$$\hat{L}^2|lm\rangle = \hbar^2 l(l+1)|lm\rangle$$
$$\hat{L}_z|lm\rangle = \hbar m|lm\rangle \; ; \; \langle\theta\varphi|lm\rangle = Y_{lm}(\theta;\varphi)$$

$$-i\frac{\partial}{\partial\varphi} Y_{lm}(\theta;\varphi) = m Y_{lm}(\theta;\varphi)$$
$$-\left(\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2} + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta}\right)\cancel{e^{im\varphi}} f_{lm}(\theta) = l(l+1)\cancel{e^{im\varphi}} f_{lm}(\theta)$$

$$Y_{lm}(\theta,\varphi) = \phi_m(\varphi)\, f_{lm}(\theta)$$
$$\frac{\partial}{\partial\varphi}\phi_m(\varphi) = im\,\phi(\varphi)$$
$$\phi_m(\varphi) = e^{im\varphi}$$

$$z = \cos\theta \to \frac{\partial}{\partial\theta} = \left(\frac{dz}{d\theta}\right)\frac{\partial}{\partial z}$$
$$-\frac{\partial}{\partial z} = \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}$$

$$\left\{(1-z^2)\frac{d^2}{dz^2} - 2z\frac{d}{dz} + l(l+1) - \frac{m^2}{1-z^2}\right\} f_{lm}(z) = 0 \to \text{Diff. vgl. van Legendre}$$

Stel $m=0 \Rightarrow \left\{(1-z^2)\frac{d^2}{dz^2} - 2z\frac{d}{dz} + l(l+1)\right\} P_l(z) = 0$ $\quad P_l(z) = \frac{1}{2^l l!}\frac{d^l}{dz^l}\left[(z^2-1)^l\right]$

Stel $v(z) = (z^2-1)^l$; $P_l(z) \sim \frac{d^l}{dz^l}v(z)$

$$(z^2-1)\frac{dv}{dz} = (z^2-1)l(z^2-1)^{l-1}2z = 2lzv$$

$l+1$ maal afleiden : $\frac{d^n}{dz^n}(fg) = \sum_{i=0}^{n}\binom{n}{i}\frac{d^i f}{dz^i}\frac{d^{n-i}g}{dz^{n-i}}$

$$(z^2-1)\left(\frac{d^{l+2}}{dz^{l+2}}v\right) + 2z\frac{d^{l+1}v}{dz^{l+1}} - l(l+1)\frac{d^l v}{dz^l} = 0$$
$$(z^2-1)P_l''(z) + 2zP_l'(z) - l(l+1)P_l(z) = 0$$

Wat met $m>0$: $(1-z^2)u'' - 2zu' + \left[l(l+1) - \frac{m^2}{1-z^2}\right]u = 0$

Leid m keer af naar z , $v = (z^2-1)^l$
$P_l(z) \sim \frac{d^l}{dz^l}v$

$$(1-z^2)u'' - 2z(m+1)u' + \left[l(l+1) - m(m+1)\right]u = 0$$

$$\omega(z) = (1-z^2)^{m/2}u(z)$$

$$(1-z^2)\omega'' - 2zm\omega' + \left[l(l+1) - \frac{m^2}{1-z^2}\right]\omega = 0$$

$P_{lm}(z) = (1-z^2)^{m/2}\frac{d^m}{dz^m}P_l(z)$: geassocieerde Legendre functies

Wat met $m<0$?
$$P_{l;-m}(z) = (-1)^m\frac{(l-m)!}{(l+m)!}P_{lm}(z)$$
$m>0$