Great Theoretical Ideas In Computer Science

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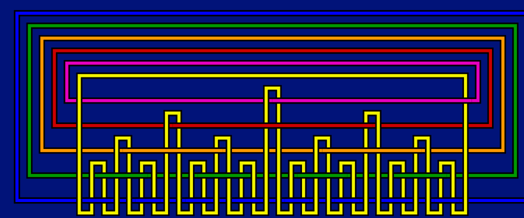
Mathematical Games II
Sums of Games

$4 \oplus 2 = 6$



This is a Brunnian Link, which shows how one loop of rope and five rings can be attached together in such a way that if any ring or rope is removed, all the remaining rinks and ropes become separated.

Formidable Fourteen Puzzle

You're given fourteen disks with the following diameters in inches:

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2.150 2.250 2.308 2.348 2.586 2.684 2.684
2.964 2.986 3.194 3.320 3.414 3.670 3.736
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Working in the plane, and without overlapping, figure out how to fit them into a circular cavity one foot in diameter.

The first person to solve this puzzle will receive an ovation from the class, and 'The Colossal Book of Short Puzzles and Problems' by Martin Gardner

Part II - Sums of Games

Consider a game called Boxing Match which was defined in a programming contest
http://potm.tripod.com/BOXINGMATCH/problem.short.html

An n x m rectangular board is initialized with 0 or 1 stone on each cell. Players alternate removing all the stones in any square subarray where all the cells are full. The player taking the last stone wins.

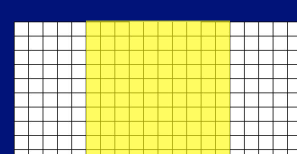
Boxing Match Example

Suppose we start with a 10 x 20 array that is completely full.

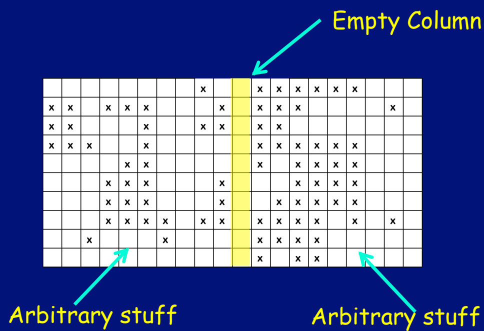
Is this a P or an N-position?

Example Contd.

The 10 x 20 full board is an N-position. A winning move is to take a 10x10 square in the middle. This leaves a 5x10 rectangle on the left and a 5x10 rectangle on the right. This is a P-position via mirroring. QED.



Sums of Games



In this kind of situation, the left and right games are completely independent games that don't interact at all. This naturally leads to the notion of the sum of two games.

A + B

A + B

A and B are games. The game A+B is a new game where the allowed moves are to pick one of the two games A or B (that is non-terminal) and make a move in that game. The position is terminal iff both A and B are terminal.

The sum operator is commutative and associative (explain).

Sums of Games*

We assign a number to any position in any game. This number is called the Nimber of the game.

(It's also called the "Nim Sum" or the "Sprague-Grundy" number of a game. But we'll call it the Nimber.)

*Only applies to normal, impartial games.

The MEX

The "MEX" of a finite set of natural numbers is the Minimum EXcluded element.

MEX {0, 1, 2, 4, 5, 6} = 3

MEX {1, 3, 5, 7, 9} = 0

MEX {} = 0

Definition of Nimber

The Nimber of a game G (denoted $N(G)$) is defined inductively as follows:

$N(G) = 0$ if G is terminal

$N(G) = MEX\{N(G_1), N(G_2), \ldots N(G_n)\}$

Where $G_1, G_2, \ldots G_n$ are the successor positions of game G. (I.e. the positions resulting from all the allowed moves.)

Another look at Nim

Let P_k denote the game that is a pile of k stones in the game of Nim.

Theorem: $N(P_k) = k$

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Proof: Use induction.

Base case is when k=0. Trivial.
When k>0 the set of moves is

$P_{k-1}, P_{k-2}, \ldots P_0$.

By induction these positions have nimbers k-1, k-2, … 0.

The MEX of these is k. QED.

Theorem: A game G is a P-position if and only if $N(G)=0$.

(i.e. Nimber = 0 iff P-position)

Proof: Induction.
Trivially true if G is a terminal position.

Suppose G is non-terminal.

If $N(G)\neq 0$, then by the MEX rule there must be a move G' in G that has $N(G')=0$. By induction this is a P-position. Thus G is an N position.

Nimber = 0 iff P-position (contd)

If $N(G)=0$, then by the MEX rule none of the successors of G have $N(G')=0$. By induction all of them are N-positions. Therefore G is a P-position.

QED.

The Nimber Theorem

Theorem: Let A and B be two impartial normal games. Then:

$$N(A+B) = N(A) \oplus N(B)$$

Proof: We'll get to this in a minute.

The beauty of Nimbers is that they completely capture what you need to know about a game in order to add it to another game. This often allows you to compute winning strategies, and can speed up game search exponentially.

Application to Nim

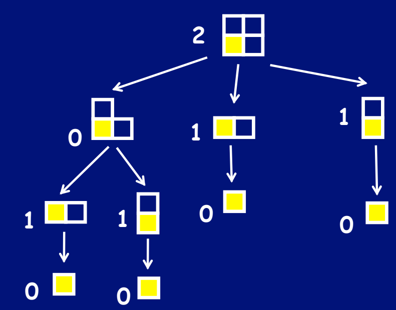
Note that the game of Nim is just the sum of several games. If the piles are of size a, b, and c, then the nim game for these piles is just $P_a + P_b + P_c$.

The nimber of this position, by the Nimber Theorem, is just $a \oplus b \oplus c$.

So it's a P-position if and only if $a \oplus b \oplus c = 0$, which is what we proved before.

Application to Chomp

What is the nimber of this chomp game?



What if we add this to a nim pile of size 4?

$4 \oplus 2 = 6$

Is this an N-position or a P-position?

N() ≠ 0 → it's an N-position. How do you win?

If we remove two chips from the nim pile, then the nimber is 0, giving a P-position. This is the unique winning move in this position.

Proof of the Nimber Theorem:

$N(A+B) = N(A) \oplus N(B)$

Let the moves in A be A1, A2, …, An

And the moves in B be B1, B2, …, Bm

We use induction. If either of these lists is empty the theorem is trivial (base case)

The moves in A+B are:

A+B1, A+B2, … A+Bm,

A1+B, A2+B, … An+B

$N(A+B) = MEX\{N(A+B1),\ldots N(A+Bm),$
$N(A1+B),\ldots N(An+B)\}$

$N(A+B) =$ (by induction)
$MEX\{N(A)\oplus N(B1),\ldots, N(A)\oplus N(Bm),$
$N(A1)\oplus N(B),\ldots, N(An)\oplus N(B)\}$

How do we prove this is $N(A)\oplus N(B)$?

We do it by proving two things:

(1) $N(A)\oplus N(B)$ is not in the list

(2) For all $y < N(A)\oplus N(B)$, y is in the list

(1) $N(A)\oplus N(B)$ is not in the list

$MEX\{N(A)\oplus N(B1),\ldots, N(A)\oplus N(Bm),$
$N(A1)\oplus N(B),\ldots, N(An)\oplus N(B)\}$

Why is $N(A)\oplus N(B)$ not in this list?

Because

$N(Bi) \neq N(B) \rightarrow N(A)\oplus N(Bi) \neq N(A)\oplus N(B)$

And

$N(Ai) \neq N(A) \rightarrow N(Ai)\oplus N(B) \neq N(A)\oplus N(B)$

(2) For all y < N(A)⊕N(B), y is in the list

N(A)⊕N(B) =	0	0	1	0	1	1	0	0	0	1	0	1	0	1	1	1
y =	0	0	1	0	1	1	0	0	0	0	.	.	.	.	.	.
N(A) =	.	.	.	.	.	.	.	.	.	0	.	.	.	.	.	.
N(B) =	.	.	.	.	.	.	.	.	.	1	.	.	.	.	.	.
N(Bi) =	‼	‼	‼	‼	‼	‼	‼	‼	‼	0	x	x	x	x	x	x

The highlighted column is the 1st where y and N(A)⊕N(B) differ. At that bit position N(A)⊕N(B) is 1 and y is 0. Therefore one of N(A) and N(B) =1. WLOG assume N(B)=1

Because N(B)=MEX{N(B1),...N(Bm)} there is a move in B such that the bits after the 1 form any desired pattern.

Therefore we can produce the desired y by moving in B to Bi. QED.

The Game of Dawson's Kayles

Start with a row of n bowling pins:

A move consists of knocking down 2 neighboring pins.

The last player to move wins.

An isolated pin is stuck and can never be removed.

How do we analyze this game?

Note that in a row of n pins there are n-1 possible moves:

(0,n-2), (1,n-3), … , (n-3,1), (n-2,0)

So the nimber of a row of n pins, denoted N(n) is:

0 if n=0
0 if n=1
MEX{N(0)⊕N(n-2), N(1)⊕N(n-3), … N(n-2)⊕N(0)}

Let's work out some small values.....

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n	0	1	2	3	4	5	6	7	8	9	10	11	12
N(n)	0	0	1	1	2	0	3	1	1	0	3	3	2

The sequence is eventually periodic with period 34 (after an initial transient block of length 53).

Time to compute to N(n) is $O(n^2)$

Note that the case n=9 is an P-position

The Game of Treblecross

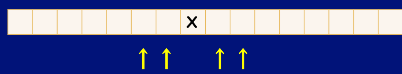
Tic-Tac-Toe on a line with only X's allowed. First player to form 3-in-a-row wins.

Let's play. I go first.

This game is equivalent to Dawson's Kayles[3] (of size n+2). The [3] means you must take 3 in a row.

(Proving equivalence of games comes up often, specially on the homework.)

First we eliminate "stupid" moves. A stupid move is one which allows the opponent to win immediately on the next move.



Stupid moves

Stupid move elimination does not change the outcome or the strategy of the game, but it converts it to a normal impartial game.

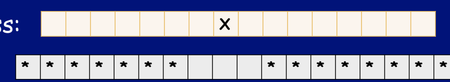
Claim: Treblecross of length n is equivalent to Dawson's Kayles[3] of length n+2.

Proof: Verify base cases (easy).

General case: We will prove that the game trees are identical.

Treblecross:

D. Kayles:



To the left of the X in the treblecross game, there is a treblecross game of size 5 (not counting stupid moves). This is equivalent (by induction) to the size 7 Dawson Kayles[3] game. The right side is the same. Therefore the game trees are identical. QED

We can now evaluate the game just as we did with regular Dawson's Kayles.

n			0	1	2	3	4	5	6	7	8	9	10
n+2	0	1	2	3	4	5	6	7	8	9	10	11	12
N(n)	0	0	0	1	1	1	2	2	0	3	3	1	1

Is this eventually periodic?

Open Problem... The first few values:

0 0 0 1 1 1 2 2 0 3 3 1 1 1 0 4 3 3 3 2 2 2 4 4 0 5 5 2 2 2 3 3 0 5 0 1 1 1 3 3 3 5 6 4 4 1 0 5 5 6 6 2 7 7 7 8 0 1 9 2 7 2 3 3 3 9 0 5 4 4 8 6 6 2 7 1 1 1 0 5 5 9 3 1 8 2 8 5 0 1 1 12 2 2 7 3 3 9 4 4 0 11 3 3 3 9 2 2 8 1 3 5 0 9 12 2 6 13 13 5 0 1 1 4 11 7 7 10 3 4 1 4 0 5 0 3 3 6 7 2 14 13 10 4 12 9 2 2 3 3 6 9 9 1 16 4 8 3 3 2 15 1 1 4 0 5 5 16 6 6 6 8 0 16 5 4 4 17 2 2 7 14 6 10 12 1 0 16 13 3 6 2 7 7 8 1 0 5 17 2 12 15 3 11 0 19 18 12 4 16 17 2 2 21 6 9 4 19 5 5 ....

Exercise: What happens in Dawson Kayles[1]?

Let's experiment....

Green Hackenbush

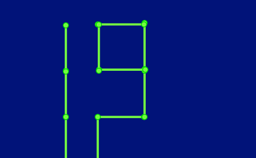
- **Definition of the game**
- **Evaluation of some simple positions**
- **Look at trees**
- **Discover the rule for evaluating a tree**

http://merganser.math.gvsu.edu/david/reed05/projects/esselstein/msri/Esselstein/

Green Hackenbush

A position is defined by an arbitrary connected graph of green colored edges, with one special vertex called the ground.

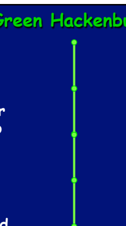
The Ground



The Ground, although shown here as two vertices, always really acts as one vertex.

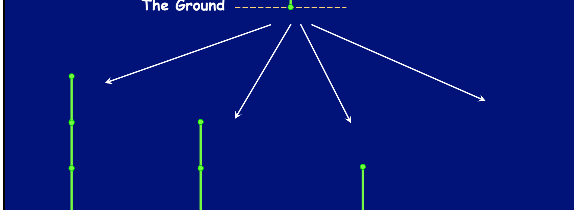
Green Hackenbush

A move: pick an edge to delete. All edges no longer connected to the ground go away.



A stalk of n edges is the same as a nim pile of size n

The Ground



Green Hackenbush

Theorem: if we elevate a green hackenbush position by one edge, we increase its nimber by one.

The proof is by induction. We know there are moves in the blob on the left which have nimbers 0, 1, 2, …, n-1 (but not n). Those moves all exist in the right picture, except elevated by one. Thus, by induction, their nimbers are 1, 2, 3, …, n. In addition, in the right picture we can remove the bottom edge, creating a position of nimber 0. Thus, on the right there are moves with nimbers 0, 1, … n (but not n+1). Therefore by the MEX rule the nimber of the position on the right is n+1. QED.

Green Hackenbush

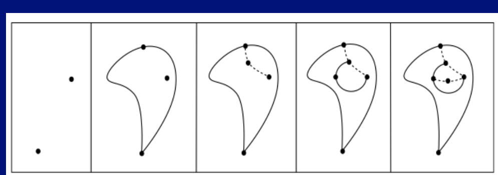
We now have the tools to evaluate the nimber of any green hackenbush tree.

If there is just one edge connected to the ground, then we can evaluate the part on top of the edge and add one.

If there is more than one edge connected to the ground, then we can evaluate each tree separately and combine them together using Nim addition.

It turns out that it's not hard to evaluate any Green Hackenbush position, using the "fusion" principle and the "colon" principle. See the reference above.

Sprouts



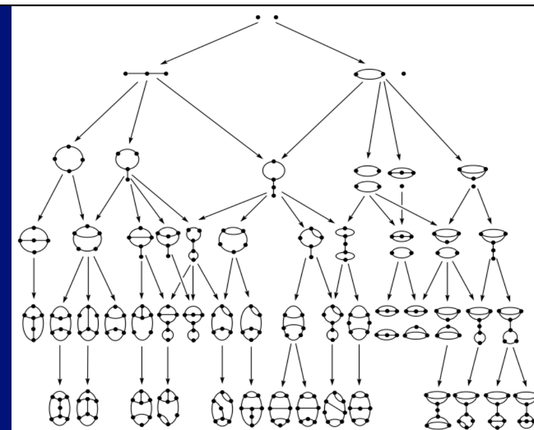


Figure 2: The complete game tree for two-spot Sprouts

number of spots	1	2	3	4	5	6	7	8	9	10	11
normal play	L	L	W	W	W	L	L*	L*	W*	W*	W*
misère play	W	L	L	L	W*	W*	L*	L*	L*		

W denotes that the game is a win for the first player;
L denotes a loss for the first player.
A "*" indicates new results obtained by our program.

See "Computer Analysis of Sprouts" by Applegate, Jacobson, and Sleator http://www.cs.cmu.edu/~sleator/papers/Sprouts.htm

Application to Boxing Match

The beauty of Nimbers is that they completely capture what you need to know about a game in order to add it to another game. This can speed up game search exponentially.

How would you use this to win in Boxing Match against an opponent who did not know about Nimbers?

(My friends Guy Jacobson and David Applegate used this to cream all the other players in the Boxing Match contest.)

Study Bee

- Sums of games
- Definition of Nimbers
- The MEX operator
- The Nimber Theorem
- Applications of the theorem:
  - Dawson's Kayles
  - Treblecross
  - Green Hackenbush
  - Sprouts
  - Boxing Match