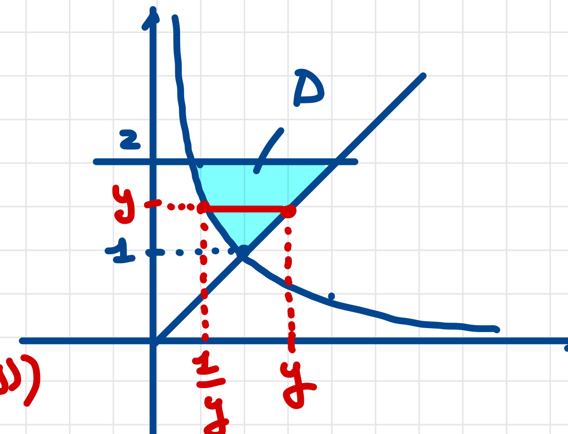
ESERCIZIO 1. 20-5-2021

Sia D la regione di piano delimitata dalla iperbole $y=\frac{1}{x}$ e dalle rette $y=2$ e $y=x$. Calcolare il volume del solido ottenuto ruotando S intorno all'asse y.

SOL.



Lo strato $\Omega(y)$

$$\Omega(y)=\left\{\frac{1}{y^2}\le x^2+z^2\le y^2\right\}$$

$$V(S)=\iiint_S dx\,dy\,dz=$$

$$=\int_1^2\Big(\overbrace{\iint_{\Omega(y)} dx\,dz}^{=\ \text{Area}(\Omega(y))}\Big)dy=$$

$$= \int_1^2 \underbrace{\pi\left(y^2 - \frac{1}{y^2}\right)}_{\text{Area}(\Omega(y))} dy = \pi\left[\frac{y^3}{3} + \frac{1}{y}\right]_1^2 =$$

$$= \pi\left(\frac{8}{3} + \frac{1}{2} - \frac{1}{3} - 1\right) = \pi\,\frac{11}{6} = \frac{11}{6}\pi.$$

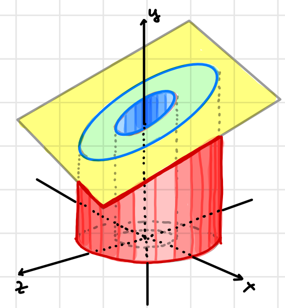
ESERCIZIO 2.

Calcolare il volume del solido

$$S = \left\{ (x,y,z) \in \mathbb{R}^3 \ / \ 1 \leq x^2 + z^2 \leq 9 \ , \ 0 \leq y \leq z+3 \right\}$$

Sol.

In coord. cilindriche:



$$\begin{cases} x = \rho\cos\theta \\ y = t \\ z = \rho\,\text{sen}\,\theta \end{cases} \qquad \theta \in [0, 2\pi]$$

$$dx\,dy\,dz = \rho\,d\rho\,d\theta\,dt$$

$$1 \le x^2 + z^2 \le 9 \longrightarrow 1 \le \rho^2 \le 9 \longrightarrow 1 \le \rho \le 3.$$

$$0 \le y \le z + 3 \longrightarrow 0 \le t \le \rho \operatorname{sen}\theta + 3.$$

$$\operatorname{Vol}(S) = \iiint_S dx\,dy\,dz = \iiint_{S'} \rho\, d\rho\, d\theta\, dt =$$

con $S' = \{ (\rho, \theta, t) \,/\, 1 \le \rho \le 3,\ 0 \le \theta \le 2\pi,\ 0 \le t \le \rho \operatorname{sen}\theta + 3 \}$

$$= \int_0^{2\pi} \left(\int_1^3 \left(\int_0^{3+\rho \operatorname{sen}\theta} \rho \, dt \right) d\rho \right) d\theta =$$

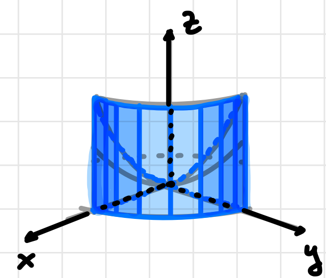
$$= \int_0^{2\pi} \left(\int_1^3 \rho \left(3 + \rho \operatorname{sen}\theta \right) d\rho \right) d\theta =$$

$$= \int_0^{2\pi} \left[\frac{3}{2}\rho^2 + \frac{\rho^3}{3} \operatorname{sen}\theta \right]_1^3 d\theta = \int_0^{2\pi} \left(\frac{27}{2} + 9\operatorname{sen}\theta - \frac{3}{2} - \frac{1}{3}\operatorname{sen}\theta \right) d\theta$$

$$= \int_0^{2\pi} \left(12 + \cancel{\frac{26}{3}\operatorname{sen}\theta} \right) d\theta = 12(2\pi) = 24\pi.$$

ESERCIZIO 3. Sia $T = \{ (x;y;z) \in \mathbb{R}^3 \,/\, 0 \le z \le x^2+y^2,$
$x^2+y^2 \le 1,\ x \ge 0,\ y \ge 0 \}$. Calcolare

$$\iiint_T y\sqrt{x}\, dx\,dy\,dz$$



SOL. $\begin{cases} x = \rho\cos\theta \\ y = \rho\operatorname{sen}\theta \\ z = t \end{cases}$

$$x^2+y^2 \leq 1 \longrightarrow 0 \leq \rho \leq 1.$$

$$\left.\begin{array}{l} x>0 \\ y>0 \end{array}\right\} \longrightarrow \theta \in \left[0, \frac{\pi}{2}\right]$$

$$0 \leq y \leq x^2+y^2 \longrightarrow t \in [0, \rho^2].$$

$$\iiint_T y\sqrt{x}\, dx\, dy\, dz = \int_0^1 \left(\int_0^{\pi/2} \left(\int_0^{\rho^2} \rho \operatorname{sen}\theta \sqrt{\rho\cos\theta}\, \rho\, dt \right) d\theta \right) d\rho$$

$$= \int_0^1 \left(\int_0^{\pi/2} \rho^2 \operatorname{sen}\theta \sqrt{\rho\cos\theta}\, \rho^2\, d\theta \right) d\rho =$$

$$= \int_0^1 \rho^{9/2} \left(-\int_0^{\pi/2} -\operatorname{sen}\theta\, (\cos\theta)^{1/2}\, d\theta \right) d\rho =$$

$$= \left[\frac{2}{11}\rho^{11/2}\right]_0^1 \cdot \left\{-\left[\frac{2}{3}(\cos\theta)^{3/2}\right]_0^{\pi/2}\right\}$$

$$= \frac{2}{11}\cdot\left(-\frac{2}{3}\right)(0-1) = \frac{4}{33}.$$

ESERCIZI A CASA:

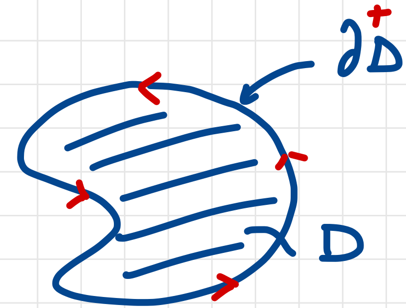
1) Con le coordinate sferiche calcolare il volume di una sfera di raggio R.

2) Calcolare il volume del solido ottenuto ruotando intorno all'asse y il triangolo di vertici $(0;0;2)$, $(0,-1,1)$, $(0,1,1)$

TEOREMA DI GAUSS-GREEN E SUE APPLICAZIONI

$\underline{F} \in C^1$, $\quad \underline{F} = F_1 \underline{i} + F_2 \underline{j}$, $\quad D \subset \mathbb{R}^2$

$$L = \int_{\partial^+ D} \underline{F} \cdot d\underline{r} = \iint_D \left(F_{2x} - F_{1y} \right) dx\,dy$$

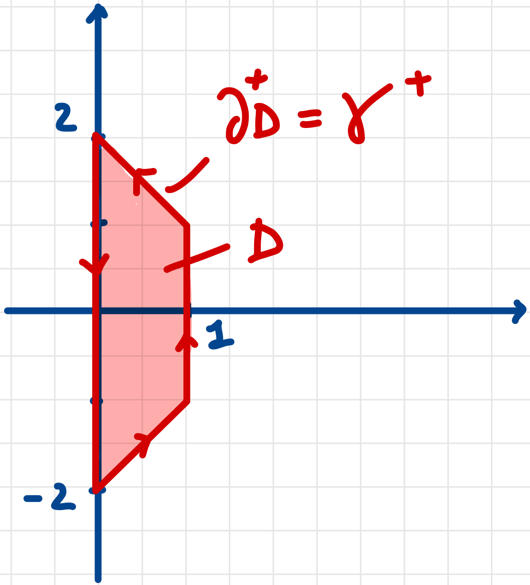


APPLICAZIONE 1. Calcolo di un lavoro mediante un integrale doppio.

ESERCIZIO 4. Sia

$$\underline{F}(x;y) = \left(x \operatorname{sen} y^2 - y^2 \right) \underline{i} + \left(x^2 y \cos y^2 + 3x \right)$$

un campo piano. Calcolare il lavoro di $\underline{F}$ lungo il percorso γ in figura:



$$L_\gamma = \int_\gamma \underline{F} \cdot d\underline{r} =$$

$$= \iint_D \left(\cancel{2xy\cos y^2} + 3 - \cancel{2xy\cos y^2} + 2y \right) dx\,dy$$

$$= \iint_D (3 + 2y)\, dx\,dy =$$

$$= \iint_D 3\, dx\,dy + \iint_D \cancel{2y\, dx\,dy} =$$

D è simmetrico rispetto asse x

$g(x;y) = 2y$ è disp. rispetto a x

$$= 3\, \text{Area}(D)$$

$$= 3 \cdot 3 = 9.$$

APPLICAZIONE 2. Calcolo di aree mediante un integrale di linea.

N.B. Si ricorda che $\text{Area}(D) = \iint_D 1 \, dx dy$, cerco un campo $\underline{F}$ t.c.

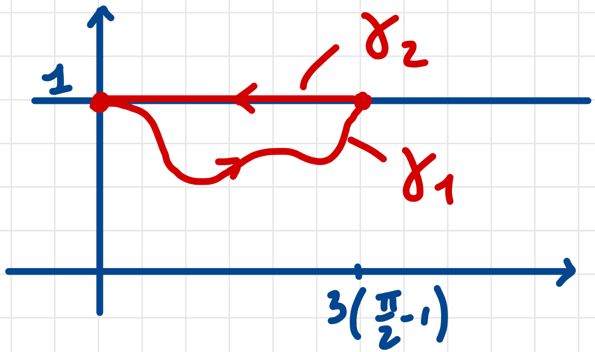
$$F_{2x} - F_{1y} = 1$$

un tale campo è $\underline{F}(x,y) = \begin{bmatrix} 0 \\ x \end{bmatrix} = 0\underline{i} + x\underline{j}$

$$\text{Area}(D) = \iint_D 1 \, dx dy = \int_{\partial^+ D} \underline{F} \cdot d\underline{r}$$

ESERCIZIO 5. Calcolare l'area della regione di piano compresa tra le curve $y = 1$ e $\underline{r}(t) = 3(t - \operatorname{sen} t)\underline{i} + (1 - \operatorname{sen} 2t)\underline{j}$ $t \in [0, \frac{\pi}{2}]$.

SOL.



$$\gamma_1 \quad \underline{r} : \begin{cases} x = 3(t - \operatorname{sen} t) \\ y = 1 - \operatorname{sen} 2t \end{cases} \quad t \in [0, \tfrac{\pi}{2}]$$

$$\underline{r}(0) = (0; 1)$$

$$\underline{r}(\tfrac{\pi}{2}) = \left(3\left(\tfrac{\pi}{2} - 1\right), 1\right)$$

$$\underline{r}' : \begin{cases} x' = 3(1 - \cos t) \\ y' = -2\cos 2t \end{cases}$$

$$\gamma_2 : \begin{cases} x = \tfrac{3}{2}\pi - 3 - t \\ y = 1 \end{cases} \qquad t \in [0, \tfrac{3}{2}\pi - 3] \qquad \gamma_2' : \begin{cases} x' = -1 \\ y' = 0 \end{cases}$$

$$\underline{F}(x,y) = \begin{bmatrix} 0 \\ x \end{bmatrix}$$

$$A(D) = \iint_D 1 \, dx dy = \int_{\partial^+ D} \underline{F} \cdot d\underline{r} = \int_{\gamma_1} \underline{F} \cdot d\underline{r} + \int_{\gamma_2} \underline{F} \cdot d\underline{r} =$$

$$= \int_0^{\pi/2} \begin{bmatrix} 0 \\ 3(t-\operatorname{sen} t) \end{bmatrix} \cdot \begin{bmatrix} 3(1-\cos t) \\ -2\cos 2t \end{bmatrix} dt + \underbrace{\cancel{\int_0^{\frac{3}{2}\pi-3} \begin{bmatrix} 0 \\ \frac{3}{2}\pi-3-t \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 0 \end{bmatrix}}}_{=0}$$

$$= \int_0^{\pi/2} -6(t-\operatorname{sen} t)\cos 2t \, dt$$

$$= -6 \int_0^{\pi/2} (t\cos 2t - \operatorname{sen} t \cos 2t) \, dt = \dots = 1.$$

APPLICAZIONE 3. Calcolo di un integrale doppio mediante un integrale di linea.

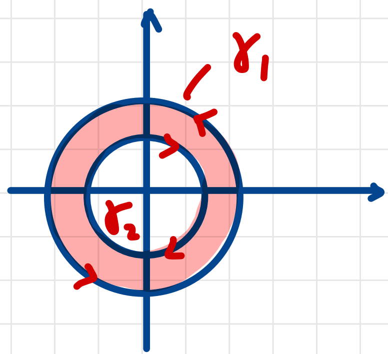
ESERCIZIO 6. Calcolare $\iint_D x^2 \, dx\, dy$ con

$$D = \{ (x;y) \in \mathbb{R}^2 / 1 \le x^2 + y^2 \le 2 \}$$

Sol.

Cerco un campo $\underline{F}(x;y)$ t.c. $F_{2x} - F_{1y} = x^2$

$$F(x;y) = \begin{bmatrix} 0 \\ x^3/3 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 0 \\ x^3 \end{bmatrix}$$



$$\gamma_1 : \begin{cases} x = \sqrt{2}\cos t \\ y = \sqrt{2}\,\text{sen}\, t \end{cases} \qquad \gamma_1' : \begin{cases} x' = -\sqrt{2}\,\text{sen}\, t \\ y' = \sqrt{2}\cos t \end{cases}$$

$$\gamma_2 : \begin{cases} x = \cos t \\ y = -\text{sen}\, t \end{cases} \qquad \gamma_2' : \begin{cases} x' = -\text{sen}\, t \\ y' = -\cos t \end{cases}$$

$$\iint_D x^2\,dx\,dy = \int_{\partial^+ D} \underline{F}\cdot d\underline{r} = \int_{\gamma_1} \underline{F}\cdot d\underline{r} + \int_{\gamma_2} \underline{F}\cdot d\underline{r} =$$

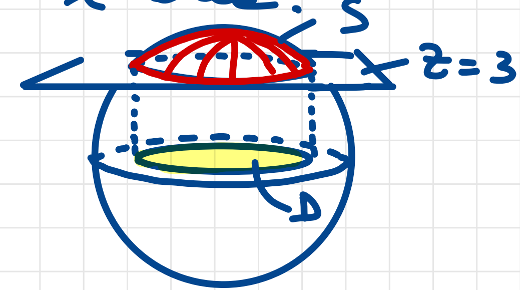
$$= \frac{1}{3}\int_0^{2\pi} \begin{bmatrix} 0 \\ (\sqrt{2}\cos t)^3 \end{bmatrix} \cdot \begin{bmatrix} -\sqrt{2}\,\text{sen}\, t \\ \sqrt{2}\cos t \end{bmatrix} dt + \frac{1}{3}\int_0^{2\pi} \begin{bmatrix} 0 \\ \cos^3 t \end{bmatrix} \cdot \begin{bmatrix} -\text{sen}\, t \\ -\cos t \end{bmatrix}$$

$$= \frac{1}{3}\int_0^{2\pi} \left(4\cos^4 t - \cos^4 t\right) dt = 3 \cdot \frac{1}{3}\int_0^{2\pi} \cos^4 t \, dt =$$

$$= \dots = \frac{3}{4}.$$

INTEGRALI DI SUPERFICIE

ESERCIZIO 7. In $\mathbb{R}^3$ sia S la calotta sferica ottenuta intersecando la sfera di centro $O(0,0,0)$ e raggio $\sqrt{10}$ con il semispazio $z \geq 3$. Si disegni S e se ne calcoli l'area.



SOL. $S: \; x^2+y^2+z^2=10$

$$z = \sqrt{10-x^2-y^2} = f(x;y)$$

$$\partial D : \begin{cases} x^2+y^2+z^2=10 \\ z=3 \end{cases} \longrightarrow \quad x^2+y^2=1 \qquad D=\{x^2+y^2\leq 1\}$$

$$\text{Area}(S) = \iint_S d\sigma = \iint_D \sqrt{1+|\nabla f|^2}\, dx\,dy = (*)$$

$$f(x;y) = \sqrt{10-x^2-y^2} \qquad \nabla f(x;y) = \begin{bmatrix} \dfrac{-x}{\sqrt{10-x^2-y^2}} \\ \dfrac{-y}{\sqrt{10-x^2-y^2}} \end{bmatrix}$$

$$|\nabla f|^2 = \frac{x^2+y^2}{10-x^2-y^2}$$

$$(*) = \iint_D \sqrt{1+\frac{x^2+y^2}{10-(x^2+y^2)}}\, dx\,dy = \text{COORD. POLARI}$$

$$= \int_0^{2\pi} \left(\int_0^1 \sqrt{1 + \frac{\rho^2}{10-\rho^2}} \cdot \rho \, d\rho \right) d\theta =$$

$$= 2\pi \int_0^1 \rho \sqrt{\frac{10}{10-\rho^2}} \, d\rho = \frac{2\pi\sqrt{10}}{-2} \int_0^1 -2\rho \left(10-\rho^2\right)^{-1/2} d\rho$$

$$= -2\pi\sqrt{10} \left[\left(10-\rho^2\right)^{1/2} \right]_0^1 = +2\pi\sqrt{10} \left(-3 + \sqrt{10}\right) =$$

$$= 2\pi\left(10 - 3\sqrt{10}\right).$$

ESERCIZIO 8. Calcolare l'area della superficie

$$\Sigma : \underline{r}(u,v) = u \cos v \, \underline{i} + u \,\mathrm{sen}\, v \, \underline{j} + u^2 \, \underline{k}$$

con $(u,v) \in [0,1] \times [0,\pi]$.

SOL.

$$\underline{r}(u,v) = \left(u\cos v,\ u\,\mathrm{sen}\,v,\ u^2\right)$$

$$\underline{r}_u(u,v) = \left(\cos v,\ \mathrm{sen}\,v,\ 2u\right)$$

$$\underline{r}_v(u,v) = \left(-u\,\mathrm{sen}\,v,\ u\cos v,\ 0\right)$$

$$d\sigma = \left|\underline{r}_u \times \underline{r}_v\right| du\,dv$$

$$\underline{r}_u \times \underline{r}_v = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \cos v & \mathrm{sen}\,v & 2u \\ -u\,\mathrm{sen}\,v & u\cos v & 0 \end{vmatrix} =$$

$$= \underline{i}\left(-2u^2\cos v\right) - \underline{j}\left(2u^2\,\mathrm{sen}\,v\right) + \underline{k}\,(u) =$$

$$= \left(-2u^2\cos v,\ -2u^2 \operatorname{sen} v,\ u\right)$$

$$|\underline{r}_u \times \underline{r}_v| = \sqrt{4u^4\cos^2 v + 4u^4 \operatorname{sen}^2 v + u^2} = \sqrt{4u^4+u^2}$$

$$= \cancel{|}u\cancel{|}\sqrt{4u^2+1}$$

$$\text{Area}(\Sigma) = \iint_{\Sigma} d\sigma = \iint_{[0,1]\times[0,\pi)} u\sqrt{4u^2+1}\,du\,dv =$$

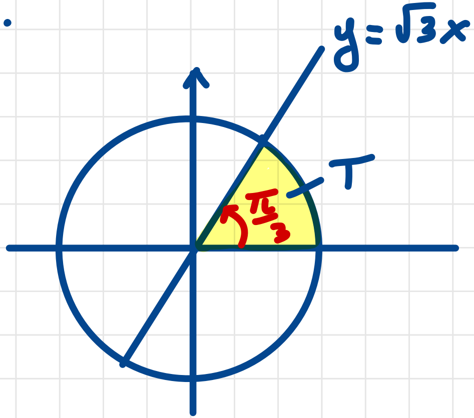
$$= \int_0^{\pi}\left(\int_0^1 u\sqrt{4u^2+1}\,du\right)dv = \frac{\pi}{8}\int_0^1 8u\,(4u^2+1)^{1/2}\,du$$

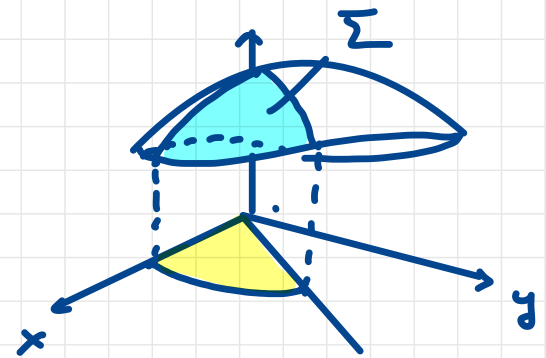
$$= \frac{\pi}{\cancel{8}\,4}\left[(4u^2+1)^{3/2}\cdot\frac{\cancel{2}}{3}\right]_0^1 = \frac{\pi}{12}\left(5^{3/2}-1\right) = \frac{\pi}{12}\left(5\sqrt{5}-1\right)$$

ESERCIZIO 9. Calcolare $\iint_{\Sigma} z \, d\sigma$ dove

Σ è la porzione di superficie di eq. $z = xy$ che si proietta in $T = \{ (x,y) \in \mathbb{R}^2 \,/\, 0 \le y \le \sqrt{3}x , \; x^2+y^2 \le 1 \}$.

SOL.





$$z = f(x,y) = xy \qquad |\nabla f| = \left\| \begin{bmatrix} y \\ x \end{bmatrix} \right\| = \sqrt{x^2+y^2}$$

$$d\sigma = \sqrt{1+x^2+y^2}\,dx\,dy$$

$$\iint_{\Sigma} z\,d\sigma = \iint_{T} xy\sqrt{1+x^2+y^2}\,dx\,dy =$$

$$= \int_0^{\pi/3}\left(\int_0^1 \rho^2 \cos\theta \,\mathrm{sen}\,\theta \sqrt{1+\rho^2}\,\rho\,d\rho\right)d\theta$$

$$= \int_0^{\pi/3} \mathrm{sen}\,\theta\cos\theta\,d\theta \cdot \int_0^1 \rho^{\cancel{3}\,2}\sqrt{1+\rho^2}\,\rho\,d\rho =$$

$$= \left[\frac{\mathrm{sen}^2\theta}{2}\right]_0^{\pi/3} \cdot \int_1^{\sqrt{2}} (t^2-1)\,t^2\,dt =$$

$$\sqrt{1+\rho^2} = t$$

$$1+\rho^2 = t^2 \longrightarrow \rho^2 = t^2-1$$

$$\cancel{2}\rho\,d\rho = \cancel{2}t\,dt$$

$$\rho\,d\rho = t\,dt$$

$$= \frac{1}{2}\left(\frac{3}{4}\right)\cdot\left[\frac{t^5}{5}-\frac{t^3}{3}\right]_1^{\sqrt{2}} = \frac{3}{8}\left(\frac{4\sqrt{2}}{5}-\frac{2\sqrt{2}}{3}-\frac{1}{5}+\frac{1}{3}\right) =$$

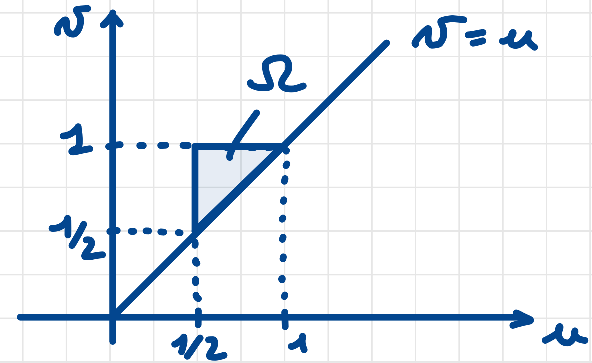
$$= \frac{\sqrt{2}+1}{20}$$

ESERCIZIO 10

Sia data la superficie $\Sigma = \{(x;y;z) \in \mathbb{R}^3 / (x;y;z) = (\operatorname{sen} uv, \cos uv, u), \ (u,v) \in \Omega\}$ essendo $\Omega = \{(u,v) \in \mathbb{R}^2 / \frac{1}{2} \leq u \leq v, \ v \leq 1\}$.

1) Calcolare Area(Σ)

2) Calcolare $\displaystyle\iint_\Sigma \frac{x^2+y^2}{z^3}\, d\sigma$



SOL.

$$\underline{r}(u,v) = (\operatorname{sen} uv, \cos uv, u)$$

$$\underline{r}_u(u,v) = (v\cos uv, -v\operatorname{sen} uv, 1)$$

$$\underline{r}_v(u,v) = (u\cos uv, -u\operatorname{sen} uv, 0)$$

$$r_u \times r_v = (u\operatorname{sen} uv, u\cos uv, 0)$$

$$|r_u \times r_v| = \sqrt{u^2} = u \qquad d\sigma = u\,du\,dv.$$

1) $$\text{Area}(\Sigma) = \iint_{\Sigma} d\sigma = \iint_{\Omega} u\,du\,dv = \int_{1/2}^{1}\left(\int_{u}^{1} u\,dv\right)du$$

$$= \int_{1/2}^{1} u(1-u)\,du = \left[\frac{u^2}{2} - \frac{u^3}{3}\right]_{1/2}^{1} = \frac{1}{2} - \frac{1}{3} - \frac{1}{8} + \frac{1}{24}$$

$$= \frac{1}{12}.$$

2) $$\iint_{\Sigma} \frac{x^2+y^2}{z^3}\, d\sigma = \iint_{\Omega} \frac{1}{u^{\cancel{3}\,2}}\, \cancel{u}\, du\, dv = \int_{\frac{1}{2}}^{1} \left(\int_{u}^{1} \frac{1}{u^2}\, dv \right) du =$$

$$= \int_{\frac{1}{2}}^{1} \frac{1}{u^2}(1-u)\, du = \int_{1/2}^{1} \left(\frac{1}{u^2} - \frac{1}{u} \right) du =$$

$$= \left[-\frac{1}{u} - \ln u \right]_{1/2}^{1} = -1 - \cancel{\ln 1} + 2 + \ln\frac{1}{2} = 1 - \ln 2.$$