Esercitazione 9: sistemi a 2-gdl lineari

Meccanica Applicata (FIS)

16/01/2017

Es. 1

Il sistema in Figura 1 é raffigurato in posizione di equilibrio statico. Si chiede di:

- Scrivere l'equazione di moto mediante equazioni di Lagrange.
- Calcolare frequenze proprie e modi di vibrare utilizzando i seguenti dati: $m_1 = 2m$, $m_2 = m$, $J_1 = \frac{2}{3}mR_1^2$, $k_1 = k$, $k_2 = \frac{4}{3}k$, $k_3 = 3k$. k=3000 N/m; m=3 kg, $r_j = k_j/1000$;
- Ampiezza di vibrazione a regime in fiunzioe di Ω, considerando $C(t) = C0\sin(\Omega t)$ e $C_0/R =$ 1000 N.

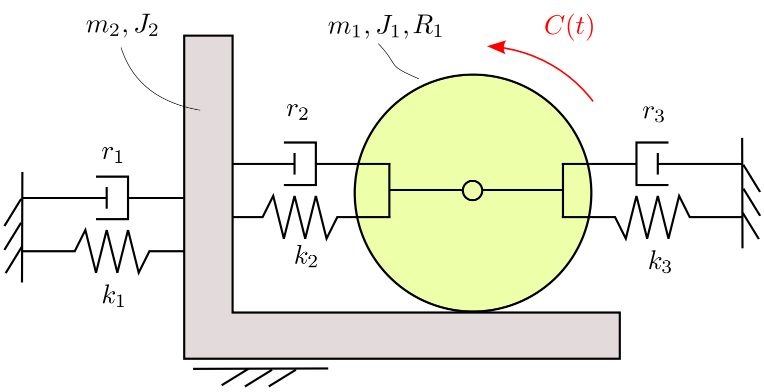
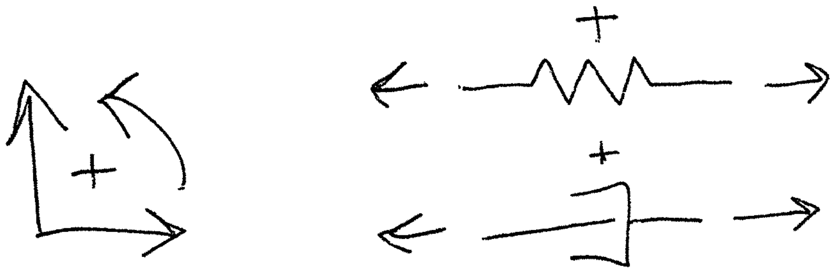


Figura 1:

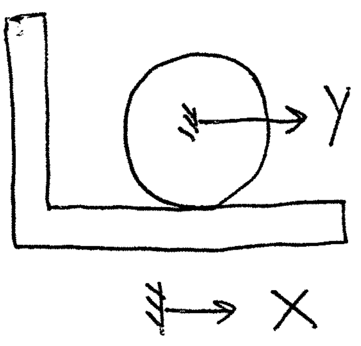
1 CONVENZIONI



2 GDL e COORDINATE LIBERE

• 2 CORPI RIGIDI	6 GDL −
• PATTINO	2
• CONTATTO PURO ROTOLAMENTO	2
TOTALE	2 GDL RESIDUI

⇒ 2 COORD LIBERE: X e Y



(TRASLAZIONE ASSOLUTA CENTRO DISCO)

(TRASLAZ. ASS.)

③ LEGAMI CINEMATICI

POICHÉ IL SISTEMA È LINEARE, APPLICO IL PRINCIPIO DI SOVRAPPOSIZIONE DEGLI EFFETTI:

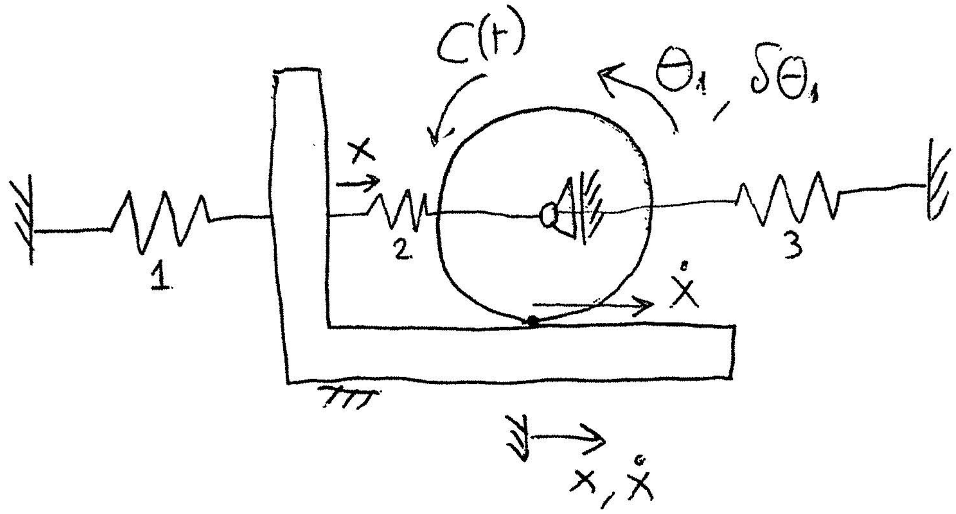
1) EFFETTO DI X (CON $y=0$)

2) EFFETTO DI y (CON $x=0$)

3) SOMMO GLI EFFETTI DI X e Y

EFFETTO DI X $(y=0)$

SISTEMA EQUIVALENTE



DISCO m_1

$V_1 = 0$

$\dot{\theta}_1 = \dfrac{\dot{x}}{R}$

SLITTA m_2

$V_2 = \dot{x}$

$\dot{\theta}_2 = 0$

MOLLA 1

$\Delta l_1 = x$

$\Delta \dot{l}_1 = \dot{x}$

MOLLA 2

$\Delta l_2 = -x$

$\Delta \dot{l}_2 = -\dot{x}$

MOLLA 3

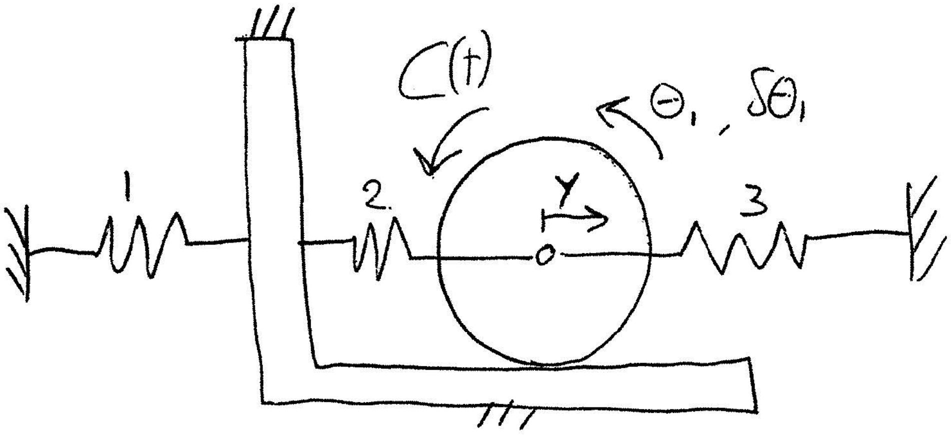
$\Delta l_3 = 0$

$\Delta \dot{l}_3 = 0$

SPOST. VIRTUALI

$\delta\theta_1 = \dfrac{\delta x}{R}$

EFFETTO DI y $(x=0)$



DISCO m_1

$V_1 = \dot{y}$

$\dot{\theta}_1 = -\frac{\dot{y}}{R}$

SLITTA m_2

$V_2 = 0$

$\dot{\theta}_2 = 0$

MOLLA 1

$\Delta l_1 = 0$

$\Delta \dot{l}_1 = 0$

MOLLA 2

$\Delta l_2 = y$

$\Delta \dot{l}_2 = \dot{y}$

MOLLA 3

$\Delta l_3 = -y$

$\Delta \dot{l}_3 = -\dot{y}$

SPOST VIRT

$\delta\theta_1 = -\frac{\delta y}{R}$

SOMMO GLI EFFETTI

DISCO m_1

$$V_1 = \mathring{y}$$

$$\mathring{\theta}_1 = \frac{\mathring{x}}{R} - \frac{\mathring{y}}{R}$$

SLITTA m_2

$$V_2 = \mathring{x}$$

$$\mathring{\theta}_2 = 0$$

MOLLA 1

$$\Delta l_1 = x \quad , \quad \Delta \mathring{l}_1 = \mathring{x}$$

MOLLA 2

$$\Delta l_2 = y - x \quad , \quad \Delta \mathring{l}_2 = \mathring{y} - \mathring{x}$$

MOLLA 3

$$\Delta l_3 = -y$$

$$\Delta \mathring{l}_3 = -\mathring{y}$$

SPOST VIRT

$$\delta\theta_1 = \frac{\delta x}{R} - \frac{\delta y}{R}$$

Eq di Moto (Lagrange)

$$\begin{cases} \dfrac{d}{dt}\left(\dfrac{\partial E_c}{\partial \dot{x}}\right) - \dfrac{\partial E_c}{\partial x} + \dfrac{\partial D}{\partial \dot{x}} + \dfrac{\partial V}{\partial x} = Q_x \\[2ex] \dfrac{d}{dt}\left(\dfrac{\partial E_c}{\partial \dot{y}}\right) - \dfrac{\partial E_c}{\partial y} + \dfrac{\partial D}{\partial \dot{y}} + \dfrac{\partial V}{\partial y} = Q_y \end{cases}$$

$$E_c = \frac{1}{2} m_1 V_1^2 + \frac{1}{2} m_2 V_2^2 + \frac{1}{2} J_1 \theta_1^2 + \frac{1}{2} J_2 \dot{\theta}_2^2$$

$$= \frac{1}{2} m_1 \dot{y}^2 + \frac{1}{2} m_2 \dot{x}^2 + \frac{1}{2} J_1 \left(\frac{\dot{x}}{R} - \frac{\dot{y}}{R}\right)^2 + \emptyset$$

$$\frac{\partial E_c}{\partial x} = \frac{\partial E_c}{\partial y} = 0$$

$$\frac{\partial E_c}{\partial \dot{x}} = m_2 \dot{x} + J_1 \left(\frac{\dot{x}}{R} - \frac{\dot{y}}{R}\right) \frac{1}{R} = m_2 \dot{x} + \frac{J_1}{R^2} (\dot{x} - \dot{y})$$

$$\frac{d}{dt}\left(\frac{\partial E_c}{\partial \dot{x}}\right) = m_2 \ddot{x} + \frac{J_1}{R^2} (\ddot{x} - \ddot{y}) = \left(m_2 + \frac{J_1}{R^2}\right) \ddot{x} - \left(\frac{J_1}{R^2}\right) \ddot{y}$$

$$\frac{\partial E_c}{\partial \dot{y}} = m_1 \dot{y} + J_1 \left(\frac{\dot{x}}{R} - \frac{\dot{y}}{R}\right)\left(-\frac{1}{R}\right)$$

$$\frac{d}{dt}\left(\frac{\partial E_c}{\partial \dot{y}}\right) = \left(m_1 + \frac{J_1}{R^2}\right) \ddot{y} - \frac{J_1}{R^2} \ddot{x}$$

Umbria experience
Distretto Integrato Turistico

Agenzia per la Coesione Territoriale

Programma Attuativo Regionale Fondo per lo Sviluppo e la Coesione
FSC 2007-2013 Umbria

Regione Umbria

$$V = \frac{1}{2}K_1 \Delta l_1^2 + \frac{1}{2}K_2 \Delta l_2^2 + \frac{1}{2}K_3 \Delta l_3^2$$

$$= \frac{1}{2}K_1 x^2 + \frac{1}{2}K_2 (y-x)^2 + \frac{1}{2}K_3(-y)^2$$

$$\frac{\partial V}{\partial x} = K_1 x + K_2 (y-x)(-1)$$

$$= (K_1 + K_2)\, x - K_2 y$$

$$\frac{\partial V}{\partial y} = K_2 (y-x) + K_3 (-y)(-1)$$

$$= -K_2 x + (K_2 + K_3) y$$

$$D = \frac{1}{2} r_1 \Delta \dot{l}_1^{\,2} + \frac{1}{2} r_2 \Delta \dot{l}_2^{\,2} + \frac{1}{2} r_3 \Delta \dot{l}_3^{\,2}$$

$$= \frac{1}{2} r_1 \dot{x}^2 + \frac{1}{2} r_2 (\dot{y} - \dot{x})^2 + \frac{1}{2} r_3 (-\dot{y})^2$$

$$\frac{\partial D}{\partial \dot{x}} = r_1 \dot{x} + r_2 (\dot{y} - \dot{x})(-1) =$$

$$= (r_1 + r_2)\dot{x} - r_2 \dot{y}$$

$$\frac{\partial D}{\partial \dot{y}} = r_2 (\dot{y} - \dot{x}) + r_3 \dot{y} =$$

$$= -r_2 \dot{x} + (r_2 + r_3)\dot{y}$$

$$\delta L = C\,\delta\theta_1 = C\left(\frac{\delta x}{R} - \frac{\delta y}{R}\right)$$

$$= \left(\frac{C}{R}\right)\delta x + \left(-\frac{C}{R}\right)\delta y$$

$$= Q_x\,\delta x + Q_y\,\delta y$$

EQ DI MOTO:

$$\begin{cases} \left(m_2 + \dfrac{J_1}{R_2}\right)\ddot{x} - \left(\dfrac{J_1}{R^2}\right)\ddot{y} + (r_1 + r_2)\dot{x} - r_2\dot{y} + (k_1 + k_2)x - k_2 y = \dfrac{C}{R} \\[2ex] -\dfrac{J_1}{R^2}\ddot{x} + \left(m_1 + \dfrac{J_1}{R^2}\right)\ddot{y} + (-r_2)\dot{x} + (r_2 + r_3)\dot{y} - k_2 x + (k_2 + k_3)y = -\dfrac{C}{R} \end{cases}$$

FORMA MATRICIALE

$$\begin{bmatrix} m_2 + \dfrac{J_1}{R^2} & -\dfrac{J_1}{R_2} \\[2ex] -\dfrac{J_1}{R^2} & m_1 + \dfrac{J_1}{R^2} \end{bmatrix} \begin{bmatrix} \ddot{x} \\[2ex] \ddot{y} \end{bmatrix} + \begin{bmatrix} r_1 + r_2 & -r_2 \\[2ex] -r_2 & r_2 + r_3 \end{bmatrix} \begin{bmatrix} \dot{x} \\[2ex] \dot{y} \end{bmatrix} +$$

$$\begin{bmatrix} K_1 + K_2 & -K_2 \\[2ex] -K_2 & K_2 + K_3 \end{bmatrix} \begin{bmatrix} x \\[2ex] y \end{bmatrix} = \begin{bmatrix} \dfrac{C}{R} \\[2ex] -\dfrac{C}{R} \end{bmatrix}$$

$$\underline{z} = \begin{bmatrix} x \\ y \end{bmatrix} \qquad \dot{\underline{z}} = \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} \qquad \ddot{\underline{z}} = \begin{bmatrix} \ddot{x} \\ \ddot{y} \end{bmatrix}$$

$$[M]\,\ddot{\underline{z}} + [R]\,\dot{\underline{z}} + [K]\,\underline{z} = \underline{F}$$

SOSTITUENDO I DATI:

$$[M] = \begin{bmatrix} 5 & -2 \\ -2 & 8 \end{bmatrix} \; ; \; [K] = \begin{bmatrix} 7000 & -4000 \\ -4000 & 13\,000 \end{bmatrix}$$

$$[R] = \begin{bmatrix} 7\cancel{0} & -4\cancel{0} \\ -4\cancel{0} & 13\cancel{0} \end{bmatrix}$$

FREQUENZE PROPRIE

$$[M]\ddot{\underline{z}} + [K]\underline{z} = \underline{0}$$

SISTEMA LIBERO NON SMORZATO

$$\underline{z} = \underline{\phi}\, e^{i\omega t}$$

$$\ddot{\underline{z}} = \underline{\phi}\,(-\omega^2)\, e^{i\omega t}$$

SOLUZIONE TEST

SOSTITUISCO

$$\underbrace{\left(-\omega^2[M] + [K]\right)}\underline{\phi}\, e^{i\omega t} = \underline{0}$$

AFFINCHÉ ESISTA LA SOLUZIONE NON BANALE ($\underline{\phi} \neq \underline{0}$)

$$\det\left(-\omega^2[M] + [K]\right) = 0 \quad \Longleftrightarrow \quad \text{EQ. CARATTERISTICA}$$

$$\begin{vmatrix} 7000 - 5\omega^2 & -4000 + 2\omega^2 \\ -4000 + 2\omega^2 & 13\,000 - 8\omega^2 \end{vmatrix} = 0$$

$$36\,\omega^4 - 105\,000\,\omega^2 + 75 \cdot 10^6 = 0$$

$$\omega^2_{I,II} = \begin{cases} 1250 \\ 1667 \end{cases}$$

$$\begin{cases} \omega_I = \pm 35 \ \frac{rad}{s} \\ \omega_{II} = \pm 41 \ \frac{rad}{s} \end{cases}$$

PULSAZIONI PROPRIE

FREQUENZE

$$\omega = 2\pi f \rightarrow f = \frac{\omega}{2\pi}$$

RISPOSTA IN FREQUENZA A REGIME

$$[M]\ddot{\underline{z}} + [R]\dot{\underline{z}} + [K]\underline{z} = \underline{F} = \begin{bmatrix} \frac{C_0}{R} \\ -\frac{C_0}{R} \end{bmatrix} \sin \Omega T =$$

$$= \underline{F}_0 \sin \Omega T$$

$$= \underline{F}_0 \, e^{i\Omega T}$$

METODO SOMIGLIANZA

$$\underline{z} = \underline{C} \, e^{i\Omega T}$$

$$\underline{z} = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} C_x \\ C_y \end{bmatrix} e^{i\Omega T}$$

$$\dot{\underline{z}} = \underline{C}(i\Omega)\, e^{i\Omega T} \; ; \quad \ddot{\underline{z}} = \underline{C}(-\Omega^2)\, e^{i\Omega T}$$

$$\underbrace{\left(-\Omega^2[M] + [R](i\Omega) + [K]\right)}_{[D]} \underline{C} = F_0$$

$$\underline{C} = [D]^{-1} F_0$$

$$\underline{z} = |\underline{C}|\, e^{i\varphi} e^{i\Omega T} = |\underline{C}| \sin(\Omega T + \varphi)$$

AD ESEMPIO

$\Omega = 0$ (FORZA COSTANTE) $\qquad \dfrac{C_0}{R} = 1000 \text{ N}$

$$\underline{C} = \begin{bmatrix} x \\ y \end{bmatrix} = [K]^{-1} F_0 = \begin{bmatrix} 0.12 \\ -0.04 \end{bmatrix}$$

$\Omega = 1000$ rad/s

$$\underline{C} = [D]^{-1} \underline{F}_0 = \begin{bmatrix} -0.166 - 0.002\, i \\ 0.083 + 0.001\, i \end{bmatrix} 10^{-3}$$

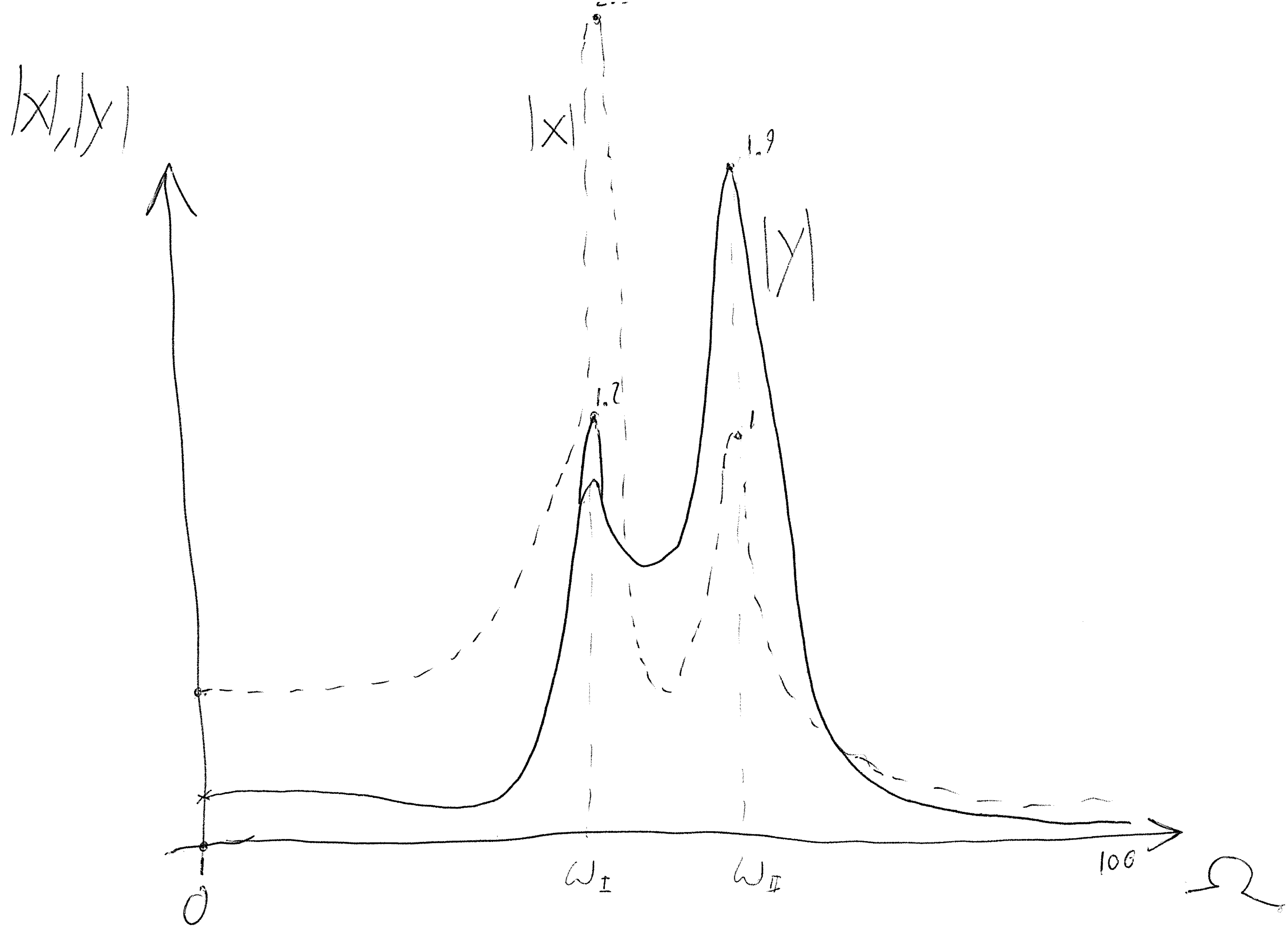
$$|\underline{C}| = \begin{bmatrix} 0.166 \\ 0.083 \end{bmatrix} 10^{-3}$$

$\Omega \approx \omega_I = 35$ rad/s

$$|\underline{C}| = \begin{bmatrix} 2.1 \\ 0.84 \end{bmatrix}$$

$\Omega \approx \omega_2 = 41$

$$|\underline{C}| = \begin{bmatrix} 1.05 \\ 1.87 \end{bmatrix}$$



|X|,|Y|
|X|
1.9
|Y|
1.2
0
ω_I
ω_II
100
Ω