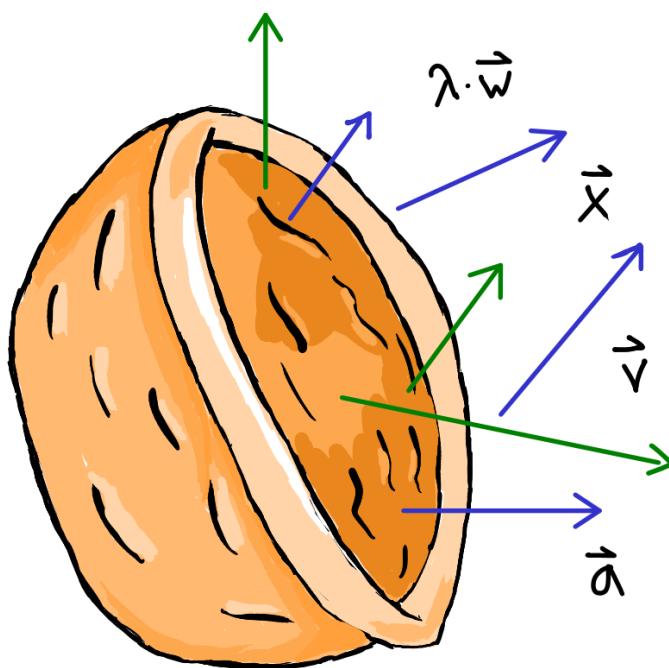


Linear Algebra

- in a nutshell -

This book was created and used for the lecture at Hamburg University of Technology in the winter term 2018/19 for General Engineering Science and Computer Science students.

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J.P.G.

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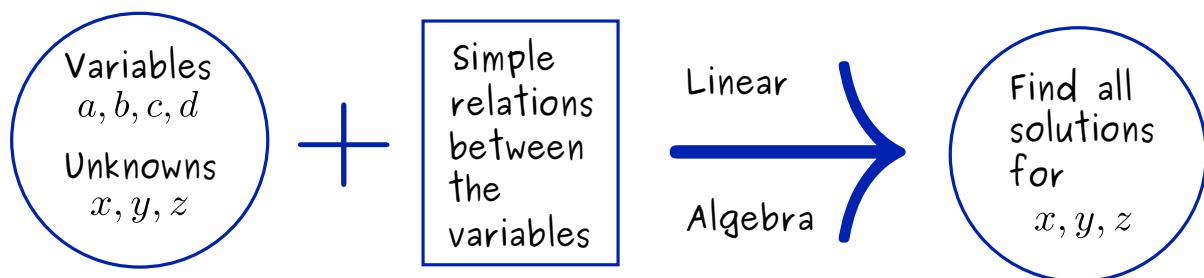
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Some words

This text should help you to understand the course Linear Algebra. To expand your knowledge, you can look into the following books:

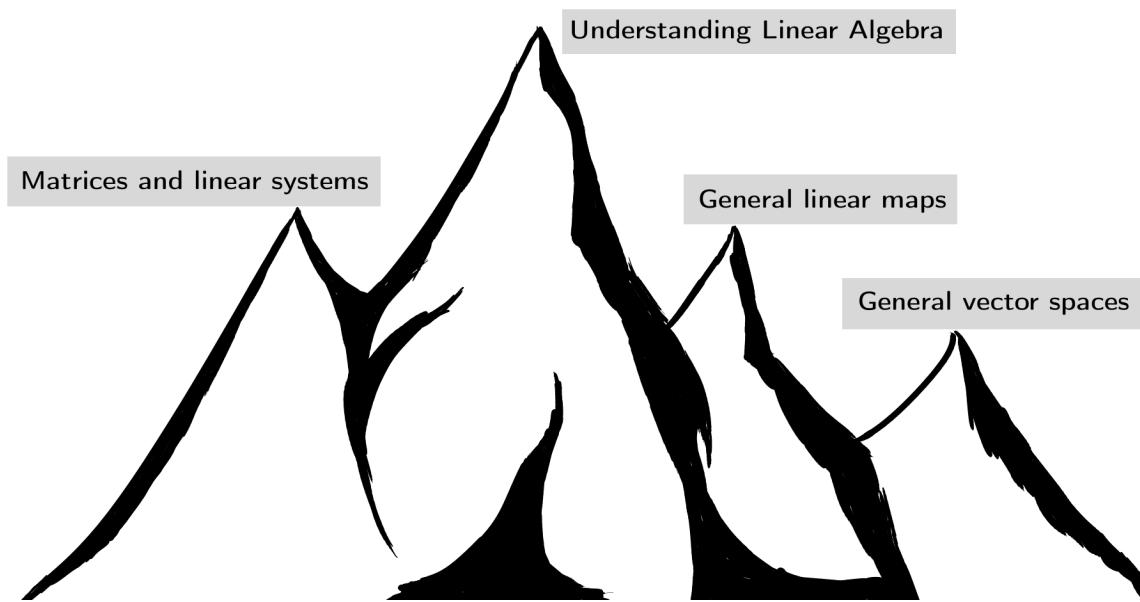
- Gilbert Strang: *Introduction to Linear Algebra*,
- Sheldon Axler: *Linear Algebra Done Right*,
- Gerald Teschl, Susanne Teschl: *Mathematik für Informatiker, Band 1*.
- Shin Takahashi, Iroha Inoue: *The Manga Guide to Linear Algebra*.
- Klaus Jänich: *Lineare Algebra*.

Linear Algebra is a very important topic and useful in different applications. We¹ discuss simple examples later. However, the main idea is that we have a problem consisting of a lot of quantities where some are fixed and others can be altered or are not known. However, if we know the relations between the quantities, we use *Linear Algebra* to find all the possible solutions for the *Unknowns*.



That would be the calculation side of the world regarding *Linear Algebra*. In this lecture, we will concentrate on understanding the field as a whole. Of course, this is not an easy task and it will be a hiking tour that we will do together. The summit and goal is to understand why solving equations is indeed a meaningful mathematical theory.

¹In mathematical texts, usually, the first-person plural is used even if there is only one author. Most of the time it simply means “we” = “I (the author) and the reader”.



We start in the valley of mathematics and will shortly scale the first hills. Always stay in shape, practise and don't hesitate to ask about the ways up. It is not an easy trip but you can do it. Maybe the following tips can guide you:

- You will need a lot of time for this course if you really want to *understand* everything you learn. Hence, make sure that you have enough time each week to do mathematics and keep these time slots clear of everything else.
- Work in groups, solve problems together and discuss your solutions. Learning mathematics is not a competition.
- Explain the content of the lectures to your fellow students. Only the things you can illustrate and explain to others are really understood by you.
- Learn the Greek letters that we use in mathematics:

α	alpha	β	beta	γ	gamma	Γ	Gamma
δ	delta	ϵ	epsilon	ε	epsilon	ζ	zeta
η	eta	θ	theta	Θ	Theta	ϑ	theta
ι	iota	κ	kappa	λ	lambda	Λ	Lambda
μ	mu	ν	nu	ξ	xi	Ξ	Xi
π	pi	Π	Pi	ρ	rho	σ	sigma
Σ	Sigma	τ	tau	υ	upsilon	Υ	Upsilon
ϕ	phi	Φ	Phi	φ	phi	χ	chi
ψ	psi	Ψ	Psi	ω	omega	Ω	Omega

This video may help you there:

<https://jp-g.de/bsom/la/greek/>



- Choosing a book is a matter of taste. Look into different ones and choose the book that really convinces you.
- Keep interested, fascinated and eager to learn. However, do not expect to understand everything at once.

1

Foundations of mathematics

It is a mistake to think you can solve any major problems just with potatoes.

Douglas Adams

Before starting with *Linear Algebra*, we first have to learn the mathematical language, which consists of symbols, logic, sets, numbers, maps and so on. We also talk about the concept of a mathematical proof. These things build up the mathematical foundation.

A little bit of knowledge about numbers and how to calculate with them is assumed but not much more than that. All symbols are introduced such that you know how to work with them. However, if you interested in a more detailed discussion, I can recommend you my video series about the foundations of mathematics:

Video: Start Learning Mathematics



Start Learning
Mathematics

\mathbb{N}

$\forall \exists$

$A \cap B$

<https://jp-g.de/bsom/la/slm/>



1.1 Logic and sets

Basic logic is something, we usually accomplish intuitively right. However, in mathematics we have to define it in an unambiguous way and it may differ a little bit from the everyday logic. It is very important and useful to bring into our attention some of the basic rules and notations of logic. For Computer Science students, logic is considered in more detail in other courses.

Let us start with a definition:

Definition 1.1. logical statement, proposition

A *logical statement* (or *proposition*) is a statement, which means a meaningful declarative sentence, that is either true or false.

Instead of *true*, one often writes T or 1 and instead of *false*, one often writes F or 0.

Not every meaningful declarative fulfils this requirement. There are opinions, alternative facts, self-contradictory statements, undecidable statements and so on. In fact, a lot of examples here, outside the mathematical world, work only if we give the words unambiguous definitions which we will implicitly do.

Example 1.2. Which of these are logical statements?

- (a) Hamburg is a city.
- (b) $1 + 1 = 2$.
- (c) The number 5 is smaller than the number 2.
- (d) Good morning!
- (e) $x + 1 = 1$.
- (f) Today is Tuesday.

The last two examples are not logical statements but so-called predicates and will be considered later.

Logical operations

For given logical statements, one can form new logical statements with so-called *logical operations*. In the following, we will consider two logical statements A and B .

Definition 1.3. Negation $\neg A$ (“not A ”)

$\neg A$ is true if and only if A is false.

Truth table	A	$\neg A$	(1.1)
	T	F	
	F	T	

Example 1.4. What are the negations of the following logical statements?

- (a) The wine bottle is full.
- (b) The number 5 is smaller than the number 2.
- (c) All students are in the lecture hall.

Definition 1.5. Conjunction $A \wedge B$ (“ A and B ”)

$A \wedge B$ is true if and only if both A and B are true.

	A	B	$A \wedge B$	
Truth table	T	T	T	(1.2)
	T	F	F	
	F	T	F	
	F	F	F	

Definition 1.6. Disjunction $A \vee B$ (“A or B”)

$A \vee B$ is true if and only if at least one of A or B is true.

	A	B	$A \vee B$	
Truth table	T	T	T	(1.3)
	T	F	T	
	F	T	T	
	F	F	F	

Definition 1.7. Conditional $A \rightarrow B$ (“If A then B”)

$A \rightarrow B$ is only false if A is true but B is false.

	A	B	$A \rightarrow B$	
Truth table	T	T	T	(1.4)
	T	F	F	
	F	T	T	
	F	F	T	

Definition 1.8. Biconditional $A \leftrightarrow B$ (“A if and only if B”)

$A \leftrightarrow B$ is true if and only if $A \rightarrow B$ and $B \rightarrow A$ is true.

	A	B	$A \leftrightarrow B$	
Truth table	T	T	T	(1.5)
	T	F	F	
	F	T	F	
	F	F	T	

If a conditional or biconditional is true, we have a short notation for this that is used throughout the whole field of mathematics:

Definition 1.9. Implication and equivalence

If $A \rightarrow B$ is true, we call this an implication and write:

$$A \Rightarrow B .$$

If $A \leftrightarrow B$ is true, we call this an equivalence and write:

$$A \Leftrightarrow B .$$

This means that we speak of *equivalence* of A and B if the truth values in the truth table are exactly the same. For example, we have

$$A \leftrightarrow B \Leftrightarrow (A \rightarrow B) \wedge (B \rightarrow A).$$

Now one can ask: *What to do with truth-tables?* Let us show that $\neg B \rightarrow \neg A$ is the same as $A \rightarrow B$.

	A	B	$\neg A$	$\neg B$	$\neg B \rightarrow \neg A$
Truth table	T	T	F	F	T
	T	F	F	T	F
	F	T	T	F	T
	F	F	T	T	T

(1.6)

Therefore:

$$(A \rightarrow B) \Leftrightarrow (\neg B \rightarrow \neg A).$$

This is the *proof by contraposition*:

“Assume that B does not hold, then we can show that A cannot hold as well”. Hence A implies B .

Contraposition

If $A \Rightarrow B$, then also $\neg B \Rightarrow \neg A$.

Rule of thumb: Contraposition

To get the contraposition $A \Rightarrow B$, you should exchange A and B and set a \neg -sign in front of both: $\neg B \Rightarrow \neg A$.

It is clear: The contraposition of the contraposition is again $A \Rightarrow B$.

The contraposition is an example of a [deduction rule](#), which basically tells us how to get new true proposition from other true propositions. The most important deduction rules are given just by using the implication.

Modus ponens

If $A \Rightarrow B$ and A is true, then also B is true.

Chain syllogism

If $A \Rightarrow B$ and $B \Rightarrow C$, then also $A \Rightarrow C$.

Reductio ad absurdum

If $A \Rightarrow B$ and $A \Rightarrow \neg B$, then $\neg A$ is true.

One can easily prove these rules by truth tables. However, here we do not state every deduction in this formal manner. We may still use deduction in the intuitive way as well. Try it here:

Exercise 1.10. Let “All birds can fly” be a true proposition (axiom). Are the following deductions correct?

- If Seagulls are birds, then Seagulls can fly.

- If Penguins are birds, then Penguins can fly.
- If Butterflies are birds, then Butterflies can fly.
- If Butterflies can fly, then Butterflies are birds.

Sets

Modern mathematics does not say what sets are, but only specifies rules. This is, however, too difficult for us right now, and we rather cite the attempt of a definition by Georg Cantor:

“Unter einer ‚Menge‘ verstehen wir jede Zusammenfassung von bestimmten wohlunterschiedenen Objekten unserer Anschauung oder unseres Denkens zu einem Ganzen.”

Definition 1.11. Set, element

A set is a collection into a whole of definite, distinct objects of our perception or of our thought. Such an object x of a set M is called an element of M and one writes $x \in M$. If x is not such an object of M , we write $x \notin M$.

A set is defined by giving all its elements $M := \{1, 4, 9\}$.

The symbol “ $:=$ ” is read as defined by and means that the symbol M is newly introduced as a set by the given elements.

Example 1.12.

- The empty set $\{\} = \emptyset = \varnothing$ is the unique set that has no elements at all.
- The set that contains the empty set $\{\emptyset\}$, which is non-empty since it has exactly one element.
- A finite set of numbers is $\{1, 2, 3\}$.

Notation 1.13.

Let A, B be sets:

- $x \in A$ means x is an element of A
- $x \notin A$ means x is not an element of A
- $A \subset B$ means A is a subset of B : every element of A is contained in B
- $A \supset B$ means A is a superset of B : every element of B is contained in A
- $A = B$ means $A \subset B \wedge A \supset B$. Note that the order of the elements does not matter in sets. If we want the order to matter, we rather define *tuples*: $(1, 2, 3) \neq (1, 3, 2)$. For sets, we always have $\{1, 2, 3\} = \{1, 3, 2\}$.
- $A \subsetneq B$ means A is a “proper” subset of B , every element of A is contained in B , but $A \neq B$.

The important number sets

- \mathbb{N} is the set of the natural numbers $1, 2, 3, \dots$;
- \mathbb{N}_0 is the set of the natural numbers and zero: $0, 1, 2, 3, \dots$;
- \mathbb{Z} is the set of the integers, which means $\dots, -3, -2, -1, 0, 1, 2, 3, \dots$;
- \mathbb{Q} is the set of the rational numbers, which means all fractions $\frac{p}{q}$ with $p \in \mathbb{Z}$ and $q \in \mathbb{N}$;
- \mathbb{R} is the set of the real numbers (see next semester).

Other ways to define sets:

$$A = \{n \in \mathbb{N} : 1 \leq n \leq 300\}$$

$$\mathbf{P}(B) = \{M : M \subset B\} \text{ power set: set of all subsets of } B$$

$$I = \{x \in \mathbb{R} : 1 \leq x < \pi\} = [1, \pi) \text{ half-open interval}$$

More about these constructions later.

Definition 1.14. Cardinality

We use vertical bars $|\cdot|$ around a set to denote the number of elements. For example, we have $|\{1, 4, 9\}| = 3$. The number of elements is called the cardinality of the set.

Example 1.15. $|\{1, 3, 3, 1\}| = 2$, $|\{1, 2, 3, \dots, n\}| = n$, $|\mathbb{N}| = \infty$ (?)

Exercise 1.16. Which of the following logical statements are true?

$$\begin{array}{llll} \boxed{3 \in \mathbb{N}}, & \boxed{12034 \in \mathbb{N}}, & \boxed{-1 \in \mathbb{N}}, & \boxed{0 \in \mathbb{N}_0} \\ \boxed{-1 \in \mathbb{Z}}, & \boxed{0 \notin \mathbb{Z}}, & \boxed{-2.7 \in \mathbb{Z}}, & \boxed{\frac{2}{3} \in \mathbb{Z}}, \\ \boxed{\frac{2}{3} \in \mathbb{Q}}, & \boxed{-3 \in \mathbb{Q}}, & \boxed{-2.7 \in \mathbb{Q}}, & \boxed{\sqrt{2} \in \mathbb{Q}}, \\ \boxed{\sqrt{2} \in \mathbb{R}}, & \boxed{\sqrt{-2} \in \mathbb{R}}, & \boxed{-\frac{2}{3} \in \mathbb{R}}, & \boxed{0 \in \mathbb{R}}. \end{array}$$

Predicates and quantifiers

Definition 1.17. Predicate

If X is any set and $A(x)$ is a logical statement depending on $x \in X$ (and true or false for every $x \in X$), we call $A(x)$ a predicate with variable x . Usually, one writes simply $A(x)$ instead of $A(x) = \text{true}$.

Example 1.18.

$$X = \mathbb{R} \quad A(x) = "x < 0"$$

Then we can define the set

$$\{x \in X : A(x)\} = \{x \in \mathbb{R} : x < 0\}$$

Definition 1.19. Quantifiers \forall and \exists

We use \forall (“for all”) and \exists (“it exists”) and call them quantifiers. Moreover, we use the double point “ $:$ ” inside the set brackets, which means “that fulfil”.

The quantifiers and predicates are very useful for a compact notation:

- $\forall x \in X : A(x)$ for all $x \in X$ $A(x)$ is true
- $\exists x \in X : A(x)$ there exists at least one $x \in X$ for which $A(x)$ is true
- $\exists! x \in X : A(x)$ there exists exactly one $x \in X$ for which $A(x)$ is true

Negation of statements with quantifiers:

- $\neg(\forall x \in X : A(x)) \Leftrightarrow \exists x \in X : \neg A(x)$
- $\neg(\exists x \in X : A(x)) \Leftrightarrow \forall x \in X : \neg A(x)$

Example 1.20. There is no greatest natural number:

$$A(n) = \text{“}n \text{ is the greatest natural number”}$$

In our notation: $\neg(\exists n \in \mathbb{N} : A(n))$ this is the same as $\forall n \in \mathbb{N} : \neg A(n)$, i.e. *Each $n \in \mathbb{N}$ is not the greatest natural number*. But this is clear, because $n + 1 > n$.

Rule of thumb: Negation of the quantifier (\forall and \exists)

$$\neg \forall = \exists \neg \quad \text{and} \quad \neg \exists = \forall \neg$$

Example 1.21. The set $M := \{x \in \mathbb{Z} : x^2 = 25\}$ is defined by the set of each integer x that squares to 25. We immediately see that this is just -5 and 5 .

$$\begin{aligned} \{x \in \mathbb{Z} : x^2 = 25\} &= \{-5, 5\}, \\ \{x \in \mathbb{N} : x^2 = 25\} &= \{5\}, \\ \{x \in \mathbb{R} : x^2 = -25\} &= \emptyset. \end{aligned}$$

In other words: The equation $x^2 = 25$ with unknown x has, depending in which number realm you want to solve it, one or two solutions, and the equation $x^2 = -25$ has no solution in the real numbers. However, we will find solutions in the complex numbers as we will see later.

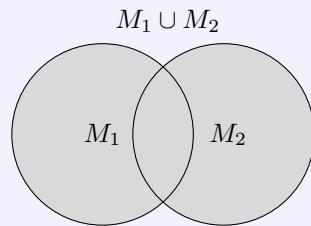
Operations on sets

We remember the important operations for sets:

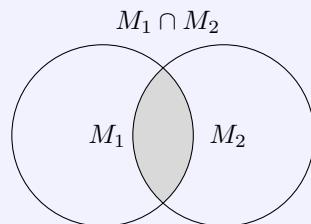
- $M_1 \cup M_2 := \{x : x \in M_1 \vee x \in M_2\}$ (union)
- $M_1 \cap M_2 := \{x : x \in M_1 \wedge x \in M_2\}$ (intersection)
- $M_1 \setminus M_2 := \{x : x \in M_1 \wedge x \notin M_2\}$ (set difference)

Definition 1.22. Set compositions

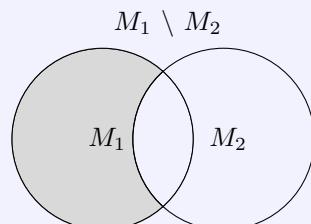
The union $M_1 \cup M_2$ is the new set that consists exactly of the objects that are elements of M_1 **or** M_2 .



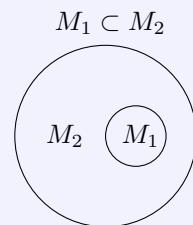
The intersection $M_1 \cap M_2$ is the new set whose elements are the objects that are elements of M_1 **and** M_2 .



We write $M_1 \setminus M_2$ for the set difference whose elements are the objects that are elements of M_1 **but not** elements of M_2 .



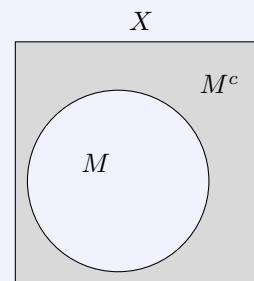
A subset of M_2 is each set whose elements are also elements of M_2 .



Definition 1.23. Complement set

Let X be a set. Then for a subset $M \subset X$ there is a unique complement of M with respect to X :

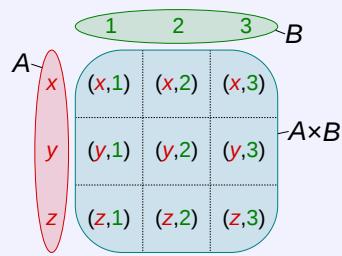
$$M^c := X \setminus M = \{x \in X : x \notin M\}$$



Definition 1.24. Product set

The Cartesian product of two sets A, B is given as the set of all pairs (two elements with order):

$$A \times B := \{(a, b) : a \in A, b \in B\}$$



(Source of the picture: Author Quartl - Wikipedia)

In the same sense, for sets A_1, \dots, A_n the set of all n -tupels is defined:

$$A_1 \times \dots \times A_n := \{(a_1, \dots, a_n) : a_1 \in A_1, \dots, a_n \in A_n\}$$

Exercise 1.25. Which statements are correct?

$$\{1, 3\} \cup \{2, 4\} = \{1, 2, 4\}, \quad \boxed{\{1, 2\} \cup \{3, 4\} = \{3, 2, 4, 2, 1\}}, \quad \boxed{\mathbb{N} \cup \mathbb{Z} = \mathbb{Z}}.$$

$$\boxed{\{1, 2, 4\} \cap \{3, 4, 5\} = \{4\}}, \quad \boxed{\{1, 3\} \cap \{2, 4\} = \emptyset}, \quad \mathbb{N} \cap \mathbb{Z} = \mathbb{N}_0.$$

$$\boxed{\{1, 2, 4\} \setminus \{3, 4, 5\} = \{1\}}, \quad \boxed{\mathbb{N}_0 \setminus \mathbb{N} = \{0\}}, \quad \boxed{\mathbb{N} \setminus \mathbb{Z} = \emptyset}.$$

$$\mathbb{Z} \setminus \mathbb{N} = \{-x : x \in \mathbb{N}\}, \quad \boxed{\mathbb{N} \subset \mathbb{N}_0}, \quad \mathbb{Z} \subset \mathbb{N}_0, \quad \boxed{(\mathbb{Z} \setminus \mathbb{Q}) \subset \mathbb{N}}.$$

$$\boxed{\mathbb{N} \subset \mathbb{N}}, \quad \boxed{-3 \in \mathbb{Z} \setminus \mathbb{N}_0}, \quad \boxed{\frac{3}{7} \in \mathbb{Q} \setminus \mathbb{Z}}, \quad \boxed{\sqrt{2} \in \mathbb{R} \setminus \mathbb{Q}},$$

Exercise 1.26. Which claims are correct? Prove or give a counterexample.

- (a) $(\mathbb{Q} \setminus \mathbb{R}) \subset \mathbb{N}_0$.
- (b) Let A, B, C be three sets. Then one has $A \cup (B \cap C) = (A \cup B) \cap C$.
- (c) Let A, B, C be three sets. Then one has $A \cap (B \cap C) = (A \cap B) \cap C$.
- (d) Let A, B, C be three sets. Then one has $A \setminus (B \cup C) = (A \setminus B) \cap (A \setminus C)$.

Exercise 1.27. Describe the following sets and calculate its cardinalities:

- (a) $X_1 := \{x \in \mathbb{N} : \exists a, b \in \{1, 2, 3\} \text{ with } x = a - b\}$
- (b) $X_2 := \{(a - b) : a, b \in \{1, 2, 3\}\}$
- (c) $X_3 := \{|a - b| : a, b \in \{1, 2, 3\}\}$
- (d) $X_4 := \{1, \dots, 20\} \setminus \{n \in \mathbb{N} : \exists a, b \in \mathbb{N} \text{ with } 2 \leq a \text{ and } 2 \leq b \text{ and } n = a \cdot b\}$.
- (e) $X_5 := \{S : S \subset \{1, 2, 3\}\}$.

1.2 Real Numbers

Everybody has got to know at school, the rational numbers, the real, and basic arithmetics. There are certain rules that we can apply, and usually we do not think about them.

In our lecture, we will get to know other objects than real numbers (vectors, matrices), where some of these laws do not apply any more. So try to have a fresh look at those well known laws:

We can add $(a+b)$ and multiply (ab) or $a \cdot b$ real numbers and use parentheses (\cdot) to describe the order of the computations. We have the notational *convention* that multiplication binds stronger than addition: $(ab + c)$ means $(ab) + c$ and not $ab + c = a(b + c)$

Some *laws* apply:

$$\begin{array}{lll} a + (b + c) = (a + b) + c, & a(bc) = (ab)c & \text{associative law} \\ a + b = b + a & ab = ba & \text{commutative law} \\ a(b + c) = ab + ac & & \text{distributive law} \end{array}$$

Furthermore, we are used to have the neutral numbers 0 and 1 with special properties:

$$a + 0 = a \quad a \cdot 1 = a$$

and additive inverse element $-a$ and also the multiplicative inverse $a^{-1} = 1/a$ for $a \neq 0$. They fulfil $a + (-a) = 0$ and $aa^{-1} = 1$.

A set with such properties is called a [field](#). Here we have the field of real numbers \mathbb{R} .

It is also well known that the real numbers can be ordered, i.e., the relation $a < b$ makes sense. It has turned out, that the following rules are sufficient to derive all known rules concerning ordering of numbers.

- For any $a \in \mathbb{R}$ exactly one of the three relations hold

$$a < 0, \text{ or } a > 0 \text{ or } a = 0$$

- For all $a, b \in \mathbb{R}$ with $a > 0$ and $b > 0$ one has $a + b > 0$ and $ab > 0$.

Then, as a definition we write:

$$a < b \iff a - b < 0$$

and

$$a \leq b \iff a - b < 0 \text{ or } a = b.$$

In particular, we have for $a \neq 0$ that always $a^2 > 0$, because $a^2 = (-a)^2 > 0$ by the last rule applied to one of these terms.

The order relations are the reason, why we can think of the real numbers as a line, the "real line".

For describing subsets of the real numbers, we will use [intervals](#). Let $a, b \in \mathbb{R}$. Then we define

$$\begin{aligned} [a, b] &:= \{x \in \mathbb{R}: a \leq x \leq b\} \\ (a, b] &:= \{x \in \mathbb{R}: a < x \leq b\} \\ [a, b) &:= \{x \in \mathbb{R}: a \leq x < b\} \\ (a, b) &:= \{x \in \mathbb{R}: a < x < b\}. \end{aligned}$$

Obviously, in the case $a > b$, all the sets above are empty. We also can define unbounded intervals:

$$\begin{aligned}[a, \infty) &:= \{x \in \mathbb{R} : a \leq x\}, & (a, \infty) &:= \{x \in \mathbb{R} : a < x\} \\ (-\infty, b] &:= \{x \in \mathbb{R} : x \leq b\}, & (-\infty, b) &:= \{x \in \mathbb{R} : x < b\}.\end{aligned}$$

Definition 1.28. Absolute value for real numbers

The absolute value of a number $x \in \mathbb{R}$ is defined by

$$|x| := \begin{cases} x & \text{if } x \geq 0, \\ -x & \text{if } x < 0. \end{cases}$$

Question 1.29. Which of the following claims are true?

$$|-3.14| = 3.14, \quad |3| = 3, \quad \left|-\frac{7}{5}\right| = \frac{7}{5}, \quad -\left|-\frac{3}{5}\right| = \frac{3}{5}, \quad |0| \text{ is not well-defined.}$$

Proposition 1.30. Two important properties

For any two real numbers $x, y \in \mathbb{R}$, one has

- (a) $|x \cdot y| = |x| \cdot |y|$, ($|\cdot|$ is multiplicative),
- (b) $|x + y| \leq |x| + |y|$, ($|\cdot|$ fulfills the triangle inequality).

(*) Supplementary details: Real numbers

The real numbers are a non-empty set \mathbb{R} together with the operations $+$: $\mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ and \cdot : $\mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ and an ordering relation $<$: $\mathbb{R} \times \mathbb{R} \rightarrow \{\text{True}, \text{False}\}$ that fulfil the following rules

- (A) Addition
 - (A1) associative: $x + (y + z) = (x + y) + z$
 - (A2) neutral element: There is a (unique) element 0 with $x + 0 = x$ for all x .
 - (A3) inverse element: For all x there is a (unique) y with $x + y = 0$. We write for this element simply $-x$.
 - (A4) commutative: $x + y = y + x$
- (M) Multiplication
 - (M1) associative: $x \cdot (y \cdot z) = (x \cdot y) \cdot z$
 - (M2) neutral element: There is a (unique) element 1 $\neq 0$ with $x \cdot 1 = x$ for all x .
 - (M3) inverse element: For all $x \neq 0$ there is a (unique) y with $x \cdot y = 1$. We write for this element simply x^{-1} .
 - (M4) commutative: $x \cdot y = y \cdot x$
- (D) Distributivity: $x \cdot (y + z) = x \cdot y + x \cdot z$.
- (O) Ordering
 - (O1) for given x, y exactly one of the following three assertions is true: $x < y$, $y < x$, $x = y$.
 - (O2) transitive: $x < y$ and $y < z$ imply $x < z$.
 - (O3) $x < y$ implies $x + z < y + z$ for all z .
 - (O4) $x < y$ implies $x \cdot z < y \cdot z$ for all $z > 0$.
 - (O5) $x > 0$ and $\varepsilon > 0$ implies $x < \varepsilon + \dots + \varepsilon$ for sufficiently many summands.

(C) Completeness: Every sequence $(a_n)_{n \in \mathbb{N}}$ with the property [For all $\varepsilon > 0$ there is an $N \in \mathbb{N}$ with $|a_n - a_m| < \varepsilon$ for all $n, m > N$] has a limit.

(*) Supplementary details: Definition: field

Every set M together with two operations $+ : M \times M \rightarrow M$ and $\cdot : M \times M \rightarrow M$ that fulfil (A), (M) and (D) is called a field.

Sums and products

We will use the following notations.

$$\begin{aligned}\sum_{i=1}^n a_i &= a_1 + a_2 + \cdots + a_{n-1} + a_n \\ \prod_{i=1}^n a_i &= a_1 \cdot a_2 \cdot \cdots \cdot a_{n-1} \cdot a_n \\ \bigcup_{i=1}^n A_i &= A_1 \cup A_2 \cup \cdots \cup A_{n-1} \cup A_n\end{aligned}$$

The union works also for an arbitrary index set \mathcal{I} :

$$\bigcup_{i \in \mathcal{I}} A_i = \{x : \exists i \in \mathcal{I} \text{ with } x \in A_i\}.$$

The first is a useful notation for a sum which is the result of an addition. Two or more summands added. Instead of using points, we use the Greek letter \sum . For example,

$$3 + 7 + 15 + \dots + 127$$

is not an unambiguous way to describe the sum. Using the sum symbol, there is no confusion:

$$\sum_{i=2}^7 (2^i - 1).$$

Of course, the parentheses are necessary here. You can read this as a **for loop**:

for loop for the sum above

```
sum := 0;
for i:=2 to 7 do {
    sum := sum + (2^i-1);
}
```

Rule of thumb: Let i run from 2 to 7, calculate $2^i - 1$ and add.

<i>index variable:</i>	$i = 2,$	<i>first summand:</i>	$2^i - 1 = 2^2 - 1 =$	$4 - 1 =$	3
<i>index variable:</i>	$i = 3,$	<i>second summand:</i>	$2^i - 1 = 2^3 - 1 =$	$8 - 1 =$	7
<i>index variable:</i>	$i = 4,$	<i>third summand:</i>	$2^i - 1 = 2^4 - 1 =$	$16 - 1 =$	15
<i>index variable:</i>	$i = 5,$	<i>fourth summand:</i>	$2^i - 1 = 2^5 - 1 =$	$32 - 1 =$	31
<i>index variable:</i>	$i = 6,$	<i>fifth summand:</i>	$2^i - 1 = 2^6 - 1 =$	$64 - 1 =$	63
<i>index variable:</i>	$i = 7,$	<i>last summand:</i>	$2^i - 1 = 2^7 - 1 =$	$128 - 1 =$	127
		<i>Sum:</i>			246

Example 1.31.

$$\sum_{i=1}^{10} (2i - 1) = 1 + 3 + 5 + \dots + 19 \stackrel{?}{=} 100$$

$$\sum_{i=-10}^{10} i = -10 - 9 - 8 - \dots - 1 + 0 + 1 + \dots + 8 + 9 + 10 \stackrel{?}{=} 0$$

With the same construction, we describe the result of a multiplication, called a [product](#), which consists of two or more [factors](#). There we use the Greek letter \prod . For example:

$$\prod_{i=1}^8 (2i) = (2 \cdot 1) \cdot (2 \cdot 2) \cdot (2 \cdot 3) \cdot \dots \cdot (2 \cdot 8) \stackrel{?}{=} 10321920.$$

Rational versus real numbers

For most practical purposes the rational numbers (all fractions)

$$\mathbb{Q} = \left\{ x : x = \frac{n}{d} \text{ with } n \in \mathbb{Z}, d \in \mathbb{N} \right\}$$

are enough. All numbers that can somehow be stored sensibly on a computer are rational.

But not all quantities can be written as a fraction, such as the zeros of the following function:

$$f(x) = x^2 - 2$$

We can, however, approximate these “numbers” (we cannot call it a number, yet) to arbitrary precision in \mathbb{Q} .

How to finally arrive at an explanation of what this number really is (we cannot just write it down) is a topic of analysis (see next semester!)

Now we just give it a name, namely $\pm\sqrt{2}$, and remark that the real numbers $\mathbb{R} \supset \mathbb{Q}$ is a larger set of numbers, that can all be approximated by \mathbb{Q} , and the other way round: if something can be approximated to arbitrary precision by rational numbers, it is in \mathbb{R} “by definition”.

Mathematicians say: \mathbb{R} is complete, \mathbb{Q} is dense in \mathbb{R} , \mathbb{R} is the completion of \mathbb{Q} .

We come back to this in the lecture Mathematical Analysis.

1.3 Maps

Definition 1.32. Function or map

Let X, Y be non-empty sets. A rule that assigns to each argument $x \in X$ a unique value $y \in Y$ is called a map or function from X into Y . One writes for this y usually $f(x)$.

Notation:

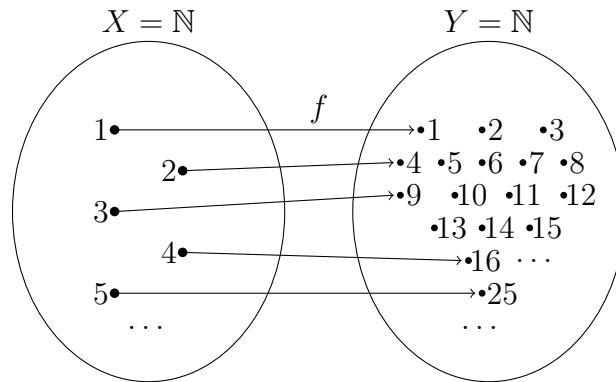
$$\begin{aligned} f : X &\rightarrow Y \\ x &\mapsto f(x) \end{aligned}$$

Here, X is called domain of f , and Y is called codomain.

Attention! Two arrows!

We use the arrow “ \rightarrow ” only between the sets, domain and codomain, and “ \mapsto ” only between the elements.

Example 1.33. (a) $f : \mathbb{N} \rightarrow \mathbb{N}$ with $f(x) = x^2$ maps each natural number to its square.



(b)

$$\begin{aligned} f : \mathbb{R}^2 &\rightarrow \mathbb{R} \\ (x_1, x_2) &\mapsto x_1^2 + x_2^2 \end{aligned}$$

(c)

$$\begin{aligned} f : \mathbb{Z} \times \mathbb{N} &\rightarrow \mathbb{Q} \\ (q, p) &\mapsto \frac{q}{p} \end{aligned}$$

Well-definedness

What can go wrong with the definition of a map? Sometimes, when defining a function, it is not completely clear, if this makes sense. Then one has to work and make this function well-defined.

Example: the square-root

Try to define a map $a \rightarrow \sqrt{a}$ in a mathematically rigorous way.

Naive definition:

$$\begin{aligned}\sqrt{-} : \mathbb{R} &\rightarrow \mathbb{R} \\ a &\mapsto \text{the solution of } x^2 = a.\end{aligned}$$

Problem of well-definedness: As we all know, the above equation has two ($a > 0$), one ($a = 0$), or zero ($a < 0$) solutions.

First way: restrict the domain of definition and the codomain

$$\mathbb{R}_0^+ = \{a \in \mathbb{R} : a \geq 0\}$$

Then:

$$\begin{aligned}\sqrt{-} : \mathbb{R}_0^+ &\rightarrow \mathbb{R}_0^+ \\ a &\mapsto \text{the non-negative solution of } x^2 = a.\end{aligned}$$

This yields the classical square-root.

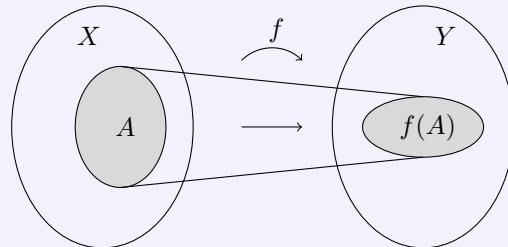
Image and preimage

For every well-defined map $f : X \rightarrow Y$ and $A \subset X$, $B \subset Y$ we are interested in the following sets:

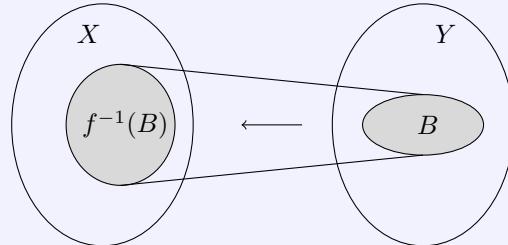
Definition 1.34.

Let $f : X \rightarrow Y$ be a function and $A \subset X$ and $B \subset Y$ some sets.

$f(A) := \{f(x) : x \in A\}$
is called the image of A under f .



$f^{-1}(B) := \{x \in X : f(x) \in B\}$
is called the preimage of B under f .



Note that the preimage can also be the empty set if none of the elements in B are “hit” by the map.

To describe the behaviour of a map, the following sets are very important:

Definition 1.35. Range and fiber

Let $f : X \rightarrow Y$ be a map. Then

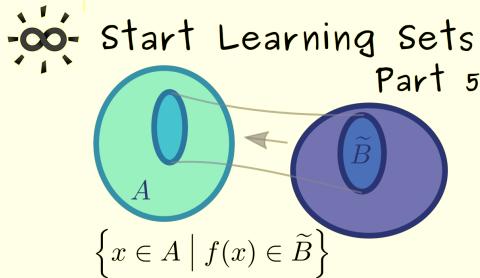
$\text{Ran}(f) := f(X) = \{f(x) : x \in X\}$ is called the range of f .

For each $y \in Y$ the set

$f^{-1}(\{y\}) := \{x \in X : f(x) = y\}$ is called a fiber of f .

If these definitions seem too abstract, the following video may help you to get used to the terms.

Video: Range, Image and Preimage



<https://jp-g.de/bsom/la/sls5/>



Injectivity, surjectivity, bijectivity, inverse

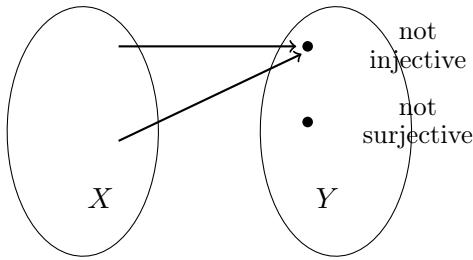
Definition 1.36. Injective, surjective and bijective

A map $f : X \rightarrow Y$ is called

- injective if every fiber of f has only one element: $x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$.
- surjective if $\text{Ran}(f) = Y$. With quantifiers: $\forall y \in Y \exists x \in X : f(x) = y$.
- bijective if f is both injective and surjective.

Example 1.37. Define the function that maps each student to her or his chair. This means that X is the set of all students in the room, and Y the set of all chairs in the room.

- well-defined: every student has a chair
- surjective: every chair is taken
- injective: on each chair there is no more than one student
- bijective: every student has his/her own chair, and no chair is empty



Rule of thumb: Surjective, injective, bijective

A map $f : X \rightarrow Y$ is

$$\begin{aligned} \text{surjective} &\Leftrightarrow \text{at each } y \in Y \text{ arrives at least one arrow} \\ &\Leftrightarrow f(X) = Y \\ &\Leftrightarrow \text{the equation } f(x) = y \text{ has for all } y \in Y \text{ a solution} \end{aligned}$$

$$\begin{aligned} \text{injective} &\Leftrightarrow \text{at each } y \in Y \text{ arrives at most one arrow} \\ &\Leftrightarrow (x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)) \\ &\Leftrightarrow (f(x_1) = f(x_2) \Rightarrow x_1 = x_2) \\ &\Leftrightarrow \text{the equation } f(x) = y \text{ has for all } y \in f(X) \text{ a unique solution} \end{aligned}$$

$$\begin{aligned} \text{bijective} &\Leftrightarrow \text{at each } y \in Y \text{ arrives exactly one arrow} \\ &\Leftrightarrow \text{the equation } f(x) = y \text{ has for all } y \in Y \text{ a unique solution} \end{aligned}$$

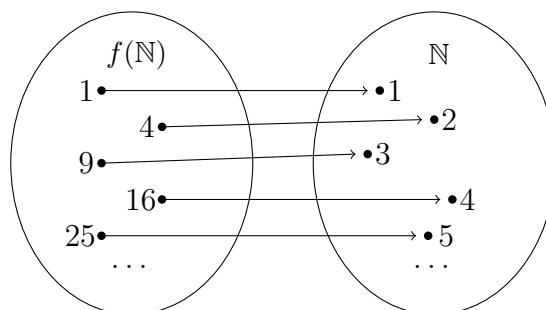
Thus, if f is bijective, there is a well defined inverse map

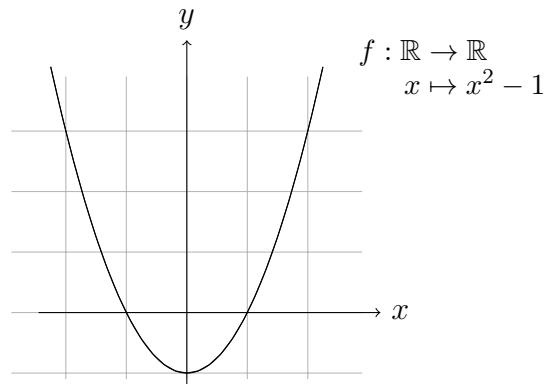
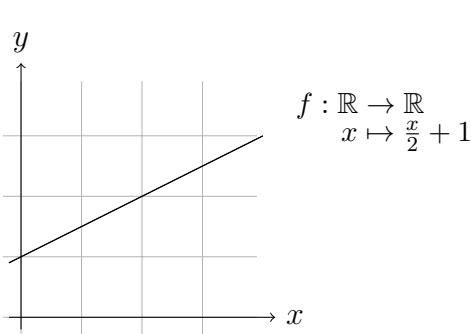
$$\begin{aligned} f^{-1} : Y &\rightarrow X \\ y &\mapsto x \text{ where } f(x) = y. \end{aligned}$$

Then f is called invertible and f^{-1} is called the inverse map of f .

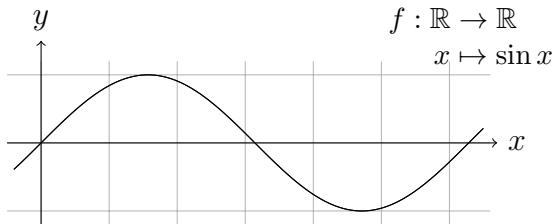
Example 1.38. Consider the function $f : \mathbb{N} \rightarrow \{1, 4, 9, 16, \dots\}$ given by $f(n) = n^2$. This is a bijective function. The inverse map f^{-1} is given by:

$$\begin{aligned} f^{-1} : \{1, 4, 9, 16, 25, \dots\} &\rightarrow \mathbb{N} \\ m &\mapsto \sqrt{m} \\ \text{or: } n^2 &\mapsto n \end{aligned}$$





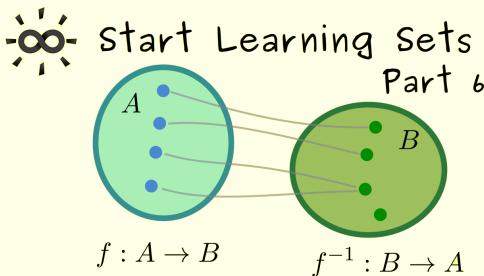
Example 1.39. For a function $f : \mathbb{R} \rightarrow \mathbb{R}$, we can sketch the graph $\{(x, f(x)) : x \in X\}$ in the x - y -plane:



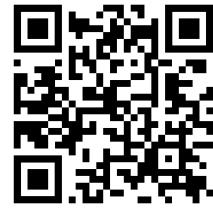
Which of the functions are injective, surjective or bijective?

These notions might seem a little bit off-putting, but we will use them so often that you need to get use to them. Maybe the following video helps you as well:

Video: Injectivity, Surjectivity and Bijectivity



<https://jp-g.de/bsom/la/sls6/>



Composition of maps

Definition 1.40.

If $f : X \rightarrow Y$ and $g : Y \rightarrow Z$, we may compose, or concatenate these maps:

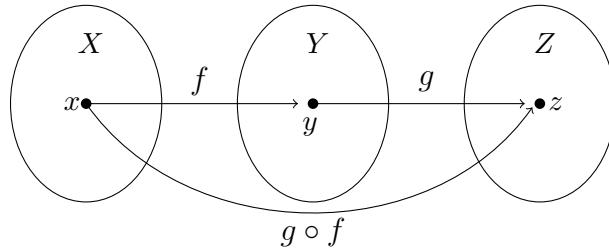
$$g \circ f : X \rightarrow Z$$

$$x \mapsto g(f(x))$$

We call $g \circ f$ the composition of the two functions.

Usually, $g \circ f \neq f \circ g$, the latter does not even make sense, in general.

$$X \rightarrow Y \rightarrow Z$$



Example 1.41. (a) $f : \mathbb{R} \rightarrow \mathbb{R}$, $x \mapsto x^2$; $g : \mathbb{R} \rightarrow \mathbb{R}$, $x \mapsto \sin(x)$

$$\begin{aligned} g \circ f : \mathbb{R} &\rightarrow \mathbb{R} \\ x &\mapsto \sin(x^2) \\ f \circ g : \mathbb{R} &\rightarrow \mathbb{R} \\ x &\mapsto (\sin(x))^2 \end{aligned}$$

(b) Let X be a set. Then $id_X : X \rightarrow X$ with $x \mapsto x$ is called the identity map. If there is no confusion, one usually writes id instead of id_X . Let $f : X \rightarrow X$ be a function. Then

$$f \circ id = f = id \circ f.$$

Algebraic vs. analytic properties of maps

Maps are a versatile tool in mathematics and often the main object of interest. Many other problems can be reformulated with maps.

We have seen here some *algebraic* properties: injectivity, surjectivity, bijectivity.

Other algebraic properties may be compatibility with operations on X and Y .

Examples:

$f(x - y) = f(x) - f(y)$	affine maps
$f(\alpha x) = \alpha f(x)$	homogenous maps
$f(\alpha x + \beta y) = \alpha f(x) + \beta f(y)$	linear maps
$f(xy) = f(x)f(y) \dots$	

These are sometimes called "homomorphisms".

In analysis next semester, we will learn about other properties, like continuity, differentiability, integrability, ... But for this, we have to define *open sets* first.

1.4 Natural numbers and induction

The natural numbers are $\mathbb{N} = \{1, 2, 3, \dots\}$.

Using natural numbers is our first mathematical abstraction. We learn this as children in the kindergarten.

What is this abstraction? A number is an abstraction for all finite sets of the same size.

- *Question 1:* When are two sets S, T of the same size? Have the same *cardinality* $|S| = |T|$? *Answer:* They have the same size if there is a bijective map $S \rightarrow T$. For example, \mathbb{N} and the set of all even numbers have the same cardinality.
- *Question 2:* When is a set S finite? *Answer:* It is finite if removing one element changes the *cardinality* of S .

In mathematical language: “Natural numbers are equivalence classes of finite sets of the same cardinality.”

Mathematical induction

Mathematical induction is an important technique of proof: Proof step by step. It is a close relative to recursion in computer science.

“Assume I can solve a problem of size n . How can I solve one of size $n + 1$?”

In mathematics:

“If an assertion is true for n , show that it is true for $n + 1$ ”

Example 1.42. What is the sum of the first n natural numbers?

$$s_n := \sum_{k=1}^n k = ?$$

To make this practical, we need three ingredients:

- An idea what the result could be. (Induction hypothesis)
- The verification that our hypothesis is true for $n = 1$ (Base case)
- A proof, that if it holds for n , then also for $n + 1$. (Induction step)

Getting the first ingredient is often the most difficult one. Often one has to try it out:

$$\begin{aligned} s_1 &= 1 \\ s_2 &= s_1 + 2 = 3 \\ s_3 &= s_2 + 3 = 6 \\ s_4 &= s_3 + 4 = 10 \\ s_5 &= s_4 + 5 = 15 \\ s_{n+1} &= s_n + n + 1 \end{aligned}$$

Ideas? Let's take the hypothesis

$$s_n = \frac{(n+1)n}{2} \quad (\text{Induction hypothesis}).$$

Very good! We can verify our formula for these examples. In particular:

$$s_1 = \frac{(1+1)1}{2} = 1 \quad (\text{Base case}).$$

Induction step: We have to show

$$\frac{(n+2)(n+1)}{2} \text{ is equal to } s_{n+1} = s_n + (n+1) = \frac{(n+1)n}{2} + n + 1$$

where we used the induction hypothesis in the last step. So let us compute:

$$s_n + (n+1) = \frac{(n+1)n}{2} + n + 1 = \frac{n^2 + n + 2n + 2}{2} = \frac{(n+2)(n+1)}{2}.$$

This proves that $s_n = \frac{(n+1)n}{2}$ for all $n \in \mathbb{N}$.

We will get plenty of other examples later.

Rule of thumb: Mathematical induction

To show that the predicate $A(n)$ is true for all $n \in \mathbb{N}$, we have to show two things:

- (1) Show that $A(1)$ is true.
- (2) Show that $A(n+1)$ is true under the assumption that $A(n)$ is true.

Sometimes it can happen that a claim $A(n)$ is indeed false for finitely many natural numbers, but it is eventually true. This means that the base case cannot be shown for $n = 1$ but for some other natural number $n_0 \in \mathbb{N}$. Then the induction proof shows that $A(n)$ is true for all natural number $n \geq n_0$.

1.5 Summary

- For doing Mathematics, we need logic and sets. A set is just a gathering of its elements.
- Important symbols: \in , \notin , \emptyset , \forall , \exists , \subset , \subsetneq , \cap , \cup , \setminus
- Implication $A \Rightarrow B$: If A holds, then also B .
- Equivalence $A \Leftrightarrow B$: The statement A holds if and only if B holds.
- Sums and products Σ, Π
- A map or function $f : X \rightarrow Y$ sends each $x \in X$ to exactly one $y \in Y$.
- f is surjective: Each $y \in Y$ is “hit” (one or more times).
- f is injective: Each $y \in Y$ is “hit” at most one time.
- f is bijective: Each $y \in Y$ is “hit” exactly once.
- Is $f : X \rightarrow Y$ bijective, then the inverse map $f^{-1} : Y \rightarrow X$ sends each $y \in Y$ to exactly one $x \in X$.
- The composition $g \circ f : X \rightarrow Z$ is the application of the function $g : Y \rightarrow Z$ to the result of another function $f : X \rightarrow Y$: $(g \circ f)(x) = g(f(x))$.
- Mathematical induction is a tool for proving mathematical statements for all natural numbers at once. You have to show a base case and then do the induction step.

1.6 Exercises

Exercise 1

Calculate the following numbers and sets:

$$(a) \prod_{j=2}^4 \frac{j}{j+1}, \quad (b) \sum_{i=0}^4 3, \quad (c) \bigcup_{n=0}^5 [2n, 2n+2), \quad (d) \sum_{k=1}^{50} k.$$

Exercise 2

- (a) Consider the two functions $f_1 : \mathbb{R} \rightarrow \mathbb{R}$, $x \mapsto x^2$ and $f_2 : [0, \infty) \rightarrow \mathbb{R}$, $x \mapsto x^2$. For both functions calculate preimages of the sets $\{1\}$, $[4, 9)$ and $(-1, 0)$.
- (b) Consider the two functions $g_1 : \mathbb{R} \rightarrow [0, 1]$, $x \mapsto |\sin(x)|$ and $g_2 : [0, 2\pi] \rightarrow \mathbb{R}$, $x \mapsto \sin(x)$. For both functions calculate images of the sets $(0, \pi/2)$, $[0, \pi)$ and $(0, 2\pi]$.
- (c) Consider the two functions $h_1 : \mathbb{R} \rightarrow \mathbb{R}$ and $h_2 : [-1, 1] \rightarrow [\sqrt{3}, 2]$ given by

$$x = (h_1(x) + 2)^2 - 2 \quad \text{and} \quad x^2 + h_2(x)^2 = 4.$$

Check whether h_1 and h_2 respectively are correctly defined.

- (d) Consider all 6 functions from above and find out which of them are injective, surjective and bijective. Try to provide proofs and counterexamples.

Exercise 3

Let X be the set of all fish in a given aquarium. Define a function $f : X \rightarrow Y$ by mapping every fish on its species where Y denotes the set of all species of fish. What does it mean if f is injective or surjective or bijective?

Exercise 4

In the lecture you already learnt about the example $(A \rightarrow B) \Leftrightarrow (\neg B \rightarrow \neg A)$ of two logically equivalent statements. Show that the following statements are also logically equivalent by using truth tables:

- (a) $\neg(A \wedge \neg B) \Leftrightarrow (A \rightarrow B)$,
- (b) $\neg(A \wedge B) \Leftrightarrow \neg A \vee \neg B$.

Exercise 5

One usually deals with subsets A , B , etc. of a given fixed set X . In such a situation it is useful to introduce $A^c := X \setminus A$ which is called the *complement* of A (with respect to (w.r.t.) the set X). Show for $A, B \subset X$

- (a) $A \setminus B = A \cap B^c$,
- (b) $(A \cap B)^c = A^c \cup B^c$.

Exercise 6

Let A, B and C be sets.

- (a) Show $A \times (B \cup C) = (A \times B) \cup (A \times C)$.
- (b) Let $|A| = n$ and $|B| = m$ where $n, m \in \mathbb{N}$. Show that

$$|A \times B| = n \cdot m.$$

2

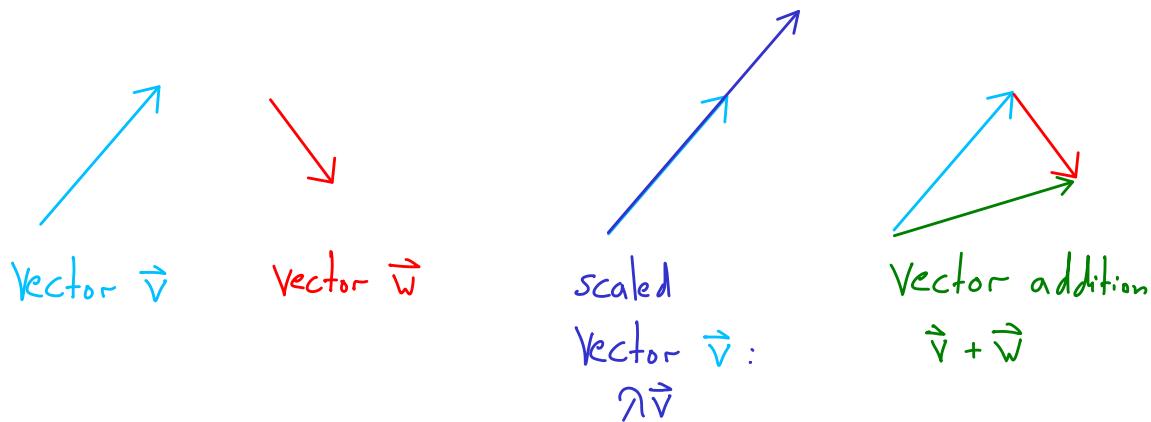
Vectors in \mathbb{R}^n

This is Frank Drebin, Police Squad. Throw down your guns, and come on out with your hands up. Or come on out, then throw down your guns, whichever way you want to do it. Just remember the two key elements here: one, guns to be thrown down; two, come on out!

2.1 What are vectors?

In this section we do some informal discussions about the objects of linear algebra. We will make all objects into rigorous definitions later.

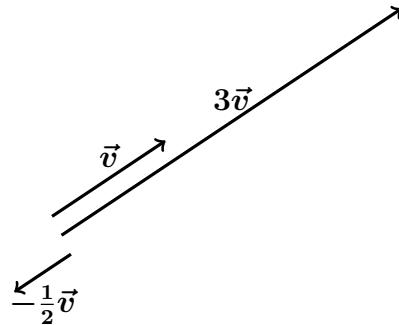
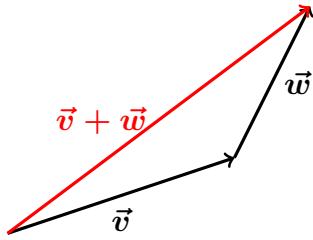
When we are talking about a vector, we often mean an object or a quantity that has a length and a direction in some sense. Therefore, we can always visualise this object as an “arrow” and we write, for example, \vec{v} and \vec{w} for two vectors.



Now we can exactly do two things to vectors. First of all, we can scale a vector \vec{v} by a number λ and get a new vector that has the same direction but a different length. The second operation is that we form two vectors into a new one. This vector addition where one sets the tail of the one arrow at the tip of the other one.

With vectors or arrows, you can do two things:

- Add the two arrows, by concatenating them and call the result $\vec{v} + \vec{w}$.
- Scale an arrow by a (positive or negative) factor λ and call the result $\lambda\vec{v}$.



With these operations we can combine \vec{v} and \vec{w} to a large number of arrows and this is what one calls a [linear combination](#):

If we scale two vectors \vec{v} and \vec{w} and add them, we get a new vector:

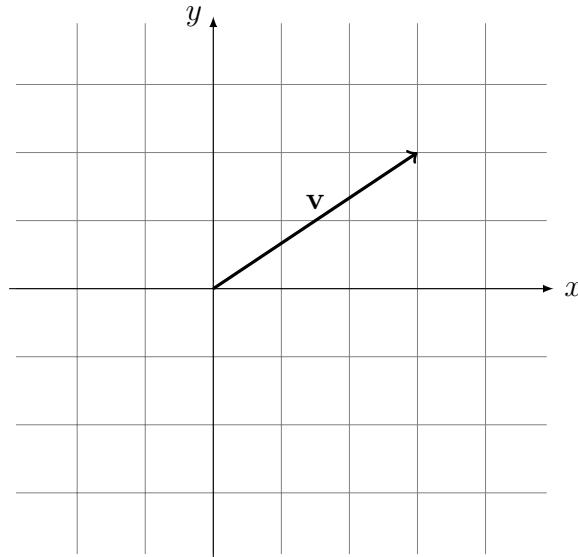
$$\lambda\vec{v} + \mu\vec{w} \quad (\text{linear combination})$$

Mostly, there is no confusion which variables are vectors and which one are just numbers such that we will omit the arrow from now on. However, we will use **bold letters** in this script to denote vectors most of the time.

2.2 Vectors in the plane

We already know that we can describe the two-dimensional plane by the cartesian product $\mathbb{R} \times \mathbb{R}$, which consists of all the pairs of real numbers. For each point in the plane, there is an arrow where the tail sits at the origin. This is what one calls a [position vector](#).

$$\mathbf{v} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$$



Our vector is given by the point in coordinate system, which means it consists of exactly two numbers, an x - and a y -coordinate. The arrow is given if we know these two numbers as in the example above we can write

$$\mathbf{v} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}.$$

The first number says how many steps we have to go to right (or left) and the second number says how many steps we have to go upwards (or downwards) parallel to the y -axis. These numerical *representations* of the arrows are called *columns* or [column vectors](#).

Now we also know how to add and scale these column vectors:

Define addition and scaling:

$$\mathbf{v} + \mathbf{w} = \begin{pmatrix} v_1 + w_1 \\ v_2 + w_2 \end{pmatrix} \quad \lambda \mathbf{v} = \begin{pmatrix} \lambda v_1 \\ \lambda v_2 \end{pmatrix}$$

These are the two things, we want to do with vectors and now we can describe such arrows in the two-dimensional plane. We have the geometric view given by arrows and the numerical view by operating on the coordinates.

Definition 2.1. Vector space \mathbb{R}^2

The set $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$ is called the [vector space \$\mathbb{R}^2\$](#) if we write the elements in column form

$$\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \quad \text{with} \quad v_1, v_2 \in \mathbb{R}$$

and use the vector addition and scaling from above. The numbers v_1 and v_2 are called the [components](#) of \mathbf{v} .

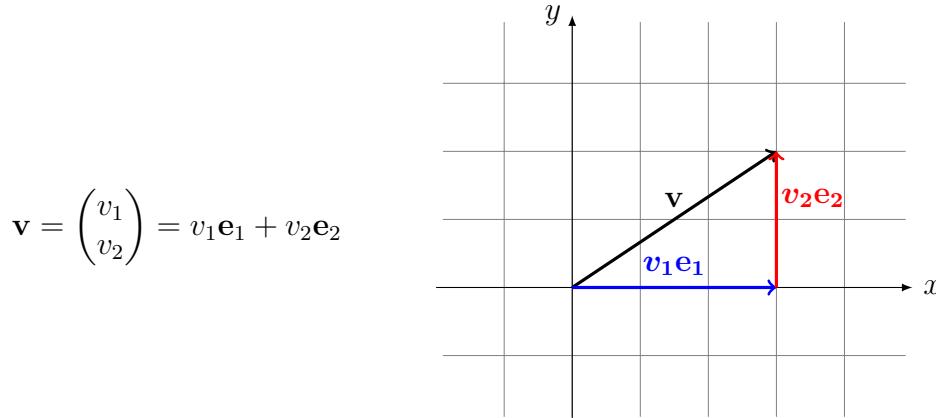
For describing each point in the plane, the following elements are useful:

Definition 2.2. Zero vector and canonical unit vectors

The two vectors $\mathbf{e}_1, \mathbf{e}_2 \in \mathbb{R}^n$ are called canonical unit vectors and \mathbf{o} is called the zero vector:

$$\mathbf{o} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \mathbf{e}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Note that we can write every vector $\mathbf{v} \in \mathbb{R}^2$ as a linear combination of the two unit vectors:



Linear combinations

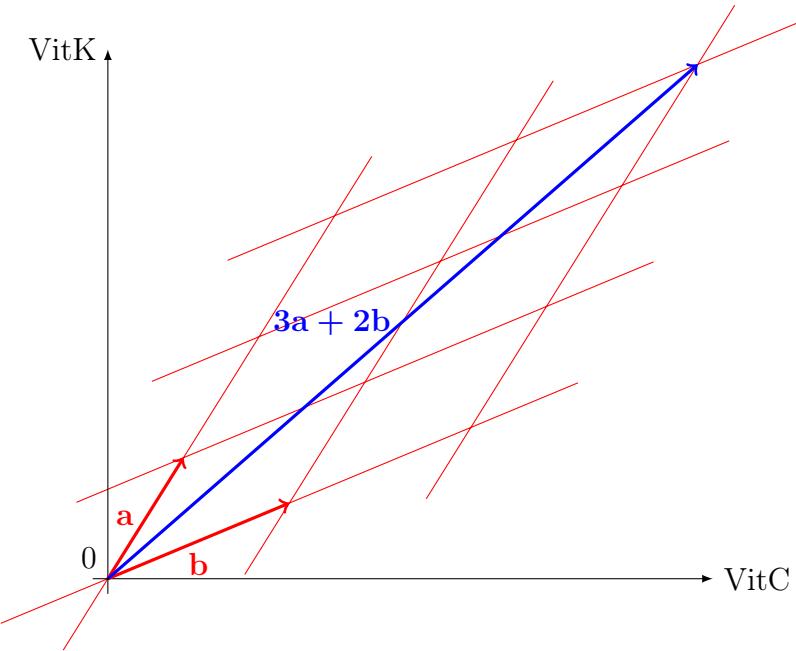
To compare apples and oranges: An apple has 8mg vitamin C and 4μg vitamin K. An orange has 85mg vitamin C and 0.5μg vitamin K:

$$\text{Apple } \mathbf{a} = \begin{pmatrix} 8 \\ 4 \end{pmatrix}_{\text{VitC VitK}}, \quad \text{Orange } \mathbf{b} = \begin{pmatrix} 85 \\ 0.5 \end{pmatrix}_{\text{VitC VitK}}$$

Fruit salad: How much vitamin C and vitamin K do I get if I eat 3 apples and 2 oranges? Answer:

$$3\mathbf{a} + 2\mathbf{b} = 3\begin{pmatrix} 8 \\ 4 \end{pmatrix} + 2\begin{pmatrix} 85 \\ 0.5 \end{pmatrix} = \begin{pmatrix} 3 \cdot 8 + 2 \cdot 85 \\ 3 \cdot 4 + 2 \cdot 0.5 \end{pmatrix} = \begin{pmatrix} 194 \\ 13 \end{pmatrix}_{\text{VitC VitK}}$$

Here, you can see a rough sketch of this vector addition:



A vector written as

$$\lambda \mathbf{a} + \mu \mathbf{b} \quad \text{with} \quad \lambda, \mu \in \mathbb{R} \quad (2.1)$$

is called a linear combination of **a** and **b**. We can expand this definition:

Definition 2.3. Linear combination

Let $\mathbf{v}_1, \dots, \mathbf{v}_k$ be vectors in \mathbb{R}^2 and $\lambda_1, \dots, \lambda_k \in \mathbb{R}$ scalars. Then

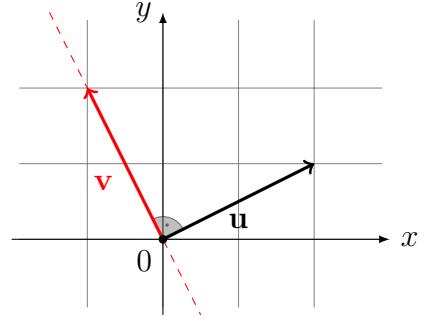
$$\sum_{j=1}^k \lambda_j \mathbf{v}_j = \lambda_1 \mathbf{v}_1 + \dots + \lambda_k \mathbf{v}_k$$

is called a linear combination of the vectors.

Orthogonal vector and inner product

Question:

Which vectors \mathbf{v} in \mathbb{R}^2 are perpendicular to the vector $\mathbf{u} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$?



Doing the sketch, one easily recognises that, for example, $\mathbf{v} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$ is perpendicular to \mathbf{u} . Of course, all multiples of this vector will also work. In general:

$$\mathbf{v} \in \mathbb{R}^2 \text{ is perpendicular to } \mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \iff \mathbf{v} = \lambda \begin{pmatrix} -u_2 \\ u_1 \end{pmatrix} \text{ for a } \lambda \in \mathbb{R} \quad (2.2)$$

Rule of thumb: orthogonal vector in \mathbb{R}^2

To find a vector that is orthogonal to $\begin{pmatrix} x \\ y \end{pmatrix}$, exchange the x and y and write a minus sign in front of **one** of the two.

Looking at (2.2), we can reformulate:

$$\begin{aligned} \mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \text{ and } \mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \text{ are orthogonal} &\iff \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \lambda \begin{pmatrix} -u_2 \\ u_1 \end{pmatrix} \text{ for a } \lambda \in \mathbb{R} \\ &\iff u_1 v_1 = -u_2 v_2 \\ &\iff u_1 v_1 + u_2 v_2 = 0 \end{aligned}$$

Hence, this means that $\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$ and $\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ are orthogonal if the calculation of $u_1 v_1 + u_2 v_2$ gives us 0. Therefore, the term $u_1 v_1 + u_2 v_2$ is used to define the so-called inner product or scalar product.

Definition 2.4. Inner product: $\langle \text{vector}, \text{vector} \rangle = \text{number}$

For two vectors

$$\mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \in \mathbb{R}^2 \quad \text{the number} \quad \langle \mathbf{u}, \mathbf{v} \rangle := u_1 v_1 + u_2 v_2 = \sum_{i=1}^2 u_i v_i$$

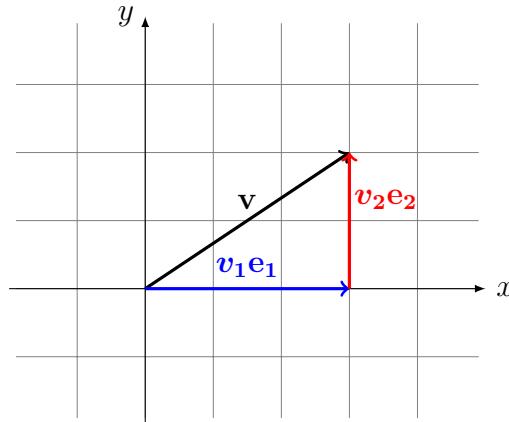
is called the (standard) inner product of \mathbf{u} and \mathbf{v} . Sometimes also called: (standard) scalar product.

Definition 2.5. Orthogonality of two vectors in \mathbb{R}^2

Two vectors \mathbf{u} and \mathbf{v} in \mathbb{R}^2 are called orthogonal (or perpendicular) if $\langle \mathbf{u}, \mathbf{v} \rangle = 0$ holds. We also denote this by $\mathbf{u} \perp \mathbf{v}$

By using Pythagoras' theorem, we can calculate the length of the arrow in the coordinate system.

$$\text{Length of } \mathbf{v} = \sqrt{v_1^2 + v_2^2}$$



Obviously, we can also define it by using the inner product:

Definition 2.6. Norm of a vector in \mathbb{R}^2

For a vector

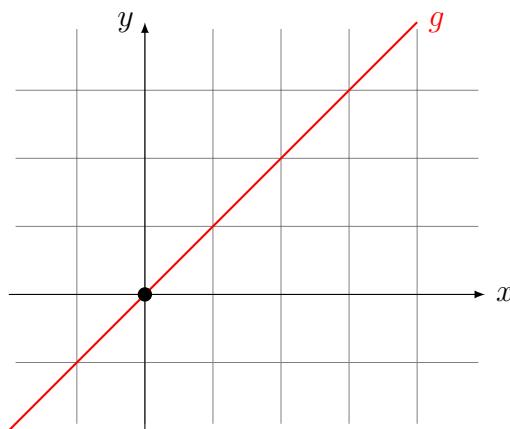
$$\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \in \mathbb{R}^2 \quad \text{the number} \quad \|\mathbf{v}\| := \sqrt{\langle \mathbf{v}, \mathbf{v} \rangle} = \sqrt{v_1^2 + v_2^2}$$

is called the norm or length of \mathbf{v} .

Lines in \mathbb{R}^2

For describing points in the plane, we can use the position vectors and just use the vector operations to define objects in the plane. One of the simplest objects is a line g inside the plane:

First case: The origin lies on the line g .

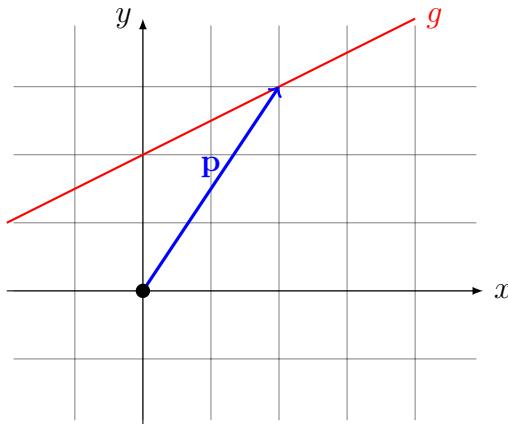


We already know that all vectors that are orthogonal to a fixed vector $\mathbf{u} \in \mathbb{R}^2$, which means that $\langle \mathbf{u}, \mathbf{v} \rangle = 0$, build a line through the origin. On the other hand, if we have a line g through the origin, we can find a vector \mathbf{n} that is perpendicular to the vectors lying on the line. Such an orthogonal vector is often called normal vector of the line.

In this first case, where g goes through the origin, we denote the normal vector by $\mathbf{n} = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \in \mathbb{R}^2$ and get:

$$g = \{\mathbf{v} \in \mathbb{R}^2 : \langle \mathbf{n}, \mathbf{v} \rangle = 0\} = \left\{ \underbrace{\begin{pmatrix} x \\ y \end{pmatrix}}_{\mathbf{v}} \in \mathbb{R}^2 : \underbrace{\alpha x + \beta y}_{\langle \mathbf{n}, \mathbf{v} \rangle} = 0 \right\}.$$

Second case: General case.



In this case, there is generally no special point on the line such that we can choose any point P with position vector $\mathbf{p} = \begin{pmatrix} p_1 \\ p_2 \end{pmatrix}$ to fix the line in the plane. Now, if we again fix a normal vector $\mathbf{n} = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$ of the line g , then we can describe all points V (with position vector $\mathbf{v} = \begin{pmatrix} x \\ y \end{pmatrix}$) on line: Such a point V lies on g if and only if the vector $\mathbf{v} - \mathbf{p}$ is inside the line, which means it is orthogonal to \mathbf{n} . Calculating this, gives us:

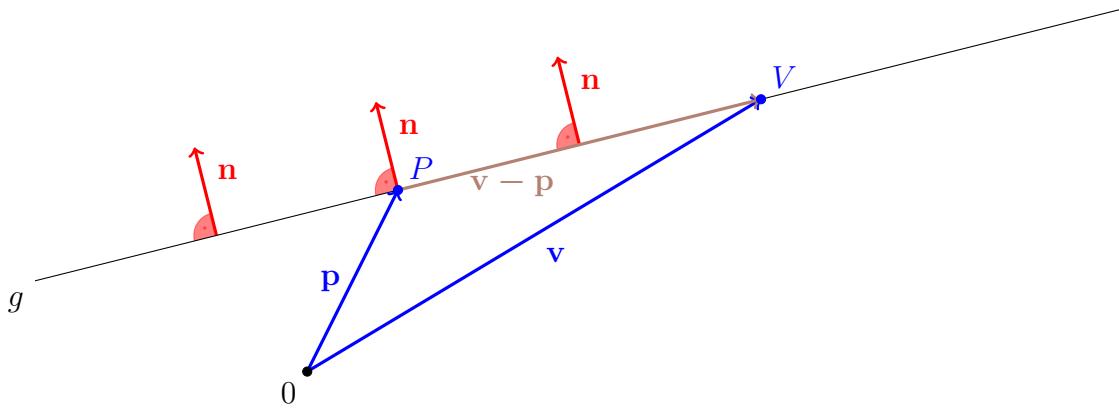
$$0 = \langle \mathbf{n}, \mathbf{v} - \mathbf{p} \rangle = \langle \begin{pmatrix} \alpha \\ \beta \end{pmatrix}, \begin{pmatrix} x-p_1 \\ y-p_2 \end{pmatrix} \rangle = \alpha(x - p_1) + \beta(y - p_2) = \alpha x + \beta y - \underbrace{(\alpha p_1 + \beta p_2)}_{=: \delta}.$$

Lines in the plane \mathbb{R}^2 (Equation in normal form)

For each line g , one has the following representation:

$$g = \{\mathbf{v} \in \mathbb{R}^2 : \langle \mathbf{n}, \mathbf{v} - \mathbf{p} \rangle = 0\} = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 : \alpha x + \beta y = \delta \right\}$$

with $\delta := \alpha p_1 + \beta p_2 = \langle \begin{pmatrix} \alpha \\ \beta \end{pmatrix}, \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} \rangle = \langle \mathbf{n}, \mathbf{p} \rangle$. If the origin lies on g , then $\delta = 0$ (choose $\mathbf{p} = \mathbf{o}$).



2.3 The vector space \mathbb{R}^n

Instead of restricting to two components, we could also imagine that we have an arbitrary number n of directions. It is easy to visualise a three-dimensional space but harder to do this for an n -dimensional space when $n > 3$. However, even without a visualisation, we can transfer the calculation from above to column vectors with n components:

$$\begin{aligned}\lambda \in \mathbb{R}, \mathbf{v} = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} \quad \Rightarrow \quad \lambda \mathbf{v} = \lambda \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} := \begin{pmatrix} \lambda v_1 \\ \vdots \\ \lambda v_n \end{pmatrix} \\ \mathbf{u} = \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix}, \mathbf{v} = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} \quad \Rightarrow \quad \mathbf{u} + \mathbf{v} = \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix} + \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} := \begin{pmatrix} u_1 + v_1 \\ \vdots \\ u_n + v_n \end{pmatrix}\end{aligned}$$

Definition 2.7. Vector space \mathbb{R}^n

The set $\mathbb{R}^n = \mathbb{R} \times \dots \times \mathbb{R}$ is called the [vector space \$\mathbb{R}^n\$](#) if we write the elements in column form

$$\mathbf{v} = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} \quad \text{with} \quad v_1, \dots, v_n \in \mathbb{R}$$

and use the vector addition and scaling from above. The numbers v_i are called the [ith component](#) of \mathbf{v} .

The same calculation rules as for \mathbb{R}^2 also hold for the general case. The most important properties we should note:

Proposition 2.8. Properties of the vector space \mathbb{R}^n

The set $V = \mathbb{R}^n$ with the addition $+$ and scalar multiplication \cdot fulfills the following:

- (1) $\forall \mathbf{v}, \mathbf{w} \in V : \mathbf{v} + \mathbf{w} = \mathbf{w} + \mathbf{v}$ (+ is commutative)
- (2) $\forall \mathbf{u}, \mathbf{v}, \mathbf{w} \in V : \mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$ (+ is associative)
- (3) There is a zero vector $\mathbf{o} \in V$ with the property: $\forall \mathbf{v} \in V$ we have $\mathbf{v} + \mathbf{o} = \mathbf{v}$.
- (4) For all $\mathbf{v} \in V$ there is a vector $-\mathbf{v} \in V$ with $\mathbf{v} + (-\mathbf{v}) = \mathbf{o}$.
- (5) For the number $1 \in \mathbb{R}$ and each $\mathbf{v} \in V$, one has: $1 \cdot \mathbf{v} = \mathbf{v}$.

- $$(6) \forall \lambda, \mu \in \mathbb{R} \quad \forall \mathbf{v} \in V : \quad \lambda \cdot (\mu \cdot \mathbf{v}) = (\lambda\mu) \cdot \mathbf{v} \quad (\cdot \text{ is associative})$$
- $$(7) \forall \lambda \in \mathbb{R} \quad \forall \mathbf{v}, \mathbf{w} \in V : \quad \lambda \cdot (\mathbf{v} + \mathbf{w}) = (\lambda \cdot \mathbf{v}) + (\lambda \cdot \mathbf{w}) \quad (\text{distributive } \cdot +)$$
- $$(8) \forall \lambda, \mu \in \mathbb{R} \quad \forall \mathbf{v} \in V : \quad (\lambda + \mu) \cdot \mathbf{v} = (\lambda \cdot \mathbf{v}) + (\mu \cdot \mathbf{v}) \quad (\text{distributive } +)$$

Each set V with an addition and scalar multiplication that satisfies the eight rules above is called a vector space. We will come back to this in an abstract sense later. First we will use this notion to talk about vector spaces inside \mathbb{R}^n .

Definition 2.9. Zero vector and canonical unit vectors

For $i = 1, \dots, n$, we denote the i th canonical unit vector by $\mathbf{e}_i \in \mathbb{R}^n$ and the zero vector by $\mathbf{o} \in \mathbb{R}^n$, which means:

$$\mathbf{o} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{e}_2 = \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \\ 0 \end{pmatrix}, \dots, \quad \mathbf{e}_{n-1} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ 0 \end{pmatrix}, \quad \mathbf{e}_n = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$$

2.4 Linear and affine subspaces (and the like)

So far, when we identified vectors in \mathbb{R}^n with points in our usual space, we considered single points, lines, planes, and the space itself. Sometimes these objects went through the origin, sometimes they did not. Let us develop a general name for these things:

Linear subspaces

Rule of thumb:

Linear subspaces correspond to lines, planes, ... through the origin.

Definition 2.10. Subspaces in \mathbb{R}^n

A nonempty subset $U \subset \mathbb{R}^n$ is called a (linear) subspace of \mathbb{R}^n if all linear combinations of vectors in U remain also in U :

$$\mathbf{u}_1, \dots, \mathbf{u}_k \in U, \quad \lambda_1, \dots, \lambda_k \in \mathbb{R} \implies \sum_{j=1}^k \lambda_j \mathbf{u}_j \in U.$$

Since we can set all λ_j to 0, the zero vector \mathbf{o} is always contained in U , and therefore $\{\mathbf{o}\}$ is the smallest possible subspace. On the other hand, \mathbb{R}^n itself is the largest possible subspace. Both are called the *trivial* subspaces.

Each linear subspace U of the vector space \mathbb{R}^n is also a vector space in the sense of the properties given in Proposition 2.8.

Linear combinations remain in U (by definition), and rules are inherited from V .

Proposition 2.11. Characterisation for subspaces

Let $U \subset \mathbb{R}^n$ with $U \neq \emptyset$, such that

$$\mathbf{u}, \mathbf{v} \in U, \lambda, \mu \in \mathbb{R} \implies \lambda\mathbf{u} + \mu\mathbf{v} \in U. \quad (2.3)$$

Then U is already a linear subspace.

Proof. We do the proof by induction for k vectors like in the definition of a subspace:

Induction hypothesis (**IH**): Linear combinations of k vectors remain in U .

Base case (**BC**): For $k = 2$. This is exactly given by equation (2.3).

Induction step (**IS**): $k \rightarrow k + 1$. Let $\mathbf{u}_1, \dots, \mathbf{u}_{k+1} \in U$ and $\lambda_1, \dots, \lambda_{k+1}$ be given. We can write:

$$\begin{aligned} \mathbf{v} := \sum_{j=1}^{k+1} \lambda_j \mathbf{u}_j &= \underbrace{\left(\sum_{j=1}^k \lambda_j \mathbf{u}_j \right)}_{=: \mathbf{w}} + \lambda_{k+1} \mathbf{u}_{k+1} \\ &= \mathbf{w} + \lambda_{k+1} \mathbf{u}_{k+1} \in U \end{aligned}$$

By our induction hypothesis, $\mathbf{w} \in U$ because it is a linear combination of k vectors. Thus, $\mathbf{v} \in U$ as well because it is a linear combination of \mathbf{w} and \mathbf{u}_{k+1} , see (2.3). \square

Thus, to check if a given set U is a linear subspace, we only have to check if linear combinations of two vectors remain in U . Or we can check it separately:

Corollary 2.12. How to check if a set is a subspace

Let $U \subset \mathbb{R}^n$ such that

- (1) $\mathbf{0} \in U,$
- (2) $\mathbf{u} \in U, \lambda \in \mathbb{R} \implies \lambda\mathbf{u} \in U,$
- (3) $\mathbf{u}, \mathbf{v} \in U \implies \mathbf{u} + \mathbf{v} \in U.$

Then U is already a linear subspace.

Rule of thumb: Subspace

A set U is a subspace if, by applying the operations $+$ and $\lambda \cdot$ on elements of U , one cannot escape the set U .

Linear hull or span

If we take a set of vectors $M \subset \mathbb{R}^n$, we can create a linear subspace by building all possible linear combinations:

Definition 2.13. Span

Let $M \subset \mathbb{R}^n$ be any non-empty subset. Then we define:

$$\text{Span}(M) := \left\{ \mathbf{u} \in \mathbb{R}^n : \exists \lambda_j \in \mathbb{R}, \mathbf{u}_j \in M \text{ such that } \mathbf{u} = \sum_{j=1}^k \lambda_j \mathbf{u}_j \right\}.$$

This subspace is called the [span](#) or the [linear hull](#) of M . For convenience, we define $\text{Span}(\emptyset) := \{\mathbf{o}\}$.

An equivalent definition would be: $\text{Span}(M)$ is the smallest linear subspace $U \subset \mathbb{R}^n$ with $M \subset U$. See Proposition 2.15.

Rule of thumb: All linear combinations form the span

Every vector in $\text{Span}(M)$ can be written (possibly in several ways) as a linear combination of elements of M . Vice versa, every linear combination of M is contained in $\text{Span}(M)$.

Most interesting is the case, where $M = \{\mathbf{u}_1, \dots, \mathbf{u}_k\}$ just consists of finitely many vectors. We say that $U := \text{Span}(M)$ is spanned by the vectors $\mathbf{u}_1, \dots, \mathbf{u}_k$ or, the other way around, that $\{\mathbf{u}_1, \dots, \mathbf{u}_k\}$ is a [generating set](#) for U (generates U , spans U). In this case, we often write $U = \text{Span}(\mathbf{u}_1, \dots, \mathbf{u}_k)$.

Example 2.14. The vector space \mathbb{R}^n is spanned by the n unit vectors:

$$\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{e}_2 = \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \\ 0 \end{pmatrix}, \quad \dots, \quad \mathbf{e}_{n-1} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ 0 \end{pmatrix}, \quad \mathbf{e}_n = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$$

because $\mathbf{v} = \sum_{i=1}^n v_i \mathbf{e}_i$ for all $\mathbf{v} \in \mathbb{R}^n$. In short: $\mathbb{R}^n = \text{Span}(\mathbf{e}_1, \dots, \mathbf{e}_n)$.

To check, if a vector space is spanned by some vectors, we only have to check this for some generating set:

Proposition 2.15. Span is smallest linear subspace

Let $U \subset \mathbb{R}^n$ be a linear subspace and $M \subset U$ any set. Then $\text{Span}(M)$ is a linear subspace and $\text{Span}(M) \subset U$.

Proof. Exercise! □

We need one further notation.

Definition 2.16. Addition for subspaces?

If U_1 and U_2 are linear subspaces in \mathbb{R}^n , then one defines

$$U_1 + U_2 := \text{Span}(U_1 \cup U_2) = \{\mathbf{u}_1 + \mathbf{u}_2 : \mathbf{u}_1 \in U_1, \mathbf{u}_2 \in U_2\}.$$

Example 2.17. Let us look at some spans:

- (a) $\text{Span}(\begin{pmatrix} 3 \\ 1 \end{pmatrix}) \subset \mathbb{R}^2$ is the line that “the vector $\begin{pmatrix} 3 \\ 1 \end{pmatrix}$ spans” going through the origin of \mathbb{R}^2 .
- (b) $\text{Span}(\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix})$ is the whole plane \mathbb{R}^2 . $\text{Span}(\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix})$ is also the whole plane.
- (c) $\text{Span}\left(\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}\right)$ this is the xy -plane in \mathbb{R}^3 .
- (d) $\text{Span}\left(\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 2 \\ 4 \\ 7 \end{pmatrix}\right)$ is a plane in \mathbb{R}^3 going through $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ 4 \\ 7 \end{pmatrix}$.
- (e) $\text{Span}\left(\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\right)$ is the whole space \mathbb{R}^3 . $\text{Span}\left(\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}\right)$ is also the whole space
- (f) $\text{Span}\left(\begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{pmatrix}\right)$ is a “line” in \mathbb{R}^5 , $\text{Span}\left(\begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{pmatrix}, \begin{pmatrix} 5 \\ 4 \\ 3 \\ 2 \\ 1 \end{pmatrix}\right)$ is a “plane”.

Affine subspaces and convex subsets

Rule of thumb:

Affine subspaces correspond to arbitrary lines, planes, In other words: translated linear subspaces.

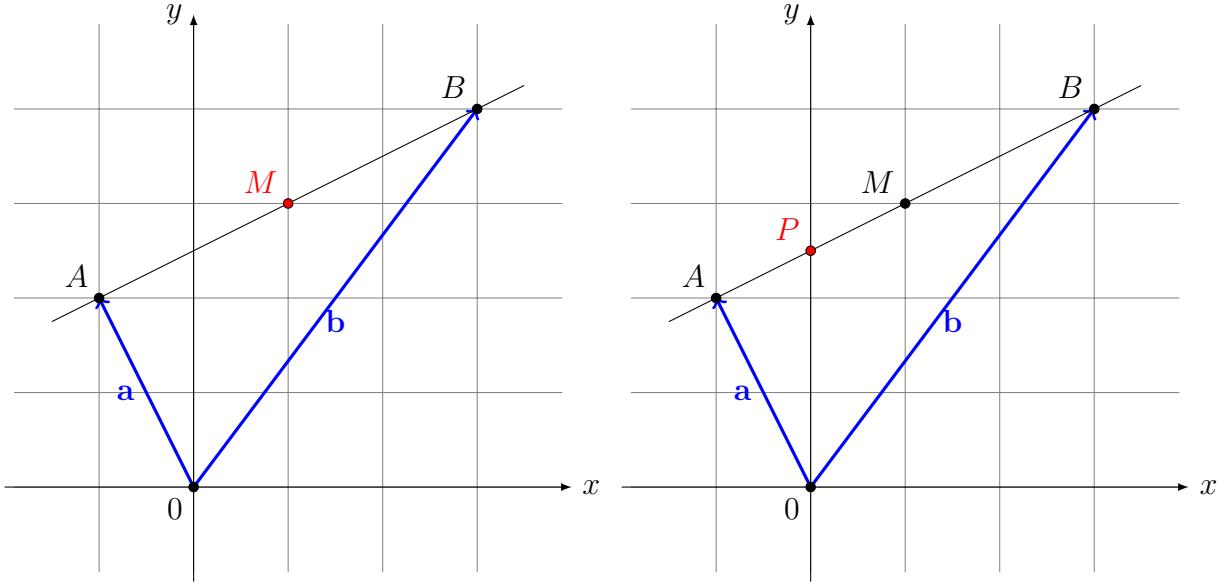
If we do not want \mathbf{o} to be part of our “generalised plane”, we have to replace linear combinations by *affine combinations*:

$$\mathbf{v} = \sum_{j=1}^k \lambda_j \mathbf{u}_j \quad \text{where} \quad \sum_{j=1}^k \lambda_j = 1.$$

Example 2.18. Consider the position vectors

$$\mathbf{a} = \begin{pmatrix} -1 \\ 2 \end{pmatrix} \quad \text{und} \quad \mathbf{b} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$$

corresponding to the points A and B . Find the centre point of the line between A and B .



The connection vector from A to B is then:

$$-\mathbf{a} + \mathbf{b} = -\begin{pmatrix} -1 \\ 2 \end{pmatrix} + \begin{pmatrix} 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \end{pmatrix} + \begin{pmatrix} 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 1+3 \\ -2+4 \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \end{pmatrix} =: \mathbf{d}$$

The center point is then given by going only half way in the direction of \mathbf{d} :

$$\mathbf{m} = \mathbf{a} + \frac{1}{2}\mathbf{d} = \begin{pmatrix} -1 \\ 2 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 4 \\ 2 \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \end{pmatrix} + \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix} \quad (2.4)$$

The point M with position vector $\mathbf{m} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$ is the wanted centre point. In general, we get the formula:

$$\mathbf{m} = \mathbf{a} + \frac{1}{2}\mathbf{d} = \mathbf{a} + \frac{1}{2}(-\mathbf{a} + \mathbf{b}) = \mathbf{a} - \frac{1}{2}\mathbf{a} + \frac{1}{2}\mathbf{b} = \frac{1}{2}\mathbf{a} + \frac{1}{2}\mathbf{b} = \frac{1}{2}(\mathbf{a} + \mathbf{b})$$

Instead of using $\frac{1}{2}$, we can choose $\lambda \in \mathbb{R}$ to divide the line from A to B . We get:

$$\mathbf{q} := \mathbf{a} + \lambda \overbrace{(-\mathbf{a} + \mathbf{b})}^{\mathbf{d}} = (1 - \lambda)\mathbf{a} + \lambda\mathbf{b} \quad (2.5)$$

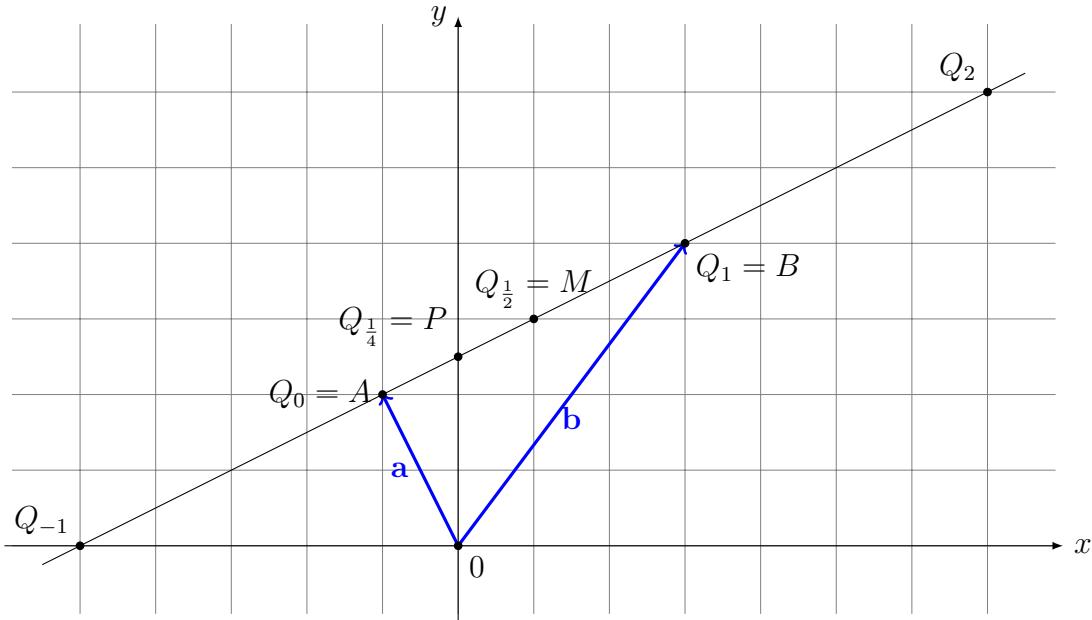
The corresponding point Q (with position vector \mathbf{q} from the equation above) lies

$$\begin{aligned} & \text{at point } A \text{ if } \lambda = 0, \\ & \text{at point } B \text{ if } \lambda = 1, \\ & \text{at the centre point } M \text{ if } \lambda = \frac{1}{2}, \\ & \text{between } A \text{ and } B \text{ if } \lambda \in [0, 1] \end{aligned} \quad (2.6)$$

$$\text{on the line through } A \text{ and } B \text{ for all } \lambda \in \mathbb{R}, \quad (2.7)$$

on the line through A and B , “in front of” A for all $\lambda < 0$,

on the line through A and B , “behind” B for all $\lambda > 1$.



This brings out to the following:

Definition 2.19. Affine Subspaces in \mathbb{R}^n

A subset $U \subset \mathbb{R}^n$ is called an affine subspace of \mathbb{R}^n if all affine combinations of vectors in U remain also in U :

$$\mathbf{u}_1, \dots, \mathbf{u}_k \in U, \lambda_1, \dots, \lambda_k \in \mathbb{R} \text{ with } \sum_{j=1}^k \lambda_j = 1 \implies \sum_{j=1}^k \lambda_j \mathbf{u}_j \in U$$

Definition 2.20. Convex subsets in \mathbb{R}^n

A subset $U \subset \mathbb{R}^n$ is called a convex subset of \mathbb{R}^n if all convex combinations of vectors in U remain also in U :

$$\mathbf{u}_1, \dots, \mathbf{u}_k \in U, \lambda_1, \dots, \lambda_k \in [0, 1] \text{ with } \sum_{j=1}^k \lambda_j = 1 \implies \sum_{j=1}^k \lambda_j \mathbf{u}_j \in U$$

The analogous formulation to the linear hull is the *affine hull*. Try to give a definition!

Proposition 2.21. Properties of affine subspaces

- (i) Each linear subspace is an affine subspace.
- (ii) If an affine subspace contains \mathbf{o} , it is a linear subspace.
- (iii) Given an affine subspace $S \subset \mathbb{R}^n$ and a vector $\mathbf{v} \in \mathbb{R}^n$, the translated set:

$$\mathbf{v} + S := \{\mathbf{x} \in \mathbb{R}^n : \mathbf{x} = \mathbf{v} + \mathbf{s} \text{ for } \mathbf{s} \in S\}$$

is also an affine subspace.

- (iv) Every nonempty affine subspace S can be written in the form $S = \mathbf{v} + U$ for some $\mathbf{v} \in S$ and U a linear subspace.

Proof. (i) : Follows from the definition because each affine combination is a linear combination.

(ii) : If we have an arbitrary linear combination, we can trivially add also the zero vector. But if the zero-vector is in a linear combination, we can make it an affine one.

(iii) : Let us write an arbitrary affine combination of elements of $\mathbf{v} + S$:

$$\sum_{j=1}^k \lambda_j (\underbrace{\mathbf{s}_j + \mathbf{v}}_{\in \mathbf{v} + S}) = \sum_{j=1}^k \lambda_j \mathbf{s}_j + \underbrace{\sum_{j=1}^k \lambda_j \mathbf{v}}_1 = \underbrace{\sum_{j=1}^k \lambda_j \mathbf{s}_j}_{\in \mathbf{v} + S} + \mathbf{v}.$$

(iv) : If S is an affine subspace, choose $\mathbf{v} \in S$ and define $U = -\mathbf{v} + S$. By (iii) U is an affine subspace and it contains \mathbf{o} . Hence, by (i) it is a linear subspace. Obviously, we have $S = \mathbf{v} + U$. \square

Proposition 2.22. Characterisation of affine subspaces

Let $S \subset \mathbb{R}^n$, such that

$$\mathbf{a}, \mathbf{b} \in S, \lambda \in \mathbb{R} \implies \lambda \mathbf{a} + (1 - \lambda) \mathbf{b} \in S$$

Then S is already an affine subspace.

Proof. We do a proof by mathematical induction:

Induction hypothesis: affine combinations of k vectors remain in S . In other words:

$$\mathbf{v} = \sum_{j=1}^k \lambda_j \mathbf{a}_j \text{ and } \sum_{j=1}^k \lambda_j = 1 \text{ implies } \mathbf{v} \in S$$

for every k and every admissible choice of λ_j and $\mathbf{a}_j \in S$.

Base case: by assumption, this is certainly true for $k = 2$.

Induction step: $k \rightarrow k+1$. Let \mathbf{a}_j and λ_j be given for all $j \in \{1, \dots, k+1\}$. By definition

$$\lambda_1 + \dots + \lambda_k = 1 - \lambda_{k+1}$$

thus we can write:

$$\begin{aligned} \mathbf{v} &= \sum_{j=1}^{k+1} \lambda_j \mathbf{a}_j = (1 - \lambda_{k+1}) \underbrace{\left(\sum_{j=1}^k \frac{\lambda_j}{\lambda_1 + \dots + \lambda_k} \mathbf{a}_j \right)}_{\text{affine combination } \mathbf{w}} + \lambda_{k+1} \mathbf{a}_{k+1} \\ &= (1 - \lambda_{k+1}) \mathbf{w} + \lambda_{k+1} \mathbf{a}_{k+1} \in S \end{aligned}$$

By our induction hypothesis, $\mathbf{w} \in S$, because it is an affine combination of k vectors. Thus, $\mathbf{v} \in S$ as well, because it is an affine combination of \mathbf{w} and \mathbf{a}_{k+1} . \square

Conical combinations (an outlook)

There are also other rules for combining vectors. they lead to different classes of sets. For example, *conical combinations* of vectors are defined as:

$$\mathbf{v} = \sum_{j=1}^k \lambda_j \mathbf{u}_j \text{ where } \lambda_j \geq 0.$$

The sets which contain all possible conical combinations of their elements are called *convex cones*, and we can define the conical hull of a set of vectors.

We can summarise this in the following table:

	no sign imposed	$\lambda_j \geq 0$
no sum imposed	linear	conical
$\sum \lambda_j = 1$	affine	convex

For all these types of sets we know "... combinations", and "... hulls".

This illustrates our strategy: describe things known from \mathbb{R}^2 and \mathbb{R}^3 algebraically, and thus generalise them to arbitrary dimensions.

2.5 Inner product and norm in \mathbb{R}^n

We transfer the notion of the inner product to define orthogonality and the length of the vector to the general \mathbb{R}^n

Definition 2.23. Inner product: $\langle \text{Vector}, \text{Vector} \rangle = \text{Number}$

For two vectors

$$\mathbf{u} = \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix}, \quad \mathbf{v} = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} \in \mathbb{R}^n \quad \text{the number} \quad \langle \mathbf{u}, \mathbf{v} \rangle := u_1 v_1 + \dots + u_n v_n = \sum_{i=1}^n u_i v_i$$

is called the (standard) inner product of \mathbf{u} and \mathbf{v} . Sometimes also called: (standard) scalar product. If $\langle \mathbf{u}, \mathbf{v} \rangle = 0$, then we call \mathbf{u} and \mathbf{v} orthogonal.

Proposition 2.24.

The standard inner product $\langle \cdot, \cdot \rangle : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$ fulfils the following: For all vectors $\mathbf{x}, \mathbf{x}', \mathbf{y} \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}$, one has

- (S1) $\langle \mathbf{x}, \mathbf{x} \rangle > 0$ for all $\mathbf{x} \neq \mathbf{0}$, (positive definite)
- (S2) $\langle \mathbf{x} + \mathbf{x}', \mathbf{y} \rangle = \langle \mathbf{x}, \mathbf{y} \rangle + \langle \mathbf{x}', \mathbf{y} \rangle$, (additive)
- (S3) $\langle \lambda \mathbf{x}, \mathbf{y} \rangle = \lambda \langle \mathbf{x}, \mathbf{y} \rangle$, (homogeneous) } (linear)

$$(S4) \quad \langle \mathbf{x}, \mathbf{y} \rangle = \langle \mathbf{y}, \mathbf{x} \rangle. \quad (\text{symmetric})$$

Definition 2.25. Norm of a vector in \mathbb{R}^n

For a vector

$$\mathbf{v} = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} \in \mathbb{R}^n \quad \text{the number} \quad \|\mathbf{v}\| := \sqrt{\langle \mathbf{v}, \mathbf{v} \rangle} = \sqrt{v_1^2 + \dots + v_n^2}$$

is called the norm or length of \mathbf{v} .

In general, we just need a map $\langle \cdot, \cdot \rangle$ with the properties given in Proposition 6.18 to define orthogonality as follows:

$$\mathbf{u} \perp \mathbf{v} \iff \langle \mathbf{u}, \mathbf{v} \rangle = 0.$$

From the first binomial formula, we obtain directly a generalisation of the Pythagorean theorem:

$$\mathbf{u} \perp \mathbf{v} \Rightarrow \|\mathbf{u} + \mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2.$$

For a linear subspace $U \subset \mathbb{R}^n$ we define the orthogonal complement:

$$U^\perp := \{ \mathbf{v} \in \mathbb{R}^n : \langle \mathbf{v}, \mathbf{u} \rangle = 0 \quad \forall \mathbf{u} \in U \}.$$

However, we come back to such constructions later.

2.6 A special product in $\mathbb{R}^3(!)$: The vector product or cross product

The three-dimensional space is in some sense special: One can define a product between two vectors and gets a vector as a result. In contrary to the inner product, this multiplication exists **only in \mathbb{R}^3** :

Definition 2.26. Cross product: Vector \times Vector = Vector

The cross product or vector product of two vectors

$$\mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}, \quad \mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \in \mathbb{R}^3 \quad \text{is given by} \quad \mathbf{u} \times \mathbf{v} := \begin{pmatrix} u_2 v_3 - u_3 v_2 \\ u_3 v_1 - u_1 v_3 \\ u_1 v_2 - u_2 v_1 \end{pmatrix} \in \mathbb{R}^3.$$

Rule of thumb: How to remember this formula?

$\overline{u_1} \quad \overline{v_1}$		
$u_2 \quad \cancel{v_2}$ $u_3 \quad \cancel{v_3}$ $u_1 \quad \cancel{v_1}$ $u_2 \quad \cancel{v_2}$	\Rightarrow	$\begin{pmatrix} +u_2 v_3 - u_3 v_2 \\ +u_3 v_1 - u_1 v_3 \\ +u_1 v_2 - u_2 v_1 \end{pmatrix}$
$\overline{u_3} \quad \overline{v_3}$		

Remark:

In some calculations it can be really helpful to use the [Levi-Civita symbol](#):

$$\varepsilon_{ijk} = \begin{cases} +1 & \text{if } (i, j, k) \text{ is an even permutation of } (1, 2, 3) \\ -1 & \text{if } (i, j, k) \text{ is an odd permutation of } (1, 2, 3) \\ 0 & \text{if } i = j, \text{ or } j = k, \text{ or } k = i \end{cases}$$

Then we find a short notation for the cross product of two vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^3$:

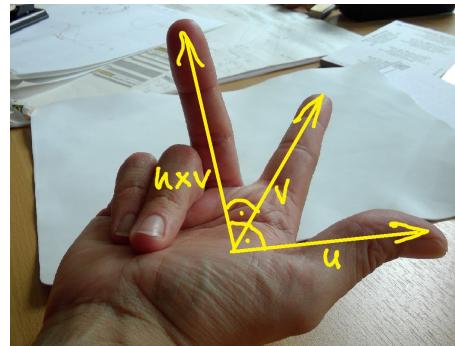
$$\mathbf{u} \times \mathbf{v} = \sum_{ijk} \varepsilon_{ijk} u_i v_j \mathbf{e}_k .$$

Since we have a good imagination for the three-dimensional space, we can interpret the result of the cross product $\mathbf{u} \times \mathbf{v}$ in a geometric way. It is the only vector in \mathbb{R}^3 that has the following three properties:

1.) $\mathbf{u} \times \mathbf{v} \perp \mathbf{u}$ and $\mathbf{u} \times \mathbf{v} \perp \mathbf{v}$

2.) $\|\mathbf{u} \times \mathbf{v}\| = \text{Area} \left(\begin{array}{c} \text{blue parallelogram} \\ \text{shaded parallelogram} \end{array} \right)$

3.) Orientation: “right-hand rule”



You can use the cross product, for example,

- to find a vector that is perpendicular to \mathbf{u} and \mathbf{v} ,
- to calculate the area of parallelogram.

Since all triangles are the half of a parallelogram, you can also use it to calculate the area of a triangle. Keep in mind that you can embed \mathbb{R}^2 into \mathbb{R}^3 to use the cross product even if you have just a two-dimensional problem.

2.7 What are complex numbers?

Once we can solve the equation

$$x^2 - 2 = 0 \text{ in } \mathbb{R}$$

we also would like to solve

$$x^2 + 1 = 0 \text{ or } x = \sqrt{-1}$$

This has no *real* solution, because for $x \in \mathbb{R}$, $x^2 \geq 0 \Rightarrow x^2 + 1 \geq 1 > 0$. However, in a bigger number set, it is solvable.

Let us see what happens if we *postulate* the existence of two solutions and call them $\pm i$, where i stands for “imaginary” (some engineers use the letter j instead). This means that

$$i^2 = -1, \text{ or } i = \sqrt{-1}$$

Of course, in general we would like to solve an arbitrary quadratic equation:

$$x^2 + 2bx + c = 0,$$

which has the solutions

$$x_{1,2} = -b \pm \sqrt{b^2 - c}$$

If $b^2 - c \geq 0$, $x_{1,2}$ can be solved as usual, but otherwise, we have to compute:

$$\sqrt{\underbrace{b^2 - c}_{<0}} = \sqrt{(-1)(\underbrace{c - b^2}_{>0})} = i\sqrt{c - b^2} \Rightarrow x_{1,2} = -b \pm i\sqrt{c - b^2}$$

Thus, to write down solutions of quadratic equations, we have to define

$$\text{Complex numbers } \mathbb{C} = \{x + iy : x, y \in \mathbb{R}\}.$$

In fact, there is the *fundamental theorem of algebra*, which says that complex numbers can even be used to solve any algebraic equation.

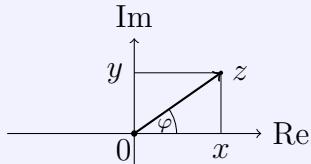
Theorem 2.27. Fundamental theorem of algebra

Every algebraic equation:

$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 = 0, \quad a_k \in \mathbb{C} : k = 0 \dots n$$

has at least one zero in \mathbb{C} if the left hand side is not constant.

Complex plane



Geometrical Identification:

$$x + iy \in \mathbb{C} \longleftrightarrow \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2$$

(*) Supplementary details: Complex numbers

The vector space \mathbb{R}^2 becomes a field by defining a multiplication $\cdot : \mathbb{R}^2 \times \mathbb{R}^2 \rightarrow \mathbb{R}^2$ in the following way:

$$\begin{pmatrix} a \\ b \end{pmatrix} \cdot \begin{pmatrix} c \\ d \end{pmatrix} := \begin{pmatrix} ac - bd \\ bc + ad \end{pmatrix}.$$

One writes $a + ib := \begin{pmatrix} a \\ b \end{pmatrix}$ and calls this field the [complex numbers](#) \mathbb{C} . The natural embedding $\mathbb{R} \rightarrow \mathbb{C}$ with $a \mapsto a + i0$ justifies the notation $\mathbb{R} \subset \mathbb{C}$.

Computations in \mathbb{C}

Business as usual, only new rule $i^2 = -1$. We use two complex numbers $w = u + iv$ and $z = x + iy$:

$$\begin{aligned} w + z &= (u + iv) + (x + iy) = (u + x) + i(v + y) \\ wz &= (u + iv)(x + iy) = ux + i(vx + uy) + i^2vy = (ux - vy) + i(vx + uy) \end{aligned}$$

$$\begin{aligned} \frac{w}{z} &= \frac{u+iv}{x+iy} = \frac{(u+iv)(x-iy)}{(x+iy)(x-iy)} = \frac{ux+ivx-iuy-i^2vy}{x^2-(iy)^2} \\ &= \frac{ux+vy}{x^2+y^2} + i \frac{vx-uy}{x^2+y^2} \end{aligned}$$

The last formula works if and only if $z \neq 0$, which means $x^2 + y^2 \neq 0$.

If $w = u$ (i.e. $v = 0$) is a real number, then: $uz = u(x+iy)$.

Thus, complex numbers can be added like vectors, and scaled by real numbers, just like vectors. So we can think of complex numbers as 2d vectors:

$$z = x + iy \cong \begin{pmatrix} x \\ y \end{pmatrix} \cong (x, y)$$

But they are more: vectors cannot be multiplied with each other, but with complex numbers we can do that. Just like the reals, they are a field (but have no ordering). This is very special. There is no 3d analogue to the complex numbers.

Definition 2.28.

We can define the following derived quantities for $z = x + iy$:

$\operatorname{Re} z = x$	<u>real part</u>
$\operatorname{Im} z = y$	<u>imaginary part</u>
$\bar{z} = x - iy$	<u>complex conjugate</u>
$r = z = \sqrt{x^2 + y^2}$	<u>absolute value, modulus</u>
$\varphi = \arg z = \text{angle of } z \text{ with positive real line}$	<u>argument</u>

A warning, concerning $\arg z$: its value is not unique: for example

$$\arg -i = -\pi/2 \text{ or } 3/2\pi.$$

Usually, one either takes $-\pi < \arg z \leq \pi$ or $0 \leq \arg z < 2\pi$. In cases of ambiguity, one has to carefully explain, what is meant.

We have:

$$z = |z|(\cos \arg z + i \sin \arg z) = |z|e^{i \arg z}$$

It holds

$$|wz| = |w||z| \quad \text{and} \quad \arg(zw) = \arg z + \arg w.$$

So we can write shortly:

$$zw = |w||z|e^{i(\arg w + \arg z)}$$

However, as usual with an angle, we would like to have $\arg w + \arg z$ in $[0, 2\pi[$. Thus,

$$zw = |w||z|e^{i\varphi}$$

where φ is chosen by using a $k \in \mathbb{Z}$ in a way that $0 \leq \varphi = \arg w + \arg z - 2k\pi < 2\pi$.

Representations of complex numbers

$$z = \underbrace{x + iy}_{\text{algebraical representation}} = \underbrace{|z| \cdot (\cos \varphi + i \sin \varphi)}_{\text{trigonometrical representation}} = \underbrace{|z| \cdot e^{i\varphi}}_{\text{exponential representation}}$$

The n^{th} square root of a complex number

It is well-known, that the equation $x^2 = a$ has two solutions $\pm\sqrt{a}$ for $a \neq 0$. What about $z^n = a$?

It follows from the multiplication rule that

$$z^n = |z|^n e^{in \arg z} = |a| e^{i \arg a}$$

Thus $|z| = |a|^{1/n}$ and $0 \leq \arg a = n \arg z - 2k\pi < 2\pi$, where $k \in \mathbb{N}$ can again be chosen.

Thus, we get the following solutions:

$$\begin{aligned}\arg z_0 &= \frac{1}{n} \arg a \\ \arg z_1 &= \frac{1}{n}(\arg a + 2\pi) \\ \arg z_2 &= \frac{1}{n}(\arg a + 2 \cdot 2\pi) \\ &\vdots\end{aligned}$$

So in general:

$$z_j = |a|^{\frac{1}{n}} e^{i \frac{1}{n}(\arg a + j \cdot 2\pi)} \quad \text{for } j = 0, \dots, n-1.$$

Thus, we have n complex n^{th} roots, which are evenly distributed on the circle around 0 with radius $|a|^{\frac{1}{n}}$.

It does not matter here, which $\arg a$ one takes, if *all* the results are written again in the form $z_k = x_k + iy_k$.

Summary

- \mathbb{R}^n denoted the set of all vectors with n real components
- You can add and scale vectors. Both operations in \mathbb{R}^n are realised by doing these inside the components.
- \mathbb{R}^n is an example of an abstract concept, called a *vector space*.
- Combinations like $\lambda \mathbf{v} + \mu \mathbf{u}$ are called *linear combinations*.
- There are linear subspaces and affine linear spaces. They are the generalisation of lines and planes one can illustrate in \mathbb{R}^3 .
- The inner product shows you orthogonality and the norm measures the length of a vector.

- In \mathbb{R}^3 we have the cross product to calculate orthogonal vectors.
- Complex numbers are given by a multiplication rule on \mathbb{R}^2 .

3

Matrices and linear systems

Arthur looked up. “Ford!” he said, “there’s an infinite number of monkeys outside who want to talk to us about this script for Hamlet they’ve worked out.”

Douglas Adams

In this chapter, we will study matrices in more detail, and after that, describe systems of linear equations. First of all, a *matrix* is just a table of numbers. One writes the numbers in a rectangle with m rows and n columns, where m, n are natural numbers.

Definition 3.1.

The set of all matrices with m rows and n columns is notated as:

$$\mathbb{R}^{m \times n} := \left\{ A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix} : a_{ij} \in \mathbb{R}, i = 1, \dots, m, j = 1, \dots, n \right\}$$

Since we can add matrices of the same size $A + B$ by adding the entries and scale a matrix $\lambda \cdot A$ by scaling the entries, the space $\mathbb{R}^{m \times n}$ is also a vector space.

Definition 3.2. Matrix + Matrix = Matrix

Let $A, B \in \mathbb{R}^{m \times n}$. The addition $A + B \in \mathbb{R}^{m \times n}$ is defined by

$$\underbrace{\begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \dots & a_{mn} \end{pmatrix}}_A + \underbrace{\begin{pmatrix} b_{11} & \dots & b_{1n} \\ \vdots & & \vdots \\ b_{m1} & \dots & b_{mn} \end{pmatrix}}_B := \underbrace{\begin{pmatrix} a_{11} + b_{11} & \dots & a_{1n} + b_{1n} \\ \vdots & & \vdots \\ a_{m1} + b_{m1} & \dots & a_{mn} + b_{mn} \end{pmatrix}}_{A+B}.$$

Example 3.3.

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 2 & -1 \end{pmatrix} = \begin{pmatrix} 1+1 & 2+0 \\ 3+2 & 4-1 \end{pmatrix} = \begin{pmatrix} 2 & 2 \\ 5 & 3 \end{pmatrix}.$$

Attention!

The addition $A + B$ is only defined for matrices with the same height and the same width.

Definition 3.4. Scalar · Matrix = Matrix

Let $A \in \mathbb{R}^{m \times n}$ and $\lambda \in \mathbb{R}$. Then the scalar multiplication $\lambda \cdot A \in \mathbb{R}^{m \times n}$ is defined by:

$$\lambda \cdot \underbrace{\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix}}_A := \underbrace{\begin{pmatrix} \lambda a_{11} & \cdots & \lambda a_{1n} \\ \vdots & & \vdots \\ \lambda a_{m1} & \cdots & \lambda a_{mn} \end{pmatrix}}_{\lambda \cdot A}.$$

Example 3.5.

$$2 \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 2 \cdot 1 & 2 \cdot 2 \\ 2 \cdot 3 & 2 \cdot 4 \end{pmatrix} = \begin{pmatrix} 2 & 4 \\ 6 & 8 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} + \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}.$$

(*) Supplementary details: $\mathbb{R}^{m \times n}$ is a vector space

The set $V := \mathbb{R}^{m \times n}$ with the addition $+$ and scalar multiplication \cdot is a [vector space](#), which means it fulfills the following:

- (1) $\forall A, B \in V : A + B = B + A$ (+ is commutative)
- (2) $\forall C, A, B \in V : C + (A + B) = (C + A) + B$ (+ is associative)
- (3) There is the zero matrix $0 \in V$ with the property: $\forall A \in V$ we have $A + 0 = A$.
- (4) For all $A \in V$ there is a matrix $-A \in V$ with $A + (-A) = 0$.
- (5) For the number $1 \in \mathbb{R}$ and each $A \in V$, one has: $1 \cdot A = A$.
- (6) $\forall \lambda, \mu \in \mathbb{R} \quad \forall A \in V : \lambda \cdot (\mu \cdot A) = (\lambda\mu) \cdot A$ (\cdot is associative)
- (7) $\forall \lambda \in \mathbb{R} \quad \forall A, B \in V : \lambda \cdot (A + B) = (\lambda \cdot A) + (\lambda \cdot B)$ (distributive $\cdot +$)
- (8) $\forall \lambda, \mu \in \mathbb{R} \quad \forall A \in V : (\lambda + \mu) \cdot A = (\lambda \cdot A) + (\mu \cdot A)$ (distributive $+ \cdot$)

To see why it is interesting to study matrices, we first have to look at linear equations.

3.1 Introduction to systems of linear equations

We start with some easy examples:

Xavier is two years older than Yasmin.

Together they are 40 years old.

How old is Xavier and how old is Yasmin?

$$x - y = 2$$

$$x + y = 40$$

$$x = ?, \quad y = ?$$

This was an example with two unknowns (x and y). Here we give an example for three

unknowns. (x, y and z):

$$\begin{array}{rcl} 2x & -3y & +4z = -7 \\ -3x & +y & -z = 0 \\ 20x & +10y & = 80 \\ & 10y & +25z = 90 \end{array}$$

You can imagine that we can have an arbitrary number of unknowns and also an arbitrary number of equations. Often these unknowns are denoted by x_1, x_2, \dots, x_n and we search for suitable values such that *all* equations are satisfied.

Here, the most important part is that the equations are linear. The exact definition will follow later. The sloppy way to say that an equation is linear is:

$$\text{constant} \cdot x_1 + \text{constant} \cdot x_2 + \cdots + \text{constant} \cdot x_n = \text{constant}. \quad (3.1)$$

As you can see, there are a lot of constants that have to be numeric.

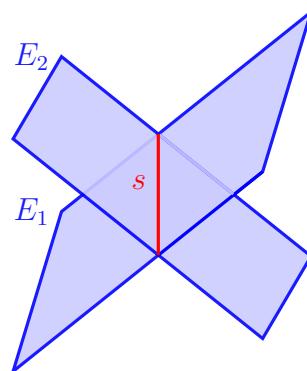
Definition 3.6. System of linear equations (LES)

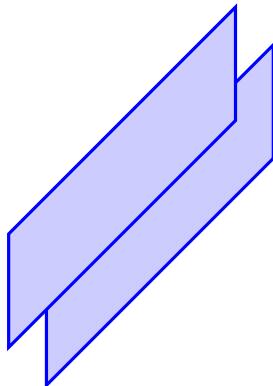
Let $m, n \in \mathbb{N}$ be two natural numbers. A system of linear equations or a linear equation system (abbreviation: LES) with m equations and n unknowns x_1, x_2, \dots, x_n is given by:

$$\left. \begin{array}{rcl} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n & = & b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n & = & b_2 \\ \vdots & & \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n & = & b_m \end{array} \right\} \quad (\text{LES})$$

Here, a_{ij} and b_i are given numbers, mostly just real numbers. A solution of the LES is a choice of values for x_1, \dots, x_n such that *all* m equations are satisfied.

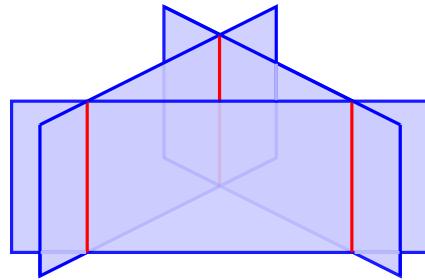
Example 3.7. Having three unknowns x_1, x_2, x_3 , we could have different cases for the set of solutions:





2 equations ↴

or



3 equations ↴

While the system with two variables was very well-arranged, the general case seems more complicated in spite of representing the same idea. At this point matrix and vector notation comes in very handy:

Definition 3.8. LES in matrix notation

Let $A \in \mathbb{R}^{m \times n}$ with entries $a_{ij} \in \mathbb{R}$ and $\mathbf{b} \in \mathbb{R}^m$ with entries $b_i \in \mathbb{R}$. Then

$$A\mathbf{x} = \mathbf{b}$$

represents (LES) from above, where $\mathbf{x} \in \mathbb{R}^n$.

The two examples from above in this notation:

$$\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ 40 \end{pmatrix}, \quad \begin{pmatrix} 2 & -3 & 4 \\ -3 & 1 & -1 \\ 20 & 10 & 0 \\ 0 & 10 & 25 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -7 \\ 0 \\ 80 \\ 90 \end{pmatrix}.$$

This can be seen as a short notation for a system of linear equations. However, this also defines a product of a matrix and a vector.

Definition 3.9. Matrix · Vector = Vector

Let $m, n \in \mathbb{N}$ and

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \in \mathbb{R}^{m \times n} \quad \text{and} \quad \mathbf{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{R}^n.$$

The product $A\mathbf{x} = A \cdot \mathbf{x}$ (where we mostly do not use a dot) is given as the vector

$$A\mathbf{x} := \begin{pmatrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \end{pmatrix} \in \mathbb{R}^m.$$

Attention!

The width of A has to be the same as the height of \mathbf{x} ! Otherwise $A\mathbf{x}$ is not defined.

3.2 Some words about matrices

For a matrix $A \in \mathbb{R}^{m \times n}$ the number m is called the number of rows and n the number of columns. The matrix A is a rectangle with height m and width n .

As special cases, we note:

- $A \in \mathbb{R}^{n \times n}$ (i.e. $m = n$) is called a square matrix or quadratic matrix
- $A \in \mathbb{R}^{m \times 1}$ is a column vector of size m
- $A \in \mathbb{R}^{1 \times n}$ is a row vector of size n
- $A \in \mathbb{R}^{1 \times 1}$ is a scalar, just a real number.

Then there are certain properties of A , concerning its diagonal:

- The entries a_{ii} are called “diagonal entries”. If outside of the diagonal entries of A are only zero entries, then A is called diagonal matrix.
- Everything above the diagonal (including it), is the upper triangle, and similarly we have the lower triangle. If A only contains non-zero entries there, we call it an upper or lower triangular matrix, respectively.
- If for all indices i, j , one has $a_{ij} = a_{ji}$. Then A is called symmetric (A is reflected over the diagonal).
- If for all indices i, j , one has $a_{ij} = -a_{ji}$ (so in particular $a_{ii} = 0$), then A is called skew-symmetric.

3.3 Looking at the columns and the associated linear map

One way to imagine a matrix in $\mathbb{R}^{m \times n}$ is to see it as a collection of n columns of size m :

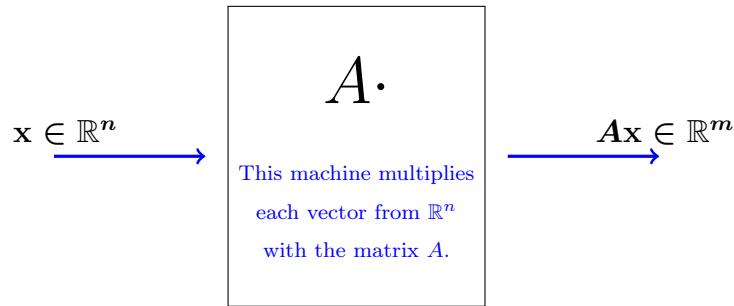
$$A = \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \dots & a_{mn} \end{pmatrix} = \begin{pmatrix} | & & | \\ \mathbf{a}_1 & \dots & \mathbf{a}_n \\ | & & | \end{pmatrix}, \text{ where } \mathbf{a}_i = \begin{pmatrix} a_{1i} \\ \vdots \\ a_{mi} \end{pmatrix}$$

In this view, the product of the matrix with a vector can just be seen as a linear combination of the columns of A :

Ax is a linear combination of the columns of A

$$Ax = \begin{pmatrix} | & & | \\ \mathbf{a}_1 & \dots & \mathbf{a}_n \\ | & & | \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = x_1 \begin{pmatrix} | \\ \mathbf{a}_1 \\ | \end{pmatrix} + \dots + x_n \begin{pmatrix} | \\ \mathbf{a}_n \\ | \end{pmatrix} \quad (3.2)$$

We can interpret this also as the following: The matrix is a machine where we can put in vectors $\mathbf{x} \in \mathbb{R}^n$ and we get a new vector $A\mathbf{x} \in \mathbb{R}^m$ as a linear combination of the columns. This machine is given by the multiplication of A by $\mathbf{x} \in \mathbb{R}^n$:



Such a machine is, of course, nothing else than “function” or “map” defined in Section 1.3. We call this function above f_A . It maps $\mathbf{x} \in \mathbb{R}^n$ to $A\mathbf{x} \in \mathbb{R}^m$.

The function f_A defined by the matrix A

$$f_A : \mathbb{R}^n \rightarrow \mathbb{R}^m, \quad \text{with} \quad f_A : \mathbf{x} \mapsto A\mathbf{x} \quad (3.3)$$

Here, the function f_A and the matrix include the same amount of information.

3.4 Looking at the rows

Above, we have considered a matrix $A \in \mathbb{R}^{m \times n}$ as a collection of columns and defined a linear map $f_A : \mathbb{R}^n \rightarrow \mathbb{R}^m$. However, we may also see A as a collection of m rows of size n :

$$A = \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \dots & a_{mn} \end{pmatrix} = \begin{pmatrix} -\boldsymbol{\alpha}_1^T- \\ \dots \\ -\boldsymbol{\alpha}_m^T- \end{pmatrix}, \quad \text{where } \boldsymbol{\alpha}_i^T = (a_{i1} \ \dots \ a_{in})$$

Here, we use the notation T for the transpose of a column vector. The result is a row vector with the same entries. We fix this as a space:

$$\mathbb{R}^{1 \times n} = \{ \mathbf{x}^T = (x_1 \ \dots \ x_n) : x_1, \dots, x_n \in \mathbb{R} \}$$

Since a row vector $\mathbf{u}^T \in \mathbb{R}^{1 \times n}$ is just a very flat matrix, the product with a column vector $\mathbf{v} \in \mathbb{R}^n$ is well-defined:

$$\mathbf{u}^T \mathbf{v} = (u_1 v_1 + \dots + u_n v_n) \in \mathbb{R}^{1 \times 1}.$$

Indeed, this is just a scalar and it coincides with standard inner product we already know: $\langle \mathbf{u}, \mathbf{v} \rangle = \mathbf{u}^T \mathbf{v}$.

In this view, the product of the matrix with a vector can just be seen as the scalar product with each row:

Ax is the scalar product of x with the rows of A

$$Ax = \begin{pmatrix} -\alpha_1^T - \\ \ddots \\ -\alpha_m^T - \end{pmatrix} \begin{pmatrix} | \\ x \\ | \end{pmatrix} = \begin{pmatrix} \alpha_1^T x \\ \vdots \\ \alpha_m^T x \end{pmatrix} \quad (3.4)$$

3.5 Matrix multiplication

Let $A \in \mathbb{R}^{m \times n}$. Recall that we can build the product of A by a vector $\mathbf{b} \in \mathbb{R}^n$. The result is a column vector in \mathbb{R}^m . We can do this for several vectors grouped into a matrix B , and group the result in a matrix again:

$$A \begin{pmatrix} | \\ \mathbf{b}_1 \dots \mathbf{b}_k \\ | \end{pmatrix} = \begin{pmatrix} | \\ A\mathbf{b}_1 & A\mathbf{b}_2 & \dots & A\mathbf{b}_k \\ | \end{pmatrix} \quad \text{for } k \text{ column vectors } \mathbf{b}_1, \dots, \mathbf{b}_k.$$

The result is a matrix with m rows and k columns and denoted by AB . It is called the *matrix product of A and B*.

Definition 3.10. Matrix product

For matrices $A \in \mathbb{R}^{m \times n}$ and $B \in \mathbb{R}^{n \times k}$, the *matrix product* is defined as

$$AB = \begin{pmatrix} -\alpha_1^T - \\ \ddots \\ -\alpha_m^T - \end{pmatrix} \begin{pmatrix} | \\ \mathbf{b}_1 \dots \mathbf{b}_k \\ | \end{pmatrix} = \begin{pmatrix} \alpha_1^T \mathbf{b}_1 & \dots & \alpha_1^T \mathbf{b}_k \\ \vdots & & \vdots \\ \alpha_m^T \mathbf{b}_1 & \dots & \alpha_m^T \mathbf{b}_k \end{pmatrix} \in \mathbb{R}^{m \times k}. \quad (3.5)$$

Or in other words: AB is the $m \times k$ -matrix that has the following entries:

$$(AB)_{ij} = \sum_{r=1}^n a_{ir} b_{rj}$$

for $i = 1, \dots, m$ and $j = 1, \dots, k$.

Attention!

The product AB is only defined if the width of A coincides with the height of B . The “inner dimensions” have to match.

Special cases:

- $A = \mathbf{a}^T \in \mathbb{R}^{1 \times n}$, $B = \mathbf{b} \in \mathbb{R}^{n \times 1}$: $AB = \mathbf{a}^T \mathbf{b} \in \mathbb{R}$
- $A = \mathbf{a} \in \mathbb{R}^{n \times 1}$, $B = \mathbf{b}^T \in \mathbb{R}^{1 \times m}$: $AB = \mathbf{a} \mathbf{b}^T \in \mathbb{R}^{n \times m}$, $(AB)_{ij} = a_i b_j$ (called rank 1 matrix)

Example 3.11. Just calculate some examples:

(a) We combine the following matrix dimensions $(2 \times 2) \cdot (2 \times 3) \Rightarrow 2 \times 3$:

$$\begin{aligned} \underbrace{\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}}_A \underbrace{\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}}_B &= \left(\underbrace{\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}}_A \underbrace{\begin{pmatrix} 1 \\ 4 \end{pmatrix}}_{\mathbf{b}_1} \quad \underbrace{\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}}_A \underbrace{\begin{pmatrix} 2 \\ 5 \end{pmatrix}}_{\mathbf{b}_2} \quad \underbrace{\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}}_A \underbrace{\begin{pmatrix} 3 \\ 6 \end{pmatrix}}_{\mathbf{b}_3} \right) \\ &= \left(\underbrace{1 \cdot 1 + 2 \cdot 4}_{A\mathbf{b}_1} \quad \underbrace{1 \cdot 2 + 2 \cdot 5}_{A\mathbf{b}_2} \quad \underbrace{1 \cdot 3 + 2 \cdot 6}_{A\mathbf{b}_3} \right) = \begin{pmatrix} 9 & 12 & 15 \\ 19 & 26 & 33 \end{pmatrix} \end{aligned}$$

(b) Let $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$. Then the product of A and B is not defined since $\text{width}(A) = 3 \neq 2 = \text{height}(B)$. The product with the other order BA is defined.

(c) Now the matrix dimensions $(3 \times 1) \cdot (1 \times 3) \Rightarrow 3 \times 3$:

$$\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} (1 \ 2 \ 3) = \left(\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \underbrace{1}_{\mathbf{b}_1} \quad \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \underbrace{2}_{\mathbf{b}_2} \quad \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \underbrace{3}_{\mathbf{b}_3} \right) = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{pmatrix}$$

(d) Now the matrix dimensions $(1 \times 3) \cdot (3 \times 1) \Rightarrow 1 \times 1$:

$$(1 \ 2 \ 3) \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = (1 \cdot 1 + 2 \cdot 2 + 3 \cdot 3) = (14) = 14$$

(e) A 2×2 -example:

$$\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} \text{Mario} & \text{Mario} \\ \text{Goomba} & \text{Goomba} \end{bmatrix} = \begin{bmatrix} \text{Mario} & \text{Mario} \\ \text{Mario} & \text{Mario} \end{bmatrix}$$

$$\begin{aligned} 1 \cdot \text{Mario} + 0 \cdot \text{Goomba} &= \text{Mario} & 1 \cdot \text{Mario} + 0 \cdot \text{Goomba} &= \text{Mario} \\ 1 \cdot \text{Mario} + 1 \cdot \text{Goomba} &= \text{Mario} & 1 \cdot \text{Mario} + 1 \cdot \text{Goomba} &= \text{Mario} \end{aligned}$$

(Source of the picture: g33ktheory.com)

We can also ask what happens if we multiply a row vector \mathbf{x}^T from the left to a matrix $B \in \mathbb{R}^{n \times k}$. By definition, we get:

$$\mathbf{x}^T B = (x_1 \ \cdots \ x_n) \begin{pmatrix} -\beta_1^T \\ \vdots \\ -\beta_n^T \end{pmatrix} = x_1 (-\beta_1^T) + \cdots + x_n (-\beta_n^T).$$

This means the product $\mathbf{x}^T B$ is a linear combination of the rows of B . This is an analogy that $A\mathbf{x}$ is a linear combination of the columns of A , cf. equation (3.2).

Remark:

Now we can see the matrix product as introduced

$$AB = \begin{pmatrix} A\mathbf{b}_1 & A\mathbf{b}_2 & \dots & A\mathbf{b}_k \end{pmatrix}$$

This means that each column of AB consists of a linear combination of the columns from A .

Seeing the product with the other eye

$$AB = \begin{pmatrix} -\boldsymbol{\alpha}_1^T - \\ \vdots \\ -\boldsymbol{\alpha}_m^T - \end{pmatrix} B = \begin{pmatrix} -\boldsymbol{\alpha}_1^T B - \\ \vdots \\ -\boldsymbol{\alpha}_m^T B - \end{pmatrix},$$

we see that each row of AB consists of a linear combination of the rows from B .

Now, we summarise the properties of the matrix multiplication.

Proposition 3.12. Properties of the matrix product

(a) For all $A, B \in \mathbb{R}^{m \times n}$ and $C \in \mathbb{R}^{n \times k}$ and $D \in \mathbb{R}^{\ell \times m}$ we have:

$$(A + B) \cdot C = A \cdot C + B \cdot C \quad \text{and} \quad D \cdot (A + B) = D \cdot A + D \cdot B.$$

(b) For all $A \in \mathbb{R}^{m \times n}$ and $B \in \mathbb{R}^{n \times k}$ and $\lambda \in \mathbb{R}$ we have:

$$\lambda \cdot (A \cdot B) = (\lambda \cdot A) \cdot B = A \cdot (\lambda \cdot B).$$

(c) Associative rule: For all $A \in \mathbb{R}^{m \times n}$ and $B \in \mathbb{R}^{n \times k}$ and $C \in \mathbb{R}^{k \times \ell}$ we have:

$$A \cdot (B \cdot C) = (A \cdot B) \cdot C.$$

Proof. All these rules follow from the definition of the matrix product of A and B ,

$$(AB)_{ij} = \sum_{r=1}^n a_{ir} b_{rj},$$

and the fact that these rules hold for the real numbers $a_{ir}, b_{rj} \in \mathbb{R}$. For example, for showing (c):

$$(A(BC))_{ij} = \sum_{r=1}^n a_{ir} (BC)_{rj} = \sum_{r=1}^n a_{ir} \sum_{z=1}^n b_{rz} c_{zj} = \sum_{z=1}^n \left(\sum_{r=1}^n a_{ir} b_{rz} \right) c_{zj} = ((AB)C)_{ij}.$$

Properties (a) and (b) are left as an exercise. □

Attention! No commutative rule

In general, we have for two matrices:

$$AB \neq BA \quad (\text{in general}).$$

Remark:

We thus have the following interpretations of the matrix vector product AB :

- the columns of B are used to build linear combinations of the columns of A ,
- the rows of A are used to build linear combinations of the rows of B ,
- each row α_i^T of A and each column \mathbf{b}_j of B are multiplied to form an entry of the product: $(AB)_{ij} = \alpha_i^T \mathbf{b}_j$,
- each column \mathbf{a}_i of A and each row β_i^T of B is combined to a rank-1 matrix $\mathbf{a}_i \beta_i^T$, and the matrices are added up,

All these interpretations are equally valid, and from situation to situation, we can change our point of view to gain additional insights.

3.6 Linear maps

Each map that conserves the structure of our vector space \mathbb{R}^n is called a [linear map](#). We already know that the only structure we have is the vector addition and the scalar multiplication.

Definition 3.13. Linearity of maps

A map $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is called [linear](#) if for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}$, we have:

$$f(\mathbf{x} + \mathbf{y}) = f(\mathbf{x}) + f(\mathbf{y}) \quad (+)$$

$$f(\lambda \mathbf{x}) = \lambda f(\mathbf{x}) \quad (\cdot)$$

Rule of thumb:

Equation (+) means: First adding, then mapping = First mapping, then adding
 Equation (·) means: First scaling, then mapping = First mapping, then scaling

We already know that for each matrix $A \in \mathbb{R}^{m \times n}$ there is an associated map f_A . This map is indeed a linear map.

Proposition 3.14. f_A is linear

Let $m, n \in \mathbb{N}$, $A \in \mathbb{R}^{m \times n}$ and $f_A : \mathbb{R}^n \rightarrow \mathbb{R}^m$ with $\mathbf{x} \xrightarrow{f_A} A\mathbf{x}$. Then the following holds:

(a) For all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ we have

$$f_A(\mathbf{x} + \mathbf{y}) = f_A(\mathbf{x}) + f_A(\mathbf{y}), \quad \text{i.e.} \quad A(\mathbf{x} + \mathbf{y}) = A\mathbf{x} + A\mathbf{y}. \quad (+)$$

(b) For all $\lambda \in \mathbb{R}$ and $\mathbf{x} \in \mathbb{R}^n$ one has

$$f_A(\lambda \mathbf{x}) = \lambda f_A(\mathbf{x}), \quad \text{i.e.} \quad A(\lambda \mathbf{x}) = \lambda A\mathbf{x}. \quad (\cdot)$$

Proof. This follows immediately from the properties of the matrix product in Proposition 3.12. However, it may be helpful to write down a direct proof for the case $n = 2$.

(a) Let $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ and $\mathbf{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$ be vectors in \mathbb{R}^2 . Then we have:

$$\begin{aligned} f_A(\mathbf{x} + \mathbf{y}) &= A(\mathbf{x} + \mathbf{y}) = \begin{pmatrix} | & | \\ \mathbf{a}_1 & \mathbf{a}_2 \\ | & | \end{pmatrix} \begin{pmatrix} x_1 + y_1 \\ x_2 + y_2 \end{pmatrix} \stackrel{(3.2)}{=} (x_1 + y_1)\mathbf{a}_1 + (x_2 + y_2)\mathbf{a}_2 \\ &= x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + y_1\mathbf{a}_1 + y_2\mathbf{a}_2 \stackrel{(3.2)}{=} \begin{pmatrix} | & | \\ \mathbf{a}_1 & \mathbf{a}_2 \\ | & | \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} | & | \\ \mathbf{a}_1 & \mathbf{a}_2 \\ | & | \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \\ &= A\mathbf{x} + A\mathbf{y} = f_A(\mathbf{x}) + f_A(\mathbf{y}). \end{aligned}$$

(b) Let $\lambda \in \mathbb{R}$ and $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2$. Then:

$$\begin{aligned} f_A(\lambda\mathbf{x}) &= A(\lambda\mathbf{x}) = \begin{pmatrix} | & | \\ \mathbf{a}_1 & \mathbf{a}_2 \\ | & | \end{pmatrix} \begin{pmatrix} \lambda x_1 \\ \lambda x_2 \end{pmatrix} \stackrel{(3.2)}{=} (\lambda x_1)\mathbf{a}_1 + (\lambda x_2)\mathbf{a}_2 \\ &= \lambda(x_1\mathbf{a}_1 + x_2\mathbf{a}_2) \stackrel{(3.2)}{=} \lambda \begin{pmatrix} | & | \\ \mathbf{a}_1 & \mathbf{a}_2 \\ | & | \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \lambda A\mathbf{x} = \lambda f_A(\mathbf{x}). \quad \square \end{aligned}$$

Now we look at the connection of the composition and matrix product:

- For $A \in \mathbb{R}^{m \times k}$, we have the linear map $f_A : \mathbb{R}^k \rightarrow \mathbb{R}^m$: $\mathbf{v} \mapsto A\mathbf{v}$
- For $B \in \mathbb{R}^{k \times n}$, we have the linear map $f_B : \mathbb{R}^n \rightarrow \mathbb{R}^k$: $\mathbf{x} \mapsto B\mathbf{x} = \mathbf{v}$.

Then we can interpret the product $AB \in \mathbb{R}^{m \times n}$ as a linear map of column vectors:

$$\begin{aligned} f_A \circ f_B : \mathbb{R}^n &\rightarrow \mathbb{R}^m \\ \mathbf{x} &\mapsto A(B\mathbf{x}) = AB\mathbf{x}. \end{aligned}$$

where we use the associativity law.

If we have a linear map $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$, we can write it as

$$f(\mathbf{x}) = f(x_1\mathbf{e}_1 + \cdots + x_n\mathbf{e}_n) = x_1f(\mathbf{e}_1) + \cdots + x_nf(\mathbf{e}_n)$$

and immediately find:

Remark: Linear maps induce matrices

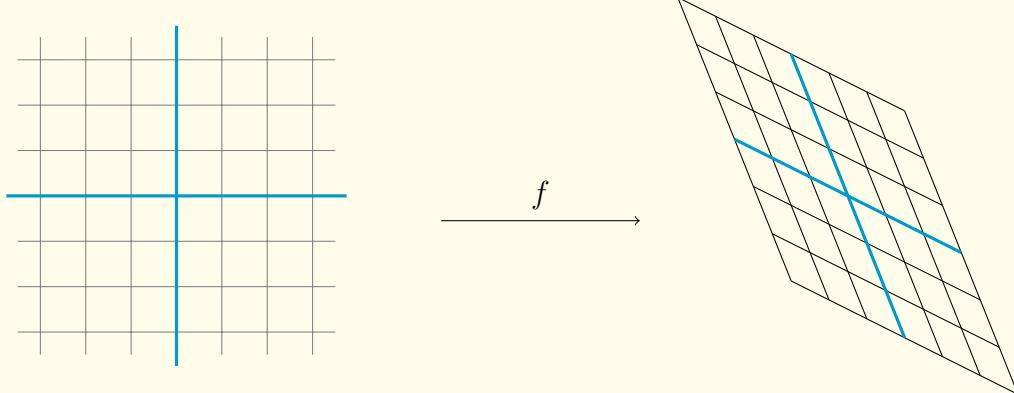
For each linear map $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$, there is exactly one matrix $A \in \mathbb{R}^{m \times n}$ with $f = f_A$. In the columns of A , one finds the images of the canonical unit vectors:

$$A := \begin{pmatrix} | & & | \\ f(\mathbf{e}_1) & \dots & f(\mathbf{e}_n) \\ | & & | \end{pmatrix} \quad (3.6)$$

A is often called the transformation matrix of f .

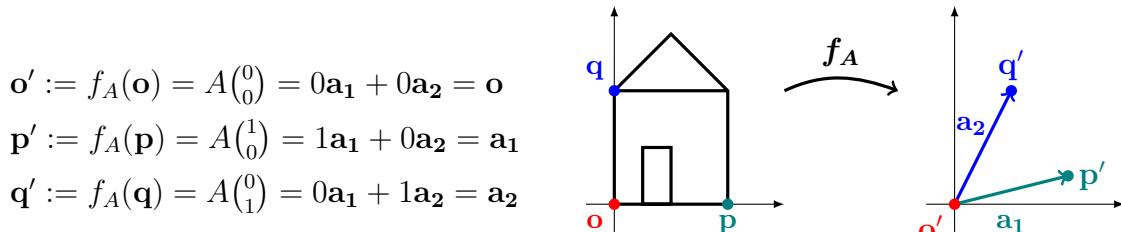
Rule of thumb: Linear map = lines stay lines

A linear map $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$ conserves the linear structure: If $U \subset \mathbb{R}^n$ is a linear subspace then also the image $f(U) \subset \mathbb{R}^m$. Or in other words: Lines on the left stay lines on the right:



(However, lines could shrink down to the origin.)

A linear map is completely determined when one knows how it acts on the canonical unit vectors $\mathbf{e}_1, \dots, \mathbf{e}_n$. Therefore, in \mathbb{R}^2 , a good visualisation is to look at “houses”: A house H is given by two points. Now what happens under a linear map f_A associated to a matrix A ? One just have to look at the corners:

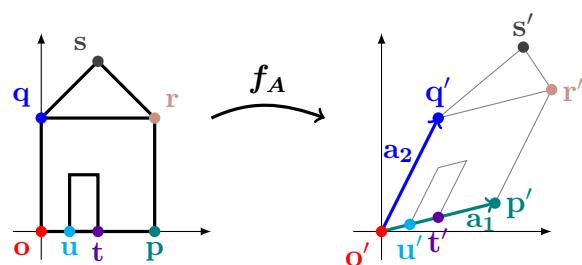


With the help of the linearity, we also know what happens with the other parts of the house, for example the corners of the door:

Since $\mathbf{t} = \frac{1}{2}\mathbf{p}$ and $\mathbf{u} = \frac{1}{4}\mathbf{p}$, we have:

$$f_A(\mathbf{t}) = f_A\left(\frac{1}{2}\mathbf{p}\right) \stackrel{(\cdot)}{=} \frac{1}{2}f_A(\mathbf{p}) = \frac{1}{2}\mathbf{p}'$$

$$f_A(\mathbf{u}) = f_A\left(\frac{1}{4}\mathbf{p}\right) \stackrel{(\cdot)}{=} \frac{1}{4}f_A(\mathbf{p}) = \frac{1}{4}\mathbf{p}'$$

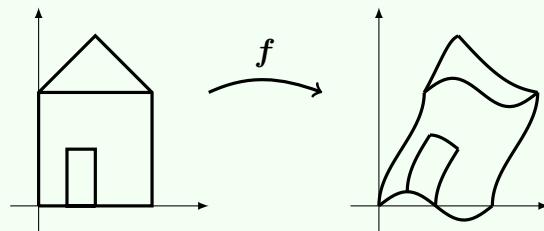


Example 3.15. A non-linear map

A map $f : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ given by

$$f : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x - \frac{1}{5}(\cos(\pi y) - 1) \\ y + \frac{1}{8}\sin(2\pi x) \end{pmatrix}.$$

is not linear!



Example 3.16. Some linear house transformations

$A = \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}$	$B = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$	$C = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}$
$D = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$	$E = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$	$F = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$
$G = \begin{pmatrix} 5 & 0 \\ 0 & 5 \end{pmatrix} \quad H' = 5H$	$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad H' = H$	$J = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$
$K = \begin{pmatrix} 3 & 1 \\ 1 & 2 \end{pmatrix}$	$L = \begin{pmatrix} 3 & 6 \\ 1 & 2 \end{pmatrix}$	$M = \begin{pmatrix} 3 & 0 \\ 1 & 0 \end{pmatrix}$
$N = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$	$O = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$	$P = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$
$Q = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$	$R = \begin{pmatrix} \cos(\frac{\pi}{6}) & -\sin(\frac{\pi}{6}) \\ \sin(\frac{\pi}{6}) & \cos(\frac{\pi}{6}) \end{pmatrix}$	$S = \begin{pmatrix} -1 & -1 \\ -3 & 3 \end{pmatrix}$

3.7 Linear dependence, linear independence, basis and dimension

We have seen that in \mathbb{R}^2 two vectors can be parallel ([colinear](#)):

There is a $\lambda \in \mathbb{R}$ with $\mathbf{a} = \lambda\mathbf{b}$.

Similarly, in \mathbb{R}^3 three vectors can be in the same plane ([coplanar](#)):

There are $\lambda, \mu \in \mathbb{R}$ with $\mathbf{a} = \lambda\mathbf{b} + \mu\mathbf{c}$.

If this is the case, we can build a loop of vectors, starting at \mathbf{o} and ending at \mathbf{o} again:

$$\mathbf{o} = (-1)\mathbf{a} + \lambda\mathbf{b} + \mu\mathbf{c}.$$

Let us generalise this:

Definition 3.17. Linear dependence and independence

A family $(\mathbf{v}_1, \dots, \mathbf{v}_k)$ of k vectors from \mathbb{R}^n is called [linearly dependent](#) if we find a non-trivial linear combination for \mathbf{o} . This means that we can find $\lambda_1, \dots, \lambda_n \in \mathbb{R}$ that are not all equal zero such that

$$\sum_{j=1}^k \lambda_j \mathbf{v}_j = \mathbf{o}.$$

If this is not possible, we call the family $(\mathbf{v}_1, \dots, \mathbf{v}_k)$ [linearly independent](#). This means that

$$\sum_{j=1}^k \lambda_j \mathbf{v}_j = \mathbf{o} \Rightarrow \lambda_1, \dots, \lambda_k = 0$$

holds.

Example 3.18. Let us look at examples:

(a) The family $((1), (1), (0))$ is linearly dependent since

$$\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

(b) $((1), (1), (2))$ is linearly dependent.

(c) Each family which includes \mathbf{o} is linearly dependent. Also each family that has the same vector twice or more is linearly dependent.

(d)

$$\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad \mathbf{e}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad \mathbf{e}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

These are linearly independent vectors, because

$$\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = \lambda_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \lambda_2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + \lambda_3 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{pmatrix}$$

yields $\lambda_1 = \lambda_2 = \lambda_3 = 0$.

If we add an arbitrary additional vector

$$\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix},$$

we can combine it from the other three by setting $\lambda_i = a_i$, which means:

$$\mathbf{a} = a_1 \mathbf{e}_1 + a_2 \mathbf{e}_2 + a_3 \mathbf{e}_3$$

So the resulting set of vectors is linearly dependent.

Proposition 3.19. Linear dependence

For a family $(\mathbf{v}_1, \dots, \mathbf{v}_k)$ of vectors from \mathbb{R}^n the following claims are equivalent:

- (i) $(\mathbf{v}_1, \dots, \mathbf{v}_k)$ is linearly dependent.
- (ii) There is a vector in $\text{Span}(\mathbf{v}_1, \dots, \mathbf{v}_k)$ that has two or more representations as a linear combination.
- (iii) At least one of the vectors in $(\mathbf{v}_1, \dots, \mathbf{v}_k)$ is a linear combination of the others.
- (iv) There is an $i \in \{1, \dots, k\}$ such that we have

$$\text{Span}(\mathbf{v}_1, \dots, \mathbf{v}_k) = \text{Span}(\mathbf{v}_1, \dots, \mathbf{v}_{i-1}, \mathbf{v}_{i+1}, \dots, \mathbf{v}_k).$$

- (v) There is an $i \in \{1, \dots, k\}$ with $\mathbf{v}_i \in \text{Span}(\mathbf{v}_1, \dots, \mathbf{v}_{i-1}, \mathbf{v}_{i+1}, \dots, \mathbf{v}_k)$.

Proof. Exercise! □

Since the opposite of linear dependence is linear independence, we can simply negate Proposition 3.19 and get the following:

Proposition 3.20. Linear independence

For a family $(\mathbf{v}_1, \dots, \mathbf{v}_k)$ of vectors from \mathbb{R}^n the following are equivalent:

- (i) $(\mathbf{v}_1, \dots, \mathbf{v}_k)$ is linearly independent.
- (ii) Every vector in $\text{Span}(\mathbf{v}_1, \dots, \mathbf{v}_k)$ can be formed by linear combinations in exactly one way.
- (iii) None of the vectors in $(\mathbf{v}_1, \dots, \mathbf{v}_k)$ is a linear combination of the others.

(iv) For all $i \in \{1, \dots, k\}$ we have:

$$\text{Span}(\mathbf{v}_1, \dots, \mathbf{v}_k) \neq \text{Span}(\mathbf{v}_1, \dots, \mathbf{v}_{i-1}, \mathbf{v}_{i+1}, \dots, \mathbf{v}_k).$$

(v) For all $i \in \{1, \dots, k\}$ we have $\mathbf{v}_i \notin \text{Span}(\mathbf{v}_1, \dots, \mathbf{v}_{i-1}, \mathbf{v}_{i+1}, \dots, \mathbf{v}_k)$.

A simple consequence is:

Corollary 3.21.

If the family $(\mathbf{v}_1, \dots, \mathbf{v}_k)$ is linearly dependent, we can subjoin vectors and the resulting family is still linearly dependent. On the other hand, if the family $(\mathbf{v}_1, \dots, \mathbf{v}_k)$ is linearly independent, we can omit vectors and the resulting family is still linearly independent.

Let now V be a subspace of \mathbb{R}^n , which is spanned by the vectors $\mathbf{v}_1, \dots, \mathbf{v}_k \in \mathbb{R}^n$. Hence $V = \text{Span}(\mathbf{v}_1, \dots, \mathbf{v}_k)$.

Question: Efficiency question:

How many vectors do we actually need to span V ?

The quick answer “ k ” is in general false since the family $(\mathbf{v}_1, \dots, \mathbf{v}_k)$ could be linearly dependent.

Our geometric intuition says that on a plane we cannot have more than two linearly independent vectors, and in three-dimensional space not more than three. We express this, by saying, a plane is two-dimensional, and space is three-dimensional. We will again formalise this:

Definition 3.22. Basis, basis vectors

Let V be a subspace of \mathbb{R}^n . A family $\mathcal{B} = (\mathbf{v}_1, \dots, \mathbf{v}_k)$ is called a basis of V if

- (a) $V = \text{Span}(\mathcal{B})$ and
- (b) \mathcal{B} is linearly independent.

The elements of \mathcal{B} are called the basis vectors of V .

We can show that each subspace $V \subset \mathbb{R}^n$ has a basis. We define:

Proposition & Definition 3.23. Coefficients with respect to a basis

Let $\mathcal{B} = (\mathbf{v}_1, \dots, \mathbf{v}_k)$ be a basis of a subspace $V \subset \mathbb{R}^n$. Each $\mathbf{x} \in V$ can be written as a linear combination $\lambda_1 \mathbf{v}_1 + \dots + \lambda_k \mathbf{v}_k$ where the coefficients $\lambda_1, \dots, \lambda_k$ are **uniquely determined**. They are called the coordinates of \mathbf{x} with respect to \mathcal{B} .

Proof. Let $\mathcal{B} = (\mathbf{v}_1, \dots, \mathbf{v}_k)$ be a basis of V . Since $\text{Span}(\mathcal{B}) = V$, we can express each $\mathbf{v} \in V$ as a linear combination $\lambda_1 \mathbf{v}_1 + \dots + \lambda_k \mathbf{v}_k$. The uniqueness of the coefficients $\lambda_1, \dots, \lambda_k$ follows from the linear independence of the basis vectors. This is an easy exercise. \square

Theorem 3.24. Steinitz's theorem

Consider a basis $\mathcal{B} = (\mathbf{v}_1, \dots, \mathbf{v}_k)$ of a subspace $V \subset \mathbb{R}^n$ and a linearly independent set of vectors $\mathcal{A} = (\mathbf{a}_1, \dots, \mathbf{a}_\ell) \subset V$. Then we can extend \mathcal{A} to a basis of V by adding $k - \ell$ elements of \mathcal{B} .

Sketch of the proof. Pack \mathcal{B} and \mathcal{A} together to a linearly dependent set, and remove vectors (starting with elements of \mathcal{B}) until it is linearly independent. One has to show now, that the resulting set has again k elements, and that \mathcal{A} remains untouched. \square

Now, we can record that all bases of V have the same number of elements.

Corollary 3.25.

Let V be a subspace of \mathbb{R}^n and let $\mathcal{B} = (\mathbf{v}_1, \dots, \mathbf{v}_k)$ be a basis of V . Then:

- (a) Each family $(\mathbf{w}_1, \dots, \mathbf{w}_m)$ consisting of vectors from V where $m > k$ is linearly dependent.
- (b) Each basis of V has exactly k elements.

So we can define:

Definition 3.26. Dimension of a linear subspace

Let V be a subspace of \mathbb{R}^n , and let \mathcal{B} be a chosen basis of V . The number of elements in \mathcal{B} is well-defined and called the dimension of V , written as $\dim(V)$. As a special case, we set $\dim(\{\mathbf{o}\}) = 0$.

The unit vectors $\mathbf{e}_1, \dots, \mathbf{e}_n$ in \mathbb{R}^n form a basis. The linear independency can be seen by:

$$x_1 \mathbf{e}_1 + \dots + x_n \mathbf{e}_n = \mathbf{o} \quad \Rightarrow \quad \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \mathbf{o} \quad \Rightarrow \quad x_1 = \dots = x_n = 0.$$

We obtain, as expected:

$$\dim(\mathbb{R}^n) = n.$$

Rule of thumb:

The dimension of a vector space V says how many independent degrees of freedom are needed to build linear combinations of all vectors in V .

Theorem 3.27.

Let U and V be two linear subspaces of \mathbb{R}^n .

- (i) One has $\dim(U) = \dim(V)$ if and only if there exists a linear bijective map between U and V .
- (ii) If $U \subset V$ and $\dim(U) = \dim(V)$, then $U = V$.

Proof. (i) (\Rightarrow): If $\dim(U) = \dim(V)$, a bijection can be defined by mapping the basis vectors of the subspace U to the basis vectors of V and extending it linearly.

(\Leftarrow): If there is a bijection between U and V , the image of the basis of U is a basis of V (linearly independent by injectivity and spanning by surjectivity). Hence, $\dim(U) = \dim(V)$.

(ii) A basis of U is an linearly independent family in V with $\dim(U) = \dim(V)$ vectors. Thus it is also a basis for V , and thus $U = V$. \square

Example 3.28.

The following subspaces of \mathbb{R}^n are very important:

- The trivial subspace $\{0\}$ with $\dim(\{0\}) = 0$
- Lines L (through the origin): $\dim(L) = 1$
- Planes P (through the origin): $\dim(P) = 2$
- Hyperplanes H (through the origin): $\dim(H) = n - 1$

The dimension of an affine subspace $W = \mathbf{u}_0 + U$ (where U is a linear subspace) is usually set to the dimension of U .

Corollary 3.29.

A family consisting of more than n vectors in \mathbb{R}^n is always linearly dependent.

Proof. Use Corollary 3.25. \square

3.8 Identity and inverses

For each $n \in \mathbb{N}$, we define the identity matrix $\mathbf{1}_n$ by

$$\mathbf{1}_n := \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & & \vdots \\ 0 & 0 & 1 & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & 1 \end{pmatrix} \in \mathbb{R}^{n \times n}.$$

The associated linear map $\mathbb{R}^n \rightarrow \mathbb{R}^n$ is, of course, the identity *id*. Other notations for the identity matrix are $\mathbf{1}_n$, I_n or E_n . If the context is clear, one usually omits the index n .

In the space of quadratic matrices $A \in \mathbb{R}^{n \times n}$, the identity matrix fulfils:

$$\mathbf{1}_n \mathbf{x} = \mathbf{x}.$$

Also we can define inverses A^{-1} , which may or may not exist:

$$AA^{-1} = A^{-1}A = \mathbf{1}_n.$$

Definition 3.30. Invertible Matrix, A^{-1}

We call a square matrix $A \in \mathbb{R}^{n \times n}$ invertible or nonsingular if the corresponding linear map $f_A : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is bijective. Otherwise, we call A singular. A matrix \tilde{A} with $f_{\tilde{A}} = (f_A)^{-1}$ is called the inverse of A and is usually denoted by A^{-1} . We have

$$f_{A^{-1}} \circ f_A = id \quad \text{and} \quad f_A \circ f_{A^{-1}} = id,$$

which means $f_{A^{-1}} = (f_A)^{-1}$.

For the matrices, this means:

$$A^{-1}(Ax) = x \quad \text{and} \quad A(A^{-1}x) = x \quad \text{for all } x \in \mathbb{R}^n.$$

In short: $A^{-1}A = \mathbf{1}$ and $AA^{-1} = \mathbf{1}$.

If A is invertible, the linear system $Ax = \mathbf{b}$ has the unique solution $x = A^{-1}\mathbf{b}$.

Theorem 3.31.

Let $A \in \mathbb{R}^{n \times n}$ be a square matrix. Then

$$f_A \text{ injective} \Leftrightarrow f_A \text{ surjective}$$

Hence, if one of these cases holds, then f_A is already bijective, i.e., invertible.

Proof. This is a classical dimension argument:

(\Rightarrow): If f_A is injective, then $(f(\mathbf{e}_1), \dots, f(\mathbf{e}_n))$ are linearly independent vectors and form a basis of \mathbb{R}^n . This means that f_A is also surjective.

(\Leftarrow): If f_A is surjective, then each $\mathbf{y} \in \mathbb{R}^n$ is given by a linear combination from the family $(f(\mathbf{e}_1), \dots, f(\mathbf{e}_n))$ and, hence, it forms a basis of \mathbb{R}^n . Therefore, f_A is also injective. \square

For two invertible matrices A and B we have the formula:

$$(AB)^{-1} = B^{-1}A^{-1}.$$

Remark:

If $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a linear map that is bijective, then $f^{-1} : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is also a linear map.

3.9 Transposition

Mathematicians found it convenient to write down most things in terms of column vectors, and mainly think about column matrices. If one wants to talk about the rows of a matrix, most of the time, one defines the transposed matrix and talks about their columns.

We already know transposition of column vectors:

$$\begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}^T = (a_1 \ \dots \ a_n)$$

and similarly, we can define:

$$(a_1 \dots a_n)^T = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix},$$

Then we have the simple formula $(\mathbf{a}^T)^T = \mathbf{a}$.

For a matrix, we can do the same:

Definition 3.32. Transpose

For a matrix $A \in \mathbb{R}^{m \times n}$, we define a matrix $A^T \in \mathbb{R}^{n \times m}$ and call it the transpose of A . The i^{th} column of A becomes the i^{th} row of A^T and the j^{th} row of A becomes the j^{th} column of A^T :

$$\text{For } A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}, \quad \text{we define } A^T := \begin{pmatrix} a_{11} & a_{21} & \dots & a_{m1} \\ a_{12} & a_{22} & \dots & a_{m2} \\ \vdots & & & \vdots \\ a_{1n} & a_{2n} & \dots & a_{mn} \end{pmatrix}.$$

Example 3.33. (a)

$$A = \begin{pmatrix} 1 & 2 & 0 & 1 \\ 2 & 0 & 3 & 0 \end{pmatrix} \in \mathbb{R}^{2 \times 4} \quad \Rightarrow \quad A^T = \begin{pmatrix} 1 & 2 \\ 2 & 0 \\ 0 & 3 \\ 1 & 0 \end{pmatrix} \in \mathbb{R}^{4 \times 2}.$$

(b)

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \in \mathbb{R}^{2 \times 2} \quad \Rightarrow \quad A^T = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix} \in \mathbb{R}^{2 \times 2}.$$

(c)

$$A = \begin{pmatrix} 1 & -2 \\ -2 & 1 \end{pmatrix} \in \mathbb{R}^{2 \times 2} \quad \Rightarrow \quad A^T = \begin{pmatrix} 1 & -2 \\ -2 & 1 \end{pmatrix} \in \mathbb{R}^{2 \times 2}.$$

(d)

$$A = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \in \mathbb{R}^{3 \times 1} \quad \Rightarrow \quad A^T = (1 \ 2 \ 3) \in \mathbb{R}^{1 \times 3}.$$

(e)

$$A = (4 \ 5 \ 6 \ 7) \in \mathbb{R}^{1 \times 4} \quad \Rightarrow \quad A^T = \begin{pmatrix} 4 \\ 5 \\ 6 \\ 7 \end{pmatrix} \in \mathbb{R}^{4 \times 1}.$$

Since we have exchanged the roles of rows and columns, the order of multiplication changes, too:

$$(Ax)^T = \mathbf{x}^T A^T \quad \mathbf{x}^T A = (A^T \mathbf{x})^T.$$

Just as with matrix-vector multiplication, transposition reverses the order of matrix-matrix multiplication:

$$(AB)^T = B^T A^T.$$

In particular, if A is invertible, then

$$\mathbb{1} = \mathbb{1}^T = (A^{-1}A)^T = A^T(A^{-1})^T \Rightarrow A^T \text{ is invertible and } (A^T)^{-1} = (A^{-1})^T.$$

Example 3.34. We find

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \cdot \begin{pmatrix} 0 & -2 & 2 \\ 4 & -1 & 2 \\ 1 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 11 & -4 & 9 \\ 26 & -13 & 24 \end{pmatrix}$$

and

$$\begin{pmatrix} 0 & 4 & 1 \\ -2 & -1 & 0 \\ 2 & 2 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix} = \begin{pmatrix} 11 & 26 \\ -4 & -13 \\ 9 & 24 \end{pmatrix}.$$

Proposition 3.35. Some rules for transposition

- (a) For all $A, B \in \mathbb{R}^{m \times n}$ we have: $(A + B)^T = A^T + B^T$.
- (b) For all $A \in \mathbb{R}^{m \times n}$ and for all $\lambda \in \mathbb{R}$, we have: $(\lambda \cdot A)^T = \lambda \cdot A^T$.
- (c) For all $A \in \mathbb{R}^{m \times n}$ we have: $(A^T)^T = A$.
- (d) For all $A \in \mathbb{R}^{m \times n}$ and for all $B \in \mathbb{R}^{n \times r}$ we have: $(A \cdot B)^T = B^T \cdot A^T$.
- (e) If $A \in \mathbb{R}^{n \times n}$ is invertible, then A^T is also invertible and we get $(A^T)^{-1} = (A^{-1})^T$.
- (f) For all $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ we have: $\mathbf{u}^T \cdot \mathbf{v} = \mathbf{v}^T \cdot \mathbf{u} = \langle \mathbf{u}, \mathbf{v} \rangle$.

Proof. If we denote the entries of a matrix A by A_{ij} . Then we have

$$(A^T)_{ij} = A_{ji} \quad \text{for all } i, j$$

and from this one can prove all properties. For example for showing (d), we see

$$(B^T \cdot A^T)_{ij} = \sum_k (B^T)_{ik} (A^T)_{kj} = \sum_k A_{jk} B_{ki} = (A \cdot B)_{ji} = ((A \cdot B)^T)_{ij}$$

for all i, j . □

Proposition 3.36. What has A^T to do with the inner product?

For $\mathbf{x} \in \mathbb{R}^n$, $\mathbf{y} \in \mathbb{R}^m$ and $A \in \mathbb{R}^{m \times n}$, we have for the standard inner product:

$$\langle A\mathbf{x}, \mathbf{y} \rangle = \langle \mathbf{x}, A^T \mathbf{y} \rangle.$$

Proof. We already know that for all $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$, we have $\langle \mathbf{u}, \mathbf{v} \rangle = \mathbf{u}^T \mathbf{v}$. Hence, we conclude that for $\mathbf{x} \in \mathbb{R}^n$, $\mathbf{y} \in \mathbb{R}^m$ and $A \in \mathbb{R}^{m \times n}$, the following holds $\langle A\mathbf{x}, \mathbf{y} \rangle = (A\mathbf{x})^T \mathbf{y} = \mathbf{x}^T A^T \mathbf{y} = \langle \mathbf{x}, A^T \mathbf{y} \rangle$. \square

Moreover, A^T is the only matrix in $B \in \mathbb{R}^{n \times m}$ that satisfies the equation $\langle A\mathbf{x}, \mathbf{y} \rangle = \langle \mathbf{x}, B\mathbf{y} \rangle$ for all $\mathbf{x} \in \mathbb{R}^n$ and $\mathbf{y} \in \mathbb{R}^m$. Therefore, some people use this as the definition for A^T .

Definition 3.37. Symmetric and skew-symmetric matrices

One typical notation for quadratic matrices:

- If $A^T = A$, then A is called symmetric.
- If $A^T = -A$, then A is called skew-symmetric.

Example 3.38. (a)

$$A = \begin{pmatrix} 1 & 3 & -4 \\ 3 & 0 & 5 \\ -4 & 5 & 3 \end{pmatrix} \quad \text{is symmetric since} \quad A^T = \begin{pmatrix} 1 & 3 & -4 \\ 3 & 0 & 5 \\ -4 & 5 & 3 \end{pmatrix} = A.$$

(b)

$$A = \begin{pmatrix} 0 & 3 & 4 \\ -3 & 0 & -5 \\ -4 & 5 & 0 \end{pmatrix} \quad \text{is skew-symmetric since} \quad A^T = \begin{pmatrix} 0 & -3 & -4 \\ 3 & 0 & 5 \\ 4 & -5 & 0 \end{pmatrix} = -A.$$

By definition, all skew-symmetric matrices have only zeros on the diagonal.

3.10 The kernel, range and rank of a matrix

Definition 3.39. Range and kernel of matrices

Let $A \in \mathbb{R}^{m \times n}$. The set

$$\text{Ran}(A) := \{A\mathbf{x} : \mathbf{x} \in \mathbb{R}^n\} \subset \mathbb{R}^m$$

is called the range or image of the matrix A .

The set

$$\text{Ker}(A) := \{\mathbf{x} \in \mathbb{R}^n : A\mathbf{x} = \mathbf{0}\} \subset \mathbb{R}^n$$

is called the kernel or nullspace of the matrix A .

Note that the range of A coincides with the range of the corresponding map f_A and that the kernel of A corresponds to the fiber of f_A for the origin \mathbf{o} . In other words, $\text{Ran}(A) = \text{Ran}(f_A)$ and $\text{Ker}(A) = f_A^{-1}(\{\mathbf{o}\})$.

In our previous study of matrices, we have already found out quite a lot of things about matrices:

- $\text{Ran}(A) = \text{Span}(\mathbf{a}_1, \dots, \mathbf{a}_n)$ where the vectors \mathbf{a}_i are the columns of A .

- $\text{Ker}(A)$ is a subspace of \mathbb{R}^n
- For a matrix A the linear mapping f_A is injective if and only if $\text{Ker}(A) = \{\mathbf{o}\}$ and surjective if and only if $\text{Ran}(A) = \mathbb{R}^m$.

Since our ultimate goal is to understand linear systems given the form:

$$A\mathbf{x} = \mathbf{b},$$

we would like to know more about $\text{Ran}(A)$ (because it tell us, for which \mathbf{b} our system has a solution) and $\text{Ker}(A)$ (because it tells us about the uniqueness of solutions).

Definition 3.40. Rank of a matrix

Let $A \in \mathbb{R}^{m \times n}$. The number

$$\text{rank}(A) := \dim(\text{Ran}(A)) = \dim(\text{Span}(\mathbf{a}_1, \dots, \mathbf{a}_n)).$$

is called the rank of the matrix A .

We obviously have:

$$\text{rank}(A) \leq \min\{m, n\}$$

A is said to have full rank, if $\text{rank}(A) = \min\{m, n\}$.

Let A be some matrix with, say m columns. Let us assume that somebody gives us $r = \text{rank}(A)$. This means that A has r linearly independent columns, and these columns are a basis for $\text{Ran}(A)$. Let us again assume that we know these columns, and we have already reordered them, so that they are the first r columns of A :

$$A = (\underbrace{B}_{r} \mid \underbrace{F}_{m-r})$$

Later, when we discuss the Gauß algorithm, we will find a way to identify these columns.

With this information, we would like to compute $\text{Ker}(A)$, i.e. all \mathbf{x} , such that $A\mathbf{x} = \mathbf{o}$. It is also of interest to obtain its dimension $\dim(\text{Ker}(A))$.

Theorem 3.41. Rank-nullity theorem

Consider a matrix A with n columns. Then

$$\dim(\text{Ker}(A)) + \dim(\text{Ran}(A)) = n.$$

Proof. Choose a basis $(\mathbf{b}_1, \dots, \mathbf{b}_k)$ of $\text{Ker}(A)$. Then choose $r := n - k$ vectors $\mathbf{c}_1, \dots, \mathbf{c}_r$ such that $(\mathbf{b}_1, \dots, \mathbf{b}_k, \mathbf{c}_1, \dots, \mathbf{c}_r)$ is a basis of \mathbb{R}^n . This is possible by Steinitz's theorem in Theorem 3.24. Then we have:

$$\text{Ran}(A) = \text{Span}(A\mathbf{c}_1, \dots, A\mathbf{c}_r).$$

This means that $(A\mathbf{c}_1, \dots, A\mathbf{c}_r)$ would be a basis of $\text{Ran}(A)$ if this family is linearly independent. Hence, assume that we have

$$\lambda_1 A\mathbf{c}_1 + \dots + \lambda_r A\mathbf{c}_r = \mathbf{o} \quad \text{for some } \lambda_i \in \mathbb{R}.$$

From this, we conclude $A(\sum_{i=1}^r \lambda_i \mathbf{c}_i) = \mathbf{o}$ which means that $\sum_{i=1}^r \lambda_i \mathbf{c}_i \in \text{Ker}(A)$. Since $(\mathbf{b}_1, \dots, \mathbf{b}_k, \mathbf{c}_1, \dots, \mathbf{c}_r)$ is linearly independent, only $\lambda_1 = \dots = \lambda_r = 0$ is possible. Therefore $\dim(\text{Ran}(A)) = r$ which proves the claim. \square

3.11 Solving systems of linear equations

Now we are coming to linear equations. For example, we want to solve the system:

$$2x_1 + 3x_2 + 4x_3 = 1$$

$$4x_1 + 6x_2 + 9x_3 = 1$$

$$2x_1 + 4x_2 + 6x_3 = 1$$

Seeing this as the matrix vector form, we can write $A\mathbf{x} = \mathbf{b}$. Later, we will also put in the right-hand side and get the [augmented matrix](#) ($A|\mathbf{b}$) that simply is

$$\left(\begin{array}{ccc|c} 2 & 3 & 4 & 1 \\ 4 & 6 & 9 & 1 \\ 2 & 4 & 6 & 1 \end{array} \right).$$

There are a lot of viewpoints for the method of solving this system. We start with the most important one.

3.11.1 Row operations and the Gauß algorithm

If you come across simple linear systems, for example, just with two unknowns, you may be tempted just randomly combining the equations to find solution. For example:

Example 3.42.

A typical 2×2 LES might be look like this

$$\begin{array}{l} E_1: \quad x_1 + 3x_2 = 7 \\ E_2: \quad 2x_1 - x_2 = 0. \end{array}$$

From equation E_2 we can conclude $x_2 = 2x_1$. Putting this equation for x_2 into the first equation E_1 , we immediately get:

$$7 \stackrel{E_1}{=} x_1 + 3x_2 = x_1 + 3 \cdot 2x_1 = 7x_1$$

and therefore $x_1 = 1$. Hence, the system has exactly one solution given by the vector $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$.

My advice here: *Please, do not do that.* It works for 2×2 LES but you will be in big trouble while solving larger systems, most of the time. Therefore, we will now learn a systematic solving recipe, called the [Gauß algorithm](#) or often just called [Gaussian elimination](#).

One idea of Gaussian elimination can be shown with the example above. One does row operations to get:

$$\begin{array}{rcl} E_1: \quad \boxed{x_1 + 3x_2 = 7} & \rightsquigarrow & E'_1 := E_1: \quad \boxed{x_1 + 3x_2 = 7} \\ E_2: \quad \boxed{2x_1 - x_2 = 0} & \rightsquigarrow & E'_2 := E_2 - 2 \cdot E_1: \quad \boxed{ - 7x_2 = -14.} \end{array}$$

The LES on the right-hand side can easily be solved.

The next idea is, as always, to rewrite the system in matrix form $A\mathbf{x} = \mathbf{b}$. Operations on the system, can now be realised by using an invertible matrix M . This reformulates the

LES:

$$A\mathbf{x} = \mathbf{b} \Leftrightarrow MA\mathbf{x} = M\mathbf{b}$$

If the product MA has a simpler form than A , this helps to solve our system.

Linear systems can be solved by building linear combinations of rows. This means multiplication of invertible matrices on A and \mathbf{b} from the left.

Recall that for each vector $\mathbf{c} \in \mathbb{R}^m$ the product $\mathbf{c}^T A$ builds linear combinations of rows of A which are always denoted by $\boldsymbol{\alpha}_1^T, \dots, \boldsymbol{\alpha}_m^T$. As a reminder:

$$A = \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \dots & a_{mn} \end{pmatrix} = \begin{pmatrix} -\boldsymbol{\alpha}_1^T- \\ \vdots \\ -\boldsymbol{\alpha}_m^T- \end{pmatrix}, \text{ where } \boldsymbol{\alpha}_i^T = (a_{i1} \ \dots \ a_{in}).$$

Now, we have the following operations, for example:

$$\begin{aligned} \mathbf{c}^T = (0 \ \dots \ c_i \ \dots \ 0) \text{ gives: } \mathbf{c}^T A = c_i \boldsymbol{\alpha}_i^T \\ \mathbf{c}^T = (0 \ \dots \ c_i \ 0 \ \dots \ 0 \ c_j \dots \ 0) \text{ gives: } \mathbf{c}^T A = c_i \boldsymbol{\alpha}_i^T + c_j \boldsymbol{\alpha}_j^T \end{aligned}$$

If we put such a vector \mathbf{c}^T as the i^{th} row into a matrix M , then the i^{th} row of MA will contain this result. This gives us the row operations for the example above.

Similar things can be done, of course, with the columns of A by right multiplication of columns. However, row operations are more important right now.

Example 3.43. Adding multiples of rows

In the following 3×3 example, λ times the first row is added to the second:

$$Z_{2+\lambda 1} A = \begin{pmatrix} 1 & 0 & \\ \lambda & 1 & \\ & & 1 \end{pmatrix} \begin{pmatrix} -\boldsymbol{\alpha}_1^T- \\ -\boldsymbol{\alpha}_2^T- \\ -\boldsymbol{\alpha}_3^T- \end{pmatrix} = \begin{pmatrix} -\boldsymbol{\alpha}_1^T- \\ \boldsymbol{\alpha}_2^T + \lambda \boldsymbol{\alpha}_1^T \\ -\boldsymbol{\alpha}_3^T- \end{pmatrix}.$$

The next example shows how the first row is added λ -times to the last:

$$Z_{3+\lambda 1} A = \begin{pmatrix} 1 & & \\ & 1 & \\ \lambda & & 1 \end{pmatrix} \begin{pmatrix} -\boldsymbol{\alpha}_1^T- \\ -\boldsymbol{\alpha}_2^T- \\ -\boldsymbol{\alpha}_3^T- \end{pmatrix} = \begin{pmatrix} -\boldsymbol{\alpha}_1^T- \\ -\boldsymbol{\alpha}_2^T- \\ \boldsymbol{\alpha}_3^T + \lambda \boldsymbol{\alpha}_1^T \end{pmatrix}.$$

Undoing this last operation means using subtraction of such a row again. Thus, for the last example, we have

$$Z_{3-\lambda 1} Z_{3+\lambda 1} A = \begin{pmatrix} 1 & & \\ & 1 & \\ -\lambda & & 1 \end{pmatrix} \begin{pmatrix} -\boldsymbol{\alpha}_1^T- \\ -\boldsymbol{\alpha}_2^T- \\ \boldsymbol{\alpha}_3^T + \lambda \boldsymbol{\alpha}_1^T \end{pmatrix} = \begin{pmatrix} -\boldsymbol{\alpha}_1^T- \\ -\boldsymbol{\alpha}_2^T- \\ \boldsymbol{\alpha}_3^T + \lambda \boldsymbol{\alpha}_1^T - \lambda \boldsymbol{\alpha}_1^T \end{pmatrix} = A.$$

In other words: $Z_{3+\lambda 1}^{-1} = Z_{3-\lambda 1}$.

In general, we can define $Z_{j+\lambda i}$ that adds the i^{th} row of A to the j^{th} row, where $i \neq j$. By the 3×3 examples, it is easy to see how one has to define this matrix in the $m \times m$ -case.

This is always an invertible matrix since the inverse of $Z_{j+\lambda i}$ is always the matrix $Z_{j-\lambda i}$ since this undoes the row addition. One gets:

$$Z_{j+\lambda i} Z_{j-\lambda i} = \mathbb{1}_m. \quad (3.7)$$

Instead of adding rows, we could also exchange rows: We replace the i^{th} row of A by its j^{th} row, and vice versa.

Example 3.44. Exchanging rows

In the following 3×3 example, the first and the second row are exchanged:

$$P_{1 \leftrightarrow 2} A = \begin{pmatrix} 0 & 1 & \\ 1 & 0 & \\ & & 1 \end{pmatrix} \begin{pmatrix} -\alpha_1^T & - \\ -\alpha_2^T & - \\ -\alpha_3^T & - \end{pmatrix} = \begin{pmatrix} -\alpha_2^T & - \\ -\alpha_1^T & - \\ -\alpha_3^T & - \end{pmatrix}.$$

The next example shows how to exchange the first and the last row:

$$P_{1 \leftrightarrow 3} A = \begin{pmatrix} 0 & & 1 \\ & 1 & \\ 1 & & 0 \end{pmatrix} \begin{pmatrix} -\alpha_1^T & - \\ -\alpha_2^T & - \\ -\alpha_3^T & - \end{pmatrix} = \begin{pmatrix} -\alpha_3^T & - \\ -\alpha_2^T & - \\ -\alpha_1^T & - \end{pmatrix}.$$

The inverse of $P_{i \leftrightarrow j}$ is $P_{i \leftrightarrow j}$ again:

$$P_{i \leftrightarrow j}^{-1} = P_{i \leftrightarrow j} \quad (3.8)$$

because exchanging of rows is undone by exchanging them again! Such an action is a special case of a *permutation* and $P_{i \leftrightarrow j}$ is a permutation matrix. Permutations of columns are also possible, just multiply $P_{i \leftrightarrow j}$ from the right: $AP_{i \leftrightarrow j}$.

Next one is scaling a row:

Example 3.45. Scaling rows

Each row is multiplied by a scalar value:

$$DA = \begin{pmatrix} d_1 & & \\ & \ddots & \\ & & d_m \end{pmatrix} \begin{pmatrix} -\alpha_1^T & - \\ \vdots & \\ -\alpha_n^T & - \end{pmatrix} = \begin{pmatrix} -d_1 \alpha_1^T & - \\ \vdots & \\ -d_m \alpha_m^T & - \end{pmatrix}.$$

D is called a diagonal matrix. D is invertible if *no diagonal entry is 0*. Then the inverse of D is a scaling matrix with reverse entries:

$$\begin{pmatrix} d_1 & & \\ & \ddots & \\ & & d_m \end{pmatrix} \Rightarrow D^{-1} = \begin{pmatrix} 1/d_1 & & \\ & \ddots & \\ & & 1/d_m \end{pmatrix}.$$

Definition 3.46. Row operations

For a given matrix $A \in \mathbb{R}^{m \times n}$, we call the multiplication with the invertible matrices $Z_{i+\lambda j}, P_{i \leftrightarrow j}, D \in \mathbb{R}^{m \times m}$ from the left simply row operations.

Proposition 3.47. Row operations do not change the kernel

Let $A \in \mathbb{R}^{m \times n}$ and $M \in \mathbb{R}^{m \times m}$ a matrix consisting of finitely many row operations (for example $M = Z_{2+\frac{1}{4}1}Z_{3+\frac{1}{2}1}P_{2 \leftrightarrow 3}$). Then we have:

$$\text{Ker}(MA) = \text{Ker}(A) \text{ and } \text{Ran}(MA) = M\text{Ran}(A).$$

Proof. For the kernel:

$$MAx = \mathbf{0} \Leftrightarrow Ax = M^{-1}\mathbf{0} \Leftrightarrow Ax = \mathbf{0}.$$

The range formula directly follows from the definition. \square

Note that the range may change a lot by row operations. What happens with the kernel and the range if you just do column operations?

3.11.2 Set of solutions

Solving a linear system $Ax = b$ (with $A \in \mathbb{R}^{m \times n}$) means:

- finding out, if a solution exists, i.e. $b \in \text{Ran}(A)$
- if yes, computation of *all* solutions of this system

We usually denote the set of all solution by $\mathcal{S} = \{\mathbf{x} \in \mathbb{R}^n : Ax = b\}$.

Proposition 3.48. The set of solutions is an affine subspace

Let $Ax = b$ be a LES with $A \in \mathbb{R}^{m \times n}$. Then for the set of solutions we have either $\mathcal{S} = \emptyset$ or

$$\mathcal{S} = \mathbf{v}_0 + \text{Ker}(A) = \{\mathbf{v}_0 + \mathbf{x}_0 : \mathbf{x}_0 \in \text{Ker}(A)\}$$

for a $\mathbf{v}_0 \in \mathbb{R}^n$.

Proof. Let $\mathbf{v}_0 \in \mathcal{S}$ be a solution $A\mathbf{v}_0 = \mathbf{b}$. By linearity of A we know:

$$\begin{aligned} \mathbf{x} = \mathbf{v}_0 + \mathbf{x}_0 \in \mathcal{S} &\iff \mathbf{b} = Ax = A(\mathbf{v}_0 + \mathbf{x}_0) = A\mathbf{v}_0 + Ax_0 = \mathbf{b} + Ax_0 \\ &\iff Ax_0 = \mathbf{0} \iff \mathbf{x}_0 \in \text{Ker}(A). \end{aligned}$$

If there is no such \mathbf{v}_0 , then $\mathcal{S} = \emptyset$. \square

In combination with Proposition 3.47, we get the most important fact:

Corollary 3.49.

Row operations do not change the set of solutions.

We are looking for a method that allows:

- to decide, if $\mathbf{b} \in \text{Ran}(A)$, and read off $\text{rank}(A) = r$.
- to compute a particular solution $A\mathbf{v}_0 = \mathbf{b}$
- to compute a basis \mathcal{F} of $\text{Ker}(A)$, so that \mathcal{S} is completely known.

Since *triangular systems* are easy and straightforward to solve, we would like to bring A into triangular form, or something similar.

3.11.3 Gaussian elimination (without pivoting)

Goal

For solving $A\mathbf{x} = \mathbf{b}$, use row operations M to bring A into upper triangular form, which is matrix that has zeros below the diagonal, or into row echelon form, which we will define later. Then, one can construct the solution set for $MA\mathbf{x} = M\mathbf{b}$.

Since we use the same row operations on A and \mathbf{b} it is useful to use *augmented matrix* $(A|\mathbf{b})$. In the end, we obtain $(MA|M\mathbf{b})$.

Example 3.50. Let us solve the system:

$$\begin{aligned} 2x_1 + 3x_2 - 1x_3 &= 4 \\ 2x_1 - x_2 + 7x_3 &= 0 \\ 6x_1 + 13x_2 - 4x_3 &= 9 \end{aligned}$$

Then we do the following steps:

$$\begin{array}{l} E_1: \left(\begin{array}{ccc|c} 2 & 3 & -1 & 4 \\ 2 & -1 & 7 & 0 \\ 6 & 13 & -4 & 9 \end{array} \right) \xrightarrow{-1 \cdot E_1} \left(\begin{array}{ccc|c} 2 & 3 & -1 & 4 \\ 0 & -4 & 8 & -4 \\ 0 & 4 & -1 & -3 \end{array} \right) \text{ (mark)} \xrightarrow{+1 \cdot E_2'} \left(\begin{array}{ccc|c} 2 & 3 & -1 & 4 \\ 0 & -4 & 8 & -4 \\ 0 & 0 & 7 & -7 \end{array} \right) \\ E_2': \left(\begin{array}{ccc|c} 2 & 3 & -1 & 4 \\ 0 & -4 & 8 & -4 \\ 0 & 4 & -1 & -3 \end{array} \right) \xrightarrow{E_3': \quad 0 \quad | \quad 4} \left(\begin{array}{ccc|c} 2 & 3 & -1 & 4 \\ 0 & -4 & 8 & -4 \\ 0 & 0 & 7 & -7 \end{array} \right) \end{array}$$

We get from E_3'' immediately $x_3 = \frac{-7}{7} = -1$. Putting this in E_2'' , we also get $-4x_2 + 8 \cdot (-1) = -4$ and, hence, $x_2 = \frac{-4+8}{-4} = -1$. The last step is then, using $x_2 = -1$ and $x_3 = -1$ in E_1'' , to get $2x_1 + 3 \cdot (-1) - (-1) = 4$, which means $x_1 = \frac{4+3-1}{2} = 3$. The unique solution is then

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 3 \\ -1 \\ -1 \end{pmatrix}.$$

Attention!



Do not use equation E_1' anymore at this point!
Otherwise, you would bring the variable x_1 back in the game.

Gauß with a bug

We start with a square matrix $A \in \mathbb{R}^{n \times n}$. Let us write $\tilde{A} := (A|\mathbf{b})$ as a row matrix:

$$\tilde{A} = (A|\mathbf{b}) = \left(\begin{array}{cccc|c} a_{11} & a_{12} & \dots & a_{1n} & b_1 \\ a_{21} & a_{22} & \dots & a_{2n} & b_2 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} & b_n \end{array} \right) = \begin{pmatrix} \text{--- } \tilde{\alpha}_1^T \text{ ---} \\ \text{--- } \tilde{\alpha}_2^T \text{ ---} \\ \vdots \\ \text{--- } \tilde{\alpha}_n^T \text{ ---} \end{pmatrix}$$

So a_{ij} is the j^{th} entry of $\tilde{\alpha}_i^T$.

We can eliminate a_{21} by adding rows: $\tilde{\alpha}_2^T \sim \tilde{\alpha}_2^T - \lambda_2 \tilde{\alpha}_1^T$, where $\lambda_2 = \frac{a_{21}}{a_{11}}$.

This can be written in terms of a matrix, and we obtain:

$$Z_{2-\lambda_2 1}(A|\mathbf{b}) = \left(\begin{array}{cccc|c} a_{11} & a_{12} & \dots & a_{1n} & b_1 \\ 0 & \tilde{a}_{22} & \dots & \tilde{a}_{2n} & \tilde{b}_2 \\ a_{31} & a_{32} & \dots & a_{3n} & b_3 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} & b_n \end{array} \right) = \left(\begin{array}{c} \tilde{\alpha}_1^T \\ \tilde{\alpha}_2^T - \lambda_2 \tilde{\alpha}_1^T \\ \tilde{\alpha}_3^T \\ \vdots \\ \tilde{\alpha}_n^T \end{array} \right)$$

Now we can do the same with all other rows, defining $\lambda_i = \frac{a_{i1}}{a_{11}}$, and computing:

$$\underbrace{Z_{n-\lambda_n 1} \dots Z_{2-\lambda_2 1}}_{L_1^{-1}}(A|\mathbf{b}) = \left(\begin{array}{cccc|c} a_{11} & a_{12} & \dots & a_{1n} & b_1 \\ 0 & \tilde{a}_{22} & \dots & \tilde{a}_{2n} & \tilde{b}_2 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & \tilde{a}_{n2} & \dots & \tilde{a}_{nn} & \tilde{b}_n \end{array} \right) = \left(\begin{array}{c} \tilde{\alpha}_1^T \\ \tilde{\alpha}_2^T - \lambda_2 \tilde{\alpha}_1^T \\ \vdots \\ \tilde{\alpha}_n^T - \lambda_n \tilde{\alpha}_1^T \end{array} \right) = L_1^{-1}(A|\mathbf{b})$$

Since L_1^{-1} subtracts the first row from all others, its inverse is easily seen to be the matrix that adds this row to all others:

$$L_1^{-1} = \begin{pmatrix} 1 & & & \\ -\lambda_2 & 1 & & \\ \vdots & & \ddots & \\ -\lambda_n & & & 1 \end{pmatrix}, \quad L_1 = \begin{pmatrix} 1 & & & \\ \lambda_2 & 1 & & \\ \vdots & & \ddots & \\ \lambda_n & & & 1 \end{pmatrix}$$

Once, we have eliminated the entries $a_{21} \dots a_{n1}$. We can do the same with $\tilde{a}_{32} \dots \tilde{a}_{n2}$. Here we use factors $\tilde{\lambda}_i = \frac{\tilde{a}_{i2}}{\tilde{a}_{22}}$

We then obtain:

$$L_2^{-1} L_1^{-1}(A|\mathbf{b}) = \left(\begin{array}{ccccc|c} a_{11} & a_{12} & a_{13} & \dots & a_{1n} & b_1 \\ 0 & \tilde{a}_{22} & \tilde{a}_{23} & \dots & \tilde{a}_{2n} & \tilde{b}_2 \\ 0 & 0 & \hat{a}_{33} & \dots & \hat{a}_{3n} & \hat{b}_3 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \hat{a}_{n3} & \dots & \hat{a}_{nn} & \hat{b}_n \end{array} \right)$$

It (luckily) turns out, that

$$L_1 L_2 = \begin{pmatrix} 1 & & & \\ \lambda_2 & 1 & & \\ \lambda_3 & \tilde{\lambda}_3 & 1 & \\ \vdots & \vdots & & \ddots \\ \lambda_n & \tilde{\lambda}_n & & 1 \end{pmatrix}$$

If we do this column by column, (and don't run out of hats) we obtain:

$$\underbrace{L_{n-1}^{-1} \dots L_1^{-1}}_{L^{-1}}(A|\mathbf{b}) = \left(\begin{array}{ccc|c} u_{11} & \dots & u_{1n} & c_1 \\ \ddots & \ddots & \vdots & \vdots \\ & u_{nn} & c_n \end{array} \right) = (U|\mathbf{c}).$$

where L is unit lower triangular:

$$L = \begin{pmatrix} 1 & & & \\ l_{21} & 1 & & \\ \vdots & \ddots & \ddots & \\ l_{n1} & \dots & l_{n(n-1)} & 1 \end{pmatrix}$$

LU-decomposition $A = LU$

We thus have $L^{-1}A = U$, and $L^{-1}\mathbf{b} = \mathbf{c}$. Multiplication by L yields the famous **LU-decomposition**:

$$A = LU, \quad \mathbf{b} = L\mathbf{c}.$$

Here L is lower triangular, and U is upper triangular. Once, we have decomposed A , we can, for given \mathbf{b} compute \mathbf{x} as follows:

solve $L\mathbf{c} = \mathbf{b}$ "forward substitution"
solve $U\mathbf{x} = \mathbf{c}$ "backward substitution"

Then $A\mathbf{x} = LU\mathbf{x} = L\mathbf{c} = \mathbf{b}$.

- Since L and U are both triangular, the above solves can be performed easily.
- Since our factorisation is done once and for all, further problems with the same matrix but different \mathbf{b} can be solved later.
- Another point of view on L is the following: we keep track of what is done during the transformation of the right hand side $\mathbf{b} \rightarrow \mathbf{c}$.

$$c_i \rightsquigarrow c_i - \lambda c_j \Leftrightarrow l_{ij} = \lambda$$

If later our system has to be solved for another right hand side \mathbf{b} , then we can use the subdiagonal entries of L to do the same computations to this new \mathbf{b} , as was done to the old one. This can be nicely written as $\mathbf{c} = L^{-1}\mathbf{b}$.

The Gauß algorithm can be performed by hand, or implemented on a computer. The following pseudo-code describes it in detail:

Here U and \mathbf{c} are overwritten, so we do not distinguish u_{ij} and \tilde{u}_{ij} , and so on.

Gaussian elimination without pivoting ($A \in \mathbb{R}^{n \times n}$)

```
(U|c) = (A|b), L = 1_n
for j = 1 ... n           (loop over columns)
    for i = j + 1 ... n   (loop over rows)
        l_ij = u_ij / u_jj
        u_ij = 0            (eliminate entry)
        for s = j + 1 ... n
            u_is = u_is - l_ij * u_js      (subtract remaining entries)
            c_i = c_i - l_ij * c_j         (subtract rhs)
```

- We recognise three nested loops, and thus, the cost of this algorithm is proportional to n^3 .

- After that, we have to perform backward substitution to compute \mathbf{x} from \mathbf{c} .
- If \mathbf{b} is only known after the decomposition, we can compute \mathbf{c} by forward substitution.
- In computer libraries A is overwritten by L and U . The upper triangular part is used to store U , the lower triangular part is used to store L :

$$\begin{matrix} u_{11} & \dots & \dots & u_{1n} \\ l_{21} & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ l_{n1} & \dots & l_{n(n-1)} & u_{nn} \end{matrix}$$

This is called in place factorisation. It is possible, since we know anyway that U is zero below the diagonal, L is zero above the diagonal, and $l_{ii} = 1$. We may call this storage matrix $L\backslash U$.

Example 3.51. LU-factorisation:

We do the row operation and save them in the matrix L :

$$A = \begin{pmatrix} 2 & 1 & 1 \\ 4 & 4 & 6 \\ -2 & 3 & 9 \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}}_{=:L} \underbrace{\begin{pmatrix} 2 & 1 & 1 \\ 0 & 2 & 4 \\ 0 & 4 & 10 \end{pmatrix}}_{=:U} = \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 2 & 1 \end{pmatrix}}_{=:L} \underbrace{\begin{pmatrix} 2 & 1 & 1 \\ 0 & 2 & 4 \\ 0 & 0 & 2 \end{pmatrix}}_{=:U}$$

Or if one uses the storage-saving notation:

$$A = \begin{pmatrix} 2 & 1 & 1 \\ 4 & 4 & 6 \\ -2 & 3 & 9 \end{pmatrix} \rightsquigarrow \left(\begin{array}{c|cc} 2 & 1 & 1 \\ \hline 2 & 2 & 4 \\ -1 & 4 & 10 \end{array} \right) \rightsquigarrow L\backslash U = \left(\begin{array}{c|cc} 2 & 1 & 1 \\ \hline 2 & 2 & 4 \\ -1 & 2 & 2 \end{array} \right),$$

For a right-hand side \mathbf{b} , one simply does the same row calculation steps:

$$\mathbf{b} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 \\ -1 \\ 4 \end{pmatrix} =: \mathbf{c}$$

Alternatively: $\mathbf{c} := L^{-1}\mathbf{b}$. For finding then the (unique) solution of $A\mathbf{x} = \mathbf{b}$, just compute $\mathbf{x} = U^{-1}\mathbf{c}$ as usual.

Video: LU decomposition - An Example



LU decomposition

<https://jp-g.de/bsom/la/lu/>

$$A = LU =$$

$$\begin{pmatrix} 1 & & & \\ -2 & 1 & & \\ 3 & -4 & 1 & \\ 2 & 1 & 3 & 1 \end{pmatrix} \begin{pmatrix} 2 & 4 & 3 & 5 \\ & 1 & 1 & 2 \\ & & -3 & 2 \\ & & & -4 \end{pmatrix}$$



The bug:

- What, if u_{jj} is zero at some stage of the computation? Then we have division by 0.
- Also, on the computer, if u_{jj} is very small, say 10^{-14} , then due to round-off error, problems may also occur.

Remark: Gaussian elimination or LU-decomposition?

For solving a system $Ax = b$ you have now two options:

- (a) Gaussian elimination of $(A|b)$ without memorising the row operations.
- (b) LU-decomposition of A with memorising the row operations in the matrix L .

If you are just interested in the solution(s) of a given LES, then you will just do the Gaussian elimination step by step until you reach the upper triangle form (or the row echelon form, see next section).

Example 3.52. Let us a look at a higher dimensional and non-square example:

$$\begin{array}{l} E_1: \quad x_1 + 2x_2 + x_4 = 3 \\ E_2: \quad 4x_1 + 8x_2 + 3x_4 + 4x_5 = 14 \\ E_3: \quad \quad \quad 2x_3 + 3x_4 + 12x_5 = 10 \\ E_4: \quad -3x_1 - 6x_2 - 6x_3 + 8x_4 + 4x_5 = 4 \end{array} \quad (3.9)$$

You should immediately rewrite this in an augmented matrix form:

$$(A|b) = \left(\begin{array}{ccccc|c} 1 & 2 & 0 & 1 & 0 & 3 \\ 4 & 8 & 2 & 3 & 4 & 14 \\ 0 & 0 & 2 & 3 & 12 & 10 \\ -3 & -6 & -6 & 8 & 4 & 4 \end{array} \right)$$

The entry in grey **1** is first one we have to consider. All entries below should get zero after the first elimination.

- multiply $\frac{4}{1} = 4$ to E_1 and subtract the result from E_2 ,
- multiply $\frac{-3}{1} = (-3)$ to E_1 and subtract the result from E_4 .

$$\begin{array}{l} E_1: \left(\begin{array}{ccccc|c} 1 & 2 & 0 & 1 & 0 & 3 \end{array} \right) \\ E_2: \left(\begin{array}{ccccc|c} 1 & 2 & 0 & 1 & 0 & 3 \end{array} \right) - 4 \cdot E_1 \\ E_3: \left(\begin{array}{ccccc|c} 1 & 2 & 0 & 1 & 0 & 3 \end{array} \right) \\ E_4: \left(\begin{array}{ccccc|c} 1 & 2 & 0 & 1 & 0 & 3 \end{array} \right) + 3 \cdot E_1 \end{array} \rightsquigarrow \begin{array}{l} E'_1: \left(\begin{array}{ccccc|c} 1 & 2 & 0 & 1 & 0 & 3 \end{array} \right) \\ E'_2: \left(\begin{array}{ccccc|c} 0 & 0 & 2 & -1 & 4 & 2 \end{array} \right) \\ E'_3: \left(\begin{array}{ccccc|c} 0 & 0 & 2 & 3 & 12 & 10 \end{array} \right) \\ E'_4: \left(\begin{array}{ccccc|c} 0 & 0 & -6 & 11 & 4 & 13 \end{array} \right) \end{array}$$

The next number on the diagonal is a zero and it seems like that our algorithm has to stop here. However, since below there are also zeros, the column is already eliminated. We can just ignore the variable x_2 at this point and just restart the algorithm with starting point **2**.

Just subtract the equation E'_2 with the right factor from the other rows: $\frac{2}{2} = 1$ times

from E'_3 and $\frac{-6}{2} = (-3)$ times from E'_4 . We get:

$$\begin{array}{l} E'_1: \left(\begin{array}{ccccc|c} x_1 & x_2 & x_3 & x_4 & x_5 & 3 \\ \hline 1 & 2 & 0 & 1 & 0 & 3 \\ 0 & 0 & 2 & -1 & 4 & 2 \end{array} \right) \xrightarrow{-1 \cdot E'_2} \\ E'_2: \left(\begin{array}{ccccc|c} x_1 & x_2 & x_3 & x_4 & x_5 & 3 \\ \hline 0 & 0 & 2 & -1 & 4 & 2 \\ 0 & 0 & 2 & 3 & 12 & 10 \end{array} \right) \xrightarrow{-1 \cdot E'_2} \\ E'_3: \left(\begin{array}{ccccc|c} x_1 & x_2 & x_3 & x_4 & x_5 & 3 \\ \hline 0 & 0 & 2 & 3 & 12 & 10 \\ 0 & 0 & -6 & 11 & 4 & 13 \end{array} \right) \xrightarrow{+3 \cdot E'_2} \\ E'_4: \left(\begin{array}{ccccc|c} x_1 & x_2 & x_3 & x_4 & x_5 & 3 \\ \hline 0 & 0 & 2 & 3 & 12 & 10 \\ 0 & 0 & -6 & 11 & 4 & 13 \end{array} \right) \end{array} \rightsquigarrow \begin{array}{l} E''_1: \left(\begin{array}{ccccc|c} x_1 & x_2 & x_3 & x_4 & x_5 & 3 \\ \hline 1 & 2 & 0 & 1 & 0 & 3 \\ 0 & 0 & 2 & -1 & 4 & 2 \end{array} \right) \\ E''_2: \left(\begin{array}{ccccc|c} x_1 & x_2 & x_3 & x_4 & x_5 & 3 \\ \hline 0 & 0 & 2 & -1 & 4 & 2 \\ 0 & 0 & 0 & 4 & 8 & 8 \end{array} \right) \\ E''_3: \left(\begin{array}{ccccc|c} x_1 & x_2 & x_3 & x_4 & x_5 & 3 \\ \hline 0 & 0 & 0 & 4 & 8 & 8 \\ 0 & 0 & 0 & 8 & 16 & 19 \end{array} \right) \end{array}$$

Next variable is x_4 . Now we consider 4 . Multiply E''_3 with $\frac{8}{4} = 2$ and subtract from equation E''_4 :

$$\begin{array}{l} E''_1: \left(\begin{array}{ccccc|c} x_1 & x_2 & x_3 & x_4 & x_5 & 3 \\ \hline 1 & 2 & 0 & 1 & 0 & 3 \end{array} \right) \\ E''_2: \left(\begin{array}{ccccc|c} x_1 & x_2 & x_3 & x_4 & x_5 & 3 \\ \hline 0 & 0 & 2 & -1 & 4 & 2 \end{array} \right) \\ E''_3: \left(\begin{array}{ccccc|c} x_1 & x_2 & x_3 & x_4 & x_5 & 3 \\ \hline 0 & 0 & 0 & 4 & 8 & 8 \end{array} \right) \\ E''_4: \left(\begin{array}{ccccc|c} x_1 & x_2 & x_3 & x_4 & x_5 & 3 \\ \hline 0 & 0 & 0 & 8 & 16 & 19 \end{array} \right) \xrightarrow{-2 \cdot E''_3} \\ E'''_1: \left(\begin{array}{ccccc|c} x_1 & x_2 & x_3 & x_4 & x_5 & 3 \\ \hline 1 & 2 & 0 & 1 & 0 & 3 \end{array} \right) \\ E'''_2: \left(\begin{array}{ccccc|c} x_1 & x_2 & x_3 & x_4 & x_5 & 3 \\ \hline 0 & 0 & 2 & -1 & 4 & 2 \end{array} \right) \\ E'''_3: \left(\begin{array}{ccccc|c} x_1 & x_2 & x_3 & x_4 & x_5 & 3 \\ \hline 0 & 0 & 0 & 4 & 8 & 8 \end{array} \right) \\ E'''_4: \left(\begin{array}{ccccc|c} x_1 & x_2 & x_3 & x_4 & x_5 & 3 \\ \hline 0 & 0 & 0 & 0 & 0 & 3 \end{array} \right) \end{array}$$

Now, we cannot use any rows for elimination and we are finished. We get the following result:

$$\begin{array}{l} E''''_1: \left(\begin{array}{ccccc|c} x_1 & x_2 & x_3 & x_4 & x_5 & 3 \\ \hline 1 & 2 & 0 & 1 & 0 & 3 \end{array} \right) \\ E''''_2: \left(\begin{array}{ccccc|c} x_1 & x_2 & x_3 & x_4 & x_5 & 3 \\ \hline 0 & 0 & 2 & -1 & 4 & 2 \end{array} \right) \\ E''''_3: \left(\begin{array}{ccccc|c} x_1 & x_2 & x_3 & x_4 & x_5 & 3 \\ \hline 0 & 0 & 0 & 4 & 8 & 8 \end{array} \right) \\ E''''_4: \left(\begin{array}{ccccc|c} x_1 & x_2 & x_3 & x_4 & x_5 & 3 \\ \hline 0 & 0 & 0 & 0 & 0 & 3 \end{array} \right) \end{array} \quad (3.10)$$

This is not a triangle matrix like in Example 3.42 but an upper triangle matrix by definition since below the diagonal, there are just zeros. This form is called the row echelon form and defined below.

3.11.4 Row echelon form

Definition 3.53. Row echelon form, pivot element

A matrix $A \in \mathbb{R}^{m \times n}$ in the form of the left-hand side of (3.10) is called row echelon form. This means that the matrix A fulfills:

- all zero rows, if any, are at the bottom of the matrix,
- for each row: the first nonzero number from the left is always strictly to the right of the first nonzero coefficient from the row above it.

This leading nonzero number in each row is called the pivot.

In the row echelon form we can put the variable into two groups:

Definition 3.54. Free and leading variables

Variables in the column of a pivot are called leading variables.

The other variables are called free variables.

Example 3.55. Looking at equation (3.10) again, we can distinguish the variables

$$\begin{array}{l}
 G_1'''': \left(\begin{array}{ccccc|c} x_1 & x_2 & x_3 & x_4 & x_5 & \\ \hline 1 & 2 & 0 & 1 & 0 & 3 \end{array} \right) \\
 G_2''''': \left(\begin{array}{ccccc|c} 0 & 0 & 2 & -1 & 4 & 2 \\ \hline 0 & 0 & 0 & 4 & 8 & 8 \end{array} \right) \\
 G_3''''': \left(\begin{array}{ccccc|c} 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right) \\
 G_4''''': \left(\begin{array}{ccccc|c} 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right)
 \end{array} \quad (3.11)$$

In this example x_1 , x_3 and x_4 are the leading variables and x_2 and x_5 are free.

- Free variables can be chosen independently in \mathbb{R} .
- The leading variables are chosen dependently of the free variables.

If you have a LES where the matrix is given in row echelon form, then the solution set is immediately given. You just have to push the free variables to the right and solve the leading variables by backward substitution.

For example, the solution of (3.11) is empty since the last row is not satisfiable. However, we can give another example:

Example 3.56.

The LES

$$\begin{array}{l}
 E_1: \left(\begin{array}{ccccc|c} x_1 & x_2 & x_3 & x_4 & x_5 & \\ \hline 1 & 2 & 0 & 1 & 0 & 3 \end{array} \right) \\
 E_2: \left(\begin{array}{ccccc|c} 0 & 0 & 2 & -1 & 4 & 2 \\ \hline 0 & 0 & 0 & 4 & 8 & 8 \end{array} \right) \\
 E_3: \left(\begin{array}{ccccc|c} 0 & 0 & 0 & 4 & 8 & 8 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right)
 \end{array} \quad (3.12)$$

is already in row echelon form and can be equivalently written as

$$\begin{array}{l}
 E_1: \left(\begin{array}{ccc|c} x_1 & x_3 & x_4 & \\ \hline 1 & 0 & 1 & 3 - 2x_2 \\ 0 & 2 & -1 & 2 - 4x_5 \\ 0 & 0 & 4 & 8 - 8x_5 \end{array} \right) \\
 E_2: \left(\begin{array}{ccc|c} x_1 & x_3 & x_4 & \\ \hline 1 & 0 & 1 & 3 - 2x_2 \\ 0 & 2 & -1 & 2 - 4x_5 \\ 0 & 0 & 4 & 8 - 8x_5 \end{array} \right) \\
 E_3: \left(\begin{array}{ccc|c} x_1 & x_3 & x_4 & \\ \hline 1 & 0 & 1 & 3 - 2x_2 \\ 0 & 2 & -1 & 2 - 4x_5 \\ 0 & 0 & 4 & 8 - 8x_5 \end{array} \right)
 \end{array} \quad (3.13)$$

and backward substitution gives us the solution set:

$$S = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 1 - 2x_2 + 2x_5 \\ x_2 \\ 2 - 3x_5 \\ 2 - 2x_5 \\ x_5 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 2 \\ 2 \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + x_5 \begin{pmatrix} 2 \\ 0 \\ -3 \\ -2 \\ 1 \end{pmatrix} : x_2, x_5 \in \mathbb{R} \right\}$$

Corollary 3.57.

If $A \in \mathbb{R}^{m \times n}$ is a row echelon matrix, then $\text{rank}(A)$ is the number of leading variables and $\dim(\text{Ker}(A))$ is the number of free variables.

Proof. Obviously, the columns with pivots are linearly independent vectors where the columns with free variables are a linear combination of the other ones. \square

In the next section, we will generalise what we did in the example before.

3.11.5 Gaussian elimination with pivoting and $PA = LK$ decomposition

Now we consider the general case of $A\mathbf{x} = \mathbf{b}$ with non-square $A \in \mathbb{R}^{m \times n}$. Here the role of the upper triangular U is played by a matrix K in *row echelon form*, see Definition 3.53.

In such a case, we need to make use of another technique, the exchange of rows of our temporary matrices. This is again a multiplication from the left by an invertible matrix. This is called pivoting.

Pen-and-paper strategy: "non-zero pivoting"

Let j by the current column and assume that we want to eliminate all entries below k_{rj} . (In the standard Gaussian elimination, we always wanted to eliminate all entries below a diagonal element k_{jj} , but in the general case, we have $r \leq j$, as already seen in Example 3.62.)

Initialise the permutations matrix $P_{row} = \mathbb{1}$ to store permutations and start with $K := A$ and $\mathbf{c} := \mathbf{b}$.

- If $k_{rj} = 0$, test for $i = r + 1 \dots m$, if $k_{ij} \neq 0$
- At the first occurrence i_{pivot} , exchange row r and row i_{pivot} of $L \setminus K$, and \mathbf{c} and P_{row} . This means that *only the subdiagonal entries* of L are exchanged, not the diagonal entries (which are 1).
- If all tested k_{ij} are zero, continue with the next column.

Remark: Pivot search

In other words: If the next entry that we want to choose as a pivot is zero, we just search the rest of the column below for a non-zero entry and switch the rows.

Question:

One may wonder, why only the entries of L are permuted, which are below the diagonal, but not on the diagonal. As you recall, L is constructed for the purpose of book-keeping, which row is subtracted from what row, and by which scaling. The unit diagonal of L has nothing to do with this book-keeping.

If we permute the subdiagonal entries of L , we update the book-keeping according to the permutations of the rows of A . It is, as if the rows would have been permuted in the beginning, before the start of the elimination.

Example 3.58. Invertible matrix (with pivoting):

$$\begin{pmatrix} 2 & 3 & 4 \\ 4 & 6 & 9 \\ 2 & 4 & 6 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 3 & 4 \\ 0 & 0 & 1 \\ 0 & 1 & 2 \end{pmatrix} = (P_{2 \leftrightarrow 3})^2 \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} (P_{2 \leftrightarrow 3})^2 \begin{pmatrix} 2 & 3 & 4 \\ 0 & 0 & 1 \\ 0 & 1 & 2 \end{pmatrix}$$

$$= P_{2 \leftrightarrow 3} \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 3 & 4 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}$$

Or if one uses the storage-saving notation:

$$\begin{pmatrix} 2 & 3 & 4 \\ 4 & 6 & 9 \\ 2 & 4 & 6 \end{pmatrix} \rightsquigarrow \left(\begin{array}{c|cc} 2 & 3 & 4 \\ \hline 2 & 0 & 1 \\ 1 & 1 & 2 \end{array} \right) \xrightarrow{(P_{2 \leftrightarrow 3})} \left(\begin{array}{c|cc} 2 & 3 & 4 \\ \hline 1 & 1 & 2 \\ 2 & 0 & 1 \end{array} \right)$$

Example 3.59. Invertible matrix (with hindsight and pivoting)

$$\begin{pmatrix} 2 & 3 & 4 \\ 4 & 6 & 9 \\ 2 & 4 & 6 \end{pmatrix} \xrightarrow{(P_{2 \leftrightarrow 3})} \begin{pmatrix} 2 & 3 & 4 \\ 2 & 4 & 6 \\ 4 & 6 & 9 \end{pmatrix} \rightsquigarrow \left(\begin{array}{c|cc} 2 & 3 & 4 \\ \hline 1 & 1 & 2 \\ 2 & 0 & 1 \end{array} \right)$$

Since we have exchanged the subdiagonal entries of L , all row exchanges, applied during the transformation $\mathbf{b} \rightsquigarrow \mathbf{c}$ can be performed at the beginning. Thus, we do not have to remember when a row exchange took place. We only need the *result* of all row exchanges, and apply it at the beginning.

$$\mathbf{b} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \rightarrow (\text{permute first}) \rightsquigarrow P_{row}\mathbf{b} = \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix} \rightsquigarrow L^{-1}P_{row}\mathbf{b} = \mathbf{c} = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}.$$

This algorithm can also be applied to non-square matrices and leads to K in the so called *row echelon form*. This is why there are now two variables r and j . The first stands for the column, the second for the "head" of the column. $r \leq j$.

Gaussian elimination with pivoting ($A \in \mathbb{R}^{m \times n}$)

```

 $K = A, L = \mathbf{1}_m, \mathbf{c} = \mathbf{b}, r = 1, P_{row} = \mathbf{1}_m$ 
for  $j = 1 \dots n$            (loop over columns)
    perform pivot search for the first non-zero element of  $K$  at or below  $k_{rj}$ 
    if  $i_{pivot}$  was found, exchange row  $r$  and row  $i_{pivot}$  of  $L \setminus K$ ,  $\mathbf{c}$ , and  $P_{row}$ 
        for  $i = r \dots m$       (loop over rows)
             $l_{ir} = \frac{k_{ij}}{k_{rj}}$ 
             $k_{ij} = 0$ 
            for  $s = r + 1 \dots m$ 
                 $k_{is} = k_{is} - l_{ir}k_{rs}$ 
                 $c_r = c_r - l_{ir}c_r$ 
             $r = r + 1$            consider the next row

```

It does, however, not work properly on the computer, because the test $u_{jj} = 0$ is unreliable, in the presence of round-off errors. For toy examples, however, it can be used to find all possible solutions of a non-square linear system.

Example 3.60. non-square matrix (no pivoting needed here)

$$\begin{pmatrix} 1 & 2 & 1 & 2 \\ 1 & 2 & 2 & 3 \\ 2 & 4 & 3 & 5 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 1 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix} \\ = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 1 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

We observe that $\text{rank}(A) = 2$.

Example 3.61. Modified Example: with pivoting

$$\begin{aligned} \begin{pmatrix} 1 & 2 & 1 & 2 \\ 1 & 2 & 1 & 3 \\ 2 & 4 & 3 & 5 \end{pmatrix} &= \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 1 & 2 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix} \\ &= P_{2 \leftrightarrow 3} \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 1 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{aligned}$$

The result of pen-and-paper pivoting

Although this exchange of rows happens during the course of elimination, the outcome of the resulting algorithm for matrices A can be written as

$$P_{row} A = LK,$$

where in P_{row} all the performed permutations, and K is in row echelon form. Hence, for a right hand side \mathbf{b} we may solve $A\mathbf{x} = \mathbf{b}$ as follows:

$$\begin{aligned} \mathbf{w} &= P_{row} \mathbf{b} && \text{(row permutations)} \\ L\mathbf{c} &= \mathbf{w} && \text{(forward substitution)} \\ K\mathbf{x} &= \mathbf{c} && \text{(backward substitution)} \end{aligned}$$

Then $P_{row} A \mathbf{x} = LK \mathbf{x} = L\mathbf{c} = \mathbf{w} = P_{row} \mathbf{b}$. Usually, P_{row} is not stored as a matrix, but rather as a vector \mathbf{p} of indices: $w_i = b_{p_i}$.

Example 3.62. We look at the example:

$$\boxed{\begin{array}{l} E_1: 2x_3 + 3x_4 + 12x_5 = 10 \\ E_2: 4x_1 + 8x_2 + 2x_3 + 3x_4 + 4x_5 = 14 \\ E_3: x_1 + 2x_2 + x_4 = 3 \\ E_4: -3x_1 - 6x_2 - 6x_3 + 8x_4 + 4x_5 = 1 \end{array}} \quad (3.14)$$

As always:

$$(A|\mathbf{b}) = \left(\begin{array}{ccccc|c} E_1: & 0 & 0 & 2 & 3 & 12 & 10 \\ E_2: & 4 & 8 & 2 & 3 & 4 & 14 \\ E_3: & 1 & 2 & 0 & 1 & 0 & 3 \\ E_4: & -3 & -6 & -6 & 8 & 4 & 1 \end{array} \right)$$

We need a row exchange.

Then we get the new pivot. Let us exchange E_1 with E_3 :

$$\left(\begin{array}{ccccc|c} E_1: & 0 & 0 & 2 & 3 & 12 & 10 \\ E_2: & 4 & 8 & 2 & 3 & 4 & 14 \\ E_3: & 1 & 2 & 0 & 1 & 0 & 3 \\ E_4: & -3 & -6 & -6 & 8 & 4 & 1 \end{array} \right) \xrightarrow{\substack{\uparrow \\ | \\ \downarrow}} \sim \left(\begin{array}{ccccc|c} E'_1: & 1 & 2 & 0 & 1 & 0 & 3 \\ E'_2: & 4 & 8 & 2 & 3 & 4 & 14 \\ E'_3: & 0 & 0 & 2 & 3 & 12 & 10 \\ E'_4: & -3 & -6 & -6 & 8 & 4 & 1 \end{array} \right)$$

Now, there is a gray 1 that we will use for the subtraction:

- Subtract $\frac{4}{1} = 4$ times E'_1 from E'_2 ,
- subtract $\frac{-3}{1} = (-3)$ times E'_1 from E'_4 .

Here the solution:

$$\begin{array}{l} E'_1: \left(\begin{array}{ccccc|c} x_1 & x_2 & x_3 & x_4 & x_5 & 3 \\ 1 & 2 & 0 & 1 & 0 & 3 \end{array} \right) \\ E'_2: \left(\begin{array}{ccccc|c} 4 & 8 & 2 & 3 & 4 & 14 \\ 0 & 0 & 2 & 3 & 12 & 10 \end{array} \right) -4 \cdot E'_1 \\ E'_3: \left(\begin{array}{ccccc|c} -3 & -6 & -6 & 8 & 4 & 1 \end{array} \right) +3 \cdot E'_1 \\ E'_4: \left(\begin{array}{ccccc|c} 1 & 2 & 0 & 1 & 0 & 3 \\ 0 & 0 & 2 & -1 & 4 & 2 \\ 0 & 0 & 2 & 3 & 12 & 10 \\ 0 & 0 & -6 & 11 & 4 & 10 \end{array} \right) \end{array} \rightsquigarrow \begin{array}{l} E''_1: \left(\begin{array}{ccccc|c} 1 & 2 & 0 & 1 & 0 & 3 \end{array} \right) \\ E''_2: \left(\begin{array}{ccccc|c} 0 & 0 & 2 & -1 & 4 & 2 \end{array} \right) \\ E''_3: \left(\begin{array}{ccccc|c} 0 & 0 & 2 & 3 & 12 & 10 \end{array} \right) \\ E''_4: \left(\begin{array}{ccccc|c} 0 & 0 & -6 & 11 & 4 & 10 \end{array} \right) \end{array}$$

Now there is no x_1 in the rows 2, 3 and 4. Now, we should not touch the first row any more since otherwise x_1 comes back in the game.

Also, x_2 remains only in row 1. Hence, we do not have to do anything with x_2 . We can go to x_3 .

There the gray 2 in E''_2 is the next pivot. Subtract E''_2 with the right multiple ($\frac{2}{2} = 1$) from E''_3 . Also subtract $\frac{-6}{2} = (-3)$ times E''_2 from E''_4 . We get:

$$\begin{array}{l} E''_1: \left(\begin{array}{ccccc|c} 1 & 2 & 0 & 1 & 0 & 3 \end{array} \right) \\ E''_2: \left(\begin{array}{ccccc|c} 0 & 0 & 2 & -1 & 4 & 2 \end{array} \right) \\ E''_3: \left(\begin{array}{ccccc|c} 0 & 0 & 2 & 3 & 12 & 10 \end{array} \right) -1 \cdot E''_2 \\ E''_4: \left(\begin{array}{ccccc|c} 0 & 0 & -6 & 11 & 4 & 10 \end{array} \right) +3 \cdot E''_2 \\ E''_1: \left(\begin{array}{ccccc|c} 1 & 2 & 0 & 1 & 0 & 3 \end{array} \right) \\ E''_2: \left(\begin{array}{ccccc|c} 0 & 0 & 2 & -1 & 4 & 2 \end{array} \right) \\ E''_3: \left(\begin{array}{ccccc|c} 0 & 0 & 0 & 4 & 8 & 8 \end{array} \right) \\ E''_4: \left(\begin{array}{ccccc|c} 0 & 0 & 0 & 8 & 16 & 16 \end{array} \right) \end{array}$$

Look at x_4 . Here, 4 is the pivot. We subtract E'''_3 $\frac{8}{4} = 2$ times from E'''_4 :

$$\begin{array}{l} G'''_1: \left(\begin{array}{ccccc|c} 1 & 2 & 0 & 1 & 0 & 3 \end{array} \right) \\ G'''_2: \left(\begin{array}{ccccc|c} 0 & 0 & 2 & -1 & 4 & 2 \end{array} \right) \\ G'''_3: \left(\begin{array}{ccccc|c} 0 & 0 & 0 & 4 & 8 & 8 \end{array} \right) \\ G'''_4: \left(\begin{array}{ccccc|c} 0 & 0 & 0 & 8 & 16 & 16 \end{array} \right) -2 \cdot G'''_3 \\ E''''_1: \left(\begin{array}{ccccc|c} 1 & 2 & 0 & 1 & 0 & 3 \end{array} \right) \\ E''''_2: \left(\begin{array}{ccccc|c} 0 & 0 & 2 & -1 & 4 & 2 \end{array} \right) \\ E''''_3: \left(\begin{array}{ccccc|c} 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right) \\ E''''_4: \left(\begin{array}{ccccc|c} 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right) \end{array}$$

The elimination algorithm ends. This is the wanted solution in row echelon form

$$\begin{array}{l} E''''_1: \left(\begin{array}{ccccc|c} 1 & 2 & 0 & 1 & 0 & 3 \end{array} \right) \\ E''''_2: \left(\begin{array}{ccccc|c} 0 & 0 & 2 & -1 & 4 & 2 \end{array} \right) \\ E''''_3: \left(\begin{array}{ccccc|c} 0 & 0 & 0 & 4 & 8 & 8 \end{array} \right) \\ E''''_4: \left(\begin{array}{ccccc|c} 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right) \end{array} \quad (3.15)$$

Can you write down the set of all solutions \mathcal{S} ?

Video: PLU decomposition - An Example
 **PLU decomposition**
<https://jp-g.de/bsom/la/plu/>

$$A = PLU =$$

$$P \begin{pmatrix} \text{blue triangle} \\ \text{green triangle} \end{pmatrix}$$



3.12 Looking at columns and maps

In the Gaussian elimination everything works in the rows. Now, we will look what we can say about the columns. As a reminder: The LES $A\mathbf{x} = \mathbf{b}$ has at least one solution \mathbf{x} if and only if \mathbf{b} can be written as $A\mathbf{x}$ for a $\mathbf{x} \in \mathbb{R}^n$, which means that $\mathbf{b} \in \text{Ran}(A)$. Looking at the columns of the matrix

$$A = \begin{pmatrix} | & & | \\ \mathbf{a}_1 & \dots & \mathbf{a}_n \\ | & & | \end{pmatrix} \in \mathbb{R}^{m \times n},$$

we can conclude

$$A\mathbf{x} = \begin{pmatrix} | & & | \\ \mathbf{a}_1 & \dots & \mathbf{a}_n \\ | & & | \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = x_1 \begin{pmatrix} | \\ \mathbf{a}_1 \\ | \end{pmatrix} + \dots + x_n \begin{pmatrix} | \\ \mathbf{a}_n \\ | \end{pmatrix}$$

In other words:

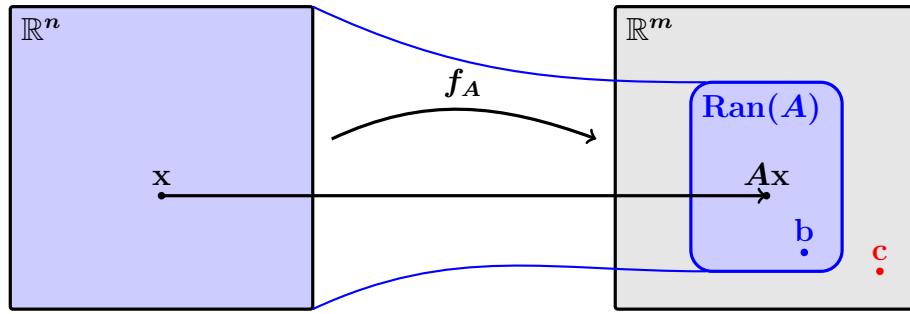
$$\text{Ran}(A) = \{A\mathbf{x} : \mathbf{x} \in \mathbb{R}^n\} = \{x_1\mathbf{a}_1 + \dots + x_n\mathbf{a}_n : x_1, \dots, x_n \in \mathbb{R}\} \subset \mathbb{R}^m. \quad (3.16)$$

Reformulating the fact from above

Corollary 3.63. Solvability in the column picture

For a matrix $A \in \mathbb{R}^{m \times n}$ and vector $\mathbf{b} \in \mathbb{R}^m$ the following claims are equivalent

- (i) $A\mathbf{x} = \mathbf{b}$ has at least one solution,
- (ii) $\mathbf{b} \in \text{Ran}(A)$,
- (iii) \mathbf{b} can be written as a linear combination of the columns from A .



$\mathbf{b} \in \text{Ran}(A) \Rightarrow A\mathbf{x} = \mathbf{b}$ has at least one solution
 $\mathbf{c} \notin \text{Ran}(A) \Rightarrow A\mathbf{x} = \mathbf{c}$ has no solution

Example 3.64. Let $A = \begin{pmatrix} 3 & 6 \\ 1 & 2 \end{pmatrix}$. Then $A\mathbf{x} = \mathbf{b}$ has at least one solution if and only if

$$\begin{aligned} \mathbf{b} \in \text{Ran}(A) &= \left\{ x_1 \begin{pmatrix} 3 \\ 1 \end{pmatrix} + x_2 \begin{pmatrix} 6 \\ 2 \end{pmatrix} : x_1, x_2 \in \mathbb{R} \right\} = \left\{ x_1 \begin{pmatrix} 3 \\ 1 \end{pmatrix} + 2x_2 \begin{pmatrix} 3 \\ 1 \end{pmatrix} : x_1, x_2 \in \mathbb{R} \right\} \\ &= \left\{ (x_1 + 2x_2) \begin{pmatrix} 3 \\ 1 \end{pmatrix} : x_1, x_2 \in \mathbb{R} \right\} = \left\{ \lambda \begin{pmatrix} 3 \\ 1 \end{pmatrix} : \lambda \in \mathbb{R} \right\} \end{aligned}$$

This means that $\mathbf{b} \in \mathbb{R}^2$ lies on the line through $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 3 \\ 1 \end{pmatrix}$.

Remember that for each matrix A there is a linear map $f_A : \mathbb{R}^n \rightarrow \mathbb{R}^m$, cf. section 3.3, defined by

$$\mathbf{x} \in \mathbb{R}^n \xrightarrow{f_A} A\mathbf{x} \in \mathbb{R}^m.$$

Of course, solving $A\mathbf{x} = \mathbf{b}$, is the same as solving $f_A(\mathbf{x}) = \mathbf{b}$. This means we want to find the preimage of the element \mathbf{b} with respect to the map f_A . Obviously the image of f_A is exactly the range of A $\text{Ran}(A)$, so we get:

$$f_A(\mathbb{R}^n) = \{f_A(\mathbf{x}) : \mathbf{x} \in \mathbb{R}^n\} = \{A\mathbf{x} : \mathbf{x} \in \mathbb{R}^n\} = \text{Ran}(A).$$

Hence, we find the following:

Proposition 3.65. Unconditional solvability needs surjectivity of f_A)

For a matrix $A \in \mathbb{R}^{m \times n}$ the following claims are equivalent:

- (i) The LES $A\mathbf{x} = \mathbf{b}$ has for **every** $\mathbf{b} \in \mathbb{R}^m$ **at least one** solution \mathbf{x} .
- (ii) All $\mathbf{b} \in \mathbb{R}^m$ lie in $\text{Ran}(A)$.
- (iii) $\text{Ran}(A) = \mathbb{R}^m$.
- (iv) $\text{rank}(A) = m \leq n$.
- (v) The row echelon form of A , denoted by A' , has a pivot in **every row**.
- (vi) f_A is surjective.

					n	
m	0	0				
	0	0	0			
	0	0	0	0		
	0	0	0	0	0	

Row echelon form A' of A :

- each row has a pivot
- There are no zero rows in A' .
- We will never have $(0 \dots 0 | c \neq 0)$ in the last row.

Proof. (i) \Leftrightarrow (ii) \Leftrightarrow (iii) \Leftrightarrow (vi) and (iv) \Leftrightarrow (v) is clear. Assume (iv), $\text{rank}(A) = m$. Then for each $\mathbf{b} \in \mathbb{R}^m$ we can use $\text{rank}(A) \leq \text{rank}(A|\mathbf{b}) \leq m$ and therefore $\text{rank}(A|\mathbf{b}) = m$. We can conclude $\text{rank}(A) = \text{rank}(A|\mathbf{b})$. This means that \mathbf{b} is a linear combination of the columns of A and, hence, this gives us a solution \mathbf{x} , which shows (i).

Show now (i) \Rightarrow (v) by contraposition. Assume $\neg(v)$. By doing the elimination $A \sim A'$ to get a row echelon form A' , we also get at least one zero row. Then, it is possible to choose $\mathbf{b} \in \mathbb{R}^m$ in such a way that we get a row in $(A'|\mathbf{b}')$ that is given by $(0 \dots 0 | \mathbf{c})$ with $\mathbf{c} \neq 0$. Such a row cannot be solved and, hence, $A\mathbf{x} = \mathbf{b}$ has no solution. Therefore, we get $\neg(i)$. \square

Example 3.66. Consider a 3×5 matrix A and calculate the row echelon form A' :

$$A = \begin{pmatrix} 1 & 4 & 0 & 2 & -1 \\ -1 & 2 & -2 & -2 & 3 \\ -3 & 0 & -4 & -3 & 8 \end{pmatrix} \quad \sim \quad \dots \quad \sim \quad A' = \left(\begin{array}{ccccc} 1 & 4 & 0 & 2 & -1 \\ 0 & 6 & -2 & 0 & 2 \\ 0 & 0 & 0 & 3 & 1 \end{array} \right)$$

Each row of A' has a pivot and (v) from Proposition 3.65 holds. One immediately sees $\text{rank}(A) = 3 = m \leq n = 5$.

(i) says that the LES $A\mathbf{x} = \mathbf{b}$ has for every right-hand sides $\mathbf{b} \in \mathbb{R}^3$ at least one solution.

Now we go to the uniqueness

Proposition 3.67. Unique solution (injectivity of f_A)

For a matrix $A \in \mathbb{R}^{m \times n}$ the following claims are equivalent:

- (i) $A\mathbf{x} = \mathbf{b}$ has for **every** $\mathbf{b} \in \mathbb{R}^m$ **at most** one solution \mathbf{x} .
- (ii) $A\mathbf{x} = \mathbf{o}$ has only the solution $\mathbf{x} = \mathbf{o}$.
- (iii) $\text{Ker}(A) = \{\mathbf{o}\}$.
- (iv) $\text{rank}(A) = n \leq m$.
- (v) The row echelon form of A , denoted by A' , has in **every column** a pivot.
- (vi) f_A is injective.

					n	
m	0					
	0	0				
	0	0	0			
	0	0	0	0		
	0	0	0	0	0	

Row echelon form A' of A :

- Each column has a pivot.
- All variables are leading variables.
- There are no free variables.
- It is not possible to have more than one solution.

Proof. Equivalence (i) \Leftrightarrow (iii) follow from $\mathcal{S} = \mathbf{v}_0 + \text{Ker}(A)$, see Proposition 3.48. (i) \Leftrightarrow (vi) holds by the definition of injectivity. The equivalence (ii) \Leftrightarrow (iii) follows from the definition of $\text{Ker}(A)$. (iii) \Leftrightarrow (iv) holds by Theorem 3.41, the Rank-nullity Theorem. (iv) \Leftrightarrow (v) holds since row operations do not change the rank of a matrix. \square

Example 3.68. Consider a 4×3 matrix A and calculate the row echelon form A' :

$$A = \begin{pmatrix} 2 & 3 & 0 \\ 2 & 2 & 5 \\ -4 & -5 & -3 \\ 4 & 7 & 1 \end{pmatrix} \rightsquigarrow \dots \rightsquigarrow A' = \begin{pmatrix} 2 & 3 & 0 \\ 0 & -1 & 5 \\ 0 & 0 & -8 \\ 0 & 0 & 0 \end{pmatrix}$$

Each column in A' has a pivot. One also sees: $\text{rank}(A) = 3 = n \leq m = 4$.

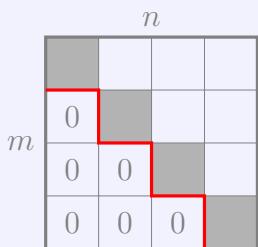
The LES $A\mathbf{x} = \mathbf{b}$ has exactly the solution $\mathbf{x} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ for $\mathbf{b} = \begin{pmatrix} 2 \\ 2 \\ -4 \\ 4 \end{pmatrix}$; but for $\mathbf{b} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$ there is no solution \mathbf{x} .

Both things together:

Proposition 3.69. Existence and Uniqueness of a solution

For a matrix $A \in \mathbb{R}^{m \times n}$ the following claims are equivalent:

- (i) The LES $A\mathbf{x} = \mathbf{b}$ has **for every** $\mathbf{b} \in \mathbb{R}^m$ a **unique** solution.
- (ii) $\text{Ker}(A) = \{\mathbf{0}\}$ and $\text{Ran}(A) = \mathbb{R}^m$.
- (iii) $\text{rank}(A) = m = n$, i.e. A is quadratic with maximal rank.
- (iv) f_A is bijective.



Row echelon form A' of A :

- Each column and row has a pivot.
- The matrix has to be quadratic.
- We have $\text{rank}(A) = m = n$.
- The row echelon form A' has triangle form.

We look at the special case of square matrices

Proposition 3.70. $m=n$: square matrices

For a quadratic $A \in \mathbb{R}^{n \times n}$ the following claims are equivalent:

- (i) The LES $A\mathbf{x} = \mathbf{b}$ has a solution **for every** $\mathbf{b} \in \mathbb{R}^n$.
- (ii) The LES $A\mathbf{x} = \mathbf{b}$ has **for some** $\mathbf{b} \in \mathbb{R}^n$ a **unique** solution.
- (iii) The LES $A\mathbf{x} = \mathbf{b}$ has a unique solution **for every** $\mathbf{b} \in \mathbb{R}^n$.
- (iv) $\text{Ker}(A) = \{\mathbf{0}\}$

- (v) $\text{Ran}(A) = \mathbb{R}^n$.
- (vi) $\text{rank}(A) = n$.
- (vii) For A , the row echelon form A' has a pivot in each **row**.
- (viii) For A , the row echelon form A' has a pivot in each **column**.
- (ix) f_A is surjective.
- (x) f_A is injective.
- (xi) f_A is bijective.

Proof. Since $m = n$, the equations $\text{rank}(A) = m$ and $\text{rank}(A) = n$ from Proposition 3.65 and Proposition 3.67 are equivalent. Therefore all the claims above are equivalent. \square

We can conclude:

Box 3.71. Fredholm alternative

For square matrices, we have either both claims below or neither of them:

- unconditional solvability (f_A ist surjective),
- unique solution ($\text{Ker}(A) = \{\mathbf{0}\}$, hence f_A is injective)

Summary

- By $\mathbb{R}^{m \times n}$ we denote number tables with m rows and n columns.
- We call these number tables *matrices* and can naturally scale them and add them. Both operations in $\mathbb{R}^{m \times n}$ are realised by doing these inside the components.
- *Linear equations* look like

$$\text{constant} \cdot x_1 + \text{constant} \cdot x_2 + \cdots + \text{constant} \cdot x_n = \text{constant}.$$

- Systems of linear equations (LES) are finitely many of these linear equations.
- A *solution* of the system is a choice of all unknowns x_1, \dots, x_n such that all equations are satisfied.
- A short notation for LES is the matrix notation: $A\mathbf{x} = \mathbf{b}$.
- This notation leads us to the general *matrix product*.
- Each matrix A induces a *linear map* $f_A : \mathbb{R}^n \rightarrow \mathbb{R}^m$. A linear map satisfies two properties (\cdot) and ($+$).
- If f_A is bijective, the corresponding matrix is *invertible* with respect to the matrix product.
- *Linearly independent vectors* are the most efficient method to describe a linear subspace.

- A linearly independent family that generates the whole subspace U is called a *basis* of U .
- Range, rank and kernel are important objects for matrices.
- For solving a LES, we use *Gaussian elimination* or equivalently *LU*-decomposition. In the general case the upper triangular matrix U is substituted by a *row echelon form*.
- Solvability and unique solvability can be equivalently formulated and, for example, read from the row echelon form.

4

Determinants

A learning experience is one of those things that says, 'You know that thing you just did? Don't do that.'

Douglas Adams, The Salmon of Doubt

Let $A \in \mathbb{R}^{n \times n}$ be a square matrix. The determinant $\det(A) \in \mathbb{R}$ of A is a special real number, associated to A .

- Geometrically, if A is a column matrix, the absolute value $|\det(A)|$ describes the (generalised) n -dimensional *volume* of the parallel-epiped, spanned by the columns of A .
- In particular, this volume is non-zero *if and only if* the columns of A are linearly independent and this hold also if and only if A is invertible
- The sign (\pm) of $\det(A)$ gives an *orientation*, where $\det(\mathbf{1}) = +1$ (column matrix of unit vectors in the usual order, unit cube). If we exchange two columns of a matrix, then the sign of its determinant changes.
- The determinant should have all other properties that one expects from a volume.

4.1 Determinant in two dimensions

We already know how to solve the system $A\mathbf{x} = \mathbf{b}$ if A is a square matrix. The determinant should then tell us if the system has a unique solution before solving it. For a 2×2 LES, we get (for $a_{11} \neq 0$):

$$\left(\begin{array}{cc|c} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \end{array} \right) \quad \sim \quad \left(\begin{array}{cc|c} a_{11} & a_{12} & b_1 \\ 0 & a_{11}a_{22} - a_{12}a_{21} & b_2a_{11} - b_1a_{21} \end{array} \right)$$

Hence, we know that the LES has a unique solution if and only if in the second column there is a pivot. This means $a_{11}a_{22} - a_{12}a_{21} \neq 0$. And that is the determinant of the system or rather the matrix A .

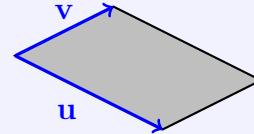
Definition 4.1.

For a matrix $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \in \mathbb{R}^{2 \times 2}$, we call $\det(A) := a_{11}a_{22} - a_{12}a_{21}$ the determinant of A .

The determinant in two dimensions has an immediate interpretation when we compare it to an area measurement:

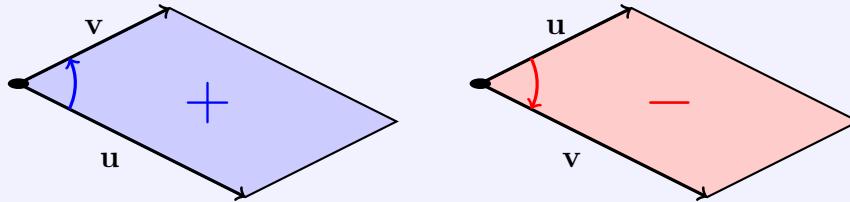
Consider two vectors $\mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ from \mathbb{R}^2 and the spanned parallelogram. We define $\text{Area}(\mathbf{u}, \mathbf{v}) \in \mathbb{R}$ as the real number that fulfills

$$|\text{Area}(\mathbf{u}, \mathbf{v})| = \text{area of parallelogram}$$



and the sign of $\text{Area}(\mathbf{u}, \mathbf{v})$ is chosen in the following way:

- **Plus sign** if then rotation by rotating \mathbf{u} such that the angle between \mathbf{u} and \mathbf{v} gets smaller is the mathematical positive sense,
- **Minus sign** if this rotation is the mathematical negative sense.



If you look back at Section 2.6, you already know a possible calculation for the area of a parallelogram. However, this only works in the three-dimensional space \mathbb{R}^3 since it involves the cross product. However, if we just embed the vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^2$ and therefore the whole parallelogram into \mathbb{R}^3 , we can calculate $|\text{Area}(\mathbf{u}, \mathbf{v})|$ in this way. A possible way is to set the supplementary third component as zero:

$$\tilde{\mathbf{u}} := \begin{pmatrix} u_1 \\ u_2 \\ 0 \end{pmatrix} \quad \text{and} \quad \tilde{\mathbf{v}} := \begin{pmatrix} v_1 \\ v_2 \\ 0 \end{pmatrix}.$$

Then we find:

$$|\text{Area}(\mathbf{u}, \mathbf{v})| = \|\tilde{\mathbf{u}} \times \tilde{\mathbf{v}}\| = \left\| \begin{pmatrix} u_2 \cdot 0 - 0 \cdot v_2 \\ 0 \cdot v_1 - u_1 \cdot 0 \\ u_1 v_2 - u_2 v_1 \end{pmatrix} \right\| = \left\| \begin{pmatrix} 0 \\ 0 \\ u_1 v_2 - u_2 v_1 \end{pmatrix} \right\| = |u_1 v_2 - u_2 v_1|.$$

Without the absolute value this coincides with the determinant of the matrix

$$A := \begin{pmatrix} \mathbf{u} & \mathbf{v} \end{pmatrix}.$$

Indeed, also the sign rule from above is fulfilled.

Proposition 4.2.

$$\text{Area}\left(\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}\right) = \det \begin{pmatrix} u_1 & v_1 \\ u_2 & v_2 \end{pmatrix} = u_1 v_2 - u_2 v_1 \quad (4.1)$$

Example 4.3. (a) If we look at $\mathbf{u} = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$, we get:

$$\text{Area}(\mathbf{u}, \mathbf{v}) = \text{Area}\left(\begin{pmatrix} 3 \\ -2 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \end{pmatrix}\right) = 3 \cdot 1 - (-2) \cdot 2 = 7.$$

The area is 7 and the orientation is positive.

(b) The other ordering gets:

$$\text{Area}\left(\begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ -2 \end{pmatrix}\right) = 2 \cdot (-2) - 1 \cdot 3 = -7.$$

The area is again 7 but the orientation is negative.

(c) Choose $\mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$ and the scaled vector $\mathbf{v} = \alpha \mathbf{u} = \begin{pmatrix} \alpha u_1 \\ \alpha u_2 \end{pmatrix}$. Then:

$$\text{Area}(\mathbf{u}, \mathbf{v}) = \text{Area}(\mathbf{u}, \alpha \mathbf{u}) = \text{Area}\left(\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \begin{pmatrix} \alpha u_1 \\ \alpha u_2 \end{pmatrix}\right) = u_1 \alpha u_2 - u_2 \alpha u_1 = 0.$$

Note that the vectors \mathbf{u} and $\mathbf{v} = \alpha \mathbf{u}$ do not span a actual parallelogram but rather just a stripe. Therefore, the area has to be zero.

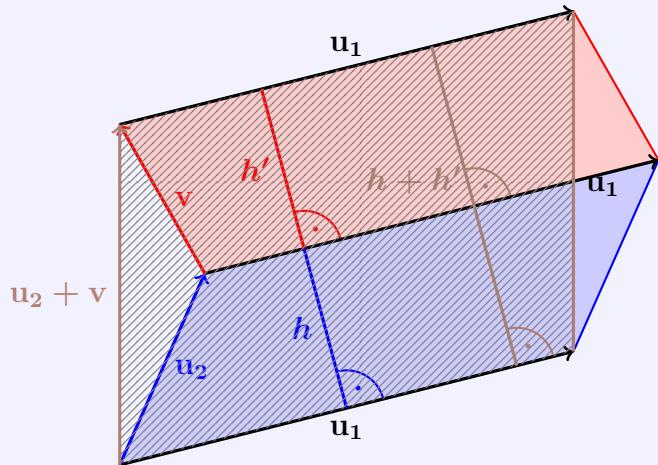
4.2 Determinant as a volume measure

In the previous section, we showed that, in two dimensions, the determinant is connected to a measuring of an area. In three dimensions, we therefore expect that the determinant will measure a volume. In general, we want that the determinant measures an generalised n -dimensional volume in \mathbb{R}^n . We use the symbol Vol_n for this. We already know that $\text{Vol}_2 = \text{Area}$. Now, we can summarise what one should demand of a meaningful volume measure.

Definition 4.4. Properties that Vol_n should have.

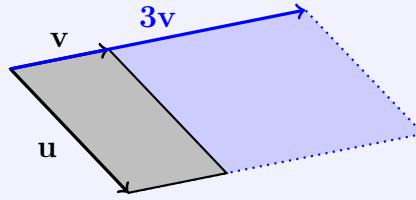
The n -dimensional volume function $\text{Vol}_n : \mathbb{R}^n \times \cdots \times \mathbb{R}^n \rightarrow \mathbb{R}$ that gets n vectors as input should fulfil:

- (1) $\text{Vol}_n(\mathbf{u}_1, \dots, \mathbf{u}_j + \mathbf{v}, \dots, \mathbf{u}_n) = \text{Vol}_n(\mathbf{u}_1, \dots, \mathbf{u}_j, \dots, \mathbf{u}_n) + \text{Vol}_n(\mathbf{u}_1, \dots, \mathbf{v}, \dots, \mathbf{u}_n)$
for all $\mathbf{u}_1, \dots, \mathbf{u}_n, \mathbf{v} \in \mathbb{R}^n$ and $j \in \{1, \dots, n\}$.



(Picture in the case $n = j = 2$)

- (2) $\text{Vol}_n(\mathbf{u}_1, \dots, \alpha \mathbf{u}_j, \dots, \mathbf{u}_n) = \alpha \text{Vol}_n(\mathbf{u}_1, \dots, \mathbf{u}_j, \dots, \mathbf{u}_n)$ for all $\mathbf{u}_1, \dots, \mathbf{u}_n \in \mathbb{R}^n$, $\alpha \in \mathbb{R}$ and $j \in \{1, \dots, n\}$.



- (3) $\text{Vol}_n(\mathbf{u}_1, \dots, \mathbf{u}_i, \dots, \mathbf{u}_j, \dots, \mathbf{u}_n) = -\text{Vol}_n(\mathbf{u}_1, \dots, \mathbf{u}_j, \dots, \mathbf{u}_i, \dots, \mathbf{u}_n)$ for all $\mathbf{u}_1, \dots, \mathbf{u}_n \in \mathbb{R}^n$, and $i, j \in \{1, \dots, n\}$ with $i \neq j$.



- (4) The unit cube ($\mathbf{u}_1 = \mathbf{e}_1, \dots, \mathbf{u}_n = \mathbf{e}_n$) has volume 1: $\text{Vol}(\mathbf{e}_1, \dots, \mathbf{e}_n) = 1$.

In mathematical terms, this means that the volume function is linear in each entry, anti-symmetric and normalised to the standard basis. For the case $n = 2$, we can show that solely these properties imply equation (4.1).

First, we show an easy consequence that follows from the two properties (2) and (3):

Proposition 4.5. Colinear vectors do not have an area.

For all $\mathbf{u} \in \mathbb{R}^2$ and $\alpha \in \mathbb{R}$, we have $\text{Vol}_2(\mathbf{u}, \alpha \mathbf{u}) = 0$.

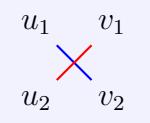
Proof. Because of (3), we find $\text{Vol}_2(\mathbf{u}, \mathbf{u}) = -\text{Vol}_2(\mathbf{u}, \mathbf{u})$ and this implies $\text{Vol}_2(\mathbf{u}, \mathbf{u}) = 0$. Since (2) holds, we get $\text{Vol}_2(\mathbf{u}, \alpha \mathbf{u}) = \alpha \text{Vol}_2(\mathbf{u}, \mathbf{u}) = \alpha 0 = 0$. \square

Now, we can prove the formula (4.1):

Proposition 4.6.

If Vol_2 fulfills (1), (2), (3), (4), then for all $\mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \in \mathbb{R}^2$ the following

holds $\text{Vol}_2(\mathbf{u}, \mathbf{v}) = \text{Vol}_2\left(\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}\right) = +u_1 v_2 - u_2 v_1.$



Proof.

$$\begin{aligned} \text{Vol}_2(\mathbf{u}, \mathbf{v}) &= \text{Vol}_2\left(\begin{pmatrix} u_1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ u_2 \end{pmatrix}, \mathbf{v}\right) \stackrel{(1)}{=} \text{Vol}_2\left(\begin{pmatrix} u_1 \\ 0 \end{pmatrix}, \mathbf{v}\right) + \text{Vol}_2\left(\begin{pmatrix} 0 \\ u_2 \end{pmatrix}, \mathbf{v}\right) \\ &\stackrel{(1)}{=} \text{Vol}_2\left(\begin{pmatrix} u_1 \\ 0 \end{pmatrix}, \begin{pmatrix} v_1 \\ 0 \end{pmatrix}\right) + \text{Vol}_2\left(\begin{pmatrix} u_1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ v_2 \end{pmatrix}\right) \\ &\quad + \text{Vol}_2\left(\begin{pmatrix} 0 \\ u_2 \end{pmatrix}, \begin{pmatrix} v_1 \\ 0 \end{pmatrix}\right) + \text{Vol}_2\left(\begin{pmatrix} 0 \\ u_2 \end{pmatrix}, \begin{pmatrix} 0 \\ v_2 \end{pmatrix}\right) \\ &\stackrel{(2),(2)}{=} \underbrace{u_1 v_1 \text{Vol}_2\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right)}_{=0 \text{ see Proposition 4.5}} + \underbrace{u_1 v_2 \text{Vol}_2\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right)}_{=1 \text{ because of (4)}} \\ &\quad + \underbrace{u_2 v_1 \text{Vol}_2\left(\begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right)}_{=-1 \text{ because of (3),(4)}} + \underbrace{u_2 v_2 \text{Vol}_2\left(\begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right)}_{=0 \text{ see Proposition 4.5}} \\ &= u_1 v_2 - u_2 v_1. \end{aligned}$$

□

Note that this proves that the volume function Vol_2 is uniquely defined by the four properties (1), (2), (3) and (4) alone. We expect the same for arbitrary dimension n and indeed we prove this now.

For $\mathbf{a}_1 = \begin{pmatrix} a_{11} \\ \vdots \\ a_{n1} \end{pmatrix}, \dots, \mathbf{a}_n = \begin{pmatrix} a_{1n} \\ \vdots \\ a_{nn} \end{pmatrix} \in \mathbb{R}^n$, it follows

$$\text{Vol}_n(\mathbf{a}_1, \dots, \mathbf{a}_n) = \text{Vol}_n(a_{11}\mathbf{e}_1 + \dots + a_{n1}\mathbf{e}_n, \dots, a_{1n}\mathbf{e}_1 + \dots + a_{nn}\mathbf{e}_n).$$

Using the linearity in each entry, we can conclude:

$$\text{Vol}_n(\mathbf{a}_1, \dots, \mathbf{a}_n) = \sum_{i_1=1}^n \sum_{i_2=1}^n \cdots \sum_{i_n=1}^n a_{i_1,1} a_{i_2,2} \cdots a_{i_n,n} \text{Vol}_n(\mathbf{e}_{i_1}, \mathbf{e}_{i_2}, \dots, \mathbf{e}_{i_n})$$

with $n \cdot \dots \cdot n = n^n$ summands, where most of them are zero since $\det(\mathbf{e}_{i_1} \cdots \mathbf{e}_{i_n}) = 0$ if two indices coincide. The possibilities (i_1, i_2, \dots, i_n) for that all entries are different can be counted. The number is the number of all permutations of the set $\{1, \dots, n\}$, which is exactly $n \cdot (n-1) \cdots 1 = n!$. Let P_n be the set of these $n!$ permutations, which can also be denoted by $\tau = (i_1, i_2, \dots, i_n)$.

For a permutation $\tau \in P_n$, we define $\text{sgn}(\tau) = 1$ if one can use an even number of exchanges of two elements to get from τ to $(1, 2, \dots, n)$. If one needs an odd number of exchanges of two elements to get from τ to $(1, 2, \dots, n)$, we define $\text{sgn}(\tau) = -1$.

Repeatable usage of (3) shows $\text{Vol}_n(\mathbf{e}_{i_1} \cdots \mathbf{e}_{i_n}) = \text{sgn}(i_1, \dots, i_n)$. In summary, we get

Proposition 4.7. Leibniz formula

The volume form Vol_n is uniquely determined by the properties (1), (2), (3) and (4) and fulfils for n vectors $\mathbf{a}_1 = \begin{pmatrix} a_{11} \\ \vdots \\ a_{n1} \end{pmatrix}, \dots, \mathbf{a}_n = \begin{pmatrix} a_{1n} \\ \vdots \\ a_{nn} \end{pmatrix} \in \mathbb{R}^n$:

$$\text{Vol}_n(\mathbf{a}_1, \dots, \mathbf{a}_n) = \sum_{(i_1, \dots, i_n) \in \mathcal{P}_n} \text{sgn}(i_1, \dots, i_n) a_{i_1,1} a_{i_2,2} \cdots a_{i_n,n}. \quad (4.2)$$

Proof. The calculation from above shows that (4.2) is the only function that fulfils all the four rules. \square

Definition 4.8. Determinant of square matrices

For a square matrix $A \in \mathbb{R}^{n \times n}$ with entries

$$A := \begin{pmatrix} | & & | \\ \mathbf{a}_1 & \dots & \mathbf{a}_n \\ | & & | \end{pmatrix} = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix},$$

we define the determinant as the volume measure of the column vectors:

$$\det(A) := \text{Vol}_n(\mathbf{a}_1, \dots, \mathbf{a}_n) = \sum_{(i_1, \dots, i_n) \in \mathcal{P}_n} \text{sgn}(i_1, \dots, i_n) a_{i_1,1} a_{i_2,2} \cdots a_{i_n,n}.$$

Remark:

You can remember the Leibniz formula of the determinant $\det(A)$ in the following way:

- (1) Build a product of n factors out of the entries in A . From each row and each column you are only allowed to choose one factor.
- (2) Sum up all the possibilities for such a product where you add a minus-sign for the odd permutations.

Example 4.9. Consider the matrix $P_{k \leftrightarrow \ell}$ that we used in the Gaussian elimination to switch the k th row with the ℓ th row. Let's denote the entries by p_{ij} and then we know

$$p_{ij} = \begin{cases} 1 & \text{if } i = j \text{ and } i, j \neq k, \ell \\ 1 & \text{if } i = k, j = \ell \text{ or } i = \ell, j = k \\ 0 & \text{else} \end{cases}$$

This means that in the Leibniz formula, there is only one non-vanishing term:

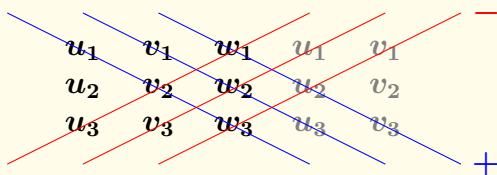
$$\det(P_{k \leftrightarrow \ell}) = \text{sgn}(\tau) p_{11} \cdots p_{k,\ell} \cdots p_{\ell,k} \cdots p_{nn} = -1$$

Since the permutation is only one single exchange, the sign is -1 . Of course, we expect this result by property (3) of the volume form.

We have used the Leibniz formula to finally define the determinant of a matrix or, equivalently, the volume measure in \mathbb{R}^n . However, despite being useful in abstract proofs, this formula is not a good one for actual calculation. Even for $n = 4$, we have to sum up $4! = 24$ terms. Only for $n = 2$ and $n = 3$, we get good calculation formulas, which can be memorised.

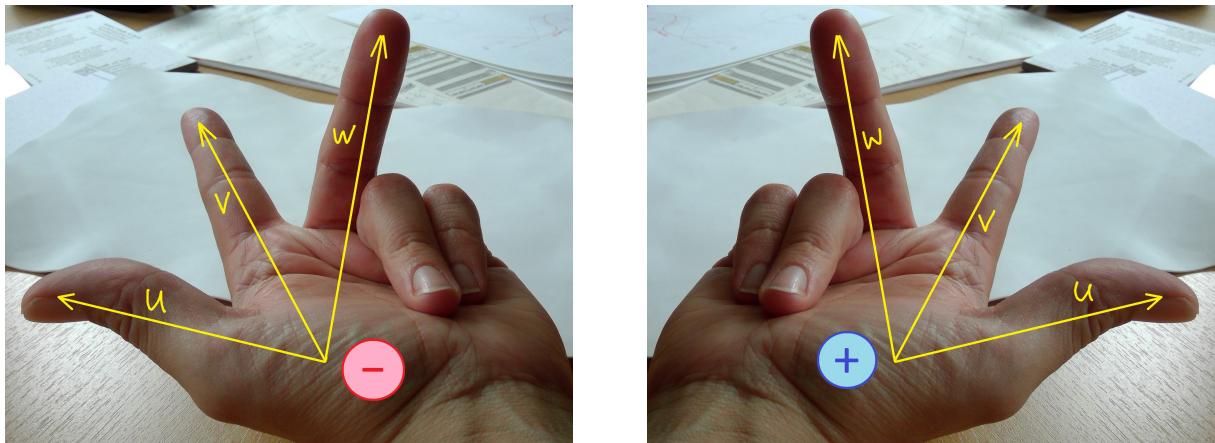
$$\det \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = + a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33}$$

Rule of thumb: Rule of Sarrus (Only for $n = 3$)



$$\text{Vol}_3(\mathbf{u}, \mathbf{v}, \mathbf{w}) = +u_1v_2w_3 + v_1w_2u_3 + w_1u_2v_3 - u_3v_2w_1 - v_3w_2u_1 - w_3u_2v_1$$

Moreover, the sign of the three-dimensional volume can be easily seen by the right-hand-rule:



4.3 The cofactor expansion

We already know that volume measure and the determinant of a matrix coincide. From now on, we will only consider the determinant of matrices and keep in mind that this is the volume spanned by the columns of the matrix. We consider a matrix

$$A = (a_{ij}) \quad \text{with} \quad i, j = 1, \dots, n.$$

We have already encountered these cases:

- $A \in \mathbb{R}^{1 \times 1}$:

$$\det(A) = a_{11}$$

- $A \in \mathbb{R}^{2 \times 2}$:

$$\det(A) = a_{11}a_{22} - a_{21}a_{12}$$

- $A \in \mathbb{R}^{3 \times 3}$:

$$\begin{aligned}\det(A) = & + a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} \\ & - a_{13}a_{22}a_{31} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33}\end{aligned}$$

Note that it would be helpful to have an algorithm that reduces an $n \times n$ -matrix to these cases.

Checkerboard of signs:

$$\begin{pmatrix} + & - & + & - & \dots \\ - & + & - & + & \dots \\ + & - & + & - & \dots \\ - & + & - & + & \dots \\ \vdots & \vdots & \vdots & \vdots & \end{pmatrix}$$

The entry (i, j) of this matrix is $(-1)^{i+j}$.

Cofactors:

For $A \in \mathbb{R}^{n \times n}$ pick one entry a_{ij} .

- Delete the i^{th} row and the j^{th} column of A .
- Call the remaining matrix $A^{(i,j)} \in \mathbb{R}^{(n-1) \times (n-1)}$.
- The cofactor c_{ij} of a_{ij} is defined as

$$c_{ij} := (-1)^{i+j} \det(A^{(i,j)})$$

Definition 4.10.

$C = (c_{ij}) = (-1)^{i+j} \det(A^{(i,j)})$ is called the cofactor matrix of A .

Using this, we find:

Proposition 4.11. Laplace's formula

Let $A \in \mathbb{R}^{n \times n}$ and choose the j^{th} column:

$$\det(A) = \sum_{i=1}^n a_{ij} c_{ij} = \sum_{i=1}^n (-1)^{i+j} a_{ij} \det(A^{(i,j)})$$

This is expanding $\det(A)$ along the j^{th} column.

If you choose the i^{th} row:

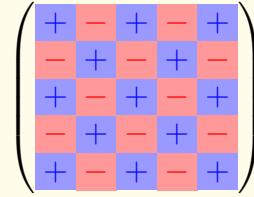
$$\det(A) = \sum_{j=1}^n a_{ij} c_{ij} = \sum_{j=1}^n (-1)^{i+j} a_{ij} \det(A^{(i,j)})$$

One calls this expanding $\det(A)$ along the i^{th} row.

To compute $\det(A^{(i,j)})$, apply the same formula recursively, until you reach 2×2 matrices, where the corresponding formula can be applied. The proof of this formula follows immediately from the Leibniz formula and is left to the reader.

Rule of thumb: Do not forget the checkerboard matrix

Remember the signs when expanding a matrix along a column or a row.



Rule of thumb: Use the nothingness.

Since it is your choice which of the rows or the columns you want to expand along, you should search for zeros. If you find a row or column with a lot of zeros, which means that one has $a_{ij} = 0$ for some indices, you do not need to calculate $\det(A_{ij})$ for these indices.

Example 4.12. Consider the matrix

$$A = \begin{pmatrix} 0 & 2 & 3 & 4 \\ 2 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 6 & 0 & 1 & 2 \end{pmatrix}.$$

Here, it would be useful first to expand along the second row since we find three zeros there:

$$\det(A) = \det \begin{pmatrix} 0 & 2 & 3 & 4 \\ 2 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 6 & 0 & 1 & 2 \end{pmatrix} = (-2) \cdot \det \begin{pmatrix} 2 & 3 & 4 \\ 1 & 0 & 0 \\ 0 & 1 & 2 \end{pmatrix} = (-2) \cdot (-1) \cdot \det \begin{pmatrix} 3 & 4 \\ 1 & 2 \end{pmatrix} = 4.$$

If C is the cofactor matrix of $A \in \mathbb{R}^{n \times n}$, then each entry of C is given by

$$c_{ij} = (-1)^{i+j} \det(A^{(i,j)}) = \det \left(\begin{array}{c|c|c|c} | & | & | & | \\ \mathbf{a}_1 & \dots & \mathbf{a}_{j-1} & \mathbf{e}_i & \mathbf{a}_{j+1} & \dots & \mathbf{a}_n \\ | & | & | & | \end{array} \right)$$

as one can easily see by the Laplace's formula. Now we can conclude:

Proposition 4.13.

Let $A \in \mathbb{R}^{n \times n}$ and C be its cofactor matrix. Then

$$C^T A = \det(A) \mathbb{1}_n.$$

In particular, if $\det(A) \neq 0$, then A is invertible and the inverse is given by

$$A^{-1} = \frac{C^T}{\det(A)}.$$

Proof. This is just a matrix multiplication where we consider the $(i, j)^{\text{th}}$ entry:

$$\begin{aligned} (C^T A)_{ij} &= \sum_{k=1}^n c_{ki} a_{kj} = \sum_{k=1}^n \det \left(\begin{array}{c|c|c|c|c} \mathbf{a}_1 & \dots & \mathbf{a}_{i-1} & \mathbf{e}_k & \mathbf{a}_{i+1} \dots \mathbf{a}_n \\ \hline & & & | & \\ & & & | & \\ & & & | & \\ & & & | & \end{array} \right) a_{kj} \\ &= \det \left(\begin{array}{c|c|c|c|c} \mathbf{a}_1 & \dots & \mathbf{a}_{i-1} & \mathbf{a}_j & \mathbf{a}_{i+1} \dots \mathbf{a}_n \\ \hline & & & | & \\ & & & | & \\ & & & | & \end{array} \right) \end{aligned}$$

This is $\det(A)$ for $i = j$ and otherwise just zero. \square

4.4 Important facts and using Gauß

Proposition 4.14.

If the square matrix $A \in \mathbb{R}^{n \times n}$ is in triangular form,

$$A = \begin{pmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ 0 & a_{2,2} & & a_{2,n} \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & a_{n,n} \end{pmatrix},$$

then the determinant is given by the multiplication of the diagonal:

$$\det(A) = a_{11}a_{22} \cdots a_{nn}.$$

Proof. Use Laplace's formula to the first column recursively. \square

Proposition 4.15. Determinants for block matrices

Let $A \in \mathbb{R}^{n \times n}$ and $C \in \mathbb{R}^{k \times k}$ two square matrices. For every matrix $B \in \mathbb{R}^{n \times k}$ define a so-called block matrix in triangular form

$$\begin{pmatrix} A & B \\ 0 & C \end{pmatrix} \in \mathbb{R}^{(n+k) \times (n+k)},$$

where 0 denotes the $(k \times n)$ zero matrix. Then, one has:

$$\det \begin{pmatrix} A & B \\ 0 & C \end{pmatrix} = \det(A) \det(C).$$

Proof. Exercise! Use the Leibniz formula. □

Proposition 4.16.

For each matrix $A \in \mathbb{R}^{n \times n}$, we have $\det(A^T) = \det(A)$.

Proof. We can use the definition by the Leibniz formula:

$$\det(A) = \sum_{\sigma \in \mathcal{P}_n} \operatorname{sgn}(\sigma) \prod_{i=1}^n a_{\sigma(i),i} = \sum_{\sigma \in \mathcal{P}_n} \operatorname{sgn}(\sigma) a_{\sigma(1),1} a_{\sigma(2),2} \cdots a_{\sigma(n),n} .$$

For the transpose A^T we find the following:

$$\det(A^T) = \sum_{\sigma \in \mathcal{P}_n} \operatorname{sgn}(\sigma) \prod_{i=1}^n a_{i,\sigma(i)} = \sum_{\sigma \in \mathcal{P}_n} \operatorname{sgn}(\sigma) a_{1,\sigma(1)} a_{2,\sigma(2)} \cdots a_{n,\sigma(n)} .$$

We only have to show that both sums consist of the same summands.

By definition of the signum function:

$$\operatorname{sgn}(\sigma \circ \omega) = \operatorname{sgn}(\sigma) \operatorname{sgn}(\omega)$$

From this we get:

$$\operatorname{sgn}(\sigma^{-1}) = \operatorname{sgn}(\sigma)$$

The multiplication is commutative and we can rearrange the product $a_{\sigma(1),1} a_{\sigma(2),2} \cdots a_{\sigma(n),n}$. Hence, we get

$$\operatorname{sgn}(\sigma) a_{\sigma(1),1} a_{\sigma(2),2} \cdots a_{\sigma(n),n} = \operatorname{sgn}(\sigma^{-1}) a_{1,\sigma^{-1}(1)} a_{2,\sigma^{-1}(2)} \cdots a_{n,\sigma^{-1}(n)},$$

Now we substitute ω for σ^{-1} and recognise that all summands are, in fact, the same. Here, it is important that \mathcal{P}_n is a so-called *group*, in which each element has exactly one inverse. Therefore, we can sum over ω instead of σ without changing anything. In summary, we have:

$$\sum_{\sigma \in \mathcal{P}_n} \operatorname{sgn}(\sigma) a_{\sigma(1),1} a_{\sigma(2),2} \cdots a_{\sigma(n),n} = \sum_{\omega \in \mathcal{P}_n} \operatorname{sgn}(\omega) a_{1,\omega(1)} a_{2,\omega(2)} \cdots a_{n,\omega(n)} . \quad \square$$

Proposition 4.17.

For $A, B \in \mathbb{R}^{n \times n}$, we have

$$\det(AB) = \det(A) \det(B).$$

In particular, if A is invertible, we have

$$\det(A^{-1}) = \frac{1}{\det(A)} \quad \text{and} \quad \det(A^{-1}BA) = \det(B)$$

Proof. Denote the column vectors of A by \mathbf{a}_j and the rows of B by $\boldsymbol{\beta}_j^T$, we can write the matrix product as $AB = \sum_j \mathbf{a}_j \boldsymbol{\beta}_j^T$ and get:

$$\det(AB) = \text{Vol}_n \left(\sum_{j_1} \mathbf{a}_{j_1} b_{j_1,1}, \dots, \sum_{j_n} \mathbf{a}_{j_n} b_{j_n,n} \right).$$

Now we can use the properties (1), (2), (3), and (4) the volume form has, see Definition 4.4. We get:

$$\begin{aligned} \det(AB) &= \sum_{j_1, \dots, j_n} b_{j_1,1} \cdots b_{j_n,n} \text{Vol}_n(\mathbf{a}_{j_1}, \dots, \mathbf{a}_{j_n}) \\ &= \sum_{\sigma \in \mathcal{P}_n} b_{\sigma(1),1} \cdots b_{\sigma(n),n} \text{Vol}_n(\mathbf{a}_{\sigma(1)}, \dots, \mathbf{a}_{\sigma(n)}) \\ &= \sum_{\sigma \in \mathcal{P}_n} b_{\sigma(1),1} \cdots b_{\sigma(n),n} \text{sgn}(\sigma) \text{Vol}_n(\mathbf{a}_1, \dots, \mathbf{a}_n) \\ &= \det(A) \sum_{\sigma \in \mathcal{P}_n} \text{sgn}(\sigma) b_{\sigma(1),1} \cdots b_{\sigma(n),n} = \det(A) \det(B). \end{aligned} \quad \square$$

The determinant function $\det : \mathbb{R}^{n \times n} \rightarrow \mathbb{R}$ is therefore *multiplicative*. This is what we can use for calculating determinants with the Gaussian elimination since this is nothing more than multiplying matrices from the left.

Corollary 4.18.

Using the row operations $Z_{i+\lambda j}$ (for $i \neq j$ and $\lambda \in \mathbb{R}$) do not change the determinant. The switching of rows only changes the sign of the determinant. However, scaling rows with a diagonal matrix D changes the determinant by the product of these scaling factors.

Since $\det(A^T) = \det(A)$, one can equivalently use column operations if you just want to calculate the determinant. It is not recommended in the case you need indeed the row echelon form in the end for other applications.

Rule of thumb: Using Gauß for determinants

For calculating $\det(A)$, you can add multiples of a row to another row or add multiples of a column to another column without changing the determinant. If you exchange two rows or two columns, you simply have to change the sign. Do not scale rows or columns since this changes the determinant.

In a formal way, we would say:

- Compute $PA = LU$, count the row permutations, to find either $\det(P) = +1$ or $\det(P) = -1$
- $\det(A) = \det(P) \det(U)$.

Example 4.19. We calculate the determinant of the following 5×5 matrix A . The third column already has three zeros but we can generate a fourth zero by using one Gaussian

step: Subtract the fifth row from the second one:

$$A = \begin{pmatrix} -1 & 1 & 0 & -2 & 0 \\ 0 & 2 & 1 & -1 & 4 \\ 1 & 0 & 0 & -3 & 1 \\ 1 & 2 & 0 & 0 & 3 \\ 0 & -2 & 1 & 1 & 2 \end{pmatrix} \sim \begin{pmatrix} -1 & 1 & 0 & -2 & 0 \\ 0 & 4 & 0 & -2 & 2 \\ 1 & 0 & 0 & -3 & 1 \\ 1 & 2 & 0 & 0 & 3 \\ 0 & -2 & 1 & 1 & 2 \end{pmatrix} =: B$$

Now, we know that $\det(A) = \det(B)$ holds. Having so many zeros, it is the best to expand $\det(B)$ along the third column:

$$\det(B) = \det \begin{pmatrix} -1 & 1 & 0 & -2 & 0 \\ 0 & 4 & 0 & -2 & 2 \\ 1 & 0 & 0 & -3 & 1 \\ 1 & 2 & 0 & 0 & 3 \\ 0 & -2 & 1 & 1 & 2 \end{pmatrix} = (-1)^{3+5} \cdot 1 \cdot \det \underbrace{\begin{pmatrix} -1 & 1 & -2 & 0 \\ 0 & 4 & -2 & 2 \\ 1 & 0 & -3 & 1 \\ 1 & 2 & 0 & 3 \end{pmatrix}}_{=:C}$$

Looking at C , we can use Gaussian elimination to get two more zeros in the second row. (Add the fourth column to the third column and subtract it from the second column two times):

$$C = \begin{pmatrix} -1 & 1 & -2 & 0 \\ 0 & 4 & -2 & 2 \\ 1 & 0 & -3 & 1 \\ 1 & 2 & 0 & 3 \end{pmatrix} \sim \begin{pmatrix} -1 & 1 & -2 & 0 \\ 0 & 0 & 0 & 2 \\ 1 & -2 & -2 & 1 \\ 1 & -4 & 3 & 3 \end{pmatrix} =: D$$

Of course, we have again: $\det(C) = \det(D)$. Now, $\det(D)$ should be expanded along the second row:

$$\det(D) = \det \begin{pmatrix} -1 & 1 & -2 & 0 \\ 0 & 0 & 0 & 2 \\ 1 & -2 & -2 & 1 \\ 1 & -4 & 3 & 3 \end{pmatrix} = (-1)^{2+4} \cdot 2 \cdot \det \underbrace{\begin{pmatrix} -1 & 1 & -2 \\ 1 & -2 & -2 \\ 1 & -4 & 3 \end{pmatrix}}_{=:E}.$$

Now, we only have a 3×3 matrix and use the formula of Sarrus:

$$\begin{aligned} \det(E) &= (-1) \cdot (-2) \cdot 3 + 1 \cdot (-2) \cdot 1 + (-2) \cdot 1 \cdot (-4) \\ &\quad - 1 \cdot (-2) \cdot (-2) - (-4) \cdot (-2) \cdot (-1) - 3 \cdot 1 \cdot 1 \\ &= 6 - 2 + 8 - 4 + 8 - 3 = 13 \end{aligned}$$

In summary:

$$\det(A) = \det(B) = 1 \cdot \det(C) = \det(D) = 2 \cdot \det(E) = 2 \cdot 13 = \underline{\underline{26}}.$$

Remark:

- $\det(A^{-1}) = \frac{1}{\det(A)}$ (if the inverse exists)
- If Q is an orthogonal matrix ($Q^T Q = \mathbb{1}$), then $\det(Q) = \pm 1$
- Let P be a row permutation matrix, then $\det(P) = 1$, if the number of row

exchanges is even, and $\det(P) = -1$ if it is odd.

- If $PA = LU$, then $\det(A) = \frac{1}{\det(P)} \det(L) \det(U) = \det(P) \det(U) = \pm \det(U)$.
- If $A = S^{-1}BS$, then $\det(A) = \frac{1}{\det(S)} \det(B) \det(S) = \det(B)$ (similar matrices have the same determinant).

Attention! Comparison: $n^3/3$ (Gauß) vs. $n!$ (Laplace/Leibniz formula)

n	2	3	4	5	6	7	8	9	10	...	20
$n^3/3$	2	9	21	42	72	114	171	243	333	...	2667
$n!$	2	6	24	120	720	5040	40320	362880	3628800	...	$2.4 \cdot 10^{18}$

4.5 Determinants for linear maps

- For each matrix A , there is the linear map $f_A : \mathbb{R}^n \rightarrow \mathbb{R}^n$.
- For each linear map $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$, there is exactly one matrix A such that $f = f_A$.
- The columns of A are then the images of the unit cube under f_A .
- Then $\det(A)$ is the *relative change of volume* (of the unit cube) caused by f_A .

Definition 4.20. Determinant for f_A

For a linear map $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$, we define

$$\det(f) := \det(A)$$

where A is the uniquely determined matrix with $f = f_A$.

In fact $\det(f)$ is the relative change of all volumes and we remind that we have the following:

Let $A, B \in \mathbb{R}^{n \times n}$. We have the formula:

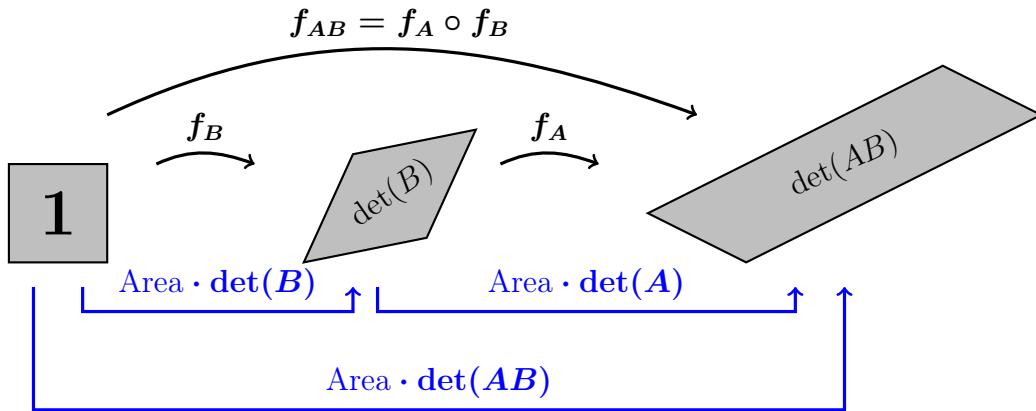
$$\det(f_A \circ f_B) = \det(f_A) \det(f_B)$$

In general, $\det(A) = \det(f_A)$ describes the change of volume for every figure:

$\det(A) = \text{factor for change of volume under } f_A$

$$\begin{array}{ccc} \text{figure } F & \xrightarrow{f_A} & \text{figure } F' = f_A(F) \\ & \underbrace{\text{volume} \cdot \det(A)} & \end{array}$$

For the composition, we get the following picture:



4.6 Determinants and systems of equations

Simple reasoning: if $\det(A) = 0$, then A is not invertible, and vice versa. A matrix with $\det(A) = 0$ is called singular. Thus, $\det(A)$ can be used to check, if a linear system $A\mathbf{x} = \mathbf{b}$ always has a unique solution (for right-hand sides \mathbf{b}).

This is helpful, if A depends on a parameter e.g. $A(\lambda)$.

Example 4.21.

$$A(\lambda) = \begin{pmatrix} \lambda & 1 & 2 \\ 1 & 2 & 3 \\ 1 & 1 & 2 \end{pmatrix} \quad \det(A(\lambda)) = \lambda(4 - 3) - 1(2 - 2) + 1(3 - 4) = \lambda - 1.$$

This matrix is singular if and only if $\lambda = 1$, and indeed, for $\lambda = 1$, we have for the column vectors $\mathbf{a}_1(\lambda) + \mathbf{a}_2 = \mathbf{a}_3$.

Conclusion: singular matrices do not appear very often. Whatever this means.

Warning: this is only good for pen-and-paper computations. In numerical computations, $\det(A + \text{round off})$ says *nothing* about invertibility of A , only about change of volume:

$$\det \begin{pmatrix} \varepsilon & 0 \\ 0 & 1/\varepsilon \end{pmatrix} = 1.$$

We summarise our knowledge:

Proposition 4.22. Nonsingular matrices and LES

Let $A \in \mathbb{R}^{n \times n}$. Then the following is equivalent

- (i) $\det(A) \neq 0$,
- (ii) the columns of A are linearly independent,
- (iii) the rows of A are linearly independent,
- (iv) $\text{rank}(A) = n$,

(v) A is invertible,

(vi) $A\mathbf{x} = \mathbf{b}$ has a unique solution for every $\mathbf{b} \in \mathbb{R}^n$,

(vii) $\text{Ker}(A) = \{\mathbf{0}\}$.

Proof. Exercise! □

4.7 Cramer's rule

Consider the linear system of equations, with full rank matrix A :

$$A\mathbf{x} = \mathbf{b}.$$

Then by our formula for the inverse we get:

$$A\mathbf{x} = \mathbf{b} \Rightarrow \mathbf{x} = A^{-1}\mathbf{b} = \frac{C^T\mathbf{b}}{\det(A)}.$$

This let us say the following about the components of a solution:

Proposition 4.23. Cramer's rule

Let $A \in \mathbb{R}^{n \times n}$ invertible and $\mathbf{b} \in \mathbb{R}^n$. Then the unique solution $\mathbf{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{R}^n$ of the LES $A\mathbf{x} = \mathbf{b}$ is given by:

$$x_i = \frac{\det \left(\begin{array}{|c|c|c|c|c|} \hline \mathbf{a}_1 & \dots & \mathbf{a}_{i-1} & \mathbf{b} & \mathbf{a}_{i+1} & \dots & \mathbf{a}_n \\ \hline \end{array} \right)}{\underbrace{\det \left(\begin{array}{|c|c|c|c|c|} \hline \mathbf{a}_1 & \dots & \mathbf{a}_{i-1} & \mathbf{a}_i & \mathbf{a}_{i+1} & \dots & \mathbf{a}_n \\ \hline \end{array} \right)}_{A}} \quad \text{for } i = 1, \dots, n.$$

Proof. Having the cofactor matrix C , we already know that the solution is given by

$$\mathbf{x} = A^{-1}\mathbf{b} = \frac{C^T\mathbf{b}}{\det(A)}$$

Therefore, we just have to look at the i th row of the matrix $C^T\mathbf{b}$ which is given by:

$$\begin{aligned} (C^T\mathbf{b})_i &= \sum_{k=1}^n c_{ki} b_k = \sum_{k=1}^n \det \left(\begin{array}{|c|c|c|c|c|} \hline \mathbf{a}_1 & \dots & \mathbf{a}_{i-1} & \mathbf{e}_k & \mathbf{a}_{i+1} & \dots & \mathbf{a}_n \\ \hline \end{array} \right) b_k \\ &= \det \left(\begin{array}{|c|c|c|c|c|} \hline \mathbf{a}_1 & \dots & \mathbf{a}_{i-1} & \mathbf{b} & \mathbf{a}_{i+1} & \dots & \mathbf{a}_n \\ \hline \end{array} \right) \end{aligned}$$
□

Attention! Do not use Cramer's rule to solve a system $Ax=b$!

Cramer's rule is less efficient than Gaussian elimination. That is noticeable for large matrices.

If you calculate the determinants by Laplace, then your work is of order $n!$. If you use Gaussian elimination for calculating the determinants, you only need $\frac{n^3}{3}$ steps for each component of \mathbf{x} , but, of course in this case, you could have solved the whole system $A\mathbf{x} = \mathbf{b}$ by using Gaussian elimination.

For computational reasons the Cramer's rule can only be used for small matrices, but the real advantage is the theoretical interest. You can use Cramer's rule in proofs if you need claims about a single component x_i of the solution \mathbf{x} .

Summary

- The determinant is the volume form.
- The determinant fulfills three defining properties:
 - (1) Linear in each column.
 - (2) Alternating when exchanging columns.
 - (3) The identity matrix has determinant 1.
- To calculate a determinant, you have the Leibniz formula, the Laplace expansion or Gaussian elimination (without scaling!).

5

General inner products, orthogonality and distances

Sadly, very little school maths focuses on how to win free drinks in a pub.

Matt Parker

We have already encountered the standard inner product (also called Euclidean scalar product) in \mathbb{R}^n :

$$\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x}^T \mathbf{y} = \sum_{i=1}^n x_i y_i, \quad \text{for } \mathbf{x}, \mathbf{y} \in \mathbb{R}^n$$

With the help of this inner product, we are able to define and compute many useful things:

- length: $\|\mathbf{x}\| = \sqrt{\langle \mathbf{x}, \mathbf{x} \rangle}$
- distances: $\text{dist}(\mathbf{x}, \mathbf{y}) = \|\mathbf{x} - \mathbf{y}\|$
- angle: $\cos(\angle(\mathbf{x}, \mathbf{y})) := \frac{\langle \mathbf{x}, \mathbf{y} \rangle}{\|\mathbf{x}\| \|\mathbf{y}\|}$
- orthogonality: $\mathbf{x} \perp \mathbf{y} : \Leftrightarrow \langle \mathbf{x}, \mathbf{y} \rangle = 0$.
- orthogonal projections, e.g. the height
- rotations about an axis by an angle
- reflections at a hyperplane

5.1 General inner products in \mathbb{R}^n

The following properties capture everything that is needed to measure angles and lengths in a useful way:

Definition 5.1. Inner product

Let V be \mathbb{R}^n or a subspace of \mathbb{R}^n . We call a map of two arguments $\langle \cdot, \cdot \rangle : V \times V \rightarrow \mathbb{R}$ an inner product if it satisfies for all $\mathbf{x}, \mathbf{y}, \mathbf{v} \in V$ and $\lambda \in \mathbb{R}$:

(S1) Positive definiteness:

$$\langle \mathbf{x}, \mathbf{x} \rangle > 0 \text{ for } \mathbf{x} \neq \mathbf{0}$$

(S2) Additivity in the first argument:

$$\langle \mathbf{x} + \mathbf{y}, \mathbf{v} \rangle = \langle \mathbf{x}, \mathbf{v} \rangle + \langle \mathbf{y}, \mathbf{v} \rangle$$

(S3) Homogeneity in the first argument:

$$\langle \lambda \mathbf{x}, \mathbf{v} \rangle = \lambda \langle \mathbf{x}, \mathbf{v} \rangle$$

(S4) Symmetry:

$$\langle \mathbf{x}, \mathbf{y} \rangle = \langle \mathbf{y}, \mathbf{x} \rangle$$

We usually summarise (S2) and (S3) to *linearity in the first argument*. Note that from (S3) always follows $\langle \mathbf{o}, \mathbf{o} \rangle = 0 \cdot \langle \mathbf{o}, \mathbf{o} \rangle = 0$. In combination with (S1), we get:

$$\langle \mathbf{x}, \mathbf{x} \rangle = 0 \Leftrightarrow \mathbf{x} = \mathbf{o}. \quad (5.1)$$

Also, due to positive definiteness, we can define a norm (or length) via

$$\|\mathbf{x}\| := \sqrt{\langle \mathbf{x}, \mathbf{x} \rangle}.$$

We call it the the associated norm with respect to $\langle \cdot, \cdot \rangle$.

- Inner products are also linear in the second argument, by symmetry.
- Later, we will define complex-valued inner products that fulfil instead of (S4):

$$\langle \mathbf{x}, \mathbf{y} \rangle = \overline{\langle \mathbf{y}, \mathbf{x} \rangle}. \quad (5.2)$$

Then the second argument actually has different properties than the first.

- In the usual real case, the binomial formulas hold:

$$\begin{aligned} \|\mathbf{x} \pm \mathbf{y}\|^2 &= \|\mathbf{x}\|^2 \pm 2\langle \mathbf{x}, \mathbf{y} \rangle + \|\mathbf{y}\|^2 \\ \langle \mathbf{x} + \mathbf{y}, \mathbf{x} - \mathbf{y} \rangle &= \|\mathbf{x}\|^2 - \|\mathbf{y}\|^2. \end{aligned}$$

Example 5.2. The standard inner product on \mathbb{R}^n :

$$\langle \mathbf{x}, \mathbf{y} \rangle_{euklid} := \mathbf{x}^T \mathbf{y} = \sum_{i=1}^n x_i y_i.$$

In contrast to Section 2.5, we now use a subscript to denote this special inner product. If there is no confusion which inner product we use, we can omit the index.

Remark:

Due to its simplicity, this inner product is prominent in theory and practice. However, in particular for very large scale problems with special structure other “specially tailored” inner products play a major role.

Definition 5.3. Positive definite matrix

A matrix $A \in \mathbb{R}^{n \times n}$ is called positive definite if it is symmetric ($A^T = A$) and satisfies

$$\langle \mathbf{x}, A\mathbf{x} \rangle_{euklid} = \mathbf{x}^T A \mathbf{x} > 0$$

for all $\mathbf{x} \neq \mathbf{o}$.

Example 5.4. Each diagonal matrix $D \in \mathbb{R}^{n \times n}$ with positive entries on the diagonal is a positive definite matrix.

Let $A \in \mathbb{R}^{n \times n}$ be a positive definite matrix. Then the following defines an inner product on \mathbb{R}^n :

$$\langle \mathbf{x}, \mathbf{y} \rangle_A := \langle \mathbf{x}, A\mathbf{y} \rangle_{euklid} = \mathbf{x}^T A \mathbf{y}$$

It is linear in the first argument, by the linearity of the matrix- (row) vector product, symmetric by symmetry of A ($a_{ij} = a_{ji}$, or $A = A^T$) and positive definite by positive definiteness of A .

The simplest case is $A = \mathbf{1}$, so $\langle \mathbf{x}, \mathbf{y} \rangle_{\mathbf{1}} = \mathbf{x}^T \mathbf{y}$ is the standard euclidean product.

A simple case, where $A = D$ is a diagonal matrix makes sense, if \mathbb{R}^3 corresponds to spatial coordinates, given in different units (say inch and centimeters).

Our abstract assumptions already yield all the useful formulas, known from our standard inner product:

Proposition 5.5.

Let $\langle \cdot, \cdot \rangle$ be an inner product on a subspace $V \subset \mathbb{R}^n$ and $\|\cdot\|$ its associated norm. Then for all $\mathbf{x}, \mathbf{y} \in V$ and $\lambda \in \mathbb{R}$, we have:

- (a) $|\langle \mathbf{x}, \mathbf{y} \rangle| \leq \|\mathbf{x}\| \|\mathbf{y}\|$ (Cauchy-Schwarz inequality). Equality holds if and only if \mathbf{x} and \mathbf{y} are colinear (written as $\mathbf{x} \parallel \mathbf{y}$).
- (b) The norm fulfils three properties:
 - (N1) $\|\mathbf{x}\| \geq 0$, and $\|\mathbf{x}\| = 0 \Leftrightarrow \mathbf{x} = \mathbf{o}$,
 - (N2) $\|\lambda \mathbf{x}\| = |\lambda| \|\mathbf{x}\|$,
 - (N3) $\|\mathbf{x} + \mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\|$.

Proof. We show the Cauchy-Schwarz inequality (CSI) in a short proof. Let $\mathbf{y} \neq \mathbf{o}$, otherwise the CSI reads $0 = 0$.

For any $\lambda \in \mathbb{R}$ the binomial formula yields:

$$0 \leq \|\mathbf{x} - \lambda \mathbf{y}\|^2 \|\mathbf{y}\|^2 = \|\mathbf{x}\|^2 \|\mathbf{y}\|^2 - 2\lambda \langle \mathbf{x}, \mathbf{y} \rangle \|\mathbf{y}\|^2 + \lambda^2 \|\mathbf{y}\|^4.$$

(This is zero, if $\mathbf{y} = \mathbf{o}$, or $\mathbf{x} = \lambda \mathbf{y}$, i.e. \mathbf{x} and \mathbf{y} are colinear). Setting $\lambda = \langle \mathbf{x}, \mathbf{y} \rangle / \|\mathbf{y}\|^2$, we obtain

$$0 \leq \|\mathbf{x}\|^2 \|\mathbf{y}\|^2 - 2\langle \mathbf{x}, \mathbf{y} \rangle^2 + \langle \mathbf{x}, \mathbf{y} \rangle^2 = \|\mathbf{x}\|^2 \|\mathbf{y}\|^2 - \langle \mathbf{x}, \mathbf{y} \rangle^2.$$

The norm properties are left as an exercise. □

No matter which inner product we are using, we can define orthogonality as follows:

$$\mathbf{x} \perp \mathbf{y} \iff \langle \mathbf{x}, \mathbf{y} \rangle = 0.$$

A norm for matrices

Once we can measure the size of a vector \mathbf{v} by a norm $\|\mathbf{v}\|$, we may think about measuring the “size” of a linear map. Consider $A \in \mathbb{R}^{m \times n}$, and $\mathbf{w} = A\mathbf{v}$. Then the following quotient

$$\frac{\|\mathbf{w}\|_{\mathbb{R}^m}}{\|\mathbf{v}\|_{\mathbb{R}^n}} = \frac{\|A\mathbf{v}\|_{\mathbb{R}^m}}{\|\mathbf{v}\|_{\mathbb{R}^n}}$$

tells us, how much longer (or shorter) $\mathbf{w} = A\mathbf{v}$ is, compared to \mathbf{v} . A should be “large”, if it produces long vectors from short ones, and “small”, if it produces short vectors from long ones. Thus, we may define

$$\|A\| := \max_{\mathbf{v} \neq 0} \frac{\|A\mathbf{v}\|_{\mathbb{R}^m}}{\|\mathbf{v}\|_{\mathbb{R}^n}},$$

and call it the matrix norm of A . Hence, we have:

$$\|\mathbf{w}\|_{\mathbb{R}^m} = \|A\mathbf{v}\|_{\mathbb{R}^m} \leq \|A\| \|\mathbf{v}\|_{\mathbb{R}^n}.$$

It is not easy to compute this norm. We will consider a possibility later.

5.2 Orthogonal projections

In this section $\langle \cdot, \cdot \rangle$ denotes an arbitrary inner product in \mathbb{R}^n .

5.2.1 Orthogonal projection onto a line

Imagine you ride a rowboat on a river. You want to go in a direction $\mathbf{r} \neq \mathbf{o}$. However water flows in direction \mathbf{x} , which is not parallel to \mathbf{r} .

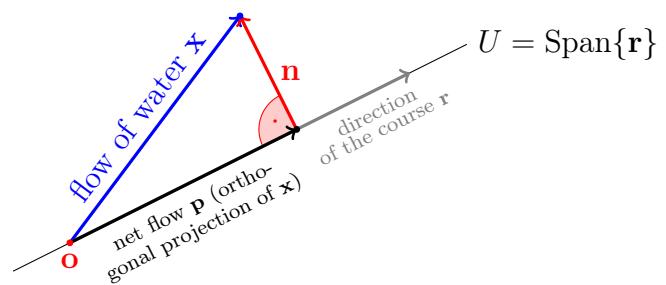
The steersman asks:

What is the component of \mathbf{x} with respect to the wanted direction \mathbf{r} ?

In mathematical language: Write the vector \mathbf{x} in a linear combination

$$\mathbf{x} = \mathbf{p} + \mathbf{n}$$

consisting of two orthogonal vectors: \mathbf{p} is parallel to the wanted direction \mathbf{r} and \mathbf{n} is orthogonal to this.



Definition 5.6. Orthogonal projection onto a line

Let $\langle \cdot, \cdot \rangle$ be an inner product in \mathbb{R}^n and $U := \text{Span}(\mathbf{r})$ for $\mathbf{r} \neq \mathbf{o}$. For a decomposition $\mathbf{x} = \mathbf{p} + \mathbf{n}$ for a vector $\mathbf{x} \in \mathbb{R}^n$ into two orthogonal vectors $\mathbf{p} \in U$ and $\mathbf{n} \perp \mathbf{r}$, we call \mathbf{p} the orthogonal projection of \mathbf{x} onto U , and \mathbf{n} is called the normal component of \mathbf{x} with respect to U .

The orthogonal projection and normal component is indeed well-defined. If there are two decompositions $\mathbf{x} = \mathbf{p} + \mathbf{n} = \mathbf{p}' + \mathbf{n}'$ with $\mathbf{p}, \mathbf{p}' \in U$ and $\mathbf{n}, \mathbf{n}' \in U^\perp$, then we can use the subspace properties to conclude $\mathbf{n} - \mathbf{n}' \in U^\perp$ and $\mathbf{p} - \mathbf{p}' \in U$. Applying the inner product onto $\mathbf{p} - \mathbf{p}' = \mathbf{n}' - \mathbf{n}$, we conclude $\|\mathbf{p} - \mathbf{p}'\| = 0$ and $\|\mathbf{n}' - \mathbf{n}\| = 0$. From the norm properties, we get $\mathbf{n} = \mathbf{n}'$ and $\mathbf{p} = \mathbf{p}'$.

Calculation of \mathbf{p} and \mathbf{n} : Because of $\mathbf{p} \in U = \text{Span}(\mathbf{r})$, we have $\mathbf{p} = \lambda \mathbf{r}$ for a $\lambda \in \mathbb{R}$, which we simply have to find. Since $\mathbf{x} = \mathbf{p} + \mathbf{n} = \lambda \mathbf{r} + \mathbf{n}$ and $\mathbf{n} \perp \mathbf{r}$, we get:

$$\langle \mathbf{x}, \mathbf{r} \rangle = \underbrace{\langle \lambda \mathbf{r} + \mathbf{n}, \mathbf{r} \rangle}_{\mathbf{p}} = \langle \lambda \mathbf{r}, \mathbf{r} \rangle + \underbrace{\langle \mathbf{n}, \mathbf{r} \rangle}_0 = \lambda \langle \mathbf{r}, \mathbf{r} \rangle \quad \text{and therefore} \quad \lambda = \frac{\langle \mathbf{x}, \mathbf{r} \rangle}{\langle \mathbf{r}, \mathbf{r} \rangle}.$$

The case $\mathbf{r} = \mathbf{o}$ (i.e. $\mathbf{p} = \mathbf{o}$ and $\mathbf{n} = \mathbf{x}$) is omitted here. In summary, we get:

Proposition 5.7. Orth. projection & and normal component w.r.t a line

Let $\mathbf{x}, \mathbf{r} \in \mathbb{R}^n$ with $\mathbf{r} \neq \mathbf{o}$. For the orthogonal projection \mathbf{p} of \mathbf{x} onto $U = \text{Span}(\mathbf{r})$ and the associated normal component \mathbf{n} of \mathbf{x} w.r.t. U , one finds:

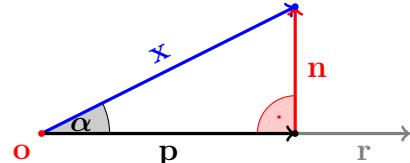
$$\mathbf{p} = \frac{\langle \mathbf{x}, \mathbf{r} \rangle}{\langle \mathbf{r}, \mathbf{r} \rangle} \mathbf{r} \quad \text{and} \quad \mathbf{n} = \mathbf{x} - \mathbf{p} = \mathbf{x} - \frac{\langle \mathbf{x}, \mathbf{r} \rangle}{\langle \mathbf{r}, \mathbf{r} \rangle} \mathbf{r}.$$

Proof. Obviously, $\mathbf{p} \in U$ and calculation shows $\mathbf{n} \in U^\perp$. □

Rule of thumb: $\|\cdot\|$ gives length and $\langle \cdot, \cdot \rangle$ gives an angle

Geometrically $\|\mathbf{x}\|$ is seen as a **length** of the vector \mathbf{x} . The inner product $\langle \mathbf{x}, \mathbf{y} \rangle$ gives back the **angle** between \mathbf{x} and \mathbf{y} .

To define a meaningful angle between vectors, we again look at the triangle, given by the vectors \mathbf{x} , \mathbf{p} and \mathbf{n} . It is right-angled since $\mathbf{p} \perp \mathbf{n}$ is our definition of 90 degree. The angle between \mathbf{x} and \mathbf{r} is called α in the picture.



If α is an acute angle, i.e. $\alpha \in [0, \pi/2]$, then $\lambda \geq 0$ and:

$$\cos(\alpha) \|\mathbf{x}\| = \|\mathbf{p}\| = \|\lambda \mathbf{r}\| = |\lambda| \|\mathbf{r}\| \stackrel{\lambda \geq 0}{=} \lambda \|\mathbf{r}\| = \frac{\langle \mathbf{x}, \mathbf{r} \rangle}{\langle \mathbf{r}, \mathbf{r} \rangle} \|\mathbf{r}\| = \frac{\langle \mathbf{x}, \mathbf{r} \rangle}{\|\mathbf{r}\|^2} \|\mathbf{r}\| = \frac{\langle \mathbf{x}, \mathbf{r} \rangle}{\|\mathbf{r}\|},$$

We reformulate this:

$$\langle \mathbf{x}, \mathbf{r} \rangle = \|\mathbf{x}\| \|\mathbf{r}\| \cos(\alpha).$$

If α is not acute, we can do an analogue calculation. In summary, we can give the following definition for an angle:

Definition 5.8. Angle between two vectors in \mathbb{R}^n

For two vectors $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n \setminus \{\mathbf{0}\}$ we write $\text{angle}(\mathbf{x}, \mathbf{y})$ for the angle $\alpha \in [0, \pi]$ between \mathbf{x} and \mathbf{y} , which is defined by

$$\cos(\alpha) = \frac{\langle \mathbf{x}, \mathbf{y} \rangle}{\|\mathbf{x}\| \|\mathbf{y}\|}. \quad (5.3)$$

Using Proposition 5.5 (Cauchy-Schwarz-inequality), we conclude that the angle is well-defined:

$$\frac{|\langle \mathbf{x}, \mathbf{y} \rangle|}{\|\mathbf{x}\| \|\mathbf{y}\|} \leq 1 \quad \text{and hence} \quad -1 \leq \frac{\langle \mathbf{x}, \mathbf{y} \rangle}{\|\mathbf{x}\| \|\mathbf{y}\|} \leq 1.$$

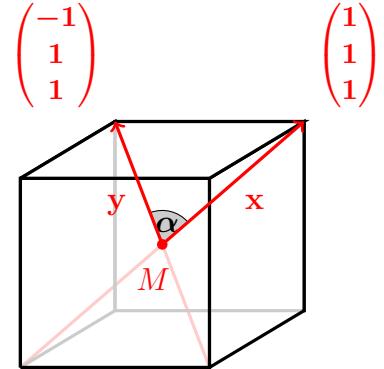
This means the right-hand side of (5.3) is indeed in the range of the cos function. Restricted to $\alpha \in [0, \pi]$, we know that $\cos(\alpha)$ is bijective, and hence, α is well-defined by equation (5.3).

Example 5.9. Consider the cube C in \mathbb{R}^3 with center M in the origin and the corners $(\pm 1, \pm 1, \pm 1)^T$, where all the combinations with \pm -signs occur.

All diagonals of C go through M and intersect with an angle α , which is calculated with the vectors $\mathbf{x} = (1, 1, 1)^T$ and $\mathbf{y} = (-1, 1, 1)^T$:

$$\cos(\alpha) = \frac{\langle \mathbf{x}, \mathbf{y} \rangle_{\text{euklid}}}{\|\mathbf{x}\| \|\mathbf{y}\|} = \frac{-1 + 1 + 1}{\sqrt{1+1+1} \sqrt{1+1+1}} = \frac{1}{3},$$

which implies $\alpha = \arccos(\frac{1}{3}) \approx 70.53^\circ$.



5.2.2 Orthogonal projection onto a subspace

Before we projected a vector onto a line, which is just a 1-dimensional subspace of \mathbb{R}^n . Now we generalise this procedure for arbitrary subspaces of \mathbb{R}^n . In order to do this, we recall the concept of orthogonal complements:

Definition 5.10. Orthogonal complement M^\perp

Let $M \subset \mathbb{R}^n$ be nonempty. Then we call

$$M^\perp := \{\mathbf{x} \in \mathbb{R}^n : \langle \mathbf{x}, \mathbf{m} \rangle = 0 \text{ for all } \mathbf{m} \in M\}$$

the orthogonal complement for M . Instead of $\mathbf{x} \in M^\perp$, we often write $\mathbf{x} \perp M$.

Example 5.11. Consider $\langle \cdot, \cdot \rangle_{\text{euklid}}$ the standard inner product in \mathbb{R}^n .

(a) For $M = \{\mathbf{0}\}$ in \mathbb{R}^n , we have $M^\perp = \mathbb{R}^n$.

- (b) For $M = \{\mathbf{e}_1\}$ in \mathbb{R}^2 , we have $M^\perp = \text{Span}(\mathbf{e}_2) \subset \mathbb{R}^2$.
- (c) For $M = \{3\mathbf{e}_1\}$ in \mathbb{R}^2 , we have $M^\perp = \text{Span}(\mathbf{e}_2) \subset \mathbb{R}^2$.
- (d) For $M = \{\mathbf{e}_1, 3\mathbf{e}_1\}$ in \mathbb{R}^2 , we have $M^\perp = \text{Span}(\mathbf{e}_2) \subset \mathbb{R}^2$.
- (e) For $M = \text{Span}(\mathbf{e}_1)$ in \mathbb{R}^2 , we have $M^\perp = \text{Span}(\mathbf{e}_2) \subset \mathbb{R}^2$.
- (f) For $M = \{\mathbf{e}_1, \mathbf{e}_2\}$ or $M = \text{Span}(\mathbf{e}_1, \mathbf{e}_2)$ in \mathbb{R}^3 , we have $M^\perp = \text{Span}(\mathbf{e}_3) \subset \mathbb{R}^3$.
- (g) For $\mathbf{n} \in \mathbb{R}^3 \setminus \{\mathbf{o}\}$, we have $\{\mathbf{n}\}^\perp$ the plane \mathbb{R}^3 through 0 with normal vector \mathbf{n} .
- (h) For $\mathbf{n} \in \mathbb{R}^3 \setminus \{\mathbf{o}\}$ and $\mathbf{p} \in \mathbb{R}^3$, we have $\mathbf{p} + \{\mathbf{n}\}^\perp$, the plane \mathbb{R}^3 through \mathbf{p} with normal vector \mathbf{n} (this is an affine space $\{\mathbf{x} \in \mathbb{R}^3 : \langle \mathbf{x} - \mathbf{p}, \mathbf{n} \rangle = 0\}$).
- (i) For $M = \text{Span}(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_5)$ in \mathbb{R}^5 , we have $M^\perp = \text{Span}(\mathbf{e}_3, \mathbf{e}_4) \subset \mathbb{R}^5$.

Proposition 5.12.

For all nonempty sets $M \subset \mathbb{R}^n$ we have:

- (a) $M^\perp = (\text{Span}(M))^\perp$,
- (b) M^\perp is a linear subspace of \mathbb{R}^n .

Proof. Exercise! □

We state one important property of the orthogonal complement. Other important one, you find at the end of this section.

Proposition 5.13. Properties of U^\perp

For a linear subspace $U \subset \mathbb{R}^n$, we have $U \cap U^\perp = \{\mathbf{o}\}$.

Proof. (a) For $\mathbf{x} \in U \cap U^\perp$, we have $\mathbf{x} \perp \mathbf{x}$, i.e. $\langle \mathbf{x}, \mathbf{x} \rangle = 0$. Using equation (5.1), we get $\mathbf{x} = \mathbf{o}$. □

Proposition 5.14. Orthogonal to a basis

Let U be a linear subspace of \mathbb{R}^n and $\mathcal{B} = (\mathbf{u}_1, \dots, \mathbf{u}_k)$ a basis of U . Then for all $\mathbf{x} \in \mathbb{R}^n$ we have:

$$\mathbf{x} \perp U \iff \mathbf{x} \perp \mathcal{B}.$$

In other words: \mathbf{x} is orthogonal to all vectors in U if and only if it is orthogonal to the basis vectors of U .

Proof. \Rightarrow If $\mathbf{x} \perp \mathbf{u}$ holds for all $\mathbf{u} \in U$, then, of course, also for all basis elements $\mathbf{u} = \mathbf{u}_i \in \mathcal{B} \subset U$.

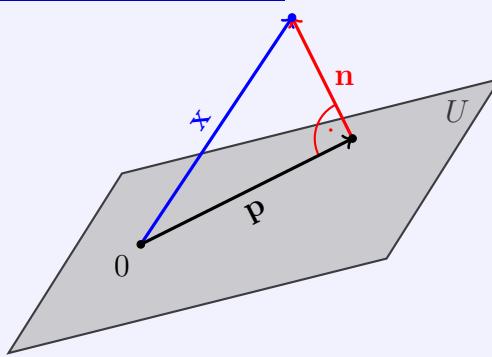
\Leftarrow If $\langle \mathbf{x}, \mathbf{u}_i \rangle = 0$ holds for $i = 1, \dots, k$ and we have $\mathbf{u} = \alpha_1 \mathbf{u}_1 + \dots + \alpha_k \mathbf{u}_k \in U$, then $\langle \mathbf{x}, \mathbf{u} \rangle = \langle \mathbf{x}, \alpha_1 \mathbf{u}_1 + \dots + \alpha_k \mathbf{u}_k \rangle = \alpha_1 \langle \mathbf{x}, \mathbf{u}_1 \rangle + \dots + \alpha_k \langle \mathbf{x}, \mathbf{u}_k \rangle = \alpha_1 0 + \dots + \alpha_k 0 = 0$. □

Definition 5.15. Orthogonal projection onto a subspace U

Let U be a linear subspace of \mathbb{R}^n and also let $\mathbf{x} \in \mathbb{R}^n$. Again, we search for a decomposition:

$$\mathbf{p} \in U \quad \text{and} \quad \mathbf{n} \perp U \quad \text{with} \quad \mathbf{x} = \mathbf{p} + \mathbf{n}.$$

In other words, we write \mathbf{x} as a sum of two vectors, where one lies in U and the other one is orthogonal to U .



The (uniquely determined) vector \mathbf{p} is called the orthogonal projection of \mathbf{x} onto U , and \mathbf{n} is called the normal component of \mathbf{x} w.r.t. U .

For the orthogonal projection \mathbf{p} of \mathbf{x} onto U , we often simply write $\mathbf{x}|_U$. In this notation, the decomposition $\mathbf{x} = \mathbf{p} + \mathbf{n}$ gets:

$$\mathbf{x} = \mathbf{x}|_U + \mathbf{x}|_{U^\perp}, \quad \text{i.e.} \quad \mathbf{p} = \mathbf{x}|_U \quad \text{and} \quad \mathbf{n} = \mathbf{x}|_{U^\perp}$$

Calculation of the orthogonal projection $\mathbf{x}|_U$: It works exactly the same as in the one-dimensional case given in Section 5.2.1. The only difference is that U is spanned by $k \geq 1$ vectors. Hence, choose a basis $\mathcal{B} = (\mathbf{u}_1, \dots, \mathbf{u}_k)$ of U . Then we can rewrite $\mathbf{p} = \mathbf{x}|_U \in U$ as $\mathbf{x}|_U = \alpha_1 \mathbf{u}_1 + \dots + \alpha_k \mathbf{u}_k$ with coefficients $\alpha_1, \dots, \alpha_k \in \mathbb{R}$, we now have to determine. For $\mathbf{n} = \mathbf{x}|_{U^\perp}$, we only need the information $\mathbf{n} \perp U$.

To find $\alpha_1, \dots, \alpha_k$ we just consider the inner product of

$$\mathbf{x} = \mathbf{x}|_U + \mathbf{n} = (\alpha_1 \mathbf{u}_1 + \dots + \alpha_k \mathbf{u}_k) + \mathbf{n}$$

with respect to all k basis vectors of U : For $i = 1, \dots, k$, we have

$$\langle \mathbf{x}, \mathbf{u}_i \rangle = \langle \alpha_1 \mathbf{u}_1 + \dots + \alpha_k \mathbf{u}_k + \mathbf{n}, \mathbf{u}_i \rangle = \alpha_1 \langle \mathbf{u}_1, \mathbf{u}_i \rangle + \dots + \alpha_k \langle \mathbf{u}_k, \mathbf{u}_i \rangle + \underbrace{\langle \mathbf{n}, \mathbf{u}_i \rangle}_0. \quad (5.4)$$

Now we have k equations and k unknowns $\alpha_1, \dots, \alpha_k$:

Proposition 5.16. Calculating the projection $\mathbf{x}|_U$

Let $\mathbf{x} \in \mathbb{R}^n$ and U be a linear subspace of \mathbb{R}^n where $\mathcal{B} = (\mathbf{u}_1, \dots, \mathbf{u}_k)$ is a basis of U . Then we get the orthogonal projection

$$\mathbf{x}|_U = \alpha_1 \mathbf{u}_1 + \dots + \alpha_k \mathbf{u}_k,$$

where $\alpha_1, \dots, \alpha_k$ are given by the (unique) solution of the LES:

$$\begin{pmatrix} \langle \mathbf{u}_1, \mathbf{u}_1 \rangle & \langle \mathbf{u}_2, \mathbf{u}_1 \rangle & \dots & \langle \mathbf{u}_k, \mathbf{u}_1 \rangle \\ \langle \mathbf{u}_1, \mathbf{u}_2 \rangle & \langle \mathbf{u}_2, \mathbf{u}_2 \rangle & \dots & \langle \mathbf{u}_k, \mathbf{u}_2 \rangle \\ \vdots & \vdots & & \vdots \\ \langle \mathbf{u}_1, \mathbf{u}_k \rangle & \langle \mathbf{u}_2, \mathbf{u}_k \rangle & \dots & \langle \mathbf{u}_k, \mathbf{u}_k \rangle \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_k \end{pmatrix} = \begin{pmatrix} \langle \mathbf{x}, \mathbf{u}_1 \rangle \\ \langle \mathbf{x}, \mathbf{u}_2 \rangle \\ \vdots \\ \langle \mathbf{x}, \mathbf{u}_k \rangle \end{pmatrix}. \quad (5.5)$$

The $(k \times k)$ matrix on the left-hand side is called the Gramian matrix $G(\mathcal{B})$. The normal component $\mathbf{n} = \mathbf{x}|_{U^\perp}$ is then given by $\mathbf{n} = \mathbf{x} - \mathbf{x}|_U$.

Proof. The result for α_i follows from equation (5.4). Now we show that the Gramian matrix $G(\mathcal{B}) =: G$ is invertible. This means that we have to show $\text{Ker}(G) = \{\mathbf{o}\}$. Let $(\beta_1, \dots, \beta_k)^T \in \text{Ker}(G)$. Then for all $i = 1, \dots, k$, we have:

$$0 = \beta_1 \langle \mathbf{u}_1, \mathbf{u}_i \rangle + \dots + \beta_k \langle \mathbf{u}_k, \mathbf{u}_i \rangle = \langle \beta_1 \mathbf{u}_1 + \dots + \beta_k \mathbf{u}_k, \mathbf{u}_i \rangle.$$

Hence, $\mathbf{u} := \beta_1 \mathbf{u}_1 + \dots + \beta_k \mathbf{u}_k$ is orthogonal to all \mathbf{u}_i . Using Proposition 5.14, we get $\mathbf{u} \in U^\perp$. Per construction, we get $\mathbf{u} \in U$, which means $\mathbf{u} \in U \cap U^\perp$. Using Proposition 5.13, we conclude $\mathbf{u} = \mathbf{o}$. The family $(\mathbf{u}_1, \dots, \mathbf{u}_k)$ is linearly independent since it is a basis. So $\mathbf{u} = \mathbf{o}$ implies $\beta_1 = \dots = \beta_k = 0$. \square

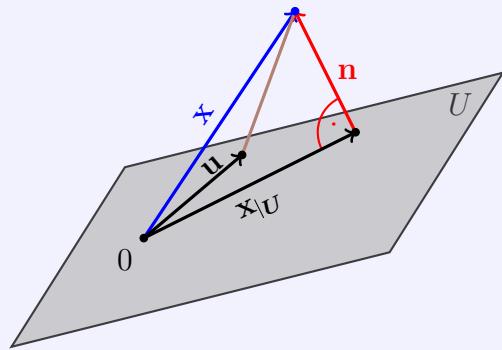
Proposition 5.17. Approximation formula

Let $\mathbf{x} \in \mathbb{R}^n$ and U be a linear subspace of \mathbb{R}^n .

The orthogonal projection $\mathbf{x}|_U$ minimises the distance between \mathbf{x} and the subspace U :

$$\|\underbrace{\mathbf{x} - \mathbf{x}|_U}_{\mathbf{n}}\| = \min_{\mathbf{u} \in U} \|\mathbf{x} - \mathbf{u}\| =: \text{dist}(\mathbf{x}, U)$$

In other words: No other vector of U is as closed to \mathbf{x} as $\mathbf{x}|_U$.



Proof. For all $\mathbf{u} \in U$, we get

$$\|\mathbf{x} - \mathbf{u}\|^2 = \left\| \underbrace{(\mathbf{x} - \mathbf{x}|_U)}_{\mathbf{n}} + \underbrace{(\mathbf{x}|_U - \mathbf{u})}_{=: \mathbf{v}} \right\|^2 = \langle \mathbf{n} + \mathbf{v}, \mathbf{n} + \mathbf{v} \rangle = \underbrace{\langle \mathbf{n}, \mathbf{n} \rangle}_{\|\mathbf{n}\|^2} + 2 \underbrace{\langle \mathbf{n}, \mathbf{v} \rangle}_{0} + \underbrace{\langle \mathbf{v}, \mathbf{v} \rangle}_{\geq 0} \geq \|\mathbf{n}\|^2,$$

and, hence, $\|\mathbf{x} - \mathbf{u}\| \geq \|\mathbf{n}\| = \|\mathbf{x} - \mathbf{x}|_U\|$. Equality holds if and only if $\mathbf{v} = \mathbf{o}$, i.e. $\mathbf{u} = \mathbf{x}|_U$. \square

Proposition 5.18.

For all nonempty sets $M \subset \mathbb{R}^n$ we have:

- (a) $\mathbb{R}^n = \text{Span}(M) + M^\perp$ and $\text{Span}(M) \cap M^\perp = \{\mathbf{o}\}$,
- (b) $(M^\perp)^\perp = \text{Span}(M)$.

Proof. (a): By Proposition 5.16, we know that for each $\mathbf{x} \in \mathbb{R}^n$ there is a decomposition $\mathbf{x} = \mathbf{p} + \mathbf{n}$ with $\mathbf{p} \in \text{Span}(M)$ and $\mathbf{n} \in (\text{Span}(M))^\perp$. Now we just have to use Proposition 5.12 and Proposition 5.13.

(b): Exercise! (Use part (a)) \square

Corollary 5.19. Properties of U^\perp

For a linear subspace $U \subset \mathbb{R}^n$, we have:

- (a) $\mathbb{R}^n = U + U^\perp$ and $U \cap U^\perp = \{\mathbf{o}\}$. Usually, one writes in this case

$$\mathbb{R}^n = U \oplus U^\perp,$$

and calls it direct sum or, more correctly, orthogonal sum of two subspaces.

$$(b) \dim(U^\perp) = \dim(\mathbb{R}^n) - \dim(U).$$

$$(c) (U^\perp)^\perp = U.$$

5.3 Orthonormal systems and bases

For some applications it is very useful to have a set of vectors $\{\mathbf{u}_1 \dots \mathbf{u}_k\} \subset \mathbb{R}^n$ which are mutually orthogonal:

$$i \neq j \Rightarrow \mathbf{u}_i \perp \mathbf{u}_j \Leftrightarrow \langle \mathbf{u}_i, \mathbf{u}_j \rangle = 0$$

and have unit norm:

$$\|\mathbf{u}_i\| = \sqrt{\langle \mathbf{u}_i, \mathbf{u}_i \rangle} = 1.$$

Using the Kronecker symbol:

$$\delta_{ij} = \begin{cases} 1 & : i = j \\ 0 & : i \neq j \end{cases}$$

we may write this in short:

$$\langle \mathbf{u}_i, \mathbf{u}_j \rangle = \delta_{ij}.$$

Definition 5.20. OS, ONS, OB, ONB

Let U be a linear subspace of \mathbb{R}^n . A family $\mathcal{F} = (\mathbf{u}_1, \dots, \mathbf{u}_k)$ consisting of vectors from U is called:

- Orthogonal system (OS) if the vectors in \mathcal{F} are mutually orthogonal: $\langle \mathbf{u}_i, \mathbf{u}_j \rangle = 0$ for all $i, j \in \{1, \dots, k\}$ with $i \neq j$;
- Orthonormal system (ONS) if $\langle \mathbf{u}_i, \mathbf{u}_j \rangle = \delta_{ij}$ for all $i, j \in \{1, \dots, k\}$;
- Orthogonal basis (OB) if it is an OS and a basis of U ;
- Orthonormal basis (ONB) if it is an ONS and a basis of U .

If \mathcal{F} is an ONB, then the Gram matrix $G(\mathcal{F})$ is the identity matrix and projections are very easily calculable.

Example 5.21. Let $\langle \cdot, \cdot \rangle = \langle \cdot, \cdot \rangle_{eukl}$ the standard inner product.

(a) The canonical unit vectors

$$\mathbf{e}_1 = (1, 0, \dots, 0)^T, \quad \mathbf{e}_2 = (0, 1, 0, \dots, 0)^T, \quad \dots, \quad \mathbf{e}_n = (0, \dots, 0, 1)^T$$

in \mathbb{R}^n define an ONB for $U = \mathbb{R}^n$.

(b) The family $\mathcal{F} = (\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3)$ given by

$$\mathbf{u}_1 = (1, 0, 1)^T, \quad \mathbf{u}_2 = (1, 0, -1)^T, \quad \mathbf{u}_3 = (0, 1, 0)^T$$

defines an OB of \mathbb{R}^3 . We show this: We immediately have $\langle \mathbf{u}_1, \mathbf{u}_3 \rangle = 0$ and $\langle \mathbf{u}_2, \mathbf{u}_3 \rangle = 0$. Moreover, we find

$$\langle \mathbf{u}_1, \mathbf{u}_2 \rangle = \left\langle \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \right\rangle = 1 + 0 - 1 = 0.$$

Hence, \mathcal{F} is an OS. It remains to show that \mathcal{F} is also a basis for \mathbb{R}^3 . Since $\dim(\mathbb{R}^3) = 3$ and \mathcal{F} consists of three linearly independent vectors, we are finished. For showing the linear independence, the next Proposition 5.22 will be always helpful.

- (c) Normalising the vectors from (b), we obtain an ONB $(\frac{1}{\sqrt{2}}\mathbf{u}_1, \frac{1}{\sqrt{2}}\mathbf{u}_2, \mathbf{u}_3)$.

Proposition 5.22. An OS is linearly independent.

Let $\mathcal{F} = (\mathbf{u}_1, \dots, \mathbf{u}_k)$ be an OS in \mathbb{R}^n with $\mathbf{u}_i \neq \mathbf{o}$ for $i = 1, \dots, k$. Then \mathcal{F} is linearly independent.

Proof. Let \mathcal{F} be an OS. To show the linear independence of \mathcal{F} , we only have to show that $\alpha_1\mathbf{u}_1 + \dots + \alpha_k\mathbf{u}_k = \mathbf{o}$ always implies $\alpha_1 = \dots = \alpha_k = 0$. Using the inner product for \mathbf{u}_i with $i = 1, \dots, k$, we get:

$$\begin{aligned} 0 &= \langle \mathbf{o}, \mathbf{u}_i \rangle = \langle \alpha_1\mathbf{u}_1 + \dots + \alpha_k\mathbf{u}_k, \mathbf{u}_i \rangle \\ &= \alpha_1 \underbrace{\langle \mathbf{u}_1, \mathbf{u}_i \rangle}_{0} + \dots + \alpha_{i-1} \underbrace{\langle \mathbf{u}_{i-1}, \mathbf{u}_i \rangle}_{0} + \alpha_i \underbrace{\langle \mathbf{u}_i, \mathbf{u}_i \rangle}_{\|\mathbf{u}_i\|^2} + \alpha_{i+1} \underbrace{\langle \mathbf{u}_{i+1}, \mathbf{u}_i \rangle}_{0} + \dots + \alpha_k \underbrace{\langle \mathbf{u}_k, \mathbf{u}_i \rangle}_{0} \\ &= \alpha_i \|\mathbf{u}_i\|^2. \end{aligned}$$

Since $\|\mathbf{u}_i\|^2 \neq 0$, the only possibility is $\alpha_i = 0$ and this holds for all $i = 1, \dots, k$. \square

Now we can show, how easy it is to calculate Gramian matrices with a basis that is orthogonal.

Proposition 5.23. Gramian matrix for OB and ONB

The Gramian matrix $G(\mathcal{B})$ for an OB $\mathcal{B} = (\mathbf{u}_1, \dots, \mathbf{u}_k)$ is a diagonal matrix:

$$G(\mathcal{B}) = \begin{pmatrix} \langle \mathbf{u}_1, \mathbf{u}_1 \rangle & \langle \mathbf{u}_2, \mathbf{u}_1 \rangle & \dots & \langle \mathbf{u}_k, \mathbf{u}_1 \rangle \\ \langle \mathbf{u}_1, \mathbf{u}_2 \rangle & \langle \mathbf{u}_2, \mathbf{u}_2 \rangle & \dots & \langle \mathbf{u}_k, \mathbf{u}_2 \rangle \\ \vdots & \vdots & & \vdots \\ \langle \mathbf{u}_1, \mathbf{u}_k \rangle & \langle \mathbf{u}_2, \mathbf{u}_k \rangle & \dots & \langle \mathbf{u}_k, \mathbf{u}_k \rangle \end{pmatrix} = \begin{pmatrix} \|\mathbf{u}_1\|^2 & 0 & & \\ 0 & \|\mathbf{u}_2\|^2 & \ddots & \\ & \ddots & \ddots & 0 \\ & & 0 & \|\mathbf{u}_k\|^2 \end{pmatrix}.$$

If \mathcal{B} actually is an ONB, then we have $G(\mathcal{B}) = \mathbb{1}$.

The orthogonal projection $\mathbf{x}|_U$ for a vector $\mathbf{x} \in \mathbb{R}^n$ onto the linear subspace $U = \text{Span}(\mathcal{B})$ is then given by the coefficients

$$\alpha_1 = \frac{\langle \mathbf{x}, \mathbf{u}_1 \rangle}{\|\mathbf{u}_1\|^2}, \quad \alpha_2 = \frac{\langle \mathbf{x}, \mathbf{u}_2 \rangle}{\|\mathbf{u}_2\|^2}, \quad \dots, \quad \alpha_k = \frac{\langle \mathbf{x}, \mathbf{u}_k \rangle}{\|\mathbf{u}_k\|^2}$$

for equation (5.5). We get:

$$\mathbf{x}|_U = \frac{\langle \mathbf{x}, \mathbf{u}_1 \rangle}{\|\mathbf{u}_1\|^2} \mathbf{u}_1 + \dots + \frac{\langle \mathbf{x}, \mathbf{u}_k \rangle}{\|\mathbf{u}_k\|^2} \mathbf{u}_k \quad \text{and} \quad \mathbf{x}|_{U^\perp} = \mathbf{x} - \mathbf{x}|_U$$

If \mathcal{B} is even an ONB, then all the denominators $\|\mathbf{u}_i\|^2$ are equal to 1.

Even if one is not interested in the projection, this can be helpful for calculating the coefficients for the linear combination.

Corollary 5.24. Fourier expansion w.r.t. an OB or ONB

Let U be a linear subspace of \mathbb{R}^n and $\mathcal{B} = (\mathbf{u}_1, \dots, \mathbf{u}_k)$ an OB of U . Then the unique linear combination for a vector $\mathbf{x} \in U$ with respect to \mathcal{B} is given by:

$$\mathbf{x} = \alpha_1 \mathbf{u}_1 + \dots + \alpha_k \mathbf{u}_k \quad \text{with} \quad \alpha_i = \frac{\langle \mathbf{x}, \mathbf{u}_i \rangle}{\|\mathbf{u}_i\|^2} \quad \text{for all } i \in \{1, \dots, k\}. \quad (5.6)$$

This formula is called the Fourier expansion of \mathbf{x} with respect to \mathcal{B} , and the numbers α_i are called the associated Fourier coefficients. If \mathcal{B} even is an ONB, then

$$\alpha_i = \langle \mathbf{x}, \mathbf{u}_i \rangle \quad \text{for all } i = 1, \dots, k.$$

Note that in the case $U = \mathbb{R}^n$, we simply set $k = n$.

Looking at the formula for $\mathbf{n} = \mathbf{x}|_{U^\perp}$ from Proposition 5.23, one recognise a general principle how to construct orthogonal vectors. We summarise this in the following algorithm

Remark: Gram-Schmidt orthonormalisation

Let U be a linear subspace of \mathbb{R}^n and $(\mathbf{u}_1, \dots, \mathbf{u}_k)$ a basis of U . The following procedure will give us an ONB $(\mathbf{w}_1, \dots, \mathbf{w}_k)$ for U .

(1) Normalise the first vector:

$$\mathbf{w}_1 := \frac{1}{\|\mathbf{u}_1\|} \mathbf{u}_1.$$

(2) Choose the normal component of \mathbf{u}_2 with respect to $\text{Span}(\mathbf{w}_1)$

$$\mathbf{v}_2 := \mathbf{u}_2 - \underbrace{\langle \mathbf{u}_2, \mathbf{w}_1 \rangle \mathbf{w}_1}_{\mathbf{u}_2|_{\text{Span}(\mathbf{w}_1)}} \quad \text{and normalise it:} \quad \mathbf{w}_2 := \frac{1}{\|\mathbf{v}_2\|} \mathbf{v}_2.$$

(3) Choose the normal component of \mathbf{u}_3 with respect to $\text{Span}(\mathbf{w}_1, \mathbf{w}_2)$

$$\mathbf{v}_3 := \mathbf{u}_3 - \underbrace{\left(\langle \mathbf{u}_3, \mathbf{w}_1 \rangle \mathbf{w}_1 + \langle \mathbf{u}_3, \mathbf{w}_2 \rangle \mathbf{w}_2 \right)}_{\mathbf{u}_3|_{\text{Span}(\mathbf{w}_1, \mathbf{w}_2)}} \quad \text{and normalise it:} \quad \mathbf{w}_3 := \frac{1}{\|\mathbf{v}_3\|} \mathbf{v}_3.$$

⋮

(k) In the last step choose the normal component of \mathbf{u}_k w.r.t. $\text{Span}(\mathbf{w}_1, \dots, \mathbf{w}_{k-1})$

$$\mathbf{v}_k := \mathbf{u}_k - \underbrace{\sum_{i=1}^{k-1} \langle \mathbf{u}_k, \mathbf{w}_i \rangle \mathbf{w}_i}_{\mathbf{u}_k|_{\text{Span}(\mathbf{w}_1, \dots, \mathbf{w}_{k-1})}} \quad \text{and normalise it: } \mathbf{w}_k := \frac{1}{\|\mathbf{v}_k\|} \mathbf{v}_k.$$

Example 5.25. Let $\mathbf{u}_1 = (1, 1, 0)^T$ and $\mathbf{u}_2 = (2, 0, 2)^T$ be two vectors in \mathbb{R}^3 and $U = \text{Span}(\mathbf{u}_1, \mathbf{u}_2)$ the spanned plane. We calculate an ONB $(\mathbf{w}_1, \mathbf{w}_2)$ for U . The first vector is

$$\mathbf{w}_1 := \frac{1}{\|\mathbf{u}_1\|} \mathbf{u}_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}.$$

For the second vector, we first need to calculate:

$$\mathbf{v}_2 := \mathbf{u}_2 - \langle \mathbf{u}_2, \mathbf{w}_1 \rangle \mathbf{w}_1 = \begin{pmatrix} 2 \\ 0 \\ 2 \end{pmatrix} - \left\langle \begin{pmatrix} 2 \\ 0 \\ 2 \end{pmatrix}, \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \right\rangle \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 2 \end{pmatrix} - \frac{1}{2} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$$

Then \mathbf{v}_2 is getting normalised:

$$\mathbf{w}_2 := \frac{1}{\|\mathbf{v}_2\|} \mathbf{v}_2 = \frac{1}{\sqrt{6}} \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}.$$

Now we have $\|\mathbf{w}_1\| = 1 = \|\mathbf{w}_2\|$ and $\langle \mathbf{w}_1, \mathbf{w}_2 \rangle = 0$ and also $\text{Span}(\mathbf{w}_1, \mathbf{w}_2) = U = \text{Span}(\mathbf{u}_1, \mathbf{u}_2)$.

We recall Corollary 5.24: Why are such ONB helpful? Usually, if we want to write a vector \mathbf{v} as a linear combination of basis vectors $\mathcal{B} = (\mathbf{b}_1, \dots, \mathbf{b}_k)$, we have to solve a linear system:

$$\mathbf{v} = \sum_{i=1}^k \lambda_i \mathbf{b}_i.$$

If we have an orthonormal basis $\mathcal{B} = (\mathbf{u}_1, \dots, \mathbf{u}_k)$, then we can dispense with this. We can simply calculate:

$$\langle \mathbf{v}, \mathbf{u}_i \rangle = \left\langle \sum_j \lambda_j \mathbf{u}_j, \mathbf{u}_i \right\rangle = \lambda_i \langle \mathbf{u}_i, \mathbf{u}_i \rangle = \lambda_i.$$

Thus, each coefficient of the linear combination results from a simple inner product.

Remark: Outlook

It is this principle the so called Fourier-Transformation is built on. It decomposes a signal $\mathbf{v}(t)$ into frequencies $\mathbf{u}_i(t) = \sin(\omega_i t)$. This is, however, a problem formulated in a more abstract vector space.

5.4 Orthogonal matrices

Let us now restrict our attention to the standard inner product

$$\langle \mathbf{x}, \mathbf{y} \rangle = \langle \mathbf{x}, \mathbf{y} \rangle_{euklid} = \mathbf{x}^T \mathbf{y},$$

and write down our results from above in terms of matrices.

Let $\mathcal{B} = (\mathbf{u}_1, \dots, \mathbf{u}_n)$ be a basis for \mathbb{R}^n . Then each $\mathbf{x} \in \mathbb{R}^n$ can be uniquely written as:

$$\mathbf{x} = \alpha_1 \begin{pmatrix} | \\ \mathbf{u}_1 \\ | \end{pmatrix} + \dots + \alpha_n \begin{pmatrix} | \\ \mathbf{u}_n \\ | \end{pmatrix} = \underbrace{\begin{pmatrix} | & & | \\ \mathbf{u}_1 & \dots & \mathbf{u}_n \\ | & & | \end{pmatrix}}_{=:A} \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix}.$$

For the so-defined matrix $A = (\mathbf{u}_1 \cdots \mathbf{u}_n)$, we get:

$$\begin{aligned} A^T A &= \begin{pmatrix} -\mathbf{u}_1^T & - \\ \vdots & \\ -\mathbf{u}_n^T & - \end{pmatrix} \begin{pmatrix} | & & | \\ \mathbf{u}_1 & \dots & \mathbf{u}_n \\ | & & | \end{pmatrix} = \begin{pmatrix} \mathbf{u}_1^T \mathbf{u}_1 & \dots & \mathbf{u}_1^T \mathbf{u}_n \\ \vdots & & \vdots \\ \mathbf{u}_n^T \mathbf{u}_1 & \dots & \mathbf{u}_n^T \mathbf{u}_n \end{pmatrix} \\ &= \begin{pmatrix} \langle \mathbf{u}_1, \mathbf{u}_1 \rangle & \dots & \langle \mathbf{u}_n, \mathbf{u}_1 \rangle \\ \vdots & & \vdots \\ \langle \mathbf{u}_1, \mathbf{u}_n \rangle & \dots & \langle \mathbf{u}_n, \mathbf{u}_n \rangle \end{pmatrix} = G(\mathcal{B}), \end{aligned} \quad (5.7)$$

This means that $A^T A$ is the Gramian Matrix $G(\mathcal{B})$ for the basis \mathcal{B} . For an ONB \mathcal{B} , the matrix $G(\mathcal{B})$ is the identity matrix $\mathbb{1}$ by Proposition 5.23. This gives us the following:

Definition 5.26. Orthogonal matrix

A matrix $A \in \mathbb{R}^{n \times n}$ with the property $A^T A = \mathbb{1}$ is called orthogonal.

We immediately see that an orthogonal matrix A has an ONB as columns and fulfils

$$\langle A\mathbf{x}, A\mathbf{y} \rangle = \langle \mathbf{x}, \mathbf{y} \rangle.$$

The last property says that the corresponding linear map f_A preserves the inner product, and thus angles and lengths.

Proposition 5.27. Defining properties of orthogonal matrices

For a matrix $A \in \mathbb{R}^{n \times n}$ the following claims are equivalent:

- (a) A is an orthogonal matrix
- (b) $A^T A = \mathbb{1}$.
- (c) $AA^T = \mathbb{1}$.
- (d) $A^{-1} = A^T$.
- (e) A^T is an orthogonal matrix.
- (f) The columns of A define an ONB of \mathbb{R}^n .

- (g) The rows of A define an ONB of \mathbb{R}^n .
- (h) For all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, we get $\langle A\mathbf{x}, A\mathbf{y} \rangle = \langle \mathbf{x}, \mathbf{y} \rangle$.
- (i) For all $\mathbf{x} \in \mathbb{R}^n$, we get $\|A\mathbf{x}\| = \|\mathbf{x}\|$.

Proof. Exercise! □

Such matrices correspond to maps of special geometric interest:

- Rotations
- Reflections
- Special case: permutation matrices

We also see that solving a LES $A\mathbf{x} = \mathbf{b}$ described by an orthogonal matrix A is easy to solve:

$$\mathbf{x} = A^{-1}\mathbf{b} = A^T\mathbf{b}$$

The inverse is computed now more easily than in the general case.

Proposition 5.28. Determinant of orthogonal matrices

For an orthogonal matrix A , we have $\det(A) = \pm 1$.

Proof. $1 = \det(\mathbb{1}) = \det(A^T A) = \det(A^T) \det(A) = \det(A)^2$. □

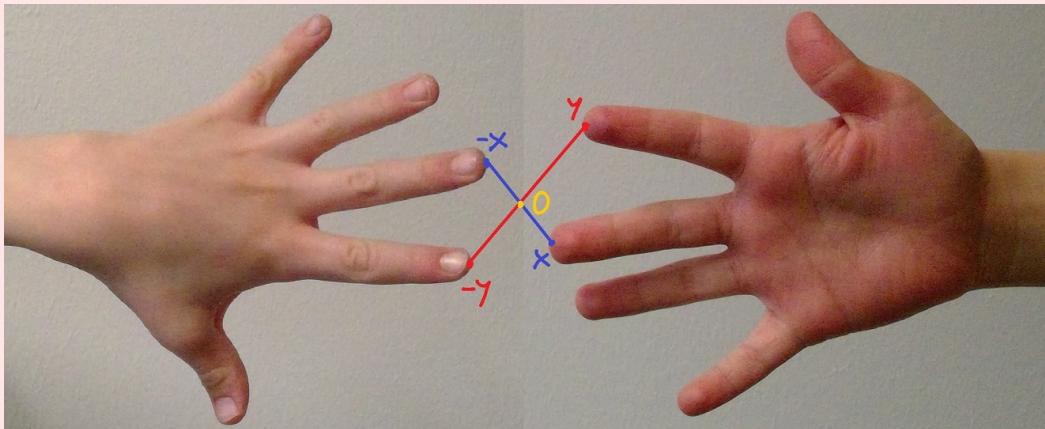
Definition 5.29. Rotations and reflections

Let $A \in \mathbb{R}^{n \times n}$ be an orthogonal matrix. If $\det(A) = 1$, we call A a rotation. If $\det(A) = -1$, we call the matrix a reflection.

Which matrices in Example 3.16 are rotations or reflections?

Attention! Notions: Rotation or reflection

- (a) Not every matrix $A \in \mathbb{R}^{n \times n}$ with $\det(A) = 1$ (or $\det(A) = -1$) is a rotation (or a reflection)!
- (b) A “reflection” from Definition 5.29 could also be a point reflection in the case $n \geq 3$.



5.5 Orthogonalisation: the QR-decomposition

Since orthonormal systems and bases are so useful, we learnt the Gram-Schmidt procedure to turn a family $(\mathbf{a}_1, \dots, \mathbf{a}_k)$ of linearly independent vectors into an orthonormal system $(\mathbf{q}_1, \dots, \mathbf{q}_k)$ that spans the same space.

Putting the the vectors \mathbf{a}_i as columns in a matrix A and the vectors \mathbf{q}_i in a matrix Q , this can be written down as a decomposition:

$$A = QR \quad \text{QR-decomposition}$$

We see that the columns of A can be seen as linear combinations of the columns of Q . Here R is a column matrix, in which the coefficients of linear combinations of the \mathbf{q}_i stand. Since the l^{th} column of A should always be a linear combination of the first l columns of Q we have:

$$\mathbf{a}_l = \sum_{i=1}^l r_{il} \mathbf{q}_i$$

This means that R is an upper triangular matrix.

There are at least three alternatives to compute this:

- “Classical Gram-Schmidt”: this is what we learn next (good for pen-and-paper computations), but instable on the computer
- “Modified Gram-Schmidt”: equivalent to our Gram-Schmidt, order of loops exchanged, numerically more stable
- “Householder reflections”: are cheaper and even more stable. This is the method of choice in numerical computations

If A is square matrix with $\text{rank}(A) = n$, we get

$$A = \begin{pmatrix} & & \\ \mathbf{a}_1 & \dots & \mathbf{a}_n \\ & & \end{pmatrix}$$

$$= \underbrace{\left(Q \underbrace{\begin{pmatrix} r_{11} \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}}_{\mathbf{a}_1} \right)}_{Q \mathbf{a}_1} \underbrace{\left(Q \underbrace{\begin{pmatrix} r_{12} \\ r_{22} \\ 0 \\ \vdots \\ 0 \end{pmatrix}}_{\mathbf{a}_2} \right)}_{Q \mathbf{a}_2} \cdots \underbrace{\left(Q \underbrace{\begin{pmatrix} r_{1n} \\ r_{2n} \\ r_{3n} \\ \vdots \\ r_{nn} \end{pmatrix}}_{\mathbf{a}_n} \right)}_{Q \mathbf{a}_n} = Q \underbrace{\begin{pmatrix} r_{11} & r_{12} & r_{13} & \cdots & r_{1n} \\ r_{22} & r_{23} & \cdots & r_{2n} \\ r_{33} & \cdots & r_{3n} \\ \ddots & & \vdots \\ r_{nn} \end{pmatrix}}_{=: R} = QR.$$

This defines the so-called [QR-decomposition](#) of a matrix A .

As a result, we get $(\mathbf{q}_1, \dots, \mathbf{q}_n)$ as an ONB for the space $\text{Span}(\mathbf{a}_1, \dots, \mathbf{a}_n) = \text{Ran}(A)$. We immediately get $Q^T Q = \mathbb{1}_n$. Since we consider square matrices, $m = n$, we can write $Q^{-1} = Q^T$.

Example 5.30. Consider

$$A = \begin{pmatrix} 2 & -1 & 8 \\ 1 & 1 & 1 \\ -2 & 4 & 4 \end{pmatrix}$$

$$\mathbf{q}_1 = \frac{\mathbf{a}_1}{\|\mathbf{a}_1\|} = \frac{\mathbf{a}_1}{3} = \begin{pmatrix} 2/3 \\ 1/3 \\ -2/3 \end{pmatrix}, \quad r_{11} = 3$$

$$r_{12} = \langle \mathbf{a}_2, \mathbf{q}_1 \rangle = -3, \quad \mathbf{q}_2 = \begin{pmatrix} 1/3 \\ 2/3 \\ 2/3 \end{pmatrix}, \quad r_{22} = 3$$

$$r_{13} = \langle \mathbf{a}_3, \mathbf{q}_1 \rangle = 3, \quad r_{23} = \langle \mathbf{a}_3, \mathbf{q}_2 \rangle = 6, \quad \mathbf{q}_3 = \begin{pmatrix} 2/3 \\ -2/3 \\ 1/3 \end{pmatrix}, \quad r_{33} = 6;$$

$$Q = \begin{pmatrix} 2/3 & 1/3 & 2/3 \\ 1/3 & 2/3 & -2/3 \\ -2/3 & 2/3 & 1/3 \end{pmatrix} \quad R = \begin{pmatrix} 3 & -3 & 3 \\ 3 & 3 & 6 \\ 6 & & \end{pmatrix}$$

Example 5.31.

$$\text{For } A = \begin{pmatrix} 2 & -1 & 8 \\ 1 & 1 & 1 \\ -2 & 4 & 4 \end{pmatrix} \quad \text{Gram-Schmidt gives us } \mathbf{q}_1 = \frac{\mathbf{a}_1}{\|\mathbf{a}_1\|} = \begin{pmatrix} 2/3 \\ 1/3 \\ -2/3 \end{pmatrix},$$

$$\mathbf{q}_2 = \frac{\mathbf{a}_2 - (\mathbf{a}_2)_{|\text{Span}(\mathbf{q}_1)}}{\|\dots\|} = \begin{pmatrix} 1/3 \\ 2/3 \\ 2/3 \end{pmatrix}, \quad \mathbf{q}_3 = \frac{\mathbf{a}_3 - (\mathbf{a}_3)_{|\text{Span}(\mathbf{q}_1, \mathbf{q}_2)}}{\|\dots\|} = \begin{pmatrix} 2/3 \\ -2/3 \\ 1/3 \end{pmatrix}$$

$$\text{Hence: } Q = \frac{1}{3} \begin{pmatrix} 2 & 1 & 2 \\ 1 & 2 & -2 \\ -2 & 2 & 1 \end{pmatrix} \quad \text{and} \quad R = Q^T A = \begin{pmatrix} 3 & -3 & 3 \\ 3 & 3 & 6 \\ 6 & & \end{pmatrix}.$$

As we have seen in the LR-decomposition, we can also use the QR-decomposition for solving an LES $Ax = \mathbf{b}$. If A is a square matrix ($m = n$), we know:

$$A\mathbf{x} = \mathbf{b} \iff QR\mathbf{x} = \mathbf{b} \stackrel{Q^{-1}=Q^T}{\iff} R\mathbf{x} = Q^T\mathbf{b} \quad (5.8)$$

The last system has a triangle form and is solved by backwards substitution. A QR -decomposition is also possible in the non-square case as we will see later in detail.

5.6 Distances: points, lines and planes

Recall that we call an affine subspace H in \mathbb{R}^n with dimension $n - 1$ a [hyperplane](#). This is, for example, a line in \mathbb{R}^2 or a plane in \mathbb{R}^3 .

Definition 5.32. Hesse normal form (HNF), distance $\text{dist}(\cdot, \cdot)$

For each hyperplane in \mathbb{R}^n , there exists a normal form

$$\{\mathbf{v} \in \mathbb{R}^n : \langle \mathbf{n}, \mathbf{v} - \mathbf{p} \rangle = 0\}$$

where $\mathbf{p} \in \mathbb{R}^n$ is one chosen point and $\mathbf{n} \in \mathbb{R}^n$ a normal vector. We call it [Hesse normal form \(HNF\)](#) if $\|\mathbf{n}\| = 1$ holds.

For a given point $\mathbf{q} \in \mathbb{R}^n$ and affine subspaces S, T in \mathbb{R}^n , we write:

$$\text{dist}(\mathbf{q}, T) := \min_{\mathbf{t} \in T} \|\mathbf{q} - \mathbf{t}\| \quad \text{and} \quad \text{dist}(S, T) := \min_{\mathbf{s} \in S} \text{dist}(\mathbf{s}, T) = \min_{\mathbf{s} \in S} \min_{\mathbf{t} \in T} \|\mathbf{s} - \mathbf{t}\|$$

for the shortest distance between \mathbf{v} and T and the shortest distance between S and T , respectively.

If we are using the HNF for a hyperplane, then the expression $\langle \mathbf{n}, \mathbf{v} - \mathbf{p} \rangle$ can indeed measure the distances:

Proposition 5.33.

For a hyperplane $T = \{\mathbf{v} \in \mathbb{R}^n : \langle \mathbf{n}, \mathbf{v} - \mathbf{p} \rangle = 0\}$ with $\|\mathbf{n}\| = 1$ (this is the HNF), we have

$$\langle \mathbf{n}, \mathbf{q} - \mathbf{p} \rangle = \pm \text{dist}(\mathbf{q}, T) \quad (5.9)$$

where the sign “+” holds if \mathbf{q} lies on the same side of T as the normal vector \mathbf{n} , and “−” holds if \mathbf{q} lies on the other side of T .

Proof. This is an exercise where you should use

$$\langle \mathbf{n}, \mathbf{v} - \mathbf{p} \rangle = \frac{\langle \mathbf{n}, \mathbf{v} - \mathbf{p} \rangle}{1} = \frac{\langle \mathbf{v} - \mathbf{p}, \mathbf{n} \rangle}{\langle \mathbf{n}, \mathbf{n} \rangle}$$

and use projections. □

Using equation (5.9), we are able to calculate distances. We summarise all possibilities for such problems for \mathbb{R}^3 :

Distances in \mathbb{R}^3

- Point/Point: $\text{dist}(\mathbf{p}, \mathbf{q}) = \|\mathbf{p} - \mathbf{q}\|$, (for completeness's sake),
- Point/Plane: $\text{dist}(\mathbf{q}, \underbrace{\mathbf{p} + \text{Span}(\mathbf{a}, \mathbf{b})}_E) = |\langle \mathbf{n}, \mathbf{q} - \mathbf{p} \rangle|$, cf. (5.9).
- Line/Plane:

$$\text{dist}(\underbrace{\mathbf{p} + \text{Span}(\mathbf{a})}_g, \underbrace{\mathbf{q} + \text{Span}(\mathbf{b}, \mathbf{c})}_E) = \text{dist}(\mathbf{p}, E),$$

if g is parallel with respect to E . In the other case, g and E intersect, and, hence, $\text{dist}(g, E) = 0$. If g is parallel with respect to E , one has \mathbf{a} in $\text{Span}(\mathbf{b}, \mathbf{c})$: the family $(\mathbf{a}, \mathbf{b}, \mathbf{c})$ is linearly dependent, i.e. $\det(\mathbf{a} \ \mathbf{b} \ \mathbf{c}) = 0$.

- Plane/Plane:

$$\text{dist}(\underbrace{\mathbf{p} + \text{Span}(\mathbf{a}, \mathbf{b})}_{E_1}, \underbrace{\mathbf{q} + \text{Span}(\mathbf{c}, \mathbf{d})}_{E_2}) = \text{dist}(\mathbf{p}, E_2),$$

if E_1 is parallel to E_2 . Otherwise, $\text{dist}(E_1, E_2) = 0$, which means that E_1 and E_2 have an intersection. If E_1 and E_2 are parallel, then the normal vectors $\mathbf{n}_1 := \mathbf{a} \times \mathbf{b}$ and $\mathbf{n}_2 := \mathbf{c} \times \mathbf{d}$ of E_1 and E_2 , respectively, are linearly dependent.

- Line/Line: Let $g_1 = \mathbf{p} + \text{Span}(\mathbf{a})$ and $g_2 = \mathbf{q} + \text{Span}(\mathbf{b})$ be lines in \mathbb{R}^3 .
 - 1st case: If \mathbf{a} and \mathbf{b} are parallel, then g_1 and g_2 are parallel. Let E the plane with $\mathbf{p} \in E$ and normal vector $\mathbf{n} := \mathbf{a}$. The intersection point of E and g_2 is denoted by \mathbf{p}' (Calculation: Put the parameter equation of g_2 into the normal form of E). Then $\text{dist}(g_1, g_2) = \text{dist}(\mathbf{p}, \mathbf{p}') = \|\mathbf{p} - \mathbf{p}'\|$.
 - 2nd case: If the vectors \mathbf{a} and \mathbf{b} are not parallel, then

$$\text{dist}(g_1, g_2) = \text{dist}(\mathbf{p}, \underbrace{\mathbf{q} + \text{Span}(\mathbf{a}, \mathbf{b})}_{=:E}).$$

- Point/Line: Let \mathbf{p} be a point in \mathbb{R}^3 and $g = \mathbf{q} + \text{Span}(\mathbf{a})$ a line in \mathbb{R}^3 . Define $\mathbf{b} := (\mathbf{p} - \mathbf{q}) \times \mathbf{a}$. If $\mathbf{b} = \mathbf{0}$, then \mathbf{p} lies in g , and, hence, $\text{dist}(\mathbf{p}, g) = 0$. On the other hand, \mathbf{b} can be perpendicular to the plane, defined by \mathbf{p} and g . In this case:

$$\text{dist}(\mathbf{p}, g) = \text{dist}(\mathbf{p}, \underbrace{\mathbf{q} + \text{Span}(\mathbf{a}, \mathbf{b})}_{=:E}).$$

Alternatively: Norm of the normal component of $\mathbf{p} - \mathbf{q}$ with respect to $\text{Span}(\mathbf{a})$ (Proposition 5.7).

Note that we always have

$$\text{dist}(\mathbf{p}, \mathbf{q} + U) = \min_{\mathbf{u} \in U} \|\mathbf{p} - (\mathbf{q} + \mathbf{u})\| = \min_{\mathbf{u} \in U} \|(\mathbf{p} - \mathbf{q}) - \mathbf{u}\| = \text{dist}(\mathbf{p} - \mathbf{q}, U),$$

and therefore, one usually just considers the case of linear subspaces instead of affine subspaces.

Summary

- Each vector $\mathbf{x} \in \mathbb{R}^n$ can be uniquely decomposed into
 - a vector $\mathbf{x}|_U$ in a given subspace U and
 - a vector \mathbf{n} that is orthogonal to U .

The vector $\mathbf{x}|_U$ is called the orthogonal projection of \mathbf{x} onto U . \mathbf{n} is equal to $\mathbf{x}|_{U^\perp}$.

- If $\dim(U) = 1$, the calculation of $\mathbf{x}|_U$ is very easy, while one can use Proposition 5.7; If $\dim(U) \geq 2$, the one has to choose a basis \mathcal{B} for U and either
 - solve an LES with the help of the Gramian matrix $G(\mathcal{B})$ (Proposition 5.16) or
 - build an ONS or ONB with the help of the Gram-Schmidt procedure and use Proposition 5.23.
- A matrix $A \in \mathbb{R}^{n \times n}$ with $A^{-1} = A^T$ is called orthogonal. The determinant is ± 1 . Depending on the sign of $\det(A)$, the matrix A describes a reflection or a rotation.

6

Eigenvalues and similar things

The first person you should be careful not to fool is yourself. Because you are the easiest person to fool.

Richard Feynman

Consider again a square matrix $A \in \mathbb{R}^{n \times n}$ and the associated linear map $f_A : \mathbb{R}^n \rightarrow \mathbb{R}^n$ which maps \mathbb{R}^n into itself.

Question:

Are there vectors \mathbf{v} which are only scaled by f_A ? This means that they satisfy:

$$A\mathbf{v} = \lambda\mathbf{v} \text{ or equivalently } (A - \lambda\mathbf{1})\mathbf{v} = \mathbf{0}$$

- λ is called eigenvalue of A ,
- \mathbf{v} is called eigenvector of A (if $\mathbf{v} \neq \mathbf{0}$).

First conclusions:

- Not very interesting (trivial): $\mathbf{v} = \mathbf{0}$.
- $\mathbf{v} \in \text{Ker}(A) \setminus \{\mathbf{0}\} \Rightarrow A\mathbf{v} = 0\mathbf{v}$, so $\lambda = 0$.
- $\mathbf{v} \in \text{Ker}(A - \lambda\mathbf{1}) \setminus \{\mathbf{0}\} \Rightarrow A\mathbf{v} = \lambda\mathbf{v}$, so λ is an eigenvalue.
- \mathbf{v} eigenvector $\Rightarrow \alpha\mathbf{v}$ is also an eigenvector (for $\alpha \neq 0$).

Example. (a) $A\mathbf{v} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} v_1 + v_2 \\ v_2 \end{pmatrix} \Rightarrow \lambda = 1, \mathbf{v} = \begin{pmatrix} v_1 \\ 0 \end{pmatrix}$

(b) $A\mathbf{v} = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 3v_1 \\ 2v_2 \end{pmatrix} \Rightarrow \lambda = 2, \mathbf{v} = \begin{pmatrix} 0 \\ v_2 \end{pmatrix}, \text{ or } \lambda = 3, \mathbf{v} = \begin{pmatrix} v_1 \\ 0 \end{pmatrix}$

(c) The eigenvalues of a diagonal matrix are the diagonal entries, its eigenvectors are the (scaled) canonical unit vectors.

(d) Suppose $A \in \mathbb{R}^{2 \times 2}$ is a rotation about an angle (not a multiple of 180°). Then “obviously” there cannot be any eigenvectors (at least no real ones).

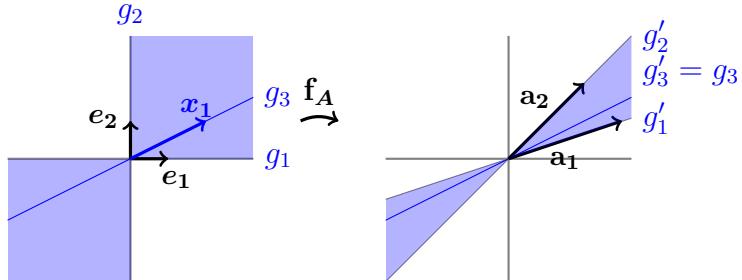
6.1 What is an eigenvalue and an eigenvector?

We start with an illustration in two-dimensional cases and consider a matrix

$$A = \begin{pmatrix} & \\ \mathbf{a}_1 & \mathbf{a}_2 \\ & \end{pmatrix} \in \mathbb{R}^{2 \times 2}$$

and the associated linear map $f_A : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ with $\mathbf{x} \mapsto A\mathbf{x}$.

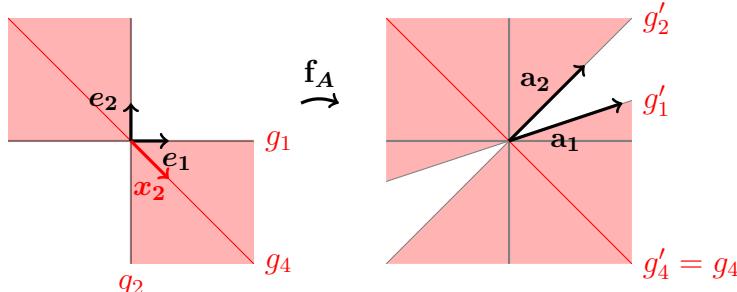
Define the two lines $g_1 := \text{Span}(\mathbf{e}_1)$ and $g_2 := \text{Span}(\mathbf{e}_2)$, and look at the images under the map f_A , denoted by $g'_1 := f_A(g_1)$ and $g'_2 := f_A(g_2)$.



We already know $g'_1 = \text{Span}(\mathbf{a}_1)$ and $g'_2 = \text{Span}(\mathbf{a}_2)$. In the picture above, you see an example what could happen under the map f_A : Both lines have been rotated but in different orientations. In this example we chose the matrix $A = \begin{pmatrix} 3 & 2 \\ 1 & 2 \end{pmatrix}$.

By the linearity of f_A , we also know that all other lines between g_1 and g_2 should also be rotated. However, then in this example, we should also find a line g_3 , which goes through the origin, that is not rotated at all when using f_A . This simply means that the image $g'_3 := f_A(g_3)$ is equal to g_3 .

Of course, this does not mean that all the points of g_3 stay fixed after using f_A but only that a point of g_3 is mapped to another point on g_3 .



In the same sense, we can look at the other quadrants of our coordinate system. There we also find such a special line:

$$g'_4 := f_A(g_4) = g_4.$$

For points \mathbf{x} on both lines, we find our defining equation for eigenvectors and eigenvalues again:

$$A\mathbf{x}_1 = f_A(\mathbf{x}_1) \in g_3, \quad \text{i.e.} \quad A\mathbf{x}_1 = \lambda_1 \mathbf{x}_1 \quad \text{with a certain} \quad \lambda_1 \in \mathbb{R}$$

and

$$A\mathbf{x}_2 = f_A(\mathbf{x}_2) \in g_4, \quad \text{i.e.} \quad A\mathbf{x}_2 = \lambda_2 \mathbf{x}_2 \quad \text{with a certain} \quad \lambda_2 \in \mathbb{R}.$$

Note that the numbers λ_1 or λ_2 cannot change for different points on the line since f_A is linear. We fix all this in a definition:

Definition 6.1. Eigenvector, spectrum

Let A be a square matrix. A vector $\mathbf{x} \neq \mathbf{0}$ is called an eigenvector of A , if $A\mathbf{x}$ is a multiple of \mathbf{x} . This scalar λ , which means $A\mathbf{x} = \lambda\mathbf{x}$, is called eigenvalue of A . The set of all eigenvalues of A is called the spectrum of A and denoted by $\text{spec}(A)$.

This is very general definition and will work later for other cases in the same manner. Here, we are first interested in matrices $A \in \mathbb{R}^{n \times n}$ and eigenvalues $\lambda \in \mathbb{R}$. However, you may already see that this can also work for complex numbers. We may also include $\lambda \in \mathbb{C}$ later.

Proposition 6.2. Multiple of eigenvector = eigenvector

Every multiple (not \mathbf{o}) of an eigenvector \mathbf{x} for A is also an eigenvector for A , corresponding to the same eigenvalue λ .

Proof. Let $A\mathbf{x} = \lambda\mathbf{x}$ with $\mathbf{x} \neq \mathbf{0}$, which means \mathbf{x} is an eigenvector of A for the eigenvalue λ . Let $\mathbf{y} = \alpha\mathbf{x}$ with $\alpha \neq 0$. Then $A\mathbf{y} = A(\alpha\mathbf{x}) = \alpha A\mathbf{x} = \alpha\lambda\mathbf{x} = \lambda(\alpha\mathbf{x}) = \lambda\mathbf{y}$, which means $\mathbf{y} (\neq \mathbf{0})$ is also an eigenvector of A associated to the same eigenvalue λ . \square

Looking again at the pictures above:

- We have $A\mathbf{x} = \lambda_1\mathbf{x}$ for all multiples \mathbf{x} of $\mathbf{x}_1 \in g_3$ (which means for all $\mathbf{x} \in g_3$).
- Also we have $A\mathbf{x} = \lambda_2\mathbf{x}$ for all multiples \mathbf{x} of $\mathbf{x}_2 \in g_4$ (which means for all $\mathbf{x} \in g_4$).
- Looking at the line g_3 , the map f_A acts like scaling with the factor λ_1 .
- Looking at the line g_4 , the map f_A acts like scaling with the factor λ_2 .

Optimal coordinate system for the map f_A

Describing \mathbb{R}^2 with a coordinate system given by the two lines g_3 and g_4 (instead of g_1 and g_2), the acting of the map f_A is very simple: The coordinate axes are only stretched: The one with factor λ_1 , and the other one with factor λ_2 .

To get this “optimal coordinate system” we need all the eigenvalues λ_1, λ_2 and the corresponding eigenvectors \mathbf{x}_1 and \mathbf{x}_2 .

Question:

- (a) How to find the eigenvalues and the eigenvectors of A ?
- (b) Do you always find n eigenvalues for an $n \times n$ matrix A ?
- (c) Do you find n different directions for eigenvectors?
- (d) How to change the coordinate system?
- (e) What are applications for this?

6.2 The characteristic polynomial

Our goal is to find $\lambda \in \mathbb{R}$ and $\mathbf{x} \neq \mathbf{o}$ such that $(A - \lambda\mathbb{1})\mathbf{x} = \mathbf{o}$, i.e., $(A - \lambda\mathbb{1})$ has a nontrivial kernel. This means that the corresponding map for $A - \lambda\mathbb{1}$ is not injective and, hence, it is a singular matrix.

Idea:

Compute $\det(A - \lambda \mathbb{1})$, which yields a polynomial of degree n in λ and determine its zeros, because

$$\begin{aligned}\det(A - \lambda \mathbb{1}) = 0 &\Leftrightarrow A - \lambda \mathbb{1} \text{ is singular} \\ &\Leftrightarrow \text{Ker}(A - \lambda \mathbb{1}) \text{ is non-trivial} \\ &\Leftrightarrow \lambda \text{ is an eigenvalue}\end{aligned}$$

Then, compute a basis for $\text{Ker}(A - \lambda \mathbb{1})$ for each eigenvalue.

Example 6.3.

$$\begin{aligned}A &= \begin{pmatrix} 3 & 2 \\ 1 & 4 \end{pmatrix} \\ \det(A - \lambda \mathbb{1}) &= \det \begin{pmatrix} 3 - \lambda & 2 \\ 1 & 4 - \lambda \end{pmatrix} = (3 - \lambda)(4 - \lambda) - 2 \cdot 1 = 10 - 7\lambda + \lambda^2 \\ \lambda_{1,2} &= \frac{7 \pm \sqrt{49 - 40}}{2} = \frac{7 \pm 3}{2} \Rightarrow \lambda_1 = 2, \lambda_2 = 5\end{aligned}$$

Thus we have the eigenvalues $\lambda_1 = 2$ and $\lambda_2 = 5$. Let us compute the eigenvectors:

$$\begin{aligned}\mathbf{o} &= (A - 2\mathbb{1})\mathbf{v} = \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} v_1 + 2v_2 \\ v_1 + 2v_2 \end{pmatrix} \Rightarrow \mathbf{v} = \alpha \begin{pmatrix} 2 \\ -1 \end{pmatrix} \\ \mathbf{o} &= (A - 5\mathbb{1})\mathbf{v} = \begin{pmatrix} -2 & 2 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} -2v_1 + 2v_2 \\ v_1 - v_2 \end{pmatrix} \Rightarrow \mathbf{v} = \alpha \begin{pmatrix} 1 \\ 1 \end{pmatrix}\end{aligned}$$

We summarise what we discovered:

Proposition 6.4. Five properties of an eigenvalue

For a square matrix A and a number λ the following is equivalent:

- (i) λ is an eigenvalue of A .
- (ii) There is a vector $\mathbf{x} \neq \mathbf{o}$ with $A\mathbf{x} = \lambda\mathbf{x}$.
- (iii) The space $\text{Ker}(A - \lambda \mathbb{1})$ contains a vector $\mathbf{x} \neq \mathbf{o}$.
- (iv) The matrix $A - \lambda \mathbb{1}$ is not invertible.
- (v) $\det(A - \lambda \mathbb{1}) = 0$

Proof. Exercise! □

Let $A \in \mathbb{R}^{n \times n}$. Then we observe that $\det(A - \lambda \mathbb{1}) = p_A(\lambda)$ is a polynomial of order n in the variable λ . For example, there could be coefficients c_i such that

$$p_A(\lambda) = (-1)^n \lambda^n + c_{n-1} \lambda^{n-1} + \dots + c_1 \lambda + c_0.$$

Definition 6.5. Characteristic polynomial

For an $n \times n$ -Matrix A , the polynomial $\lambda \mapsto \det(A - \lambda \mathbf{1})$ is called the characteristic polynomial of the matrix A and is denoted by p_A .

Example 6.6. Look at $A = \begin{pmatrix} 3 & 2 \\ 1 & 2 \end{pmatrix}$.

$$\begin{aligned} p_A(\lambda) &= \det(A - \lambda \mathbf{1}) = \det\left(\begin{pmatrix} 3 & 2 \\ 1 & 2 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right) = \det\begin{pmatrix} 3-\lambda & 2 \\ 1 & 2-\lambda \end{pmatrix} \\ &= (3-\lambda) \cdot (2-\lambda) - 2 \cdot 1 = 6 - 3\lambda - 2\lambda + \lambda^2 - 2 = \lambda^2 - 5\lambda + 4 \end{aligned}$$

Solving the quadratic equation:

$$\lambda_{1,2} = -\frac{-5}{2} \pm \sqrt{\frac{25}{4} - 4} = \frac{5}{2} \pm \sqrt{\frac{9}{4}} = \frac{5 \pm 3}{2} \in \{1, 4\}, \quad \text{hence } \underline{\lambda_1 = 4}, \underline{\lambda_2 = 1}.$$

If we observe the polynomial in the largest number space we know, the complex numbers, we recall the fundamental theorem of algebra:

Theorem 6.7. Fundamental theorem of algebra (Gauß 1799)

Let $a_0, a_1, \dots, a_n \in \mathbb{C}$ with $a_n \neq 0$. Then the polynomial equation

$$\underbrace{a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x^1 + a_0}_{=: p(x)} = 0$$

has n (not necessarily different) solutions x_1, \dots, x_n in \mathbb{C} . Moreover, we find for $x \in \mathbb{C}$:

$$p(x) = a_n(x - x_1)(x - x_2) \cdots (x - x_n).$$

For the characteristic polynomial, this means:

- Every polynomial has at least one (possibly) *complex* root λ_1 , so we have at least one eigenvalue, but not always a real one. (e.g. $p_A(\lambda) = \lambda^2 + 1$, $\lambda_{1,2} = \pm i$).
- Finding roots of a polynomial means factorisation into linear factors:

$$p_A(\lambda) = (\pm 1)(\lambda - \lambda_1)(\lambda - \lambda_2) \cdots (\lambda - \lambda_n).$$

Sometimes some of the values $\lambda_1 \dots \lambda_n$ are equal (e.g. $p_A(\lambda) = (\lambda - 1)^2$, and then $\lambda_1 = \lambda_2 = 1$), so we have a multiple root.

Definition 6.8.

If the same eigenvalue λ appears $\alpha(\lambda)$ times in this factorisation, we say:

$$\lambda \text{ has } \underline{\text{algebraic multiplicity }} \alpha(\lambda) .$$

- If we have k *different* eigenvalues $\lambda_1, \dots, \lambda_k \in \mathbb{C}$, then $\alpha(\lambda_1) + \dots + \alpha(\lambda_k) = n$, because polynomials of degree n can be factorised into n linear factors.
- If λ is an eigenvalue, then $A - \lambda \mathbf{1}$ is singular, so $\gamma(\lambda) := \dim(\text{Ker}(A - \lambda \mathbf{1})) \geq 1$.

Since we can calculate eigenvalues by calculating determinants, we immediately get the eigenvalues if A has a triangular form or a block triangular form:

Proposition 6.9. Spectrum for triangular matrices

Let $A \in \mathbb{R}^{n \times n}$ be a square matrix.

(a) For a matrix in triangular form

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & & a_{2n} \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & a_{nn} \end{pmatrix},$$

we get $\text{spec}(A) = \{a_{11}, a_{22}, \dots, a_{nn}\}$.

(b) For a square block matrix in triangular form

$$A = \begin{pmatrix} B & C \\ 0 & D \end{pmatrix}$$

with square matrices B and D , we get $\text{spec}(A) = \text{spec}(B) \cup \text{spec}(D)$.

(c) Also $\text{spec}(A) = \text{spec}(A^T)$. Hence (a) and (b) also hold for lower triangular matrices.

Proof. For (b): This immediately follows from Proposition 4.15 since $\lambda \in \text{spec}(A)$ if and only if

$$0 = \det\left(\begin{pmatrix} B & C \\ 0 & D \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right) = \det\begin{pmatrix} B - \lambda \mathbf{1} & C \\ 0 & D - \lambda \mathbf{1} \end{pmatrix} = \det(B - \lambda \mathbf{1}) \det(D - \lambda \mathbf{1}),$$

which means that $\lambda \in \text{spec}(B)$ or $\lambda \in \text{spec}(D)$.

For (a): Use repeatedly (b):

$$0 = \det(A - \lambda \mathbf{1}) = \det\begin{pmatrix} a_{11} - \lambda & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} - \lambda & & a_{2n} \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & a_{nn} - \lambda \end{pmatrix} = (a_{11} - \lambda) \cdot \dots \cdot (a_{nn} - \lambda)$$

if and only if $\lambda \in \{a_{11}, a_{22}, \dots, a_{nn}\}$.

For (c): Using Proposition 4.16, one gets $0 = \det(A - \lambda \mathbf{1}) = \det((A - \lambda \mathbf{1})^T) = \det(A^T - \lambda \mathbf{1}^T) = \det(A^T - \lambda \mathbf{1})$. \square

Example 6.10. We give some examples for Proposition 6.9.

$$(a) \text{ spec} \begin{pmatrix} 1 & 2 & 3 & 4 \\ & 5 & 6 & 7 \\ & & 8 & 9 \\ & & & 10 \end{pmatrix} = \{1, 5, 8, 10\}$$

$$(b) \text{ spec} \begin{pmatrix} 1 & 2 & | & 3 & 4 & 5 \\ 6 & | & 7 & 8 & 9 \\ \hline & | & 10 & & & \\ & | & 11 & 12 & & \\ & | & 13 & 14 & 15 & \end{pmatrix} = \text{spec} \begin{pmatrix} 1 & 2 \\ & 6 \end{pmatrix} \cup \text{spec} \begin{pmatrix} 10 & & \\ 11 & 12 & \\ 13 & 14 & 15 \end{pmatrix} = \{1, 6, 10, 12, 15\}$$

$$(c) \text{ spec} \begin{pmatrix} 1 & 2 & | & 3 & 4 & 5 & 6 \\ 7 & | & 8 & 9 & 10 & 11 \\ \hline & | & 12 & & & \\ & | & 13 & 14 & & \\ & | & 15 & 16 & 17 & 18 \\ & | & 19 & 20 & & 21 \end{pmatrix} = \text{spec} \begin{pmatrix} 1 & 2 \\ & 7 \end{pmatrix} \cup \text{spec} \begin{pmatrix} 12 & & \\ 13 & 14 & \\ 15 & 16 & 17 & 18 \\ 19 & 20 & & 21 \end{pmatrix}$$

$$= \text{spec} \begin{pmatrix} 1 & 2 \\ & 7 \end{pmatrix} \cup \text{spec} \begin{pmatrix} 12 & \\ 13 & 14 \end{pmatrix} \cup \text{spec} \begin{pmatrix} 17 & 18 \\ & 21 \end{pmatrix} = \{1, 7, 12, 14, 17, 21\}$$

From the Leibniz formula of the determinant, we conclude:

Remark:

The characteristic polynomial for $A \in \mathbb{R}^{n \times n}$ is of the following form

$$p_A(\lambda) = (-1)^n \lambda^n + \text{tr}(A)(-1)^{n-1} \lambda^{n-1} + \cdots + \det(A), \quad (6.1)$$

where $\text{tr}(A) := \sum_{j=1}^n a_{jj}$ is the sum of the diagonal, the so-called [trace](#) of A .

6.3 Complex matrices and vectors

We have seen that we need complex numbers when we are talking about eigenvalues. Eigenvalues given in Definition 6.1 should always be complex numbers. Then we also have to see the matrix A and the corresponding eigenvector \mathbf{x} as general complex entities.

Definition 6.11. Complex matrices

For $m, n \in \mathbb{N}$, the set of all $m \times n$ matrices with entries in \mathbb{C} is denoted by $\mathbb{C}^{m \times n}$. Analogously, \mathbb{C}^n denotes the set of all (column-)vectors with n entries in \mathbb{C} .

Naturally, we define the addition and scalar multiplication in \mathbb{C}^n and $\mathbb{C}^{m \times n}$ as we did for the objects with real entries. Indeed, everything works the same and we find:

Proposition 6.12. Properties of the vector space \mathbb{C}^n

The set $V = \mathbb{C}^n$ with the addition $+$ and scalar multiplication \cdot fulfils the following:

- (1) $\forall \mathbf{v}, \mathbf{w} \in V : \mathbf{v} + \mathbf{w} = \mathbf{w} + \mathbf{v}$ ($+$ is commutative)
- (2) $\forall \mathbf{u}, \mathbf{v}, \mathbf{w} \in V : \mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$ ($+$ is associative)
- (3) There is a zero vector $\mathbf{o} \in V$ with the property: $\forall \mathbf{v} \in V$ we have $\mathbf{v} + \mathbf{o} = \mathbf{v}$.
- (4) For all $\mathbf{v} \in V$ there is a vector $-\mathbf{v} \in V$ with $\mathbf{v} + (-\mathbf{v}) = \mathbf{o}$.
- (5) For the number $1 \in \mathbb{C}$ and each $\mathbf{v} \in V$, one has: $1 \cdot \mathbf{v} = \mathbf{v}$.
- (6) $\forall \lambda, \mu \in \mathbb{C} \quad \forall \mathbf{v} \in V : \lambda \cdot (\mu \cdot \mathbf{v}) = (\lambda\mu) \cdot \mathbf{v}$ (\cdot is associative)

- $$(7) \forall \lambda \in \mathbb{C} \quad \forall \mathbf{v}, \mathbf{w} \in V : \quad \lambda \cdot (\mathbf{v} + \mathbf{w}) = (\lambda \cdot \mathbf{v}) + (\lambda \cdot \mathbf{w}) \quad (\text{distributive } \cdot +)$$
- $$(8) \forall \lambda, \mu \in \mathbb{C} \quad \forall \mathbf{v} \in V : \quad (\lambda + \mu) \cdot \mathbf{v} = (\lambda \cdot \mathbf{v}) + (\mu \cdot \mathbf{v}) \quad (\text{distributive } + \cdot)$$

We already mentioned that a set V with an addition and scalar multiplication that fulfils the rules above is called a [vector space](#). However, note that we now can also scale vectors by using complex numbers. To make this clear, we often speak of the [complex vector space](#) \mathbb{C}^n .

Recall the notions of linear dependence, linear independence and basis, which we still use in the complex vector space \mathbb{C}^n .

Definition 6.13. Subspaces in \mathbb{C}^n

A nonempty subset $U \subset \mathbb{C}^n$ is called a [\(linear\) subspace](#) of \mathbb{C}^n if all linear combinations of vectors in U remain also in U . This means:

- $$(1) \quad \mathbf{o} \in U,$$
- $$(2) \quad \mathbf{u} \in U, \quad \lambda \in \mathbb{C} \implies \lambda \mathbf{u} \in U,$$
- $$(3) \quad \mathbf{u}, \mathbf{v} \in U \implies \mathbf{u} + \mathbf{v} \in U.$$

Definition 6.14. Span

Let $M \subset \mathbb{C}^n$ be any non-empty subset. Then we define:

$$\text{Span}(M) := \{\lambda_1 \mathbf{u}_1 + \dots + \lambda_k \mathbf{u}_k : \mathbf{u}_1, \dots, \mathbf{u}_k \in M, \lambda_1, \dots, \lambda_k \in \mathbb{C}, k \in \mathbb{N}\}.$$

This subspace is called the [span](#) or the [linear hull](#) of M . For convenience, we define $\text{Span}(\emptyset) := \{\mathbf{o}\}$.

Definition 6.15. Linear dependence and independence

A family $(\mathbf{v}_1, \dots, \mathbf{v}_k)$ of k vectors from \mathbb{C}^n is called [linearly dependent](#) if we find a non-trivial linear combination for \mathbf{o} . This means that we can find $\lambda_1, \dots, \lambda_n \in \mathbb{C}$ that are not all equal zero such that

$$\sum_{j=1}^k \lambda_j \mathbf{v}_j = \mathbf{o}.$$

If this is not possible, we call the family $(\mathbf{v}_1, \dots, \mathbf{v}_k)$ [linearly independent](#). This means that

$$\sum_{j=1}^k \lambda_j \mathbf{v}_j = \mathbf{o} \Rightarrow \lambda_1, \dots, \lambda_k = 0$$

holds.

Definition 6.16. Basis, basis vectors

Let V be a subspace of \mathbb{C}^n . A family $\mathcal{B} = (\mathbf{v}_1, \dots, \mathbf{v}_k)$ is called a [basis](#) of V if

- $$(a) \quad V = \text{Span}(\mathcal{B}) \text{ and}$$

(b) \mathcal{B} is linearly independent.

The elements of \mathcal{B} are called the basis vectors of V .

Even in the complex vector space \mathbb{C}^n , we are able speak of geometry when endowing the space with an inner product. We try to generalise what we know from the complex plane \mathbb{C} and the real vector space \mathbb{R}^n .

Definition 6.17. Inner product in \mathbb{C}^n

For the vectors

$$\mathbf{u} = \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix}, \quad \mathbf{v} = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} \in \mathbb{C}^n \quad \text{the number} \quad \langle \mathbf{u}, \mathbf{v} \rangle := u_1 \bar{v}_1 + \dots + u_n \bar{v}_n = \sum_{i=1}^n u_i \bar{v}_i$$

is called the (standard) inner product of \mathbf{u} and \mathbf{v} . Moreover, we define the real number

$$\|\mathbf{v}\| := \sqrt{\langle \mathbf{v}, \mathbf{v} \rangle} = \sqrt{|v_1|^2 + \dots + |v_n|^2}$$

and call it the norm of \mathbf{v} .

Attention!

In some other books, you might find an alternative definition of the standard inner product in \mathbb{C}^n where the first argument is the complex conjugated one.

Note that $\langle \mathbf{v}, \mathbf{v} \rangle$ is always a real number with ≥ 0 such it gives us indeed a length. Again, we find the important property: $\langle \mathbf{v}, \mathbf{v} \rangle = 0$ if and only if $\mathbf{v} = \mathbf{o}$. Hence, $\sqrt{\langle \mathbf{v}, \mathbf{v} \rangle}$ is well-defined and the norm $\|\cdot\|$ has the same properties as in \mathbb{R}^n , see Proposition 6.19 below.

Proposition 6.18.

The standard inner product $\langle \cdot, \cdot \rangle : \mathbb{C}^n \times \mathbb{C}^n \rightarrow \mathbb{C}$ fulfils the following: For all vectors $\mathbf{x}, \mathbf{x}', \mathbf{y} \in \mathbb{C}^n$ and $\lambda \in \mathbb{C}$, one has

- (S1) $\langle \mathbf{x}, \mathbf{x} \rangle > 0$ for $\mathbf{x} \neq \mathbf{o}$, (positive definite)
- (S2) $\langle \mathbf{x} + \mathbf{x}', \mathbf{y} \rangle = \langle \mathbf{x}, \mathbf{y} \rangle + \langle \mathbf{x}', \mathbf{y} \rangle$, (additive)
- (S3) $\langle \lambda \mathbf{x}, \mathbf{y} \rangle = \lambda \langle \mathbf{x}, \mathbf{y} \rangle$, (homogeneous) } (linear)
- (S4) $\langle \mathbf{x}, \mathbf{y} \rangle = \overline{\langle \mathbf{y}, \mathbf{x} \rangle}$. (conjugate symmetric)

Proposition 6.19. Norm

The norm $\|\cdot\| : \mathbb{C}^n \rightarrow \mathbb{R}$ defined by using the standard inner product satisfies for all $\mathbf{x}, \mathbf{y} \in \mathbb{C}^n$ and $\alpha \in \mathbb{C}$:

- (N1) $\|\mathbf{x}\| \geq 0$, and $\|\mathbf{x}\| = 0 \Leftrightarrow \mathbf{x} = \mathbf{o}$, (positive definite)
- (N2) $\|\alpha \mathbf{x}\| = |\alpha| \|\mathbf{x}\|$, (absolutely homogeneous)
- (N3) $\|\mathbf{x} + \mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\|$. (triangle inequality).

Looking at the standard inner product in \mathbb{R}^n , we know that A^T satisfies the equation $\langle Ax, y \rangle = \langle x, A^T y \rangle$ for all $x \in \mathbb{R}^n, y \in \mathbb{R}^m$ (Proposition 3.36). In the standard inner product in \mathbb{C}^n , we also have the complex conjugation involved.

Proposition & Definition 6.20. Adjoint matrix

For a given matrix

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix} \in \mathbb{C}^{m \times n}, \text{ the matrix } A^* := \overline{A^T} = \begin{pmatrix} \overline{a_{11}} & \cdots & \overline{a_{m1}} \\ \vdots & & \vdots \\ \overline{a_{1n}} & \cdots & \overline{a_{mn}} \end{pmatrix} \in \mathbb{C}^{n \times m}$$

is called the adjoint matrix of A . It is the uniquely determined matrix that fulfills the equation

$$\langle Ax, y \rangle = \langle x, A^* y \rangle$$

for all $x \in \mathbb{C}^n$ and $y \in \mathbb{C}^m$.

Proof. For $\mathbf{u}, \mathbf{v} \in \mathbb{C}^n$ with

$$\mathbf{u} = \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix}, \mathbf{v} = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} \quad \text{we have} \quad \langle \mathbf{u}, \mathbf{v} \rangle = u_1 \overline{v_1} + \dots + u_n \overline{v_n} = (\overline{v_1} \ \dots \ \overline{v_n}) \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix} = \mathbf{v}^* \mathbf{u}.$$

Hence for all: $\mathbf{x} \in \mathbb{C}^n$ and $\mathbf{y} \in \mathbb{C}^m$

$$\langle Ax, y \rangle = y^*(Ax) = (y^* A)x = (A^* y)^* x = \langle x, A^* y \rangle. \quad \square$$

In analogy to Proposition 6.9 (c), we get the following for complex matrices:

Proposition 6.21. Spectrum of A^*

For all $A \in \mathbb{C}^{n \times n}$, we have $\text{spec}(A^*) = \{\bar{\lambda} : \lambda \in \text{spec}(A)\}$.

Proof. $\det(A^* - \lambda \mathbf{1}) = \det((A - \bar{\lambda} \mathbf{1})^*) = \overline{\det(A - \bar{\lambda} \mathbf{1})}$ by Proposition 4.15 and $\overline{z+w} = \bar{z} + \bar{w}$ and $\overline{z \cdot w} = \bar{z} \cdot \bar{w}$. \square

Some important notions:

Definition 6.22.

A complex matrix $A \in \mathbb{C}^{n \times n}$ is called

- selfadjoint if $A = A^*$ (complex version of “symmetric”),
- skew-adjoint if $A = -A^*$ (complex version of “skew-symmetric”),
- unitary if $AA^* = \mathbf{1} = A^*A$ (complex version of “orthogonal”),
- normal if $AA^* = A^*A$.

Beispiel 6.23. (a) $A = \begin{pmatrix} 1 & 2i \\ -2i & 0 \end{pmatrix} \Rightarrow A^* = \begin{pmatrix} \bar{1} & \overline{-2i} \\ \bar{2i} & \bar{0} \end{pmatrix} = \begin{pmatrix} 1 & 2i \\ -2i & 0 \end{pmatrix} = A$

$$(b) A = \begin{pmatrix} i & -1+2i \\ 1+2i & 3i \end{pmatrix} \Rightarrow A^* = \begin{pmatrix} \overline{i} & \overline{1+2i} \\ \overline{-1+2i} & \overline{3i} \end{pmatrix} = \begin{pmatrix} -i & 1-2i \\ -1-2i & -3i \end{pmatrix} = -A$$

$$(c) A = \begin{pmatrix} 1+i & 3-2i \\ 2i & -1 \end{pmatrix} \Rightarrow A^* = \begin{pmatrix} \overline{1+i} & \overline{3-2i} \\ \overline{2i} & \overline{-1} \end{pmatrix} = \begin{pmatrix} 1-i & -2i \\ 3+2i & -1 \end{pmatrix} \notin \{A, -A\}$$

If $A \in \mathbb{C}^{n \times n}$ is a real matrix, i.e. $a_{ij} \in \mathbb{R}$ for all i, j , then we get:

adjoint matrix A^*	=	transpose matrix A^T ,
selfadjoint	=	symmetric,
skew-adjoint	=	skew-symmetric,
unitary	=	orthogonal.

Proposition 6.24. Where are the eigenvalues?

- (a) If $A^* = A$ (selfadjoint), then all eigenvalues of A lie on the real line.
- (b) If $A^* = -A$ (skew-adjoint), then all eigenvalues of A lie on the imaginary axis.
- (c) If $A^*A = \mathbb{1}$ (unitary), then all eigenvalues of A lie on the unit circle in \mathbb{C} .

Proof. If λ is an eigenvalue of A , then we find an eigenvalue $\mathbf{x} \neq \mathbf{0}$ with $A\mathbf{x} = \lambda\mathbf{x}$. Using Proposition 6.2, we normalise \mathbf{x} , which means that we multiply it by $\alpha = \frac{1}{\|\mathbf{x}\|}$. The resulting vector is an eigenvector with norm 1. Therefore, consider the case $\|\mathbf{x}\| = 1$. Then:

$$\lambda = \lambda\|\mathbf{x}\|^2 = \lambda\langle \mathbf{x}, \mathbf{x} \rangle = \langle \lambda\mathbf{x}, \mathbf{x} \rangle = \langle A\mathbf{x}, \mathbf{x} \rangle. \quad (6.2)$$

(a): If $A = A^*$, then:

$$\lambda \stackrel{(6.2)}{=} \langle A\mathbf{x}, \mathbf{x} \rangle = \langle \mathbf{x}, A^*\mathbf{x} \rangle \stackrel{A^*=A}{=} \langle \mathbf{x}, A\mathbf{x} \rangle \stackrel{(S4)}{=} \overline{\langle A\mathbf{x}, \mathbf{x} \rangle} = \bar{\lambda},$$

which implies $\lambda \in \mathbb{R}$.

(b): If $A = -A^*$, then:

$$\lambda \stackrel{(6.2)}{=} \langle A\mathbf{x}, \mathbf{x} \rangle = \langle \mathbf{x}, A^*\mathbf{x} \rangle \stackrel{A^*=-A}{=} \langle \mathbf{x}, -A\mathbf{x} \rangle \stackrel{(S4)}{=} \overline{\langle -A\mathbf{x}, \mathbf{x} \rangle} = -\overline{\langle A\mathbf{x}, \mathbf{x} \rangle} = -\bar{\lambda},$$

which implies $\lambda = iy$ with $y \in \mathbb{R}$.

(c): If $A^*A = \mathbb{1}$, then:

$$1 = \|\mathbf{x}\|^2 = \langle \mathbf{x}, \mathbf{x} \rangle = \langle \mathbf{x}, \underbrace{A^*A}_{\mathbb{1}} \mathbf{x} \rangle = \langle A\mathbf{x}, A\mathbf{x} \rangle = \|A\mathbf{x}\|^2 = \|\lambda\mathbf{x}\|^2 = (|\lambda|\|\mathbf{x}\|)^2 = |\lambda|^2. \quad \square$$

6.4 Eigenvalues and similarity

Definition 6.25. Similarity

Two matrices $A, B \in \mathbb{C}^{n \times n}$ are called similar if there is an invertible $S \in \mathbb{C}^{n \times n}$ with $A = S^{-1}BS$.

Proposition 6.26.

Similar matrices have the same characteristic polynomial and thus the same eigenvalues.

Proof. Direct computation. Let $A = S^{-1}BS$, so

$$A - \lambda\mathbb{1} = S^{-1}BS - \lambda S^{-1}S = S^{-1}(B - \lambda\mathbb{1})S.$$

Thus, $A - \lambda\mathbb{1}$ and $B - \lambda\mathbb{1}$ are similar matrices. Similar matrices have the same determinant:

$$p_A(\lambda) = \det(A - \lambda\mathbb{1}) = \det(B - \lambda\mathbb{1}) = p_B(\lambda). \quad \square$$

Remark:

Later, we will see that any matrix $A \in \mathbb{C}^{n \times n}$ is similar to a triangular matrix.

6.5 Calculating eigenvectors

Even for matrices $A \in \mathbb{R}^{n \times n}$, we now consider the eigenvalues in \mathbb{C} and the eigenvectors in \mathbb{C}^n . This means that we now consider all square matrices as matrices in $\mathbb{C}^{n \times n}$.

Example 6.27. Consider $A \in \mathbb{R}^{2 \times 2}$ with $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. Then $p_A(\lambda) = \lambda^2 + 1$ and $\text{spec}(A) = \{-i, i\}$.

The corresponding map $f_A : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ rotates \mathbf{e}_1 and \mathbf{e}_2 , and hence any vector in \mathbb{R}^2 , by an angle of $\frac{\pi}{2}$ (or 90°) in positive sense. In this sense, no line is sent to itself again. However, this is only a problem if we look at the “real” picture.

We fix two important properties:

Proposition 6.28. Spectrum is not empty

For a square matrix $A \in \mathbb{C}^{n \times n}$ holds:

- (a) $\text{spec}(A) \neq \emptyset$.
- (b) A is invertible if and only if $0 \notin \text{spec}(A)$.

Proof. A is not invertible $\iff 0 = \det(A) = \det(A - 0\mathbb{1}) \iff 0 \in \text{spec}(A)$. \square

Looking at Proposition 6.4, we see what we have to do in order to calculate the eigenvectors of a given matrix A if we already know the eigenvalues λ :

Definition 6.29. Eigenspace

The solution set of the LES $(A - \lambda\mathbb{1})\mathbf{x} = \mathbf{0}$, which means $\text{Ker}(A - \lambda\mathbb{1})$, is called the eigenspace with respect to the eigenvalue λ and denoted by $\text{Eig}(\lambda)$. Each nonzero

vector $\mathbf{x} \in \text{Eig}(\lambda) \setminus \{\mathbf{o}\}$ is an eigenvector w.r.t. the eigenvalue λ .

Note that $\text{Eig}(\lambda)$ is always a linear subspace and makes also sense in the case when λ is not an eigenvalue of A . In this instance, we simply have $\text{Eig}(\lambda) = \{\mathbf{o}\}$.

Example 6.30. Consider $A = \begin{pmatrix} 3 & 2 \\ 1 & 2 \end{pmatrix}$: $\mathbf{x}_i \neq \mathbf{o}$ is an eigenvector for λ_i with $i \in \{1, 2\}$ if

$$A\mathbf{x}_i = \lambda_i \mathbf{x}_i, \quad \text{i.e.} \quad (A - \lambda_i \mathbb{1})\mathbf{x}_i = \mathbf{o}.$$

Hence, we have to solve the LES $(A - \lambda_1 \mathbb{1})\mathbf{x}_1 = \mathbf{o}$ and $(A - \lambda_2 \mathbb{1})\mathbf{x}_2 = \mathbf{o}$.

$$\underline{\lambda_1 = 4}: \quad A - \lambda_1 \mathbb{1} = \begin{pmatrix} 3 - \lambda_1 & 2 \\ 1 & 2 - \lambda_1 \end{pmatrix} = \begin{pmatrix} -1 & 2 \\ 1 & -2 \end{pmatrix},$$

$$(A - \lambda_1 \mathbb{1})\mathbf{x}_1 = \mathbf{o} \iff \begin{pmatrix} -1 & 2 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \iff \underline{\mathbf{x}_1 = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}}$$

In the same manner:

$$\underline{\lambda_2 = 1}: \quad A - \lambda_2 \mathbb{1} = \begin{pmatrix} 3 - \lambda_2 & 2 \\ 1 & 2 - \lambda_2 \end{pmatrix} = \begin{pmatrix} 2 & 2 \\ 1 & 1 \end{pmatrix},$$

$$(A - \lambda_2 \mathbb{1})\mathbf{x}_2 = \mathbf{o} \iff \begin{pmatrix} 2 & 2 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \iff \underline{\mathbf{x}_2 = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}}$$

Definition 6.31. Multiplicities

Let $A \in \mathbb{C}^{n \times n}$ be square matrix. Then the characteristic polynomial can be written as:

$$p_A(z) = (\lambda_1 - z)^{\alpha_1} \cdot (\lambda_2 - z)^{\alpha_2} \cdots (\lambda_k - z)^{\alpha_k} \quad (6.3)$$

where $\lambda_1, \dots, \lambda_k$ are pairwise different. The natural number α_j above is called:

$$\alpha(\lambda_j) := \alpha_j \quad \text{algebraic multiplicity of } \lambda_j$$

and tells you how often the eigenvalue λ_j occurs in the characteristic polynomial. We also define

$$\gamma(\lambda_j) := \dim(\text{Eig}(\lambda_j)) = \dim(\text{Ker}(A - \lambda_j \mathbb{1})) \quad \text{geometric multiplicity of } \lambda_j$$

Remark: Recipe for calculating eigenvectors

Let $A \in \mathbb{C}^{n \times n}$ be a square matrix.

(1) The eigenvalues λ are the zeros of the characteristic polynomial p_A of A . In other words, the solutions of

$$p_A(\lambda) = \det(A - \lambda \mathbb{1}) = 0.$$

(2) If A is real, then $p_A(\lambda)$ is a real polynomial. If it has a complex zero $\lambda \notin \mathbb{R}$, then its conjugate $\bar{\lambda}$ is also a zero,

- (3) If one eigenvalue is found, we can reduce the characteristic polynomial by equating coefficients (or polynomial division).
- (4) The eigenvectors \mathbf{x} are given by the solutions of the LES $(A - \lambda \mathbb{1})\mathbf{x} = \mathbf{0}$ for each eigenvalue, where only the nonzero solutions $\mathbf{x} \neq \mathbf{0}$ are interesting.

Example 6.32.

$$p(\lambda) = -\lambda^3 + 5\lambda^2 - 8\lambda + 6$$

- $n = 3$ is odd: “ $-\lambda^3$ ”
- Try some values and find: $\lambda_1 = 3$.

Equating coefficients (or polynomial division):

We derive four equations for three unknowns in the following way:

$$\begin{aligned} -\lambda^3 + 5\lambda^2 - 8\lambda + 6 &= (a\lambda^2 + b\lambda + c)(\lambda - 3) = a\lambda^3 + (b - 3a)\lambda^2 + (c - 3b)\lambda - 3c \\ -1 &= a \quad \Rightarrow \quad a = -1 \\ 5 &= b - 3a \quad \Rightarrow \quad 5 = b + 3 \Rightarrow b = 2 \\ -8 &= c - 3b \quad \Rightarrow \quad c - 6 = -8 \Rightarrow c = -2 \\ 6 &= -3c \quad \text{fulfilled, so } \lambda_1 = 3 \text{ is really a root of } p. \end{aligned}$$

Factorisation:

$$p(\lambda) = (-\lambda^2 + 2\lambda - 2)(\lambda - 3)$$

Solution of quadratic equation:

$$\lambda_{2,3} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-2 \pm \sqrt{4 - 8}}{-2} = 1 \pm i.$$

Result:

$$p(\lambda) = -(\lambda - 3)(\lambda - (i + 1))(\lambda - (i - 1)).$$

Exercise 6.33.

Let A be a square matrix and λ_1, λ_2 two different eigenvalues. Show that

$$\text{Eig}(\lambda_1) \cap \text{Eig}(\lambda_2) = \{\mathbf{0}\}$$

6.6 The spectral mapping theorem

Let $\lambda \in \mathbb{C}$ be an eigenvalue of $A \in \mathbb{C}^{n \times n}$ corresponding to the eigenvector $\mathbf{x} \in \mathbb{C}^n$, which means $A\mathbf{x} = \lambda\mathbf{x}$. Then we get for the powers:

$$\begin{aligned} A^2\mathbf{x} &= A(A\mathbf{x}) = A(\lambda\mathbf{x}) = \lambda A\mathbf{x} = \lambda\lambda\mathbf{x} = \lambda^2\mathbf{x}, \\ A^3\mathbf{x} &= A(A^2\mathbf{x}) = A(\lambda^2\mathbf{x}) = \lambda^2 A\mathbf{x} = \lambda^2\lambda\mathbf{x} = \lambda^3\mathbf{x}, \\ &\vdots \\ A^m\mathbf{x} &= \lambda^m\mathbf{x} \quad \text{for all } m \in \mathbb{N}, \end{aligned} \tag{6.4}$$

We conclude that A^m has also the eigenvector \mathbf{x} but now it corresponds to the eigenvalue λ^m instead of λ .

Now we could also bring in the addition of the matrices A^0, A^1, A^2 , and so on, and get a similar result.

Proposition 6.34. Polynomial spectral mapping theorem

Let $p(\lambda) = p_m \lambda^m + p_{m-1} \lambda^{m-1} + \dots + p_1 \lambda + p_0$ be a polynomial and $A \in \mathbb{C}^{n \times n}$ a square matrix. Putting the matrix A into p (formally), we get the following matrix:

$$p(A) := p_m A^m + p_{m-1} A^{m-1} + \dots + p_1 A + p_0 \mathbb{1}.$$

It is again an $n \times n$ matrix, and we get

$$\text{spec}(p(A)) = \{p(\lambda) : \lambda \in \text{spec}(A)\}.$$

Moreover, each eigenvector of A is also an eigenvector of $p(A)$.

Proof. Let us denote $\{p(\lambda) : \lambda \in \text{spec}(A)\}$ by M . Then we have to show two inclusions to prove $\text{spec}(p(A)) = M$.

(\supset): For $\lambda \in \text{spec}(A)$ with eigenvector \mathbf{x} we use (6.4) and get

$$\begin{aligned} p(A)\mathbf{x} &= (p_m A^m + p_{m-1} A^{m-1} + \dots + p_1 A + p_0 \mathbb{1})\mathbf{x} \\ &= p_m A^m \mathbf{x} + p_{m-1} A^{m-1} \mathbf{x} + \dots + p_1 A \mathbf{x} + p_0 \mathbb{1} \mathbf{x} \\ &= p_m \lambda^m \mathbf{x} + p_{m-1} \lambda^{m-1} \mathbf{x} + \dots + p_1 \lambda \mathbf{x} + p_0 \mathbf{x} \\ &= (p_m \lambda^m + p_{m-1} \lambda^{m-1} + \dots + p_1 \lambda + p_0) \mathbf{x} = p(\lambda) \mathbf{x}. \end{aligned}$$

Hence, the number $p(\lambda) \in M$ is an eigenvalue of the matrix $p(A)$ with the same eigenvector \mathbf{x} .

(\subset): First, assume that p is a constant polynomial $p(z) = p_0 \in \mathbb{C}$. Then let $\lambda \in \text{spec}(p(A))$, which means $0 = \det(p(A) - \lambda \mathbb{1}) = (p_0 - \lambda)^n$. We conclude $\lambda = p_0$ and $\lambda \in M$.

Now assume that p is not constant and $\mu \notin M = \{p(\lambda) : \lambda \in \text{spec}(A)\}$. (*We do a contraposition*). Then the polynomial $q(z) := p(z) - \mu$ can be written in linear factors

$$q(z) = c \prod_{j=1}^m (z - a_j)$$

with $c \neq 0$ and $a_j \in \mathbb{C}$. We get $p(a_j) = \mu$ for all j , and, hence, $a_j \notin \text{spec}(A)$. This means that $\det(A - a_j \mathbb{1}) \neq 0$ for all j , which also implies

$$\det(p(A) - \mu \mathbb{1}) = \det(q(A)) = c^n \prod_{j=1}^m \det(A - a_j \mathbb{1}) \neq 0.$$

Therefore, μ is not an eigenvalue of $p(A)$ and we conclude $\mu \notin \text{spec}(p(A))$. \square

Example 6.35. Let $A = \begin{pmatrix} 3 & 2 \\ 1 & 2 \end{pmatrix}$. We want to know the eigenvalues of the following matrix $B = 3A^3 - 7A^2 + A - 2\mathbb{1}$.

We do not need to calculate the matrix B since we can use the spectral mapping theorem: We know the eigenvalues of A from before: $\lambda_1 = 4$ and $\lambda_2 = 1$. Using the Proposition 6.34, you only need to put these the numbers into: $p(\lambda) = 3\lambda^3 - 7\lambda^2 + \lambda - 2$. Hence, for B , we find the following eigenvalues

$$\begin{aligned} p(\lambda_1) &= 3\lambda_1^3 - 7\lambda_1^2 + \lambda_1 - 2 = 3 \cdot 64 - 7 \cdot 16 + 4 - 2 = 82 \quad \text{and} \\ p(\lambda_2) &= 3\lambda_2^3 - 7\lambda_2^2 + \lambda_2 - 2 = 3 \cdot 1 - 7 \cdot 1 + 1 - 2 = -5. \end{aligned}$$

The eigenvectors $\mathbf{x}_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ and $\mathbf{x}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$, we found for A , are also eigenvectors for the matrix B . Moreover, \mathbf{x}_1 corresponds to the eigenvalue 82 and \mathbf{x}_2 to the eigenvalue -5 .

We can expand the spectral mapping theorem also to other functions besides polynomials. For example, it also works for the negative powers. This means that we can calculate the eigenvalues of A^{-1} if you know the eigenvalues of A . In this case, you do not have to calculate the inverse A^{-1} :

Let all the eigenvalues of A be nonzero (in this case, recalling Proposition 6.28 the inverse A^{-1} exists) and fix one of them as λ with corresponding eigenvector $\mathbf{x} \neq \mathbf{0}$. Then we can multiply the equation

$$A\mathbf{x} = \lambda\mathbf{x}$$

from the left by A^{-1} . Hence we get $\mathbf{x} = \lambda A^{-1}\mathbf{x}$ and also:

$$A^{-1}\mathbf{x} = \lambda^{-1}\mathbf{x}. \quad (6.5)$$

Rule of thumb:

A^{-1} has the same eigenvector \mathbf{x} as A – but for the eigenvalue λ^{-1} instead of λ .

We simply get:

$$\text{spec}(A^{-1}) = \{\lambda^{-1} : \lambda \in \text{spec}(A)\}.$$

Of course, λ^{-1} is always well-defined since $\lambda \neq 0$.

Example 6.36. Let $A = \begin{pmatrix} 3 & 2 \\ 1 & 2 \end{pmatrix}$. Now, one could calculate the inverse, using formulas from Chapter 4:

$$A^{-1} = \begin{pmatrix} 1/2 & -1/2 \\ -1/4 & 3/4 \end{pmatrix}$$

This matrix has the eigenvalues $\mu_1 = 1/4$ and $\mu_2 = 1$ and the eigenvectors $\mathbf{x}_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ and $\mathbf{x}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

If one is only interested in the eigenvalues, we do not have to calculate the matrix A^{-1} . We just use the rule from above and know that the eigenvalues of A^{-1} are the reciprocals of $\lambda_1 = 4$ and $\lambda_2 = 1$. The eigenvectors are the same as the eigenvectors of A .

We do not have to stop here. We can multiply A^{-1} again from the left to equation (6.5) and, doing this repeatedly, we get

$$A^{-2}\mathbf{x} = \lambda^{-2}\mathbf{x}, \quad A^{-3}\mathbf{x} = \lambda^{-3}\mathbf{x}, \quad \text{etc.},$$

where A^{-n} means $(A^{-1})^n$.

Hence, we can expand the equation (6.4) to all numbers $m \in \mathbb{Z}$:

$$A^m \mathbf{x} = \lambda^m \mathbf{x} \quad \text{for all } m \in \mathbb{Z}.$$

Of course, if we can also expand it to linear combinations $\dots, A^{-2}, A^{-1}, A^0, A^1, A^2, \dots$ which shows that our spectral mapping theorem is only a special case of a more general one.

6.7 Diagonalisation – the optimal coordinates

We startet this chapter with a two-dimensional picture. Now, we again revisit the 2×2 -example $A = \begin{pmatrix} 3 & 2 \\ 1 & 2 \end{pmatrix}$. We know that $\lambda_1 = 4$ and $\lambda_2 = 1$ are the eigenvalues with associated eigenvectors $\mathbf{x}_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ and $\mathbf{x}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

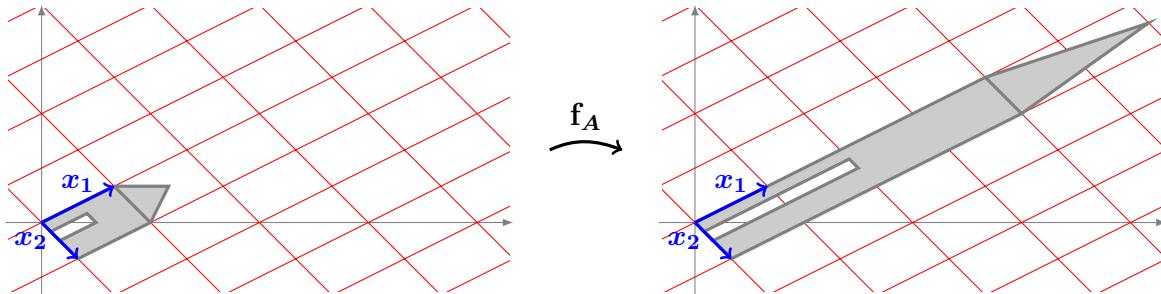
Hence we know the two vectors, \mathbf{x}_1 and \mathbf{x}_2 , that span the lines g_3 and g_4 from the start of the chapter, respectively. Also, we know the “stretch factors” $\lambda_1 = 4$ and $\lambda_2 = 1$ that describe the acting of f_A in the direction g_3 and g_4 , respectively. Therefore, choosing a coordinate system with respect to $(\mathbf{x}_1, \mathbf{x}_2)$ -coordinates makes calculations a lot easier:

Optimal coordinates for A

By using for $\mathbf{u} \in \mathbb{R}^2$ the linear combination $\mathbf{u} = \alpha_1 \mathbf{x}_1 + \alpha_2 \mathbf{x}_2$ with coefficients $\alpha_1, \alpha_2 \in \mathbb{R}$, we get

$$A\mathbf{u} = A(\alpha_1 \mathbf{x}_1 + \alpha_2 \mathbf{x}_2) = \alpha_1(A\mathbf{x}_1) + \alpha_2(A\mathbf{x}_2) = \alpha_1(4\mathbf{x}_1) + \alpha_2(1\mathbf{x}_2) = 4\alpha_1 \mathbf{x}_1 + 1\alpha_2 \mathbf{x}_2.$$

The component in \mathbf{x}_1 -direction, which is α_1 , is scaled by the factor $\lambda_1 = 4$, and the \mathbf{x}_2 -component α_2 is scaled by the factor $\lambda_2 = 1$.



We immediately see a big advantage for this coordinate system: We can apply A several times without effort. For example the operation A^{100} is directly given by: $A^{100}\mathbf{u} = 4^{100}\alpha_1 \mathbf{x}_1 + 1^{100}\alpha_2 \mathbf{x}_2$

However, we already know that in general we cannot expect to stay in the real numbers. If the eigenvalues are strictly complex numbers, the picture gets a little bit more complicated but the properties remain. Let $A \in \mathbb{C}^{n \times n}$ be a $n \times n$ matrix with complex entries. Let $\lambda_1, \dots, \lambda_n \in \mathbb{C}$ denote the n eigenvalues A (which means the zeros p_A) counted with algebraic multiplicities, and let $\mathbf{x}_1, \dots, \mathbf{x}_n \in \mathbb{C}^n$ be the corresponding eigenvectors. Then we already know:

$$A\mathbf{x}_1 = \lambda_1 \mathbf{x}_1, \dots, A\mathbf{x}_n = \lambda_n \mathbf{x}_n. \quad (6.6)$$

This is what we can put together into a matrix equation:

$$\begin{aligned}
 A \underbrace{\begin{pmatrix} | & & | \\ \mathbf{x}_1 & \dots & \mathbf{x}_n \\ | & & | \end{pmatrix}}_{=:X} &= \begin{pmatrix} | & & | \\ A\mathbf{x}_1 & \dots & A\mathbf{x}_n \\ | & & | \end{pmatrix} \\
 \stackrel{(6.6)}{=} \begin{pmatrix} | & & | \\ \lambda_1\mathbf{x}_1 & \dots & \lambda_n\mathbf{x}_n \\ | & & | \end{pmatrix} &= \begin{pmatrix} | & & | \\ \mathbf{x}_1 & \dots & \mathbf{x}_n \\ | & & | \end{pmatrix} \underbrace{\begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}}_{=:D},
 \end{aligned}$$

or in short: $AX = XD$. This means that A is similar to a diagonal matrix if X is invertible.

Now we can look what happens if AX and XD act on a given vector $\mathbf{v} = (\alpha_1 \dots \alpha_n)^T \in \mathbb{C}^n$. The equation $AX\mathbf{v} = XD\mathbf{v}$ can be written in the following form:

$$A(\alpha_1\mathbf{x}_1 + \dots + \alpha_n\mathbf{x}_n) = \lambda_1\alpha_1\mathbf{x}_1 + \dots + \lambda_n\alpha_n\mathbf{x}_n.$$

Hence, if $\mathbf{u} \in \mathbb{C}^n$ is given as a linear combination $\alpha_1\mathbf{x}_1 + \dots + \alpha_n\mathbf{x}_n$ using only the eigenvectors $\mathbf{x}_1, \dots, \mathbf{x}_n$ and a coefficient vector $(\alpha_1 \dots \alpha_n)^T$, then $A\mathbf{u}$ is also a linear combination of the eigenvectors $\mathbf{x}_1, \dots, \mathbf{x}_n$, now with coefficient vector $(\lambda_1\alpha_1 \dots \lambda_n\alpha_n)^T$. Hence in this coordinate system A acts in this way:

$$\begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix} \mapsto \begin{pmatrix} \lambda_1\alpha_1 \\ \vdots \\ \lambda_n\alpha_n \end{pmatrix}.$$

This is exactly the multiplication with the diagonal matrix D .

In mathematical terms: By changing the basis of \mathbb{C}^n from the canonical unit vectors $(\mathbf{e}_1, \dots, \mathbf{e}_n)$ to a basis consisting of eigenvectors $(\mathbf{x}_1, \dots, \mathbf{x}_n)$, the matrix A changes to a simple diagonal matrix D .

Diagonalisation of A

Choose $X = \begin{pmatrix} | & | \\ \mathbf{x}_1 & \dots & \mathbf{x}_n \\ | & | \end{pmatrix}$ and

$$D = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}. \text{ Then:}$$

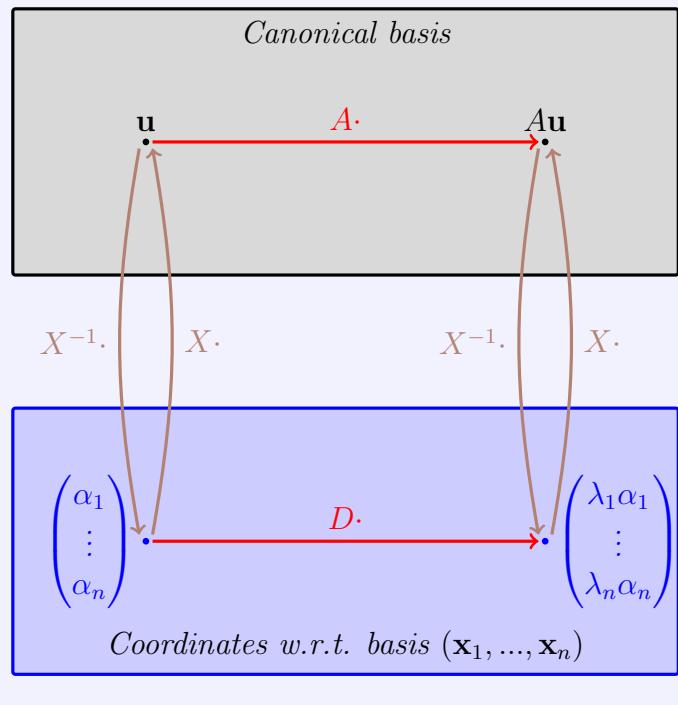
$$AX = XD. \quad (6.7)$$

Multiplication (6.7) · X^{-1} gives:

$$A = XDX^{-1} \quad (6.8)$$

and in the same ways $X^{-1} \cdot (6.7)$ gives:

$$X^{-1}AX = D.$$



The important question “Is that even possible?” is equivalent to the following:

- Can we write all $\mathbf{u} \in \mathbb{C}^n$ as $\alpha_1\mathbf{x}_1 + \dots + \alpha_n\mathbf{x}_n$?
- $\text{Span}(\mathbf{x}_1, \dots, \mathbf{x}_n) = \mathbb{C}^n$?
- Is $(\mathbf{x}_1, \dots, \mathbf{x}_n)$ a basis of \mathbb{C}^n ?
- Is X invertible?

Definition 6.37. Diagonalisability

A square matrix $A \in \mathbb{C}^{n \times n}$ is called diagonalisable if one can find n different eigenvectors $\mathbf{x}_1, \dots, \mathbf{x}_n \in \mathbb{C}^n$ that form a basis of \mathbb{C}^n .

Example 6.38. (a) The matrix $A = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$ has \mathbf{e}_1 and \mathbf{e}_2 as eigenvectors and they form a basis of \mathbb{C}^2 . Hence, A is diagonalisable.

(b) The matrix $B = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}$ has $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ as eigenvectors and they form a basis of \mathbb{C}^2 . Hence, B is diagonalisable.

(c) The matrix $C = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ has only eigenvectors in direction $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and they cannot form a basis of \mathbb{C}^2 . Hence, C is not diagonalisable.

Choosing a basis consisting of eigenvectors, we know that A acts like a diagonal matrix.

Proposition 6.39. Different eigenvalues \Rightarrow linearly ind. eigenvectors

If $\lambda_1, \dots, \lambda_k$ are k different eigenvalues of A , then each family $(\mathbf{x}_1, \dots, \mathbf{x}_k)$ of corresponding eigenvectors is linearly independent.

Proof. We use mathematical induction over k . The case $k = 2$ was proven Exercise 6.33. Now the induction hypothesis is that $(\mathbf{x}_1, \dots, \mathbf{x}_k)$ is linearly independent for k different eigenvalues. In the induction step, we now look at $k+1$ different eigenvalues $\lambda_1, \dots, \lambda_k, \lambda_{k+1}$ and corresponding eigenvectors $(\mathbf{x}_1, \dots, \mathbf{x}_k, \mathbf{x}_{k+1})$. Now assume that this family is linearly dependent. Then we can choose coefficients β_i such that $\mathbf{x}_{k+1} = \beta_1 \mathbf{x}_1 + \dots + \beta_k \mathbf{x}_k$ holds. Hence,

$$\begin{aligned} & (\lambda_{k+1} - \lambda_1) \beta_1 \mathbf{x}_1 + \dots + (\lambda_{k+1} - \lambda_k) \beta_k \mathbf{x}_k \\ &= \lambda_{k+1}(\beta_1 \mathbf{x}_1 + \dots + \beta_k \mathbf{x}_k) - (\lambda_1 \beta_1 \mathbf{x}_1 + \dots + \lambda_k \beta_k \mathbf{x}_k) \\ &= \lambda_{k+1} \mathbf{x}_{k+1} - A \mathbf{x}_{k+1} = \mathbf{0} \end{aligned}$$

By the induction hypothesis, we conclude $(\lambda_{k+1} - \lambda_i) \beta_i = 0$ for all i . Since not all β_i can be zero, we find at least one $j \in \{1, \dots, k\}$ with $\lambda_{k+1} = \lambda_j$, which is a contradiction. \square

Therefore, we conclude:

Corollary 6.40. n different eigenvalues \Rightarrow diagonalisable

If $A \in \mathbb{C}^{n \times n}$ has n different eigenvalues, then A is a diagonalisable.

Proof. A linearly independent family of n eigenvectors forms a basis for \mathbb{C}^n . \square

Example 6.41. (a) $A = \begin{pmatrix} 3 & 2 \\ 1 & 2 \end{pmatrix}$ has eigenvalues $\lambda_1 = 4$ and $\lambda_2 = 1$. Corollary 6.40 tells us that A is diagonalisable. We also verify this by looking at the eigenvectors $\mathbf{x}_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ and $\mathbf{x}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$, which form a basis of \mathbb{C}^2 . Hence, $\begin{pmatrix} 3 & 2 \\ 1 & 2 \end{pmatrix} = A = XDX^{-1} = \begin{pmatrix} 2 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 1 & -1 \end{pmatrix}^{-1}$.

(b) The 90° -rotation $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ has eigenvalues $\lambda_{1,2} = \pm i$. From $A - \lambda_{1,2} \mathbb{1} = \begin{pmatrix} \mp i & -1 \\ 1 & \mp i \end{pmatrix}$ we conclude the eigenvectors $\mathbf{x}_1 = \begin{pmatrix} i \\ 1 \end{pmatrix}$ and $\mathbf{x}_2 = \begin{pmatrix} 1 \\ i \end{pmatrix}$, which span \mathbb{C}^2 . Hence, $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = A = XDX^{-1} = \begin{pmatrix} i & 1 \\ 1 & i \end{pmatrix} \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \begin{pmatrix} i & 1 \\ 1 & i \end{pmatrix}^{-1}$. X and D are strictly complex, while A is a real matrix.

(c) Look at the 3×3 matrices:

$$A = \begin{pmatrix} 4 & 0 & 0 \\ 1 & 6 & 3 \\ -2 & -4 & -2 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 8 & 8 & 4 \\ -1 & 2 & 1 \\ -2 & -4 & -2 \end{pmatrix}.$$

If you calculate the characteristic polynomials, you find

$$p_A(\lambda) = -\lambda^3 + 8\lambda^2 - 16\lambda = -\lambda(\lambda - 4)^2 = p_B(\lambda)$$

and, hence, the same eigenvalues $\lambda_1 = 0$, $\lambda_2 = 4$ and $\lambda_3 = 4$. The eigenvalues $\lambda_{2,3} = 4$ have algebraic multiplicity 2. Since λ_1 and $\lambda_{2,3}$ are different, we know by Proposition 6.39 that we have at least two linearly independent eigenvectors: one corresponding to $\lambda_1 = 0$ and one corresponding to $\lambda_{2,3} = 4$. If we actually find three linearly independent vectors, the eigenvalue $\lambda_{2,3} = 4$ is the crucial one.

For A , the eigenspaces are:

$$\text{Ker}(A - \lambda_1 \mathbb{1}) = \text{Span}\left(\begin{pmatrix} 0 \\ -1 \\ 2 \end{pmatrix}\right) \quad \text{and} \quad \text{Ker}(A - \lambda_2 \mathbb{1}) = \text{Span}\left(\begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 2 \\ -1 \end{pmatrix}\right),$$

However for B , the eigenspaces are

$$\text{Ker}(B - \lambda_1 \mathbb{1}) = \text{Span}\left(\begin{pmatrix} 0 \\ -1 \\ 2 \end{pmatrix}\right) \quad \text{and} \quad \text{Ker}(B - \lambda_2 \mathbb{1}) = \text{Span}\left(\begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}\right).$$

While A has three different directions for eigenvectors and is diagonalisable, the matrix B has for $\lambda_{2,3} = 4$ only one direction for eigenvectors. There are too few vectors for a basis and B is not diagonalisable.

- (d) Let $A = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$. Using Proposition 6.9 the eigenvalues are $\lambda_1 = \lambda_2 = 0$. Hence, $D = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = 0$. If A was diagonalisable, then there would be an X with $A = XDX^{-1} = X0X^{-1} = 0$. Contradiction to $A \neq 0$. Therefore, A cannot be diagonalisable. In fact, $\text{Ker}(A - 0\mathbb{1}) = \text{Ker}(A) = \text{Span}(\begin{pmatrix} 0 \\ 1 \end{pmatrix})$. Alternatively: $\dim(\text{Ker}(A - 0\mathbb{1})) = \dim(\text{Ker}(A)) = 2 - \text{rank}(A) = 2 - 1 = 1$.

Reminder: Algebraic and geometric multiplicity

For each eigenvalue λ of A we consider

- the algebraic multiplicity of λ , denoted by $\alpha(\lambda)$, given by the multiplicity of λ as zero of p_A , and
- the geometric multiplicity of λ , denoted by $\gamma(\lambda)$, given by the dimension of the eigenspace $\text{Ker}(A - \lambda\mathbb{1})$.

For A from Example 6.41 (c), we find $\alpha(0) = 1 = \gamma(0)$, $\alpha(4) = 2 = \gamma(4)$.

For B from Example 6.41 (c), we get $\alpha(0) = 1 = \gamma(0)$, $\alpha(4) = 2 \neq 1 = \gamma(4)$.

Proposition 6.42. Algebraic vs. geometric multiplicity

Let $A \in \mathbb{C}^{n \times n}$ be a square matrix, and let $\lambda_1, \dots, \lambda_k \in \mathbb{C}$ be all eigenvalues of A (not counted with multiplicities). Then:

- $\alpha(\lambda_1) + \dots + \alpha(\lambda_k) = n$.
- For all $i = 1, \dots, k$, we have $1 \leq \gamma(\lambda_i) \leq \alpha(\lambda_i)$.

Therefore, the following claims are equivalent:

- A is diagonalisable,
- $\gamma(\lambda_1) + \dots + \gamma(\lambda_k) = n$,
- $\gamma(\lambda_i) = \alpha(\lambda_i)$ for all $i = 1, \dots, k$.

Proof. Exercise. □

For symmetric or selfadjoint matrices, we can improve Proposition 6.39 even more:

Proposition 6.43. $A=A^*$: orthogonal eigenvectors

Let $A \in \mathbb{C}^{n \times n}$ be selfadjoint, which means $A = A^*$, and let $\lambda, \lambda' \in \mathbb{C}$ be two different eigenvalues of A with corresponding eigenvectors \mathbf{x} and \mathbf{x}' , respectively. Then $\mathbf{x} \perp \mathbf{x}'$.

Proof. Since $\langle \mathbf{x}, \lambda' \mathbf{x}' \rangle \stackrel{(S4)}{=} \overline{\langle \lambda' \mathbf{x}', \mathbf{x} \rangle} = \overline{\lambda'} \overline{\langle \mathbf{x}', \mathbf{x} \rangle} = \overline{\lambda'} \overline{\langle \mathbf{x}', \mathbf{x} \rangle} \stackrel{(S4)}{=} \overline{\lambda'} \langle \mathbf{x}, \mathbf{x}' \rangle$, we have

$$\lambda \langle \mathbf{x}, \mathbf{x}' \rangle = \langle \lambda \mathbf{x}, \mathbf{x}' \rangle = \langle A \mathbf{x}, \mathbf{x}' \rangle \stackrel{A=A^*}{=} \langle \mathbf{x}, A \mathbf{x}' \rangle = \langle \mathbf{x}, \lambda' \mathbf{x}' \rangle \stackrel{\text{see above}}{=} \overline{\lambda'} \langle \mathbf{x}, \mathbf{x}' \rangle \stackrel{\text{Prop. 6.24}}{=} \lambda' \langle \mathbf{x}, \mathbf{x}' \rangle$$

and, hence, $(\lambda - \lambda') \langle \mathbf{x}, \mathbf{x}' \rangle = 0$. This means that the second factor has to be zero. \square

Proposition 6.44. $A=A^*$: diagonalisable - ONB of eigenvectors

Let $A \in \mathbb{C}^{n \times n}$ be selfadjoint, which means $A = A^*$. Then A is diagonalisable, where there is an ONB $(\mathbf{x}_1, \dots, \mathbf{x}_n)$ for \mathbb{C}^n consisting of eigenvectors of A . The matrix

$$X = \begin{pmatrix} & & \\ \mathbf{x}_1 & \dots & \mathbf{x}_n \\ & & \end{pmatrix}$$

is unitary, i.e. $X^{-1} = X^*$. Therefore, we have:

$$A = XDX^{-1} = XDX^* \quad \text{and} \quad D = X^{-1}AX = X^*AX. \quad (6.9)$$

Sketch of proof. Use Proposition 6.43 and Gram-Schmidt for each eigenspace to find an ONB of \mathbb{C}^n . Then $X^*X = \mathbb{1}$ and also $X^* = X^{-1}$. \square

Remark: Important!

Actually, we could generalise the Proposition from above and equation (6.9). It holds if and only if the matrix A is normal (i.e. $AA^* = A^*A$).

Proposition 6.45.

For a diagonalisable $A \in \mathbb{C}^{n \times n}$, let $\lambda_1, \dots, \lambda_n$ be the eigenvalues counted with algebraic multiplicities. Then

$$\det(A) = \prod_{i=1}^n \lambda_i \quad \text{and} \quad \text{tr}(A) = \sum_{i=1}^n \lambda_i,$$

where $\text{tr}(A) := \sum_{j=1}^n a_{jj}$ is the sum of the diagonal, the so-called trace of A .

Proof. Exercise! \square

Remark:

Later, we will see that the result of Proposition 6.45 actually holds for all matrices $A \in \mathbb{C}^{n \times n}$.

6.8 Some applications

Here, we look at some of very many possible applications.

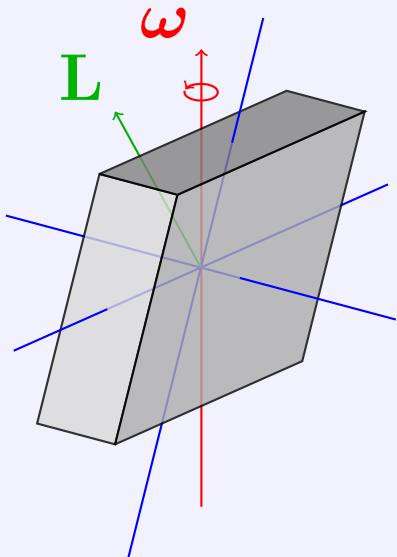
Rotation of boxes

A box of the size $10\text{cm} \times 20\text{cm} \times 30\text{cm}$ rotates around a axis given by the vector $\omega \in \mathbb{R}^3$. The whole box has an angular momentum $\mathbf{L} \in \mathbb{R}^3$.

\mathbf{L} is given by a linear equation using ω , which means

$$\mathbf{L} = J\omega$$

with a symmetric matrix $J \in \mathbb{R}^{3 \times 3}$, which is called the inertia tensor of the box. The rotation “wobbles” if \mathbf{L} , which means $J\omega$, is not parallel to the rotation axis ω . Of course, we have three special rotation axes given by the eigenvectors of J . They are called the principal axes of the box.



Curves and areas

Which points $\mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2$ satisfy the equation $3x^2 + 2\sqrt{3}xy + y^2 + x - \sqrt{3}y = 2$?
Solution: Rewrite the equation as a vector-matrix equation

$$2 = 3x^2 + 2\sqrt{3}xy + y^2 + x - \sqrt{3}y = \underbrace{\begin{pmatrix} x & y \end{pmatrix}}_{\mathbf{x}^T} \underbrace{\begin{pmatrix} 3 & \sqrt{3} \\ \sqrt{3} & 1 \end{pmatrix}}_{=:A (=A^T)} \begin{pmatrix} x \\ y \end{pmatrix} + \underbrace{\begin{pmatrix} 1 & -\sqrt{3} \end{pmatrix}}_{=:b^T} \begin{pmatrix} x \\ y \end{pmatrix}$$

and diagonalise the symmetric matrix A : $\lambda_1 = 4$, $\lambda_2 = 0$, $\mathbf{x}_1 = \frac{1}{2} \begin{pmatrix} \sqrt{3} \\ 1 \end{pmatrix}$, $\mathbf{x}_2 = \frac{1}{2} \begin{pmatrix} -1 \\ \sqrt{3} \end{pmatrix}$

$$\begin{pmatrix} 3 & \sqrt{3} \\ \sqrt{3} & 1 \end{pmatrix} = A = XDX^* = XDX^T = \underbrace{\frac{1}{2} \begin{pmatrix} \sqrt{3} & -1 \\ 1 & \sqrt{3} \end{pmatrix}}_X \underbrace{\begin{pmatrix} 4 & 0 \end{pmatrix}}_D \underbrace{\frac{1}{2} \begin{pmatrix} \sqrt{3} & 1 \\ -1 & \sqrt{3} \end{pmatrix}}_{X^T}.$$

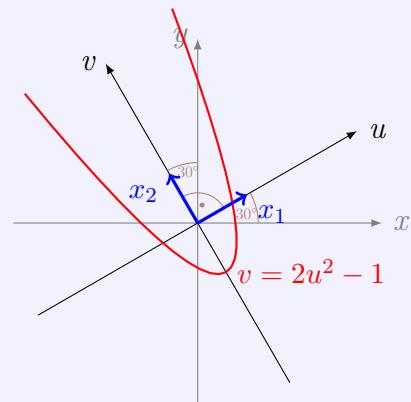
Then we get $2 = \mathbf{x}^T A \mathbf{x} + \mathbf{b}^T \mathbf{x} = \mathbf{x}^T (XDX^T) \mathbf{x} + \mathbf{b}^T \mathbf{x} = (\mathbf{x}^T X) D (X^T \mathbf{x}) + \mathbf{b}^T X (X^T \mathbf{x})$. Setting $\begin{pmatrix} u \\ v \end{pmatrix} = \mathbf{u} := X^T \mathbf{x} = \frac{1}{2} \begin{pmatrix} \sqrt{3} & 1 \\ -1 & \sqrt{3} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$ simplifies the equation to

$$2 = \mathbf{u}^T D \mathbf{u} + \mathbf{b}^T X \mathbf{u} = (u \ v) \begin{pmatrix} 4 & 0 \\ 0 & -2 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} + \underbrace{(0 \ -2)}_{\mathbf{b}^T X} \begin{pmatrix} u \\ v \end{pmatrix} = 4u^2 - 2v.$$

The more complicated equation from above looks at lot simpler in the “optimal” $(\mathbf{x}_1, \mathbf{x}_2)$ -coordinate system:

$$2 = 4u^2 - 2v, \quad \text{also} \quad v = 2u^2 - 1,$$

There you immediately see that it is a **parabola**. The transformation we did, $\begin{pmatrix} x \\ y \end{pmatrix} = \mathbf{x} \mapsto \begin{pmatrix} u \\ v \end{pmatrix} = \mathbf{u} = X^T \mathbf{x}$, was just a rotation by 30° .



A simple criterion for definiteness

$$n = 2: \det(A) = \lambda_1 \lambda_2.$$

- $\det(A) > 0 \Rightarrow$ eigenvalues have the same sign $\Rightarrow A$ (pos. or neg.) definite. If $a_{11} = \mathbf{e}_1^T A \mathbf{e}_1 > 0$, then pos., otherwise neg. definite.
- $\det(A) < 0 \Rightarrow A$ indefinite

In general: A symmetric A is positive definite if all *left upper subdeterminants* are positive.

Summary

- All matrices A we considered here were square matrices.
- A vector $\mathbf{x} \neq \mathbf{o}$, which A only scales, which means $A\mathbf{x} = \lambda\mathbf{x}$, is called an *eigenvector*; the corresponding scaling factor λ is called an *eigenvalue*. The set of all eigenvalues of A is called the *spectrum*.
- λ is an eigenvalue of A if and only if $(A - \lambda\mathbf{1})\mathbf{x} = \mathbf{o}$ has non-trivial solutions $\mathbf{x} \neq \mathbf{o}$ (namely the eigenvectors). This is fulfilled if and only if $\det(A - \lambda\mathbf{1}) = 0$.
- For $A \in \mathbb{C}^{n \times n}$, we define $p_A(\lambda) := \det(A - \lambda\mathbf{1})$, the *characteristic polynomial of A*, which is a polynomial of degree n in the variable λ . It has exactly n complex zeros: the eigenvalues of A .
- The eigenvalues λ are in general complex numbers, also the eigenvectors are complex $\mathbf{x} \in \mathbb{C}^n$. All matrices should be considered as $A \in \mathbb{C}^{n \times n}$.
- Also in \mathbb{C}^n , we can define inner products. Here, we only use the standard inner product $\langle \mathbf{x}, \mathbf{y} \rangle$, defined by $x_1\bar{y}_1 + \dots + x_n\bar{y}_n$. Hence we get a new operation for matrices: $A^* := \overline{A^T} = (\bar{a}_{ji})$. It satisfies $\langle A\mathbf{x}, \mathbf{y} \rangle = \langle \mathbf{x}, A^*\mathbf{y} \rangle$ for all \mathbf{x}, \mathbf{y} .
- Checking eigenvalues: Product of all eigenvalues of A is equal to $\det(A)$; the sum is equal to $\text{tr}(A)$.
- The matrix A is invertible if and only if all eigenvalues are nonzero.
- The eigenvalues of a triangular matrix are the diagonal entries. The eigenvalues of a block matrix in triangular form are given by the eigenvalues of the blocks on the diagonal.

- The eigenvalues of A^m are given by the eigenvalues of A to the power of m where $m \in \mathbb{Z}$. The eigenvectors stay the same as for A . For example, $3A^{17} - 2A^3 + 5A^{-6}$ has the eigenvalues $3\lambda^{17} - 2\lambda^3 + 5\lambda^{-6}$, where λ goes through all eigenvalues of A . The eigenvectors still stay the same.
- A is called *diagonalisable* if it can be written as XDX^{-1} , where D is a diagonal matrix consisting of eigenvalues of A and X gets eigenvectors in the columns. This only works if there are enough eigenvectors directions such that X is invertible.
- A is diagonalisable if and only if for all eigenvalues λ the *algebraic multiplicity* of λ is the same as the *geometric multiplicity*.
- If $A = A^*$, then A is diagonalisable and the eigenvalues are real and eigenvectors can be chosen to be orthonormal.

7

General vector spaces

'Oh man, capitalism sucks!' cries the Kangaroo as it flips the Monopoly board.

Marc-Uwe Kling

We already mentioned the general definition of an abstract [vector space](#). However, most of the time, we focused on the vector spaces \mathbb{R}^n and \mathbb{C}^n . Now, we really start walking into this new abstract terrain.

7.1 Vector space in its full glory

We already considered different *vector spaces* in the previous chapters. Recall that a vector space is a set where scaling and adding makes sense. The standard example was always \mathbb{R}^n . However, we also saw that the matrices $\mathbb{R}^{m \times n}$ form the same structure and we expanded this notion to \mathbb{C}^n and $\mathbb{C}^{m \times n}$. In this section, we will finally consider all these spaces in a general context and get a lot more different examples.

We will also recap important definitions like linear combinations, $\text{Span}(\dots)$, basis, dimension, linearity and other things we learnt in *Linear Algebra*, however, now in a general and abstract context.

The vector space needs only a notion of adding elements and scaling elements. This scalar for stretching a vector was either a real number or a complex number. We put both cases together and use the symbol \mathbb{F} that stands either for \mathbb{R} or \mathbb{C} and stands for a so-called [number field](#).

Definition 7.1. Real or complex vector spaces

Let \mathbb{F} be either \mathbb{R} or \mathbb{C} . A nonempty set V together with two operations,

- a [vector addition](#) $+ : V \times V \rightarrow V$,
- and a [scalar multiplication](#) $\cdot : \mathbb{F} \times V \rightarrow V$,

where the rules below are satisfied, is called an \mathbb{F} -[vector space](#). The elements of V are called [vectors](#), and the elements \mathbb{F} are called [scalars](#). The two operations have

to satisfy the following rules:

- (1) $\forall \mathbf{v}, \mathbf{w} \in V : \mathbf{v} + \mathbf{w} = \mathbf{w} + \mathbf{v}$ (+ is commutative)
- (2) $\forall \mathbf{u}, \mathbf{v}, \mathbf{w} \in V : \mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$ (+ is associative)
- (3) There is a zero vector $\mathbf{o} \in V$ with the property: $\forall \mathbf{v} \in V$ we have $\mathbf{v} + \mathbf{o} = \mathbf{v}$.
- (4) For all $\mathbf{v} \in V$ there is a vector $-\mathbf{v} \in V$ with $\mathbf{v} + (-\mathbf{v}) = \mathbf{o}$.
- (5) For the number $1 \in \mathbb{F}$ and each $\mathbf{v} \in V$, one has: $1 \cdot \mathbf{v} = \mathbf{v}$.
- (6) $\forall \lambda, \mu \in \mathbb{F} \quad \forall \mathbf{v} \in V : \lambda \cdot (\mu \cdot \mathbf{v}) = (\lambda\mu) \cdot \mathbf{v}$ (\cdot is associative)
- (7) $\forall \lambda \in \mathbb{F} \quad \forall \mathbf{v}, \mathbf{w} \in V : \lambda \cdot (\mathbf{v} + \mathbf{w}) = (\lambda \cdot \mathbf{v}) + (\lambda \cdot \mathbf{w})$ (distributive $\cdot +$)
- (8) $\forall \lambda, \mu \in \mathbb{F} \quad \forall \mathbf{v} \in V : (\lambda + \mu) \cdot \mathbf{v} = (\lambda \cdot \mathbf{v}) + (\mu \cdot \mathbf{v})$ (distributive $+ \cdot$)

Example 7.2. \mathbb{R}^n and \mathbb{C}^n . At this point, we are very familiar with the space \mathbb{F}^n , where the vectors have n components consisting of numbers from \mathbb{F} and the addition and scalar multiplication is done componentwise:

$$\begin{aligned} \lambda \in \mathbb{F}, \mathbf{v} = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} \Rightarrow \lambda \mathbf{v} = \lambda \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} := \begin{pmatrix} \lambda v_1 \\ \vdots \\ \lambda v_n \end{pmatrix} \\ \mathbf{u} = \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix}, \mathbf{v} = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} \Rightarrow \mathbf{u} + \mathbf{v} = \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix} + \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} := \begin{pmatrix} u_1 + v_1 \\ \vdots \\ u_n + v_n \end{pmatrix} \end{aligned}$$

Now, this is now just a special case of an \mathbb{F} -vector space.

Of course: Vectors from \mathbb{R}^n and \mathbb{C}^n are also vectors in this new sense. However, now we have a lot more examples:

Example 7.3. Matrices. The set of matrices $V := \mathbb{F}^{m \times n}$ together with the matrix addition and scalar multiplication

$$\underbrace{\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix}}_A + \underbrace{\begin{pmatrix} b_{11} & \cdots & b_{1n} \\ \vdots & & \vdots \\ b_{m1} & \cdots & b_{mn} \end{pmatrix}}_B := \underbrace{\begin{pmatrix} a_{11} + b_{11} & \cdots & a_{1n} + b_{1n} \\ \vdots & & \vdots \\ a_{m1} + b_{m1} & \cdots & a_{mn} + b_{mn} \end{pmatrix}}_{A+B}.$$

$$\lambda \cdot \underbrace{\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix}}_A := \underbrace{\begin{pmatrix} \lambda a_{11} & \cdots & \lambda a_{1n} \\ \vdots & & \vdots \\ \lambda a_{m1} & \cdots & \lambda a_{mn} \end{pmatrix}}_{\lambda \cdot A}.$$

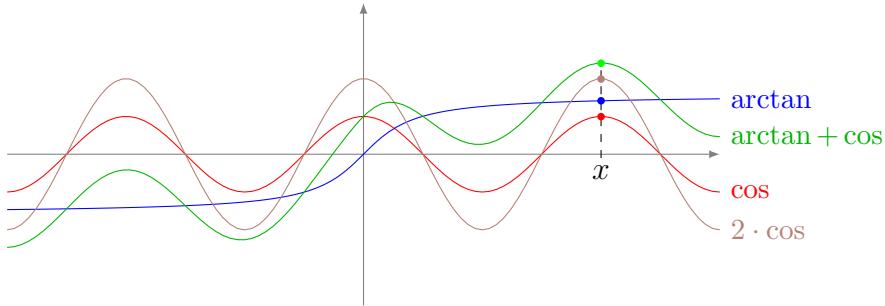
defines also an \mathbb{F} -vector space.

Example 7.4. Functions. Let $\mathcal{F}(\mathbb{R})$ be the set of functions $\mathbf{f} : \mathbb{R} \rightarrow \mathbb{R}$.

For all $\alpha \in \mathbb{R}$ and $\mathbf{f}, \mathbf{g} \in \mathcal{F}(\mathbb{R})$ define $\alpha \cdot \mathbf{f}$ and $\mathbf{f} + \mathbf{g}$ by

$$\begin{aligned}(\alpha \cdot \mathbf{f})(x) &:= \alpha \cdot \mathbf{f}(x) \\ (\mathbf{f} + \mathbf{g})(x) &:= \mathbf{f}(x) + \mathbf{g}(x)\end{aligned}$$

for all $x \in \mathbb{R}$.



This is a natural definition for the α -multiple of a function and the sum of two functions. Obviously, $\alpha \cdot \mathbf{f}$ and $\mathbf{f} + \mathbf{g}$ are again well-defined functions $\mathbb{R} \rightarrow \mathbb{R}$ and hence elements in $\mathcal{F}(\mathbb{R})$. Now we have to check to rules: (1)–(8). This is an exercise.

Hence, $\mathcal{F}(\mathbb{R})$ with $+$ and \cdot is an \mathbb{F} -vector space.

So, we see functions also as vectors since we have the same calculation rules.

Lemma 7.5. $\mathbf{o}=0 \cdot \mathbf{f}$ and $-\mathbf{f}=(-1) \cdot \mathbf{f}$

Let V be an \mathbb{F} -vector space with the operations $+$ and \cdot . For all $\mathbf{f} \in V$, we have

$$\mathbf{o} = 0 \cdot \mathbf{f} \quad \text{and} \quad -\mathbf{f} = (-1) \cdot \mathbf{f}.$$

Proof. We have $0 \cdot \mathbf{f} = (0+0) \cdot \mathbf{f} \stackrel{(8)}{=} (0 \cdot \mathbf{f}) + (0 \cdot \mathbf{f})$. Add on both sides the vector $-(0 \cdot \mathbf{f})$, we get $\mathbf{o} = 0 \cdot \mathbf{f}$. For the second claim, we have

$$\mathbf{o} = 0 \cdot \mathbf{f} = (1+(-1)) \cdot \mathbf{f} \stackrel{(8)}{=} (1 \cdot \mathbf{f}) + ((-1) \cdot \mathbf{f}) \stackrel{(5)}{=} \mathbf{f} + ((-1) \cdot \mathbf{f}).$$

Add $-\mathbf{f}$ on both sides, we get $-\mathbf{f} = (-1) \cdot \mathbf{f}$. □

7.2 Linear subspaces

We already defined the term *linear subspace* in \mathbb{R}^n and \mathbb{C}^n and, of course, it can now be generalised for the general vector spaces.

Let us look at a special subset from Example 7.4:

Example 7.6. Polynomial functions. Let $\mathcal{P}(\mathbb{R})$ denote the set of polynomial functions $\mathbf{f} : \mathbb{R} \rightarrow \mathbb{R}$. We know that $\mathcal{P}(\mathbb{R})$ is a nonempty subset of $\mathcal{F}(\mathbb{R})$ (set of all functions $\mathbb{R} \rightarrow \mathbb{R}$) from Example 7.4. The addition $+$ and scalar multiplication \cdot are just inherited from $\mathcal{F}(\mathbb{R})$.

Is $\mathcal{P}(\mathbb{R})$ also a vector space? Before checking (1)–(8), we have to prove that the vector addition $+$ and scalar multiplication \cdot is well-defined inside $\mathcal{P}(\mathbb{R})$, which means that you cannot leave $\mathcal{P}(\mathbb{R})$ by these operations:

$$\mathbf{f} + \mathbf{g} \text{ and } \alpha \cdot \mathbf{f} \text{ lie in } \mathcal{P}(\mathbb{R}) \text{ for arbitrary } \mathbf{f}, \mathbf{g} \in \mathcal{P}(\mathbb{R}) \text{ and } \alpha \in \mathbb{R}. \quad (7.1)$$

We already know equation (7.1): It means that $\mathcal{P}(\mathbb{R})$ is closed under the addition $+$ and the scalar multiplication \cdot . We can easily show that (7.1) is correct for polynomials.

Now checking (1)–(8) is very fast because:

- (1) $\mathbf{f} + \mathbf{g} = \mathbf{g} + \mathbf{f}$,
- (2) $\mathbf{f} + (\mathbf{g} + \mathbf{h}) = (\mathbf{f} + \mathbf{g}) + \mathbf{h}$,
- (5) $1 \cdot \mathbf{f} = \mathbf{f}$,
- (6) $\alpha \cdot (\beta \cdot \mathbf{f}) = (\alpha\beta) \cdot \mathbf{f}$,
- (7) $\alpha \cdot (\mathbf{f} + \mathbf{g}) = (\alpha \cdot \mathbf{f}) + (\alpha \cdot \mathbf{g})$,
- (8) $(\alpha + \beta) \cdot \mathbf{f} = (\alpha \cdot \mathbf{f}) + (\beta \cdot \mathbf{f})$

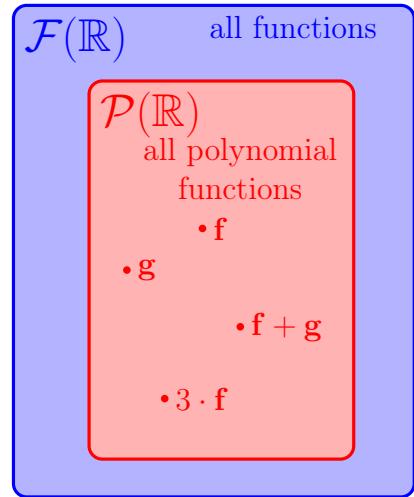
hold for all $\mathbf{f}, \mathbf{g}, \mathbf{h} \in \mathcal{F}(\mathbb{R})$ and $\alpha, \beta \in \mathbb{F} = \mathbb{R}$.

Hence, they also hold for $\mathbf{f}, \mathbf{g}, \mathbf{h} \in \mathcal{P}(\mathbb{R}) \subset \mathcal{F}(\mathbb{R})$. Great!

One also says: “ $\mathcal{P}(\mathbb{R})$ inherits (1),(2),(5)–(8) from $\mathcal{F}(\mathbb{R})$.”

And what is about property (3) $\mathbf{o} \in \mathcal{P}(\mathbb{R})$ and property (4) $\forall \mathbf{f} \in \mathcal{P}(\mathbb{R}) : -\mathbf{f} \in \mathcal{P}(\mathbb{R})$?

Both immediately follow from Lemma 7.5 and (7.1)!



This finishes the proof that $\mathcal{P}(\mathbb{R})$ is a vector space with respect to $+$ and \cdot .

Refresh that this proof could be done without much work. The only time when you have to look at $\mathcal{P}(\mathbb{R})$, which means at the **polynomial** functions, was equation (7.1). All the other things are inherited from the superset $\mathcal{F}(\mathbb{R})$.

Proposition & Definition 7.7. Linear subspace

Let V be an \mathbb{F} -vector space and let U be a non-empty subset of V , which is closed under vector addition and scalar multiplication of V , which means

- (a) for all $\mathbf{u}, \mathbf{v} \in U$, we have $\mathbf{u} + \mathbf{v} \in U$ and
- (b) for all $\alpha \in \mathbb{F}$ and $\mathbf{u} \in U$, we have $\alpha \cdot \mathbf{u} \in U$.

Then U is also an \mathbb{F} -vector space. In this case, U is called a linear subspace of V or in short a subspace of V

Recall that we introduced the notion of a subspace before for $V = \mathbb{R}^n$ and $V = \mathbb{C}^n$. In the same way, we know that $U = \{\mathbf{o}\}$ and $U = V$ are always subspaces of V . However, in all non-trivial cases, we have more than these two subspaces.

Example 7.8. Quadratic polynomials. Let $\mathcal{P}_2(\mathbb{R})$ be the set of all polynomials with degree ≤ 2 , which means

$$\text{all functions } \mathbf{p} : \mathbb{R} \rightarrow \mathbb{R}, \quad \mathbf{p}(x) = a_2 x^2 + a_1 x + a_0 \quad \text{with } a_2, a_1, a_0 \in \mathbb{R}.$$

Is $\mathcal{P}_2(\mathbb{R})$ with the vector addition $+$ and \cdot from $\mathcal{F}(\mathbb{R})$ a vector space?

Obviously, $\mathcal{P}_2(\mathbb{R}) \subset \mathcal{F}(\mathbb{R})$ and $\mathcal{P}_2(\mathbb{R}) \neq \emptyset$. Using Proposition 7.7 we only have to check that $\mathcal{P}_2(\mathbb{R})$ is closed under $+$ and \cdot , which means that we have to check (a) and (b):

Let $\mathbf{p}, \mathbf{q} \in \mathcal{P}_2(\mathbb{R})$ and $\alpha \in \mathbb{R}$. Then, there are $a_2, a_1, a_0, b_2, b_1, b_0 \in \mathbb{R}$ such that

$$\mathbf{p}(x) = a_2x^2 + a_1x + a_0 \quad \text{and} \quad \mathbf{q}(x) = b_2x^2 + b_1x + b_0.$$

Hence:

$$\begin{aligned} (\mathbf{p} + \mathbf{q})(x) &= \mathbf{p}(x) + \mathbf{q}(x) &= (a_2x^2 + a_1x + a_0) + (b_2x^2 + b_1x + b_0) \\ &&= (a_2 + b_2)x^2 + (a_1 + b_1)x + (a_0 + b_0), \\ (\alpha \cdot \mathbf{p})(x) &= \alpha \cdot \mathbf{p}(x) &= \alpha \cdot (a_2x^2 + a_1x + a_0) \\ &&= (\alpha a_2)x^2 + (\alpha a_1)x + (\alpha a_0) \end{aligned}$$

We conclude that $\mathbf{p} + \mathbf{q} \in \mathcal{P}_2(\mathbb{R})$ and $\alpha \cdot \mathbf{p} \in \mathcal{P}_2(\mathbb{R})$. The set $\mathcal{P}_2(\mathbb{R})$ is a subspace of $\mathcal{F}(\mathbb{R})$ and a vector space for its own.

Analogously, for $n \in \mathbb{N}_0$, we define $\mathcal{P}_n(\mathbb{R})$ as the set of all polynomials with degree $\leq n$. It forms again a vector space with the operations $+$ and \cdot from $\mathcal{F}(\mathbb{R})$.

Here, another vector space:

Example 7.9. Upper triangular matrices Let $n \in \mathbb{N}$ and $\nabla \mathbb{R}^{n \times n}$ the set of all upper triangular matrices $A \in \mathbb{R}^{n \times n}$. The operations $+$ and \cdot are the same as before for all matrices. Since $\mathbf{0} \in \nabla \mathbb{R}^{n \times n} \neq \emptyset$, the sum of two upper triangular matrices is again a triangular matrix and scaled triangular matrices are again triangular matrices, we know that $\nabla \mathbb{R}^{n \times n}$ is a subspace of the \mathbb{R} -vector space $\mathbb{R}^{n \times n}$ (from Example 7.3) and hence a vector space itself.

Example 7.10. The set of all matrices U in the following form:

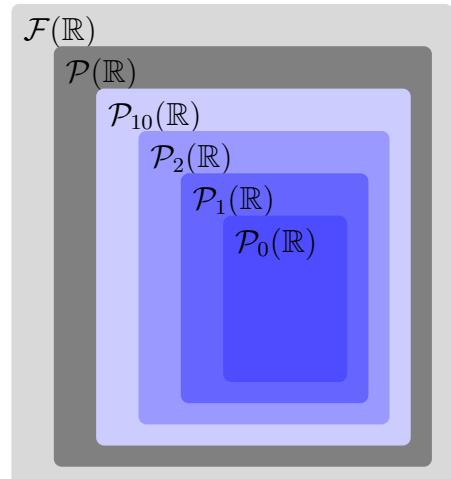
$$\begin{pmatrix} a & 0 & a \\ 0 & b & 0 \\ a & 0 & a \end{pmatrix} \quad \text{with } a, b \in \mathbb{C}$$

is closed under matrix addition and the multiplication with scalars $\alpha \in \mathbb{C}$. Therefore, U is a subspace of $\mathbb{C}^{3 \times 3}$ and a \mathbb{C} -vector space for itself.

If we look back at the polynomial spaces, we notice that we have the following inclusions:

$$\mathcal{P}_0(\mathbb{R}) \subset \mathcal{P}_1(\mathbb{R}) \subset \mathcal{P}_2(\mathbb{R}) \subset \cdots \subset \mathcal{P}(\mathbb{R}) \subset \mathcal{F}(\mathbb{R})$$

The vector space $\mathcal{F}(\mathbb{R})$ is the largest of these. The polynomial space $\mathcal{P}_n(\mathbb{R})$ gets smaller if we choose n smaller. When we talk about sizes in vector spaces, we remember the definition of a dimension of a subspace. We suppose that the dimension of $\mathcal{P}_n(\mathbb{R})$ is $n+1$. But first of all, we have to define all the notions again.



7.3 Recollection: basis, dimension and other stuff

Let $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$ and V be an \mathbb{F} -vector space with vector addition $+$ and scalar multiplication \cdot .

As we did for \mathbb{R}^n and later for \mathbb{C}^n , we introduce notions like *linear independence*, *basis*, *dimension* and related definitions. In spite of considering abstract vector spaces, the notions still work exactly the same.

Definition 7.11. Same as before: Basis, dimension, and so on

Let V be an \mathbb{F} -vector space with operations $+$ and \cdot .

- For $k \in \mathbb{N}$, vectors $\mathbf{v}_1, \dots, \mathbf{v}_k \in V$ and scalars $\alpha_1, \dots, \alpha_k \in \mathbb{F}$ the vector

$$\alpha_1\mathbf{v}_1 + \dots + \alpha_k\mathbf{v}_k = \sum_{i=1}^k \alpha_i\mathbf{v}_i \in V$$

is called a linear combination of the vectors $\mathbf{v}_1, \dots, \mathbf{v}_k$.

- The set of all possible linear combinations for the vectors of a subset $M \subset V$ is called the linear hull or span of M :

$$\text{Span}(M) := \{\lambda_1\mathbf{u}_1 + \dots + \lambda_k\mathbf{u}_k : \mathbf{u}_1, \dots, \mathbf{u}_k \in M, \lambda_1, \dots, \lambda_k \in \mathbb{F}, k \in \mathbb{N}\}.$$

- A family $\mathcal{E} = (\mathbf{v}_1, \dots, \mathbf{v}_k)$ consisting of k vectors from V is called a generating system for the subspace $U \subset V$, if $U = \text{Span}(\mathbf{v}_1, \dots, \mathbf{v}_k)$.
- A family $\mathcal{E} = (\mathbf{v}_1, \dots, \mathbf{v}_k)$ consisting of k vectors from V is called linearly dependent if one can be represented by a non-trivial linear combination of vectors from \mathcal{E} . If there is no such non-trivial linear combination, the family is called linearly independent.
- A family \mathcal{E} that is a generating system for $U \subset V$ and linearly independent is called a basis of U .
- The number of elements for a basis of U is called the dimension of U . We just write $\dim(U)$.

The definitions and proofs for related propositions are literally the same as in Section 3.7 for the vector space \mathbb{R}^n and its subspace. Therefore, we just summarise the facts in this more abstract case:

Rule of thumb: Basis, dimension and similar things

- A generating family $\mathcal{E} = (\mathbf{v}_1, \dots, \mathbf{v}_k)$ of U is called this way because we can reach each point in U with linear combinations of vectors from \mathcal{E} and no other points.
- A family \mathcal{E} is linear independent if we need all “family members” to span (or generate) the subspace $\text{Span}(\mathcal{E})$.

- A basis \mathcal{B} of U is a smallest generating set U . (We cannot omit a vector from \mathcal{B} .)
- The dimension of a subspace U
 - = the number of elements of a basis of U . (All bases have the same number of elements, just redo the proof in Proposition 3.25.)
 - = the smallest possible size for a generating system of U . (With less vectors it is not possible to span the whole space U .)
 - = the maximal number of vectors from U that form a linearly independent family. (If you choose more vectors, there are always linearly dependent.)

Let us look at different examples:

Example 7.12. – Matrix vector spaces

(a) The vector space $\mathbb{C}^{2 \times 3}$ of all complex 2×3 -matrices can be written in the following way:

$$\begin{aligned}\mathbb{C}^{2 \times 3} &= \left\{ \begin{pmatrix} \alpha & \beta & \gamma \\ \delta & \varepsilon & \varphi \end{pmatrix} : \alpha, \beta, \gamma, \delta, \varepsilon, \varphi \in \mathbb{C} \right\} \\ &= \left\{ \alpha \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \beta \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \gamma \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} + \delta \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} + \varepsilon \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} + \varphi \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} : \alpha, \beta, \gamma, \delta, \varepsilon, \varphi \in \mathbb{C} \right\} \\ &= \text{Span} \left(\left(\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 0 & 0 \end{array} \right), \left(\begin{array}{ccc} 0 & 1 & 0 \\ 0 & 0 & 0 \end{array} \right), \left(\begin{array}{ccc} 0 & 0 & 1 \\ 0 & 0 & 0 \end{array} \right), \left(\begin{array}{ccc} 0 & 0 & 0 \\ 1 & 0 & 0 \end{array} \right), \left(\begin{array}{ccc} 0 & 0 & 0 \\ 0 & 1 & 0 \end{array} \right), \left(\begin{array}{ccc} 0 & 0 & 0 \\ 0 & 0 & 1 \end{array} \right) \right)\end{aligned}$$

Hence

$$\mathcal{B} = \left(\left(\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 0 & 0 \end{array} \right), \left(\begin{array}{ccc} 0 & 1 & 0 \\ 0 & 0 & 0 \end{array} \right), \left(\begin{array}{ccc} 0 & 0 & 1 \\ 0 & 0 & 0 \end{array} \right), \left(\begin{array}{ccc} 0 & 0 & 0 \\ 1 & 0 & 0 \end{array} \right), \left(\begin{array}{ccc} 0 & 0 & 0 \\ 0 & 1 & 0 \end{array} \right), \left(\begin{array}{ccc} 0 & 0 & 0 \\ 0 & 0 & 1 \end{array} \right) \right)$$

is a generating system for $\mathbb{C}^{2 \times 3}$. \mathcal{B} is also linearly independent: From

$$\underbrace{\alpha \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \beta \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \gamma \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} + \delta \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} + \varepsilon \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} + \varphi \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{\left(\begin{smallmatrix} \alpha & \beta & \gamma \\ \delta & \varepsilon & \varphi \end{smallmatrix} \right)} = \mathbf{0},$$

we conclude $\alpha = \beta = \gamma = \delta = \varepsilon = \varphi = 0$. Hence, \mathcal{B} is a basis of $\mathbb{C}^{2 \times 3}$ and the dimension of $\mathbb{C}^{2 \times 3}$ is $|\mathcal{B}| = 6$. Analogously, one can prove: $\dim(\mathbb{F}^{m \times n}) = m \cdot n$.

(b) In a similar way, we can prove that

$$\mathcal{B} = \left(\left(\begin{array}{cc} 1 & 0 \\ 0 & 0 \end{array} \right), \left(\begin{array}{cc} 0 & 1 \\ 0 & 0 \end{array} \right), \left(\begin{array}{cc} 0 & 0 \\ 0 & 1 \end{array} \right) \right) \quad (7.2)$$

forms a basis of $\mathbb{N}\mathbb{R}^{2 \times 2}$. Hence:

$$\left\{ \begin{pmatrix} \alpha & \beta \\ 0 & \gamma \end{pmatrix} : \alpha, \beta, \gamma \in \mathbb{R} \right\} = \left\{ \alpha \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \beta \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + \gamma \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} : \alpha, \beta, \gamma \in \mathbb{R} \right\}$$

We conclude: $\dim(\mathbb{N}\mathbb{R}^{2 \times 2}) = 2 + 1 = 3$.

Analogously for given $n \in \mathbb{N}$, one can prove $\dim(\mathbb{N}\mathbb{R}^{n \times n}) = n + (n - 1) + \dots + 1 = \frac{n(n+1)}{2}$.

(c) As a special vector space, we look at:

$$\begin{aligned} U &= \left\{ \begin{pmatrix} \alpha & 0 & \alpha \\ 0 & \beta & 0 \\ \alpha & 0 & \alpha \end{pmatrix} : \alpha, \beta \in \mathbb{C} \right\} = \left\{ \alpha \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix} + \beta \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} : \alpha, \beta \in \mathbb{C} \right\} \\ &= \text{Span}\left(\underbrace{\begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix}}_{=: A}, \underbrace{\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}}_{=: B}\right). \end{aligned} \quad (7.3)$$

Hence, $\mathcal{B} := (A, B)$ is a generating system for U . Again, we show that \mathcal{B} is also linearly independent. From $\alpha A + \beta B = \mathbf{o}$, one gets

$$\begin{pmatrix} \alpha & 0 & \alpha \\ 0 & \beta & 0 \\ \alpha & 0 & \alpha \end{pmatrix} = \alpha A + \beta B = \mathbf{o} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

and concludes $\alpha = \beta = 0$. Therefore, \mathcal{B} is a basis of U and $\dim(U) = 2$.

However, the examples above were all well-known matrix spaces. It would be more interesting to look at our new function spaces like the polynomial space $\mathcal{P}_2(\mathbb{R})$ from Example 7.8:

Example 7.13. – **Polynomial space $\mathcal{P}_2(\mathbb{R})$.** We define the special polynomials $\mathbf{m}_0, \mathbf{m}_1, \mathbf{m}_2 \in \mathcal{P}_2$ by

$$\mathbf{m}_0(x) := 1, \quad \mathbf{m}_1(x) := x, \quad \text{and} \quad \mathbf{m}_2(x) := x^2 \quad \text{for all } x \in \mathbb{R}$$

and see:

$$\begin{aligned} \mathcal{P}_2(\mathbb{R}) &= \{x \mapsto a_2 x^2 + a_1 x + a_0 : a_2, a_1, a_0 \in \mathbb{R}\} = \{a_2 \mathbf{m}_2 + a_1 \mathbf{m}_1 + a_0 \mathbf{m}_0 : a_2, a_1, a_0 \in \mathbb{R}\} \\ &= \text{Span}(\mathbf{m}_0, \mathbf{m}_1, \mathbf{m}_2) \end{aligned}$$

Hence, $\mathcal{B} := (\mathbf{m}_0, \mathbf{m}_1, \mathbf{m}_2)$ is a generating system for $\mathcal{P}_2(\mathbb{R})$.

In order to show that \mathcal{B} is also linearly independent, we choose a linear combination $\alpha \mathbf{m}_0 + \beta \mathbf{m}_1 + \gamma \mathbf{m}_2$ for the zero vector \mathbf{o} and find conditions for possible coefficients $\alpha, \beta, \gamma \in \mathbb{R}$. Note that the zero vector \mathbf{o} in this vector space is the zero function, which means the function $x \mapsto 0$. In other words, the equation $\alpha \mathbf{m}_0 + \beta \mathbf{m}_1 + \gamma \mathbf{m}_2 = \mathbf{o}$ should be read as: For all $x \in \mathbb{R}$, we have

$$\alpha + \beta x + \gamma x^2 = 0. \quad (7.4)$$

Since the equation (7.4) above holds for **all** $x \in \mathbb{R}$, we can choose some suitable values for x . We start with trying $x = 0$, $x = 1$ and $x = -1$. The equation (7.4) gives us then three different linear equations:

$$\left. \begin{array}{l} \underline{x=0:} \quad \alpha + \beta \cdot 0 + \gamma \cdot 0 = 0 \\ \underline{x=1:} \quad \alpha + \beta \cdot 1 + \gamma \cdot 1 = 0 \\ \underline{x=-1:} \quad \alpha + \beta \cdot (-1) + \gamma \cdot 1 = 0 \end{array} \right\} \quad (7.5)$$

Indeed, (7.5) is an LES 3 equations and three unknowns α, β, γ , which we can just solve with our known methods. We get $\alpha = \beta = \gamma = 0$. Hence, $\mathcal{B} = (\mathbf{m}_0, \mathbf{m}_1, \mathbf{m}_2)$ is linearly independent and a basis of $\mathcal{P}_2(\mathbb{R})$. We also get $\dim(\mathcal{P}_2(\mathbb{R})) = 3$.

Proposition & Definition 7.14. Monomial basis of $\mathcal{P}_n(\mathbb{R})$

Let $n \in \mathbb{N}_0$. The particular polynomials $\mathbf{m}_0, \mathbf{m}_1, \dots, \mathbf{m}_n \in \mathcal{P}_n(\mathbb{R})$ defined by

$$\mathbf{m}_0(x) = 1, \quad \mathbf{m}_1(x) = x, \quad \dots, \quad \mathbf{m}_{n-1}(x) = x^{n-1}, \quad \mathbf{m}_n(x) = x^n \quad \text{for all } x \in \mathbb{R}$$

are called monomials. The family $\mathcal{B} = (\mathbf{m}_0, \mathbf{m}_1, \dots, \mathbf{m}_n)$ forms a basis of $\mathcal{P}_n(\mathbb{R})$ and is called the monomial basis. Hence $\dim(\mathcal{P}_n(\mathbb{R})) = n + 1$.

Sketch of proof. This works the same as for $\mathcal{P}_2(\mathbb{R})$, see Example 7.13. In order to show the linear independence, we have to choose $n+1$ different values for x . The $(n+1) \times (n+1)$ -LES always has a unique solution. \square

Now, by knowing that the monomials are linearly independent, we can always solve equations by equating coefficients:

Corollary 7.15. The method of equating the coefficients

Let \mathbf{p} and \mathbf{q} be two real polynomials with degree $n \in \mathbb{N}$, which means

$$\mathbf{p}(x) = a_n x^n + \dots + a_1 x + a_0 \quad \text{and} \quad \mathbf{q}(x) = b_n x^n + \dots + b_1 x + b_0$$

for some coefficients $a_n, \dots, a_1, a_0, b_n, \dots, b_1, b_0 \in \mathbb{R}$.

If we have the equality $\mathbf{p} = \mathbf{q}$, which means

$$a_n x^n + \dots + a_1 x + a_0 = b_n x^n + \dots + b_1 x + b_0, \tag{7.6}$$

for all $x \in \mathbb{R}$, then we can conclude $a_n = b_n, \dots, a_1 = b_1$ and $a_0 = b_0$.

Proof. Equation (7.6) means $(a_n - b_n)\mathbf{m}_n + \dots + (a_1 - b_1)\mathbf{m}_1 + (a_0 - b_0)\mathbf{m}_0 = \mathbf{0}$. Because of the linear independence, we have $(a_n - b_n) = \dots = (a_1 - b_1) = (a_0 - b_0) = 0$. \square

Remark:

Since $\dim(\mathcal{P}_n(\mathbb{R})) = n + 1$ and we have the inclusions

$$\mathcal{P}_0(\mathbb{R}) \subset \mathcal{P}_1(\mathbb{R}) \subset \mathcal{P}_2(\mathbb{R}) \subset \dots \subset \mathcal{P}(\mathbb{R}) \subset \mathcal{F}(\mathbb{R}),$$

we conclude that $\dim(\mathcal{P}(\mathbb{R}))$ and $\dim(\mathcal{F}(\mathbb{R}))$ cannot be finite natural numbers. Symbolically, we write $\dim(\mathcal{P}(\mathbb{R})) = \infty$ in such a case.

7.4 Coordinates with respect to a basis

7.4.1 Basis implies coordinates

Again, we deal with the case $\mathbb{F} = \mathbb{R}$ and $\mathbb{F} = \mathbb{C}$ simultaneously. Therefore, let V be an \mathbb{F} -vector space with the two operations $+$ and \cdot . Let also $n := \dim(V) < \infty$ and choose a basis $\mathcal{B} = (\mathbf{b}_1, \dots, \mathbf{b}_n)$ of V .

Since \mathcal{B} is a generating system and linearly independent, each \mathbf{v} from V has a linear combination

$$\mathbf{v} = \alpha_1 \mathbf{b}_1 + \dots + \alpha_n \mathbf{b}_n \quad (7.7)$$

where the coefficients $\alpha_1, \dots, \alpha_n \in \mathbb{F}$ are uniquely determined. We call these numbers the coordinates of \mathbf{v} with respect to the basis \mathcal{B} and sometimes write $\mathbf{v}^{\mathcal{B}}$ for the vector consisting of these numbers:

A vector \mathbf{v} in V and its coordinate vector $\mathbf{v}^{\mathcal{B}}$ in \mathbb{F}^n

$$\mathbf{v} = \alpha_1 \mathbf{b}_1 + \dots + \alpha_n \mathbf{b}_n \in V \quad \longleftrightarrow \quad \mathbf{v}^{\mathcal{B}} = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix} \in \mathbb{F}^n. \quad (7.8)$$

One also sees the notation $[\mathbf{x}]_{\mathcal{B}}$ for the coordinate vector. When fixing a basis \mathcal{B} in V , then each vector $\mathbf{v} \in V$ uniquely determines a coordinate vector $\mathbf{v}^{\mathcal{B}} \in \mathbb{F}^n$ – and vice versa.

Forming the coordinate vector is a linear map

The translation of a vector $\mathbf{v} \in V$ into the coordinate vector $\mathbf{v}^{\mathcal{B}} \in \mathbb{F}^n$ defines a linear map:

$$\Phi_{\mathcal{B}} : V \rightarrow \mathbb{F}^n, \quad \Phi_{\mathcal{B}}(\mathbf{v}) = \mathbf{v}^{\mathcal{B}}$$

More concretely:

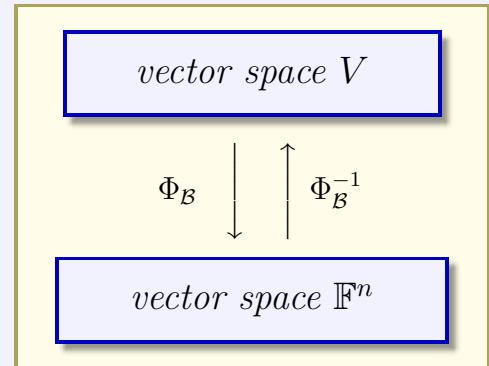
$$\Phi_{\mathcal{B}}(\alpha_1 \mathbf{b}_1 + \dots + \alpha_n \mathbf{b}_n) = \alpha_1 \mathbf{e}_1 + \dots + \alpha_n \mathbf{e}_n$$

For all $\mathbf{x}, \mathbf{y} \in V$ and $\lambda \in \mathbb{F}$, the map Φ satisfies two properties:

$$\Phi_{\mathcal{B}}(\mathbf{x} + \mathbf{y}) = \Phi_{\mathcal{B}}(\mathbf{x}) + \Phi_{\mathcal{B}}(\mathbf{y}) \quad (+)$$

$$\Phi_{\mathcal{B}}(\lambda \mathbf{x}) = \lambda \Phi_{\mathcal{B}}(\mathbf{x}) \quad (\cdot)$$

$$\mathbf{v} = \alpha_1 \mathbf{b}_1 + \dots + \alpha_n \mathbf{b}_n \in V \quad \longleftrightarrow \quad \Phi_{\mathcal{B}}(\mathbf{v}) = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix} \in \mathbb{F}^n. \quad (7.9)$$



The linear map $\Phi_{\mathcal{B}}$ is called the basis isomorphism with respect to the basis \mathcal{B} and is completely defined by $\Phi(\mathbf{b}_j) = \mathbf{e}_j$ for $j = 1, \dots, n$.

Example 7.16. An abstract vector is represented by numbers

The three functions \sin , \cos and \arctan from the vector space $\mathcal{F}(\mathbb{R})$, cf. Example 7.4, span a subspace:

$$V := \text{Span}(\sin, \cos, \arctan) \subset \mathcal{F}(\mathbb{R}).$$

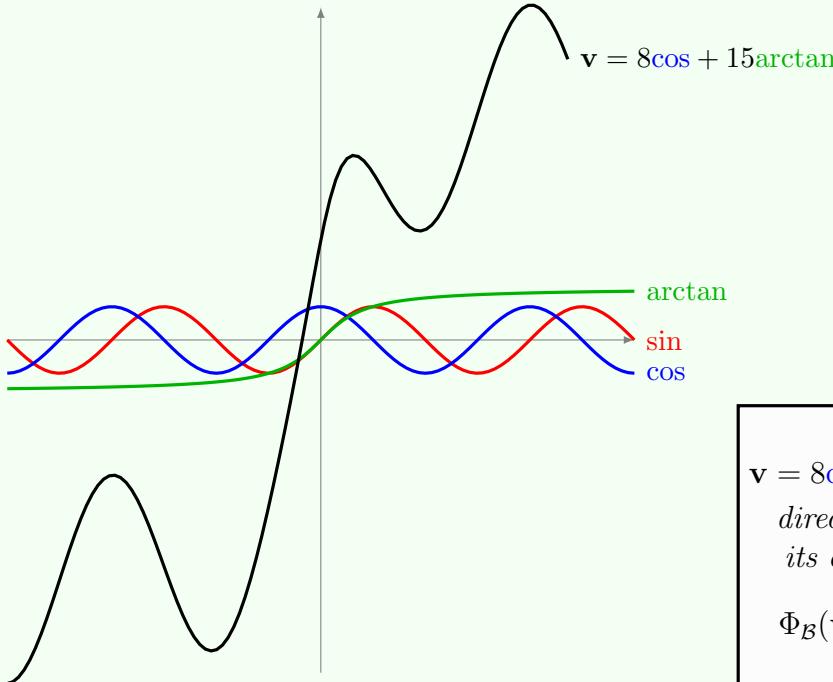
In the same manner as before, we can show that the three functions are linearly independent. Hence, they form a basis of V :

$$\mathcal{B} := (\sin, \cos, \arctan).$$

Now, if we look at another function $\mathbf{v} \in V$ given by

$$\mathbf{v}(x) = 8\cos(x) + 15\arctan(x) \text{ for all } x \in \mathbb{R}, \text{ hence } \mathbf{v} = 8\cos + 15\arctan.$$

Then:



We find
 $\mathbf{v} = 8\cos + 15\arctan$
directly by using
its coordinates:
 $\Phi_{\mathcal{B}}(\mathbf{v}) = \begin{pmatrix} 0 \\ 8 \\ 15 \end{pmatrix}$.

Rule of thumb: V is completely represented by \mathbb{F}^n

Each \mathbb{F} -vector space V with $n := \dim(V) < \infty$ is represented by \mathbb{F}^n if you fix a basis $\mathcal{B} = (\mathbf{b}_1, \dots, \mathbf{b}_n)$

For each vector $\mathbf{v} \in V$, there is exactly one coordinate vector $\Phi_{\mathcal{B}}(\mathbf{v}) \in \mathbb{F}^n$. Instead of using $\mathbf{v} \in V$, one can also do calculations with $\Phi_{\mathcal{B}}(\mathbf{v}) = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix} \in \mathbb{F}^n$.

Of course, calculations in \mathbb{F}^n might be simpler and more suitable for a computer than the calculations in an abstract vector space.

Example 7.17. The polynomials $\mathbf{p}, \mathbf{q} \in \mathcal{P}_3(\mathbb{R})$ given by $\mathbf{p}(x) = 2x^3 - x^2 + 7$ and $\mathbf{q}(x) = x^2 + 3$ can be represented with the monomial basis $\mathcal{B} = (\mathbf{m}_0, \mathbf{m}_1, \mathbf{m}_2, \mathbf{m}_3)$ by the coordinate vectors:

$$\Phi_{\mathcal{B}}(\mathbf{p}) = \begin{pmatrix} 7 \\ 0 \\ -1 \\ 2 \end{pmatrix} \in \mathbb{R}^4 \quad \text{and} \quad \Phi_{\mathcal{B}}(\mathbf{q}) = \begin{pmatrix} 3 \\ 0 \\ 1 \\ 0 \end{pmatrix} \in \mathbb{R}^4, \quad (7.10)$$

since $\mathbf{p} = 7\mathbf{m}_0 + 0\mathbf{m}_1 + (-1)\mathbf{m}_2 + 2\mathbf{m}_3$ and $\mathbf{q} = 3\mathbf{m}_0 + 0\mathbf{m}_1 + 1\mathbf{m}_2 + 0\mathbf{m}_3$.

In the same manner the two polynomials $(2\mathbf{p})(x) = 4x^3 - 2x^2 + 14$ and $(\mathbf{p} + \mathbf{q})(x) = 2x^3 + 10$,

have the following coordinate vectors:

$$\Phi_{\mathcal{B}}(2\mathbf{p}) = \begin{pmatrix} 14 \\ 0 \\ -2 \\ 4 \end{pmatrix} = 2\Phi_{\mathcal{B}}(\mathbf{p}) \quad \text{and} \quad \Phi_{\mathcal{B}}(\mathbf{p} + \mathbf{q}) = \begin{pmatrix} 10 \\ 0 \\ 0 \\ 2 \end{pmatrix} = \Phi_{\mathcal{B}}(\mathbf{p}) + \Phi_{\mathcal{B}}(\mathbf{q}).$$

This shows that we can also calculate with the coordinate vectors from equation (7.10).

Example 7.18. The matrix $A = \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix} \in \mathbb{R}^{2 \times 2}$ has the following coordinate vector with respect to the basis \mathcal{B} from equation (7.2):

$$\Phi_{\mathcal{B}}(A) = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}.$$

The matrix $3A$ has the coordinate vector $\begin{pmatrix} 3 \\ 6 \\ 9 \end{pmatrix}$.

The matrix

$$C = \begin{pmatrix} 5 & 0 & 5 \\ 0 & 2 & 0 \\ 5 & 0 & 5 \end{pmatrix}$$

has the following coordinate vector with respect to the basis from equation (7.3): $\begin{pmatrix} 5 \\ 2 \end{pmatrix}$. The matrix $2C$ has then the coordinate vector $\begin{pmatrix} 10 \\ 4 \end{pmatrix}$.

7.4.2 Change of basis

We have seen that we can represent an abstract vector $\mathbf{v} \in V$ with a very concrete vector in \mathbb{F}^n . However, this representation is heavily dependent on the chosen basis \mathcal{B} for V . If we choose another basis \mathcal{C} of V , then the coordinate vector $\Phi_{\mathcal{C}}(\mathbf{v})$ might be different to the old coordinate vector $\Phi_{\mathcal{B}}(\mathbf{v})$. Here, we will talk what happens if we switch the bases.

For a given basis $\mathcal{B} = (\mathbf{b}_1, \dots, \mathbf{b}_n)$ of the vector space V , we have the linear map

$$\Phi_{\mathcal{B}} : V \rightarrow \mathbb{F}^n, \quad \Phi_{\mathcal{B}}(\mathbf{b}_j) = \mathbf{e}_j \quad \text{for all } j$$

which is also invertible. We have called it the basis isomorphism. For a given element $\mathbf{v} = \alpha_1 \mathbf{b}_1 + \dots + \alpha_n \mathbf{b}_n$, we can write:

$$\mathbf{v} = \alpha_1 \mathbf{b}_1 + \dots + \alpha_n \mathbf{b}_n = \alpha_1 \Phi_{\mathcal{B}}^{-1}(\mathbf{e}_1) + \dots + \alpha_n \Phi_{\mathcal{B}}^{-1}(\mathbf{e}_n) = \Phi_{\mathcal{B}}^{-1}(\Phi_{\mathcal{B}}(\mathbf{v})).$$

Also remember the formula for the inverse:

$$\Phi_{\mathcal{B}}^{-1} : \mathbb{F}^n \rightarrow V, \quad \Phi_{\mathcal{B}}^{-1}(\mathbf{e}_j) = \mathbf{b}_j \quad \text{for all } j$$

Example 7.19. Consider the already introduced monomial basis (now with different order!) $\mathcal{B} = (\mathbf{m}_2, \mathbf{m}_1, \mathbf{m}_0) = (x \mapsto x^2, x \mapsto x, x \mapsto 1)$ of the space $\mathcal{P}_2(\mathbb{R})$ and the polynomial $\mathbf{p} \in \mathcal{P}_2(\mathbb{R})$ defined by $\mathbf{p}(x) = 4x^2 + 3x - 2$. Then:

$$\mathbf{p} = \Phi_{\mathcal{B}}^{-1} \begin{pmatrix} 4 \\ 3 \\ -2 \end{pmatrix} = \Phi_{\mathcal{B}}^{-1}(\Phi_{\mathcal{B}}(\mathbf{p})) \quad \text{since} \quad \mathbf{p} = 4\mathbf{m}_2 + 3\mathbf{m}_1 - 2\mathbf{m}_0.$$

Example 7.20. Let $V = \mathbb{R}^2$ and choose the basis $\mathcal{B} = ((1), (0))$. The vector $\mathbf{x} = (3) \in V$ can then be represented as:

$$\mathbf{x} = \begin{pmatrix} 3 \\ 7 \end{pmatrix} = 3 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + 4 \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 3 \\ 4 \end{pmatrix} = \Phi_{\mathcal{B}}^{-1}(\Phi_{\mathcal{B}}(\mathbf{x}))$$

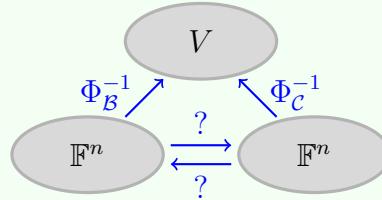
Obviously, in this case, the linear map $\Phi_{\mathcal{B}}^{-1} : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ has a corresponding 2×2 -matrix $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$.

Now, let $\mathcal{C} = (\mathbf{c}_1, \dots, \mathbf{c}_n)$ be another basis of V . Then a given vector $\mathbf{v} \in V$ can also be given as a linear combination with this new basis, which means $\mathbf{v} = \alpha'_1 \mathbf{c}_1 + \dots + \alpha'_n \mathbf{c}_n = \Phi_{\mathcal{C}}^{-1}(\Phi_{\mathcal{C}}(\mathbf{v}))$ with some coefficients $\alpha'_j \in \mathbb{F}$.

Question: Old versus new coordinates

How to switch between both coordinate vectors?

$$\mathcal{B}\text{-coordinates } \Phi_{\mathcal{B}}(\mathbf{v}) = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix} \quad \leftrightarrow ? \quad \mathcal{C}\text{-coordinates } \Phi_{\mathcal{C}}(\mathbf{v}) = \begin{pmatrix} \alpha'_1 \\ \vdots \\ \alpha'_n \end{pmatrix}$$



We see that we want to create a map $\mathbb{F}^n \rightarrow \mathbb{F}^n$ that makes this translation from one coordinate system into the other. Looking at the linear maps $\Phi_{\mathcal{B}}$ and $\Phi_{\mathcal{C}}$, we see that this is just a composition of two linear maps. To get the map from “left to right” by $f : \mathbb{F}^n \rightarrow \mathbb{F}^n$, $f := \Phi_{\mathcal{C}} \circ \Phi_{\mathcal{B}}^{-1}$. More concretely for all canonical unit vectors $\mathbf{e}_j \in \mathbb{F}^n$, we get:

$$f(\mathbf{e}_j) = \Phi_{\mathcal{C}}(\Phi_{\mathcal{B}}^{-1}(\mathbf{e}_j)) = \Phi_{\mathcal{C}}(\mathbf{b}_j) \tag{7.11}$$

Since f is a linear map, we find a uniquely determined matrix A such that $f(\mathbf{x}) = A\mathbf{x}$ for all $\mathbf{x} \in \mathbb{F}^n$. This matrix is determined by equation (7.11) and given a suitable name:

Transformation matrix or change-of-basis matrix

$$T_{\mathcal{C} \leftarrow \mathcal{B}} := \left(\Phi_{\mathcal{C}} \left| \begin{array}{c} \mathbf{b}_1 \\ \vdots \\ \mathbf{b}_n \end{array} \right. \right) \in \mathbb{F}^{n \times n} \tag{7.12}$$

is called the [transformation matrix](#) or [change-of-basis matrix](#) from \mathcal{B} to \mathcal{C} .

The corresponding linear map gives us a sense of switching from basis \mathcal{B} to the basis \mathcal{C} . Also a good mnemonic is:

$$\Phi_{\mathcal{C}}^{-1} T_{\mathcal{C} \leftarrow \mathcal{B}} \mathbf{x} = \Phi_{\mathcal{B}}^{-1} \mathbf{x} \quad \text{for all } \mathbf{x} \in \mathbb{F}^n \quad (7.13)$$

Now, if we have a vector $\mathbf{v} \in V$ and its coordinate vector $\Phi_{\mathcal{B}}(\mathbf{v})$ and $\Phi_{\mathcal{C}}(\mathbf{v})$, respectively, then we can calculate:

$$T_{\mathcal{C} \leftarrow \mathcal{B}} \Phi_{\mathcal{B}}(\mathbf{v}) = \Phi_{\mathcal{C}}(\Phi_{\mathcal{B}}^{-1}(\Phi_{\mathcal{B}}(\mathbf{v}))) = \Phi_{\mathcal{C}}(\mathbf{v}).$$

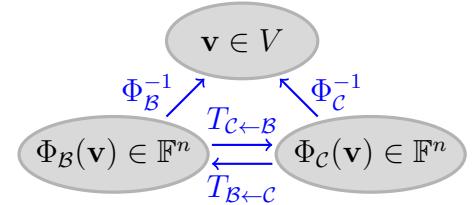
We fix our result:

Transformation formula

$$\Phi_{\mathcal{C}}(\mathbf{v}) = T_{\mathcal{C} \leftarrow \mathcal{B}} \Phi_{\mathcal{B}}(\mathbf{v}). \quad (7.14)$$

If we exchange the roles of \mathcal{B} and \mathcal{C} , we also get a transformation matrix: $T_{\mathcal{B} \leftarrow \mathcal{C}} := (\Phi_{\mathcal{B}}(\mathbf{c}_1) \ \dots \ \Phi_{\mathcal{B}}(\mathbf{c}_n))$. Analogously to (7.13) and (7.14), we get

$$\Phi_{\mathcal{B}}^{-1} T_{\mathcal{B} \leftarrow \mathcal{C}} \mathbf{x} = \Phi_{\mathcal{C}}^{-1} \mathbf{x} \quad \text{and} \quad \Phi_{\mathcal{B}}(\mathbf{v}) = T_{\mathcal{B} \leftarrow \mathcal{C}} \Phi_{\mathcal{C}}(\mathbf{v}).$$



Of course by definition or by looking at (7.13) and (7.14), we get:

$$T_{\mathcal{B} \leftarrow \mathcal{C}} = (T_{\mathcal{C} \leftarrow \mathcal{B}})^{-1}. \quad (7.15)$$

Rule of thumb: How to get the transformation matrix $T_{\mathcal{C} \leftarrow \mathcal{B}}$

The notation $T_{\mathcal{C} \leftarrow \mathcal{B}}$ means: We put the vector in \mathcal{B} -coordinates in (from the right) and get out the vector in \mathcal{C} -coordinates. To get the transformation matrix $T_{\mathcal{C} \leftarrow \mathcal{B}}$ write the basis vectors of \mathcal{B} in \mathcal{C} -coordinates and put them as columns in a matrix.

Example 7.21. We already know the monomial basis

$$\mathcal{B} = (\underbrace{\mathbf{m}_2}_{=: \mathbf{b}_1}, \underbrace{\mathbf{m}_1}_{=: \mathbf{b}_2}, \underbrace{\mathbf{m}_0}_{=: \mathbf{b}_3}) = (x \mapsto x^2, x \mapsto x, x \mapsto 1)$$

in $\mathcal{P}_2(\mathbb{R})$. Now, we can easily show that

$$\mathcal{C} = (\underbrace{\mathbf{m}_2 - \frac{1}{2}\mathbf{m}_1}_{=: \mathbf{c}_1}, \underbrace{\mathbf{m}_2 + \frac{1}{2}\mathbf{m}_1}_{=: \mathbf{c}_2}, \underbrace{\mathbf{m}_0}_{=: \mathbf{c}_3}).$$

defines also a basis of $\mathcal{P}_2(\mathbb{R})$. Now we know how to change between these two bases. Therefore, we calculate the transformation matrices. The first thing you should note is that the basis \mathcal{C} is already given in linear combinations of the basis vectors from \mathcal{B} . Hence we get:

$$\Phi_{\mathcal{B}}(\mathbf{c}_1) = \begin{pmatrix} 1 \\ -\frac{1}{2} \\ 0 \end{pmatrix}, \Phi_{\mathcal{B}}(\mathbf{c}_2) = \begin{pmatrix} 1 \\ \frac{1}{2} \\ 0 \end{pmatrix}, \Phi_{\mathcal{B}}(\mathbf{c}_3) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \implies T_{\mathcal{B} \leftarrow \mathcal{C}} = \begin{pmatrix} 1 & 1 & 0 \\ -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Then $\Phi_{\mathcal{B}}(\mathbf{p}) = T_{\mathcal{B} \leftarrow \mathcal{C}} \Phi_{\mathcal{C}}(\mathbf{p})$ gives us the wanted translation. If we want to calculate the reverse translation, we have to calculate the inverse matrix of $T_{\mathcal{B} \leftarrow \mathcal{C}}$.

By calculating the inverse, we get the other transformation matrix $T_{\mathcal{C} \leftarrow \mathcal{B}} = (T_{\mathcal{B} \leftarrow \mathcal{C}})^{-1}$:

$$T_{\mathcal{C} \leftarrow \mathcal{B}} = \begin{pmatrix} \Phi_{\mathcal{C}}(\mathbf{b}_1) & \Phi_{\mathcal{C}}(\mathbf{b}_2) & \Phi_{\mathcal{C}}(\mathbf{b}_3) \end{pmatrix} = \begin{pmatrix} 1/2 & -1 & 0 \\ 1/2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

For an arbitrary polynomial $\mathbf{p}(x) = ax^2 + bx + c$ with $a, b, c \in \mathbb{R}$, we get:

$$\Phi_{\mathcal{C}}(\mathbf{p}) = T_{\mathcal{C} \leftarrow \mathcal{B}} \Phi_{\mathcal{B}}(\mathbf{p}) = \begin{pmatrix} 1/2 & -1 & 0 \\ 1/2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} a/2 - b \\ a/2 + b \\ c \end{pmatrix}.$$

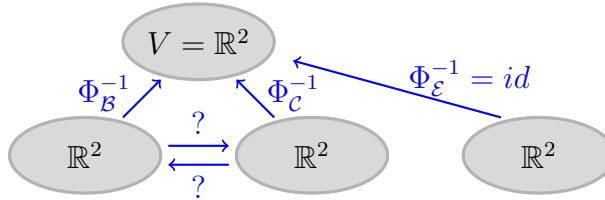
Hence, for our example $\mathbf{p}(x) = 4x^2 + 3x - 2$, $a = 4$, $b = 3$ and $c = -2$, we get:

$$\Phi_{\mathcal{C}}(\mathbf{p}) = \begin{pmatrix} 4/2 - 3 \\ 4/2 + 3 \\ -2 \end{pmatrix} = \begin{pmatrix} -1 \\ 5 \\ -2 \end{pmatrix}.$$

Let us check again if this was all correct:

$$(-1) \underbrace{(x^2 - \frac{1}{2}x)}_{\mathbf{c}_1(x)} + 5 \underbrace{(x^2 + \frac{1}{2}x)}_{\mathbf{c}_2(x)} + (-2) \underbrace{1}_{\mathbf{c}_3(x)} = (-1 + 5)x^2 + (\frac{1}{2} + \frac{5}{2})x + (-2)1 = 4x^2 + 3x - 2.$$

Example 7.22. Now, we look at \mathbb{R}^2 with the two bases $\mathcal{B} = ((1), (3))$ and $\mathcal{C} = ((0), (2))$. In this case neither the matrix $T_{\mathcal{B} \leftarrow \mathcal{C}}$ nor $T_{\mathcal{C} \leftarrow \mathcal{B}}$ is obviously given. In such a case, it might be helpful to include a third basis, which is well-known. In \mathbb{R}^2 this third basis should be of course the standard basis: $\mathcal{E} = (\mathbf{e}_1, \mathbf{e}_2)$.



The idea is to calculate first the transformation matrices $T_{\mathcal{E} \leftarrow \mathcal{B}}$ and $T_{\mathcal{E} \leftarrow \mathcal{C}}$ and the inverses and then compose the maps in a way to get the transformation matrices $T_{\mathcal{B} \leftarrow \mathcal{C}}$ and $T_{\mathcal{C} \leftarrow \mathcal{B}}$.

The basis elements of \mathcal{B} and \mathcal{C} are already given in the coordinates of the standard basis. Hence:

$$T_{\mathcal{E} \leftarrow \mathcal{B}} = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix} \quad \text{and} \quad T_{\mathcal{E} \leftarrow \mathcal{C}} = \begin{pmatrix} 1 & 2 \\ 0 & 2 \end{pmatrix}.$$

Now, for getting $T_{\mathcal{C} \leftarrow \mathcal{B}}$, we have to combine:

$$T_{\mathcal{C} \leftarrow \mathcal{B}} = T_{\mathcal{C} \leftarrow \mathcal{E}} T_{\mathcal{E} \leftarrow \mathcal{B}} = (T_{\mathcal{E} \leftarrow \mathcal{C}})^{-1} T_{\mathcal{E} \leftarrow \mathcal{B}}.$$

This means, we have to calculate the inverse of $T_{\mathcal{E} \leftarrow \mathcal{C}}$ first and to multiply with the matrix $T_{\mathcal{E} \leftarrow \mathcal{B}}$. However, we can do both things together when doing the Gaussian elimination with

more than one right-hand-side (this is often called [Gauß-Jordan algorithm](#)). In this case, we want to generate the unit matrix on the left:

$$(T_{\mathcal{E} \leftarrow \mathcal{C}} \mid T_{\mathcal{E} \leftarrow \mathcal{B}}) \rightsquigarrow (\mathbb{1} \mid T_{\mathcal{C} \leftarrow \mathcal{B}}), \quad \text{so} \quad \left(\begin{array}{cc|cc} 1 & 2 & 1 & 3 \\ 0 & 2 & 2 & 4 \end{array} \right) \rightsquigarrow \left(\begin{array}{cc|cc} 1 & 0 & -1 & -1 \\ 0 & 1 & 1 & 2 \end{array} \right).$$

In more details, this means: We solve two LES simultaneously and also do the backward substitution in the matrix notation. The advantage is that we read the solution as a matrix directly from the right-hand side after all calculation steps.

Question: Can we do a similar thing in the polynomial space? Consider bases \mathcal{B} and \mathcal{C} that is not the simple monomial basis:

$$\mathcal{B} = (\underbrace{2\mathbf{m}_2 - 1\mathbf{m}_1}_{=: \mathbf{b}_1}, \underbrace{-8\mathbf{m}_1 - 2\mathbf{m}_0}_{=: \mathbf{b}_2}, \underbrace{1\mathbf{m}_2 + 4\mathbf{m}_1 + 1\mathbf{m}_0}_{=: \mathbf{b}_3})$$

and $\mathcal{C} = (\underbrace{1\mathbf{m}_1 + 1\mathbf{m}_0}_{=: \mathbf{c}_1}, \underbrace{2\mathbf{m}_2 + 2\mathbf{m}_1}_{=: \mathbf{c}_2}, \underbrace{1\mathbf{m}_2 + 1\mathbf{m}_0}_{=: \mathbf{c}_3}).$

Answer: Yes, we can do the same by adding the the monomial basis (or a other well-known basis) in the middle. We call the monomial basis by \mathcal{A} , which means $\mathcal{A} = (\mathbf{m}_2, \mathbf{m}_1, \mathbf{m}_0)$. Then $T_{\mathcal{A} \leftarrow \mathcal{B}}$ and $T_{\mathcal{A} \leftarrow \mathcal{C}}$ are immediately given:

$$T_{\mathcal{A} \leftarrow \mathcal{B}} = \begin{pmatrix} 2 & 0 & 1 \\ -1 & -8 & 4 \\ 0 & -2 & 1 \end{pmatrix} \quad \text{and} \quad T_{\mathcal{A} \leftarrow \mathcal{C}} = \begin{pmatrix} 0 & 2 & 1 \\ 1 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix},$$

and then we get $T_{\mathcal{B} \leftarrow \mathcal{C}}$:

$T_{\mathcal{B} \leftarrow \mathcal{C}}$ by using an additional “nice” basis \mathcal{A}

$$T_{\mathcal{A} \leftarrow \mathcal{C}} = T_{\mathcal{A} \leftarrow \mathcal{B}} T_{\mathcal{B} \leftarrow \mathcal{C}} \quad \text{and hence} \quad T_{\mathcal{B} \leftarrow \mathcal{C}} = (T_{\mathcal{A} \leftarrow \mathcal{B}})^{-1} T_{\mathcal{A} \leftarrow \mathcal{C}}.$$

$$\Phi_{\mathcal{A}}(\mathbf{x}) \xleftarrow{T_{\mathcal{A} \leftarrow \mathcal{B}}} \Phi_{\mathcal{B}}(\mathbf{x}) \xleftarrow{T_{\mathcal{B} \leftarrow \mathcal{C}}} \Phi_{\mathcal{C}}(\mathbf{x})$$

Since we again have to find an inverse of a matrix, we can use the Gauß-Jordan algorithm again:

$$(T_{\mathcal{A} \leftarrow \mathcal{B}} \mid T_{\mathcal{A} \leftarrow \mathcal{C}}) \rightsquigarrow (\mathbb{1} \mid T_{\mathcal{B} \leftarrow \mathcal{C}}). \quad (7.16)$$

For our example, this gives us:

$$\left(\begin{array}{ccc|ccc} 2 & 0 & 1 & 0 & 2 & 1 \\ -1 & -8 & 4 & 1 & 2 & 0 \\ 0 & -2 & 1 & 1 & 0 & 1 \end{array} \right) \rightsquigarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & -2 & 4 \\ 0 & 1 & 0 & -7/2 & 3 & -4 \\ 0 & 0 & 1 & -6 & 6 & -7 \end{array} \right).$$

The boxed matrix is indeed $T_{\mathcal{B} \leftarrow \mathcal{C}}$.

Change of basis for audio: WAV vs. MP3

Assume you have an audio signal \mathbf{f} given at finite time steps $t = 1, 2, \dots, 50$ (e.g. milliseconds). Hence, you have some measure values $f_1, f_2, \dots, f_{50} \in \mathbb{R}$.

$$\mathbf{b}_1: \begin{array}{c} 1 \\ \hline 1 \end{array}$$

$$\mathbf{b}_2: \begin{array}{c} 1 \\ \hline 2 \end{array}$$

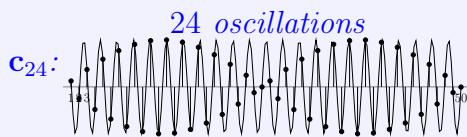
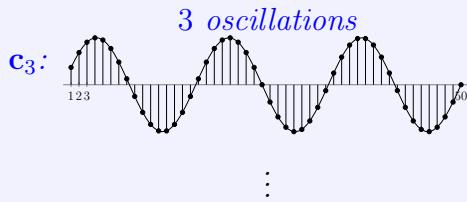
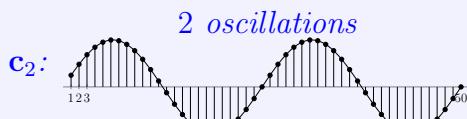
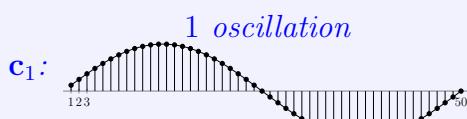
$$\mathbf{b}_{50}: \begin{array}{c} 1 \\ \vdots \\ 1 \end{array}$$

$$\mathbf{f} = \begin{array}{c} 123 \\ \hline \dots \end{array}$$

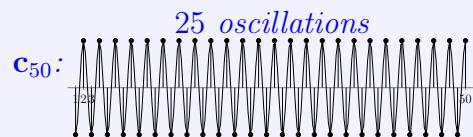
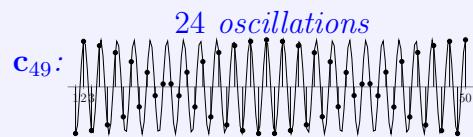
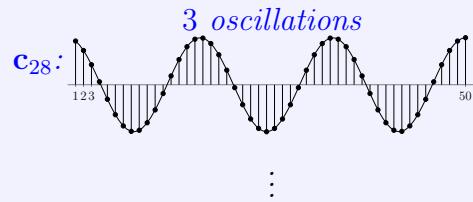
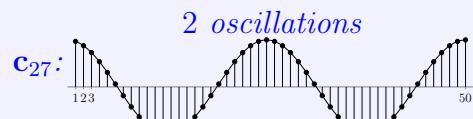
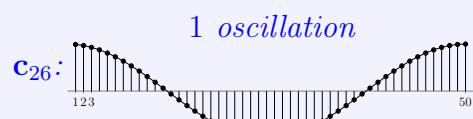
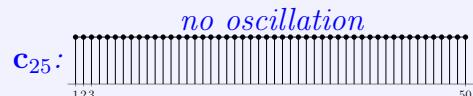
At this point you know that the audio signal \mathbf{f} is a vector in a 50-dimensional space, which can be represented with respect to the canonical basis $\mathcal{B} = (\mathbf{b}_1, \dots, \mathbf{b}_{50})$ of \mathbb{R}^{50} . The coordinates of \mathbf{f} are exactly the values f_1, f_2, \dots, f_{50} . For describing tones (so oscillations) this basis is not optimal!

We want to change to a basis \mathcal{C} of \mathbb{R}^{50} , which is better fitting for tones.

Sine waves



Cosine waves



One can show: $\mathcal{C} = (\mathbf{c}_1, \dots, \mathbf{c}_{50})$ is also linearly independent and hence a basis of \mathbb{R}^{50} .



The signal \mathbf{f} from above has the following form:

$$\mathbf{f} = \mathbf{c}_3 - \mathbf{c}_{25} + 3\mathbf{c}_{26} + \frac{1}{4}\mathbf{c}_{40}.$$

We reckon that most signals \mathbf{f} are a superposition of some “basic tones” \mathbf{c}_i .

Compression: One stores only the coordinates in $\Phi_{\mathcal{C}}(\mathbf{f})$. One can also focus on the (for humans) important frequencies and ignore the higher and lower ones (e.g. MP3 file format). All this saves storage space instead of storing the coordinates $\Phi_{\mathcal{B}}(\mathbf{f}) = (f_1, \dots, f_{50})^T$ (e.g. WAV file format). Similar ideas exist for two-dimensional signals like pictures: \Rightarrow BMP vs. JPG.

Information: The change of basis from \mathcal{B} to \mathcal{C} is important for a lot of applications and known as the Fourier transform. We will consider it in more detail in the analysis lecture.

7.5 General vector space with inner product and norms

Recall that in the vector spaces \mathbb{R}^n and \mathbb{C}^n , besides the algebraic structure given by

vector addition + and the scalar multiplication \cdot ,

we also defined a geometric structure by choosing

an inner product $\langle \cdot, \cdot \rangle$ and also a norm $\| \cdot \|$

for measuring angles and lengths.

Now we want to expand such a geometric structure to general \mathbb{F} -vector spaces.

Attention! Convention for $\mathbb{F} = \mathbb{R}$ and $\mathbb{F} = \mathbb{C}$

Since we handle the cases $\mathbb{F} = \mathbb{R}$ and $\mathbb{F} = \mathbb{C}$ simultaneously, we also use the notion of the complex conjugation in the real case. Hence, for $\alpha \in \mathbb{F}$ we write:

$$\bar{\alpha} := \begin{cases} \alpha & \text{if } \mathbb{F} = \mathbb{R}, \\ \bar{\alpha} & \text{if } \mathbb{F} = \mathbb{C} \end{cases} \quad (\text{complex conjugate number}).$$

Analogously, for a matrix $A \in \mathbb{F}^{m \times n}$ with $m, n \in \mathbb{N}$:

$$A^* := \begin{cases} A^T & \text{if } \mathbb{F} = \mathbb{R} \quad (\text{transpose}), \\ A^* & \text{if } \mathbb{F} = \mathbb{C} \quad (\text{adjoint}). \end{cases}$$

7.5.1 Inner products

Let $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$ and V be an \mathbb{F} -vector space.

Definition 7.23. Inner product

A map $\langle \cdot, \cdot \rangle : V \times V \rightarrow \mathbb{F}$ is called an inner product for V if it fulfills: For all $\mathbf{x}, \mathbf{x}', \mathbf{y} \in V$ and $\alpha \in \mathbb{F}$:

- (S1) $\langle \mathbf{x}, \mathbf{x} \rangle > 0$ for all $\mathbf{x} \neq \mathbf{0}$, (positive definite)
- (S2) $\langle \mathbf{x} + \mathbf{x}', \mathbf{y} \rangle = \langle \mathbf{x}, \mathbf{y} \rangle + \langle \mathbf{x}', \mathbf{y} \rangle$, (additive)
- (S3) $\langle \alpha \mathbf{x}, \mathbf{y} \rangle = \alpha \langle \mathbf{x}, \mathbf{y} \rangle$, (homogeneous) } (linear)
- (S4) $\langle \mathbf{x}, \mathbf{y} \rangle = \overline{\langle \mathbf{y}, \mathbf{x} \rangle}$. ((conj.) symmetric)

A vector space with an inner product is often called a pre-Hilbert space.

Recall all the properties we could derive from these four rules. For example:

$$\langle \mathbf{x}, \alpha \mathbf{y} \rangle = \overline{\alpha} \langle \mathbf{x}, \mathbf{y} \rangle \quad \text{for all } \alpha \in \mathbb{F}, \mathbf{x}, \mathbf{y} \in V.$$

The proof goes like: $\langle \mathbf{x}, \alpha \mathbf{y} \rangle \stackrel{(S4)}{=} \overline{\langle \alpha \mathbf{y}, \mathbf{x} \rangle} \stackrel{(S3)}{=} \overline{\alpha \langle \mathbf{y}, \mathbf{x} \rangle} = \overline{\alpha} \overline{\langle \mathbf{y}, \mathbf{x} \rangle} \stackrel{(S4)}{=} \overline{\alpha} \langle \mathbf{x}, \mathbf{y} \rangle$.

Example 7.24. (a) Let $V = \mathbb{F}^n$.

Standard inner product \mathbb{F}^n

$$\begin{aligned} \langle \mathbf{x}, \mathbf{y} \rangle &= \left\langle \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}, \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} \right\rangle = x_1 \overline{y_1} + \dots + x_n \overline{y_n} = (\overline{y_1} \ \dots \ \overline{y_n}) \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \quad (7.17) \\ &= \mathbf{y}^* \mathbf{x} =: \langle \mathbf{x}, \mathbf{y} \rangle_{\text{euclid}}, \quad \mathbf{x}, \mathbf{y} \in \mathbb{F}^n \end{aligned}$$

Again, the standard inner product is the most important one in \mathbb{R}^n and \mathbb{C}^n . Since it describes the usual euclidean geometry, we denote it by $\langle \mathbf{x}, \mathbf{y} \rangle_{\text{euclid}}$ in both cases.

(b) For $V = \mathbb{F}^2$ and $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, $\mathbf{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \in \mathbb{F}^2$ we define an inner product by

$$\langle \mathbf{x}, \mathbf{y} \rangle = \langle \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \rangle := x_1 \overline{y_1} + x_1 \overline{y_2} + x_2 \overline{y_1} + 4x_2 \overline{y_2}.$$

(c) For $V = \mathbb{F}^2$ and $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, $\mathbf{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \in \mathbb{F}^2$, we could also define

$$\langle \mathbf{x}, \mathbf{y} \rangle = \langle \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \rangle := x_1 \overline{y_2} + x_2 \overline{y_1}.$$

This is symmetric and linear in the first argument but not positive definite. For example, $\mathbf{x} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ gives us $\langle \mathbf{x}, \mathbf{x} \rangle = \langle \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix} \rangle = -2$.

- (d) Let $V = \mathcal{P}([0, 1], \mathbb{F})$ be the \mathbb{F} -vector space of all polynomial functions $\mathbf{f} : [0, 1] \rightarrow \mathbb{F}$. Then, we define for $\mathbf{f}, \mathbf{g} \in V$ the inner product:

$$\langle \mathbf{f}, \mathbf{g} \rangle := \int_0^1 \mathbf{f}(x) \overline{\mathbf{g}(x)} dx$$

You should see the analogy to $\langle \mathbf{x}, \mathbf{y} \rangle_{\text{euclid}}$ in \mathbb{F}^n . All data is now continuously distributed over $[0, 1]$, and we need an integral instead of a sum. Often, we are in the case $\mathbb{F} = \mathbb{R}$ and can ignore the complex conjugation $\overline{\mathbf{g}(x)}$.

Recall that for a general inner product on \mathbb{R}^n , there is a uniquely determined positive matrix A such that:

$$\langle \mathbf{x}, \mathbf{y} \rangle = \langle A\mathbf{x}, \mathbf{y} \rangle_{\text{euclid}} \quad (7.18)$$

for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$.

In the same way this also works for the complex vector space \mathbb{C}^n . We just have to expand the definition of *positive definite* matrices in this case:

Definition 7.25. Positive definite matrix

A matrix $A \in \mathbb{F}^{n \times n}$ is called *positive definite* if it is selfadjoint ($A^* = A$) and satisfies

$$\langle A\mathbf{x}, \mathbf{x} \rangle_{\text{euclid}} > 0, \quad \text{i.e.} \quad \mathbf{x}^* A \mathbf{x} > 0 \quad (7.19)$$

for all $\mathbf{x} \in \mathbb{F}^n \setminus \{\mathbf{0}\}$.

Attention! Positive definite needs selfadjointness

By our definition a positive definite matrix is always selfadjoint. In the complex case this follows from equation (7.19). However, in the real case, you cannot drop this assumption. Moreover, $\langle A\mathbf{x}, \mathbf{x} \rangle_{\text{euclid}}$ is always real, even in the case $\mathbb{F} = \mathbb{C}$,

$$\langle A\mathbf{x}, \mathbf{x} \rangle_{\text{euclid}} = \langle \mathbf{x}, A^* \mathbf{x} \rangle_{\text{euclid}} \stackrel{A=A^*}{=} \langle \mathbf{x}, A\mathbf{x} \rangle_{\text{euclid}} = \overline{\langle A\mathbf{x}, \mathbf{x} \rangle_{\text{euclid}}}.$$

Some authors might be using only equation (7.19) for defining positive definite matrices in the real case. Therefore to play it safe, we often talk about matrices that are “selfadjoint and positive definite”.

We fix the general result:

Proposition 7.26. Positive definite matrix $A \Rightarrow \langle A\mathbf{x}, \mathbf{y} \rangle_{\text{euclid}}$ inner product

If $A \in \mathbb{F}^{n \times n}$ is selfadjoint and positive definite, then

$$\langle \mathbf{x}, \mathbf{y} \rangle := \langle A\mathbf{x}, \mathbf{y} \rangle_{\text{euclid}}, \quad \mathbf{x}, \mathbf{y} \in \mathbb{F}^n$$

defines an inner product in \mathbb{F}^n .

Example 7.27. Let us look at the examples from before:

(a) The identity matrix $\mathbb{1}$ is positive definite since $\langle \mathbb{1}\mathbf{x}, \mathbf{x} \rangle_{\text{euclid}} = \langle \mathbf{x}, \mathbf{x} \rangle_{\text{euclid}} > 0$ for all $\mathbf{x} \neq \mathbf{0}$.

(b) The matrix $A = \begin{pmatrix} 1 & 1 \\ 1 & 4 \end{pmatrix} \in \mathbb{R}^{2 \times 2}$ is positive definite since for all $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2$ we have

$$\left\langle \begin{pmatrix} 1 & 1 \\ 1 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \right\rangle_{\text{euclid}} = x_1x_1 + x_2x_1 + x_1x_2 + 4x_2x_2 = (x_1 + x_2)^2 + 3(x_2)^2 \geq 0.$$

This can be only 0 if $x_1 = -x_2$ and $x_2 = 0$, hence only for $\mathbf{x} = \mathbf{0}$.

(c) The matrix $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ is selfadjoint but not positive definite. For example, for $\mathbf{x} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ the value $\langle A\mathbf{x}, \mathbf{x} \rangle_{\text{euclid}}$ is negative.

Testing a matrix $A \in \mathbb{F}^{n \times n}$ for positive definiteness can be much work, even in the case $n = 2$. Therefore, the next criterion is very useful:

Proposition 7.28. 4 recognition features for a positive definite matrix

Let $A = (a_{ij}) \in \mathbb{F}^{n \times n}$ be a selfadjoint matrix. Then the following claims are equivalent:

- (i) A is positive definite.
- (ii) All eigenvalues of A are positive.
- (iii) After using Gaussian elimination only with the matrices $Z_{i-\lambda j}$, all pivots are positive.
- (iv) The determinants of the so-called leading principal minors of A , which means $\det(H_1), \dots, \det(H_n)$, are positive.

Here

$$H_1 = (a_{11}), \quad H_2 = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \quad H_3 = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}, \quad \dots, \quad H_n = A.$$

We skip the proof here. Keep in mind that “positive” always means strictly greater than zero (> 0)! Claim (iv) is called Sylvester’s criterion.

Example 7.29. Let us check the proposition for the matrix $A = \begin{pmatrix} 1 & 1 \\ 1 & 4 \end{pmatrix}$. It is positive definite by Example 7.27 (b). The eigenvalues of A are given by solving

$$0 = \det(A - \lambda \mathbb{1}) = (1 - \lambda)(4 - \lambda) - 1 = \lambda^2 - 5\lambda + 3, \quad \text{so} \quad \lambda_{1,2} = \frac{5}{2} \pm \sqrt{\left(\frac{5}{2}\right)^2 - 3}.$$

Both eigenvalues, λ_1 and λ_2 , are positive. The Gaussian elimination gives us:

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 4 \end{pmatrix} \quad \sim \quad \begin{pmatrix} 1 & 1 \\ 0 & 3 \end{pmatrix}.$$

Both pivots, 1 and 3 , are positive. At last the minors:

$$\det(H_1) = \det(1) = 1 > 0 \quad \text{and} \quad \det(H_2) = \det \begin{pmatrix} 1 & 1 \\ 1 & 4 \end{pmatrix} = 3 > 0.$$

We have already seen in Proposition 7.26 that $\langle \mathbf{x}, \mathbf{y} \rangle := \langle A\mathbf{x}, \mathbf{y} \rangle_{\text{euclid}}$ defines an inner product in \mathbb{F}^n if A is positive definite. In some sense, also the converse is correct:

Proposition 7.30. Inner products are related to pos. definite matrices

Let V be an \mathbb{F} -vector space with inner product $\langle \cdot, \cdot \rangle$ and $\dim(V) = n$. Let \mathcal{B} be a basis of V . Then for all $\mathbf{x}, \mathbf{y} \in V$ we have

$$\langle \mathbf{x}, \mathbf{y} \rangle = \langle A\Phi_{\mathcal{B}}(\mathbf{x}), \Phi_{\mathcal{B}}(\mathbf{y}) \rangle_{\text{euclid}},$$

where $\langle \cdot, \cdot \rangle_{\text{euclid}}$ is the standard inner product in \mathbb{F}^n and

$$A = G(\mathcal{B}) = \begin{pmatrix} \langle \mathbf{b}_1, \mathbf{b}_1 \rangle & \cdots & \langle \mathbf{b}_n, \mathbf{b}_1 \rangle \\ \vdots & & \vdots \\ \langle \mathbf{b}_1, \mathbf{b}_n \rangle & \cdots & \langle \mathbf{b}_n, \mathbf{b}_n \rangle \end{pmatrix}$$

is the Gramian matrix w.r.t. \mathcal{B} .

Proof. Exercise! □

Example 7.31. Look at the \mathbb{R} -vector space $\mathcal{P}_2([0, 1])$ of all real polynomial functions $f : [0, 1] \rightarrow \mathbb{R}$ with degree ≤ 2 . The integral

$$\langle \mathbf{p}, \mathbf{q} \rangle := \int_0^1 \mathbf{p}(x)\mathbf{q}(x) dx, \quad \mathbf{p}, \mathbf{q} \in \mathcal{P}_2$$

defines an inner product. Let us check how to use Proposition 7.30 in this case. Choose a basis \mathcal{B} of \mathcal{P}_2 , for example the monomial basis $\mathcal{B} = (\mathbf{m}_0, \mathbf{m}_1, \mathbf{m}_2)$, and calculate the associated Gramian matrix:

$$\langle \mathbf{m}_i, \mathbf{m}_j \rangle = \int_0^1 x^i x^j dx = \int_0^1 x^{i+j} dx = \frac{x^{i+j+1}}{i+j+1} \Big|_0^1 = \frac{1^{i+j+1} - 0^{i+j+1}}{i+j+1} = \frac{1}{i+j+1} \quad (7.20)$$

and

$$G(\mathcal{B}) = \begin{pmatrix} \langle \mathbf{m}_0, \mathbf{m}_0 \rangle & \langle \mathbf{m}_1, \mathbf{m}_0 \rangle & \langle \mathbf{m}_2, \mathbf{m}_0 \rangle \\ \langle \mathbf{m}_0, \mathbf{m}_1 \rangle & \langle \mathbf{m}_1, \mathbf{m}_1 \rangle & \langle \mathbf{m}_2, \mathbf{m}_1 \rangle \\ \langle \mathbf{m}_0, \mathbf{m}_2 \rangle & \langle \mathbf{m}_1, \mathbf{m}_2 \rangle & \langle \mathbf{m}_2, \mathbf{m}_2 \rangle \end{pmatrix} \stackrel{(7.20)}{=} \begin{pmatrix} \frac{1}{0+0+1} & \frac{1}{1+0+1} & \frac{1}{2+0+1} \\ \frac{1}{0+1+1} & \frac{1}{1+1+1} & \frac{1}{2+1+1} \\ \frac{1}{0+2+1} & \frac{1}{1+2+1} & \frac{1}{2+2+1} \end{pmatrix} = \begin{pmatrix} 1/1 & 1/2 & 1/3 \\ 1/2 & 1/3 & 1/4 \\ 1/3 & 1/4 & 1/5 \end{pmatrix}.$$

Then, by Proposition 7.30: For all $a, b, c, d, e, f \in \mathbb{R}$, we get:

$$\begin{aligned} \langle a\mathbf{m}_0 + b\mathbf{m}_1 + c\mathbf{m}_2, d\mathbf{m}_0 + e\mathbf{m}_1 + f\mathbf{m}_2 \rangle &= \left\langle \begin{pmatrix} 1/1 & 1/2 & 1/3 \\ 1/2 & 1/3 & 1/4 \\ 1/3 & 1/4 & 1/5 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix}, \begin{pmatrix} d \\ e \\ f \end{pmatrix} \right\rangle_{\text{euclid}} \\ &= ad + \frac{1}{2}(ae + bd) + \frac{1}{3}(af + be + cd) + \frac{1}{4}(bf + ce) + \frac{1}{5}cf. \end{aligned}$$

Let's check this:

$$\begin{aligned} \langle a\mathbf{m}_0 + b\mathbf{m}_1 + c\mathbf{m}_2, d\mathbf{m}_0 + e\mathbf{m}_1 + f\mathbf{m}_2 \rangle &= \int_0^1 (a + bx + cx^2)(d + ex + fx^2) dx \\ &= \int_0^1 (ad + (ae + bd)x + (af + be + cd)x^2 + (bf + ce)x^3 + cf x^4) dx \\ &= ad \int_0^1 dx + (ae + bd) \int_0^1 x dx + (af + be + cd) \int_0^1 x^2 dx + (bf + ce) \int_0^1 x^3 dx + cf \int_0^1 x^4 dx \\ &\stackrel{(7.20)}{=} ad + \frac{1}{2}(ae + bd) + \frac{1}{3}(af + be + cd) + \frac{1}{4}(bf + ce) + \frac{1}{5}cf. \end{aligned}$$

Corollary 7.32. Gramian matrix is positive definite.

For a basis \mathcal{B} of a vector space V with inner product $\langle \cdot, \cdot \rangle$, the Gramian matrix $G(\mathcal{B})$ is selfadjoint and positive definite.

Proof. $G(\mathcal{B}) = G(\mathcal{B})^*$ follows from $\langle \mathbf{b}_i, \mathbf{b}_j \rangle = \overline{\langle \mathbf{b}_j, \mathbf{b}_i \rangle}$. Using Proposition 7.30, we know $\langle G(\mathcal{B})\Phi_{\mathcal{B}}(\mathbf{x}), \Phi_{\mathcal{B}}(\mathbf{x}) \rangle_{\text{euclid}} = \langle \mathbf{x}, \mathbf{x} \rangle > 0$ for all $\mathbf{x} \in V \setminus \{\mathbf{o}\}$ and hence also for all vectors $\Phi_{\mathcal{B}}(\mathbf{x}) \in \mathbb{F}^n \setminus \{\mathbf{o}\}$. \square

7.5.2 Norms

As always, let $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$ and V be an \mathbb{F} -vector space. Even in the case V not having an inner product, we can talk about the length of vectors if we define a length measure:

Definition 7.33. Norm

A map $\|\cdot\| : V \rightarrow \mathbb{R}$ with the following properties is called a norm on V . For all $\mathbf{x}, \mathbf{y} \in V$ and $\alpha \in \mathbb{F}$, we have:

- (N1) $\|\mathbf{x}\| \geq 0$, and $\|\mathbf{x}\| = 0 \Leftrightarrow \mathbf{x} = \mathbf{o}$, *(positive definite)*
- (N2) $\|\alpha\mathbf{x}\| = |\alpha| \|\mathbf{x}\|$, *(absolutely homogeneous)*
- (N3) $\|\mathbf{x} + \mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\|$ *(triangle inequality).*

An \mathbb{F} -vector space with such a norm is called a normed space.

Example 7.34. (a) We already know that the euclidean norm for \mathbb{F}^n , given by

$$\|\mathbf{x}\| = \left\| \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \right\| = \sqrt{|x_1|^2 + \cdots + |x_n|^2}, \quad \mathbf{x} \in \mathbb{F}^n, \quad (7.21)$$

satisfies (N1-3) from Definition 7.33.

- (b) In equation (7.21), you see squares and a square root that cancel themselves in some sense. This would also work for cubes and the third root. Or even in general:

The p -norm

For each real number $p \geq 1$, we set:

$$\|\mathbf{x}\|_p = \left\| \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \right\|_p := \sqrt[p]{|x_1|^p + \cdots + |x_n|^p}, \quad \mathbf{x} \in \mathbb{F}^n. \quad (7.22)$$

This defines the so-called p -norm. In fact, proving the triangle inequality (N3) is not trivial. The euclidean norm (7.21) is hence also called 2-norm.

- (c) Another related norm is given by:

$$\lim_{p \rightarrow \infty} \sqrt[p]{|x_1|^p + \cdots + |x_n|^p} = \max\{|x_1|, \dots, |x_n|\}$$

Therefore, we define

Maximum norm or ∞ -norm

$$\begin{aligned}\|\mathbf{x}\|_{\infty} &= \left\| \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \right\|_{\infty} := \max\{|x_1|, \dots, |x_n|\} \\ &= \lim_{p \rightarrow \infty} \|\mathbf{x}\|_p, \quad \mathbf{x} \in \mathbb{F}^n.\end{aligned}\tag{7.23}$$

Let us check for $n = 2$ that the three properties in Definition 7.33. Let $\alpha \in \mathbb{F}$ and $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \mathbf{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \in \mathbb{F}^2$.

- (N1) $\|\mathbf{x}\|_{\infty} = \max\{|x_1|, |x_2|\}$ is only 0 if $x_1 = 0$ and $x_2 = 0$, hence $\mathbf{x} = \mathbf{0}$.
- (N2) $\|\alpha\mathbf{x}\|_{\infty} = \max\{|\alpha x_1|, |\alpha x_2|\} = \max\{|\alpha| |x_1|, |\alpha| |x_2|\} = |\alpha| \max\{|x_1|, |x_2|\} = |\alpha| \|\mathbf{x}\|_{\infty}$
- (N3) The triangle inequality:

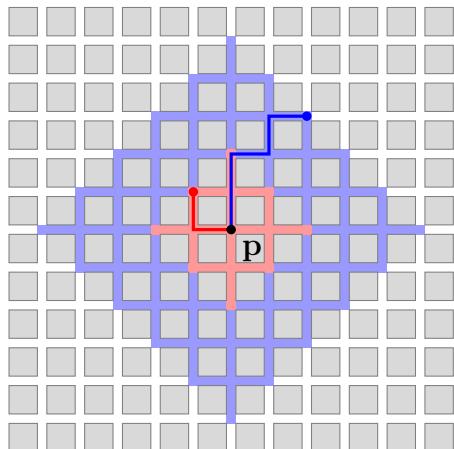
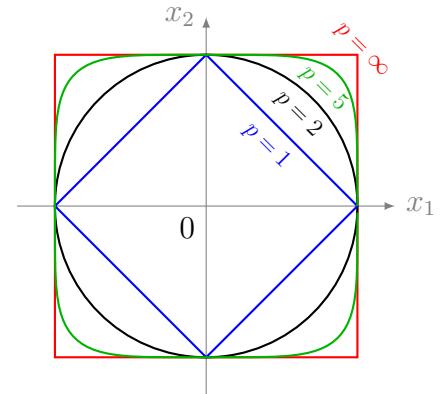
$$\begin{aligned}\|\mathbf{x} + \mathbf{y}\|_{\infty} &= \max\{|x_1 + y_1|, |x_2 + y_2|\} \leq \max\{|x_1| + |y_1|, |x_2| + |y_2|\} \\ &\stackrel{(*)}{\leq} \max\{|x_1|, |x_2|\} + \max\{|y_1|, |y_2|\} = \|\mathbf{x}\|_{\infty} + \|\mathbf{y}\|_{\infty}\end{aligned}$$

On the right-hand side, you see the geometric picture for different norms. Usually, one calls it the “unit circles”, which means the sets

$$\{\mathbf{x} \in \mathbb{R}^2 : \|\mathbf{x}\|_p = 1\}$$

Such a subset of \mathbb{R}^2 consists of all vectors with length 1, for different $p = 1, 2, 5$ and ∞ .

For $p = 2$, this is indeed a usual circle. However, also the different geometric views for other p are interesting:



Assume you are in Manhattan inside a taxicab at point \mathbf{p} . Driving one block costs you \$1. If you have \$2 in your pocket, you can reach all the red points in the map. If you have \$5, you can get to all the red and the blue points. The ε -neighbourhoods

$$\{\mathbf{x} \in \mathbb{R}^2 : \|\mathbf{x} - \mathbf{p}\| < \varepsilon\}$$

in Manhattan are just squares, which stay at one corner, and not real circles. This exactly the 1-norm $\|\cdot\|_1$, which is often alternatively called “taxicab norm”.

Rule of thumb: Norm gives you lengths and distances

You should imagine $\|\mathbf{x}\|$ as the length of the “vector arrow” \mathbf{x} . Hence, $\|\mathbf{x} - \mathbf{y}\|$ is the length of the connection vector between \mathbf{x} and \mathbf{y} – or in other words: The distance between \mathbf{x} and \mathbf{y} .

Example 7.35. – **p -norm for polynomials.** The p -norms in \mathbb{F}^n , which we defined above, can be generalised for functions. For example, for the \mathbb{R} -vector space $\mathcal{P}([a, b])$, which means all polynomial functions $\mathbf{f} : [a, b] \rightarrow \mathbb{R}$, we can also define such norms:

Norms for polynomials on $[a, b]$

$$\|\mathbf{f}\|_p := \sqrt[p]{\int_a^b |\mathbf{f}(x)|^p dx} \quad \text{for } p \in [1, \infty) \quad \text{and} \quad \|\mathbf{f}\|_\infty := \max_{x \in [a, b]} |\mathbf{f}(x)|$$

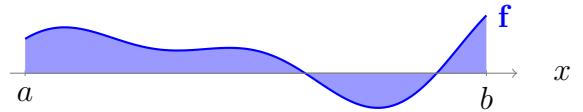
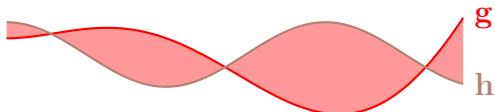
We can use the same symbol $\|\cdot\|_p$ as before since the context is always clear.

On the right-hand side, you see some polynomial functions $\mathbf{f}, \mathbf{g}, \mathbf{h} \in \mathcal{P}([a, b])$. The area of the blue region is

$$\|\mathbf{f}\|_1 = \int_a^b |\mathbf{f}(x)| dx$$

and the area of the red region is:

$$\|\mathbf{g} - \mathbf{h}\|_1 = \int_a^b |\mathbf{g}(x) - \mathbf{h}(x)| dx.$$



In later lectures, like mathematical analysis, we will prove the three properties (N1), (N2) and (N3) for all these norms.

7.5.3 Norm in pre-Hilbert spaces

If a \mathbb{F} -vector space V is equipped with an inner product $\langle \cdot, \cdot \rangle$, then we automatically get a norm $\|\cdot\|$, that is associated to $\langle \cdot, \cdot \rangle$ in such a way:

Proposition & Definition 7.36. Induced or associated norm

Let V be a pre-Hilbert space, which is an \mathbb{F} -vector space with an inner product $\langle \cdot, \cdot \rangle$. Then

$$\|\mathbf{x}\| := \sqrt{\langle \mathbf{x}, \mathbf{x} \rangle}, \quad \mathbf{x} \in V$$

defines a norm and it is called the induced norm or associated norm w.r.t. $\langle \cdot, \cdot \rangle$.

For a proof, we need the next Proposition.

Proposition 7.37. Cauchy-Schwarz inequality

Let V be a pre-Hilbert space. For all $\mathbf{x}, \mathbf{y} \in V$:

$$|\langle \mathbf{x}, \mathbf{y} \rangle|^2 \leq \langle \mathbf{x}, \mathbf{x} \rangle \langle \mathbf{y}, \mathbf{y} \rangle.$$

With the associated norm from Proposition 7.36, we get:

$$|\langle \mathbf{x}, \mathbf{y} \rangle| \leq \|\mathbf{x}\| \|\mathbf{y}\|.$$

Equality holds if and only if \mathbf{x} and \mathbf{y} are linearly dependent.

Proof. Look again at the proof of Proposition 5.5. □

We look at the examples for inner products from above and calculate the associated norms

Example 7.38. (a) The standard inner product $\langle \mathbf{x}, \mathbf{y} \rangle_{\text{euclid}} = x_1\bar{y}_1 + \cdots + x_n\bar{y}_n$ in \mathbb{F}^n induced the 2-norm $\|\mathbf{x}\| = \sqrt{|x_1|^2 + \cdots + |x_n|^2}$ in \mathbb{F}^n .

(b) The associated norm with respect to the inner product $\langle \mathbf{x}, \mathbf{y} \rangle := \langle A\mathbf{x}, \mathbf{y} \rangle_{\text{euclid}}$ in \mathbb{F}^n where $A \in \mathbb{F}^{n \times n}$ is a selfadjoint and positive definite matrix is given by

$$\|\mathbf{x}\| = \sqrt{\langle \mathbf{x}, \mathbf{x} \rangle} = \sqrt{\langle A\mathbf{x}, \mathbf{x} \rangle_{\text{euclid}}}.$$

For the example $A = \begin{pmatrix} 1 & 1 \\ 1 & 4 \end{pmatrix}$, we get

$$\left\| \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \right\| = \sqrt{\left\langle \begin{pmatrix} 1 & 1 \\ 1 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \right\rangle_{\text{euclid}}} = \sqrt{|x_1|^2 + x_1\bar{x}_2 + x_2\bar{x}_1 + 4|x_2|^2}.$$

(c) Looking at the \mathbb{F} -vector space $\mathcal{P}([a, b])$ of all polynomial functions $\mathbf{f} : [a, b] \rightarrow \mathbb{F}$, we defined the inner product

$$\langle \mathbf{f}, \mathbf{g} \rangle = \int_a^b \mathbf{f}(x)\overline{\mathbf{g}(x)} dx. \quad (7.24)$$

The associated norm in $\mathcal{P}([a, b])$ is the already introduced 2-norm since

$$\|\mathbf{f}\| = \sqrt{\langle \mathbf{f}, \mathbf{f} \rangle} = \sqrt{\int_a^b \mathbf{f}(x)\overline{\mathbf{f}(x)} dx} = \sqrt{\int_a^b |\mathbf{f}(x)|^2 dx} = \|\mathbf{f}\|_2.$$

7.5.4 Recollection: Angles, orthogonality and projection

Let V be a pre-Hilbert space, which means an \mathbb{F} -vector space with given inner product $\langle \cdot, \cdot \rangle$, and let $\|\cdot\|$ be the associated norm.

In this case, we have again the geometric structure and can talk about angles, orthogonal vectors and orthogonal projections:

Proposition & Definition 7.39. Still the same about orthogonality:

- For $\mathbf{x}, \mathbf{y} \in V$ we write $\mathbf{x} \perp \mathbf{y}$ if $\langle \mathbf{x}, \mathbf{y} \rangle = 0$.
- For $\mathbb{F} = \mathbb{R}$ and $\mathbf{x}, \mathbf{y} \in V \setminus \{\mathbf{0}\}$ we define:

$$\text{angle}(\mathbf{x}, \mathbf{y}) := \arccos \left(\frac{\langle \mathbf{x}, \mathbf{y} \rangle}{\|\mathbf{x}\| \|\mathbf{y}\|} \right).$$

- For a nonempty set $M \subset V$ we call

$$M^\perp := \{ \mathbf{x} \in V : \mathbf{x} \perp \mathbf{m} \text{ for all } \mathbf{m} \in M \}$$

the orthogonal complement of M . This is always a subspace of V .

Instead of $\mathbf{x} \in M^\perp$, we often write $x \perp M$.

- For $\mathbf{x} \in V$ and a subspace U of V there is a unique decomposition

$$\mathbf{x} = \mathbf{p} + \mathbf{n} =: \mathbf{x}|_U + \mathbf{x}|_{U^\perp}$$

into the orthogonal projection $\mathbf{p} =: \mathbf{x}|_U \in U$ and the normal component $\mathbf{n} = \mathbf{x}|_{U^\perp} \in U^\perp$ with respect to U . The calculation is given by

$$G(\mathcal{B}) \Phi_{\mathcal{B}}(\mathbf{p}) = \begin{pmatrix} \langle \mathbf{x}, \mathbf{b}_1 \rangle \\ \vdots \\ \langle \mathbf{x}, \mathbf{b}_n \rangle \end{pmatrix} \quad (7.25)$$

for any basis $\mathcal{B} = (\mathbf{b}_1, \dots, \mathbf{b}_n)$ of U , and $\mathbf{n} = \mathbf{x} - \mathbf{p}$.

- A family $\mathcal{B} = (\mathbf{u}_1, \dots, \mathbf{u}_n)$ with vectors from V is called:
 - Orthogonal system (OS) if $\mathbf{u}_i \perp \mathbf{u}_j$ for all $i, j = 1, \dots, n$ with $i \neq j$;
 - Orthonormal system (ONS) if, in addition, $\|\mathbf{u}_i\| = 1$ for all $i = 1, \dots, n$;
 - Orthogonal basis (OB) if it an OS and a basis of V ;
 - Orthonormal basis (ONB) if it an ONS and a basis of V .
- OS that do not own the zero vector \mathbf{o} are always linearly independent.
- If $\mathcal{B} = (\mathbf{b}_1, \dots, \mathbf{b}_n)$ is an OB of U , then the equation (7.25) is much simpler:

$$\Phi_{\mathcal{B}}(\mathbf{x}|_U) = \begin{pmatrix} \frac{\langle \mathbf{x}, \mathbf{b}_1 \rangle}{\|\mathbf{b}_1\|^2} \\ \vdots \\ \frac{\langle \mathbf{x}, \mathbf{b}_n \rangle}{\|\mathbf{b}_n\|^2} \end{pmatrix}, \quad i.e. \quad \mathbf{x}|_U = \frac{\langle \mathbf{x}, \mathbf{b}_1 \rangle}{\|\mathbf{b}_1\|^2} \mathbf{b}_1 + \dots + \frac{\langle \mathbf{x}, \mathbf{b}_n \rangle}{\|\mathbf{b}_n\|^2} \mathbf{b}_n. \quad (7.26)$$

If \mathcal{B} is an ONB, then it gets also easier $\|\mathbf{b}_i\|^2 (= 1)$.

Example 7.40. (a) The vectors $\mathbf{x} = \begin{pmatrix} 1 \\ i \end{pmatrix}$ and $\mathbf{y} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ from \mathbb{C}^2 are not orthogonal w.r.t. the standard inner product $\langle \cdot, \cdot \rangle_{\text{euclid}}$ since

$$\left\langle \begin{pmatrix} 1 \\ i \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\rangle_{\text{euclid}} = 1 \cdot \bar{0} + i \cdot \bar{1} = i \neq 0.$$

However, there are orthogonal w.r.t. the inner product given by $\langle \mathbf{x}, \mathbf{y} \rangle := \langle A\mathbf{x}, \mathbf{y} \rangle_{\text{euclid}}$ with $A = \begin{pmatrix} 2 & i \\ -i & 1 \end{pmatrix}$, since

$$\langle \mathbf{x}, \mathbf{y} \rangle = \left\langle \begin{pmatrix} 1 \\ i \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\rangle = \left\langle \begin{pmatrix} 2 & i \\ -i & 1 \end{pmatrix} \begin{pmatrix} 1 \\ i \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\rangle_{\text{euclid}} = \left\langle \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\rangle_{\text{euclid}} = 0.$$

The orthogonal projection of \mathbf{x} onto $\text{Span}(\mathbf{y})$ can be different for different inner products. W.r.t. $\langle \cdot, \cdot \rangle$ it is \mathbf{o} (since $\mathbf{x} \perp \mathbf{y}$), but w.r.t. $\langle \cdot, \cdot \rangle_{\text{euclid}}$ it is

$$\mathbf{x}|_{\text{Span}(\mathbf{y})} = \frac{\langle \mathbf{x}, \mathbf{y} \rangle_{\text{euclid}}}{\langle \mathbf{y}, \mathbf{y} \rangle_{\text{euclid}}} \mathbf{y} = \frac{\langle \begin{pmatrix} 1 \\ i \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \rangle_{\text{euclid}}}{\langle \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \rangle_{\text{euclid}}} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{i}{1} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = i \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ i \end{pmatrix}.$$

(b) Looking at the vector space $\mathcal{F}([0, 2\pi])$, which contains function $\mathbf{f} : [0, 2\pi] \rightarrow \mathbb{R}$, we define a subspace V that is spanned by the family $\mathcal{B} = (1, \cos, \sin)$. Then w.r.t the

inner product defined by

$$\langle \mathbf{f}, \mathbf{g} \rangle := \int_0^1 \mathbf{f}(x) \mathbf{g}(x) dx,$$

the family \mathcal{B} is an OS:

$$\begin{aligned}\langle 1, \cos \rangle &= \int_0^{2\pi} \cos x dx = 0, & \langle 1, \sin \rangle &= \int_0^{2\pi} \sin x dx = 0, & \text{and} \\ \langle \cos, \sin \rangle &= \int_0^{2\pi} \cos x \sin x dx = \frac{1}{2} \sin^2 x \Big|_0^{2\pi} = \frac{\sin^2 2\pi - \sin^2 0}{2} = 0.\end{aligned}$$

Because of

$$\langle 1, 1 \rangle = \int_0^{2\pi} 1 dx = 2\pi, \quad \langle \cos, \cos \rangle = \int_0^{2\pi} \cos^2 x dx = \pi, \quad \langle \sin, \sin \rangle = \int_0^{2\pi} \sin^2 x dx = \pi$$

the new family $\left(\frac{1}{\sqrt{2\pi}}, \frac{\cos}{\sqrt{\pi}}, \frac{\sin}{\sqrt{\pi}} \right)$ is an ONB of V .

Recall also the Gram-Schmidt orthonormalisation from Remark 5.3.

Remark: Gram-Schmidt orthonormalisation

Given: Let V be a pre-Hilbert space and \mathcal{C} a family of vectors from V .

To Find: An ONB \mathcal{B} of $\text{Span}(\mathcal{C})$.

Algorithm:

Initialise \mathcal{B} as the empty set ();

For all \mathbf{u} in \mathcal{C} :

Set $\mathbf{v} := \mathbf{u} - \mathbf{u}_{|\text{Span}(\mathcal{B})}$;

If $\mathbf{v} \neq \mathbf{0}$:

Set $\mathbf{w} := \frac{\mathbf{v}}{\|\mathbf{v}\|}$;

Add \mathbf{w} to \mathcal{B}

If you cancel the algorithm at some point, the family at this point, $\mathcal{B} = (\mathbf{w}_1, \dots, \mathbf{w}_k)$, is a ONB of the $\text{Span}(\mathbf{w}_1, \dots, \mathbf{w}_k)$.

Recall that for this ONB $\mathcal{B} = (\mathbf{w}_1, \dots, \mathbf{w}_k)$ the orthogonal projection $\mathbf{u}_{|\text{Span}(\mathcal{B})}$ is calculated by

$$\mathbf{u}_{|\text{Span}(\mathcal{B})} = \langle \mathbf{u}, \mathbf{w}_1 \rangle \mathbf{w}_1 + \dots + \langle \mathbf{u}, \mathbf{w}_k \rangle \mathbf{w}_k.$$

Example 7.41. The monomials $\mathcal{C} = (\mathbf{m}_0, \mathbf{m}_1, \mathbf{m}_2)$ do not form an ONB in $\mathcal{P}([-1, 1])$ w.r.t. $\langle \mathbf{f}, \mathbf{g} \rangle = \int_{-1}^1 \mathbf{f}(x) \mathbf{g}(x) dx$. We can apply the Gram-Schmidt procedure for \mathcal{C} . Here it is useful to start with the numbering indices 0, 1, 2, ...

$$\mathbf{v}_0 = \mathbf{m}_0 = 1, \quad \Rightarrow \quad \mathbf{w}_0(x) = \frac{\mathbf{v}_0(x)}{\|\mathbf{v}_0\|} = \frac{1}{\sqrt{2}},$$

$$\begin{aligned} \mathbf{v}_1 &= \mathbf{m}_1 - \underbrace{\langle \mathbf{m}_1, \mathbf{w}_0 \rangle}_{0} \mathbf{w}_0 = \mathbf{m}_1, & \Rightarrow \mathbf{w}_1(x) &= \frac{\mathbf{v}_1(x)}{\|\mathbf{v}_1\|} = \sqrt{\frac{3}{2}}x, \\ \mathbf{v}_2 &= \mathbf{m}_2 - \underbrace{\langle \mathbf{m}_2, \mathbf{w}_0 \rangle}_{\frac{\sqrt{2}}{3}} \mathbf{w}_0 - \underbrace{\langle \mathbf{m}_2, \mathbf{w}_1 \rangle}_{0} \mathbf{w}_1, & \Rightarrow \mathbf{w}_2(x) &= \frac{\mathbf{v}_2(x)}{\|\mathbf{v}_2\|} = \sqrt{\frac{45}{8}} \left(x^2 - \frac{1}{3} \right). \end{aligned}$$

$\mathcal{B} = (\mathbf{w}_0, \mathbf{w}_1, \mathbf{w}_2)$ is an ONB for $\text{Span}(\mathcal{C}) = \mathcal{P}_2([-1, 1])$. The polynomials $\mathbf{w}_0, \mathbf{w}_1, \mathbf{w}_2$ (or also with other normalisation factors) are called the [Legendre polynomials](#). If we add the other monomials $\mathbf{m}_3, \mathbf{m}_4, \dots$, we get the next Legendre polynomials.

Summary

- [Vectors](#) are elements in a set, called a [vector space](#) V , that one can add together and scale with numbers α from \mathbb{R} or \mathbb{C} , without leaving the set V . The addition and scalar multiplication just have to satisfy the rules (1)–(8) from Definition 7.1.
- If you know that a set V with two operations $+$ and $\alpha \cdot$ is a vector space and if you want to show that also a subset $U \neq \emptyset$ of V form a vector space, then you do not have to check (1)–(8) again, but only (a) and (b) from Proposition 7.7. This is called a *subspace* of V .
- The definitions [linear combination](#), [span](#), [generating system](#), [linearly \(in\)dependent](#), [basis](#) and [dimension](#) are literally the same in Chapter 3.
- If you fix a basis $\mathcal{B} = (\mathbf{b}_1, \dots, \mathbf{b}_n)$ in V , then each $\mathbf{x} \in V$ has a uniquely determined linear combination $\mathbf{x} = \alpha_1 \mathbf{b}_1 + \dots + \alpha_n \mathbf{b}_n$. The numbers $\alpha_1, \dots, \alpha_n \in \mathbb{F}$ (\mathbb{F} is either \mathbb{R} or \mathbb{C}) are called the [coordinates](#) of \mathbf{x} w.r.t. \mathcal{B} . This defines the vector $\Phi_{\mathcal{B}}(\mathbf{x}) \in \mathbb{F}^n$.
- Changing the basis of V from \mathcal{B} to \mathcal{C} also changes the coordinate vector from $\Phi_{cB}(\mathbf{x}) \in \mathbb{F}^n$ to $\Phi_{\mathcal{C}}(\mathbf{x}) \in \mathbb{F}^n$. This change can be described by the [transformation matrix](#) $T_{\mathcal{C} \leftarrow \mathcal{B}}$.
- One always has $T_{\mathcal{B} \leftarrow \mathcal{C}} = T_{\mathcal{C} \leftarrow \mathcal{B}}^{-1}$. Sometimes, it is helpful to go a detour $T_{\mathcal{B} \leftarrow \mathcal{A}} T_{\mathcal{A} \leftarrow \mathcal{C}}$ where \mathcal{A} is a simple and well-known basis.
- An [inner product](#) $\langle \cdot, \cdot \rangle$ is a map, which takes two vectors $\mathbf{x}, \mathbf{y} \in V$ and gives out a number $\langle \mathbf{x}, \mathbf{y} \rangle$ in \mathbb{F} . It has to satisfy the rules (S1)–(S4) from Definition 7.23.
- If $A \in \mathbb{F}^{n \times n}$ is selfadjoint and [positive definite](#), then $\langle \mathbf{x}, \mathbf{y} \rangle := \langle A\mathbf{x}, \mathbf{y} \rangle_{\text{euclid}}$ defines an inner product in \mathbb{F}^n . Here $\langle \cdot, \cdot \rangle_{\text{euclid}}$ is the well-known standard inner product in \mathbb{R}^n (Chapter 2) or \mathbb{C}^n (Chapter 6).
- A [norm](#) $\|\cdot\|$ is a map that sends a vector $\mathbf{x} \in V$ to a number $\|\mathbf{x}\| \in \mathbb{R}$ and satisfies the rules (N1)–(N3) from Definition 7.33.
- An inner product $\langle \cdot, \cdot \rangle$ always defines a norm $\|\mathbf{x}\| := \sqrt{\langle \mathbf{x}, \mathbf{x} \rangle}$.
- By having an inner product, we can talk about orthogonal projection $\mathbf{x}|_U$ for a vector $\mathbf{x} \in V$ w.r.t. a subspace $U \subset V$.

8

General linear maps

It's dangerous to go alone! Take this.

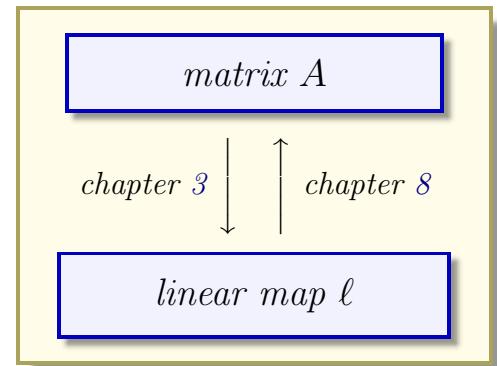
Old man in a cave

In Chapter 3, we already introduced matrices and linear maps. We have also seen that for a given matrix $A \in \mathbb{R}^{m \times n}$ there is an associated map

$$f_A : \mathbb{R}^n \rightarrow \mathbb{R}^m \quad \text{with} \quad \mathbf{x} \mapsto A\mathbf{x}$$

which fulfills two properties (+) and (\cdot) and therefore is called a linear map.

We also discovered that for any map $\ell : \mathbb{R}^n \rightarrow \mathbb{R}^m$ with these properties there is exactly one matrix $A \in \mathbb{R}^{m \times n}$, such that the associated map f_A coincides with ℓ . In short: In A , the action of ℓ is written down.



Now in Chapter 8, with the power of general vector spaces, we also can consider general linear maps between arbitrary \mathbb{F} -vector spaces V and W . Then it is not clear what a suitable matrix that captures all the information for such a map would be. Later we will see that the matrix comes from $\mathbb{F}^{\dim(W) \times \dim(V)}$ and is built in a similar way as before.

8.1 Definition: Linear maps

Let \mathbb{F} be either \mathbb{R} or \mathbb{C} again. Let V and W be two \mathbb{F} -vector spaces. It is important that for both the same field \mathbb{F} is chosen.

Definition 8.1. Linear map

A map $\ell : V \rightarrow W$ is called a linear map, linear function or linear operator if ℓ satisfies the two following properties. For all $\mathbf{x}, \mathbf{y} \in V$ and $\alpha \in \mathbb{F}$:

$$(L+) \quad \ell(\mathbf{x} + \mathbf{y}) = \ell(\mathbf{x}) + \ell(\mathbf{y}), \quad (\text{additive})$$

$$(L \cdot) \quad \ell(\alpha \mathbf{x}) = \alpha \ell(\mathbf{x}). \quad (\text{homogeneous})$$

If $W = \mathbb{F}$, one often calls ℓ a linear functional.

Proposition 8.2. Linear maps send \mathbf{o} to \mathbf{o} .

For a linear map $\ell : V \rightarrow W$, we have $\ell(\mathbf{o}_V) = \mathbf{o}_W$.

Proof. For arbitrary $\mathbf{x} \in V$, we use (L·): $\ell(\mathbf{o}_V) = \ell(0\mathbf{x}) = 0\ell(\mathbf{x}) = \mathbf{o}_W$. \square

In the following examples \mathbb{F} stands for \mathbb{R} or \mathbb{C} .

Example 8.3. (a) For $V = W = \mathbb{F}$, let $\ell(\mathbf{x}) = 3\mathbf{x}$. We can easily check (L+) and (L·).

(b) For $V = \mathbb{F}$ and $W = \mathbb{F}^2$, let $\ell(x) = x \begin{pmatrix} 3 \\ 1 \end{pmatrix}$. Obviously, ℓ satisfies (L+) and (L·).

(c) Let $\ell : \mathbb{F}^3 \rightarrow \mathbb{F}$ defined by $\ell(\mathbf{x}) = \langle \mathbf{x}, \mathbf{a} \rangle_{\text{euclid}} = \mathbf{a}^* \mathbf{x}$ with fixed $\mathbf{a} \in \mathbb{F}^3$, e.g.

$$\mathbf{a} = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}, \quad \text{hence} \quad \ell : \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \mapsto \left\langle \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} \right\rangle_{\text{euclid}} = (2 \ 1 \ 3) \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}.$$

Using the definition of an inner product, we know that ℓ is linear.

(d) Define $\ell : \mathbb{F}^3 \rightarrow \mathbb{F}$ by $\ell(\mathbf{x}) = \det(\mathbf{x} \ \mathbf{a}_2 \ \mathbf{a}_3)$ with fixed $\mathbf{a}_2, \mathbf{a}_3 \in \mathbb{F}^3$, e.g.

$$\mathbf{a}_2 = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}, \quad \mathbf{a}_3 = \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix}, \quad \text{hence} \quad \ell : \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \mapsto \det \begin{pmatrix} | & | & | \\ \mathbf{x} & \mathbf{a}_2 & \mathbf{a}_3 \\ | & | & | \end{pmatrix} = \det \begin{pmatrix} x_1 & 1 & 3 \\ x_2 & 0 & 1 \\ x_3 & 2 & 1 \end{pmatrix}.$$

We know from the definition of the determinant that ℓ is linear. Using Proposition 4.11 (Laplace's formula), we can rewrite ℓ :

$$\ell(\mathbf{x}) = x_1 \underbrace{\det \begin{pmatrix} 0 & 1 \\ 2 & 1 \end{pmatrix}}_{-2} - x_2 \underbrace{\det \begin{pmatrix} 1 & 3 \\ 2 & 1 \end{pmatrix}}_{-5} + x_3 \underbrace{\det \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}}_1 = (-2 \ 5 \ 1) \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

(e) The map $\ell : \mathbb{F}^2 \rightarrow \mathbb{F}^2$ defined by $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mapsto \begin{pmatrix} 4x_1 + 3x_2 \\ x_2 + 7 \end{pmatrix}$ is not linear because $\ell(\mathbf{o}) = \begin{pmatrix} 0 \\ 7 \end{pmatrix} \neq \mathbf{o}$.

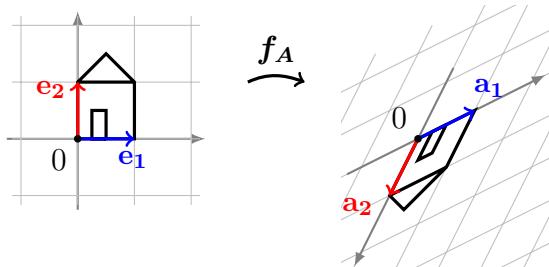
(f) For $A \in \mathbb{F}^{m \times n}$ define $f_A : \mathbb{F}^n \rightarrow \mathbb{F}^m$ by $f_A : \mathbf{x} \mapsto A\mathbf{x}$. This is a linear map by Proposition 3.14. For example, $\mathbb{F} = \mathbb{R}$ and $m = n = 2$, look at how f_A acts on houses. Let

$$A = \begin{pmatrix} | & | \\ \mathbf{a}_1 & \mathbf{a}_2 \\ | & | \end{pmatrix} \in \mathbb{R}^{2 \times 2}.$$

We know:

$$\mathbf{o} \xrightarrow{f_A} \mathbf{o}, \quad \mathbf{e}_1 \xrightarrow{f_A} \mathbf{a}_1, \quad \mathbf{e}_2 \xrightarrow{f_A} \mathbf{a}_2$$

and the rest of the plane is given by linearity.



The last example (f) includes all the other examples (a)–(e): We always find a corresponding matrix $\ell(\mathbf{x}) = A\mathbf{x}$. We find: (a) $A = 3$, (b) $A = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$, (c) $A = (2 \ 1 \ 3)$, (d) $A = (-2 \ 5 \ 1)$.

Now let us look for some abstract vector spaces:

Example 8.4. (a) Let $V = \mathcal{F}(\mathbb{R})$, $W = \mathbb{R}$ and $\delta_0 : V \rightarrow W$ the evaluation for a function $\mathbf{f} \in V$ in the origin 0, which means $\delta_0 : \mathbf{f} \mapsto \mathbf{f}(0)$. Then δ_0 is linear. (Show it!) Another example would be a evaluation at different points and using linear combinations: $\ell : \mathbf{f} \mapsto 3\mathbf{f}(0) - 7\mathbf{f}(\frac{1}{4}) + 5\mathbf{f}(1)$.

- (b) Let ∂ be the differential operator from $V = \mathcal{P}_3(\mathbb{R})$ to $W = \mathcal{P}_2(\mathbb{R})$, which means ∂ sends a polynomial $\mathbf{f} \in \mathcal{P}_3(\mathbb{R})$ to its derivative $\mathbf{f}' \in \mathcal{P}_2(\mathbb{R})$. Because of $(\mathbf{f} + \mathbf{g})' = \mathbf{f}' + \mathbf{g}'$ and $(\alpha \mathbf{f})' = \alpha \mathbf{f}'$, the map ∂ is linear. (Derivatives and the rules above, we will consider next semester in mathematical analysis. Here, you can see it as a strictly algebraic procedure, e.g. $x^n = nx^{n-1}$.)
- (c) In the same manner, we can look at the map $\mathcal{P}_3(\mathbb{R}) \rightarrow \mathcal{P}_1(\mathbb{R})$ with $\mathbf{f} \mapsto \mathbf{f}''$ given by the second derivative. In the same way, a combination is possible, $\mathbf{f} \mapsto \mathbf{f}''' + 3\mathbf{f}'' - 2\mathbf{f}' + 4\mathbf{f}$ as a map $\mathcal{P}_3(\mathbb{R}) \rightarrow \mathcal{P}_3(\mathbb{R})$.
- (d) Instead of using the derivative of a polynomial $\mathbf{f} \in \mathcal{P}([a, b]) =: V$ or evaluating it in one point, we can use the integration, hence the map $i : \mathbf{f} \mapsto \int_a^b \mathbf{f}(x) dx$. Therefore, in this case, we have $V = \mathcal{P}([a, b])$ and $W = \mathbb{R}$. Again, we get a linear map:

$$\int_a^b (\mathbf{f}(x) + \mathbf{g}(x)) dx = \int_a^b \mathbf{f}(x) dx + \int_a^b \mathbf{g}(x) dx \quad \text{and} \quad \int_a^b \alpha \mathbf{f}(x) dx = \alpha \int_a^b \mathbf{f}(x) dx.$$

(We also talk about the integration in mathematical analysis next semester.)

The linear maps Example 8.4 are not directly given by a matrix vector multiplication. However, we will see that this is possible if we go over to the representation of the vector spaces when fixing a basis. Recall the coordinate vectors and the basis isomorphism Φ_B . We will do this in Section 8.3.

8.2 Combinations of linear maps

8.2.1 Sum and multiples of a linear map

As seen in Example 7.4, we have seen that we can add and scale functions $\mathbf{f} : \mathbb{R} \rightarrow \mathbb{R}$. This can be generalised for linear maps:

Definition 8.5. Sum and scaled linear maps

Let V and W be two \mathbb{F} -vector space (with same \mathbb{F} !) and let $k : V \rightarrow W$ and $\ell : V \rightarrow W$ be linear maps. Then we define $k + \ell : V \rightarrow W$ by

$$(k + \ell)(\mathbf{x}) := k(\mathbf{x}) + \ell(\mathbf{x}) \quad \text{for all } \mathbf{x} \in V,$$

and for $\alpha \in \mathbb{F}$, we define $\alpha \cdot \ell$ by

$$(\alpha \cdot \ell)(\mathbf{x}) := \alpha \cdot \ell(\mathbf{x}) \quad \text{for all } \mathbf{x} \in V.$$

The operations $+$ and $\alpha \cdot$ on the right-hand side are the operations in W .

Proposition & Definition 8.6. Vector space of linear maps $V \rightarrow W$

The maps $k + \ell$ and $\alpha \cdot \ell$ from Definition 8.5 are again linear maps from V to W .

The set of all linear maps from V to W equipped with the two operations $+$ and $\alpha \cdot$ form again an \mathbb{F} -vector space. We denote this vector space by $\mathcal{L}(V, W)$.

The zero vector in $\mathcal{L}(V, W)$ is the zero map $o : V \rightarrow W$ defined by $o(\mathbf{x}) = \mathbf{o}$ for all $\mathbf{x} \in V$.

Proof. Let $k, \ell : V \rightarrow W$ be linear and let $\mathbf{x}, \mathbf{y} \in V$ and $\alpha \in \mathbb{F}$. Then:

$$\begin{aligned} (k + \ell)(\mathbf{x} + \mathbf{y}) &\stackrel{\text{Def. 8.5}}{=} k(\mathbf{x} + \mathbf{y}) + \ell(\mathbf{x} + \mathbf{y}) \stackrel{(L+)}{=} k(\mathbf{x}) + k(\mathbf{y}) + \ell(\mathbf{x}) + \ell(\mathbf{y}) \\ &= k(\mathbf{x}) + \ell(\mathbf{x}) + k(\mathbf{y}) + \ell(\mathbf{y}) \stackrel{\text{Def. 8.5}}{=} (k + \ell)(\mathbf{x}) + (k + \ell)(\mathbf{y}) \end{aligned}$$

$$\text{and } (k + \ell)(\alpha \mathbf{x}) \stackrel{\text{Def. 8.5}}{=} k(\alpha \mathbf{x}) + \ell(\alpha \mathbf{x}) \stackrel{(L\cdot)}{=} \alpha k(\mathbf{x}) + \alpha \ell(\mathbf{x}) = \alpha(k(\mathbf{x}) + \ell(\mathbf{x})) \stackrel{\text{Def. 8.5}}{=} \alpha(k + \ell)(\mathbf{x}),$$

which means $k + \ell$ has two properties (L+) and (L·) and is also linear. In the same manner, we see that $\alpha \cdot \ell$ is linear. Showing the properties (1)-(8) is an exercise, for you. I am serious. It could be an exam question. \square

From now on, we do not write the two operations $+$ and $\alpha \cdot$ in $\mathcal{L}(V, W)$ in red anymore. However, keep in mind that these are different operations than $+$ and $\alpha \cdot$ in W .

Example 8.7. – Projection and reflection. Let $\mathbf{n} \in \mathbb{R}^n$ be a vector $\|\mathbf{n}\| = 1$ and $G := \text{Span}(\mathbf{n})$ the spanned line. For all $\mathbf{x} \in \mathbb{R}^n$, we can calculate the orthogonal projection

$$\mathbf{x}|_G = \frac{\langle \mathbf{x}, \mathbf{n} \rangle_{\text{euclid}}}{\langle \mathbf{n}, \mathbf{n} \rangle_{\text{euclid}}} \mathbf{n} = \langle \mathbf{x}, \mathbf{n} \rangle_{\text{euclid}} \mathbf{n} = \mathbf{n} \langle \mathbf{x}, \mathbf{n} \rangle_{\text{euclid}} = \mathbf{n}(\mathbf{n}^\top \mathbf{x}) = (\mathbf{n} \mathbf{n}^\top) \mathbf{x}$$

(cf. Proposition 5.7).

Hence the map

$$\text{proj}_G : \mathbb{R}^n \rightarrow \mathbb{R}^n \quad \text{with} \quad \text{proj}_G(\mathbf{x}) := \mathbf{x}|_G = (\mathbf{n} \mathbf{n}^\top) \mathbf{x}, \quad (8.1)$$

defines a linear map $\mathbb{R}^n \rightarrow \mathbb{R}^n$. We also know that is given by the associated matrix: $\text{proj}_G = f_{\mathbf{n} \mathbf{n}^\top}$.

Using the orthogonal decomposition

$$\mathbf{x} = \mathbf{x}|_G + \mathbf{x}|_E,$$

we also can also define the linear map

$$\text{proj}_E : \mathbb{R}^n \rightarrow \mathbb{R}^n$$

which is the orthogonal projection onto $E := G^\perp = \{\mathbf{n}\}^\perp$:

$$\text{proj}_E(\mathbf{x}) := \mathbf{x}|_E = \mathbf{x} - \mathbf{x}|_G.$$

Subtracting the orthogonal projection $\mathbf{x}|_G$ again, we get the reflection of \mathbf{x} with respect to the hyperplane E .

Hence, we define:

$$\text{refl}_E : \mathbb{R}^n \rightarrow \mathbb{R}^n \quad \text{with} \quad \text{refl}_E(\mathbf{x}) := \mathbf{x}|_E - \mathbf{x}|_G = \mathbf{x} - 2\mathbf{x}|_G.$$

In other words:

$$\text{proj}_E = id - \text{proj}_G \quad \text{and} \quad \text{refl}_E = id - 2\text{proj}_G. \quad (8.2)$$

Here, $id : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is the identity map $id : \mathbf{x} \mapsto \mathbf{x}$. By these formulas, we can conclude, $\text{proj}_E, \text{refl}_E \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)$.

8.2.2 Composition and inverses

Recall that you can form the composition of two maps $\ell : U \rightarrow V$ and $k : V \rightarrow W$ by setting:

$$(k \circ \ell)(\mathbf{x}) = k(\ell(\mathbf{x})) \quad \text{for all } \mathbf{x} \in U. \quad (8.3)$$

Proposition 8.8. Composition of linear maps is linear.

Let U, V, W be \mathbb{F} -vector spaces and let $\ell : U \rightarrow V$ and $k : V \rightarrow W$ be linear maps. Then, the composition $k \circ \ell : U \rightarrow W$ is also linear. In short:

$$\ell \in \mathcal{L}(U, V), \quad k \in \mathcal{L}(V, W) \Rightarrow k \circ \ell \in \mathcal{L}(U, W).$$

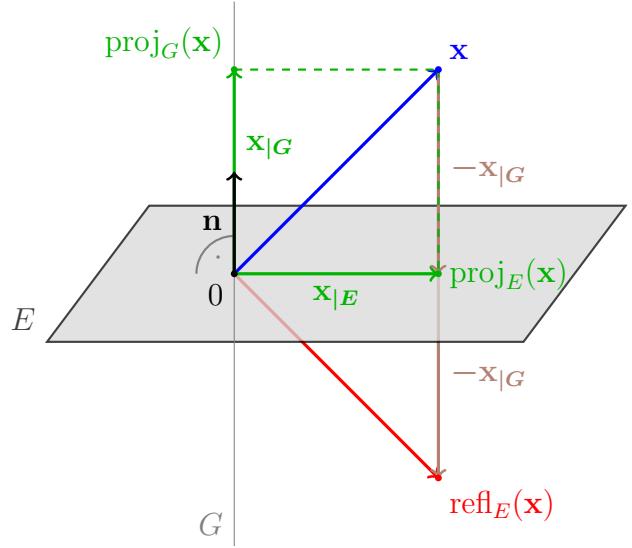
Proof. For all $\mathbf{x}, \mathbf{y} \in U$ and $\alpha \in \mathbb{F}$, we find:

$$\begin{aligned} (k \circ \ell)(\mathbf{x} + \mathbf{y}) &\stackrel{(8.3)}{=} k(\ell(\mathbf{x} + \mathbf{y})) \stackrel{(L+)}{=} k(\ell(\mathbf{x}) + \ell(\mathbf{y})) \stackrel{(L+)}{=} k(\ell(\mathbf{x})) + k(\ell(\mathbf{y})) \\ &\stackrel{(8.3)}{=} (k \circ \ell)(\mathbf{x}) + (k \circ \ell)(\mathbf{y}) \end{aligned}$$

$$\text{and } (k \circ \ell)(\alpha \mathbf{x}) \stackrel{(8.3)}{=} k(\ell(\alpha \mathbf{x})) \stackrel{(L\cdot)}{=} k(\alpha \ell(\mathbf{x})) \stackrel{(L\cdot)}{=} \alpha k(\ell(\mathbf{x})) \stackrel{(8.3)}{=} \alpha (k \circ \ell)(\mathbf{x}).$$

Hence $k \circ \ell$ also have the properties (L+) and (L·) and is linear. \square

Let compose the maps from Example 8.7:



Example 8.9. Recall both projections

$$\text{proj}_G : \mathbf{x} \mapsto \mathbf{n}\mathbf{n}^\top \mathbf{x} \quad \text{and} \quad \text{proj}_E = id - \text{proj}_G$$

and the reflection

$$\text{refl}_E = id - 2\text{proj}_G.$$

All three maps act between $\mathbb{R}^n \rightarrow \mathbb{R}^n$ and can be composed in all possible ways. We already know that projecting more than once does not change anything:

$$\text{proj}_G \circ \text{proj}_G = \text{proj}_G \quad \text{and} \quad \text{proj}_E \circ \text{proj}_E = \text{proj}_E. \quad (8.4)$$

For the reflection, we expect that using it two times brings us back to the beginning, which means that we should get the identity map:

$$\begin{aligned} \text{refl}_E \circ \text{refl}_E &= (id - 2\text{proj}_G) \circ (id - 2\text{proj}_G) \\ &= \underbrace{id \circ id}_{id} - \underbrace{id \circ 2\text{proj}_G}_{2\text{proj}_G} - \underbrace{2\text{proj}_G \circ id}_{2\text{proj}_G} + \underbrace{2\text{proj}_G \circ 2\text{proj}_G}_{4\text{proj}_G} = id. \end{aligned}$$

Composition of both projections gives us the zero map:

$$\text{proj}_G \circ \text{proj}_E = \text{proj}_G \circ (id - \text{proj}_G) = \underbrace{\text{proj}_G \circ id}_{\text{proj}_G} - \underbrace{\text{proj}_G \circ \text{proj}_G}_{\text{proj}_G} = o. \quad (8.5)$$

In the same way, $\text{proj}_E \circ \text{proj}_G = o$. We also can calculate:

$$\text{refl}_E \circ \text{proj}_G = -\text{proj}_G \quad \text{and} \quad \text{refl}_E \circ \text{proj}_E = \text{proj}_E. \quad (8.6)$$

Changing the order gives us the same result.

We again look at more abstract examples:

Example 8.10. (a) Let $\delta_0 : \mathcal{P}_2(\mathbb{R}) \rightarrow \mathbb{R}$ given by $\delta_0 : \mathbf{f} \mapsto \mathbf{f}(0)$ the point evaluation and $\partial : \mathcal{P}_3(\mathbb{R}) \rightarrow \mathcal{P}_2(\mathbb{R})$ the differential operator $\partial : \mathbf{f} \mapsto \mathbf{f}'$ from Example 8.4 (a) and (b). Then, the composition $\delta_0 \circ \partial$ from $\mathcal{P}_3(\mathbb{R})$ to \mathbb{R} is given by

$$\mathbf{f} \xrightarrow{\partial} \mathbf{f}' \xrightarrow{\delta_0} \mathbf{f}'(0), \quad \text{hence} \quad \delta_0 \circ \partial : \mathbf{f} \mapsto \mathbf{f}'(0).$$

The reverse composition $\partial \circ \delta_0$ is not defined!

(b) Let $\partial : \mathcal{P}_3(\mathbb{R}) \rightarrow \mathcal{P}_2(\mathbb{R})$ be the differentiation $\mathbf{f} \mapsto \mathbf{f}'$ and, in addition, $\int : \mathcal{P}_2(\mathbb{R}) \rightarrow \mathcal{P}_3(\mathbb{R})$ the map that sends $\mathbf{f} \in \mathcal{P}_2(\mathbb{R})$ to the function \mathbf{F} with

$$\mathbf{F}(x) = \int_0^x \mathbf{f}(t) dt \quad \text{for all } x \in [0, 1].$$

We get:

$$\mathbf{f} \xrightarrow{\int} \mathbf{F} \xrightarrow{\partial} \mathbf{F}' = \mathbf{f}, \quad \text{hence} \quad \partial \circ \int : \mathbf{f} \mapsto \mathbf{f}, \quad \text{which means} \quad \partial \circ \int = id : \mathcal{P}_2(\mathbb{R}) \rightarrow \mathcal{P}_2(\mathbb{R}).$$

We can also build the converse composition of ∂ and \int . Is $\int \circ \partial$ then the identity map $id : \mathcal{P}_3(\mathbb{R}) \rightarrow \mathcal{P}_3(\mathbb{R})$?

Let $\mathbf{f} \in \mathcal{P}_3(\mathbb{R})$ be arbitrary, which means $\mathbf{f}(x) = ax^3 + bx^2 + cx + d$ with some $a, b, c, d \in \mathbb{R}$. Then $\partial(\mathbf{f}) = \mathbf{f}'$ with $\mathbf{f}'(x) = 3ax^2 + 2bx + c$. Now, we use \int : The function $\mathbf{g} := (\int \circ \partial)(\mathbf{f}) = \int(\partial(\mathbf{f})) = \int(\mathbf{f}')$ satisfies:

$$\mathbf{g}(x) = \int_0^x \mathbf{f}'(t) dt = \int_0^x (3at^2 + 2bt + c) dt = at^3 + bt^2 + ct \Big|_0^x = ax^3 + bx^2 + cx$$

for all x . Hence, $(\int \circ \partial)(\mathbf{f}) \neq \mathbf{f}$ if $d \neq 0$. We see that “ $+d$ ” is lost. We conclude $\int \circ \partial \neq id$.

Reminder: Inverse maps

We call a map $f : V \rightarrow W$ invertible if there is another map $g : W \rightarrow V$ with

$$f \circ g = id_W \quad \text{and} \quad g \circ f = id_V$$

Since g uniquely determined, it is called the inverse map of f and denoted by f^{-1} .

Recall that bijective and invertible are equivalent notions for maps.

However, here, we are only interested in linear maps between vector spaces. As mentioned in Chapter 3, we have the following interesting result:

Proposition 8.11. Inverses are again linear.

If $\ell : V \rightarrow W$ is a linear map that is bijective, then its inverse $\ell^{-1} : W \rightarrow V$ is also linear

Proof. Let $\mathbf{u}, \mathbf{v} \in W$ be arbitrary and set $\mathbf{x} := \ell^{-1}(\mathbf{u})$ and $\mathbf{y} := \ell^{-1}(\mathbf{v})$. Since ℓ is linear, we have $\ell(\mathbf{x} + \mathbf{y}) = \ell(\mathbf{x}) + \ell(\mathbf{y}) = \mathbf{u} + \mathbf{v}$. Hence,

$$\ell^{-1}(\mathbf{u} + \mathbf{v}) = \mathbf{x} + \mathbf{y} = \ell^{-1}(\mathbf{u}) + \ell^{-1}(\mathbf{v}),$$

which means ℓ^{-1} has the property (L+). In the same manner, we can show that ℓ^{-1} satisfies (L·) as well. \square

Example 8.12. Recall that we already considered a linear map in Section 7.4, namely the map $\Phi_{\mathcal{B}} : \mathbf{v} \mapsto \mathbf{v}^{\mathcal{B}}$, which maps a vector \mathbf{v} from an \mathbb{F} -vector space V to its coordinate vector $\mathbf{v}^{\mathcal{B}} \in \mathbb{F}^n$ with respect to a basis \mathcal{B} . The map $\Phi_{\mathcal{B}}$ is invertible: It is surjective because \mathcal{B} and the standard basis of \mathbb{F}^n are generating families, and it is injective because \mathcal{B} and the standard basis of \mathbb{F}^n are linearly independent. By Proposition 8.11, we also know that $\Phi_{\mathcal{B}}^{-1} : \mathbf{v}^{\mathcal{B}} \mapsto \mathbf{v}$ is linear.

Remark:

A linear map $\ell : V \rightarrow W$ exactly conserves the structure of the vector spaces, meaning vector addition and scalar multiplication. Therefore, mathematicians call a linear map a homomorphism. A homomorphism ℓ that is invertible and has an inverse ℓ^{-1} that is also a homomorphism is called an isomorphism.

8.3 Finding the matrix for a linear map

8.3.1 Just know what happens to a basis

We have already seen it for a linear map $f_A : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ associated to a matrix $A \in \mathbb{R}^{2 \times 2}$ and the houses. When you know what f_A does to the ground side of the house, the first basis vector \mathbf{e}_1 , and the left side of the house, the second basis vector \mathbf{e}_2 of \mathbb{R}^2 , then we know what happens to the other parts of the house and indeed to the whole space \mathbb{R}^2 under the map f_A .

Rule of thumb: Linearity makes it easy

For a linear map, you only have to know what happens to a basis. The remaining part of space “tags along”.

Let $\ell : V \rightarrow W$ be a linear map and $\mathcal{B} = (\mathbf{b}_1, \dots, \mathbf{b}_n)$ some basis of V . For each $\mathbf{x} \in V$, we denote by $\Phi_{\mathcal{B}}(\mathbf{x}) \in \mathbb{F}^n$ its coordinate vector, which means

$$\Phi_{\mathcal{B}}(\mathbf{x}) = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix} \in \mathbb{F}^n \quad \text{with} \quad \mathbf{x} = \alpha_1 \mathbf{b}_1 + \cdots + \alpha_n \mathbf{b}_n = \Phi_{\mathcal{B}}^{-1} \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix}.$$

Then:

$$\ell(\mathbf{x}) = \ell(\alpha_1 \mathbf{b}_1 + \cdots + \alpha_n \mathbf{b}_n) = \alpha_1 \ell(\mathbf{b}_1) + \cdots + \alpha_n \ell(\mathbf{b}_n)$$

Equation (8.7) says everything: If you know the images of the all basis elements, which means $\ell(\mathbf{b}_1), \dots, \ell(\mathbf{b}_n)$, then you know all images $\ell(\mathbf{x})$ for each $\mathbf{x} \in V$ immediately.

Example 8.13. Let $V = \mathcal{P}_3(\mathbb{R})$ with the monomial basis $\mathcal{B} = (\mathbf{m}_0, \mathbf{m}_1, \mathbf{m}_2, \mathbf{m}_3)$ where $\mathbf{m}_k(x) = x^k$. For the differential operator $\partial \in \mathcal{L}(\mathcal{P}_3(\mathbb{R}), \mathcal{P}_2(\mathbb{R}))$ where $\partial : \mathbf{f} \mapsto \mathbf{f}'$, we have

$$\partial(\mathbf{m}_0) = \mathbf{0}, \quad \partial(\mathbf{m}_1) = \mathbf{m}_0, \quad \partial(\mathbf{m}_2) = 2\mathbf{m}_1, \quad \partial(\mathbf{m}_3) = 3\mathbf{m}_2, \quad (8.7)$$

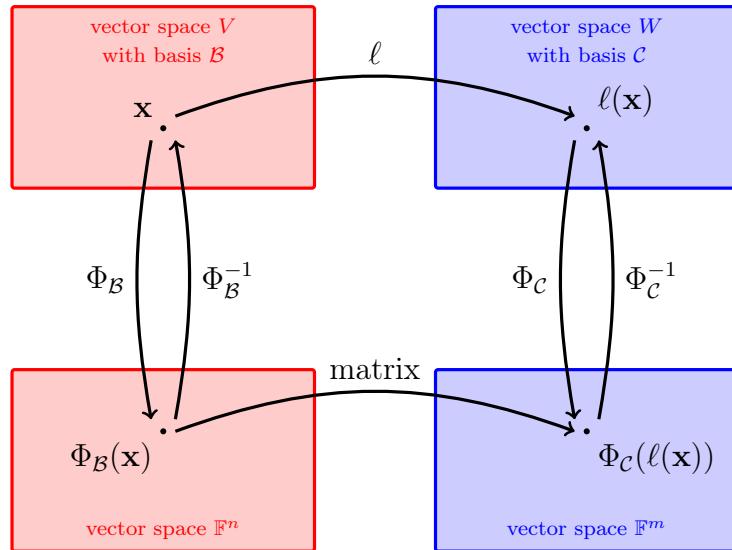
For an arbitrary $\mathbf{p} \in \mathcal{P}_3(\mathbb{R})$, which means $\mathbf{p}(x) = ax^3 + bx^2 + cx + d$ for $a, b, c, d \in \mathbb{R}$ or $\mathbf{p} = d\mathbf{m}_0 + c\mathbf{m}_1 + b\mathbf{m}_2 + a\mathbf{m}_3$, we have

$$\mathbf{p}^{\mathcal{B}} = \begin{pmatrix} d \\ c \\ b \\ a \end{pmatrix} \quad \text{and hence } \partial(\mathbf{p}) = d\partial(\mathbf{m}_0) + c\partial(\mathbf{m}_1) + b\partial(\mathbf{m}_2) + a\partial(\mathbf{m}_3) = c\mathbf{m}_0 + 2b\mathbf{m}_1 + 3a\mathbf{m}_2.$$

Checking this: $\mathbf{p}'(x) = 3ax^2 + 2bx + c$, hence $\partial(\mathbf{p}) = \mathbf{p}' = 3a\mathbf{m}_2 + 2b\mathbf{m}_1 + c\mathbf{m}_0$.

8.3.2 Matrix of a linear map with respect to bases

Let us consider again two arbitrary finite-dimensional \mathbb{F} -vector spaces V and W and linear maps between them. Set $n := \dim(V)$ and $m := \dim(W)$. Fix $\mathcal{B} = (\mathbf{b}_1, \dots, \mathbf{b}_n)$ as a basis for V and $\mathcal{C} = (\mathbf{c}_1, \dots, \mathbf{c}_m)$ as a basis for W . The idea is now to use both bases to represent vectors in the vector spaces and also to represent the linear map $\ell : V \rightarrow W$ as a matrix $A \in \mathbb{F}^{m \times n}$. The following picture shows this idea:


Question:

How to get the map or the matrix in the bottom. How to send the coordinate vector $\Phi_{\mathcal{B}}(x)$ to the coordinate vector $\Phi_{\mathcal{C}}(\ell(x))$?

Of course, this is given by composing the three maps:

$$\Phi_{\mathcal{C}}(\ell(x)) = (\Phi_{\mathcal{C}} \circ \ell \circ \Phi_{\mathcal{B}}^{-1})(\Phi_{\mathcal{B}}(x))$$

So, $f := \Phi_{\mathcal{C}} \circ \ell \circ \Phi_{\mathcal{B}}^{-1}$ is a linear map from \mathbb{F}^n to \mathbb{F}^m . We already know that there is always a corresponding matrix A with $f = f_A$. We get the columns of the matrix by putting the canonical vectors into the map:

$$(\Phi_{\mathcal{C}} \circ \ell \circ \Phi_{\mathcal{B}}^{-1})(\mathbf{e}_j) = \Phi_{\mathcal{C}}(\ell(\Phi_{\mathcal{B}}^{-1}(\mathbf{e}_j))) = \Phi_{\mathcal{C}}(\ell(\mathbf{b}_j))$$

This gives us a matrix that really represents the abstract linear map. It depends, of course, on the chosen bases \mathcal{B} and \mathcal{C} in the vector spaces V and W , respectively. Therefore, we choose a good name:

Matrix representation of the linear map

For the linear map $\ell: V \rightarrow W$, we define the matrix

$$\ell_{\mathcal{C} \leftarrow \mathcal{B}} := \begin{pmatrix} \Phi_{\mathcal{C}}(\ell(\mathbf{b}_1)) & \dots & \Phi_{\mathcal{C}}(\ell(\mathbf{b}_n)) \end{pmatrix} \in \mathbb{F}^{m \times n} \quad (8.8)$$

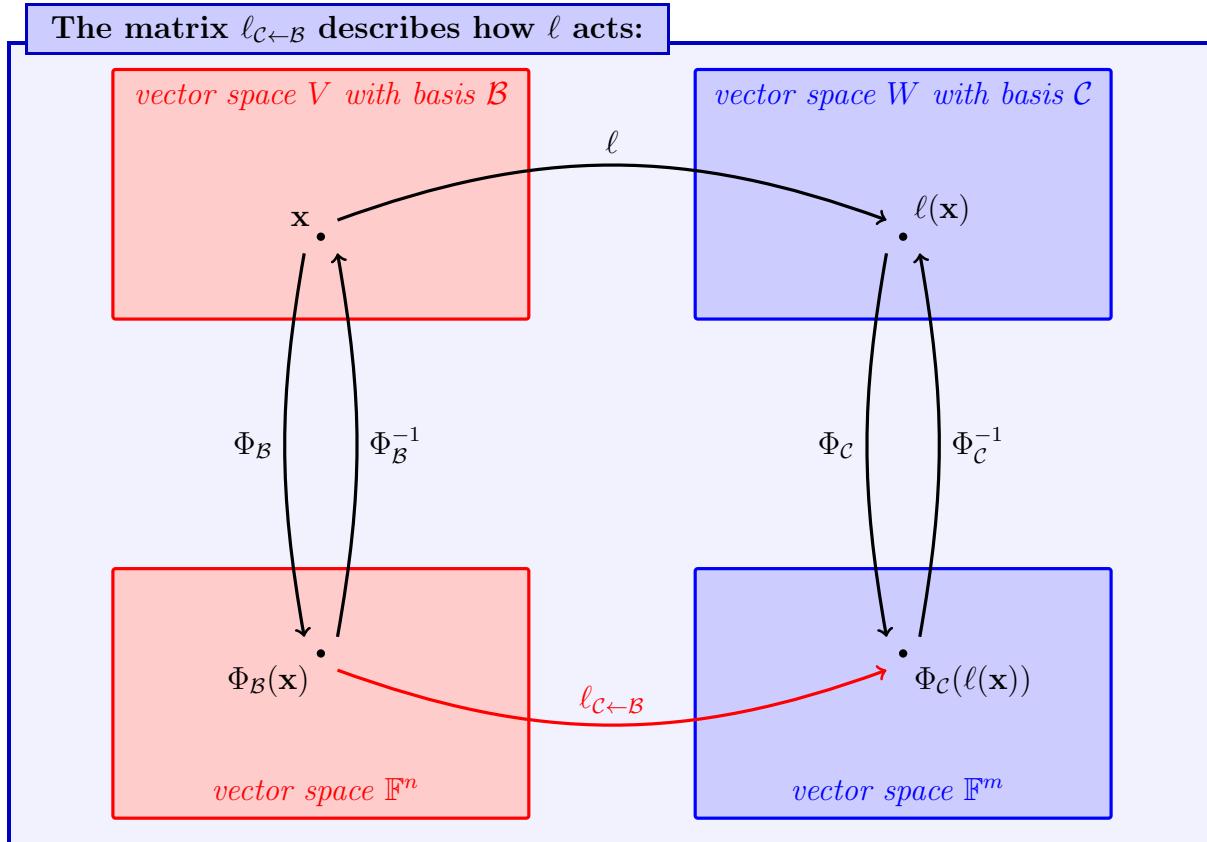
and call it the matrix representation of the linear map ℓ with respect to the basis \mathcal{B} and \mathcal{C} .

This gets us to:

How to map the coordinates

$$\Phi_{\mathcal{C}}(\ell(x)) = \ell_{\mathcal{C} \leftarrow \mathcal{B}} \Phi_{\mathcal{B}}(x). \quad (8.9)$$

This completes our picture:



Example 8.14. (a) Let $\partial : \mathcal{P}_3(\mathbb{R}) \rightarrow \mathcal{P}_2(\mathbb{R})$ with $\mathbf{f} \mapsto \mathbf{f}'$ the differential operator. We use in $\mathcal{P}_3(\mathbb{R})$ and $\mathcal{P}_2(\mathbb{R})$ the respective monomial basis:

$$\mathcal{B} = (\mathbf{m}_3, \mathbf{m}_2, \mathbf{m}_1, \mathbf{m}_0) \quad \text{and} \quad \mathcal{C} = (\mathbf{m}_2, \mathbf{m}_1, \mathbf{m}_0).$$

We already know:

$$\begin{aligned} \Phi_C(\partial(\mathbf{m}_3)) &= \Phi_C(3\mathbf{m}_2) = \begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix}, & \Phi_C(\partial(\mathbf{m}_2)) &= \Phi_C(2\mathbf{m}_1) = \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}, \\ \Phi_C(\partial(\mathbf{m}_1)) &= \Phi_C(\mathbf{m}_0) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, & \Phi_C(\partial(\mathbf{m}_0)) &= \Phi_C(\mathbf{0}) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}. \end{aligned}$$

The column vectors from above give us the columns of the matrix $\partial_{C \leftarrow B}$:

$$\partial_{C \leftarrow B} = \left(\Phi_C(\partial(\mathbf{m}_3)) \ \Phi_C(\partial(\mathbf{m}_2)) \ \Phi_C(\partial(\mathbf{m}_1)) \ \Phi_C(\partial(\mathbf{m}_0)) \right) = \begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}. \quad (8.10)$$

Now we can use the map ∂ just on the coordinate level: For $\mathbf{f} \in \mathcal{P}_3(\mathbb{R})$ given by $\mathbf{f}(x) = ax^3 + bx^2 + cx + d$ with $a, b, c, d \in \mathbb{R}$, we have

$$\Phi_B(\mathbf{f}) = \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} \quad \text{hence} \quad \Phi_B(\partial(\mathbf{f})) = \partial_{C \leftarrow B} \Phi_B(\mathbf{f}) = \begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} 3a \\ 2b \\ c \\ d \end{pmatrix}.$$

So we get:

$$\partial(\mathbf{f}) = \Phi_{\mathcal{C}}^{-1} \begin{pmatrix} 3a \\ 2b \\ c \end{pmatrix} = 3a\mathbf{m}_2 + 2b\mathbf{m}_1 + c\mathbf{m}_0.$$

We check this again by $\partial(\mathbf{f}) = \mathbf{f}'$ and $\mathbf{f}'(x) = 3ax^2 + 2bx + c$ for all x . Therefore, $\partial(\mathbf{f}) = 3a\mathbf{m}_2 + 2b\mathbf{m}_1 + c\mathbf{m}_0$. Great!

- (b) Looking again at the map $\int : \mathcal{P}_2([0, 1]) \rightarrow \mathcal{P}_3([0, 1])$ which sends \mathbf{f} to its antiderivative \mathbf{F} given by

$$\mathbf{F}(x) = \int_0^x \mathbf{f}(t) dt \quad \text{for all } x \in [0, 1].$$

Take again the monomial basis $\mathcal{B} = (\mathbf{m}_2, \mathbf{m}_1, \mathbf{m}_0)$ for $\mathcal{P}_2([0, 1])$ and $\mathcal{C} = (\mathbf{m}_3, \mathbf{m}_2, \mathbf{m}_1, \mathbf{m}_0)$ for $\mathcal{P}_3([0, 1])$. For getting the matrix $\int_{\mathcal{C} \leftarrow \mathcal{B}}$, we need the images of \mathcal{B} . Because of

$$\int(\mathbf{m}_k)(x) = \int_0^x t^k dt = \frac{t^{k+1}}{k+1} \Big|_0^x = \frac{x^{k+1}}{k+1} = \frac{1}{k+1} \mathbf{m}_{k+1}(x) \quad \text{for } k = 2, 1, 0,$$

we get

$$\begin{aligned} \Phi_{\mathcal{C}}(\int(\mathbf{m}_2)) &= \Phi_{\mathcal{C}}\left(\frac{1}{3}\mathbf{m}_3\right) = \begin{pmatrix} 1/3 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \\ \Phi_{\mathcal{C}}(\int(\mathbf{m}_1)) &= \Phi_{\mathcal{C}}\left(\frac{1}{2}\mathbf{m}_2\right) = \begin{pmatrix} 0 \\ 1/2 \\ 0 \\ 0 \end{pmatrix}, \\ \Phi_{\mathcal{C}}(\int(\mathbf{m}_0)) &= \Phi_{\mathcal{C}}\left(\frac{1}{1}\mathbf{m}_1\right) = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}. \end{aligned}$$

The matrix representation $\int_{\mathcal{C} \leftarrow \mathcal{B}}$ is now given by the coordinate vectors with respect to the basis \mathcal{C} :

$$\int_{\mathcal{C} \leftarrow \mathcal{B}} = \begin{pmatrix} \Phi_{\mathcal{C}}(\int(\mathbf{m}_2)) & \Phi_{\mathcal{C}}(\int(\mathbf{m}_1)) & \Phi_{\mathcal{C}}(\int(\mathbf{m}_0)) \end{pmatrix} = \begin{pmatrix} 1/3 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}. \quad (8.11)$$

- (c) Let $V = \mathcal{P}_2(\mathbb{R})$ with monomial basis $\mathcal{B} = (\mathbf{m}_2, \mathbf{m}_1, \mathbf{m}_0)$ and $W = \mathbb{R}$ with basis $\mathcal{C} = (1)$. Look at the map $\delta_0 : \mathbf{f} \mapsto \mathbf{f}(0)$ as a linear map $V \rightarrow W$. For the basis vectors from \mathcal{B} , we get:

$$\delta_0(\mathbf{m}_2) = \mathbf{m}_2(0) = 0, \quad \delta_0(\mathbf{m}_1) = \mathbf{m}_1(0) = 0, \quad \delta_0(\mathbf{m}_0) = \mathbf{m}_0(0) = 1$$

and hence $(\delta_0)_{\mathcal{C} \leftarrow \mathcal{B}} = (0 \ 0 \ 1)$. If we look at another map give by the evaluation at $x = 1$, meaning $\delta_1 : \mathbf{f} \mapsto \mathbf{f}(1)$, the we get $(\delta_1)_{\mathcal{C} \leftarrow \mathcal{B}} = (1 \ 1 \ 1)$. Let us check the calculations for a vector $\mathbf{f} \in \mathcal{P}_2(\mathbb{R})$, which means $\mathbf{f}(x) = ax^2 + bx + c$ with $a, b, c \in \mathbb{R}$. Then:

$$\mathbf{f} = a\mathbf{m}_2 + b\mathbf{m}_1 + c\mathbf{m}_0 = \Phi_{\mathcal{B}}^{-1} \begin{pmatrix} a \\ b \\ c \end{pmatrix}, \quad \text{hence} \quad \Phi_{\mathcal{B}}(\mathbf{f}) = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

and also

$$\Phi_{\mathcal{C}}(\delta_0(\mathbf{f})) = (0 \ 0 \ 1) \begin{pmatrix} a \\ b \\ c \end{pmatrix} = c \quad \text{and} \quad \Phi_{\mathcal{C}}(\delta_1(\mathbf{f})) = (1 \ 1 \ 1) \begin{pmatrix} a \\ b \\ c \end{pmatrix} = a + b + c.$$

In fact: $\delta_0(\mathbf{f}) = \mathbf{f}(0) = a0^2 + b0 + c = c$ and $\delta_1(\mathbf{f}) = \mathbf{f}(1) = a1^2 + b1 + c = a + b + c$.

(d) Let $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$ and $m, n \in \mathbb{N}$. Choose

$$A = \begin{pmatrix} | & & | \\ \mathbf{a}_1 & \dots & \mathbf{a}_n \\ | & & | \end{pmatrix} \in \mathbb{F}^{m \times n}$$

and the associated linear map $f_A : \mathbb{F}^n \rightarrow \mathbb{F}^m$ with $f_A : \mathbf{x} \mapsto A\mathbf{x}$. For a basis in $V = \mathbb{F}^n$, we choose $\mathcal{B} = (\mathbf{e}_1, \dots, \mathbf{e}_n)$ and in $W = \mathbb{F}^m$ canonical basis $\mathcal{C} = (\hat{\mathbf{e}}_1, \dots, \hat{\mathbf{e}}_m)$, where we choose the hats just to distinguish this basis from \mathcal{B} . For getting the matrix representation $(f_A)_{\mathcal{C} \leftarrow \mathcal{B}}$ we look what f_A does with the basis \mathcal{B} :

$$f_A(\mathbf{e}_1) = A\mathbf{e}_1 = \mathbf{a}_1 \stackrel{(*)}{=} \Phi_{\mathcal{C}}^{-1}\mathbf{a}_1, \quad \dots, \quad f_A(\mathbf{e}_n) = A\mathbf{e}_n = \mathbf{a}_n \stackrel{(*)}{=} \Phi_{\mathcal{C}}^{-1}\mathbf{a}_n.$$

In the step (*), we just use that $\Phi_{\mathcal{C}}^{-1} = id$. For the matrix representation $(f_A)_{\mathcal{C} \leftarrow \mathcal{B}}$, we write the images into the columns and get:

$$(f_A)_{\mathcal{C} \leftarrow \mathcal{B}} = \begin{pmatrix} | & & | \\ \mathbf{a}_1 & \dots & \mathbf{a}_n \\ | & & | \end{pmatrix} = A.$$

The matrix representation $(f_A)_{\mathcal{C} \leftarrow \mathcal{B}}$ of the linear map f_A with respect to the canonical basis is the associated matrix A .

(e) Let $d : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be the rotation by angle φ . Choose in $V = W = \mathbb{R}^2$ the canonical basis $\mathcal{B} = (\mathbf{e}_1, \mathbf{e}_2)$. We use the rotation d for the basis elements $\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\mathbf{e}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$:

$$\begin{aligned} d(\mathbf{e}_1) &= d\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right) = \begin{pmatrix} \cos \varphi \\ \sin \varphi \end{pmatrix} = \Phi_{\mathcal{B}}^{-1}\begin{pmatrix} \cos \varphi \\ \sin \varphi \end{pmatrix}, \\ d(\mathbf{e}_2) &= d\left(\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right) = \begin{pmatrix} -\sin \varphi \\ \cos \varphi \end{pmatrix} = \Phi_{\mathcal{B}}^{-1}\begin{pmatrix} -\sin \varphi \\ \cos \varphi \end{pmatrix}. \end{aligned}$$

The matrix representation of d with respect to the standard basis is a so-called [rotation matrix](#)

“Rotation matrix” = matrix representation of rotation with φ

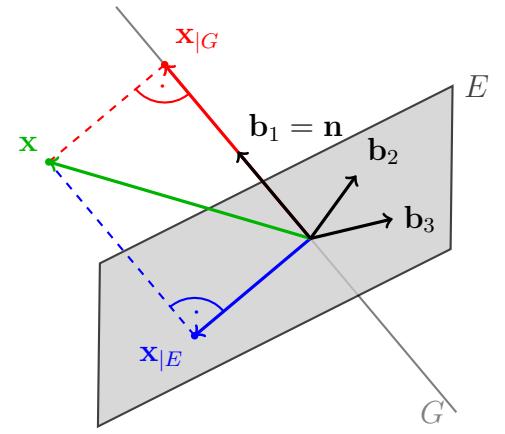
$$d_{\mathcal{B} \leftarrow \mathcal{B}} = \begin{pmatrix} \cos \varphi & -\sin \varphi \\ \sin \varphi & \cos \varphi \end{pmatrix}. \quad (8.12)$$

- (f) Let $\mathbf{n} \in \mathbb{R}^3$ with $\|\mathbf{n}\| = 1$ and $\text{proj}_G : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ the linear map given by the orthogonal projection onto $G := \text{Span}(\mathbf{n})$. We choose a basis $\mathcal{B} = (\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3)$, in both basis \mathbb{R}^3 , which fits our problem: Let $\mathbf{b}_1 := \mathbf{n}$ and \mathbf{b}_2 and \mathbf{b}_3 orthogonal to \mathbf{n} . Then:

$$\text{proj}_G : \mathbf{x} = \underbrace{\alpha \mathbf{b}_1}_{\mathbf{x}|_G} + \underbrace{\beta \mathbf{b}_2 + \gamma \mathbf{b}_3}_{\mathbf{x}|_E} \mapsto \underbrace{\alpha \mathbf{b}_1}_{\mathbf{x}|_G}$$

or in the coordinate language:

$$(\text{proj}_G)_{\mathcal{B} \leftarrow \mathcal{B}} : \Phi_{\mathcal{B}}(\mathbf{x}) = \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} \mapsto \Phi_{\mathcal{B}}(\mathbf{x}|_G) = \begin{pmatrix} \alpha \\ 0 \\ 0 \end{pmatrix}.$$



There, we can immediately see the matrix representation $(\text{proj}_G)_{\mathcal{B} \leftarrow \mathcal{B}}$:

$$(\text{proj}_G)_{\mathcal{B} \leftarrow \mathcal{B}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \quad (8.13)$$

Alternatively, you would calculate the images:

$$\begin{aligned} \Phi_{\mathcal{B}}(\text{proj}_G(\mathbf{b}_1)) &= \Phi_{\mathcal{B}}(\mathbf{b}_1) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \\ \Phi_{\mathcal{B}}(\text{proj}_G(\mathbf{b}_2)) &= \Phi_{\mathcal{B}}(\mathbf{o}) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \\ \Phi_{\mathcal{B}}(\text{proj}_G(\mathbf{b}_3)) &= \Phi_{\mathcal{B}}(\mathbf{o}) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}. \end{aligned}$$

8.3.3 Matrix representation for compositions

In Chapter 3, we introduced the addition, scalar multiplication and matrix multiplication for matrices, and later we generalised it to complex matrices as well. Now, we show that these operations are compatible with the operations $+$, $\alpha \cdot$ and \circ for linear maps.

Proposition 8.15. Operations for matrix representations

- (a) Let V and W be two \mathbb{F} -vector spaces with bases \mathcal{B} and \mathcal{C} , respectively. For linear maps $k, \ell \in \mathcal{L}(V, W)$ and $\alpha \in \mathbb{F}$, we have

$$(k + \ell)_{\mathcal{C} \leftarrow \mathcal{B}} = k_{\mathcal{C} \leftarrow \mathcal{B}} + \ell_{\mathcal{C} \leftarrow \mathcal{B}} \quad \text{and} \quad (\alpha \ell)_{\mathcal{C} \leftarrow \mathcal{B}} = \alpha \ell_{\mathcal{C} \leftarrow \mathcal{B}}.$$

- (b) Let U be a third \mathbb{F} -vector space with chosen basis \mathcal{A} . For all $\ell \in \mathcal{L}(U, V)$ and $k \in \mathcal{L}(V, W)$, we have

$$(k \circ \ell)_{\mathcal{C} \leftarrow \mathcal{A}} = k_{\mathcal{C} \leftarrow \mathcal{B}} \ell_{\mathcal{B} \leftarrow \mathcal{A}}.$$

Please note that on the left-hand side there are the operations $+$, $\alpha \cdot$ and \circ for linear maps and on the right-hand side there are the operations for matrices.

The zero matrix 0 and the identity matrix 1 are exactly the matrix representations of the zero map $o : V \rightarrow W$ with $\mathbf{x} \mapsto \mathbf{o}$ and of the identity map $id : V \rightarrow V$ with $\mathbf{x} \mapsto \mathbf{x}$, respectively.

$$o_{\mathcal{C} \leftarrow \mathcal{B}} = 0 \quad \text{and} \quad id_{\mathcal{B} \leftarrow \mathcal{B}} = 1.$$

However, for the last equality, you really have to choose the same basis in V . Otherwise, see equation (8.15) in section 8.3.4 later.

Now choose ℓ again as a linear map $V \rightarrow W$ and also a basis \mathcal{B} in V and a basis \mathcal{C} in W . If ℓ is invertible, we immediately get:

$$(\ell^{-1})_{\mathcal{B} \leftarrow \mathcal{C}} \ell_{\mathcal{C} \leftarrow \mathcal{B}} = (\ell^{-1} \circ \ell)_{\mathcal{B} \leftarrow \mathcal{B}} = id_{\mathcal{B} \leftarrow \mathcal{B}} = 1 \quad \text{and} \quad \ell_{\mathcal{C} \leftarrow \mathcal{B}} (\ell^{-1})_{\mathcal{B} \leftarrow \mathcal{C}} = 1.$$

Hence:

Matrix representation of inverse = inverse matrix

$$(\ell^{-1})_{\mathcal{B} \leftarrow \mathcal{C}} = (\ell_{\mathcal{C} \leftarrow \mathcal{B}})^{-1}. \quad (8.14)$$

From this, we can conclude a very important result:

Corollary 8.16. Bijectivity not possible, if $\dim(V) \neq \dim(W)$

If $\dim(V) \neq \dim(W)$, then all linear maps $\ell : V \rightarrow W$ are not invertible.

Proof. If ℓ is invertible, then (8.14) says the $m \times n$ -matrix $\ell_{\mathcal{C} \leftarrow \mathcal{B}}$ is invertible. This means that the matrix is a square one, hence $\dim(V) = n = m = \dim(W)$. \square

Example 8.17. (a) Let $\text{proj}_G \in \mathcal{L}(\mathbb{R}^3, \mathbb{R}^3)$ be the linear operator given by the orthogonal projection onto $G := \text{Span}(\mathbf{n})$. We choose the same basis \mathcal{B} in both \mathbb{R}^3 like in Example 8.14 (f). For the projection proj_E and the reflection refl_E with respect to the plane $E := \{\mathbf{n}\}^\perp$, Proposition 8.15 gives us:

$$\begin{aligned} (\text{proj}_E)_{\mathcal{B} \leftarrow \mathcal{B}} &\stackrel{(8.2)}{=} (id - \text{proj}_G)_{\mathcal{B} \leftarrow \mathcal{B}} \\ &= id_{\mathcal{B} \leftarrow \mathcal{B}} - (\text{proj}_G)_{\mathcal{B} \leftarrow \mathcal{B}} \stackrel{(8.13)}{=} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ (\text{refl}_E)_{\mathcal{B} \leftarrow \mathcal{B}} &\stackrel{(8.2)}{=} (id - 2 \text{proj}_G)_{\mathcal{B} \leftarrow \mathcal{B}} \\ &= id_{\mathcal{B} \leftarrow \mathcal{B}} - 2 (\text{proj}_G)_{\mathcal{B} \leftarrow \mathcal{B}} \stackrel{(8.13)}{=} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - 2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \end{aligned}$$

(b) Next, we again consider the differential operator $\partial : \mathcal{P}_3(\mathbb{R}) \rightarrow \mathcal{P}_2(\mathbb{R})$ and the anti-derivative operator $\int : \mathcal{P}_2(\mathbb{R}) \rightarrow \mathcal{P}_3(\mathbb{R})$. In $\mathcal{P}_2(\mathbb{R})$ and $\mathcal{P}_3(\mathbb{R})$ choose the monomial basis \mathcal{B} and \mathcal{C} , respectively. From Proposition 8.15 and the equations (8.10) and

(8.11), we conclude

$$(\partial \circ \int)_{\mathcal{B} \leftarrow \mathcal{B}} = \partial_{\mathcal{B} \leftarrow \mathcal{C}} \int_{\mathcal{C} \leftarrow \mathcal{B}} = \begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1/3 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = id_{\mathcal{B} \leftarrow \mathcal{B}}$$

and

$$(\int \circ \partial)_{\mathcal{C} \leftarrow \mathcal{C}} = \int_{\mathcal{C} \leftarrow \mathcal{B}} \partial_{\mathcal{B} \leftarrow \mathcal{C}} = \begin{pmatrix} 1/3 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \textcolor{red}{0} \end{pmatrix} \neq id_{\mathcal{C} \leftarrow \mathcal{C}}.$$

8.3.4 Change of basis

Let $\mathcal{B} = (\mathbf{b}_1, \dots, \mathbf{b}_n)$ and $\mathcal{C} = (\mathbf{c}_1, \dots, \mathbf{c}_n)$ be two bases of V . Then, the identity map $id : \mathbf{x} \mapsto \mathbf{x}$ of V with respect to \mathcal{B} and \mathcal{C} has the following matrix representation:

$$id_{\mathcal{C} \leftarrow \mathcal{B}} = \begin{pmatrix} \Phi_{\mathcal{C}}(id(\mathbf{b}_1)) & \dots & \Phi_{\mathcal{C}}(id(\mathbf{b}_n)) \end{pmatrix} = \begin{pmatrix} \mathbf{b}_1^{\mathcal{C}} & \dots & \mathbf{b}_n^{\mathcal{C}} \end{pmatrix} = T_{\mathcal{C} \leftarrow \mathcal{B}}. \quad (8.15)$$

The transformation matrix $T_{\mathcal{C} \leftarrow \mathcal{B}}$ from equation (7.12) for the change of basis is in fact the matrix representation id (each \mathbf{x} stays where it is).

Now, we can expand the notion of the change of basis: Let ℓ be a linear map $V \rightarrow W$. With respect to the bases \mathcal{B} in V and \mathcal{C} in W , we find the matrix representation $\ell_{\mathcal{C} \leftarrow \mathcal{B}}$ of ℓ . In addition, we now choose second bases \mathcal{B}' in V and \mathcal{C}' in W ask the following:

Question:

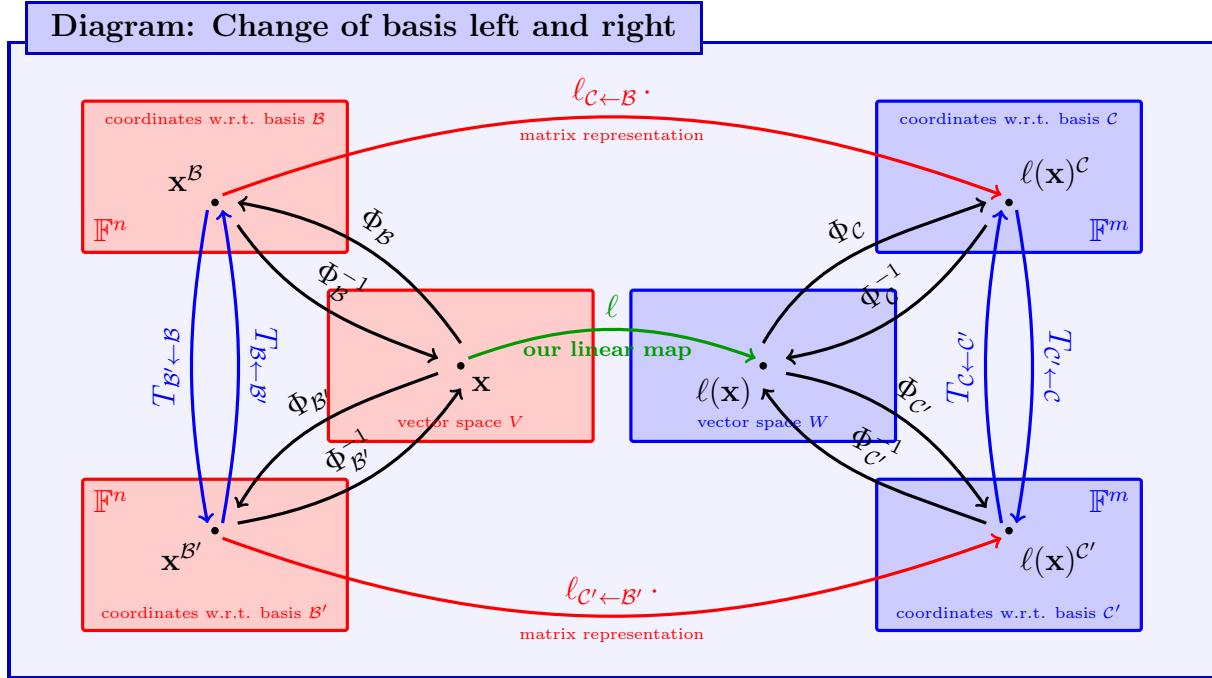
What is the relation between $\ell_{\mathcal{C} \leftarrow \mathcal{B}}$ and $\ell_{\mathcal{C}' \leftarrow \mathcal{B}'}$?

Let us try to calculate the matrices $\ell_{\mathcal{C}' \leftarrow \mathcal{B}'}$ with the help of $\ell_{\mathcal{C} \leftarrow \mathcal{B}}$:

Change of basis left and right

$$\ell_{\mathcal{C}' \leftarrow \mathcal{B}'} = (id \circ \ell \circ id)_{\mathcal{C}' \leftarrow \mathcal{B}'} = id_{\mathcal{C}' \leftarrow \mathcal{C}} \ell_{\mathcal{C} \leftarrow \mathcal{B}} id_{\mathcal{B} \leftarrow \mathcal{B}'} = T_{\mathcal{C}' \leftarrow \mathcal{C}} \ell_{\mathcal{C} \leftarrow \mathcal{B}} T_{\mathcal{B} \leftarrow \mathcal{B}'} \quad (8.16)$$

This gives us a nice picture:



Example 8.18. Let us consider the differential operator $\partial : \mathcal{P}_3(\mathbb{R}) \rightarrow \mathcal{P}_2(\mathbb{R})$ where $V = \mathcal{P}_3(\mathbb{R})$ carries the monomial basis $\mathcal{B} = (\mathbf{m}_3, \mathbf{m}_2, \mathbf{m}_1, \mathbf{m}_0)$ and an additional basis

$$\mathcal{B}' = (2\mathbf{m}_3 - \mathbf{m}_1, \mathbf{m}_2 + \mathbf{m}_0, \mathbf{m}_1 + \mathbf{m}_0, \mathbf{m}_1 - \mathbf{m}_0) =: (\mathbf{b}'_1, \mathbf{b}'_2, \mathbf{b}'_3, \mathbf{b}'_4).$$

Moreover, $W = \mathcal{P}_2(\mathbb{R})$ carries the monomial basis $\mathcal{C} = (\mathbf{m}_2, \mathbf{m}_1, \mathbf{m}_0)$ and another basis

$$\mathcal{C}' = (\mathbf{m}_2 - \frac{1}{2}\mathbf{m}_1, \mathbf{m}_2 + \frac{1}{2}\mathbf{m}_1, \mathbf{m}_0) =: (\mathbf{c}'_1, \mathbf{c}'_2, \mathbf{c}'_3).$$

We already know the transformation $T_{\mathcal{C}' \leftarrow \mathcal{C}}$ for the change of basis, see Example 7.21. The one matrix representation $\partial_{\mathcal{C} \leftarrow \mathcal{B}}$ is also known by equation (8.10). The change of basis $T_{\mathcal{B} \leftarrow \mathcal{B}'}$ is directly given by \mathcal{B}' :

$$\begin{aligned} \Phi_{\mathcal{B}}(\mathbf{b}'_1) &= \Phi_{\mathcal{B}}(2\mathbf{m}_3 - \mathbf{m}_1) = \begin{pmatrix} 2 \\ 0 \\ -1 \\ 0 \end{pmatrix}, & \Phi_{\mathcal{B}}(\mathbf{b}'_2) &= \Phi_{\mathcal{B}}(\mathbf{m}_2 + \mathbf{m}_0) = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}, \\ \Phi_{\mathcal{B}}(\mathbf{b}'_3) &= \Phi_{\mathcal{B}}(\mathbf{m}_1 + \mathbf{m}_0) = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}, & \Phi_{\mathcal{B}}(\mathbf{b}'_4) &= \Phi_{\mathcal{B}}(\mathbf{m}_1 - \mathbf{m}_0) = \begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \end{pmatrix}. \end{aligned}$$

In summary, we have:

$$T_{\mathcal{C}' \leftarrow \mathcal{C}} = \begin{pmatrix} 1/2 & -1 & 0 \\ 1/2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \partial_{\mathcal{C} \leftarrow \mathcal{B}} = \begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad T_{\mathcal{B} \leftarrow \mathcal{B}'} = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \end{pmatrix}.$$

Using (8.16), we know that the matrix representation $\partial_{\mathcal{C}' \leftarrow \mathcal{B}'}$ is given by the product of these three matrices:

$$\partial_{\mathcal{C}' \leftarrow \mathcal{B}'} = T_{\mathcal{C}' \leftarrow \mathcal{C}} \partial_{\mathcal{C} \leftarrow \mathcal{B}} T_{\mathcal{B} \leftarrow \mathcal{B}'} = \begin{pmatrix} 3 & -2 & 0 & 0 \\ 3 & 2 & 0 & 0 \\ -1 & 0 & 1 & 1 \end{pmatrix}. \quad (8.17)$$

Alternatively, we could directly calculate $\partial_{\mathcal{C}' \leftarrow \mathcal{B}'}$ from ∂ and the bases \mathcal{B}' and \mathcal{C}' . In order to do this, we apply ∂ to the basis elements from \mathcal{B}' and represent the results with respect to the basis \mathcal{C}' :

$$\begin{aligned}\Phi_{\mathcal{C}'}(\partial(\mathbf{b}'_1)) &= \Phi_{\mathcal{C}'}\left(\partial(\underbrace{2\mathbf{m}_3 - \mathbf{m}_1}_{\mathbf{b}'_1})\right) = \Phi_{\mathcal{C}'}(6\mathbf{m}_2 - \mathbf{m}_0) = \begin{pmatrix} 3 \\ 3 \\ -1 \end{pmatrix}, \\ \Phi_{\mathcal{C}'}(\partial(\mathbf{b}'_2)) &= \Phi_{\mathcal{C}'}\left(\partial(\underbrace{\mathbf{m}_2 + \mathbf{m}_0}_{\mathbf{b}'_2})\right) = \Phi_{\mathcal{C}'}(2\mathbf{m}_1) = \begin{pmatrix} -2 \\ 2 \\ 0 \end{pmatrix}, \\ \Phi_{\mathcal{C}'}(\partial(\mathbf{b}'_3)) &= \Phi_{\mathcal{C}'}\left(\partial(\underbrace{\mathbf{m}_1 + \mathbf{m}_0}_{\mathbf{b}'_3})\right) = \Phi_{\mathcal{C}'}(\mathbf{m}_0) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \\ \Phi_{\mathcal{C}'}(\partial(\mathbf{b}'_4)) &= \Phi_{\mathcal{C}'}\left(\partial(\underbrace{\mathbf{m}_1 - \mathbf{m}_0}_{\mathbf{b}'_4})\right) = \Phi_{\mathcal{C}'}(\mathbf{m}_0) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.\end{aligned}$$

This gives us, as expected, the same matrix as in (8.17).

However, we can also do another alternative computation. Choose $a, b, c, d \in \mathbb{R}$ arbitrarily. Then:

$$\begin{aligned}\mathbf{f}^{\mathcal{B}'} &= \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} \xrightarrow{\Phi_{\mathcal{B}'}^{-1}} \mathbf{f} = a(2\mathbf{m}_3 - \mathbf{m}_1) + b(\mathbf{m}_2 + \mathbf{m}_0) + c(\mathbf{m}_1 + \mathbf{m}_0) + d(\mathbf{m}_1 - \mathbf{m}_0) \\ &\quad = 2am_3 + b\mathbf{m}_2 + (-a + c + d)\mathbf{m}_1 + (b + c - d)\mathbf{m}_0 \\ &\xrightarrow{\partial} \partial(\mathbf{f}) = 6am_2 + 2b\mathbf{m}_1 + (-a + c + d)\mathbf{m}_0 \\ &\quad = 6a\underbrace{\left(\frac{1}{2}\mathbf{c}'_1 + \frac{1}{2}\mathbf{c}'_2\right)}_{\mathbf{m}_2} + 2b\underbrace{(-\mathbf{c}'_1 + \mathbf{c}'_2)}_{\mathbf{m}_1} + (-a + c + d)\underbrace{\mathbf{c}'_3}_{\mathbf{m}_0} \\ &\quad = (3a - 2b)\mathbf{c}'_1 + (3a + 2b)\mathbf{c}'_2 + (-a + c + d)\mathbf{c}'_3 \\ &\xrightarrow{\Phi_{\mathcal{C}'}} \partial(\mathbf{f})^{\mathcal{C}'} = \begin{pmatrix} 3a - 2b \\ 3a + 2b \\ -a + c + d \end{pmatrix} \stackrel{\leq}{=} \begin{pmatrix} 3 & -2 & 0 & 0 \\ 3 & 2 & 0 & 0 \\ -1 & 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}.\end{aligned}$$

8.3.5 Equivalent and similar matrices

Both matrices

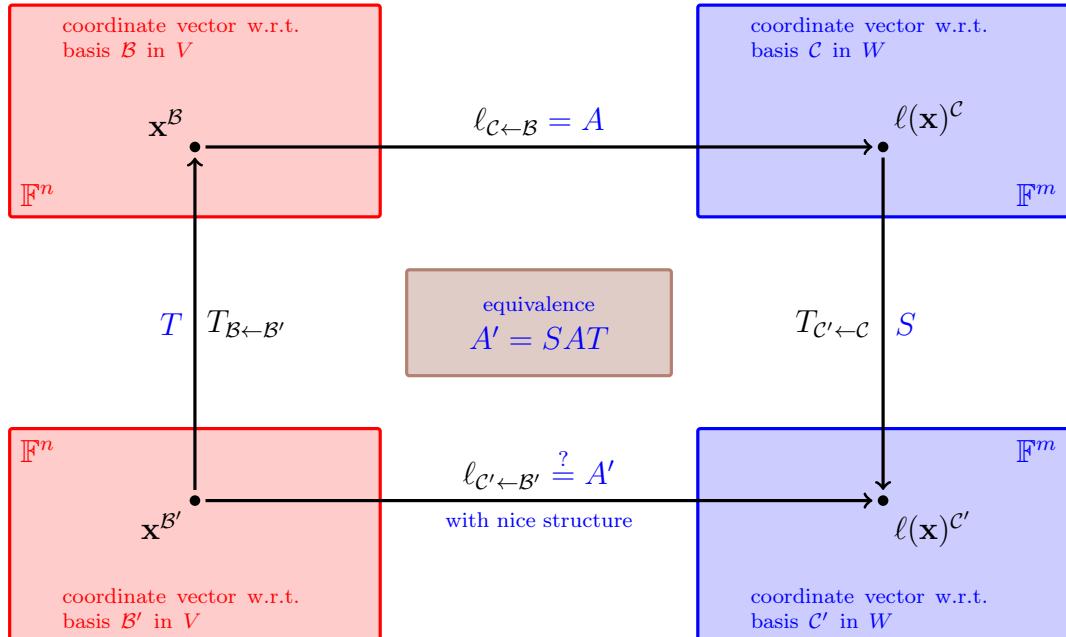
$$\partial_{\mathcal{C} \leftarrow \mathcal{B}} = \begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \quad \text{and} \quad \partial_{\mathcal{C}' \leftarrow \mathcal{B}'} = \begin{pmatrix} 3 & -2 & 0 & 0 \\ 3 & 2 & 0 & 0 \\ -1 & 0 & 1 & 1 \end{pmatrix}$$

from Example 8.18 look completely different although they describe the same linear map $\partial \in \mathcal{L}(\mathcal{P}_3(\mathbb{R}), \mathcal{P}_2(\mathbb{R}))$, however, with respect two different bases. We will show that the *rank* of the matrix is exactly the criterion for being a matrix representation for the same linear map. Let V and W be two \mathbb{F} -vector spaces with dimension n and m , respectively. Choose a basis \mathcal{B} in V and a basis \mathcal{C} in W . Let $\ell \in \mathcal{L}(V, W)$ given with a matrix representation $A := \ell_{\mathcal{C} \leftarrow \mathcal{B}} \in \mathbb{F}^{m \times n}$, and let $A' \in \mathbb{F}^{m \times n}$ be another matrix (maybe much simpler than A).

Question:

Are there bases \mathcal{B}' and \mathcal{C}' in V and W , respectively, such that A' is the matrix representation of ℓ ,

$$A' = \ell_{\mathcal{C}' \leftarrow \mathcal{B}'} ?$$



We already know, cf. (8.16),

$$\ell_{C' \leftarrow B'} = \underbrace{T_{C' \leftarrow C}}_{=: S} \underbrace{\ell_{C \leftarrow B}}_{=: A} \underbrace{T_{B \leftarrow B'}}_{=: T} = SAT.$$

Choosing all possible bases \mathcal{B}' and \mathcal{C}' in V and W , respectively, we get all possible invertible matrices S and T and hence with $\ell_{C' \leftarrow B'}$ all matrices that are equivalent to A :

Proposition & Definition 8.19. Equivalent matrices

A matrix $B \in \mathbb{F}^{m \times n}$ is called equivalent to another matrix $A \in \mathbb{F}^{m \times n}$ if there are invertible matrices $S \in \mathbb{F}^{m \times m}$ and $T \in \mathbb{F}^{n \times n}$ with

$$B = SAT.$$

In this case, we write $B \sim A$. For arbitrary matrices $A, B, C \in \mathbb{F}^{m \times n}$, the following holds:

$$A \sim A, \quad A \sim B \Rightarrow B \sim A, \quad A \sim B \wedge B \sim C \Rightarrow A \sim C.$$

Equivalent matrices describe the same linear map, just with respect to different bases. Here, we have a simple test for equivalence:

Proposition 8.20. Equivalence is given by the rank

For two matrices $A, B \in \mathbb{F}^{m \times n}$, we have: $A \sim B \iff \text{rank}(A) = \text{rank}(B)$.

Proof. For a given matrix A , we can use the Gaussian elimination to bring it into a row echelon form K , namely $PA = LK$. By using even more row operation (backward substitution) and column exchanges, we can bring the matrix into a so-called [normal form](#), given as a block matrix:

$$A \sim \begin{pmatrix} \mathbb{1}_r & 0 \\ 0 & 0 \end{pmatrix}_{m \times n} =: \mathbb{1}_{r,m,n} \quad \text{with } r = \text{rank}(A).$$

Note that in all these steps, we used invertible matrices from left and from right and, therefore, did not change the rank of the matrix. In other words, there are invertible matrices S and T with $\mathbb{1}_{r,m,n} = SAT$, which means $A \sim \mathbb{1}_{r,m,n}$.

\Leftarrow Is $\text{rank}(A) =: r$ the same as $\text{rank}(B) =: r'$, then we immediately get: $A \sim \mathbb{1}_{r,m,n}$ and $\mathbb{1}_{r',m,n} \sim B$ and therefore $A \sim B$.

\Rightarrow Now, let $A \sim B$. Then, there are invertible matrices S and T with $B = SAT$. Because of

$$\text{Ran}(AT) = \{A \underbrace{T\mathbf{x}}_{=: \mathbf{y}} : \mathbf{x} \in \mathbb{F}^n\} = \{Ay : \mathbf{y} \in \underbrace{\text{Ran}(T)}_{=\mathbb{F}^n}\} = \text{Ran}(A),$$

we get $\text{rank}(AT) = \dim(\text{Ran}(AT)) = \dim(\text{Ran}(A)) = \text{rank}(A)$ and can conclude:

$$\begin{aligned} \text{rank}(A) &= \text{rank}(AT) = \text{rank}((AT)^*) = \text{rank}(T^*A^*) \\ &\stackrel{(*)}{=} \text{rank}(T^*A^*S^*) = \text{rank}((SAT)^*) = \text{rank}(B^*) = \text{rank}(B). \end{aligned}$$

In $(*)$, we used that invertible matrices do not change the rank. □

In particular, the natural number $\text{rank}(\ell_{\mathcal{C} \leftarrow \mathcal{B}})$ is only dependent upon ℓ and not upon the bases \mathcal{B} and \mathcal{C} . Hence the proposition below works as an alternative definition for the rank of a linear map.

Definition 8.21. Rank of a linear map

Let $\ell \in \mathcal{L}(V, W)$. The number

$$\text{rank}(\ell) := \dim(\text{Ran}(\ell)).$$

is called the [rank](#) of the linear map ℓ .

Recall that the range of any map is the set of all elements that are “hit” by the map.

Proposition 8.22. Rank of linear map or matrix

For $\ell \in \mathcal{L}(V, W)$, the rank is the same as the rank of $\ell_{\mathcal{C} \leftarrow \mathcal{B}}$ where \mathcal{B} and \mathcal{C} are any bases in V and W :

$$\text{rank}(\ell) = \text{rank}(\ell_{\mathcal{C} \leftarrow \mathcal{B}}).$$

Example 8.23. The operator $\text{proj}_G \in \mathcal{L}(\mathbb{R}^3, \mathbb{R}^3)$ given by the orthogonal projection onto $G := \text{Span}(\mathbf{n})$ with $\mathbf{n} \in \mathbb{R}^3$, $\|\mathbf{n}\| = 1$ has the matrix representation with respect to \mathcal{B} given

by (8.13), cf Example 8.14 (f). On the other hand, with respect to the standard basis $\mathcal{E} = (\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$, the operator proj_G is given by $\mathbf{n}\mathbf{n}^\top$, cf. equation (8.1), which means:

$$(\text{proj}_G)_{\mathcal{B} \leftarrow \mathcal{B}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad (\text{proj}_G)_{\mathcal{E} \leftarrow \mathcal{E}} = \mathbf{n}\mathbf{n}^\top \in \mathbb{R}^{3 \times 3}.$$

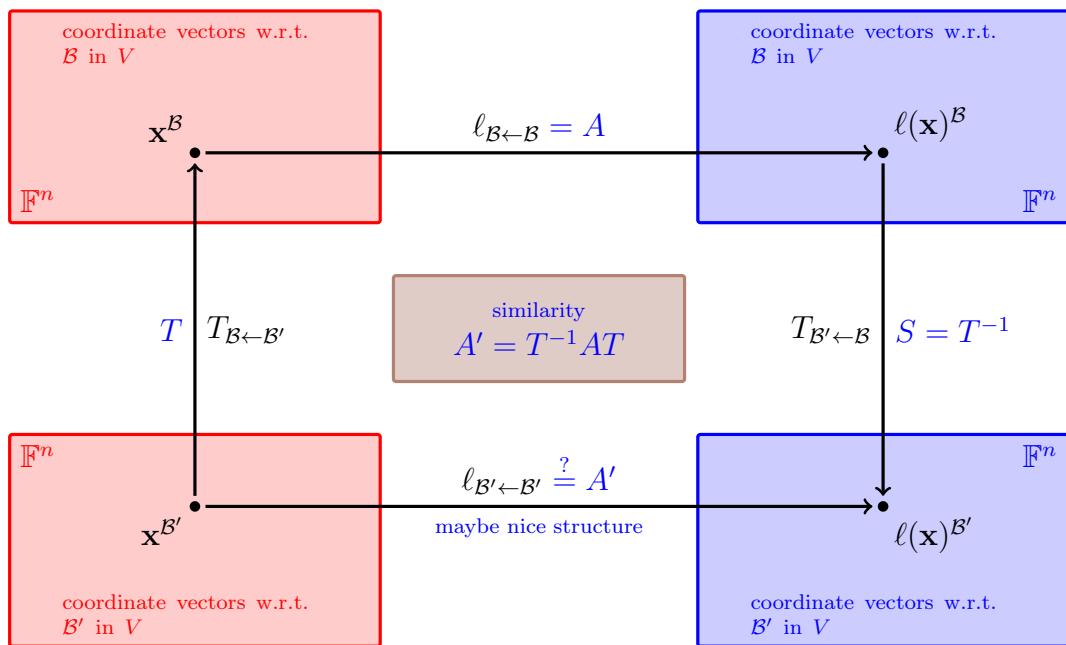
Both are matrix representations of the same linear map proj_G . Hence, they are equivalent and have the same rank.

Now, we go back to a linear map $\ell : V \rightarrow V$, which means $V = W$. There, we can choose the same basis on the left and right V , in other words $\mathcal{B} = \mathcal{C}$ and $\mathcal{B}' = \mathcal{C}'$.

Let $\ell \in \mathcal{L}(V, V)$ and $A := \ell_{\mathcal{B} \leftarrow \mathcal{B}}$ the matrix representation w.r.t. to \mathcal{B} and \mathcal{B} .

Question:

Which matrices A' do we get by $\ell_{\mathcal{B}' \leftarrow \mathcal{B}'}$ when \mathcal{B}' is any basis of V ? What is the connection to A ?



In the same way as above, we have:

$$\ell_{\mathcal{B}' \leftarrow \mathcal{B}'} = \underbrace{T_{\mathcal{B}' \leftarrow \mathcal{B}}}_{=: S} \underbrace{\ell_{\mathcal{B} \leftarrow \mathcal{B}}}_{=: A} \underbrace{T_{\mathcal{B} \leftarrow \mathcal{B}'}}_{=: T} = SAT = T^{-1}AT$$

because $S = T^{-1}$ by equation (7.15). This means the new matrix representation is given by multiplying with a suitable matrix T from right and with the inverse T^{-1} from the left. Hence, we define:

Proposition & Definition 8.24. Similar matrices

A square matrix $B \in \mathbb{F}^{n \times n}$ is called similar to another matrix $A \in \mathbb{F}^{n \times n}$ if there is an invertible $T \in \mathbb{F}^{n \times n}$ with:

$$B = T^{-1}AT.$$

In this case, we write $B \approx A$. For $A, B, C \in \mathbb{F}^{n \times n}$, we get:

$$A \approx A, \quad A \approx B \Rightarrow B \approx A, \quad A \approx B \wedge B \approx C \Rightarrow A \approx C.$$

For square matrices A and B , we have the following implication:

$$A \approx B \Rightarrow A \sim B$$

The converse implication is in general wrong: $\not\Rightarrow$. For example, we have $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ but $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \not\approx \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$.

The classification by \sim is much coarser than the one by \approx . Similarity \approx is completely determined by the Jordan normal form (cf. tutorial and Section 9.1).

8.4 Solutions of linear equations

Let V and W be two \mathbb{F} -vector spaces. For a linear map $\ell \in \mathcal{L}(V, W)$ and a vector $\mathbf{b} \in W$, we can ask about solutions of the equation

$$\ell(\mathbf{x}) = \mathbf{b}.$$

We are interested in *existence* and *uniqueness* of solutions.

Kernel and range of ℓ

$$\text{Ker}(\ell) := \{\mathbf{x} \in V : \ell(\mathbf{x}) = \mathbf{0}\} \subset V \quad \text{and} \quad \text{Ran}(\ell) := \{\ell(\mathbf{x}) : \mathbf{x} \in V\} \subset W.$$

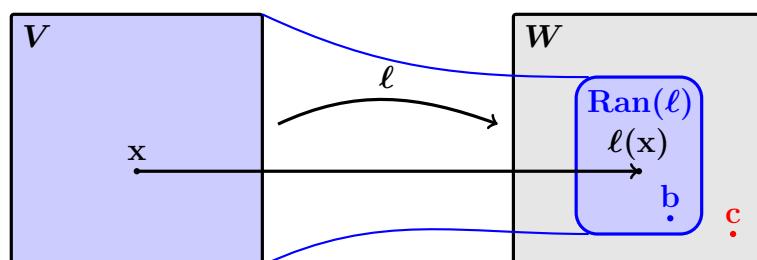
8.4.1 Existence for solutions

Clearly: The equation

$$\ell(\mathbf{x}) = \mathbf{b}$$

has solutions $\mathbf{x} \in V$ if $\mathbf{b} \in W$ is given of the form $\ell(\mathbf{x})$ for some $\mathbf{x} \in V$, which means:

$$\mathbf{b} \in \text{Ran}(\ell).$$



$$\begin{aligned} \mathbf{b} \in \text{Ran}(\ell) &\Rightarrow \ell(\mathbf{x}) = \mathbf{b} \text{ has a solution} \\ \mathbf{c} \notin \text{Ran}(\ell) &\Rightarrow \ell(\mathbf{x}) = \mathbf{c} \text{ has no solution} \end{aligned}$$

The existence of solutions is independent of the right-hand side \mathbf{b} if and only if all $\mathbf{b} \in W$ lie in $\text{Ran}(\ell)$, which means:

$$\text{Ran}(\ell) = W \tag{8.18}$$

We call this, as before, unconditional solvability of the equation $\ell(\mathbf{x}) = \mathbf{b}$ and this is equivalent to the surjectivity of ℓ .

8.4.2 Uniqueness and solution set

If $\ell(\mathbf{x}) = \mathbf{b}$ has solutions $\mathbf{x} \in V$, then the best case scenario is when there is exactly one such solution.

For $\mathbf{b} = \mathbf{o}$ (the [homogeneous equation](#)), the solution space is given by $\text{Ker}(\ell)$. Because of Proposition 8.2, $\mathbf{o} \in V$ is always a solution and always in $\text{Ker}(\ell)$. Uniqueness of the solution is then given if \mathbf{o} is the only element in $\text{Ker}(\ell)$, hence $\text{Ker}(\ell) = \{\mathbf{o}\}$.

For $\mathbf{b} \neq \mathbf{o}$ (the [inhomogeneous equation](#)) we get the same criterion for uniqueness: Let $\mathbf{x}_p \in V$ be a solution, which means $\ell(\mathbf{x}_p) = \mathbf{b}$, and $\mathbf{x} \in V$ another solution, which means $\ell(\mathbf{x}) = \mathbf{b}$, then:

$$\ell(\mathbf{x} - \mathbf{x}_p) = \ell(\mathbf{x}) - \ell(\mathbf{x}_p) = \mathbf{b} - \mathbf{b} = \mathbf{o}, \quad \text{and} \quad \mathbf{x} - \mathbf{x}_p \in \text{Ker}(\ell).$$

On the other hand, if $\mathbf{x} - \mathbf{x}_p \in \text{Ker}(\ell)$ and $\ell(\mathbf{x}_p) = \mathbf{b}$, then also \mathbf{x} is a solution because

$$\ell(\mathbf{x}) = \ell(\mathbf{x}_p + (\underbrace{\mathbf{x} - \mathbf{x}_p}_{=: \mathbf{k} \in \text{Ker}(\ell)})) = \underbrace{\ell(\mathbf{x}_p)}_{\mathbf{b}} + \underbrace{\ell(\mathbf{k})}_{\mathbf{o}} = \mathbf{b}.$$

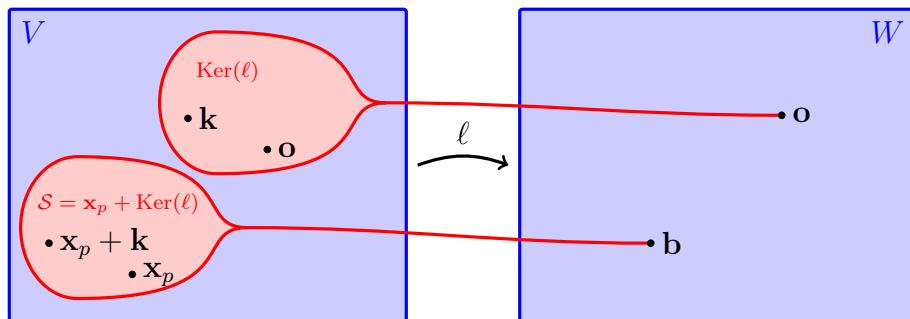
In summary:

Solution set of $\ell(\mathbf{x}) = \mathbf{b}$ is an affine subspace

Let $\mathbf{x}_p \in V$ be a solution of the equation with right-hand side $\mathbf{b} \in W$, which means $\ell(\mathbf{x}_p) = \mathbf{b}$. The solution set $\mathcal{S} = \{\mathbf{x} \in V : \ell(\mathbf{x}) = \mathbf{b}\}$ is given as an affine subspace:

$$\mathcal{S} = \{\mathbf{x} = \mathbf{x}_p + \mathbf{k} : \ell(\mathbf{k}) = \mathbf{o}\} = \{\mathbf{x}_p + \mathbf{k} : \mathbf{k} \in \text{Ker}(\ell)\} = \mathbf{x}_p + \text{Ker}(\ell)$$

While $\text{Ker}(\ell)$ is a linear subspace of V , the solution set $\mathcal{S} = \mathbf{x}_p + \text{Ker}(\ell)$ is an affine subspace of V . The kernel is translated by one particular solution \mathbf{x}_p :



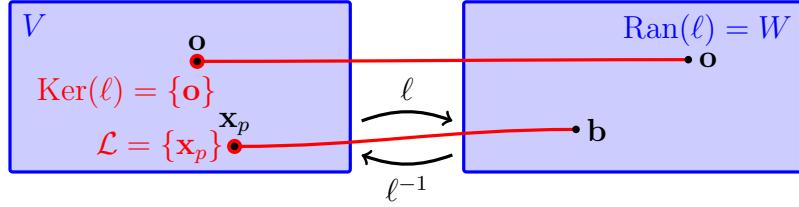
Therefore, also in the inhomogeneous case, the solution is unique if and only if

$$\text{Ker}(\ell) = \{\mathbf{o}\} \tag{8.19}$$

holds since \mathcal{S} contains at most 1 element in this case, namely \mathbf{x}_p if it exists. If there is no solution $\mathbf{x}_p \in V$ at all, then $\mathcal{L} = \emptyset$. We call this [unique solvability](#), and for a linear map ℓ this is exactly the [injectivity](#).

8.4.3 Invertibility: unconditional and unique solvability

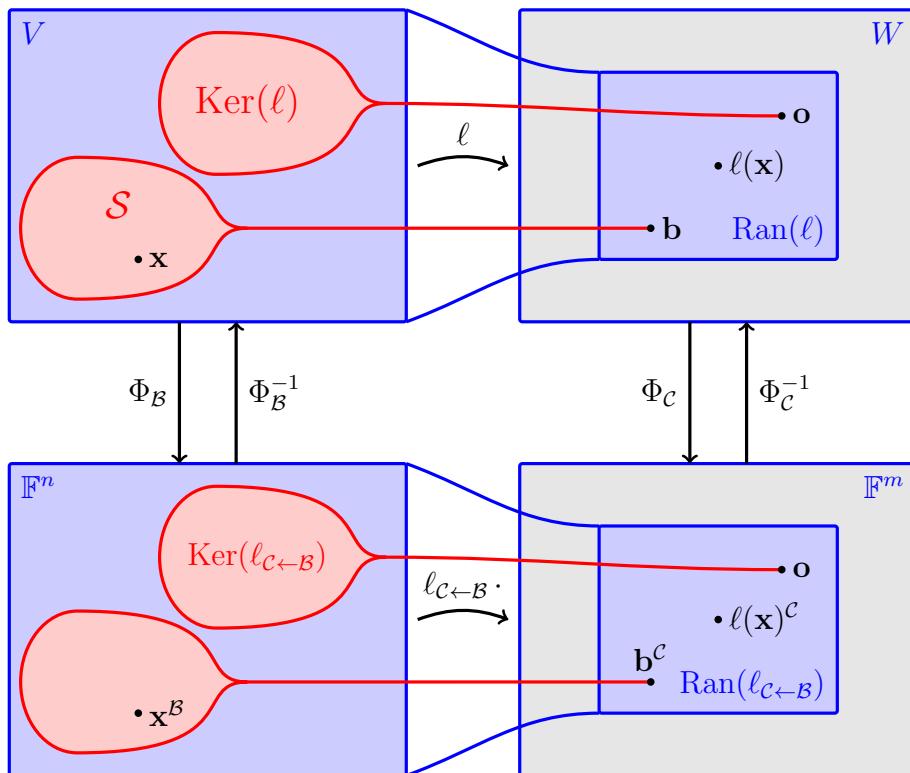
The map ℓ is *bijective* if and only if it is surjective and injective. This is again equivalent to the existence of an inverse map $\ell^{-1} : W \rightarrow V$.



ℓ^{-1} maps each right-hand side $\mathbf{b} \in W$ to the unique solution $\mathbf{x} = \mathbf{x}_p \in V$.

8.4.4 A link to the matrix representation

Let \mathcal{B} and \mathcal{C} be bases in V and W , respectively, and also denote the dimensions by n and m . Then for a linear map $\ell : V \rightarrow W$, we get a matrix representation $\ell_{\mathcal{C} \leftarrow \mathcal{B}} \in \mathbb{K}^{m \times n}$.



Our equation $\ell(\mathbf{x}) = \mathbf{b}$ can be translated to the coordinate level to $\ell_{\mathcal{C} \leftarrow \mathcal{B}} \Phi_{\mathcal{B}}(\mathbf{x}) = \Phi_{\mathcal{C}}(\mathbf{b})$. We immediately get

$$\mathbf{x} \in \text{Ker}(\ell) \iff \Phi_{\mathcal{B}}(\mathbf{x}) \in \text{Ker}(\ell_{\mathcal{C} \leftarrow \mathcal{B}})$$

and

$$\mathbf{b} \in \text{Ran}(\ell) \iff \Phi_{\mathcal{C}}(\mathbf{b}) \in \text{Ran}(\ell_{\mathcal{C} \leftarrow \mathcal{B}}).$$

This means:

$$\text{Ker}(\ell) = \Phi_{\mathcal{B}}^{-1} \text{Ker}(\ell_{\mathcal{C} \leftarrow \mathcal{B}}) \quad \text{and} \quad \text{Ran}(\ell) = \Phi_{\mathcal{C}}^{-1} \text{Ran}(\ell_{\mathcal{C} \leftarrow \mathcal{B}}). \quad (8.20)$$

Hence $\text{Ker}(\ell)$ and $\text{Ker}(\ell_{C \leftarrow B})$ can only be simultaneously be trivial, which means $\{\mathbf{0}\}$, and in the same manner $\text{Ran}(\ell)$ and $\text{Ran}(\ell_{C \leftarrow B})$ can also only be simultaneously be the whole space. For matrices $\ell_{C \leftarrow B}$, we can restate our Propositions 3.65 and 3.67 from Section 3.12:

Proposition 8.25. Unconditional solvability (surjectivity of ℓ)

The following claims are equivalent:

- (i) The equation $\ell(\mathbf{x}) = \mathbf{b}$ has **at least** one solution \mathbf{x} for each $\mathbf{b} \in W$.
- (ii) ℓ is surjective.
- (iii) $\text{Ran}(\ell) = W$.
- (iv) $\text{Ran}(\ell_{C \leftarrow B}) = \mathbb{F}^m$.
- (v) $\text{rank}(\ell) = \text{rank}(\ell_{C \leftarrow B}) = m \leq n$.
- (vi) If using Gaussian elimination to bring $\ell_{C \leftarrow B}$ into row echelon form, then each **row** has a pivot.

Proposition 8.26. Unique solvability (injectivity of ℓ)

The following claims are equivalent:

- (i) The equation $\ell(\mathbf{x}) = \mathbf{b}$ has **at most** one solution \mathbf{x} for each $\mathbf{b} \in W$.
- (ii) ℓ is injective.
- (iii) $\text{Ker}(\ell) = \{\mathbf{0}\}$.
- (iv) $\text{Ker}(\ell_{C \leftarrow B}) = \{\mathbf{0}\}$.
- (v) $\text{rank}(\ell) = \text{rank}(\ell_{C \leftarrow B}) = n \leq m$.
- (vi) If using Gaussian elimination to bring $\ell_{C \leftarrow B}$ into row echelon form, then each **column** has a pivot.

In the case $m = n$, we get:

Proposition 8.27. Fredholm alternative

In the case $m = n$, which means $\dim(V) = \dim(W) < \infty$, then either **all** claims from the Propositions 8.25 and 8.26 are true or **none** of them are true.

Also the rank-nullity theorem can now be transformed to the abstract case:

Rank-nullity theorem

For two \mathbb{F} -vector spaces V, W with $\dim(V) < \infty$ and a linear map $\ell : V \rightarrow W$, we have:

$$\dim(\text{Ker}(\ell)) + \dim(\text{Ran}(\ell)) = \dim(V).$$

Example 8.28. (a) The projection operator $\text{proj}_G \in \mathcal{L}(\mathbb{R}^3, \mathbb{R}^3)$ with $G = \text{Span}(\mathbf{n})$ has the $\text{Ker}(\text{proj}_G) = E := \{\mathbf{n}\}^\perp$ and $\text{Ran}(\text{proj}_G) = G$. The rank-nullity theorem gives us $2+1=3$. The map is neither injective nor surjective. The equations $\text{proj}_G(\mathbf{x}) = \mathbf{b}$ have only solutions if $\mathbf{b} \in G$. The solution set \mathcal{S} is then a line that is parallel to E , hence a translation of the kernel by a particular solution \mathbf{x}_p .

(b) The reflection $\text{refl}_E = id - 2 \text{proj}_G$ is an invertible map where the inverse is refl_E again.

The kernel is $\{\mathbf{o}\}$ and the range \mathbb{R}^3 . The rank-nullity theorem says $0 + 3 = 3$. The equation $\text{refl}_E(\mathbf{x}) = \mathbf{b}$ has a unique solution for all $\mathbf{b} \in \mathbb{R}^3$.

- (c) Consider the differential operator $\partial : \mathbf{f} \mapsto \mathbf{f}'$, defined as $\mathcal{P}_3(\mathbb{R}) \rightarrow \mathcal{P}_3(\mathbb{R})$, where $\mathcal{P}_3(\mathbb{R})$ carries the monomial basis \mathcal{B} . The matrix representation is:

$$\partial_{\mathcal{B} \leftarrow \mathcal{B}} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 3 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}. \quad (8.21)$$

Obviously,

$$\text{Ker}(\partial_{\mathcal{B} \leftarrow \mathcal{B}}) = \text{Span}\left(\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}\right) \quad \text{and} \quad \text{Ran}(\partial_{\mathcal{B} \leftarrow \mathcal{B}}) = \text{Span}\left(\begin{pmatrix} 0 \\ 3 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}\right),$$

and hence by (8.20), we have

$$\text{Ker}(\partial) = \Phi_{\mathcal{B}}^{-1} \text{Ker}(\partial_{\mathcal{B} \leftarrow \mathcal{B}}) = \text{Span}(\mathbf{m}_0) = \mathcal{P}_0(\mathbb{R})$$

and

$$\text{Ran}(\partial) = \Phi_{\mathcal{B}}^{-1} \text{Ran}(\partial_{\mathcal{B} \leftarrow \mathcal{B}}) = \text{Span}(\mathbf{m}_2, \mathbf{m}_1, \mathbf{m}_0) = \mathcal{P}_2(\mathbb{R}).$$

This makes sense since $\mathbf{f}' = \mathbf{o}$ is solved exactly by the constant functions, which are all functions $\mathbf{f} \in \mathcal{P}_0(\mathbb{R})$, and the set of all possible outcomes \mathbf{f}' for $\mathbf{f} \in \mathcal{P}_3(\mathbb{R})$ is exactly $\mathcal{P}_2(\mathbb{R})$.

The equation $\partial(\mathbf{f}) = \mathbf{g}$, which means searching antiderivative for a given function $\mathbf{g} \in \mathcal{P}_3(\mathbb{R})$, has the solution set $\mathcal{S} = \emptyset$ if $\mathbf{g} \notin \mathcal{P}_2(\mathbb{R}) = \text{Ran}(\partial)$. For $\mathbf{g} \in \mathcal{P}_2(\mathbb{R})$, we have

$$\mathcal{S} = \mathbf{f}_p + \text{Ker}(\partial) = \mathbf{f}_p + \mathcal{P}_0(\mathbb{R})$$

with a particular solution $\mathbf{f}_p \in \mathcal{P}_3(\mathbb{R})$.

- (d) We want to find all solutions $\mathbf{f} \in \mathcal{P}_3(\mathbb{R})$ for the differential equation $\mathbf{f}'' + 2\mathbf{f}' - 8\mathbf{f} = 2\mathbf{m}_2 - 3\mathbf{m}_1$. In order to do this, we define the linear map $\ell := \partial \circ \partial + 2\partial - 8id$ by $\mathcal{P}_3(\mathbb{R}) \rightarrow \mathcal{P}_3(\mathbb{R})$ and $\ell(\mathbf{f}) = \mathbf{f}'' + 2\mathbf{f}' - 8\mathbf{f}$. Then, we only have to solve the equation $\ell(\mathbf{f}) = 2\mathbf{m}_2 - 3\mathbf{m}_1$ in $\mathcal{P}_3(\mathbb{R})$. Choose for $\mathcal{P}_3(\mathbb{R})$ the monomial basis \mathcal{B} , namely on both sides:

$$\begin{aligned} \ell_{\mathcal{B} \leftarrow \mathcal{B}} &= (\partial \circ \partial + 2\partial - 8id)_{\mathcal{B} \leftarrow \mathcal{B}} = (\partial_{\mathcal{B} \leftarrow \mathcal{B}})^2 + 2\partial_{\mathcal{B} \leftarrow \mathcal{B}} - 8id_{\mathcal{B} \leftarrow \mathcal{B}} \\ &\stackrel{(8.21)}{=} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 3 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}^2 + 2 \begin{pmatrix} 0 & 0 & 0 & 0 \\ 3 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} - 8 \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} -8 & 0 & 0 & 0 \\ 6 & -8 & 0 & 0 \\ 6 & 4 & -8 & 0 \\ 0 & 2 & 2 & -8 \end{pmatrix}. \end{aligned}$$

Since the matrix $\ell_{\mathcal{B} \leftarrow \mathcal{B}}$ is invertible (look at the determinant $(-8)^4 \neq 0$), we can use Proposition 8.25 and Proposition 8.26, which say that we have a unique solution for all right-hand sides in $\mathcal{P}_3(\mathbb{R})$.

Let us now solve our equation from above: Seeing it in the coordinate language, we can rewrite $\ell(\mathbf{f}) = 2\mathbf{m}_2 - 3\mathbf{m}_1$ as $\ell_{\mathcal{B} \leftarrow \mathcal{B}} \mathbf{f}^{\mathcal{B}} = (2\mathbf{m}_2 - 3\mathbf{m}_1)^{\mathcal{B}}$, hence:

$$\begin{array}{l} \text{m}_3 \quad \text{m}_2 \quad \text{m}_1 \quad \text{m}_0 \\ \text{m}_3 \begin{pmatrix} -8 & 0 & 0 & 0 \\ 6 & -8 & 0 & 0 \\ 6 & 4 & -8 & 0 \\ 0 & 2 & 2 & -8 \end{pmatrix} \mathbf{f}^{\mathcal{B}} = \begin{pmatrix} \text{m}_3 \\ \text{m}_2 \\ \text{m}_1 \\ \text{m}_0 \end{pmatrix} \begin{pmatrix} 0 \\ 2 \\ -3 \\ 0 \end{pmatrix}. \quad \text{Solution:} \quad \mathbf{f}^{\mathcal{B}} = \begin{pmatrix} 0 \\ -1/4 \\ 1/4 \\ 0 \end{pmatrix} \end{array}$$

and hence $\mathbf{f} = \Phi_{\mathcal{B}}^{-1} \mathbf{f}^{\mathcal{B}} = -\frac{1}{4}\mathbf{m}_2 + \frac{1}{4}\mathbf{m}_1$, i.e. $\mathbf{f}(x) = -\frac{1}{4}x^2 + \frac{1}{4}x$.

8.5 Determinants and eigenvalues for linear maps

Reminder: Determinant and eigenvalues for similar matrices

Let A and B be two square matrices that are similar, $A \approx B$. Then:

$$\det(A) = \det(B) \quad \text{and} \quad \text{spec}(A) = \text{spec}(B).$$

For this last section, we consider a linear map $\ell \in \mathcal{L}(V, V)$, which means $V = W$. All matrix representations $\ell_{\mathcal{B} \leftarrow \mathcal{B}}$, where we have the same basis left and right, are similar matrices and have the same determinant and eigenvalues by Proposition 8.5.

Hence, we define:

Definition 8.29. Determinant of a linear map ℓ

The determinant of a linear map $\ell \in \mathcal{L}(V, V)$ is defined as the determinant of a matrix representation $\ell_{\mathcal{B} \leftarrow \mathcal{B}}$ for any basis \mathcal{B} in V :

$$\det(\ell) := \det(\ell_{\mathcal{B} \leftarrow \mathcal{B}}).$$

Knowing the matrix representations, we immediately get the rules for the determinant:

$$\begin{aligned} \det(k \circ \ell) &= \det((k \circ \ell)_{\mathcal{B} \leftarrow \mathcal{B}}) = \det(k_{\mathcal{B} \leftarrow \mathcal{B}} \ell_{\mathcal{B} \leftarrow \mathcal{B}}) = \det(k_{\mathcal{B} \leftarrow \mathcal{B}}) \det(\ell_{\mathcal{B} \leftarrow \mathcal{B}}) = \det(k) \det(\ell) \\ \text{and} \quad \det(id) &= \det(id_{\mathcal{B} \leftarrow \mathcal{B}}) = \det(\mathbb{1}) = 1. \end{aligned}$$

Combining these two rules, we get:

$$\det(\ell^{-1}) \det(\ell) = \det(\ell^{-1} \circ \ell) = \det(id) = 1 \quad \text{and} \quad \det(\ell^{-1}) = 1 / \det(\ell).$$

The notion of eigenvalues and eigenvectors were introduced for matrices and the associated linear maps. Indeed the whole definition makes also sense in the abstract setting, so that we can also use it for linear maps $\ell : V \rightarrow V$. In the end, this will be totally connected to the eigenvalues of the matrix representations.

Definition 8.30. Eigenvalue and eigenvector for linear map ℓ

Let $\ell \in \mathcal{L}(V, V)$. A vector $\mathbf{x} \in V \setminus \{\mathbf{0}\}$ is called an eigenvector of ℓ if $\ell(\mathbf{x})$ is a multiple of \mathbf{x} . The number $\lambda \in \mathbb{F}$ with $\ell(\mathbf{x}) = \lambda\mathbf{x}$ is called an eigenvalue of ℓ associated to the eigenvector \mathbf{x} .

Again, we get the following equivalences:

λ is an eigenvalue of ℓ

$$\begin{aligned} &\iff \text{the homogeneous equation } (\ell - \lambda id)(\mathbf{x}) = \mathbf{0} \text{ has a solution } \mathbf{x} \neq \mathbf{0} \\ &\iff \ell - \lambda id \text{ is not injective} \\ &\stackrel{P.8.27}{\iff} \ell - \lambda id \text{ is not invertible} \\ &\stackrel{(8.20)}{\iff} (\ell - \lambda id)_{\mathcal{B} \leftarrow \mathcal{B}} = \ell_{\mathcal{B} \leftarrow \mathcal{B}} - \lambda id_{\mathcal{B} \leftarrow \mathcal{B}} = \ell_{\mathcal{B} \leftarrow \mathcal{B}} - \lambda \mathbb{1} \\ &\quad \text{is not invertible for any basis } \mathcal{B} \text{ of } V \\ &\iff \lambda \text{ is an eigenvalue of } \ell_{\mathcal{B} \leftarrow \mathcal{B}} \text{ for all bases } \mathcal{B} \text{ of } V \\ &\iff \det((\ell - \lambda id)_{\mathcal{B} \leftarrow \mathcal{B}}) = 0 \text{ for all bases } \mathcal{B} \text{ of } V \\ &\iff \det(\ell - \lambda id) = 0 \end{aligned}$$

Example 8.31. (a) The rotation $d \in \mathcal{L}(\mathbb{R}^2, \mathbb{R}^2)$ from Example 8.3 (e) has the determinant 1 since the associated matrix representation (8.12) w.r.t. the standard basis \mathcal{B} in \mathbb{R}^2 :

$$\det(d) = \det(d_{\mathcal{B} \leftarrow \mathcal{B}}) = \det \begin{pmatrix} \cos \varphi & -\sin \varphi \\ \sin \varphi & \cos \varphi \end{pmatrix} = (\cos \varphi)^2 + (\sin \varphi)^2 = 1.$$

For $\mathbb{F} = \mathbb{R}$, we only find eigenvalues and eigenvectors if φ is an integer multiple of π . For example, for $\varphi = \pi$, we have $d = -id$ and hence each vector in \mathbb{R}^2 is an eigenvector for the eigenvalue $\lambda = -1$.

(b) For the orthogonal projection $\text{proj}_G \in \mathcal{L}(\mathbb{R}^3, \mathbb{R}^3)$ onto the line $G := \text{Span}(\mathbf{n})$ and both variants

$$\text{proj}_E = id - \text{proj}_G \quad \text{and} \quad \text{refl}_E = id - 2 \text{proj}_G$$

from Example 8.7, 8.9, 8.14 (f) and 8.17 (a), we find with the help of equation (8.13):

$$\det(\text{proj}_G) = \det \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = 0$$

Using Example 8.17 (a), we get:

$$\det(\text{proj}_E) = \det \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = 0 \quad \text{and} \quad \det(\text{refl}_E) = \det \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = -1.$$

For proj_G each vector from G is an eigenvector for the eigenvalue 1, and each vector from E is an eigenvector for the eigenvalue 0 since E is the kernel of proj_G . For proj_E we have the same with $G \leftrightarrow E$. For refl_E each vector from G is an eigenvector for the eigenvalue -1 , and each vector from E is an eigenvector for the eigenvalue 1.

Summary

- A map ℓ from one \mathbb{F} -vector space V to another \mathbb{F} -vector space W is called *linear* if $\ell(\mathbf{x} + \mathbf{y}) = \ell(\mathbf{x}) + \ell(\mathbf{y})$ and $\ell(\alpha\mathbf{x}) = \alpha\ell(\mathbf{x})$ for all $\mathbf{x}, \mathbf{y} \in V$ and $\alpha \in \mathbb{F}$. We write: $\ell \in \mathcal{L}(V, W)$.
- Linear maps $\mathcal{L}(V, W)$ can be added and scaled with $\alpha \in \mathbb{F}$. Hence, $\mathcal{L}(V, W)$ gets an \mathbb{F} -vector space.
- The composition $k \circ \ell$ of linear maps $\ell : U \rightarrow V$ and $k : V \rightarrow W$ is linear.
- The inverse map of a bijective linear map is again linear. Therefore a bijective linear map is called an *isomorphism*.
- Each linear map $\ell \in \mathcal{L}(V, W)$, between finite dimensional vector spaces V and W , can be identified with a matrix. In order to do this, choose a basis $\mathcal{B} = (\mathbf{b}_1, \dots, \mathbf{b}_n)$ in V and a basis $\mathcal{C} = (\mathbf{c}_1, \dots, \mathbf{c}_m)$ in W . By using the basis isomorphisms $\Phi_{\mathcal{B}}$ and $\Phi_{\mathcal{C}}$, we get a linear map $\mathbb{F}^n \rightarrow \mathbb{F}^m$. Such a linear map is also represented by a $m \times n$ matrix $\ell_{\mathcal{C} \leftarrow \mathcal{B}} := (\ell(\mathbf{b}_1)^{\mathcal{C}} \cdots \ell(\mathbf{b}_n)^{\mathcal{C}})$. It is called the *matrix representation* of ℓ w.r.t. \mathcal{B} and \mathcal{C} .
- The matrix representation of $k + \ell$ is the sum of both matrix representations.
- The matrix representation of $\alpha\ell$ is α times the matrix representation of ℓ .
- The matrix representation of $k \circ \ell$ is the product of both matrix representations.
- The matrix representation of ℓ^{-1} is the inverse of the matrix representation of ℓ .
- Kernel and range of a linear map ℓ can be calculated by $\ell_{\mathcal{C} \leftarrow \mathcal{B}}$.
- By changing the basis of V from \mathcal{B} to \mathcal{B}' and changing the basis of W from \mathcal{C} to \mathcal{C}' , the matrix representation of $\ell : V \rightarrow W$ changes from $\ell_{\mathcal{C} \leftarrow \mathcal{B}}$ to $\ell_{\mathcal{C}' \leftarrow \mathcal{B}'}$. In this case, we have $\ell_{\mathcal{C}' \leftarrow \mathcal{B}'} = T_{\mathcal{C}' \leftarrow \mathcal{C}} \ell_{\mathcal{C} \leftarrow \mathcal{B}} T_{\mathcal{B} \leftarrow \mathcal{B}'}$.
- We call two matrices A and B *equivalent* and write $A \sim B$ if there are invertible matrices S and T with $B = SAT$.
- We have $A \sim B$ if and only if $\text{rank}(A) = \text{rank}(B)$.
- For the special case $\ell : V \rightarrow V$, one often chooses the same basis \mathcal{B} left and right. How does the matrix $\ell_{\mathcal{B} \leftarrow \mathcal{B}}$ change when changing the basis \mathcal{B} to \mathcal{B}' ? Then, we have $S = T^{-1}$ in the formula above.
- Two matrices A and B are called *similar* and one writes $A \approx B$ if there is an invertible matrix T with $B = T^{-1}AT$.
- From $A \approx B$ follows $\det(A) = \det(B)$ and $\text{spec}(A) = \text{spec}(B)$ but the converse is in general false.
- $\det(\ell)$ for a linear map $\ell : V \rightarrow V$ is defined by $\det(\ell_{\mathcal{B} \leftarrow \mathcal{B}})$ for any basis \mathcal{B} in V .
- $\lambda \in \mathbb{F}$ is an *eigenvalue* of $\ell : V \rightarrow V$ if $\ell(\mathbf{x}) = \lambda\mathbf{x}$ for some $\mathbf{x} \in V \setminus \{\mathbf{0}\}$.

9

Some matrix decompositions

The story so far: In the beginning the Universe was created. This has made a lot of people very angry and been widely regarded as a bad move.

Douglas Adams

In the previous chapters 7 and 8, we considered general vector spaces and linear maps between them. We could show that we are able to decode abstract vectors into vectors from \mathbb{F}^n or \mathbb{F}^m and also abstract linear maps into matrices from $\mathbb{F}^{m \times n}$. Therefore, even in this new general setting, it is important to know how to deal with matrices. Accordingly, in this chapter, we only consider matrices and want to find some simpler structures for a given matrix. We already observed such transformations or decompositions of a matrix into simpler forms:

- In Section 3.11.3, we discovered how to decompose a square matrix into a lower and an upper triangular matrix: $A = LU$. We also generalised this for rectangular matrices as $A = PLK$ where K is the row echelon form.
- In Section 5.5, we discovered how to decompose a square matrix A with full rank into an orthogonal matrix and an upper triangular matrix: $A = QR$. It is not hard to generalise this method for complex matrices and for rectangular matrices as well.
- In Section 6.7, we found that some matrices A can be decomposed into three parts

$$A = XDX^{-1}$$

where D is a diagonal matrix with eigenvalues on the diagonal and X consists of eigenvectors in the columns. Note that *diagonalisable* actually means *similar to a diagonal matrix*.

The mentioned diagonalisation aspect is the one we want to generalise. At first, we do this for all square matrices, which brings us to the Jordan normal form, and then we do this even for rectangular matrices, which brings us to the singular value decomposition.

9.1 Jordan normal form

We are searching for the best substitute of the usual diagonalisation $A = XDX^{-1}$ such that it works for all matrices $A \in \mathbb{C}^{n \times n}$. A good thing would be to use a triangular matrix instead of D if A is not diagonalisable. The next Proposition tells us that we only need some 1s above the diagonal:

Proposition & Definition 9.1. Jordan normal form

Let $A \in \mathbb{C}^{n \times n}$ with pairwise different eigenvalues $\lambda_1, \dots, \lambda_r \in \mathbb{C}$, where $\alpha_1, \dots, \alpha_r$ denote the corresponding algebraic multiplicities and $\gamma_1, \dots, \gamma_r$ the corresponding geometric multiplicities. Then, there is an invertible matrix $X \in \mathbb{C}^{n \times n}$ such that

$$A = XJX^{-1} \quad \text{or equivalently} \quad X^{-1}AX = J$$

and $J \in \mathbb{C}^{n \times n}$ has the following block diagonal form:

$$J = \begin{pmatrix} J_1 & & \\ & \ddots & \\ & & J_r \end{pmatrix}.$$

J is called a Jordan normal form (JNF) of A . The entries J_i are again block matrices, which are called Jordan blocks, and have the following structure:

$$J_i = \begin{pmatrix} J_{i,1} & & \\ & \ddots & \\ & & J_{i,\gamma_i} \end{pmatrix} \in \mathbb{C}^{\alpha_i \times \alpha_i},$$

where the matrices $J_{i,\ell}$ are called Jordan boxes and have the following form:

$$J_{i,\ell} = \begin{pmatrix} \lambda_i & 1 & & \\ & \lambda_i & \ddots & \\ & & \ddots & 1 \\ & & & \lambda_i \end{pmatrix}.$$

Note that $J_{i,\ell}$ could also be a 1×1 -matrix.

Example 9.2. If you have a matrix $A \in \mathbb{C}^{9 \times 9}$ and find an invertible matrix X with $A = XJX^{-1}$ such that

$$J = \left(\begin{array}{c|cc} \begin{array}{|cc|} \hline 4 & 1 \\ 4 & 1 \\ 4 & \\ \hline \end{array} & & \\ \hline & \begin{array}{|cc|} \hline 4 & 1 \\ & 4 \\ \hline \end{array} & \\ & & \overbrace{\hspace{10em}}^{J_2} \\ & & \begin{array}{|cc|} \hline -3 & 1 \\ -3 & \\ \hline \end{array} \\ & & \overbrace{\hspace{10em}}^{J_1} \\ & & \begin{array}{|cc|} \hline -3 & \\ & -3 \\ \hline \end{array} \\ & & \end{array} \right),$$

then you immediately find the following informations for A :

- On the diagonal of J , we find the eigenvalues A (counted with algebraic multiplicity), namely $\lambda_1 = 4$ and $\lambda_2 = -3$.
- The Jordan block J_1 , which corresponds to the eigenvalue $\lambda_1 = 4$, has the size 5×5 , hence $\alpha_1 = 5$. We conclude that $\lambda_1 = 4$ is an eigenvalue of A with algebraic multiplicity 5. The block J_1 consists of two boxes and therefore $\gamma_1 = 2$, which means the eigenspace for $\lambda_1 = 4$ is 2-dimensional. Here, the Jordan box $J_{1,1}$ has the size 3×3 and the Jordan box $J_{1,2}$ has the size 2×2 .
- The Jordan block J_2 , which corresponds to the eigenvalue $\lambda_2 = -3$, has the size 4×4 , hence $\alpha_2 = 4$. We conclude that $\lambda_2 = -3$ is an eigenvalue of A with multiplicity 4. The block J_2 owns three boxes, which means $\gamma_2 = 3$. Here, the box $J_{2,1}$ has the size 2×2 and the two boxes $J_{2,2}$ and $J_{2,3}$ have size 1×1 .

On the other hand, we learn that J is not determined solely by eigenvalues and multiplicities because also the matrix

$$\left(\begin{array}{cc|c} 4 & 1 & \\ & 4 & 1 \\ & & 4 & 1 \\ & & & 4 \\ \hline & & & \\ & & & 4 \\ \hline & & & \\ & & & -3 & 1 \\ & & & & -3 \\ \hline & & & & \\ & & & & -3 \\ & & & & \\ & & & & -3 \\ & & & & \\ & & & & -3 \end{array} \right)$$

would fit to these parameters above

$$\lambda_1 = 4, \alpha_1 = 5, \gamma_1 = 2, \quad \lambda_2 = -3, \alpha_2 = 4, \gamma_2 = 3.$$

Construction of J and X : How and why?

First we need the eigenvalues $\lambda_1, \dots, \lambda_r$ of A because on a triangular matrix they have to be on the diagonal counted with the algebraic multiplicities. So we also determine $\alpha_1, \dots, \alpha_r$. For each λ_i , we do the following procedure.

Rule of thumb: Treat the problem for all λ_i separately.

Each λ_i has its own Jordan block J_i and corresponding columns in X . Therefore, we can deal with the problem for each eigenvalue separately and put it together in the end.

Since $A \approx J$, we already know that the characteristic polynomial of A and J coincide (cf. Proposition 6.26). Hence, both matrices have the same eigenvalues with the same algebraic multiplicities. They have to be on the diagonal of J by Proposition 6.9.

Size of J_i

The Jordan block J_i for the eigenvalue λ_i has the size $\alpha_i \times \alpha_i$ because we need λ_i as often on the diagonal of J as the algebraic multiplicity says.

The n columns of X have to be linearly independent vectors from \mathbb{C}^n in order that X is invertible. Just looking at the $\alpha_i \times \alpha_i$ -block J_i , we need α_i columns from this matrix X . How to get them?

Recall that for the diagonalisation, in the case that A is diagonalisable, we had enough eigenvectors corresponding to the eigenvalue λ_i , which means vectors from $\text{Ker}(A - \lambda_i \mathbf{1})$. We could choose them as a linearly independent family because

$$\dim(\text{Ker}(A - \lambda_i \mathbf{1})) =: \gamma_i = \alpha_i.$$

In the case $\gamma_i < \alpha_i$ (which means A is not diagonalisable), we are missing some columns in X .

To shorten everything: $A - \lambda_i \mathbf{1} =: N$

Let us look at an example with $\alpha_i = 8$ and $\gamma_i = 4$. Choose $\mathbf{x}_1, \dots, \mathbf{x}_4 \in \text{Ker}(N)$, which are eigenvectors of A .

Ker(N)	\cdot	\cdot	\cdot	\cdot
dimension: $4 = \gamma_i$	$\mathbf{x}_{1,1}$	$\mathbf{x}_{2,1}$	$\mathbf{x}_{3,1}$	$\mathbf{x}_{4,1}$

We need $\alpha_i = 8$ linearly independent vectors for X but at this point we only have $\gamma_i = 4$. How to get the missing four vectors?

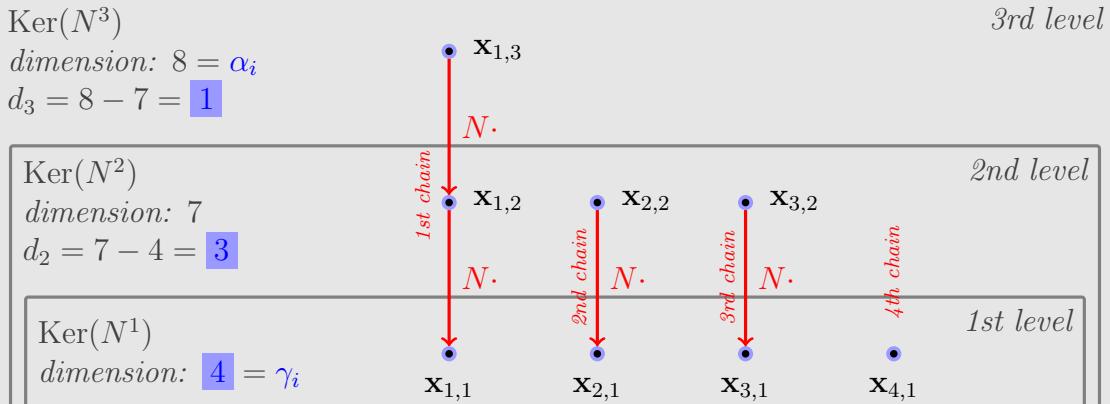
Answer: Since we have not found enough vectors in the kernel of N , we can look at the kernels of N^2, N^3, \dots until we have found 8 vectors in total. Clearly:

$$\text{Ker}(N) \subset \text{Ker}(N^2) \subset \text{Ker}(N^3) \subset \dots, \text{ since } N\mathbf{x} = \mathbf{o} \Rightarrow N^2\mathbf{x} = N(N\mathbf{x}) = \mathbf{o}, \dots$$

Recall: $\text{Ker}(N)$ has the dimension $\gamma_i = 4$. Suppose that $\text{Ker}(N^2)$ is of dimension 7 and that $\text{Ker}(N^3)$ has dimension $8 = \alpha_i$. The difference

$$d_k := \dim(\text{Ker}(N^k)) - \dim(\text{Ker}(N^{k-1}))$$

of the dimensions shows us where to find the four missing vectors. Elements from the spaces $\text{Ker}(N^k)$ are called generalised eigenvectors. To be more clear, we call an element from $\text{Ker}(N^k) \setminus \text{Ker}(N^{k-1})$ a generalised eigenvector of rank k . In this sense, the ordinary eigenvectors are now generalised eigenvector of rank 1.

Box 9.3. Levels and Jordan chains


As you can see in the picture, the vectors form “chains”, from top to bottom. We call each of these sequences a [Jordan chain](#) and it will be related to a Jordan box.

Box 9.4. Number and size of the Jordan boxes

Each Jordan chain ends at an ordinary eigenvector $\mathbf{x}_{j,1} \in \text{Ker}(N)$. Therefore, we have exactly γ_i Jordan boxes inside the chosen Jordan block J_i . The length of a Jordan chain is the size of the corresponding Jordan box. All sizes add up to α_i (here: 8), which is exactly the size of the Jordan block J_i .

Looking at our example, we have 4 Jordan boxes of size 3, 2, 2 and 1. Hence:

$$J_i = \text{Diag}\left(\begin{pmatrix} \lambda_i & 1 \\ & \lambda_i & 1 \\ & & \lambda_i \end{pmatrix}, \begin{pmatrix} \lambda_i & 1 \\ & \lambda_i \end{pmatrix}, \begin{pmatrix} \lambda_i & 1 \\ & \lambda_i \end{pmatrix}, (\lambda_i)\right) \in \mathbb{C}^{8 \times 8}.$$

At this point, we now know the whole block J_i . The next step is to find the corresponding columns of X , which means that we have to calculate the generalised eigenvectors $\mathbf{x}_{j,k}$:

Box 9.5. Generalised eigenvectors: Start the Jordan chain

The starting point $\mathbf{x}_{j,k}$ for the j th Jordan chain can be chose in an almost arbitrary way from the k th level: Let $\mathbf{x}_{j,k} \in \text{Ker}(N^k)$, but

$$\mathbf{x}_{j,k} \notin \text{Span}(\text{Ker}(N^{k-1}) \cup \{\mathbf{x}_{1,k}, \dots, \mathbf{x}_{j-1,k}\}), \quad (9.1)$$

where $\mathbf{x}_{1,k}, \dots, \mathbf{x}_{j-1,k}$ are the vectors from the chains before, 1 to $j-1$, which lie on the same level k . Now you can build the whole chain to the bottom $\mathbf{x}_{j,1}$. We just have to multiply with N in each step:

For $\mathbf{x} \in \text{Ker}(N^k)$, we have $N\mathbf{x} \in \text{Ker}(N^{k-1})$ since $\mathbf{o} = N^k\mathbf{x} = N^{k-1}(N\mathbf{x})$.

Note that equation (9.1) guarantees that all generalised eigenvectors on the k th level are linearly independent and that the linear independence remains on the levels below. All these α_i generalised eigenvectors are put as columns into X .

Box 9.6. Columns of X regarding λ_i

Let $X_i \in \mathbb{C}^{n \times \alpha_i}$ the matrix with columns filled out from left to right:

1st Jordan chain (bottom to top), \dots , γ_i th Jordan chain (bottom to top).

For our example, this means: $X_i = (\mathbf{x}_{1,1}, \mathbf{x}_{1,2}, \mathbf{x}_{1,3}, \mathbf{x}_{2,1}, \mathbf{x}_{2,2}, \mathbf{x}_{3,1}, \mathbf{x}_{3,2}, \mathbf{x}_{4,1}) \in \mathbb{C}^{n \times 8}$. After we did the whole procedure for all eigenvalues $\lambda_1, \dots, \lambda_r$, the only thing that remains is:

Put everything together

$$J := \begin{pmatrix} J_1 & & & \\ & \ddots & & \\ & & J_r & \end{pmatrix} \in \mathbb{C}^{n \times n} \quad \text{and} \quad X := (X_1, \dots, X_r) \in \mathbb{C}^{n \times n}. \quad (9.2)$$

This is all. Let us summarise the whole story:

Algorithm for calculating a Jordan normal form of A

Given: An arbitrary matrix $A \in \mathbb{C}^{n \times n}$.

Wanted: Jordan normal form J and X in $\mathbb{C}^{n \times n}$ with $A = XJX^{-1}$.

Algorithm

- Calculate all eigenvalues $\lambda_1, \dots, \lambda_r$ (pairwise distinct) of A and the algebraic multiplicities $\alpha_1, \dots, \alpha_r$.
- For $i = 1, \dots, r$:
 - Set $N := A - \lambda_i \mathbb{1}$.
 - Calculate $\text{Ker}(N), \text{Ker}(N^2), \dots, \text{Ker}(N^m)$ to $\dim(\cdot) = \alpha_i$.
 - Calculate all $d_k := \dim(\text{Ker}(N^k)) - \dim(\text{Ker}(N^{k-1}))$.
 - Draw the levels $1, \dots, m$ and Jordan chains. (Box ??)
 - Write down the Jordan block J_i . (Box 9.4)
 - Calculate all generalised eigenvectors. (Box 9.5)
 - Define X_i with all generalised eigenvectors. (Box 9.6)
- Set $J := \text{Diag}(J_1, \dots, J_r)$ and $X := (X_1, \dots, X_r)$ as in (9.2).

Why does this work? Let us look at the X -columns regarding one Jordan chain and its corresponding Jordan box. Choose the first chain from our example. The chain was given by $\mathbf{x}_{1,2} = N\mathbf{x}_{1,3}$ and $\mathbf{x}_{1,1} = N\mathbf{x}_{1,2}$. In this way, we get.

$$\mathbf{x}_{1,2} = N\mathbf{x}_{1,3} = (A - \lambda_1 \mathbb{1})\mathbf{x}_{1,3} = A\mathbf{x}_{1,3} - \lambda_1 \mathbf{x}_{1,3}, \quad \text{hence} \quad A\mathbf{x}_{1,3} = \mathbf{x}_{1,2} + \lambda_1 \mathbf{x}_{1,3}$$

and $\mathbf{x}_{1,1} = N\mathbf{x}_{1,2} = (A - \lambda_1 \mathbb{1})\mathbf{x}_{1,2} = A\mathbf{x}_{1,2} - \lambda_1 \mathbf{x}_{1,2}, \quad \text{hence} \quad A\mathbf{x}_{1,2} = \mathbf{x}_{1,1} + \lambda_1 \mathbf{x}_{1,2}$.

In summary:

$$\begin{aligned} A \begin{pmatrix} \mathbf{x}_{1,1} & \mathbf{x}_{1,2} & \mathbf{x}_{1,3} \end{pmatrix} &= \begin{pmatrix} A\mathbf{x}_{1,1} & A\mathbf{x}_{1,2} & A\mathbf{x}_{1,3} \end{pmatrix} = \begin{pmatrix} \lambda_i \mathbf{x}_{1,1} & \mathbf{x}_{1,1} + \lambda_i \mathbf{x}_{1,2} & \mathbf{x}_{1,2} + \lambda_i \mathbf{x}_{1,3} \end{pmatrix} \\ &= \begin{pmatrix} \mathbf{x}_{1,1} & \mathbf{x}_{1,2} & \mathbf{x}_{1,3} \end{pmatrix} \begin{pmatrix} \lambda_i & 1 & \\ & \lambda_i & 1 \\ & & \lambda_i \end{pmatrix} =: \begin{pmatrix} \mathbf{x}_{1,1} & \mathbf{x}_{1,2} & \mathbf{x}_{1,3} \end{pmatrix} J_{i,1}. \end{aligned}$$

By using the definition of the Jordan chain, we get the 1s above the diagonal in the matrix $J_{i,1}$. Only at the ordinary eigenvectors (here: $\mathbf{x}_{1,1}$), the chain stops. There, you do not find a 1 but only λ_i since $A\mathbf{x}_{1,1} = \lambda_i \mathbf{x}_{1,1}$.

By putting all Jordan boxes together into a Jordan block, we get γ_i equations (one per Jordan box), given by

$$A(\mathbf{x}_{j,1} \ \mathbf{x}_{j,2} \cdots \mathbf{x}_{j,k}) = (\mathbf{x}_{j,1} \ \mathbf{x}_{j,2} \cdots \mathbf{x}_{j,k}) J_{i,j}, \quad j = 1, \dots, \gamma_i,$$

one matrix equation $AX_i = X_i J_i$ for the i th Jordan block. The final assembling, cf. (9.2), of the Jordan blocks J_i to the whole matrix J gives us then $AX = XJ$, which is exactly the factorisation $A = XJX^{-1}$.

Now let us practise:

Example 9.7. Let

$$A = \begin{pmatrix} 5 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 4 \end{pmatrix}.$$

- The characteristic polynomial is

$$\det(A - \lambda \mathbb{1}) = (4 - \lambda)^3(1 - \lambda)^2.$$

We see that $\lambda_1 = 4$ with $\alpha_1 = 3$ and $\lambda_2 = 1$ with $\alpha_2 = 2$.

- Let us start the work (and fun) with the eigenvalue $\lambda_1 = 4$. For the matrix

$$N := A - \lambda_1 \mathbb{1} = A - 4\mathbb{1} = \begin{pmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & -3 & 0 & 0 & 0 \\ -1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

we get (after solving the LES $N\mathbf{x} = \mathbf{0}$) that

$$\text{Ker}(N) = \{\mathbf{x} = (-x_3, 0, x_3, 0, x_5)^\top : x_3, x_5 \in \mathbb{C}\}$$

and hence $\gamma_1 = \dim(\text{Ker}(N)) = 2$. Since $\alpha_1 = 3$, we have to calculate

$$N^2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 9 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 9 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

and we get:

$$\text{Ker}(N^2) = \{\mathbf{x} = (x_1, 0, x_3, 0, x_5)^\top : x_1, x_3, x_5 \in \mathbb{C}\}.$$

From this, we conclude $\dim(\text{Ker}(N^2)) = 3$. Now we have reached the algebraic multiplicity $\alpha_1 = 3$ and do not need to consider any higher powers of N , hence $m = 2$.

- For the differences of the dimension, we get

$$\begin{aligned} d_1 &:= \dim(\text{Ker}(N^1)) - \dim(\text{Ker}(N^0)) = 2 - 0 = 2, \\ d_2 &:= \dim(\text{Ker}(N^2)) - \dim(\text{Ker}(N^1)) = 3 - 2 = 1. \end{aligned}$$

Note that $\text{Ker}(N^0) = \text{Ker}(\mathbb{1}) = \{\mathbf{o}\}$ always have dimension 0.

- We have $m = 2$ levels whereas the second level owns $d_2 = 1$ vectors and the first level has $d_1 = 2$ vectors:

2nd level:	$\mathbf{x}_{1,2}$
1st level:	$\mathbf{x}_{1,1}$ $\mathbf{x}_{2,1}$

- Since we have a Jordan chain with length 2 and another one with length 1, we know that the first Jordan block J_1 has two Jordan blocks with different sizes:

$$J_1 = \left(\begin{array}{cc|c} 4 & 1 & \\ 0 & 4 & \\ \hline & & 4 \end{array} \right).$$

- Let us now find and correctly choose the generalised eigenvectors: We have to choose $\mathbf{x}_{1,2}$ from $\text{Ker}(N^2) \setminus \text{Ker}(N^1)$. Since we already calculated both kernels, we can choose $\mathbf{x}_{1,2} = (1, 0, 0, 0, 0)^\top$ for the top level of the first chain. We calculate the chain to the bottom: $\mathbf{x}_{1,1} := N\mathbf{x}_{1,2} = (1, 0, -1, 0, 0)^\top$.

Now, let us do the second chain. On the top level, we find $\mathbf{x}_{2,1} \in \text{Ker}(N^1)$. By choosing this vector, we have to respect equation (9.1), which means it is not a linear combination of vectors from $\text{Ker}(N^0) \cup \{\mathbf{x}_{1,1}\} = \{\mathbf{o}, \mathbf{x}_{1,1}\}$. In this case, this means that $\mathbf{x}_{2,1}$ comes from $\text{Ker}(N^1)$ and is not a multiple of $\mathbf{x}_{1,1}$. Hence, we choose $\mathbf{x}_{2,1} := (0, 0, 0, 0, 1)^\top$.

- We have finished the second chain and can give the matrix

$$X_1 = (\mathbf{x}_{1,1} \ \mathbf{x}_{1,2} \ \mathbf{x}_{2,1}) = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Now, we have done everything for the eigenvalue $\lambda_1 = 4$. Next thing is the eigenvalue $\lambda_2 = 1$.

- For the matrix

$$N := A - \lambda_2 \mathbb{1} = A - 1\mathbb{1} = \begin{pmatrix} 4 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 \end{pmatrix}$$

we get (after solving $N\mathbf{x} = \mathbf{0}$) that

$$\text{Ker}(N) = \{\mathbf{x} = (0, x_2, 0, x_4, 0)^\top : x_2, x_4 \in \mathbb{C}\}$$

and hence $\gamma_2 = \dim(\text{Ker}(N)) = 2$. Since $\alpha_2 = 2$, we do not need to calculate higher powers of N and set $m = 1$.

- We denote

$$d_1 = \dim(\text{Ker}(N^1)) - \dim(\text{Ker}(N^0)) = 2 - 0 = 2 \dots$$

- ...and get a bit boring picture with only $m = 1$ level and $d_1 = 2$ vectors:

1st level:	$\mathbf{x}_{1,1}$	$\mathbf{x}_{2,1}$
------------	--------------------	--------------------

Here, we see two chains with length 1.

- The Jordan block J_2 has two Jordan boxes of size 1 and looks like:

$$J_2 = \begin{pmatrix} 1 & \\ & 1 \end{pmatrix}.$$

- Let us determine the generalised eigenvectors: $\mathbf{x}_{1,1}$ comes from $\text{Ker}(N^1) \setminus \text{Ker}(N^0)$ and we could choose $\mathbf{x}_{1,1} = (0, 1, 0, 0, 0)^\top$. Now, for the second chain, choose $\mathbf{x}_{2,1} \in \text{Ker}(N^1)$ such that is not given by a linear combination of vectors from $\text{Ker}(N^0) \cup \{\mathbf{x}_{1,1}\} = \{\mathbf{0}, \mathbf{x}_{1,1}\}$, cf. (9.1). Let us set $\mathbf{x}_{2,1} = (0, 0, 0, 1, 0)^\top$.
- Hence, we have the matrix

$$X_2 = (\mathbf{x}_{1,1} \ \mathbf{x}_{2,1}) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$$

and also finished the work for the eigenvalue λ_2 .

- In summary, we get:

$$J = \begin{pmatrix} J_1 & \\ & J_2 \end{pmatrix} = \left(\begin{array}{c|ccccc} \boxed{4} & 1 & & & & \\ \hline 0 & 4 & & & & \\ & & \boxed{4} & & & \\ & & & \boxed{1} & & \\ & & & & \boxed{1} & \end{array} \right) \quad \text{and} \quad X = \begin{pmatrix} X_1 & X_2 \end{pmatrix} = \left(\begin{array}{cc|cc} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ \hline -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \end{array} \right),$$

hence, $A = X J X^{-1}$.

Video: Jordan normal form

Jordan normal form
<https://jp-g.de/bsom/la/jor/>

(4 videos)

$$J = \begin{pmatrix} 2 & & & \\ & 4 & 1 & \\ & 4 & 4 & \\ & & & 4 \end{pmatrix}$$


Corollary 9.8. Eigenvalues give determinant and trace

For $A \in \mathbb{C}^{n \times n}$, let $\lambda_1, \dots, \lambda_n$ be the eigenvalues counted with algebraic multiplicities. Then

$$\det(A) = \prod_{i=1}^n \lambda_i \quad \text{and} \quad \text{tr}(A) = \sum_{i=1}^n \lambda_i,$$

where $\text{tr}(A) := \sum_{j=1}^n a_{jj}$ is the sum of the diagonal, the so-called trace of A .

9.2 Singular value decomposition

In the diagonalisation and in the Jordan decomposition, we had three parts in the form $A = UDV$. There we had

U and V are inverse to each other.

Now, if we drop that condition, we can actually fulfil the following two properties even for rectangle matrices

- (1) *D is diagonal,*
- (2) *U and V are unitary square matrices.*

Since we allow U and V to be any square matrices, which means $U \in \mathbb{F}^{n \times n}$ and $V \in \mathbb{F}^{m \times m}$, we can consider an arbitrary rectangular matrix $A \in \mathbb{F}^{m \times n}$.

We will later show that each matrix A has such a decomposition. Since we want to use the common notations, we will denote the diagonal matrix D by Σ and use V^* on the right-hand side instead of V . We will see below why this is indeed a good idea. Hence, the wanted decomposition of A is now written as $A = U\Sigma V^*$ and looks like this:

Singular value decomposition of A

$$\begin{array}{c}
 \begin{array}{c} n \\ m \end{array} & \boxed{A} & = & \begin{array}{c} m \\ m \end{array} & \boxed{U} & \cdot & \begin{array}{c} m \\ m \end{array} & \boxed{\Sigma} & \cdot & \begin{array}{c} n \\ n \end{array} & \boxed{V^*} \\
 & \text{arbitrary} & & & \text{unitary} & & & \text{diagonal} & & & \text{unitary}
 \end{array} \quad (9.3)$$

The word “diagonal” for a rectangular matrix Σ is of course not literally correct. It means the following here:

$$\begin{array}{ccc}
 \Sigma = \begin{array}{c} n \\ m \end{array} & \boxed{s_1 \quad \ddots \quad s_n} & \text{or} & \Sigma = \begin{array}{c} n \\ m \end{array} & \boxed{s_1 \quad \ddots \quad s_m} & , \\
 & \text{if } m \geq n & & & \text{if } m \leq n & \quad (9.4)
 \end{array}$$

where every other entry is zero.

The equation $A = U\Sigma V^*$ tells us that A and Σ are equivalent matrices, $A \sim \Sigma$. The matrix A is the matrix representation of the linear map $\ell := f_A : \mathbb{F}^n \rightarrow \mathbb{F}^m$ given by $\mathbf{x} \mapsto A\mathbf{x}$ with respect to the standard bases \mathcal{B} in \mathbb{F}^n and \mathcal{C} in \mathbb{F}^m . The change of basis to an ONB $\mathcal{V} = (\mathbf{v}_1, \dots, \mathbf{v}_n)$ in \mathbb{F}^n and an ONB $\mathcal{U} = (\mathbf{u}_1, \dots, \mathbf{u}_m)$ in \mathbb{F}^m gives us another matrix representation $\ell_{\mathcal{U} \leftarrow \mathcal{V}}$ which is the “diagonal” matrix from (9.4):

$$\underbrace{\ell_{\mathcal{C} \leftarrow \mathcal{B}}}_{A} = \underbrace{T_{\mathcal{C} \leftarrow \mathcal{U}}}_{U} \underbrace{\ell_{\mathcal{U} \leftarrow \mathcal{V}}}_{\Sigma} \underbrace{T_{\mathcal{V} \leftarrow \mathcal{B}}}_{V^*}$$

with

$$U = \begin{pmatrix} | & & | \\ \mathbf{u}_1 & \cdots & \mathbf{u}_m \\ | & & | \end{pmatrix} = T_{\mathcal{C} \leftarrow \mathcal{U}}, \quad V = \begin{pmatrix} | & & | \\ \mathbf{v}_1 & \cdots & \mathbf{v}_n \\ | & & | \end{pmatrix} = T_{\mathcal{B} \leftarrow \mathcal{V}}, \quad V^* = V^{-1} = T_{\mathcal{V} \leftarrow \mathcal{B}}.$$

Because of $A \sim \Sigma$ and the characterisation of equivalences given by Proposition 8.20, we know $\text{rank}(\Sigma) = \text{rank}(A) =: r \leq m, n$. Hence, exactly r of the entries s_i in (9.4) are non-zero. Of course, we can choose $\mathbf{u}_i, \mathbf{v}_i$ in such an order that we have s_1, \dots, s_r as the non-zero elements.

Hence, we can see the matrix Σ from (9.4) as the following matrix representation:

$$\Sigma = \ell_{\mathcal{U} \leftarrow \mathcal{V}} = \begin{array}{c} \mathbf{v}_1 \quad \cdots \quad \mathbf{v}_r \quad \mathbf{v}_{r+1} \quad \cdots \quad \mathbf{v}_n \\ \hline \mathbf{u}_1 \quad \vdots \quad \mathbf{u}_r \quad \hline \mathbf{u}_{r+1} \quad \vdots \quad \mathbf{u}_m \end{array} \left(\begin{array}{c|c} s_1 & & & \\ \hline & \ddots & & \\ \hline & & s_r & \\ \hline & & & \end{array} \right) \in \mathbb{F}^{m \times n} \quad (9.5)$$

Multiplying $A = U\Sigma V^*$ from the right with V gets us $AV = U\Sigma$. Let us look at this in more detail:

$$\begin{aligned} \begin{pmatrix} | & & | \\ A\mathbf{v}_1 & \dots & A\mathbf{v}_n \\ | & & | \end{pmatrix} &= A \begin{pmatrix} | & & | \\ \mathbf{v}_1 & \dots & \mathbf{v}_n \\ | & & | \end{pmatrix} = AV = U\Sigma \\ &= \begin{pmatrix} | & & | \\ \mathbf{u}_1 & \dots & \mathbf{u}_m \\ | & & | \end{pmatrix} \begin{pmatrix} s_1 & & & \\ \vdots & \ddots & & \\ s_r & & & \end{pmatrix} = \begin{pmatrix} | & & | & & | \\ s_1\mathbf{u}_1 & \dots & s_r\mathbf{u}_r & \mathbf{0} & \dots & \mathbf{0} \\ | & & | & & | & \\ & & & & & \end{pmatrix}. \end{aligned}$$

Therefore, we have:

$$A\mathbf{v}_1 = s_1\mathbf{u}_1, \quad A\mathbf{v}_2 = s_2\mathbf{u}_2, \quad \dots, \quad A\mathbf{v}_r = s_r\mathbf{u}_r, \quad A\mathbf{v}_{r+1} = \mathbf{0}, \quad \dots, \quad A\mathbf{v}_n = \mathbf{0}. \quad (9.6)$$

Analogously, we get from $A^* = (U\Sigma V^*)^* = V\Sigma^* U^*$ that $A^*U = V\Sigma^*$ and hence:

$$A^*\mathbf{u}_1 = \overline{s_1}\mathbf{v}_1, \quad \dots, \quad A^*\mathbf{u}_r = \overline{s_r}\mathbf{v}_r, \quad A^*\mathbf{u}_{r+1} = \mathbf{0}, \quad \dots, \quad A^*\mathbf{u}_m = \mathbf{0}. \quad (9.7)$$

From (9.5) or (9.6), we get:

Proposition 9.9. Kernel and range of A

$$\begin{aligned} \text{Ker}(A) &= \text{Ker}(\ell) = \text{Span}(\mathbf{v}_{r+1}, \dots, \mathbf{v}_n), & \text{Ker}(A)^\perp &= \text{Span}(\mathbf{v}_1, \dots, \mathbf{v}_r) \\ \text{Ran}(A) &= \text{Ran}(\ell) = \text{Span}(\mathbf{u}_1, \dots, \mathbf{u}_r), & \text{Ran}(A)^\perp &= \text{Span}(\mathbf{u}_{r+1}, \dots, \mathbf{u}_m) \end{aligned}$$

Please recognise the rank-nullity theorem $\dim(\text{Ker}(A)) + \dim(\text{Ran}(A)) = \dim(\mathbb{F}^n)$.

Of course, we see that the decomposition $A = U\Sigma V^*$ from (9.3) is useful as a representation of the corresponding linear map. We will later see how we can use this in applications. However, the question remains how to get U , V and Σ ?

Let us go back to the result: From $A = U\Sigma V^*$, we would get:

$$A^*A = (U\Sigma V^*)^*(U\Sigma V^*) = V\Sigma^*U^*U\Sigma V^* = V\Sigma^*\Sigma V^* \quad (9.8)$$

$$\text{and} \quad AA^* = (U\Sigma V^*)(U\Sigma V^*)^* = U\Sigma V^*V\Sigma^*U^* = U\Sigma\Sigma^*U^*. \quad (9.9)$$

Because of (9.5) and $\overline{s_i}s_i = s_i\overline{s_i} = |s_i|^2$, we have square matrices

$$\Sigma^*\Sigma = \begin{array}{|c|c|} \hline |s_1|^2 & & \\ \hline & \ddots & \\ \hline & & |s_r|^2 \\ \hline \end{array}_n \quad \text{and} \quad \Sigma\Sigma^* = \begin{array}{|c|c|} \hline |s_1|^2 & & \\ \hline & \ddots & \\ \hline & & |s_r|^2 \\ \hline \end{array}_m,$$

that are also diagonal. Hence, (9.8) and (9.9) show us the unitary diagonalisations of the square matrices AA^* and A^*A . Recall that both matrices are self-adjoint and have by Proposition 6.44 in fact an ONB consisting of eigenvectors. These orthonormal eigenvectors (w.r.t to standard inner product!) are chosen as the columns of the matrices U and V .

Therefore, we find the eigenvalues of A^*A on the diagonal of $\Sigma^*\Sigma$ and the eigenvalues of AA^* on the diagonal of $\Sigma\Sigma^*$. Interestingly, with the exception of $|m - n|$ zeros, these are the same eigenvalues: $|s_1|^2, \dots, |s_r|^2, 0, \dots, 0$. To see this, just consider the non-zero eigenvalues of A^*A and AA^* . So choose $\lambda \neq 0$ and calculate

$$A^*A\mathbf{v} = \lambda\mathbf{v} \quad \xrightarrow{\mathbf{u} := A\mathbf{v}} \quad AA^*\mathbf{u} = AA^*(A\mathbf{v}) = A(A^*A\mathbf{v}) = A(\lambda\mathbf{v}) = \lambda(A\mathbf{v}) = \lambda\mathbf{u},$$

hence $\text{spec}(A^*A) \subset \text{spec}(AA^*)$. The converse works the same. As you know from the homework, all eigenvalues are non-negative. This also fits in with the diagonal entries $|s_1|^2, \dots, |s_r|^2, 0, \dots, 0$ of $\Sigma^*\Sigma$ and $\Sigma\Sigma^*$. Actually, we can choose the number s_i as we want in \mathbb{F} as long as $|s_i|^2$ is the i th eigenvalue of A^*A or AA^* . A simple choice is, of course, s_1, \dots, s_r being real and positive numbers.

In summary, we now have everything for U, V and Σ :

Definition 9.10. Singular values and singular vectors

Let $A \in \mathbb{F}^{m \times n}$. The (non-negative) square roots from the eigenvalues of A^*A are called the singular values of A and we order them from highest to lowest (counted with multiplicities):

$$s_1 \geq s_2 \geq \dots \geq s_n \geq 0.$$

The vectors from an orthonormal family $(\mathbf{v}_1, \dots, \mathbf{v}_n)$ consisting of eigenvectors of A^*A , with the same order as for s_1^2, \dots, s_n^2 , are called the right-singular vectors of A . In the same way, the vectors from an orthonormal family $(\mathbf{u}_1, \dots, \mathbf{u}_m)$ consisting of eigenvectors of AA^* , with the same order as for s_1^2, \dots, s_n^2 , are called the left-singular vectors of A .

The factorisation

$$A = U\Sigma V^*,$$

given by $U = (\mathbf{u}_1 \cdots \mathbf{u}_m)$, $V = (\mathbf{v}_1 \cdots \mathbf{v}_n)$ and Σ from (9.4), is called the singular value decomposition of A . In short: SVD.

To summarise everything, let us state the whole algorithm:

Algorithm for the SVD of A

Given: An arbitrary matrix $A \in \mathbb{F}^{m \times n}$.

Wanted: Unitary matrices $U \in \mathbb{F}^{m \times m}$, $V \in \mathbb{F}^{n \times n}$ and diagonal matrix $\Sigma \in \mathbb{R}^{m \times n}$ for the singular value decomposition $A = U\Sigma V^*$ of A .

Algorithm:

- Calculate the matrix A^*A and all eigenvalues $\lambda_1, \dots, \lambda_n$ (counted with multiplicities) and use the ordering such that $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n \geq 0$.
- Find an ONB $(\mathbf{v}_1, \dots, \mathbf{v}_n)$ consisting of eigenvectors for A^*A .
- Define r for number of the last non-zero eigenvalue λ_i .
- For $i = 1, \dots, r$:
 - Set $s_i := \sqrt{\lambda_i}$.
 - Set $\mathbf{u}_i := \frac{1}{s_i} A \mathbf{v}_i$, cf. (9.6).
- Set $s_{r+1}, \dots, s_m := 0$.
- Add to $(\mathbf{u}_1, \dots, \mathbf{u}_r)$ a family $(\mathbf{u}_{r+1}, \dots, \mathbf{u}_m)$ such that it is an ONB of \mathbb{F}^m .
- Set $U := (\mathbf{u}_1 \cdots \mathbf{u}_m)$, $V := (\mathbf{v}_1 \cdots \mathbf{v}_n)$ and Σ as in (9.4) or (9.5).

Rule of thumb: Calculating the singular vectors

- Alternatively, one finds $\mathbf{u}_1, \dots, \mathbf{u}_m$ as the eigenvectors of AA^* . However, that is more costly than using \mathbf{v}_i if we already have them.
- In the case $m < n$, AA^* is smaller than A^*A , hence it is better to calculate the eigenvalues λ_i and eigenvectors \mathbf{u}_i of the matrix A^*A and to use (9.7) for getting \mathbf{v}_i .
- Depending on the application, the eigenvectors \mathbf{u}_i and \mathbf{v}_i for $i > r$ might not be important (cf. (9.11)).

Example 9.11. Consider the matrix $A = \begin{pmatrix} 1 & \sqrt{3} \\ -2 & 0 \end{pmatrix}$. We have $m = n = 2$ and $\mathbb{F} = \mathbb{R}$.

- Calculate

$$A^*A = \begin{pmatrix} 1 & -2 \\ \sqrt{3} & 0 \end{pmatrix} \begin{pmatrix} 1 & \sqrt{3} \\ -2 & 0 \end{pmatrix} = \begin{pmatrix} 5 & \sqrt{3} \\ \sqrt{3} & 3 \end{pmatrix}$$

and get $\det(A^*A - \lambda \mathbb{1}) = (5 - \lambda)(3 - \lambda) - 3 = \lambda^2 - 8\lambda + 12 = (\lambda - 2)(\lambda - 6)$.

The eigenvalues of A^*A are (in decreasing order) $\lambda_1 = 6$ and $\lambda_2 = 2$.

- The eigenvector \mathbf{v}_1 for $\lambda_1 = 6$: We solve the linear equation: $(A^*A - 6\mathbb{1})\mathbf{v}_1 = \mathbf{0}$,

$$\text{hence } \begin{pmatrix} -1 & \sqrt{3} \\ \sqrt{3} & -3 \end{pmatrix} \mathbf{v}_1 = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \iff \mathbf{v}_1 = \frac{1}{2} \begin{pmatrix} \sqrt{3} \\ 1 \end{pmatrix} \quad (\text{normalised}).$$

Eigenvector \mathbf{v}_2 for $\lambda_2 = 2$: We solve the equation $(A^*A - 2\mathbb{1})\mathbf{v}_2 = \mathbf{0}$,

$$\text{and get } \begin{pmatrix} 3 & \sqrt{3} \\ \sqrt{3} & 1 \end{pmatrix} \mathbf{v}_2 = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \iff \mathbf{v}_2 = \frac{1}{2} \begin{pmatrix} -1 \\ \sqrt{3} \end{pmatrix} \quad (\text{normalised}).$$

Both vectors \mathbf{v}_1 and \mathbf{v}_2 are automatically orthogonal (Proposition 6.43).

- Both eigenvalues, $\lambda_1 = 6$ and $\lambda_2 = 2$, are non-zero. Hence $r = 2$.

- Next thing, we calculate $s_1 := \sqrt{\lambda_1} = \sqrt{6}$ and

$$\mathbf{u}_1 := \frac{1}{s_1} A \mathbf{v}_1 = \frac{1}{\sqrt{6}} \begin{pmatrix} 1 & \sqrt{3} \\ -2 & 0 \end{pmatrix} \frac{1}{2} \begin{pmatrix} \sqrt{3} \\ 1 \end{pmatrix} = \frac{1}{2\sqrt{6}} \begin{pmatrix} 2\sqrt{3} \\ -2\sqrt{3} \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix},$$

- Then $s_2 := \sqrt{\lambda_2} = \sqrt{2}$ and

$$\mathbf{u}_2 := \frac{1}{s_2} A \mathbf{v}_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & \sqrt{3} \\ -2 & 0 \end{pmatrix} \frac{1}{2} \begin{pmatrix} -1 \\ \sqrt{3} \end{pmatrix} = \frac{1}{2\sqrt{2}} \begin{pmatrix} 2 \\ 2 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

- In summary, we get:

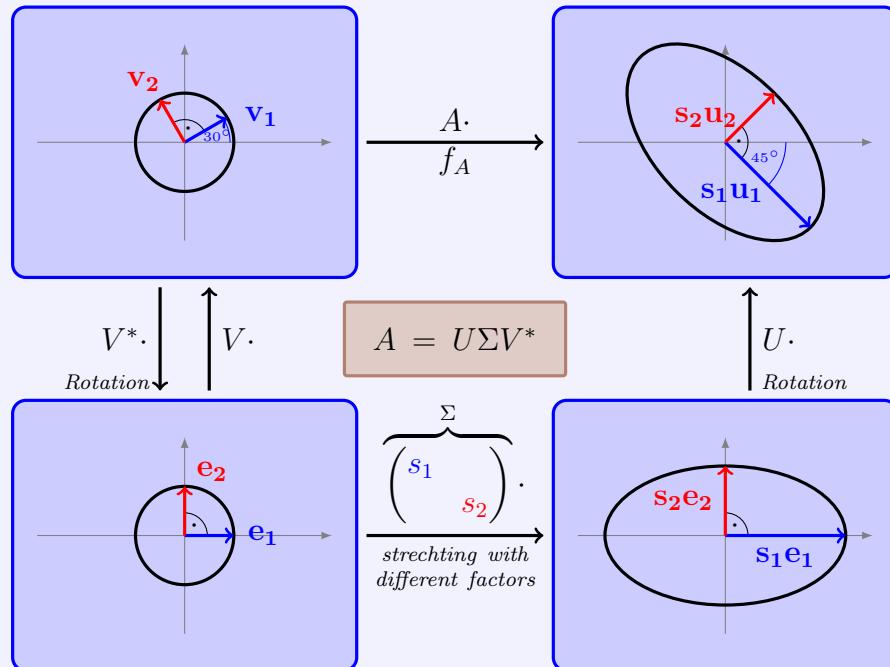
$$U = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}, \quad V = \frac{1}{2} \begin{pmatrix} \sqrt{3} & -1 \\ 1 & \sqrt{3} \end{pmatrix} \quad \text{and} \quad \Sigma = \begin{pmatrix} \sqrt{6} & 0 \\ 0 & \sqrt{2} \end{pmatrix}.$$

Because we started in $\mathbb{F} = \mathbb{R}$, we could do the whole calculation inside \mathbb{R} .

The unitary matrices U and V are indeed orthogonal matrices if all entries are real and, by our Definition in 5.29, describe rotations (if $\det(\cdot) = 1$) or reflections (if $\det(\cdot) = -1$). In the example above, both matrices are rotations: U rotates \mathbb{R}^2 by -45° and V rotates it by 30° .

In Chapter 3, we have seen that linear maps $f_A : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ with $\mathbf{x} \mapsto A\mathbf{x}$ can only stretch, rotate and reflect. Hence, a linear map changes the unit circle into an ellipse or it collapses into a line or point. By using the SVD, we can explain in more details what happens exactly. Let us look at our Example 9.11:

$A = U\Sigma V^*$ means: $f_A = \text{rotating, stretching, rotating}$



The factorisation $A = U\Sigma V^*$ decompose f_A in a composition into three maps:

- (1) a rotation by -30° (that is multiplication by $V^* = V^{-1}$),

- (2) stretching separately into two directions (with factors $s_1 = \sqrt{6}$ in direction of the x_1 -axis and with factor $s_2 = \sqrt{2}$ in direction of the x_2 -axis) and
(3) a rotation by -45° (multiplication by U).

The major axis and minor axis of the ellipse, which is construed by f_A from the unit circle, are given by the eigenvectors $\mathbf{u}_1, \mathbf{u}_2$ and the lengths are given by the singular values s_1 and s_2 .

The singular values s_i give us the *stretching factors* in certain (orthogonal) directions. For the largest singular value, s_1 , we have

$$s_1 = \|s_1 \mathbf{u}_1\| = \|A \mathbf{v}_1\| = \max\{\|A \mathbf{x}\| : \mathbf{x} \in \mathbb{F}^n, \|\mathbf{x}\| = 1\} =: \|A\|. \quad (9.10)$$

The here defined number $\|A\|$ is the already introduced *matrix norm* of A . It says how long the vector $A \mathbf{x} \in \mathbb{F}^m$ can be at most when $\mathbf{x} \in \mathbb{F}^n$ has length 1. The matrix norm fulfills the three properties of a norm.

Now we look at an important application of the SVD. We start with a calculation:

A as sum of r dyadic products

$$\begin{aligned} A &= U \Sigma V^* = U \left(\begin{array}{c|c} s_1 & \\ \hline & \ddots \\ & & s_r \end{array} \right) V^* = U \left(\begin{array}{c|c} s_1 & \\ \hline & \vdots \\ & & s_r \end{array} \right) + \cdots + \left(\begin{array}{c|c} & \\ \hline & \vdots \\ & & s_r \end{array} \right) V^* \\ &= U \left(\begin{array}{c|c} s_1 & \\ \hline & \vdots \end{array} \right) V^* + \cdots + U \left(\begin{array}{c|c} & \\ \hline & s_r \end{array} \right) V^* \\ &= s_1 \begin{pmatrix} | \\ \mathbf{u}_1 \\ | \end{pmatrix} (\mathbf{v}_1^* \cdot) + \cdots + s_r \begin{pmatrix} | \\ \mathbf{u}_r \\ | \end{pmatrix} (\mathbf{v}_r^* \cdot) = \sum_{i=1}^r s_i \mathbf{u}_i \mathbf{v}_i^* \quad (9.11) \end{aligned}$$

As we know, A has rank r . Each of the r terms in (9.11) has rank 1. Depending on the rate of decay of the singular values

$$s_1 \geq s_2 \geq \cdots \geq s_r > 0$$

we could omit some terms in the sum (9.11) without changing the matrix so much. We call this a *low-rank matrix approximation* of A .

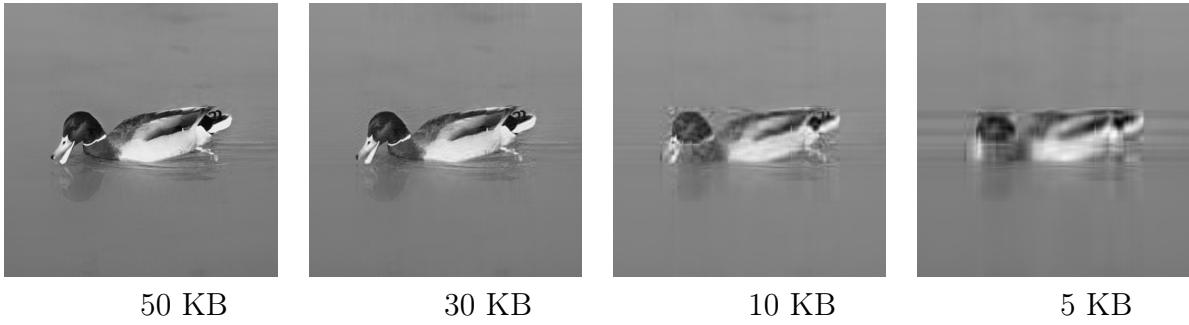
Example 9.12. Let us look at an 8-bit-grey picture with 500×500 pixels (which shows a beautiful duck on water):



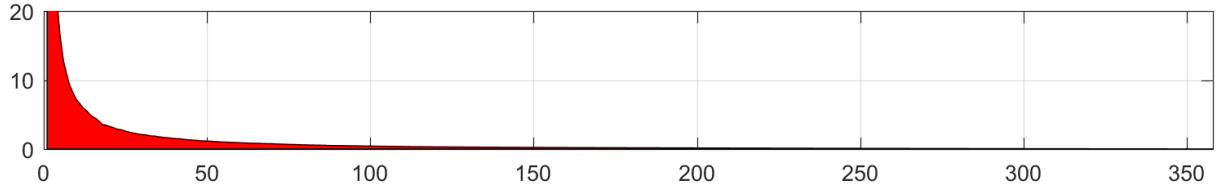
This can be saved as a matrix $A \in \mathbb{R}^{500 \times 500}$ where, in the entries, only integer values $0, 1, \dots, 255$ are allowed.

$$A = \begin{pmatrix} 1 & 3 & 7 & 5 & \dots & 16 & 8 \\ 3 & 7 & 3 & 3 & \dots & 11 & 12 \\ \vdots & \vdots & \vdots & \vdots & & & \\ 6 & 7 & \dots & 248 & \dots & 7 & 6 \\ \vdots & \vdots & \vdots & \vdots & & & \\ 4 & 8 & 8 & 5 & \dots & 4 & 8 \\ 4 & 8 & 3 & 3 & \dots & 6 & 6 \\ 2 & 3 & 3 & 4 & \dots & 9 & 9 \end{pmatrix} \quad 187 \text{ KB}$$

For calculations, we convert the number entries into the range $[0, 1]$ instead of $[0, 255]$. Most pictures should be full-rank matrices and here we can calculate the rank and actually get $r = n$. Now let us write A in the representation given by equation (9.11). Now we stop the summation instead of after $r = 500$ steps at $k = 50, 30, 10$ or 5 terms and we get the following pictures:



The decay of the singular values $s_1 \geq \dots \geq s_{500}$ below shows us why we already have at only 30 terms in the sum a very good approximation.



The first singular values ($s_1 \approx 262$, $s_2 \approx 28$ and $s_3 \approx 21$) are not shown in the picture, for obvious reason.

In Example 9.12, we have seen that a given matrix A with rank $r = 500$ can be well approximated by matrices A_k with rank $k = 50, 30, 10$ or 5 . For

$$A = \sum_{i=1}^r s_i \mathbf{u}_i \mathbf{v}_i^* \quad \text{and } k \in \{1, \dots, r\} \quad \text{we set} \quad A_k := \sum_{i=1}^k s_i \mathbf{u}_i \mathbf{v}_i^*.$$

A_k has rank k and is in fact the best $m \times n$ -matrix with rank k for the approximation of A . We measure the error of approximation by using the matrix norm in equation (9.10):

$$\|A - A_k\| = \left\| U \left(\begin{array}{c|c} s_1 & \\ \ddots & \\ s_k & \\ \hline \dots & \\ s_r & \end{array} \right) - \left(\begin{array}{c|c} s_1 & \\ \ddots & \\ s_k & \\ \hline \dots & \\ s_r & \end{array} \right) V^* \right\|$$

$$= \left\| U \begin{pmatrix} s_{k+1} & & & \\ & \ddots & & \\ & & s_r & \\ \hline & & & \end{pmatrix} V^* \right\| = s_{k+1} \text{ (largest singular value left).}$$

In short:

$$s_{k+1} = \text{distance of } A \text{ to the set of all matrices with rank } k \quad (9.12)$$

In particular, s_1 is the distance of A to the set of all matrices with rank 0, which consists only of the zero matrix 0.

At the end, let us take a look at the special case $m = n$, which means A and Σ are square matrices. In this case, eigenvalues and singular values are related in the following sense:

- A is invertible if and only if all the singular values are non-zero (see also Proposition 6.28 for the same claim with eigenvalues).

The smallest singular value of A , s_n , gives the distance of A to the set of all $n \times n$ -matrices with rank $n - 1$ or smaller (which are exactly the singular matrices) by equation (9.12).

The equation $A^{-1} = (U\Sigma V^*)^{-1} = V\Sigma^{-1}U^*$ gives the SVD of A^{-1} . Therefore, the singular values of A^{-1} are $1/s_1, \dots, 1/s_n$. The largest of these, meaning $1/s_n$, is $\|A^{-1}\|$.

- We know from Corollary 9.8 that the product of all eigenvalues of a given matrix A is exactly $\det(A)$. Since

$$\det(A) = \det(U\Sigma V^*) = \det(U) \det(\Sigma) \det(V^*) \Rightarrow |\det(A)| = \det(\Sigma),$$

we know that the product of all singular values, which is $\det(\Sigma)$, is equal to the absolute value of $\det(A)$.

- If A is normal, which means $A^*A = AA^*$, then A can be diagonalised by using a unitary matrix: $A = XDX^*$. Then $D = \text{diag}(d_1, \dots, d_n)$ is a diagonal matrix with the eigenvalues of A as entries and $X = (\mathbf{x}_1 \cdots \mathbf{x}_n)$ consists of eigenvectors for A . Hence $A^*A = XD^*DX^* = X \text{diag}(|d_1|^2, \dots, |d_n|^2) X^*$. The eigenvalues λ_i of A^*A are, on the one hand, given by $\lambda_i = \overline{d_i}d_i = |d_i|^2$ and, on the other hand, they can be written as $\lambda_i = s_i^2$ by using the singular values $s_i \geq 0$ of A . Therefore, we get:

$$s_i = |d_i|.$$

The singular values of A are exactly the absolute values of the eigenvalues of A .

Summary

- A lot of techniques in Linear Algebra deal with suitable factorisations of a given matrix A :
- From Section 3.11.5: The Gaussian elimination are summarised by a left multiplication with a lower triangular matrix Δ and a permutation matrix P . Hence, ΔPA is the row echelon form K of A and we have $PA = LK$ with lower triangular matrix $L := \Delta^{-1}$.

- From Section 5.5: A linearly independent family of vectors $(\mathbf{a}_1, \dots, \mathbf{a}_n)$ from \mathbb{F}^m can be transformed into an ONS $(\mathbf{q}_1, \dots, \mathbf{q}_n)$ by using the Gram-Schmidt procedure. Therefore, we have for $k = 1, \dots, n$ always $\mathbf{a}_k \in \text{Span}(\mathbf{q}_1, \dots, \mathbf{q}_k)$. For the matrices $A := (\mathbf{a}_1 \dots \mathbf{a}_n)$ and $Q := (\mathbf{q}_1 \dots \mathbf{q}_n)$ we find $A = QR$, where $R \in \mathbb{F}^{n \times n}$ is an invertible upper triangular matrix.
- If we decompose A into a product UDV , then we have different approaches.
- For diagonalisable matrices, we can choose $U = X$ and $V = X^{-1}$ where in X the columns are eigenvectors of A and form a basis. Then D has the eigenvalues of A on the diagonal, counted with multiplicities. See Chapter 6. We also know that selfadjoint and even normal matrices A are always diagonalisable, we can choose eigenvectors in such a way that they form an ONB, which means $X^* = X^{-1}$.
- For non-diagonalisable matrices we still can write $A = XDX^{-1}$ but now D is not diagonal. We use the Jordan normal form as a substitute. We get the important result that all (square) matrices $A \in \mathbb{C}^{n \times n}$ have such a Jordan normal form and therefore this decomposition. Note that we actually need the complex numbers here.
- For the singular value decomposition, the two matrices U and V are not connected such that we can also bring rectangular matrices A into “diagonal” structure. On the diagonal D (that is often denoted by Σ), we find the so-called *singular values* of A . The singular value decomposition is used for low rank approximation.

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