Lagrangiaan $L(q_e; \dot q_e; t)$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot q_e}\right) = \frac{\partial L}{\partial q_e} = 0 \ (q_e \text{ cyclisch})$$

$$p_e = \frac{\partial L}{\partial \dot q_e} \text{ is behouden}$$

Voorbeelden symmetrie

1) Carthesisch: $x_1; x_2; x_3$

$$\frac{\partial L}{\partial x_1} = 0$$

$$L = \frac{1}{2}m(\dot x_1^2 + \dot x_2^2 + \dot x_3^2) - V(x_2; x_3)$$

$p_1 = \frac{\partial L}{\partial \dot x_1} = m\dot x_1$ is behouden

(2) $L = \frac{1}{2}m(\dot\rho^2 + \dot z^2 + \rho^2\dot\theta^2) - V(\rho; z; \cancel{\theta})$

$$\frac{\partial V}{\partial \theta} = 0 = \frac{\partial L}{\partial \theta} \Rightarrow \frac{\partial L}{\partial \dot\theta} = p_\theta = m\rho^2\dot\theta = L_z$$

Symmetrie-operator: $\hat U$, unitair: $\hat U^\dagger \hat U = \hat 1$

$$|\psi'\rangle = \hat U|\psi\rangle$$

$$\langle\psi'|\hat O'|\phi'\rangle = \langle\psi|\hat O|\phi\rangle \Rightarrow \langle\psi|\hat U^\dagger \hat O' \hat U|\phi\rangle$$

$$\hat O' = \hat U \hat O \hat U^\dagger$$

$\hookrightarrow \hat H' = \hat U \hat H \hat U^\dagger = \hat H$

$\Rightarrow \hat H \hat U = \hat U \hat H$

$\Rightarrow [\hat U, \hat H] = 0$

Als $i\hbar \frac{d}{dt}|\psi(t)\rangle = \hat H|\psi(t)\rangle$

$\Rightarrow i\hbar \frac{d}{dt}(\hat U^\dagger|\psi'(t)\rangle) = \hat H \hat U^\dagger|\psi'(t)\rangle$

$\Rightarrow i\hbar \hat U^\dagger \frac{d}{dt}(|\psi'(t)\rangle) = \hat H \hat U^\dagger|\psi'(t)\rangle$

$\Rightarrow i\hbar \frac{d}{dt}(|\psi'(t)\rangle) = \underbrace{\hat U \hat H \hat U^\dagger}_{\hat H}|\psi'(t)\rangle$

Groep! $[\hat U, \hat H] = 0$

$\hat U_1 [\hat U_2, \hat H] + [\hat U_1, \hat H]\hat U_2$

$[\hat U_1, H] = 0$
$[\hat U_2, H] = 0$ $\Rightarrow [\hat U_1 \hat U_2, \hat H] = 0$

Stel $\hat H|\psi_n\rangle = E_n|\psi_n\rangle$

en $[\hat U, \hat H] = 0$ of $\hat U \hat H = \hat H \hat U$

Dan $|\psi_n'\rangle = \hat U|\psi_n\rangle$ is ook eigentoestand, met zelfde energie

$$\hat H|\psi_n'\rangle = \hat H \hat U|\psi_n\rangle = \hat U \hat H|\psi_n\rangle$$
$$= E_n \hat U|\psi_n\rangle$$
$$= E_n|\psi_n'\rangle$$

Als $|\psi_n\rangle, \hat U|\psi_n\rangle$ lineair onafhankelijk.

Dan: E_n ontaard

Infinitesimale unitaire operator $\to$ generator

$$\hat U = \hat 1 - i\epsilon\hat G; \ \epsilon \to 0$$

$$\hat U \hat U^\dagger = (\hat 1 - i\epsilon\hat G)(\hat 1 + i\epsilon\hat G^\dagger)$$
$$= \hat 1 - i\epsilon\underbrace{(\hat G - \hat G^\dagger)} + O(\epsilon^2)$$
$$= \hat 1 \quad \text{Als } \hat G = \hat G^\dagger$$

Translatie:

$\hat T(d\vec r) = \hat 1 - i\vec G \cdot d\vec r$

$= \hat 1 - i(\hat G_x dx + \hat G_y dy + \hat G_z dz)$

$\hat G_x = \frac{\hat p_x}{\hbar}$, etc.

$\hat T(d\vec r) = \hat 1 - \frac{i}{\hbar}\hat{\vec p} \cdot d\vec r$

Eindige translatie: $\vec a = N\left(\frac{\vec a}{N}\right)$

$d\vec r = \frac{\vec a}{N}$

$$\hat T(\vec a) = \lim_{N\to\infty}\left(\hat 1 - \frac{i\hat{\vec p}}{\hbar} \cdot \frac{\vec a}{N}\right)^N$$
$$= \exp\left(-\frac{i}{\hbar}\hat{\vec p} \cdot \vec a\right)$$

$\to \hat T(d\vec r')|\vec r'\rangle = |\vec r' + d\vec r'\rangle$

$\hat{\vec r}\,\hat T(d\vec r')|\vec r'\rangle = \hat{\vec r}|\vec r' + d\vec r'\rangle$
$= (\vec r' + d\vec r')|\vec r' + d\vec r'\rangle$

Ook: $\hat{\vec r}|\vec r'\rangle = \vec r'|\vec r'\rangle$; linksmult $\hat T(d\vec r')$

$\hat T(d\vec r')\hat{\vec r}|\vec r'\rangle = \vec r'|\vec r' + d\vec r'\rangle$

$(\hat{\vec r}\hat T(d\vec r') - \hat T(d\vec r')\hat{\vec r})|\vec r'\rangle = d\vec r'|\vec r' + d\vec r'\rangle = d\vec r'|\vec r'\rangle$

$[\hat{\vec r}, \hat T(d\vec r')] = d\vec r'$, $[\hat r_i, \hat T(d\vec r')] = dr_i'$

$$\hat T(d\vec r') = \hat 1 - \frac{i\hat{\vec p}}{\hbar} \cdot d\vec r' = 1 - \frac{i}{\hbar}\sum_j \hat p_j dr_j'$$

$$\left[\hat r_i, \frac{-i}{\hbar}\sum_j \hat p_j dr_j'\right] = \sum_j \delta_{ij} dr_j'$$

$$\boxed{[\hat r_i, \hat p_j] = i\hbar\delta_{ij}}$$