j}\,(-2-2) + \underline{k}\,(2+6) = \begin{bmatrix}2\\4\\8\end{bmatrix} \qquad \underline{n} = \begin{bmatrix}1\\2\\4\end{bmatrix}$$



$Q(x,y,z)$ $\quad P(-2,4,1)$

$$\overrightarrow{PQ} = \begin{bmatrix} x+2 \\ y-4 \\ z-1 \end{bmatrix} \quad \underline{n} = \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$$

$$\pi : \overrightarrow{PQ} \cdot \underline{n} = 0$$

$$\pi : \begin{bmatrix} x+2 \\ y-4 \\ z-1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix} = 0$$

$$(x+2) + 2(y-4) + 4(z-1) = 0$$

$$x + 2 + 2y - 8 + 4z - 4 = 0$$

$$\pi : x + 2y + 4z = 10$$

$O(0,0,0)$

$$d(\pi; O) = \frac{|-10|}{\sqrt{1+4+16}} = \frac{10}{\sqrt{21}}.$$

LUOGHI GEOMETRICI

SFERA

ESERCIZIO 3. Dopo aver scritto l'eq. della sfera di centro $C(0,1,1)$ e raggio 1, determinare le equazioni di almeno 2 piani tangenti alla sfera passanti per $(0,0,0)$.

SOL. $P(x,y,z)$ $\quad C(0,1,1)$

SFERA: $d(P,C) = R$

$$\cancel{\sqrt{}}\; x^2+(y-1)^2+(z-1)^2 = 1^2$$

$$x^2+y^2-2y+1+z^2-2z+\cancel{1} = \cancel{1}$$

$$S: x^2+y^2+z^2-2y-2z+1=0$$

Sia π il generico piano passante per $O(0,0,0)$

$$\pi: a(x-0)+b(y-0)+c(z-0)=0$$

$$\pi: ax+by+cz=0$$

$$d(\pi;G)=1 \qquad G(0,1,1)$$

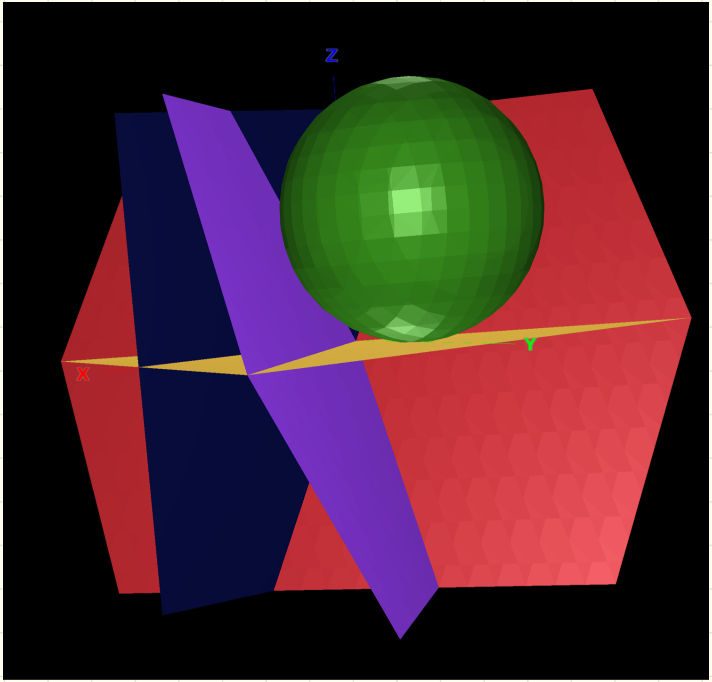
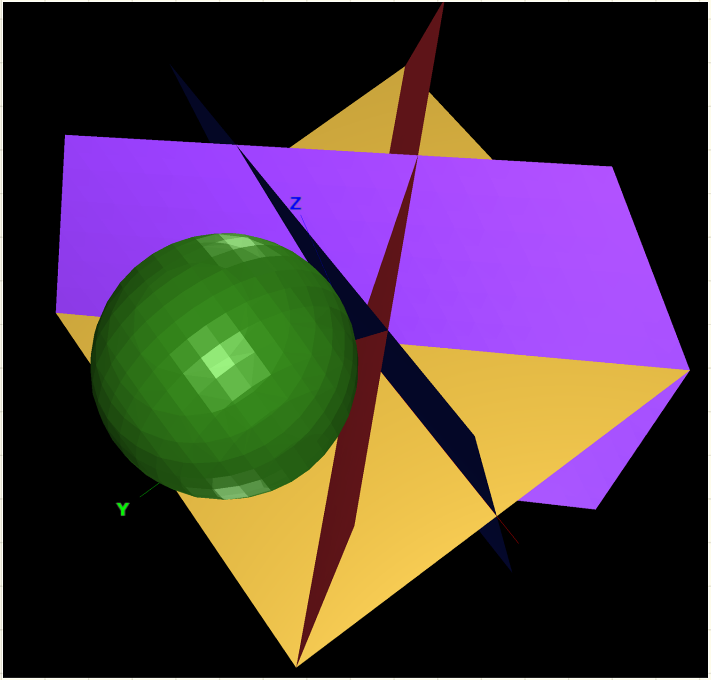
$$\frac{|b+c|}{\sqrt{a^2+b^2+c^2}}=1 \Rightarrow (b+c)^2=a^2+b^2+c^2$$

$$\cancel{b^2}+\cancel{c^2}+2bc=a^2+\cancel{b^2}+\cancel{c^2}$$

$$a^2=2bc$$

$$b=2 \quad c=1 \Rightarrow a^2=4 \Rightarrow a=\pm 2$$

$$\pi_1: 2x+2y+z=0 \qquad \pi_2: -2x+2y+z=0$$


Z
X
Y

Z
Y

CILINDRI (CON ASSE PARALLELO AGLI ASSI CARTES.)

ESERCIZIO 4

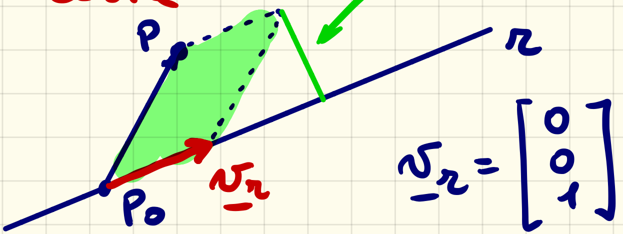
Scrivere l'equazione del luogo geometrico dei punti che hanno distanza 1 da

$$r: \begin{cases} x = 1 \\ y = 1 \\ z = t \end{cases}$$

SOL.

<u>NOTA</u>: distanza punto - retta

$$d(P, r) = \frac{|\overrightarrow{P_0P} \times \vec{v}_r|}{|\vec{v}_r|}$$



$$\underline{v}_r = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \underline{k}$$

EQ. LUOGO:

$$r: \begin{cases} x = 1 \\ y = 1 \\ z = t \end{cases} \qquad \begin{matrix} P(x, y, z) \\ P_0(1, 1, 0) \end{matrix}$$

$$\overrightarrow{P_0P} = P - P_0 = \begin{bmatrix} x-1 \\ y-1 \\ z-0 \end{bmatrix}$$

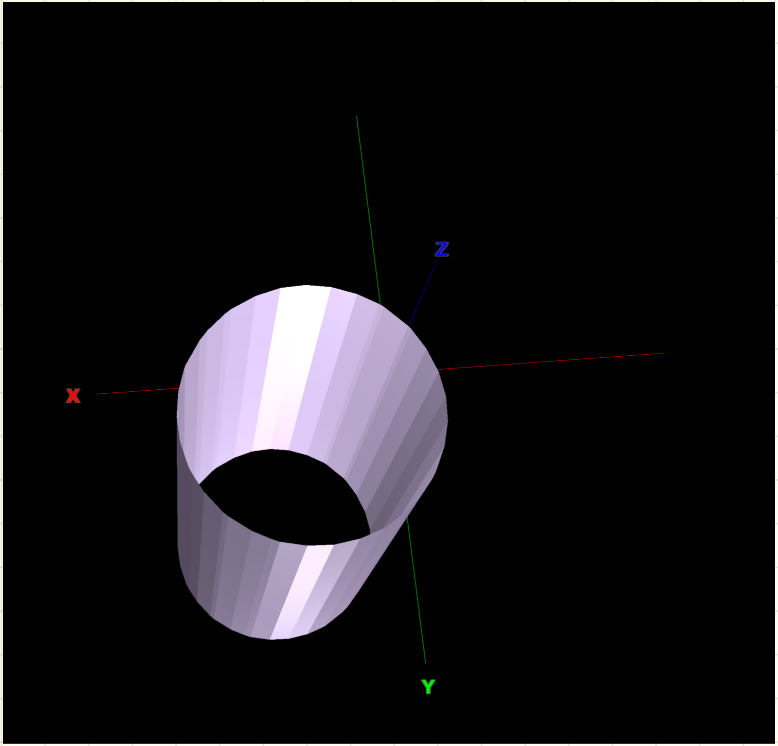
$$\overrightarrow{P_0P} \times \underline{v}_r = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ x-1 & y-1 & z \\ 0 & 0 & 1 \end{vmatrix} = \underline{i}(y-1) - \underline{j}(x-1) + \underline{k}(0) = \begin{bmatrix} y-1 \\ -x+1 \\ 0 \end{bmatrix}$$

$$1 = d(P, r) = \frac{\sqrt{(y-1)^2+(1-x)^2}}{1} = \sqrt{(y-1)^2+(1-x)^2}$$

$$(y-1)^2+(1-x)^2 = 1^2$$

$$\mathcal{C}: (x-1)^2+(y-1)^2 = 1$$ CILINDRO!

Sembra l'eq. di una circonferenza ma non in $\mathbb{R}^3$! Quindi è valida $\forall z$

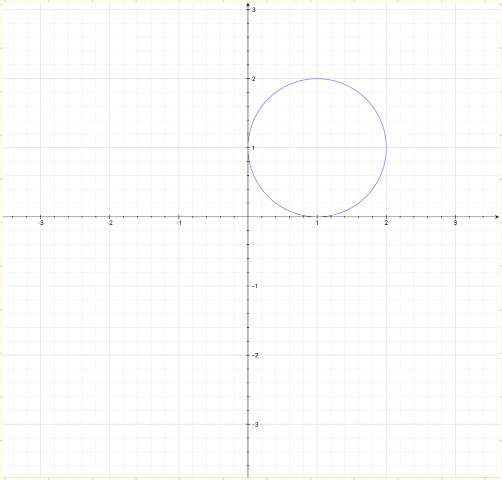


$$\left((x-1)^2 + (y-1)^2 = 1 \right.$$

in $\mathbb{R}^3$

$$(x-1)^2 + (y-1)^2 = 1$$

in $\mathbb{R}^2$



ESERCIZIO 5. (VELOCE) Scrivere l'eq. del cilindro di asse $r: \begin{cases} x = t \\ y = 1 \\ z = 2 \end{cases}$ e raggio 1.

SOL.

$$r: \begin{bmatrix} x \\ y \\ z \end{bmatrix} = t \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$$ (ASSE // asse x)

$$\text{Cil}: (y-1)^2 + (z-2)^2 = 1^2$$

ESERCIZIO 6. Siano dati i vettori di $\mathbb{R}^3$ $\underline{v} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$ e $\underline{w}_k = \begin{bmatrix} 0 \\ k \\ 1 \end{bmatrix}$.

1) Per quali valori di $k \in \mathbb{R}$ l'angolo tra essi formato è $\theta = \frac{\pi}{3}$?

2) In corrispondenza del valore di k del punto 1 calcolare in due modi diversi l'area del parallelogramma formato

3) Scrivere in funzione di k il piano passante per l'origine e contenente i due vettori. Per quali valori di k è ortogonale alla retta $r: (1+t, t, 1-t)$ e

Per quali valori di k è parallelo a r.

SOL.

1) $\underline{v} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} \qquad \underline{w}_k = \begin{bmatrix} 0 \\ k \\ 1 \end{bmatrix}$

$$\underline{v} \cdot \underline{w}_k = 2k \qquad \theta = \frac{\pi}{3}$$

$$|\underline{v}| = \sqrt{5} \qquad |\underline{w}_k| = \sqrt{k^2+1}$$

$$\frac{1}{2} = \cos\theta = \frac{\underline{v} \cdot \underline{w}_k}{|\underline{v}|\,|\underline{w}_k|} = \frac{2k}{\sqrt{5}\sqrt{k^2+1}} \qquad \theta = \frac{\pi}{3}$$

$$\Rightarrow \sqrt{5}\sqrt{k^2+1} = 4k$$

C.E. $\forall k$

C.C.S. $4k \geq 0$

$\underline{k \geq 0}$

$$5(k^2+1) = 16k^2$$

$$11k^2 = \frac{5}{11} \qquad k = \cancel{\pm}\sqrt{\frac{5}{11}}$$

2)

$$\underline{v} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} \qquad \underline{w} = \begin{bmatrix} 0 \\ \sqrt{5/11} \\ 1 \end{bmatrix}$$

$$|\underline{v}| = \sqrt{5} \qquad |\underline{w}| = \sqrt{\frac{5}{11} + 1} = \frac{4}{\sqrt{11}}$$

A) $$\text{Aree} = |\underline{v} \times \underline{w}| = |\underline{v}|\,|\underline{w}| \operatorname{sen}\theta =$$

$$= \sqrt{5}\,\frac{\cancel{4}^{\,2}}{\sqrt{11}} \cdot \frac{\sqrt{3}}{\cancel{2}} = 2\sqrt{\frac{15}{11}}$$

B) $$\underline{v} \times \underline{w} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 1 & 2 & 0 \\ 0 & \sqrt{5/11} & 1 \end{vmatrix} = \underline{i}(2) - \underline{j}(1) + \underline{k}\left(\sqrt{\frac{5}{11}}\right)$$

$$\text{Aree} = \left| \begin{bmatrix} 2 \\ -1 \\ \sqrt{5/11} \end{bmatrix} \right| = \sqrt{4 + 1 + \frac{5}{11}} = \sqrt{\frac{60}{11}} = 2\sqrt{\frac{15}{11}}$$

3) $\underline{v} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$ $\quad \underline{w}_k = \begin{bmatrix} 0 \\ k \\ 1 \end{bmatrix}$ $\quad r : \begin{cases} x = 1 + t \\ y = t \\ z = 1 - t \end{cases}$

$$\underline{n}_k = \underline{v} \times \underline{w}_k = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 1 & 2 & 0 \\ 0 & k & 1 \end{vmatrix} =$$

$$= \underline{i}(2) - \underline{j}(1) + \underline{k}(k) = \begin{bmatrix} 2 \\ -1 \\ k \end{bmatrix}$$

$$\pi_k : \ 2(x-0) - 1(y-0) + k(z-0) = 0$$

$$2x - y + kz = 0$$ (FASCIO DI PIANI)

GENERATRICI:

$k = 0 \Rightarrow 2x - y = 0$

$k \to \infty \Rightarrow z = 0$

$\boxed{\pi_k \parallel r} \iff \underline{u}_k \perp \underline{v}_r \iff \underline{u}_k \cdot \underline{v}_r = 0$

$$\begin{bmatrix} 2 \\ -1 \\ k \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} = 0 \qquad 2-1-k=0$$

$$\boxed{k=1}$$

$\pi_k \perp r \iff \underline{u}_k \parallel \underline{v}_r \iff \underline{u}_k \times \underline{v}_r = \underline{0}$

$$\underline{u}_k \times \underline{v}_r = \begin{bmatrix} 1-k \\ 2+k \\ 3 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \forall k \in \mathbb{R}$$

$\Longrightarrow$ IMPOSSIBILE.