1| Bekijk 1-D systeem met Hamiltoniaan $\hat H = \frac{\hat p^2}{2m} + V(\hat x)$

Toon aan dat: $\frac{\hbar^2}{2m} = \sum_{n'} (E_{n'} - E_n)|\langle \psi_n|\hat x|\psi_{n'}\rangle|^2$

$(\hat H|\psi_n\rangle = E_n|\psi_n\rangle)$

Tip: $[[\hat H,\hat x],\hat x] = ?$ $\Rightarrow$ $[\hat H,\hat x] = [\frac{\hat p^2}{2m}+V(\hat x),\hat x] = [\frac{\hat p^2}{2m},\hat x] + [V(\hat x),\hat x]$

$[\hat x,\hat p] = i\hbar$

$$= [\frac{\hat p\hat p}{2m},\hat x]$$
$$= \hat p[\frac{\hat p}{2m},\hat x] + [\frac{\hat p}{2m},\hat x]\hat p$$
$$= \frac{1}{2m}\{\hat p[\hat p,\hat x] + [\hat p,\hat x]\hat p\} = -\frac{i\hbar}{m}\hat p$$

$$\Rightarrow [[\hat H,\hat x],\hat x] = [-\frac{\hat p}{m}i\hbar,\hat x] = -[\hat x, -\frac{\hat p}{m}i\hbar]$$
$$= \frac{i\hbar}{m}[\hat x,\hat p]$$
$$= \frac{i\hbar\cdot i\hbar}{m}$$
$$= -\frac{\hbar^2}{m}$$

$$[[\hat H,\hat x],\hat x] = [\hat H\hat x - \hat x\hat H,\hat x]$$
$$= \hat H\hat x^2 - \hat x\hat H\hat x - \hat x\hat H\hat x + \hat x^2\hat H$$
$$= \hat H\hat x^2 + \hat x^2\hat H - 2\hat x\hat H\hat x$$

$$\langle\psi_n|[[\hat H,\hat x],\hat x]|\psi_n\rangle = \langle\psi_n|\hat H\hat x^2 + \hat x^2\hat H - 2\hat x\hat H\hat x|\psi_n\rangle$$

E_n E_n

$\left(\sum_{n'}|\psi_{n'}\rangle\langle\psi_{n'}| = \hat 1\right)$

$$= 2\langle\psi_n|E_n\hat x\hat x - \hat x\hat H\hat x|\psi_n\rangle$$

$\langle\psi_n|\hat x|\psi_{n'}\rangle^*$

$$= 2\sum_{n'}\{E_n\langle\psi_n|\hat x|\psi_{n'}\rangle\langle\psi_{n'}|\hat x|\psi_n\rangle - E_{n'}\langle\psi_n|\hat x|\psi_{n'}\rangle\langle\psi_{n'}|\hat x|\psi_n\rangle\}$$

$$= 2\sum_{n'}(E_n - E_{n'})|\langle\psi_n|\hat x|\psi_{n'}\rangle|^2$$

2| Bekijk de stappen in 2.3.10 en werk uit

a) Stel $\hat L_+$ en $\hat L_-$ op in sferische coördinaten (vertrekkend van $\hat L_x$ en $\hat L_y$)

$$\hat L_x = -i\hbar\left(-\sin\varphi\frac{\partial}{\partial\theta} - \cot\theta\cos\varphi\frac{\partial}{\partial\varphi}\right)$$
$$\hat L_y = -i\hbar\left(\cos\varphi\frac{\partial}{\partial\theta} - \cot\theta\sin\varphi\frac{\partial}{\partial\varphi}\right)$$
$$\hat L_\pm = \hat L_x \pm i\hat L_y = -i\hbar\Big(\underbrace{(-\sin\varphi \pm i\cos\varphi)}_{\pm i(\cos\varphi \pm i\sin\varphi) = \pm i e^{\pm i\varphi}}\frac{\partial}{\partial\theta} - \cot\theta\underbrace{(\cos\varphi \pm i\sin\varphi)}_{e^{\pm i\varphi}}\frac{\partial}{\partial\varphi}\Big)$$
$$= (-i\hbar)e^{\pm i\varphi}\left(\pm i\frac{\partial}{\partial\theta} - \cot\theta\frac{\partial}{\partial\varphi}\right)$$
$$= \hbar e^{\pm i\varphi}\left(\pm\frac{\partial}{\partial\theta} + i\cot\theta\frac{\partial}{\partial\varphi}\right)$$

b) Vertaal $\hat L_-|\ell,-\ell\rangle = 0$ in coördinatenrepresentatie. Noteer de oplossing opl.

$$Y_{\ell m}(\theta,\varphi) = e^{im\varphi}\,\Theta_{\ell m}(\theta)$$

Check: $\Theta_{\ell,\ell}(\theta) = c_\ell(\sin\theta)^\ell$

$$\hat L_- = \hbar e^{-i\varphi}\left(-\frac{\partial}{\partial\theta} + i\cot\theta\frac{\partial}{\partial\varphi}\right)$$
$$\Rightarrow \hat L_-|\ell,-\ell\rangle = \cancel{\hbar e^{-i\varphi}}\left(-\frac{\partial}{\partial\theta} + i\cot\theta\frac{\partial}{\partial\varphi}\right)\cancel{e^{i\ell\varphi}}\,\Theta_{\ell\ell}(\theta) = 0$$

$-i\ell$

$$\frac{\partial}{\partial\theta}\Theta_{\ell,\ell} - \ell\cot\theta\,\Theta_{\ell,\ell}(\theta) = 0$$
$$Y_{\ell,-\ell}(\theta,\varphi) = e^{-i\ell\varphi}c_\ell(\sin\theta)^\ell$$

$$\int d\Omega\, Y^*_{\ell,-\ell}(\theta,\varphi)\,Y_{\ell,-\ell}(\theta,\varphi) = 1$$
$$\int_0^{2\pi}d\varphi\int_0^\pi d\theta\sin\theta\, Y^*_{\ell,-\ell}(\theta,\varphi)\,Y_{\ell,-\ell}(\theta,\varphi) = 1$$
$$= (2\pi)|c_\ell|^2\int_0^\pi d\theta\sin\theta(\sin\theta)^{2\ell} = 1$$

$$I_\ell = \int_0^\pi d\theta(\sin\theta)^{2\ell+1}$$
$$= \int_0^\pi d\theta\left(-\frac{d}{d\theta}\cos\theta\right)(\sin\theta)^{2\ell}$$
$$= +\int_0^\pi d\theta\cos\theta\,(2\ell)(\sin\theta)^{2\ell-1}\cancel{\cos\theta}$$
$$-\cancel{\left[\cos\theta(\sin\theta)^{2\ell}\right]_0^\pi}$$
$$= 2\ell\int_0^\pi d\theta(1-\sin^2\theta)(\sin\theta)^{2\ell-1}$$
$$I_\ell = 2\ell(I_{\ell-1} - I_\ell)$$

$$I_\ell = \frac{2\ell}{2\ell+1}I_{\ell-1}$$
$$= \frac{(2\ell)(2\ell-2)(2\ell-4)\dots 2\, I_0}{(2\ell+1)(2\ell-1)(2\ell-3)\dots 1}$$
$$= \frac{2^\ell\ell!}{(2\ell+1)!!} = \frac{2^{2\ell}(\ell!)^2}{(2\ell+1)!}$$
$$(2\ell+1)!! = (2\ell+1)(2\ell-1)\dots 1$$
$$= \frac{(2\ell+1)(2\ell)(2\ell-1)(2\ell-2)\dots 1}{(2\ell)(2\ell-2)\dots 1}$$
$$= \frac{(2\ell+1)!}{2^\ell\ell!}$$