21–499
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Describing the Game
This is a game for two players, who take turns making moves.
We have two points P_1 and P_2 in a plane, connected by an edge.
Conisder a function $s : \{P_1, P_2\} \to \mathbb{N}$.
Initially, $s(P_1) \geq s(P_2)$.
There are $s(P_1)$ stones at P_1 and $s(P_2)$ stones at P_2.
A player can move stones from P_1 to P_2, but not from P_2 to P_1, but the invariant $s(P_1) \geq s(P_2)$ should always be maintained.
If a player starts her turn with $s(P_1) = s(P_2)$, then she loses.
What is the nimber of this game?
It ought to be $\frac{s(P_1)-s(P_2)}{2}$.

A Modified Version
Suppose that a few things change from before:

1. Either $s(P_1) > s(P_2)$ or $s(P_2) > s(P_1)$ initially
2. We can move stones from P_1 to P_2 or P_2 to P_1
3. Suppose at the beginning of some turn, $|s(P_1) - s(P_2)| = d$, and at the end of that turn $|s(P_1) - s(P_2)| = d'$. Then it must be that $d' < d$.
4. If a player starts with $s(P_1) = s(P_2)$ then the player loses.

How does that change the nimber of the game?
If we start with $s(P_1) = 10$ and $s(P_2) = 5$ then $G \cong *2$.
The nimber ought to be $|\frac{s(P_1)-s(P_2)}{2}|$.

Three Points
Our point set $\mathbb{P}$ is now $\{P_1, P_2, P_3\}$, with directed edges (P_1, P_2) and (P_2, P_3).
Start with the configuration $s(P_1) > s(P_2) > s(P_3)$.
Maintain the following invariants:

1. $s(P_1) \geq s(P_2)$
2. $s(P_2) \geq s(P_3)$

If a player cannot make any moves, then the player loses.
This means we can describe a game of this form with a tuple (a, b, c) which means $s(P_1) = a, s(P_2) = b, s(P_3) = c$ initially.
Examples:

1. $(3, 2, 1) \cong *0$
2. $(3, 2, 2) \cong *0$
3. $(3, 3, 0) \cong *1$
4. $(3, 3, 1) \cong *1$
5. $(3, 3, 2) \cong *0$
6. $(4, 1, 1) \cong *1$
7. $(4, 2, 0) \cong *0$
8. $(4, 2, 1) \cong *0$
9. $(4, 2, 2) \cong *1$
10. $(4, 3, 0) \cong *1$
11. $(4, 3, 1) \cong *0$
12. $(4, 4, 0) \cong *2$

13. $(5, 1, 1) \cong *2$

14. $(5, 2, 0) \cong *0$

15. $(5, 2, 1) \cong *1$

16. $(5, 3, 0) \cong *0$

17. $(6, 1, 1) \cong *2$

18. $(6, 2, 0) \cong *1$

A New Three Point Situation

The point set $\mathbb{P}$ is again $\{P_1, P_2, P_3\}$ where the directed edge set $\mathbb{E}$ is $\{(P_2, P_1), (P_2, P_3)\}$.
The initial state is of the form $s(P_1) = 0, s(P_2) = n, s(P_3) = 0$.
The rest of the rules are as before.
What is the nimber for some given value of n?
According to the program:

- $(0, 1, 0) \cong *0$
- $(0, 2, 0) \cong *1$
- $(0, 3, 0) \cong *0$
- $(0, 4, 0) \cong *2$
- $(0, e, 0) \cong *1$ for $e = 2i$ where $i \in \mathbb{N}, i > 2$
- $(0, o, 0) \cong *0$ for $o = 2i - 1$ where $i \in \mathbb{N}, i > 2$

A helpful identity to prove

$$(l, \quad m, \quad r) \ \cong \ (l + i, \quad m + i, \quad r + i)$$

Three Point Situation, 1-Position Proof, Case 8
We want to prove the following proposition:

$$P(n) : (0, \quad 6n + 8, \quad 0) \ \cong \ *1 \ \text{ where } \ n \geq 0$$

We will use strong induction to prove the above statement.

Base Cases:
Using our program, we have verified that $P(n)$ holds for $n \in \{0, 1, 2, 3\}$.
The fact that $(0, \quad 2, \quad 0) \ \cong *1$ will also be useful.

Induction Hypothesis:
Assume for some $k \geq 1$ that $\forall\, n < k,\ P(n)$ holds.

Induction Step:
We need to prove that $P(k)$ holds. In other words, we need to show:

$$(0, \quad 6k + 8, \quad 0) \ \cong \ *1$$

Simplifying the problem,
To show that $(0, \quad 6k + 8, \quad 0)$ has nimber 1, it suffices to show the following:

1. In one move, we can reach a state with nimber 0
2. In one move, we cannot reach a state with nimber 1

Looking at the initial state's children,
Notice that in one move from the original state we can reach any state of the form:

$$(m, \quad 6k + 8 - m, \quad 0) \ \text{where } 1 \leq m \leq 3k + 4$$

the remaining states that we can reach from the original state are of the form:

$$(0, \quad 6k + 8 - m, \quad m) \ \text{where } 1 \leq m \leq 3k + 4$$

Since the latter form is symmetric to the former, focusing our analysis on the former is enough to complete the proof.

When $m = 3k + 4$:
Observe that when $m = 3k + 4$, we obtain the state $(3k + 4, \quad 3k + 4, \quad 0)$ which clearly has nimber 0, so we have proved the first fact we need.

When m is odd and $1 \leq m \leq 2k + 2$:
When m is an odd number satisfying $1 \leq m \leq 2k + 2$, notice that

$$(m + 1, \quad 3k + 4 - \tfrac{m+1}{2}, \quad 3k + 4 - \tfrac{m+1}{2}) \ \cong \ *0$$

Work one step backwards from this to get

$$(m, \quad 3k + 5 - \tfrac{m+1}{2}, \quad 3k + 4 - \tfrac{m+1}{2}) \ \cong \ *1$$

Observe that

$$(m, \quad 6k + 8 - m, \quad 0) \xrightarrow{\text{one move}} (m, \quad 3k + 5 - \tfrac{m+1}{2}, \quad 3k + 4 - \tfrac{m+1}{2})$$

It follows that the minimum excluded value of the nimbers of all the child states of $(m, \quad 6k + 8 - m, \quad 0)$ cannot be 1, and thus the nimber of this state cannot be 1.

When m is even and $1 \le m \le 2k+2$:
When m is an even number satisfying $1 \le m \le 2k+2$, observe that

$$(m, \quad 6k+8-m, \quad 0) \xrightarrow{\text{one move}} (m, \quad 6k+8-2m, \quad m)$$

We can replace m with $2i$ where $1 \le i \le k+1$ to get

$$(2i, \quad 6k+8-2i, \quad 0) \xrightarrow{\text{one move}} (2i, \quad 6k+8-4i, \quad 2i)$$

In the resultant state, the difference between the left and middle values, also the difference between the right and middle values, is

$$6(k-i)+8$$

Which intuitively suggests that

$$(2i, \quad 6k+8-4i, \quad 2i) \;\cong\; (0, \quad 6(k-i)+8, \quad 0) \;\text{ for } 1 \le i \le k+1$$

and although we lack a rigorous argument for this we will assume it is true.
Our knowledge that $(0, \quad 2, \quad 0) \;\cong\; *1$ and also our induction hypothesis, tell us:

$$(0, \quad 6(k-i)+8, \quad 0) \;\cong\; *1 \text{ for } 1 \le i \le k+1$$

Consequently, $(2i, \quad 6k+8-4i, \quad 2i) \;\cong\; *1$.
It follows that the minimum excluded value of the nimbers of all the child states of $(m, \quad 6k+8-m, \quad 0)$ cannot be 1, and thus the nimber of this state cannot be 1.

When $m \ge 2k+3$:
If $m \ge 2k+3$, it is clear that

$$(m, \quad m, \quad 6k+8-2m) \;\cong\; *0$$

Working backwards from this, we get that

$$(m, \quad m+1, \quad 6k+7-2m) \;\cong\; *1$$

We know that

$$(m, \quad 6k+8-m, \quad 0) \xrightarrow{\text{one move}} (m, \quad m+1, \quad 6k+7-2m)$$

It follows that the minimum excluded value of the nimbers of all the child states of $(m, \quad 6k+8-m, \quad 0)$ cannot be 1, and thus the nimber of this state cannot be 1.

Conclusion:
1 must be the mex of the nimbers of the children of $(0, \quad 6k+8, \quad 0)$, so we have proved that $(0, \quad 6k+8, \quad 0) \;\cong\; *1$.

Three Point Situation, 1-Position Proof, Case 6 Assume the same inductive 'skeleton' and reasoning style as the previous case. This proof is mainly an illustration of the differences between the two cases:

Consider the initial position

$$(0, \quad 6n+6, \quad 0) \quad \text{where } n \geq 0$$

The first move will take the initial position to a position of the form

$$(m, \quad 6n+6-m, \quad 0) \quad \text{where } 1 \leq m \leq 3n+3$$

<u>If $m = 3n$:</u>
This is the state $(3n, \quad 3n, \quad 0)$ which clearly has nimber 0.

<u>If $m \leq 2n+2$ and m is odd:</u>
Then, consider the following move sequence,

$$\begin{aligned} & (m, \quad 6n+6-m, \quad 0) \\ \xrightarrow{\text{one possible child}} \; & (m, \quad 3n+4-\tfrac{m+1}{2}, \quad 3n+3-\tfrac{m+1}{2}) \\ \xrightarrow{\text{only child}} \; & (m+1, \quad 3n+3-\tfrac{m+1}{2}, \quad 3n+3-\tfrac{m+1}{2}) \end{aligned}$$

<u>If $m \leq 2n+2$ and m is even:</u>
Stuff.

<u>If $m \geq 2n+3$:</u>
Stuff.

Which provides us with everything we need to fill in the proof skeleton.

Three Point Situation, 1-Position Proof, Case 10