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Three Point Situation, 1-Position Proof, Case 8
We want to prove the following proposition:

$$P(n) : (0, \quad 6n + 8, \quad 0) \ \cong \ *1 \ \text{ where } \ n \geq 0$$

We will use strong induction to prove the above statement.

Base Cases:
Using our program, we have verified that $P(n)$ holds for $n \in \{0, 1, 2, 3\}$.
The fact that $(0, \quad 2, \quad 0) \ \cong \ *1$ will also be useful.

Induction Hypothesis:
Assume for some $k \geq 1$ that $\forall\, n < k,\ P(n)$ holds.

Induction Step:
We need to prove that $P(k)$ holds. In other words, we need to show:

$$(0, \quad 6k + 8, \quad 0) \ \cong \ *1$$

Simplifying the problem,
To show that $(0, \quad 6k + 8, \quad 0)$ has nimber 1, it suffices to show the following:

1. In one move, we can reach a state with nimber 0
2. In one move, we cannot reach a state with nimber 1

Looking at the initial state's children,
Notice that in one move from the original state we can reach any state of the form:

$$(m, \quad 6k + 8 - m, \quad 0) \ \text{where } 1 \leq m \leq 3k + 4$$

the remaining states that we can reach from the original state are of the form:

$$(0, \quad 6k + 8 - m, \quad m) \ \text{where } 1 \leq m \leq 3k + 4$$

Since the latter form is symmetric to the former, focusing our analysis on the former is enough to complete the proof.

When $m = 3k + 4$:
Observe that when $m = 3k + 4$, we obtain the state $(3k + 4, \quad 3k + 4, \quad 0)$ which clearly has nimber 0, so we have proved the first fact we need.

When m is odd and $1 \leq m \leq 2k + 2$:
When m is an odd number satisfying $1 \leq m \leq 2k + 2$, notice that

$$(m + 1, \quad 3k + 4 - \tfrac{m+1}{2}, \quad 3k + 4 - \tfrac{m+1}{2}) \ \cong \ *0$$

Work one step backwards from this to get

$$(m, \quad 3k + 5 - \tfrac{m+1}{2}, \quad 3k + 4 - \tfrac{m+1}{2}) \ \cong \ *1$$

Observe that

$$(m, \quad 6k + 8 - m, \quad 0) \xrightarrow{\text{one move}} (m, \quad 3k + 5 - \tfrac{m+1}{2}, \quad 3k + 4 - \tfrac{m+1}{2})$$

It follows that the minimum excluded value of the nimbers of all the child states of $(m, \quad 6k + 8 - m, \quad 0)$ cannot be 1, and thus the nimber of this state cannot be 1.

When m is even and $1 \leq m \leq 2k+2$:
When m is an even number satisfying $1 \leq m \leq 2k+2$, observe that

$$(m, \quad 6k+8-m, \quad 0) \xrightarrow{\text{one move}} (m, \quad 6k+8-2m, \quad m)$$

We can replace m with $2i$ where $1 \leq i \leq k+1$ to get

$$(2i, \quad 6k+8-2i, \quad 0) \xrightarrow{\text{one move}} (2i, \quad 6k+8-4i, \quad 2i)$$

In the resultant state, the difference between the left and middle values, also the difference between the right and middle values, is

$$6(k-i)+8$$

Which intuitively suggests that

$$(2i, \quad 6k+8-4i, \quad 2i) \;\cong\; (0, \quad 6(k-i)+8, \quad 0) \;\text{ for } 1 \leq i \leq k+1$$

and although we lack a rigorous argument for this we will assume it is true.
Our knowledge that $(0, \quad 2, \quad 0) \;\cong\; *1$ and also our induction hypothesis, tell us:

$$(0, \quad 6(k-i)+8, \quad 0) \;\cong\; *1 \text{ for } 1 \leq i \leq k+1$$

Consequently, $(2i, \quad 6k+8-4i, \quad 2i) \;\cong\; *1$.
It follows that the minimum excluded value of the nimbers of all the child states of $(m, \quad 6k+8-m, \quad 0)$ cannot be 1, and thus the nimber of this state cannot be 1.

When $m \geq 2k+3$:
If $m \geq 2k+3$, it is clear that

$$(m, \quad m, \quad 6k+8-2m) \;\cong\; *0$$

Working backwards from this, we get that

$$(m, \quad m+1, \quad 6k+7-2m) \;\cong\; *1$$

We know that

$$(m, \quad 6k+8-m, \quad 0) \xrightarrow{\text{one move}} (m, \quad m+1, \quad 6k+7-2m)$$

It follows that the minimum excluded value of the nimbers of all the child states of $(m, \quad 6k+8-m, \quad 0)$ cannot be 1, and thus the nimber of this state cannot be 1.

Conclusion:
1 must be the mex of the nimbers of the children of $(0, \quad 6k+8, \quad 0)$, so we have proved that $(0, \quad 6k+8, \quad 0) \;\cong\; *1$.

Three Point Situation, 1-Position Proof, Case 6
Assume the same inductive 'skeleton' and reasoning style as the previous case. This proof is mainly an illustration of the differences between the two cases:

Consider the initial position

$$(0, \quad 6n+6, \quad 0) \quad \text{where } n \geq 0$$

The first move will take the initial position to a position of the form

$$(m, \quad 6n+6-m, \quad 0) \quad \text{where } 1 \leq m \leq 3n+3$$

<u>If $m = 3n$:</u>
This is the state $(3n, \quad 3n, \quad 0)$ which clearly has nimber 0.

<u>If $m \leq 2n+2$ and m is odd:</u>
Then, consider the following move sequence,

$$\begin{aligned} & (m, \quad 6n+6-m, \quad 0) \\ \xrightarrow{\text{one possible child}} \quad & (m, \quad 3n+4-\tfrac{m+1}{2}, \quad 3n+3-\tfrac{m+1}{2}) \\ \xrightarrow{\text{only child}} \quad & (m+1, \quad 3n+3-\tfrac{m+1}{2}, \quad 3n+3-\tfrac{m+1}{2}) \end{aligned}$$

<u>If $m \leq 2n+2$ and m is even:</u>
Consider the following move sequence:

$$\begin{aligned} & (m, \quad 6n+6-m, \quad 0) \\ \xrightarrow{\text{one possible child}} \quad & (m, \quad 6n+6-2m, \quad m) \\ \cong \quad & (0, \quad 6n+6-3m, \quad 0) \end{aligned}$$

Since m is even, we can write it as $2i$ for some $1 \leq i \leq n+1$.

$$\begin{aligned} & (0, \quad 6n+6-3m, \quad 0) \\ \cong \quad & (0, \quad 6n+6-6i, \quad 0) \\ \cong \quad & (0, \quad 6(n-i)+6, \quad 0) \end{aligned}$$

By our induction hypothesis, $(0, \quad 6(n-i)+6, \quad 0)$ has nimber 1.

<u>If $m \geq 2n+3$:</u>
The move sequence below works:

$$\begin{aligned} & (m, \quad 6n+6-m, \quad 0) \\ \xrightarrow{\text{one possible child}} \quad & (m, \quad m+1, \quad 6n+5-2m) \\ \xrightarrow{\text{only child}} \quad & (m, \quad m, \quad 6n+6-2m) \end{aligned}$$

Which provides us with everything we need to fill in the proof skeleton.

Three Point Situation, 1-Position Proof, Case 10

Assume the same inductive 'skeleton' and reasoning style as the previous case. This proof is mainly an illustration of the differences between the two cases:

Consider the initial position

$$(0, \quad 6n+10, \quad 0) \quad \text{where } n \geq 0$$

The first move will take the initial position to a position of the form

$$(m, \quad 6n+6-m, \quad 0) \quad \text{where } 1 \leq m \leq 3n+3$$

If $m = 3n$:

This is the state $(3n, \quad 3n, \quad 0)$ which clearly has nimber 0.

If $m \leq 2n+3$ and m is odd:

Then, consider the following move sequence,

$$\begin{aligned} & (m, \quad 6n+10-m, \quad 0) \\ \xrightarrow{\text{one possible child}} \; & (m, \quad 3n+6-\tfrac{m+1}{2}, \quad 3n+5-\tfrac{m+1}{2}) \\ \xrightarrow{\text{only child}} \; & (m+1, \quad 3n+5-\tfrac{m+1}{2}, \quad 3n+5-\tfrac{m+1}{2}) \end{aligned}$$

If $m \leq 2n+3$ and m is even:

Consider the following move sequence:

$$\begin{aligned} & (m, \quad 6n+10-m, \quad 0) \\ \xrightarrow{\text{one possible child}} \; & (m, \quad 6n+10-2m, \quad m) \\ \cong \; & (0, \quad 6n+10-3m, \quad 0) \end{aligned}$$

Since m is even, we can write it as $2i$ for some $1 \leq i \leq n+1$.

$$\begin{aligned} & (0, \quad 6n+10-3m, \quad 0) \\ \cong \; & (0, \quad 6n+10-6i, \quad 0) \\ \cong \; & (0, \quad 6(n-i)+10, \quad 0) \end{aligned}$$

By our induction hypothesis, $(0, \quad 6(n-i)+10, \quad 0)$ has nimber 1.

If $m \geq 2n+4$:

The move sequence below works:

$$\begin{aligned} & (m, \quad 6n+10-m, \quad 0) \\ \xrightarrow{\text{one possible child}} \; & (m, \quad m+1, \quad 6n+9-2m) \\ \xrightarrow{\text{only child}} \; & (m, \quad m, \quad 6n+10-2m) \end{aligned}$$

Which provides us with everything we need to fill in the proof skeleton.

Three Point Situation, 1-Position Proof, Case 7
Consider the initial position

$$(0, \quad 6n+6, \quad 0) \quad \text{where } n \geq 0$$

The first move brings $(0, \quad 6n+7, \quad 0)$ to a state of the form $(m, \quad 6n+7-m, \quad 0)$, where $1 \leq m \leq 3n+3$.
If $m = 3n+3$,

$$\begin{aligned} & (3n+3, \quad 3n+4, \quad 0) \\ \xrightarrow{\text{only child}} \; & (3n+3, \quad 3n+3, \quad 1) \end{aligned}$$

<u>If $m \leq 2n+2$ and m is odd:</u>
Then, consider the following move sequence,

$$\begin{aligned} & (m, \quad 6n+7-m, \quad 0) \\ \xrightarrow{\text{one possible child}} \; & (m, \quad \tfrac{6n+7-m}{2}, \quad \tfrac{6n+7-m}{2}) \end{aligned}$$

<u>If $m \leq 2n+2$ and m is even:</u>
Consider the following move sequence:

$$\begin{aligned} & (m, \quad 6n+7-m, \quad 0) \\ \xrightarrow{\text{one possible child}} \; & (m, \quad 6n+7-2m, \quad m) \\ \cong \; & (0, \quad 6n+7-3m, \quad 0) \end{aligned}$$

Since m is odd, we can write it as $2i$ for some $1 \leq i \leq n+1$.

$$\begin{aligned} & (0, \quad 6n+7-3m, \quad 0) \\ \cong \; & (0, \quad 6n+7-6i, \quad 0) \\ \cong \; & (0, \quad 6(n-i)+7, \quad 0) \end{aligned}$$

By our induction hypothesis, $(0, \quad 6(n-i)+6, \quad 0)$ has nimber 0.

<u>If $m \geq 2n+3$:</u>
The move sequence below works:

$$\begin{aligned} & (m, \quad 6n+7-m, \quad 0) \\ \xrightarrow{\text{one possible child}} \; & (m, \quad m, \quad 6n+7-2m) \end{aligned}$$

Which provides us with everything we need to fill in the proof skeleton.

Three Point Situation, 1-Position Proof, Case 9
Consider the initial position

$$(0, \quad 6n+6, \quad 0) \quad \text{where } n \geq 0$$

The first move brings $(0, \quad 6n+7, \quad 0)$ to a state of the form $(m, \quad 6n+7-m, \quad 0)$, where $1 \leq m \leq 3n+3$.
If $m = 3n+3$,

$$\begin{aligned} & (3n+3, \quad 3n+4, \quad 0) \\ \xrightarrow{\text{only child}} \; & (3n+3, \quad 3n+3, \quad 1) \end{aligned}$$

<u>If $m \leq 2n+2$ and m is odd:</u>
Then, consider the following move sequence,

$$\begin{aligned} & (m, \quad 6n+7-m, \quad 0) \\ \xrightarrow{\text{one possible child}} \; & (m, \quad \tfrac{6n+7-m}{2}, \quad \tfrac{6n+7-m}{2}) \end{aligned}$$

If $m \leq 2n+2$ and m is even:
Consider the following move sequence:

$$\begin{aligned} & (m, \quad 6n+7-m, \quad 0) \\ \xrightarrow{\text{one possible child}} & (m, \quad 6n+7-2m, \quad m) \\ \cong & (0, \quad 6n+7-3m, \quad 0) \end{aligned}$$

Since m is odd, we can write it as $2i$ for some $1 \leq i \leq n+1$.

$$\begin{aligned} & (0, \quad 6n+7-3m, \quad 0) \\ \cong & (0, \quad 6n+7-6i, \quad 0) \\ \cong & (0, \quad 6(n-i)+7, \quad 0) \end{aligned}$$

By our induction hypothesis, $(0, \quad 6(n-i)+6, \quad 0)$ has nimber 0.

If $m \geq 2n+3$:
The move sequence below works:

$$\begin{aligned} & (m, \quad 6n+7-m, \quad 0) \\ \xrightarrow{\text{one possible child}} & (m, \quad m, \quad 6n+7-2m) \end{aligned}$$

Which provides us with everything we need to fill in the proof skeleton.

Three Point Situation, 1-Position Proof, Case 11
Consider the initial position

$$(0, \quad 6n+6, \quad 0) \quad \text{where } n \geq 0$$

The first move brings $(0, \quad 6n+7, \quad 0)$ to a state of the form $(m, \quad 6n+7-m, \quad 0)$, where $1 \leq m \leq 3n+3$.
If $m = 3n+3$,

$$\begin{aligned} & (3n+3, \quad 3n+4, \quad 0) \\ \xrightarrow{\text{only child}} & (3n+3, \quad 3n+3, \quad 1) \end{aligned}$$

If $m \leq 2n+2$ and m is odd:
Then, consider the following move sequence,

$$\begin{aligned} & (m, \quad 6n+7-m, \quad 0) \\ \xrightarrow{\text{one possible child}} & (m, \quad \tfrac{6n+7-m}{2}, \quad \tfrac{6n+7-m}{2}) \end{aligned}$$

If $m \leq 2n+2$ and m is even:
Consider the following move sequence:

$$\begin{aligned} & (m, \quad 6n+7-m, \quad 0) \\ \xrightarrow{\text{one possible child}} & (m, \quad 6n+7-2m, \quad m) \\ \cong & (0, \quad 6n+7-3m, \quad 0) \end{aligned}$$

Since m is odd, we can write it as $2i$ for some $1 \leq i \leq n+1$.

$$\begin{aligned} & (0, \quad 6n+7-3m, \quad 0) \\ \cong & (0, \quad 6n+7-6i, \quad 0) \\ \cong & (0, \quad 6(n-i)+7, \quad 0) \end{aligned}$$

By our induction hypothesis, $(0, \quad 6(n-i)+6, \quad 0)$ has nimber 0.

<u>If $m \geq 2n+3$:</u>
The move sequence below works:

$$\begin{aligned} & (m, \quad 6n+7-m, \quad 0) \\ \xrightarrow{\text{one possible child}} \; & (m, \quad m, \quad 6n+7-2m) \end{aligned}$$

Which provides us with everything we need to fill in the proof skeleton.