$[\hat{r}_i ; \hat{p}_j] = i\hbar\delta_{ij}$

$\psi(\vec{r}) = \langle \vec{r} | \psi \rangle$

Inf. translatieoperator op $|\psi\rangle$

$|\psi'\rangle = \hat{T}(d\vec{r})|\psi\rangle$

$\langle \vec{r} | \psi' \rangle = \langle \vec{r} | \hat{T}(d\vec{r}) | \psi \rangle$

$= \langle \psi | \hat{T}^\dagger(d\vec{r}) | \vec{r} \rangle^*$

en $\hat{T}^\dagger(d\vec{r}) = \hat{T}^{-1}(d\vec{r}) = \hat{T}(-d\vec{r})$

$\langle \vec{r}' | \psi \rangle = \langle \psi | \hat{T}(-d\vec{r}) | \vec{r} \rangle^*$

$= \langle \psi | \vec{r} - d\vec{r} \rangle^*$

$= \langle \vec{r} - d\vec{r} | \psi \rangle$

$\hat{\vec{r}} | \vec{r} \rangle = \vec{r} | \vec{r} \rangle$

$\hat{x} | x; y; z \rangle = x | xyz \rangle$

$\hat{y} | x; y; z \rangle = y | xyz \rangle$

$\vdots$

$\psi'(\vec{r}) = \psi(\vec{r} - d\vec{r})$

$= \psi(\vec{r}) - d\vec{r} \cdot \vec{\nabla} \psi(\vec{r})$

Anderzijds: $\psi'(\vec{r}) = \langle \vec{r} | \hat{T}(d\vec{r}) | \psi \rangle$

$= \langle \vec{r} | \hat{1} - \frac{i}{\hbar} \hat{\vec{p}} \cdot d\vec{r} | \psi \rangle$

$= \psi(\vec{r}) - \frac{i}{\hbar} d\vec{r} \cdot \langle \vec{r} | \hat{\vec{p}} | \psi \rangle$

$\langle \vec{r} | \hat{\vec{p}} | \psi \rangle = -i\hbar \vec{\nabla} \psi(\vec{r}) \rightarrow \hat{\vec{p}} = \left(-i\hbar \frac{\partial}{\partial x}, -i\hbar \frac{\partial}{\partial y}, -i\hbar \frac{\partial}{\partial z}\right)$

$\hat{A}, \hat{B}$ hermitisch

$\langle \psi | \hat{A} | \psi \rangle = \langle \hat{A} \rangle$

$\Delta A = \langle (\hat{A} - \langle \hat{A} \rangle)^2 \rangle^{\frac{1}{2}}$

$\hat{\hat{A}} = \hat{A} - \langle \hat{A} \rangle, \hat{\hat{B}} = \hat{B} - \langle \hat{B} \rangle$

$\Delta A \Delta B \geq \frac{1}{2} |\langle [\hat{A}; \hat{B}] \rangle|$

$(\Delta A)^2 = \langle \hat{\hat{A}}^2 \rangle, (\Delta B)^2 = \langle \hat{\hat{B}}^2 \rangle$

$[\hat{\hat{A}}, \hat{\hat{B}}] = [\hat{A}; \hat{B}]$

Bekijk: $\hat{C}(\lambda) = \hat{\hat{A}} + i\lambda \hat{\hat{B}}$

$\hat{C}^\dagger(\lambda) = \hat{\hat{A}} - i\lambda \hat{\hat{B}}$

$\langle \hat{C}\hat{C}^\dagger \rangle = \langle \psi | \hat{C}\hat{C}^\dagger | \psi \rangle$

$= \langle C^\dagger \psi | \hat{C}^\dagger \psi \rangle \geq 0$

$= \sum_n \langle \psi | \hat{C} | \psi_n \rangle \langle \psi_n | \hat{C}^\dagger | \psi \rangle$

$= \sum_n \langle \psi | \hat{C} | \psi_n \rangle \langle \psi | \hat{C} | \psi_n \rangle^*$

$= \sum_n | \langle \psi | C | \psi_n \rangle |^2 \geq 0$

$\langle (\hat{\hat{A}} + i\lambda \hat{\hat{B}})(\hat{\hat{A}} - i\lambda \hat{\hat{B}}) \rangle = \langle \hat{\hat{A}}^2 + \lambda^2 \hat{\hat{B}}^2 - i\lambda (\hat{\hat{A}}\hat{\hat{B}} - \hat{\hat{B}}\hat{\hat{A}}) \rangle$

$= (\Delta A)^2 + \lambda^2 (\Delta B)^2 - i\lambda \langle [\hat{A}, \hat{B}] \rangle \geq 0$

$= f(\lambda)$

$\frac{df}{d\lambda} = 2\lambda (\Delta B)^2 - i\lambda \underbrace{\langle [\hat{A}, \hat{B}] \rangle}_{\text{puur imaginair}}$

Minimum voor $\lambda_0 = \frac{i}{2} \frac{\langle [\hat{A}, \hat{B}] \rangle}{(\Delta B)^2}$

$\Rightarrow f(\lambda_0) = (\Delta A)^2 + \frac{1}{4} \frac{\langle [\hat{A}, \hat{B}] \rangle^2}{(\Delta B)^2} \geq 0$

$\Rightarrow (\Delta A)^2 (\Delta B)^2 \geq -\frac{1}{4} \langle [\hat{A}, \hat{B}] \rangle^2$ $\rightarrow$ imaginair

$\Delta A \Delta B \geq \frac{1}{2} | \langle [\hat{A}; \hat{B}] \rangle |$

$\hat{A} = \hat{X}; \hat{B} = \hat{p}_x$

$[\hat{A}; \hat{B}] = i\hbar \rightarrow (\Delta x)(\Delta p_x) \geq \frac{1}{2}\hbar$

Inf. rotatie. $\delta\vec{\varphi} = (\delta\varphi)\vec{n}$

$\hookrightarrow$ symmetrie-operator

$\hat{R}(\delta\vec{\varphi}) = \hat{1} - \frac{i}{\hbar} \hat{\vec{L}} \cdot \delta\vec{\varphi}$

$= \hat{1} - \frac{i}{\hbar} \hat{\vec{L}} \cdot \vec{n} \delta\varphi$

$\hat{L}_x$: rotatie rond x-as

$\hat{R}^\dagger(\delta\vec{\varphi}) = \hat{R}^{-1}(\delta\vec{\varphi}) = \hat{R}(-\delta\vec{\varphi})$

$\langle \vec{r} | \hat{R}(\delta\vec{\varphi}) | \psi \rangle = \langle \hat{R}(-\delta\vec{\varphi}) \vec{r} | \psi \rangle = \langle \vec{r} - \delta\vec{\varphi} \times \vec{r} | \psi \rangle$

$= \psi(\vec{r} - \delta\vec{\varphi} \times \vec{r})$

$\vec{r}' = \vec{r} + \delta\vec{\varphi} \times \vec{r}$

$\simeq \psi(\vec{r}) - (\delta\vec{\varphi} \times \vec{r}) \cdot \vec{\nabla} \psi(\vec{r})$

$= \psi(\vec{r}) - \delta\vec{\varphi} \cdot (\vec{r} \times \vec{\nabla}) \psi(\vec{r})$

$\langle \vec{r} | \hat{R}(\delta\vec{\varphi}) | \psi \rangle = \langle \vec{r} | 1 - \frac{i}{\hbar} \delta\varphi \vec{L} \cdot \vec{n} | \psi \rangle$

$= \psi(\vec{r}) - \frac{i}{\hbar} \delta\vec{\varphi} \langle \vec{r} | \vec{L} | \psi \rangle$

$\rightarrow \langle \vec{r} | \hat{\vec{L}} | \psi \rangle = -i\hbar (\vec{r} \times \vec{\nabla}) \psi(\vec{r})$

$\hat{\vec{L}} = \hat{\vec{r}} \times \hat{\vec{p}}$

$\hat{L}_x = \hat{y}\hat{p}_z - \hat{z}\hat{p}_y$

$\hat{L}_y = \hat{z}\hat{p}_x - \hat{x}\hat{p}_z$

$\hat{L}_z = \hat{x}\hat{p}_y - \hat{y}\hat{p}_x$

$[\hat{L}_x, \hat{L}_y] = [\hat{y}\hat{p}_z - \hat{z}\hat{p}_y, \hat{z}\hat{p}_x - \hat{x}\hat{p}_z]$

$= [\hat{y}\hat{p}_z, \hat{z}\hat{p}_x] + [\hat{z}\hat{p}_y, \hat{x}\hat{p}_z] - [zp_y, zp_x] - [yp_z, xp_z]$

$= i\hbar \hat{L}_z$

$[L_i, L_j] = i\hbar \sum_k \epsilon_{ijk} \hat{L}_k$