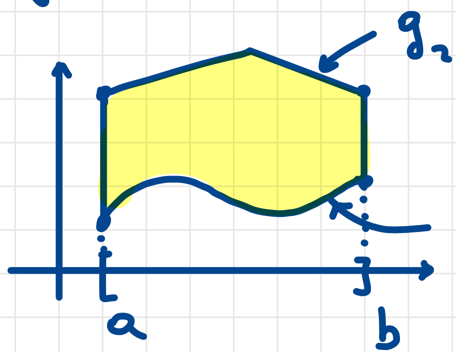
PREMESSA.

10-5-2021

Sia $E \subseteq \mathbb{R}^2$. E è y-semplice se

$$E = \{ (x,y) \in \mathbb{R}^2 \mid x \in [a,b],\ g_1(x) \leq y \leq g_2(x) \}$$

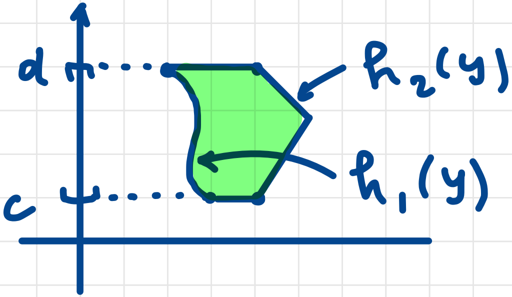
$g_1, g_2 : [a,b] \to \mathbb{R}$ continue.



$$\iint_E f(x,y)\,dx\,dy = \int_a^b \left(\int_{g_1(x)}^{g_2(x)} f(x,y)\,dy \right) dx$$

E è x-semplice se

$$E = \{ (x;y) \in \mathbb{R}^2 \mid y \in [c,d]\ \ h_1(y) \leq x \leq h_2(y) \}$$

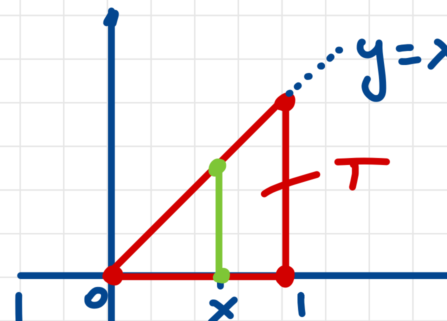


$$\iint_E f(x,y)\,dx\,dy = \int_c^d \left(\int_{h_1(y)}^{h_2(y)} f(x,y)\,dx \right) dy$$

ESERCIZIO 1. Sia T il triangolo di vertici $(0;0)$, $(1;1)$, $(1,0)$. Calcolare $\iint_T xy\,dx\,dy$.

SOL.

$$T = \{(x;y) \in \mathbb{R}^2 \,/\, x \in [0,1],\ 0 \le y \le x\}$$



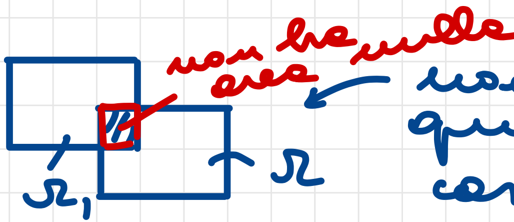
$$\iint_T xy\,dx\,dy = \int_0^1 \left(\int_0^x xy\,dy \right) dx = \int_0^1 x \left[\frac{y^2}{2} \right]_0^x dx =$$

$$= \int_0^1 x\left(\frac{x^2}{2}-0\right)dx = \frac{1}{2}\int_0^1 x^3\,dx = \frac{1}{2}\cdot\left[\frac{x^4}{4}\right]_0^1 = \frac{1}{8}.$$

NOTA. Sia Ω un dominio tale che

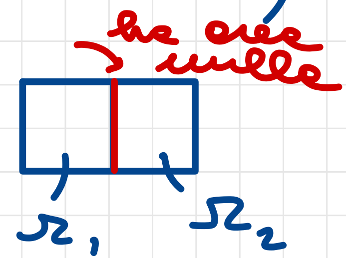
$\Omega = \Omega_1 \cup \Omega_2 \cup \dots \cup \Omega_n$; $|\Omega_i \cap \Omega_j| = 0 \quad \forall\, i \neq j$.

(gli Ω_i formano una partizione di Ω)



non è questo il caso

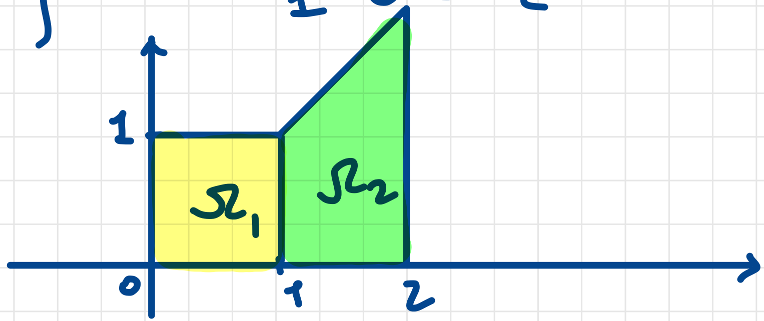
lo è:



$$\iint_{\Omega} f(x,y)\,dx\,dy = \sum_{i=1}^{n} \iint_{\Omega_i} f(x;y)\,dx\,dy.$$

ESERCIZIO 2. Calcolare $\iint_{\Omega} xy\,dx\,dy$ con:

$$\Omega = \{(x;y) \in \mathbb{R}^2 / \ x \in [0,1],\ y \in [0,1]\} \cup \{(x;y) \in \mathbb{R}^2 / \ 1 \le x \le 2,\ 0 \le y \le x\} = \Omega_1 \cup \Omega_2$$



$$\iint_{\Omega} xy\,dx\,dy = \iint_{\Omega_1} xy\,dx\,dy + \iint_{\Omega_2} xy\,dx\,dy =$$

$$= \int_0^1 \left(\int_0^1 xy\,dy \right) dx + \int_1^2 \left(\int_0^x xy\,dy \right) dx =$$

$$= \int_0^1 x\left[\frac{y^2}{2}\right]_0^1 dx + \int_1^2 x\left[\frac{y^2}{2}\right]_0^x dx = \frac{1}{2}\int_0^1 x\,dx + \frac{1}{2}\int_1^2 x^3 dx =$$

$$= \frac{1}{2}\left[\frac{x^2}{2}\right]_0^1 + \frac{1}{2}\left[\frac{x^4}{4}\right]_1^2 = \frac{1}{2}\cdot\frac{1}{2} + \frac{1}{2}\cdot\left(4-\frac{1}{4}\right) = \frac{1}{4} + \frac{15}{8} = \frac{17}{8}.$$

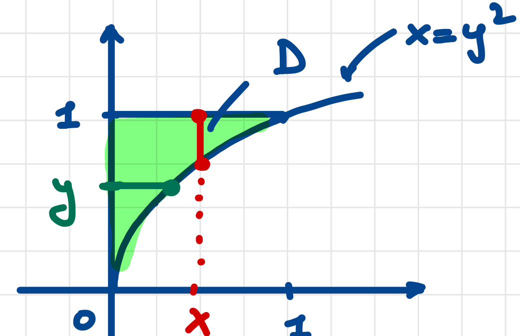
ESERCIZIO 3. Sia $D = \{(x;y) \in \mathbb{R}^2 \,/\, 0 \le x \le 1,\ \sqrt{x} \le y \le 1\}$. Calcolare $\iint_D \operatorname{sen} y^3\, dx\, dy$.

SOL.

D è y-semplice:

$$\iint_D \operatorname{sen} y^3\, dx\, dy = \int_0^1 \left(\int_{\sqrt{x}}^1 \operatorname{sen} y^3\, dy\right) dx$$

NON LO SO CALCOLARE!



Rendo il dominio x-semplice:

$$D' = \{ (x;y) \in \mathbb{R}^2 \ / \ y \in [0,1], \ 0 \le x \le y^2 \}$$

$$\iint_D \operatorname{sen} y^3 \, dx dy = \iint_{D'} \operatorname{sen} y^3 \, dx dy = \int_0^1 \left(\int_0^{y^2} \operatorname{sen} y^3 \, dx \right) dy =$$

$$= \int_0^1 \operatorname{sen} y^3 \cdot \left[x \right]_0^{y^2} dy = \frac{1}{3} \int_0^1 3 y^2 \operatorname{sen} y^3 \, dy = \frac{1}{3} \left[-\cos y^3 \right]_0^1$$

$$= -\frac{1}{3} \left(\cos 1 - 1 \right)$$

ESERCIZIO 4. Sia

$$\iint_D f \, dx dy = \underbrace{\int_0^1 \left(\int_{-3x}^{x^4} f \, dy \right) dx}_{1} + \underbrace{\int_1^2 \left(\int_{x^2-4}^{(2-x)^3} f \, dy \right) dx}_{2}$$

Rappresentare D e scambiare l'ordine di integrazione.

Sol.

$D_1 = \{ (x;y) \in \mathbb{R}^2 / x \in [0,1], -3x \le y \le x^4 \}$

$D_2 = \{ (x;y) \in \mathbb{R}^2 / x \in [1;2], x^2-4 \le y \le (2-x)^3 \}$

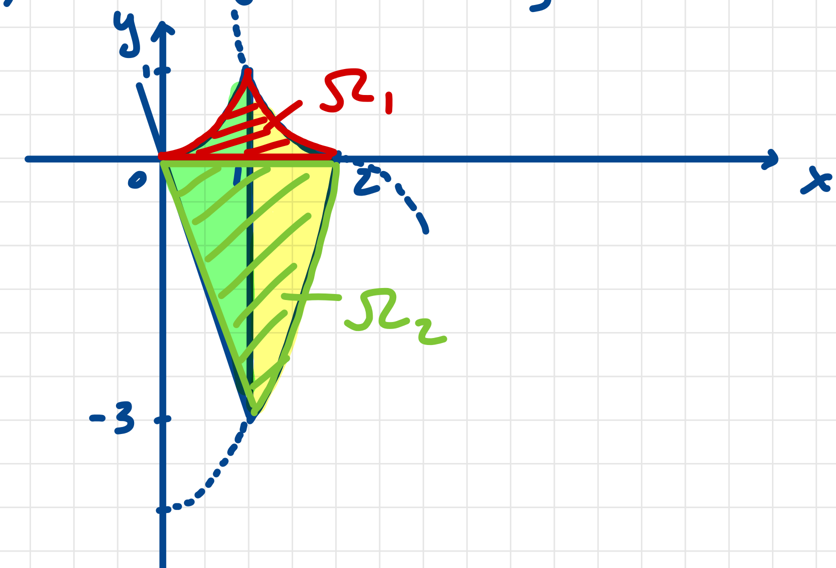
$\Rightarrow D = D_1 \cup D_2$

$y = -3x \longrightarrow x = -\frac{1}{3} y$

$y = x^4 \longrightarrow x = \sqrt[4]{y}$

$y = x^2 - 4 \longrightarrow x = \sqrt{y+4}$

$y = (2-x)^3 \longrightarrow x = 2 - \sqrt[3]{y}$



$$\Omega_1 = \left\{ (x;y) \in \mathbb{R}^2 / \ y \in [0,1], \ \sqrt[4]{y} \le x \le 2 - \sqrt[3]{y} \right\}$$

$$\Omega_2 = \left\{ (x;y) \in \mathbb{R}^2 / \ y \in [-3;0], \ -\frac{1}{3} y \le x \le \sqrt{y+4} \right\}$$

$$\iint_D f \, dx dy = \iint_\Omega f \, dx dy = \int_0^1 \left(\int_{\sqrt[4]{y}}^{2-\sqrt[3]{y}} f \, dx \right) dy + \int_{-3}^{0} \left(\int_{-\frac{1}{3}y}^{\sqrt{y+4}} f \, dx \right) dy$$

INTEGRALI DOPPI E PARTICOLARI SIMMETRIE

- f è dispari rispetto all'asse y se $f(-x;y) = -f(x;y)$
- f è " " " x se $f(x;-y) = -f(x;y)$
- f è simmetrica rispetto a $(0;0)$ se $f(-x;-y) = -f(x;y)$

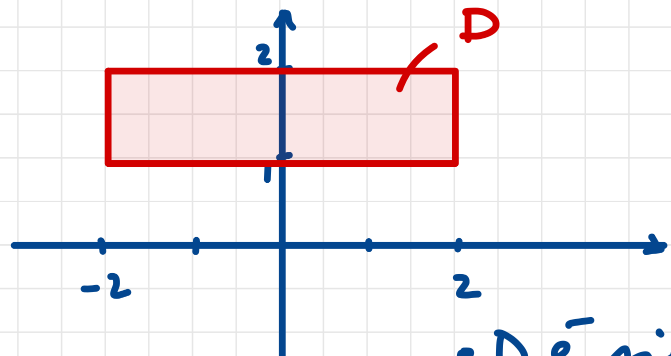
Se D è un dominio simmetrico rispetto all'asse y (x) e se f è dispari rispetto all'asse y (x) allora $\iint_D f\,dx\,dy = 0$.

Se D è simmetrico rispetto a (0,0) e lo è anche f allora $\iint_D f = 0$.

ESERCIZIO 5. Calcolare $\iint_D \left(2+3x\log(x^4+y^3)\right)dx\,dy$

essendo $D=[-2;2]\times[1,2]$.

SOL.



$$\iint_D \left(2+3x\log(x^4+y^3)\right)dx\,dy =$$

$$= \iint_D 2\,dx\,dy + \iint_D \underbrace{3x\log(x^4+y^3)}_{g(x;y)}\,dx\,dy = (*)$$

- D è simm. risp. asse y
- g è disp. risp. asse y

$g(-x;y) = -3x\log(x^4+y^3) = -g(x,y)$

$$\Longrightarrow \quad \iint_D g \, dx\,dy = 0.$$

$(*) = 2\iint_D dx\,dy = 2\,\text{Area}(D) = 2(4\cdot 1) = 8.$

INTEGRALI DOPPI E VALORI ASSOLUTI

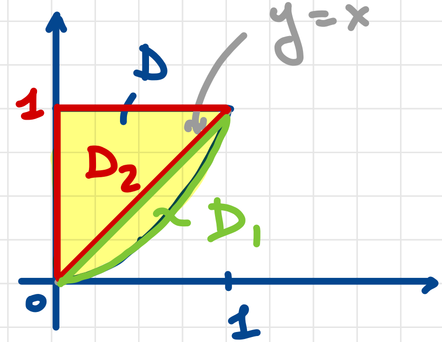
ESERCIZIO 6. Calcolare

$$\iint_D |x-y| \, dx\,dy$$

essendo $D = \{(x;y)\in\mathbb{R}^2 \,/\, 0\le x\le 1 \wedge x^2\le y\le 1\}$

SOL.

$$|x-y| = \begin{cases} x-y & \text{se } \; x-y \geq 0 \rightarrow \underline{y \leq x} \\ y-x & \text{se } \; x-y < 0 \rightarrow \underline{y > x} \end{cases}$$



$$\iint_D f \, dx\,dy = \iint_{D_1} (x-y)\, dx\,dy + \iint_{D_2} (y-x)\, dx\,dy$$

$$= \int_0^1 \left(\int_{x^2}^{x} (x-y)\, dy \right) dx + \int_0^1 \left(\int_x^1 (y-x)\, dy \right) dx =$$

$$= \int_0^1 \left[xy - \frac{y^2}{2} \right]_{x^2}^{x} dx + \int_0^1 \left[\frac{y^2}{2} - xy \right]_x^1 dx =$$

$$= \int_0^1 \left(x^2 - \cancel{\frac{x^2}{2}} - x^3 + \frac{x^4}{2} \right) dx + \int_0^1 \left(\frac{1}{2} - x \cancel{- \frac{x^2}{2}} + \cancel{x^2} \right) dx$$

$$= \left[\frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{10} + \frac{1}{2}x - \frac{x^2}{2} \right]_0^1 = \frac{1}{3} - \frac{1}{4} + \frac{1}{10} + \cancel{\frac{1}{2}} - \cancel{\frac{1}{2}} = \frac{11}{60}.$$

CAMBIO DI VARIABILI NEGLI INTEGRALI DOPPI

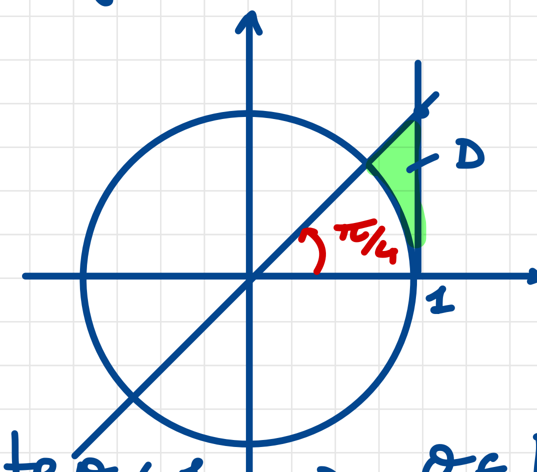
ESERCIZIO 7. Calcolare $\iint_D \frac{x}{x^2+y^2}\,dx\,dy$ con

$$D = \{ (x;y) \in \mathbb{R}^2 \,/\, x^2+y^2 \geq 1,\ 0 \leq y \leq x,\ 0 \leq x \leq 1 \}$$

SOL.

$$D: \begin{cases} x^2+y^2 \geq 1 \\ 0 \leq y \leq x \\ 0 \leq x \leq 1 \end{cases} \qquad \begin{cases} x = \rho\cos\theta \\ y = \rho\,\mathrm{sen}\,\theta \end{cases}$$

$$dx\,dy = \rho\,d\rho\,d\theta$$



$$D': \begin{cases} \rho^2 \geq 1 \to \rho \geq 1 \\ 0 \leq \rho\,\mathrm{sen}\,\theta \leq \rho\cos\theta \to 0 \leq \mathrm{tg}\,\theta \leq 1 \to \theta \in [0, \frac{\pi}{4}] \\ 0 \leq \rho\cos\theta \leq 1 \to 0 \leq \rho \leq \frac{1}{\cos\theta} \end{cases}$$

$$D' = \left\{ (\rho,\theta) \,/\, 0 \le \theta \le \frac{\pi}{4} \,,\ 1 \le \rho \le \frac{1}{\cos\theta} \right\} \rightarrow \rho\text{-semplice}$$

$$\iint_D f \, dx\,dy = \iint_{D'} \frac{\cancel{\rho}\cos\theta}{\cancel{\rho^2}} \cancel{\rho}\, d\rho\, d\theta = \int_0^{\pi/4} \left(\int_1^{\frac{1}{\cos\theta}} \cos\theta \, d\rho \right) d\theta$$

$$= \int_0^{\pi/4} \cos\theta \cdot [\rho]_1^{\frac{1}{\cos\theta}} d\theta = \int_0^{\pi/4} \cos\theta \left(\frac{1}{\cos\theta} - 1 \right) d\theta =$$

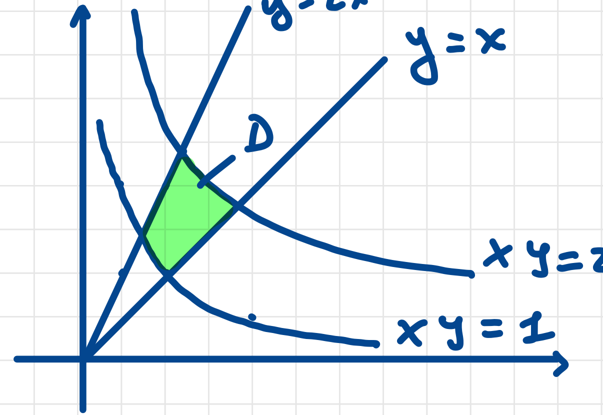
$$= \int_0^{\pi/4} (1 - \cos\theta)\, d\theta = \left[\theta - \operatorname{sen}\theta \right]_0^{\pi/4} = \frac{\pi}{4} - \frac{\sqrt{2}}{2}$$

ESERCIZIO 8.

Calcolare $\displaystyle\iint_D \frac{x}{y} \operatorname{sen} xy \, dx\,dy$ con D regione in figura.

SOL.

- $y = x \rightarrow \frac{x}{y} = 1$
- $y = 2x \longrightarrow \frac{x}{y} = \frac{1}{2}$
- $xy = 1$
- $xy = 2$



$$F^{-1}: \begin{cases} u = xy \\ v = \frac{x}{y} \end{cases}$$

$u \in [1;2]$ $\quad v \in [\frac{1}{2};1]$

$$F: \begin{cases} x = g(u,v) \\ y = h(u,v) \end{cases}$$

NON SERVE CALCOLARLA

$$J_{F^{-1}} = \begin{bmatrix} y & x \\ \frac{1}{y} & -\frac{x}{y^2} \end{bmatrix} \qquad \det J_{F^{-1}} = -\frac{x}{y} - \frac{x}{y} = -2\frac{x}{y}$$

$$\det J_F(u,v) = \frac{1}{\det J_{F^{-1}}} = -\frac{y}{2x} = -\frac{1}{2}\left(\frac{y}{x}\right) = -\frac{1}{2v}$$

$$dx\,dy = \left| -\frac{1}{2v} \right| du\,dv = \frac{1}{2v}\,du\,dv.$$

$$\iint_D \frac{x}{y} \operatorname{sen} xy \, dx\,dy = \iint_{D'} \cancel{v} \operatorname{sen} u \cdot \frac{1}{2\cancel{v}}\, du\,dv$$

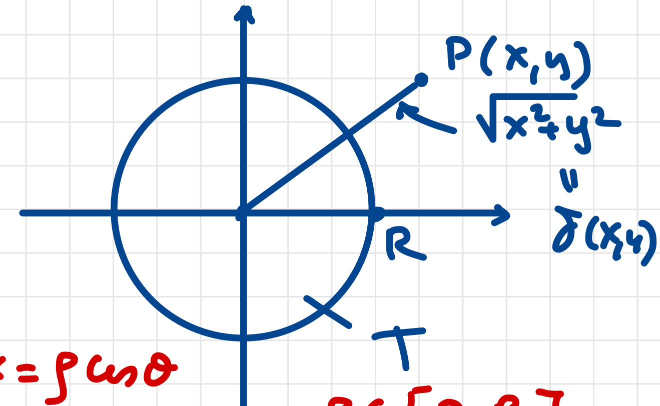
$$D' = [1,2]\times\left[\tfrac{1}{2};1\right]$$

$$= \frac{1}{2}\int_{1/2}^{1}\left(\int_1^2 \operatorname{sen} u \, du\right) dv = \frac{1}{2}\int_{1/2}^{1} dv \cdot \int_1^2 \operatorname{sen} u \, du =$$

$$= \frac{1}{2}\cdot\frac{1}{2}\left[-\cos u\right]_1^2 = \frac{1}{4}\left(-\cos 2 + \cos 1\right) = \frac{\cos 1 - \cos 2}{4}$$

ESERCIZIO 9. Dimostrare che il momento di inerzia di una lamina omogenea di massa M e raggio R è $\frac{1}{2}MR^2$ in rotazione attorno all'asse z.

SOL.



$$I=\frac{M}{Area(T)}\iint_T \delta^2(x;y)\,dx\,dy=$$

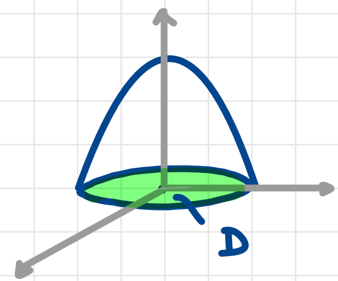
$$=\frac{M}{\pi R^2}\iint_T (x^2+y^2)\,dx\,dy=$$

$$\begin{cases} x=\rho\cos\theta \\ y=\rho\,\mathrm{sen}\,\theta \end{cases}\qquad \rho\in[0,R] \quad \theta\in[0,2\pi]$$

$$=\frac{M}{\pi R^2}\int_0^{2\pi}\left(\int_0^R \rho^2\rho\,d\rho\right)d\theta=$$

$$=\frac{M}{\cancel{\pi}R^2}\cdot 2\cancel{\pi}\cdot\left[\frac{\rho^4}{4}\right]_0^R=\frac{\cancel{2}M}{\cancel{R^2}}\cdot\frac{R^{\cancel{4}\,2}}{\cancel{4}\,2}=\frac{1}{2}MR^2.$$

ESERCIZIO 10. Calcolare il volume della porzione di spazio racchiusa tra il paraboloide $z = \underbrace{1 - x^2 - y^2}_{f(x;y)}$ e il piano x, y $(z = 0)$.



SOL.

$$\partial D : \begin{cases} z = 1 - x^2 - y^2 \\ z = 0 \end{cases} \to x^2 + y^2 = 1$$

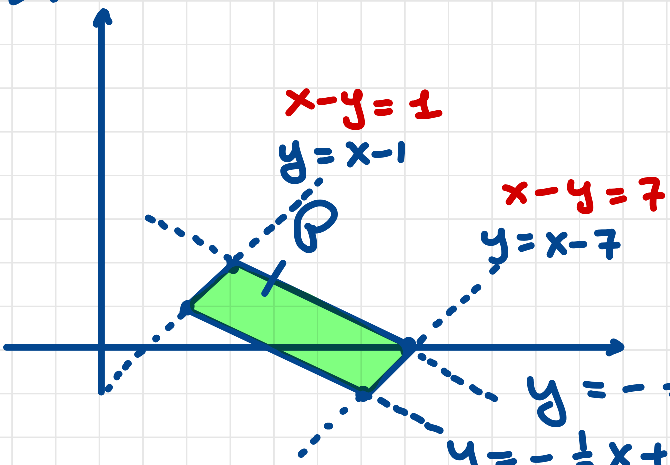
$$D = \{ (x;y) \in \mathbb{R}^2 \,/\, x^2 + y^2 \leq 1 \}$$

$$Vol = \iint_D f \, dx\,dy = \iint_D [1 - (x^2 + y^2)] \, dx\,dy = \int_0^{2\pi} \left(\int_0^1 (1 - \rho^2) \rho \, d\rho \right) d\theta$$

$$= 2\pi \int_0^1 (\rho - \rho^3) \, d\rho = 2\pi \left[\frac{\rho^2}{2} - \frac{\rho^4}{4} \right]_0^1 = 2\pi \left(\frac{1}{2} - \frac{1}{4} \right) = \frac{\pi}{2}.$$

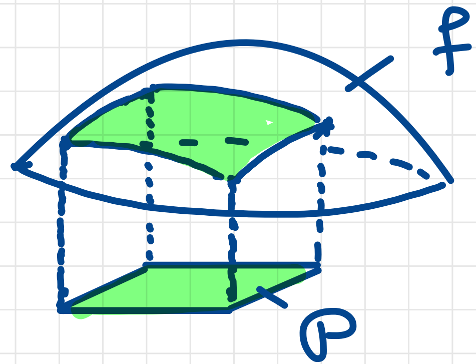
ESERCIZIO 11. Calcolare il volume del cilindroide con generatrici parallele all'asse z compreso tra il parallelogramma P di vertici $(2;1)$, $(6;-1)$, $(7;0)$, $(3;2)$ e la porzione di superficie di eq. $z = e^{x+2y}$ che si proietta su P.

SOL.



$y = -\frac{1}{2}x + \frac{7}{2} \longrightarrow x+2y = 7$

$y = -\frac{1}{2}x + 2 \longrightarrow x+2y = 4$



$$\text{Vol} = \iint_P f \, dx\,dy$$, cambio di coordinate:

$$\begin{cases} x+2y = u \\ x-y = v \end{cases} \qquad \begin{matrix} u \in [4;7] \\ v \in [1;7] \end{matrix}$$

$$F^{-1}: \begin{cases} u = x+2y \\ v = x-y \end{cases} \qquad R = [4;7]\times[1;7]$$

$$J_{F^{-1}}(x;y) = \begin{bmatrix} 1 & 2 \\ 1 & -1 \end{bmatrix} \qquad \det J_{F^{-1}} = -3$$

$$\det J_F = -\frac{1}{3}$$

$$dx\,dy = \frac{1}{3}\,du\,dv.$$

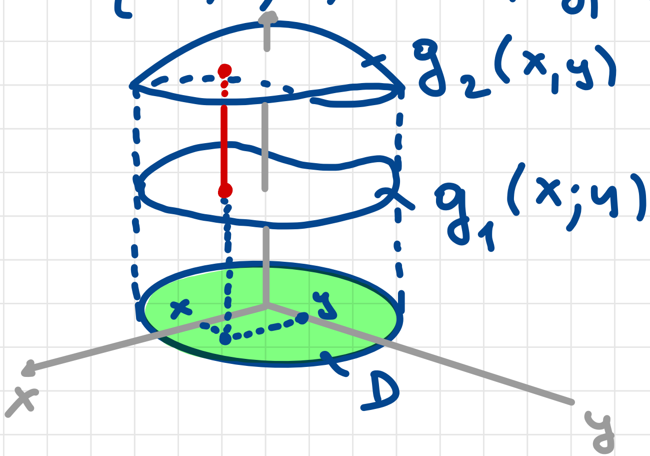
$$\text{Vol} = \iint_P e^{x+2y}\,dx\,dy = \iint_R e^{u} \cdot \frac{1}{3}\,du\,dv = \frac{1}{3}\int_1^7 dv \cdot \int_4^7 e^u du$$

$$= \frac{1}{3}\cdot 6\cdot\left[e^u\right]_4^7 = 2\left(e^7-e^4\right).$$

INTEGRALI TRIPLI

1) "PER FILI"

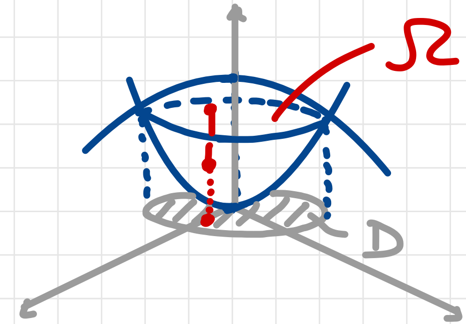
$$\Omega=\left\{(x;y;z)\in\mathbb{R}^3 \,/\, g_1(x;y)\le z\le g_2(x;y),\quad (x;y)\in D\right\}$$



$$\iiint_\Omega f\,dx\,dy\,dz=\iint_D\left(\int_{g_1(x;y)}^{g_2(x;y)} f\,dz\right)dx\,dy$$

ESERCIZIO 12. Calcolare il volume della porzione di spazio racchiusa tra i paraboloidi $z = x^2+y^2$ e $z = \frac{4}{3} - \frac{x^2+y^2}{3}$.

Sol.



$$\partial D : \begin{cases} z = x^2+y^2 \\ z = \frac{4}{3} - \frac{x^2+y^2}{3} \end{cases}$$

$$x^2+y^2 + \frac{x^2+y^2}{3} = \frac{4}{3}$$

$$\frac{4x^2+4y^2}{\cancel{3}} = \frac{4}{\cancel{3}} \rightarrow x^2+y^2 = 1 \qquad D = \{ x^2+y^2 \leq 1 \}$$

$$Vol(\Omega) = \iiint_\Omega 1 \, dx\,dy\,dz = \iint_D \left(\int_{x^2+y^2}^{\frac{4}{3} - \frac{x^2+y^2}{3}} dz \right) dx\,dy =$$

$$= \iint_D \left(\frac{4}{3} - \frac{x^2+y^2}{3} - (x^2+y^2) \right) dx\,dy =$$

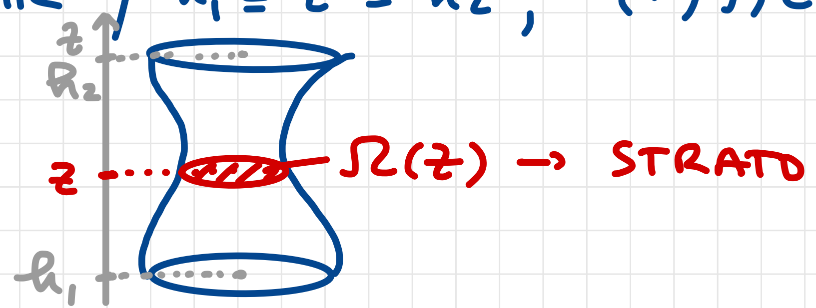
$$= \iint_D \left(\frac{4}{3} - \frac{4x^2+4y^2}{3} \right) dx\,dy = \frac{4}{3} \iint_D \left(1-(x^2+y^2)\right) dx\,dy =$$

$$= \frac{4}{3} \int_0^{2\pi} \left(\int_0^1 (1-\rho^2)\rho\, d\rho \right) d\theta = \frac{4}{3} \cdot 2\pi \left[\frac{\rho^2}{2} - \frac{\rho^4}{4} \right]_0^1 =$$

$$= \frac{8}{3}\pi \left(\frac{1}{2} - \frac{1}{4} \right) = \frac{8}{3}\pi \cdot \frac{1}{4} = \frac{2}{3}\pi .$$

2) "PER STRATI"

$$\Omega = \left\{ (x;y;z) \in \mathbb{R}^3 \,/\, h_1 \le z \le h_2 ,\quad (x;y) \in \Omega_z \right\}$$



$$\iiint_{\Omega} f\,dx\,dy\,dz = \int_{h_1}^{h_2} \left(\iint_{\Omega(z)} f\,dx\,dy \right) dz$$

ESERCIZIO 13. Calcolare $\iiint_{\Omega} z\,dx\,dy\,dz$ con

$$\Omega = \{ (x;y;z) \in \mathbb{R}^3 \,/\, \underbrace{x^2 + (y-z)^2 \leq 1}_{\text{STRATO } \Omega(z)}, \quad 0 \leq z \leq 1 \}.$$

SOL.

$$\iiint_{\Omega} z\,dx\,dy\,dz = \int_0^1 \left(\iint_{\Omega(z)} z\,dx\,dy \right) dz = (*)$$

$\Omega(z)$: cerchio di centro $(0;z)$ e raggio 1

$$(*) = \int_0^1 z \overbrace{\left(\iint_{\Omega(z)} dx\,dy \right)}^{= \text{Area}(\Omega(z)) = \pi} dz = \int_0^1 z \cdot \pi \, dz = \pi \left[\frac{z^2}{2} \right]_0^1 = \frac{\pi}{2}.$$