$$\hat{L}^2 = -\hbar^2\left(\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2} + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)\right)$$

$$\hat{L}_z = -i\hbar\frac{\partial}{\partial\varphi}$$

$$\hat{L}^2|\ell m\rangle = \hbar^2\ell(\ell+1)|\ell m\rangle$$

$$\hat{\vec{L}} = \hat{\vec{r}}\times\hat{\vec{p}}$$

$$\hat{L}_z|\ell m\rangle = m\hbar|\ell m\rangle$$

$$\langle\theta\varphi|\ell m\rangle = Y_{\ell m}(\theta;\varphi)$$

$$\langle\vec{r}|\psi\rangle = \psi(\vec{r})$$

$$\begin{cases} -i\frac{\partial}{\partial\varphi}Y_{\ell m}(\theta;\varphi) = m\,Y_{\ell m}(\theta;\varphi) \\ -\left(\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2} + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)\right)Y_{\ell m}(\theta;\varphi) = \ell(\ell+1)\,Y_{\ell m}(\theta;\varphi) \end{cases}$$

Product: $Y_{\ell m}(\theta;\varphi) = \Phi_m(\varphi)\,f_{\ell m}(\theta)$

$$(1)\left(-i\frac{\partial}{\partial\varphi}\Phi_m(\varphi)\right)\cancel{f_{\ell m}(\theta;\varphi)} = m\,\Phi_m(\varphi)\cancel{f_{\ell m}(\theta)}$$

$$\Phi_m(\varphi) = C_m e^{im\varphi}$$

$$\left\{\frac{1}{\sin^2\theta}(-m^2) + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \ell(\ell+1)\right\}f_{\ell m}(\theta) = 0$$

$$z = \cos\theta$$

$$\frac{\partial}{\partial\theta} = \left(\frac{\partial z}{\partial\theta}\right)\frac{\partial}{\partial z}$$

$$\frac{\partial}{\partial\theta} = -\sin\theta\frac{\partial}{\partial z}$$

$$\frac{1}{\sin\theta}\frac{\partial}{\partial\theta} = -\frac{\partial}{\partial z}$$

$$\left\{\frac{\partial}{\partial z}\left((1-z^2)\frac{\partial}{\partial z}\right) - \frac{m^2}{1-z^2} + \ell(\ell+1)\right\}F_{\ell m}(z) = 0$$

$$\left\{(1-z^2)\frac{d^2}{dz^2} - 2z\frac{d}{dz} + \ell(\ell+1) - \frac{m^2}{1-z^2}\right\}F_{\ell m}(z) = 0$$

$$\theta\in[0;\pi];\ z\in[-1;1]$$

Neem $m=0$

$$\left\{(1-z^2)\frac{d^2}{dz^2} - 2z\frac{d}{dz} + \ell(\ell+1)\right\}F_{\ell 0}(z) = 0 \quad \to P_\ell(z)$$

Diff. vgl. van Legendre

$$P_\ell(z) = \frac{1}{2^\ell\,\ell!}\frac{d^\ell}{dz^\ell}(z^2-1)^\ell$$

$$v(z) = (z^2-1)^\ell$$

$$(z^2-1)\,v'(z) = (z^2-1)\,\ell(z^2-1)^{\ell-1}(2z)$$

$$= 2\ell z\cdot v(z)$$

$$\frac{d^n}{dz^n}(f(z)g(z)) = \sum_{i=0}^{n}\binom{n}{i}\left(\frac{d^i}{dz^i}f\right)\left(\frac{d^{n-i}}{dz^{n-i}}g\right)$$

Linkerlid: $(z^2-1)\,v^{(\ell+2)} + (\ell+1)(2z)\,v^{(\ell+1)} + \cancel{\frac{1}{2}}(\ell+1)(\ell)\cancel{2}\,v^{(\ell)}$

Rechterlid: $2\ell z\,v^{(\ell+1)} + (\ell+1)2\ell\,v^{(\ell)}$

$$(z^2-1)\,v^{(\ell+2)} + 2z\,v^{(\ell+1)} - \ell(\ell+1)\,v^{(\ell)} = 0$$

$$(z^2-1)\,P_\ell'' + 2z\,P_\ell'(z) - \ell(\ell+1)\,P_\ell(z) = 0$$

$$(1-z^2)\,v^{(2)} - 2z\,v^{(1)} + \ell(\ell+1)\,v = 0$$

Leid m keer af naar z: $\binom{m}{0}=1;\ \binom{m}{1}=m,\ \binom{m}{2}=\frac{1}{2}m(m-1)$

$$\begin{cases} (1-z^2)\,v^{(m+2)} - 2z\,v^{(m+1)}m - \cancel{2}\,v^{(m)}\cancel{\tfrac{1}{2}}m(m-1) \\ -2z\,v^{(m+1)} - 2v^{(m)}m + \ell(\ell+1)\,v^{(m)} = 0 \end{cases}$$

$$\to (1-z^2)\,u^{(2)} - 2(m+1)z\,u^{(1)} + [\ell(\ell+1) - m(m+1)]\,u = 0$$

$$-m(m-1) - 2m = -m(m+1)$$

Stel: $w = (1-z^2)^{m/2}\,u$