

# Introduction to Set Theory

“CS103 students”

“All the computers on the  
Stanford network”

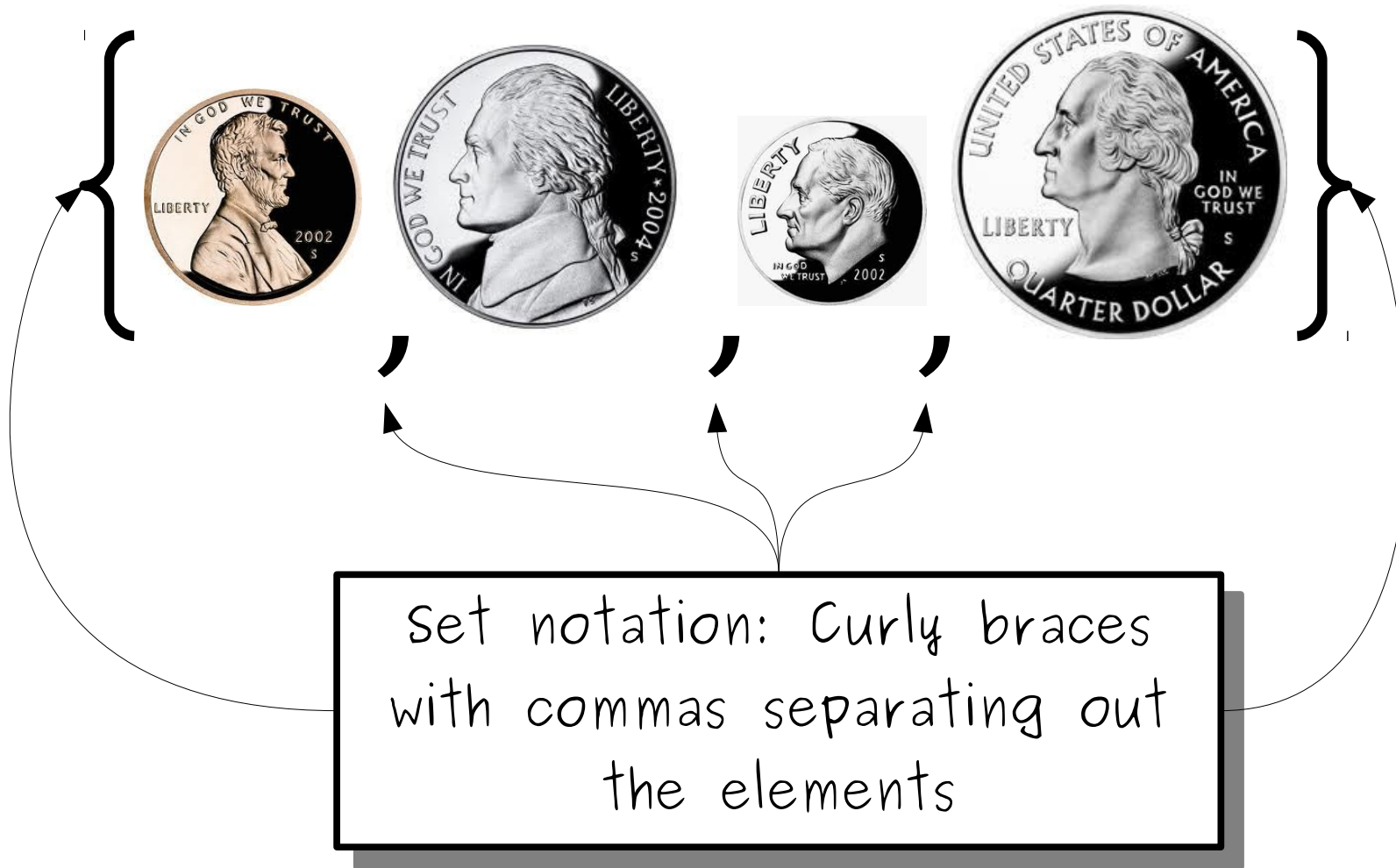
“Cool people”

“The chemical elements”

“Cute animals”

“US coins”

A ***set*** is an unordered collection of distinct objects, which may be anything (including other sets).



A **set** is an unordered collection of distinct objects, which may be anything (including other sets).



Two sets are equal when they have exactly the same contents, ignoring order.



These are the  
same set!



Sets cannot contain the same object twice.  
Repeated elements are ignored.

$\{ \}$  $=$  $\emptyset$ 

The **empty set**  
contains no elements.

We use this symbol to  
denote the empty set.

This is a  
number.

1

$\neq$

This is a set.  
It contains a  
number.

{ 1 }

Are these objects equal to one another?



This set contains  
nothing at all.

$\emptyset$

$\neq$

This set has one  
element, which  
happens to be the  
empty set.

$\{\emptyset\}$

Are these objects equal to one another?

# Membership



Is  in this set?

# Membership



Is



in this set?

# Set Membership

- Given a set  $S$  and an object  $x$ , we write

$$x \in S$$

if  $x$  is contained in  $S$ , and

$$x \notin S$$

otherwise.

- If  $x \in S$ , we say that  $x$  is an ***element*** of  $S$ .
- Given any object  $x$  and any set  $S$ , either  $x \in S$  or  $x \notin S$ .

# Infinite Sets

- Some sets contain *infinitely many* elements!
- The set  $\mathbb{N} = \{ 0, 1, 2, 3, \dots \}$  is the set of all the ***natural numbers***.
  - Some mathematicians don't include zero; in this class, assume that 0 is a natural number.
- The set  $\mathbb{Z} = \{ \dots, -2, -1, 0, 1, 2, \dots \}$  is the set of all the ***integers***.
  - Z is from German “Zahlen.”
- The set  $\mathbb{R}$  is the set of all ***real numbers***.
  - $e \in \mathbb{R}$ ,  $\pi \in \mathbb{R}$ ,  $4 \in \mathbb{R}$ , etc.

# Describing Complex Sets

- Here are some English descriptions of infinite sets:
  - “The set of all even numbers.”
  - “The set of all real numbers less than 137.”
  - “The set of all negative integers.”
- To describe complex sets like these mathematically, we'll use ***set-builder notation***.

# Even Natural Numbers

$$\{ n \mid n \in \mathbb{N} \text{ and } n \text{ is even} \}$$

# Even Natural Numbers

$$\{ n \mid n \in \mathbb{N} \text{ and } n \text{ is even} \}$$

The set of all  $n$

where

$n$  is a natural  
number

and  $n$  is even

$$\{ 0, 2, 4, 6, 8, 10, 12, 14, 16, \dots \}$$



# Set Builder Notation

- A set may be specified in ***set-builder notation***:

$$\{ x \mid \textit{some property } x \textit{ satisfies} \}$$

- For example:

$$\{ r \mid r \in \mathbb{R} \text{ and } r < 137 \}$$

$$\{ n \mid n \text{ is an even natural number} \}$$

$$\{ S \mid S \text{ is a set of US currency} \}$$

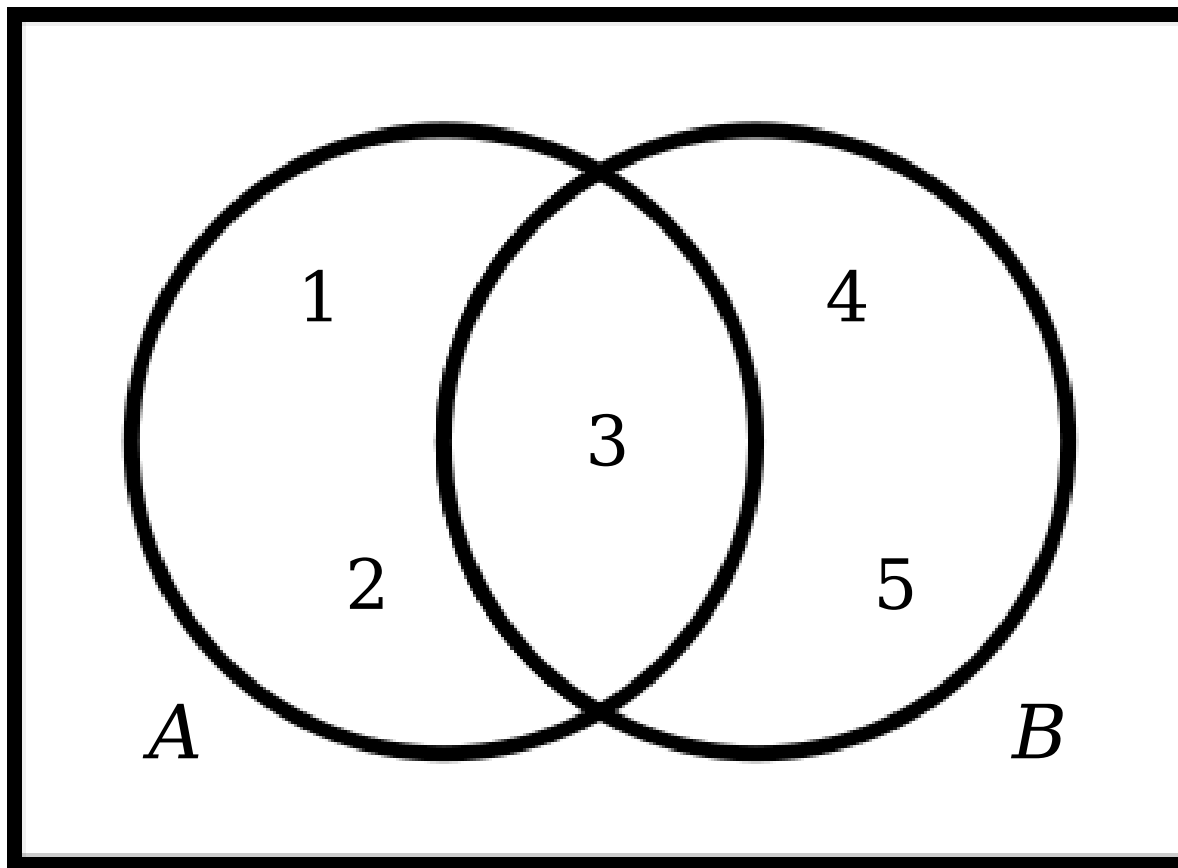
$$\{ a \mid a \text{ is cute animal} \}$$

$$\{ r \in \mathbb{R} \mid r < 137 \}$$

$$\{ n \in \mathbb{N} \mid n \text{ is odd} \}$$

# Combining Sets

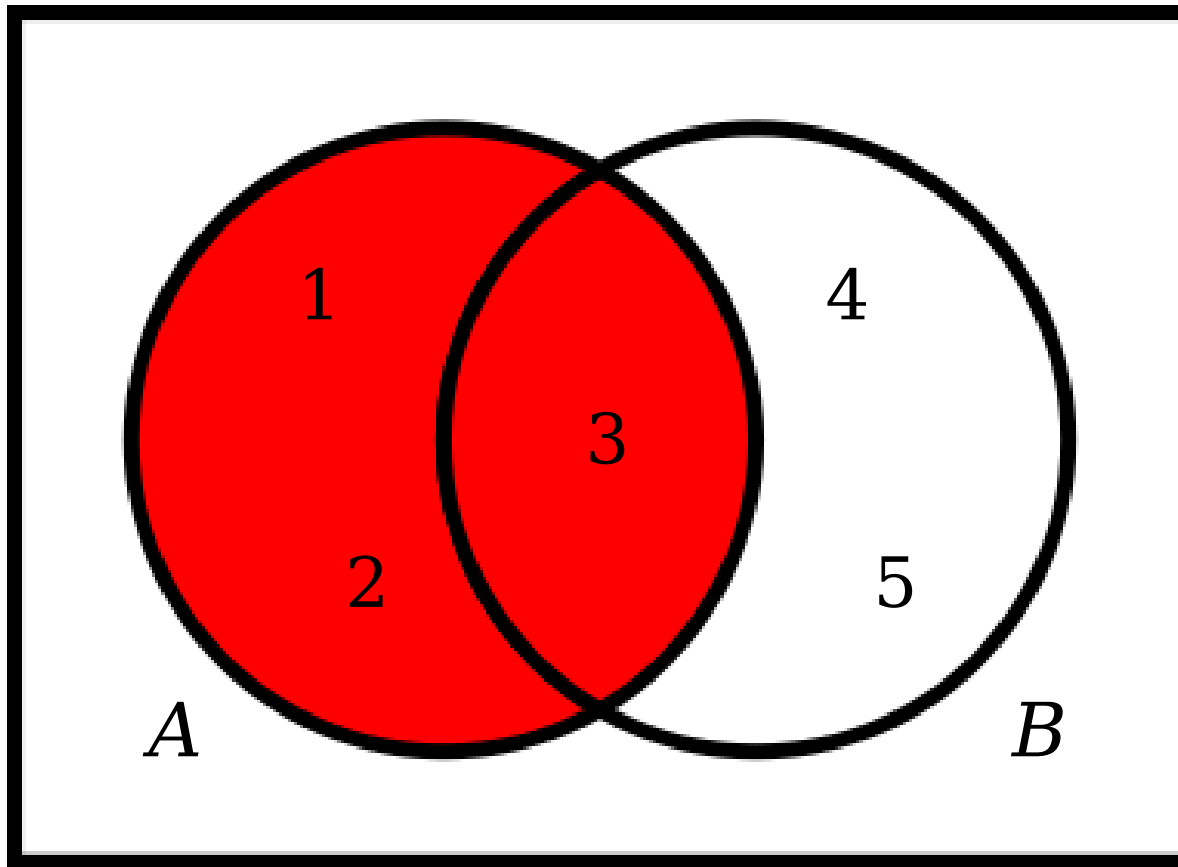
# Venn Diagrams



$$A = \{ 1, 2, 3 \}$$

$$B = \{ 3, 4, 5 \}$$

# Venn Diagrams

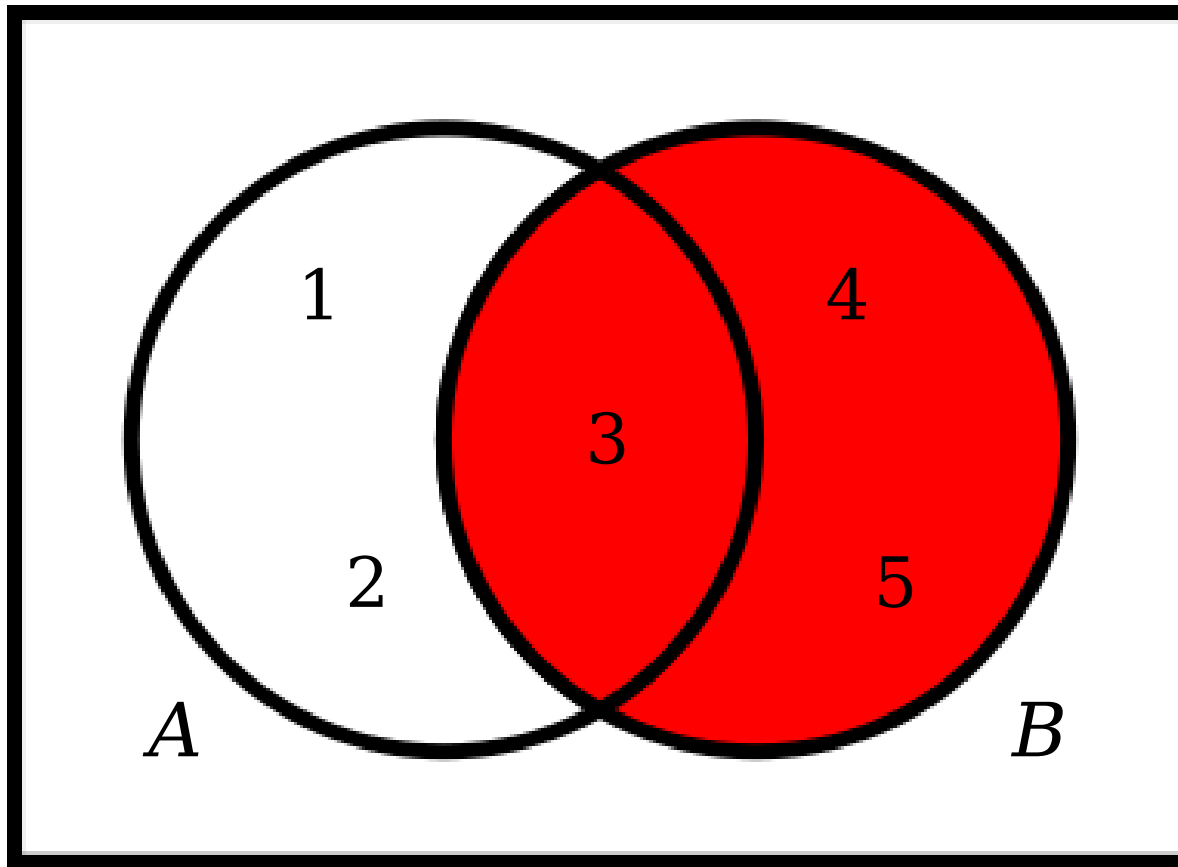


*A*

$$A = \{ 1, 2, 3 \}$$

$$B = \{ 3, 4, 5 \}$$

# Venn Diagrams

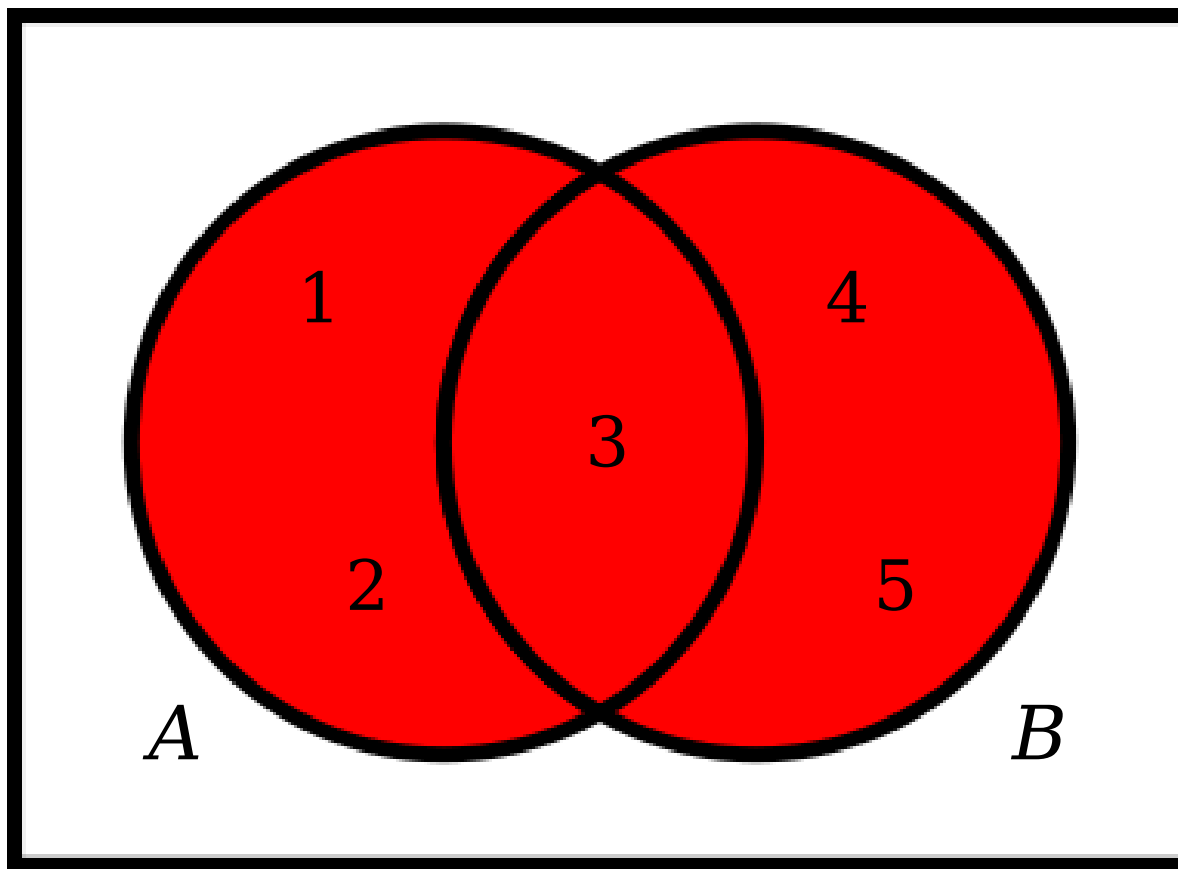


*B*

$$A = \{ 1, 2, 3 \}$$

$$B = \{ 3, 4, 5 \}$$

# Venn Diagrams

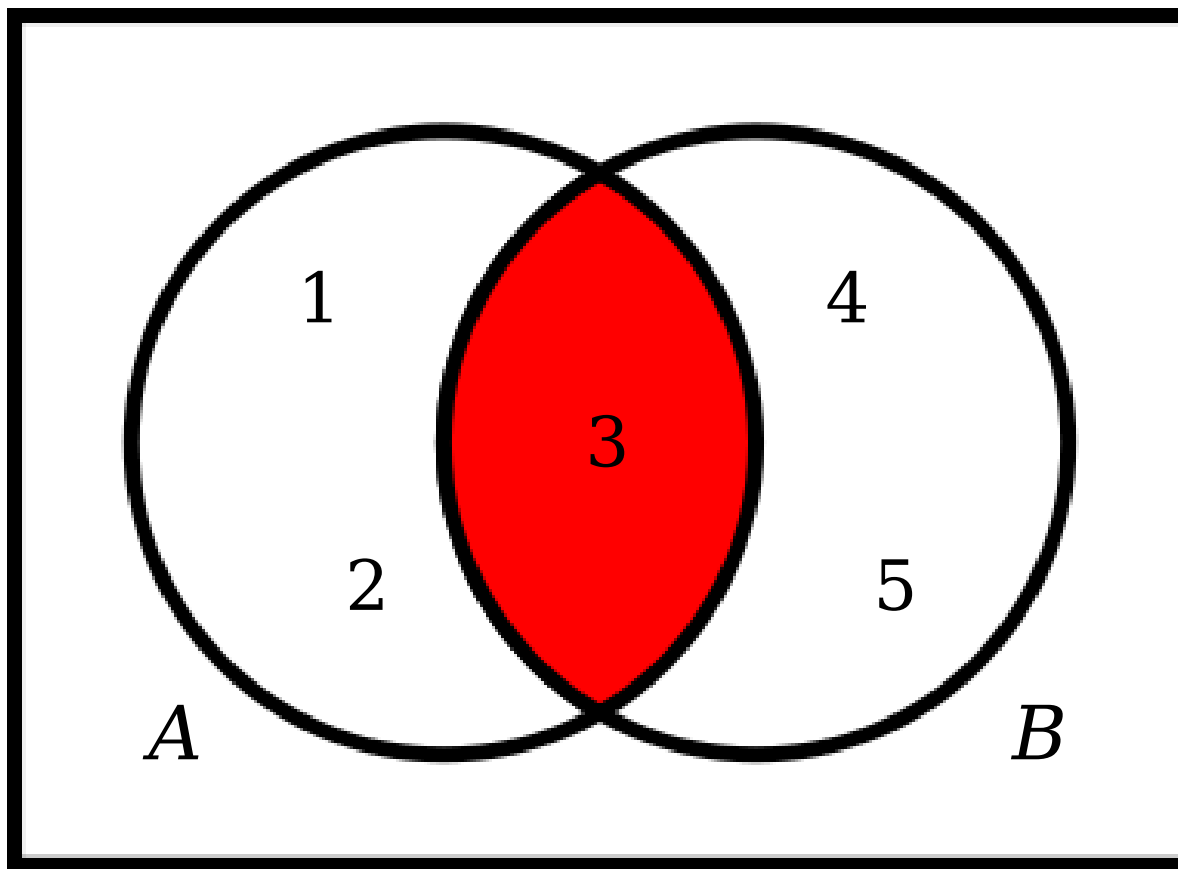


Union  
 $A \cup B$   
 $\{ 1, 2, 3, 4, 5 \}$

$$A = \{ 1, 2, 3 \}$$

$$B = \{ 3, 4, 5 \}$$

# Venn Diagrams



Intersection

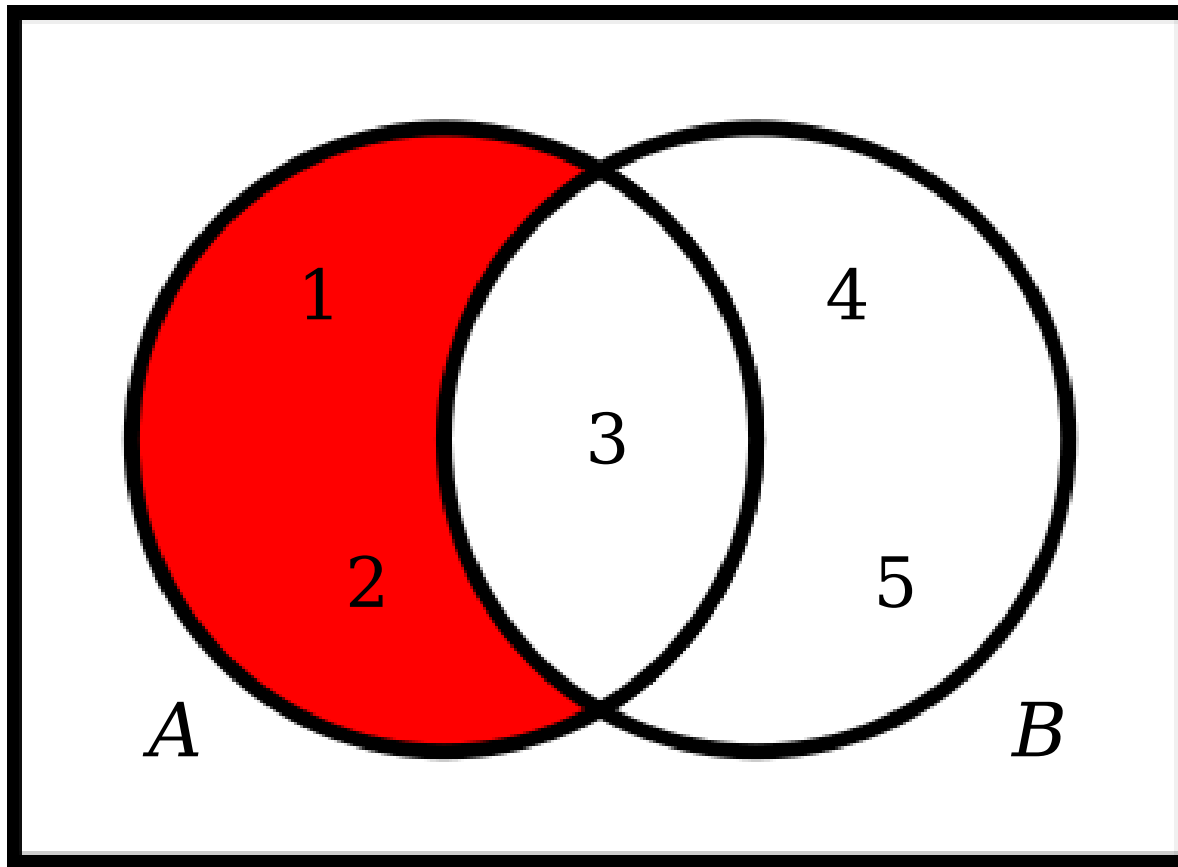
$$A \cap B$$

$$\{ 3 \}$$

$$A = \{ 1, 2, 3 \}$$

$$B = \{ 3, 4, 5 \}$$

# Venn Diagrams



Difference

$$A - B$$

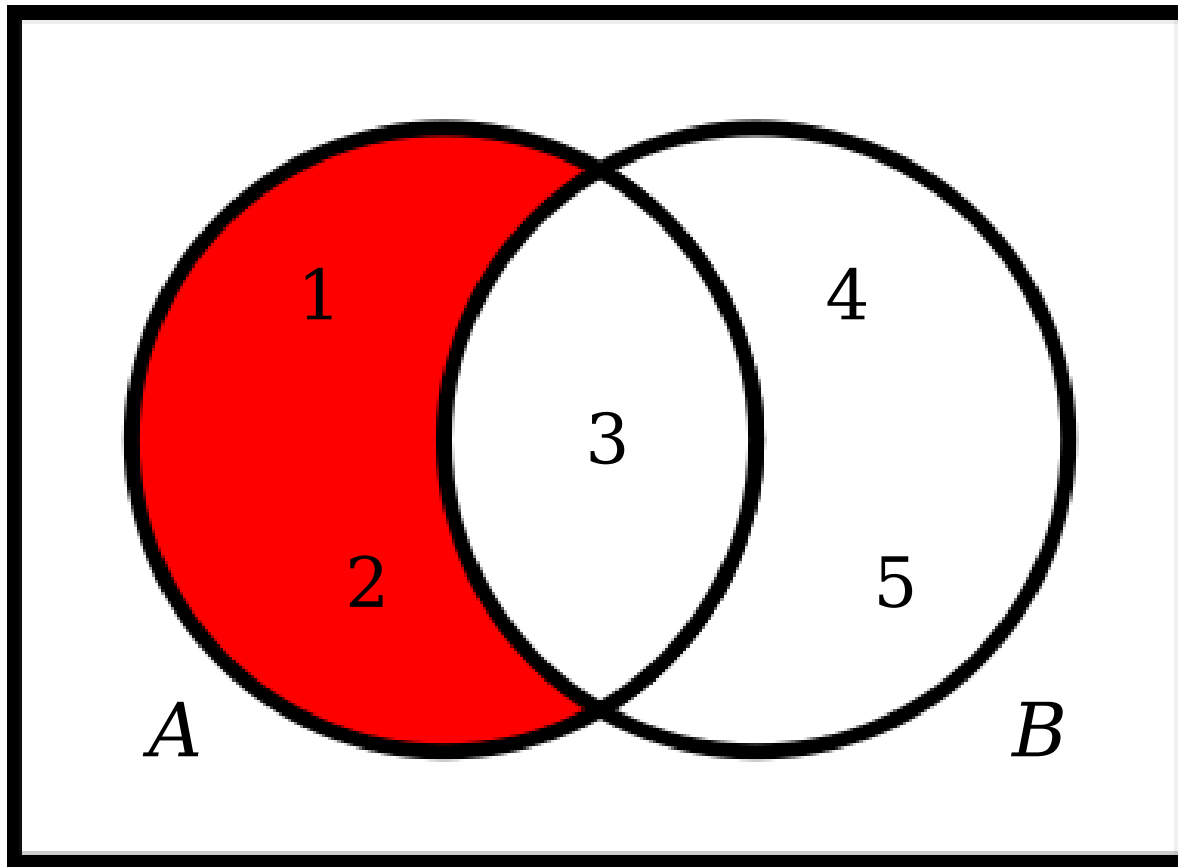
$$\{ 1, 2 \}$$

$$A = \{ 1, 2, 3 \}$$

$$B = \{ 3, 4, 5 \}$$



# Venn Diagrams



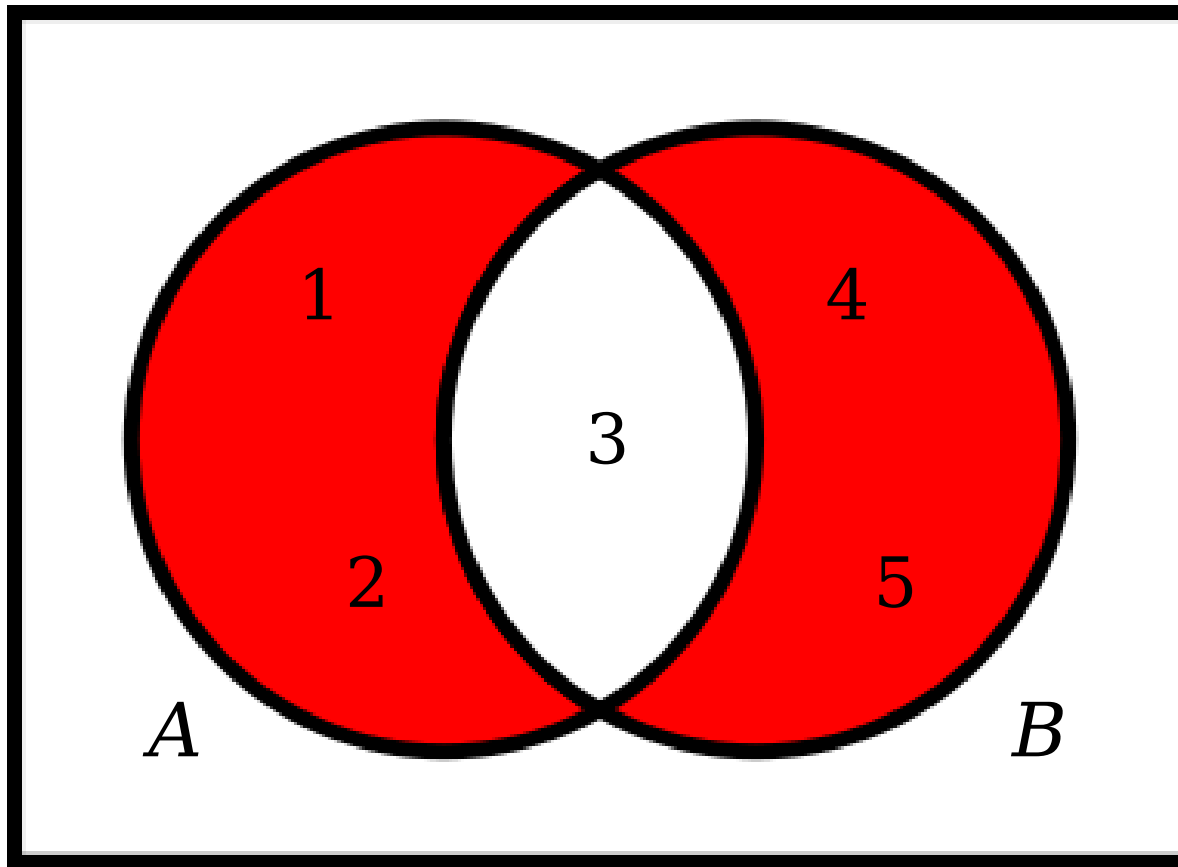
Difference

$$A \setminus B$$
$$\{ 1, 2 \}$$

$$A = \{ 1, 2, 3 \}$$

$$B = \{ 3, 4, 5 \}$$

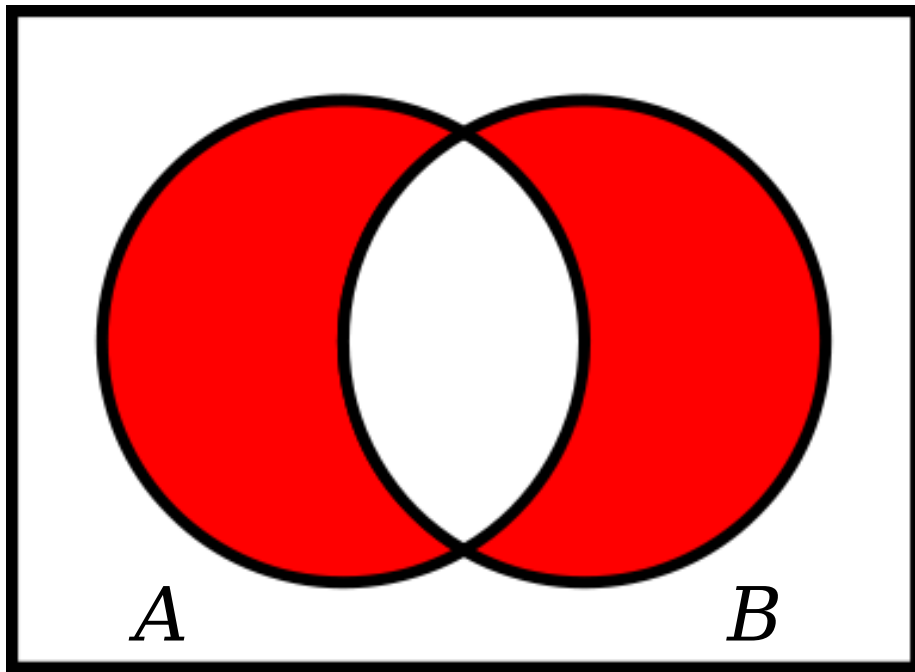
# Venn Diagrams



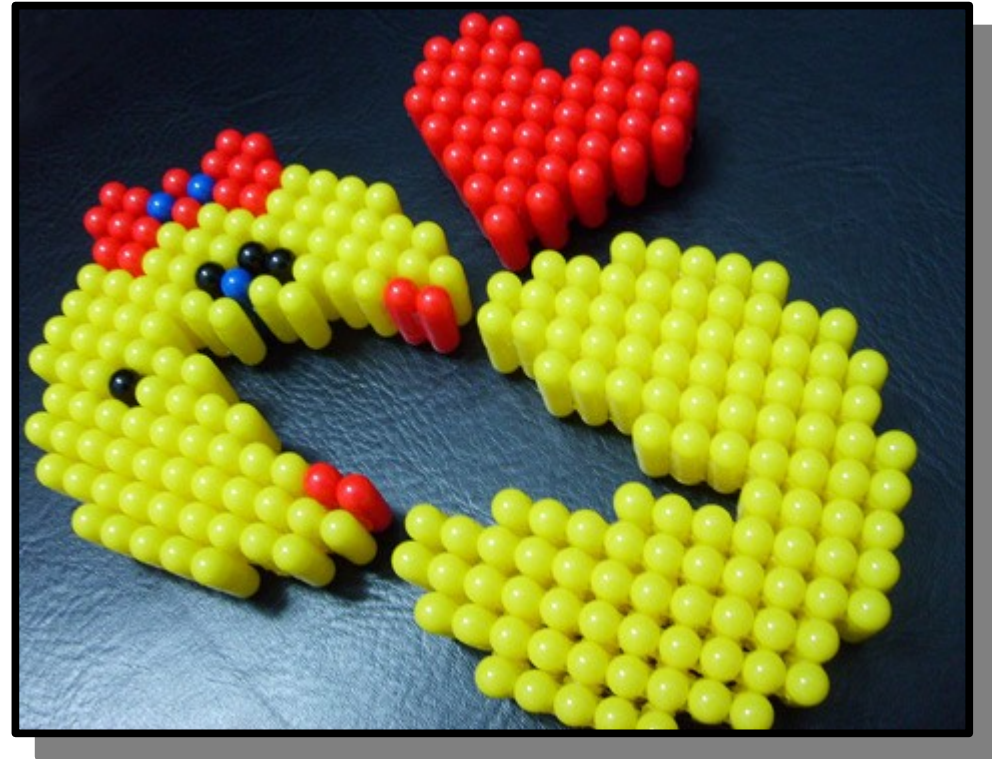
Symmetric  
Difference  
 $A \Delta B$   
 $\{ 1, 2, 4, 5 \}$

$$A = \{ 1, 2, 3 \}$$
$$B = \{ 3, 4, 5 \}$$

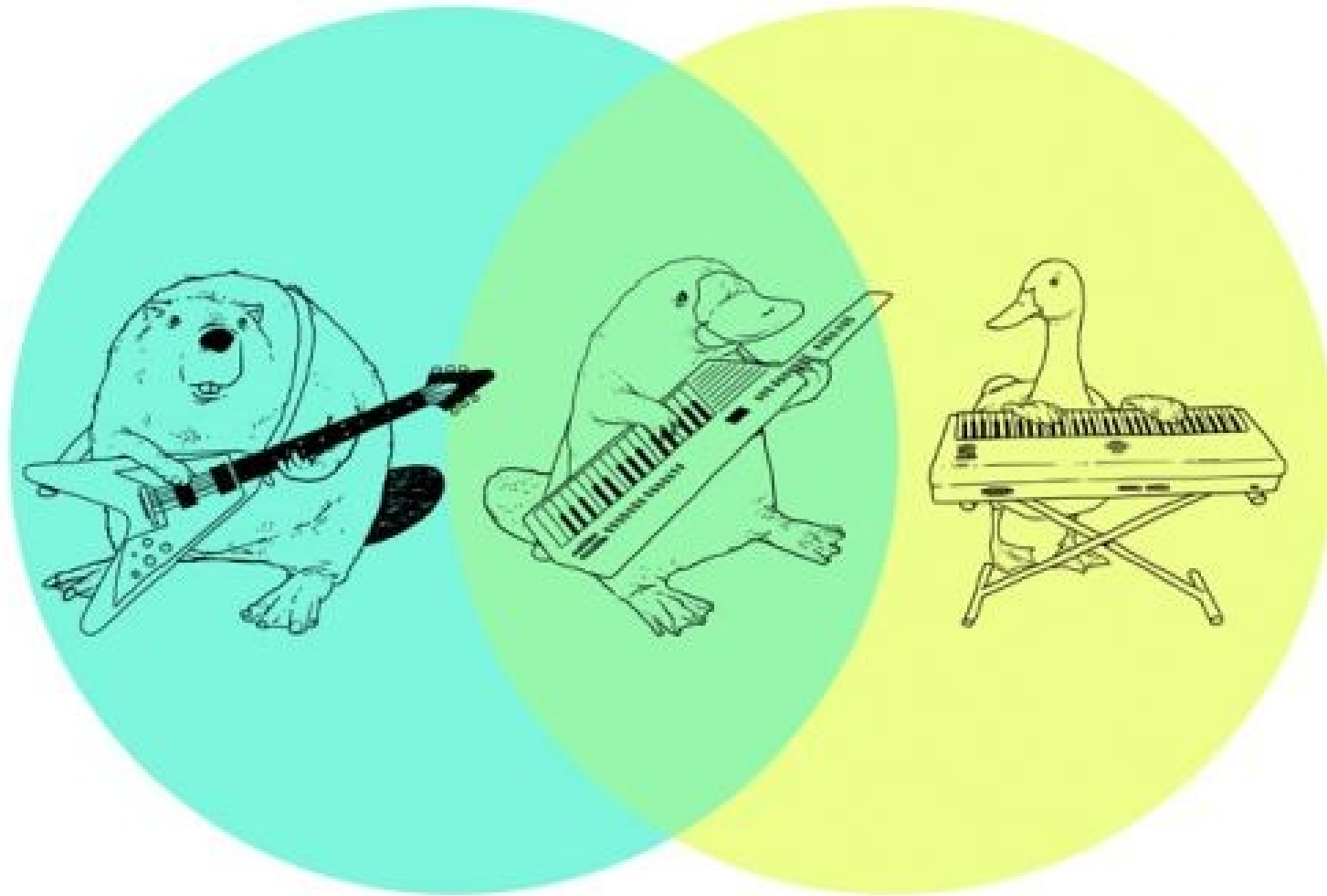
# Venn Diagrams



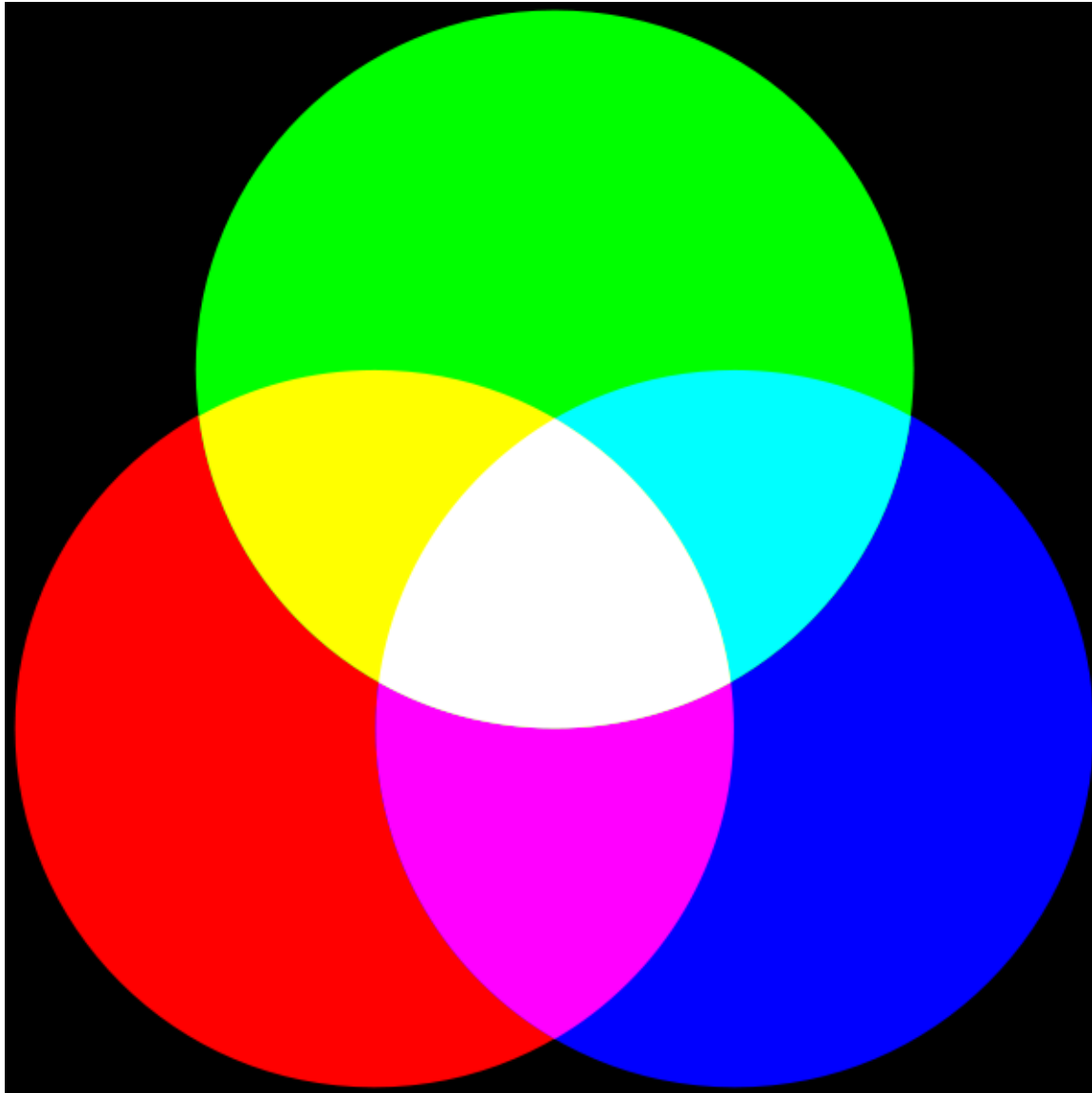
$$A \Delta B$$



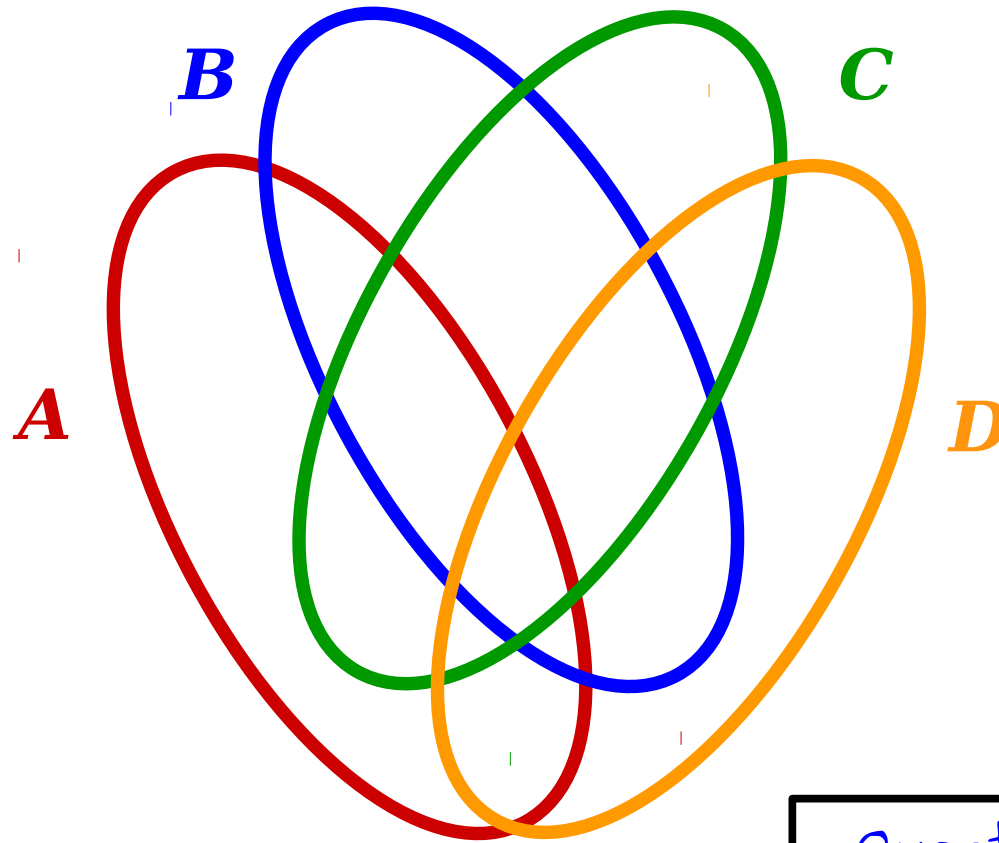
# Venn Diagrams



# Venn Diagrams for Three Sets

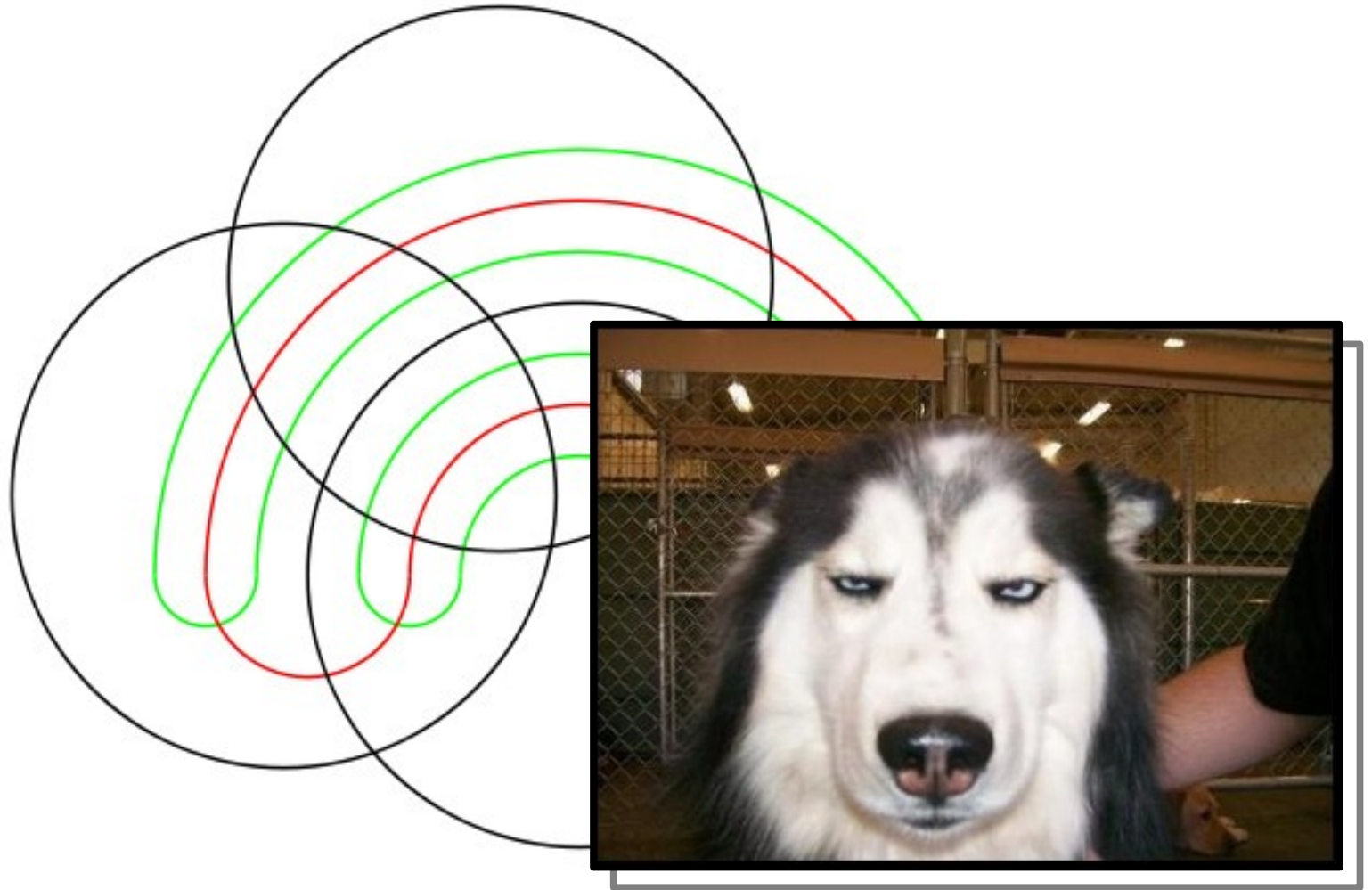


# Venn Diagrams for Four Sets



Question to ponder:  
why don't we just  
draw four circles?

# Venn Diagrams for Five Sets



# Venn Diagrams for Seven Sets

**<http://moebio.com/research/sevensets/>**



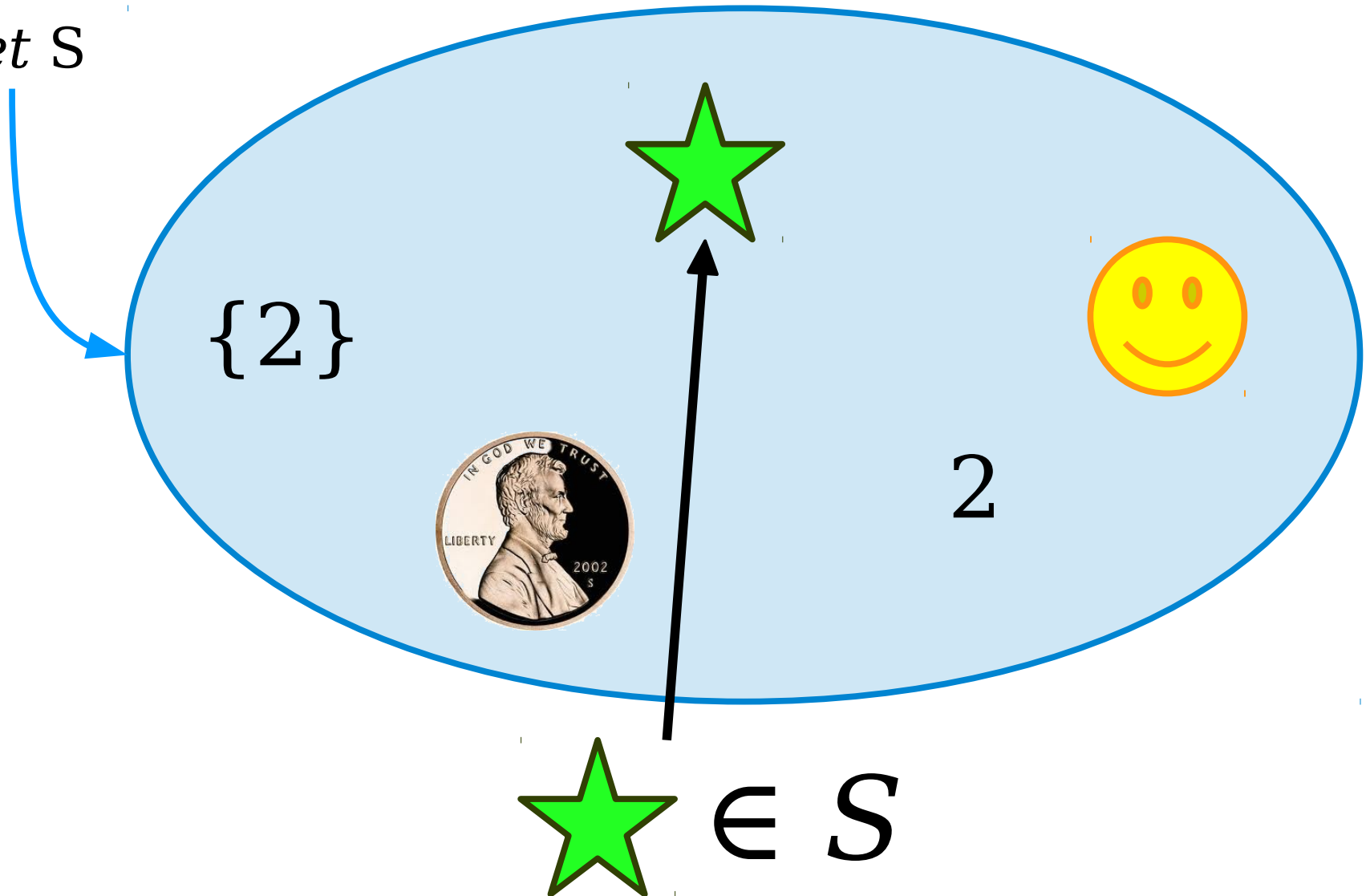
# Subsets and Power Sets

# Subsets

- A set  $S$  is called a **subset** of a set  $T$  (denoted  **$S \subseteq T$** ) if all elements of  $S$  are also elements of  $T$ .
- Examples:
  - $\{ 1, 2, 3 \} \subseteq \{ 1, 2, 3, 4 \}$
  - $\{ b, c \} \subseteq \{ a, b, c, d \}$
  - $\{ \text{H}, \text{He}, \text{Li} \} \subseteq \{ \text{H}, \text{He}, \text{Li} \}$
  - $\mathbb{N} \subseteq \mathbb{Z}$  (*every natural number is an integer*)
  - $\mathbb{Z} \subseteq \mathbb{R}$  (*every integer is a real number*)

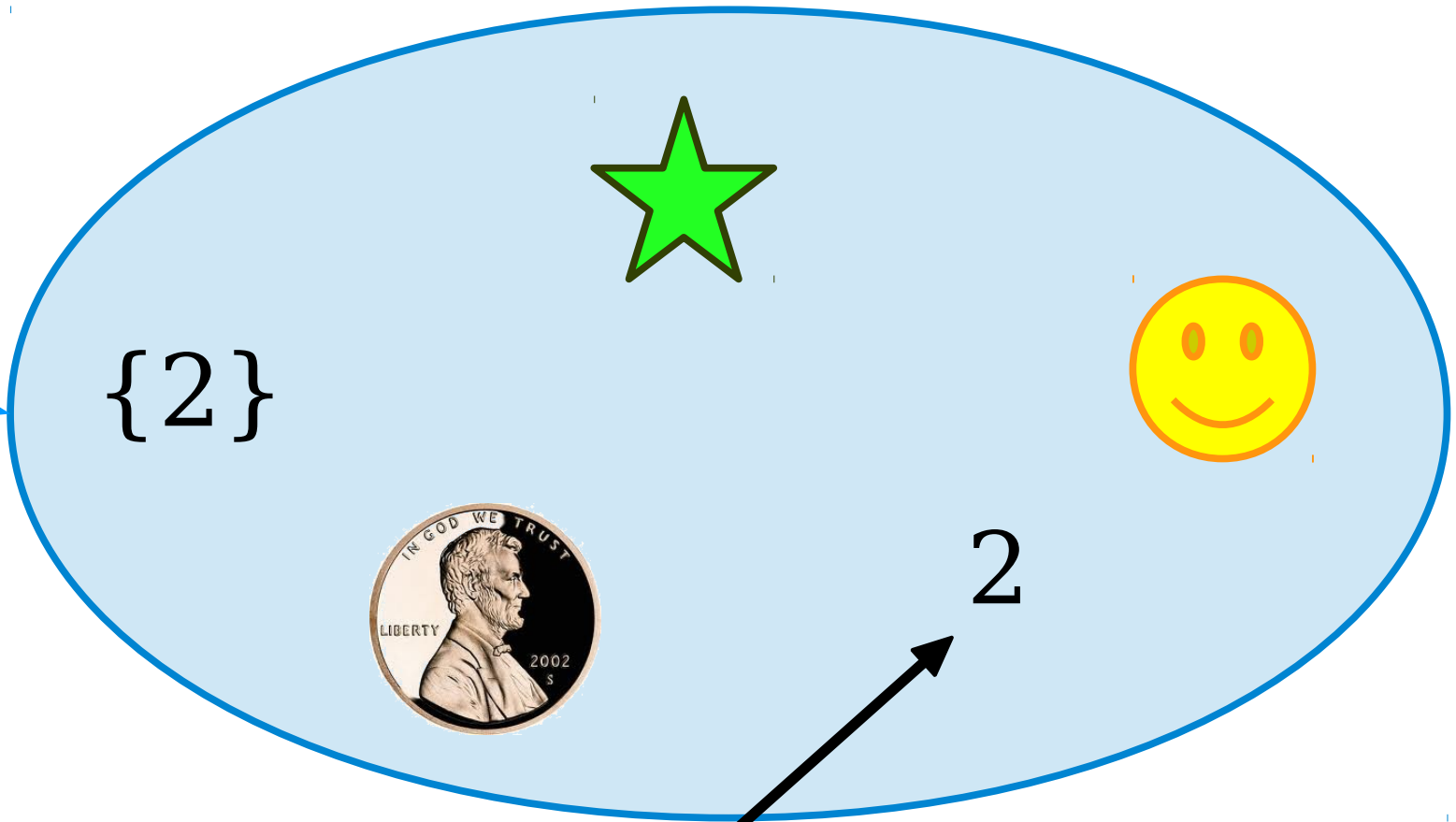
# Subsets and Elements

*Set S*



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*Set S*

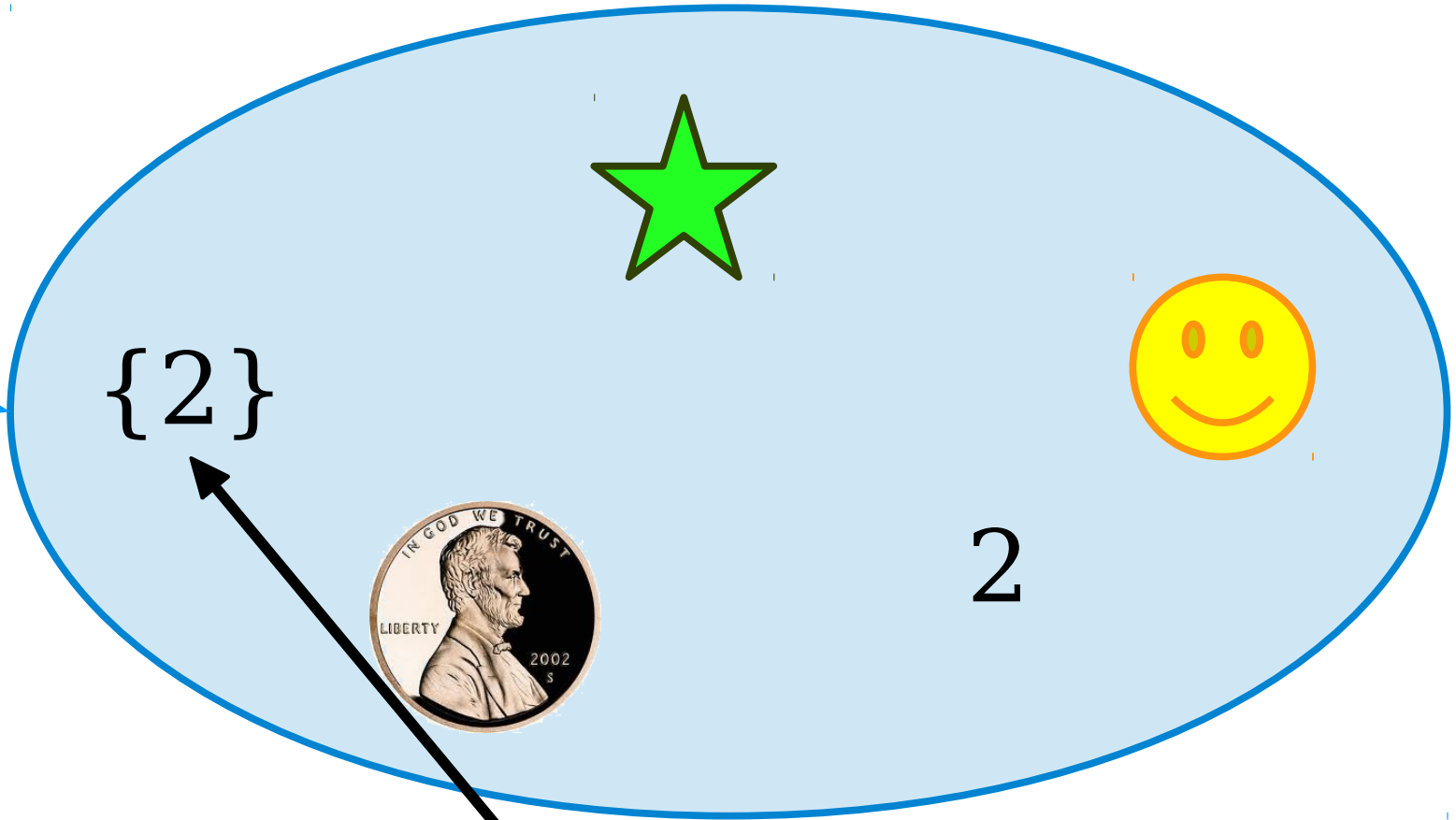


2

$2 \in S$

# Subsets and Elements

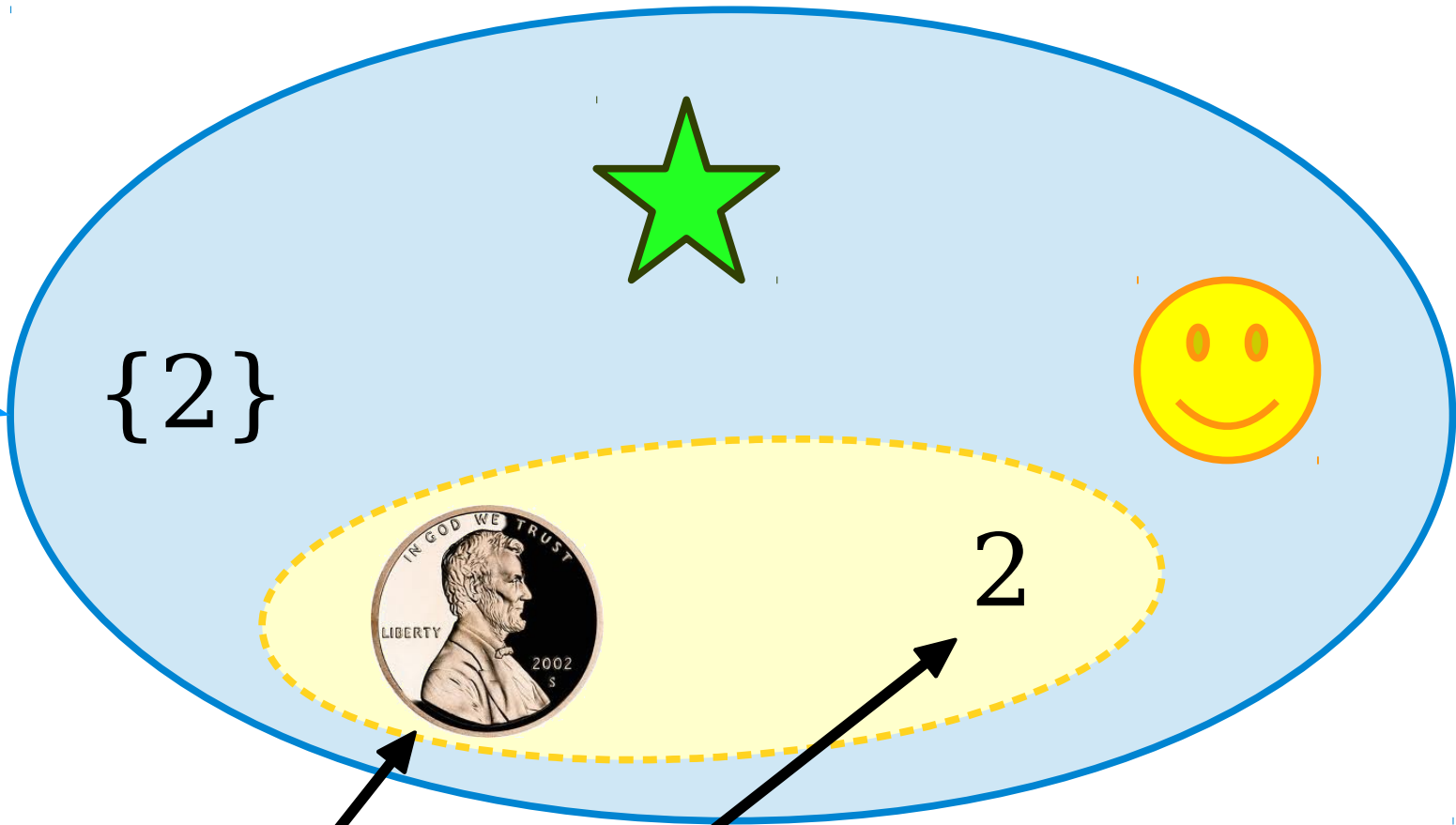
*Set S*



$\{2\} \in S$

# Subsets and Elements

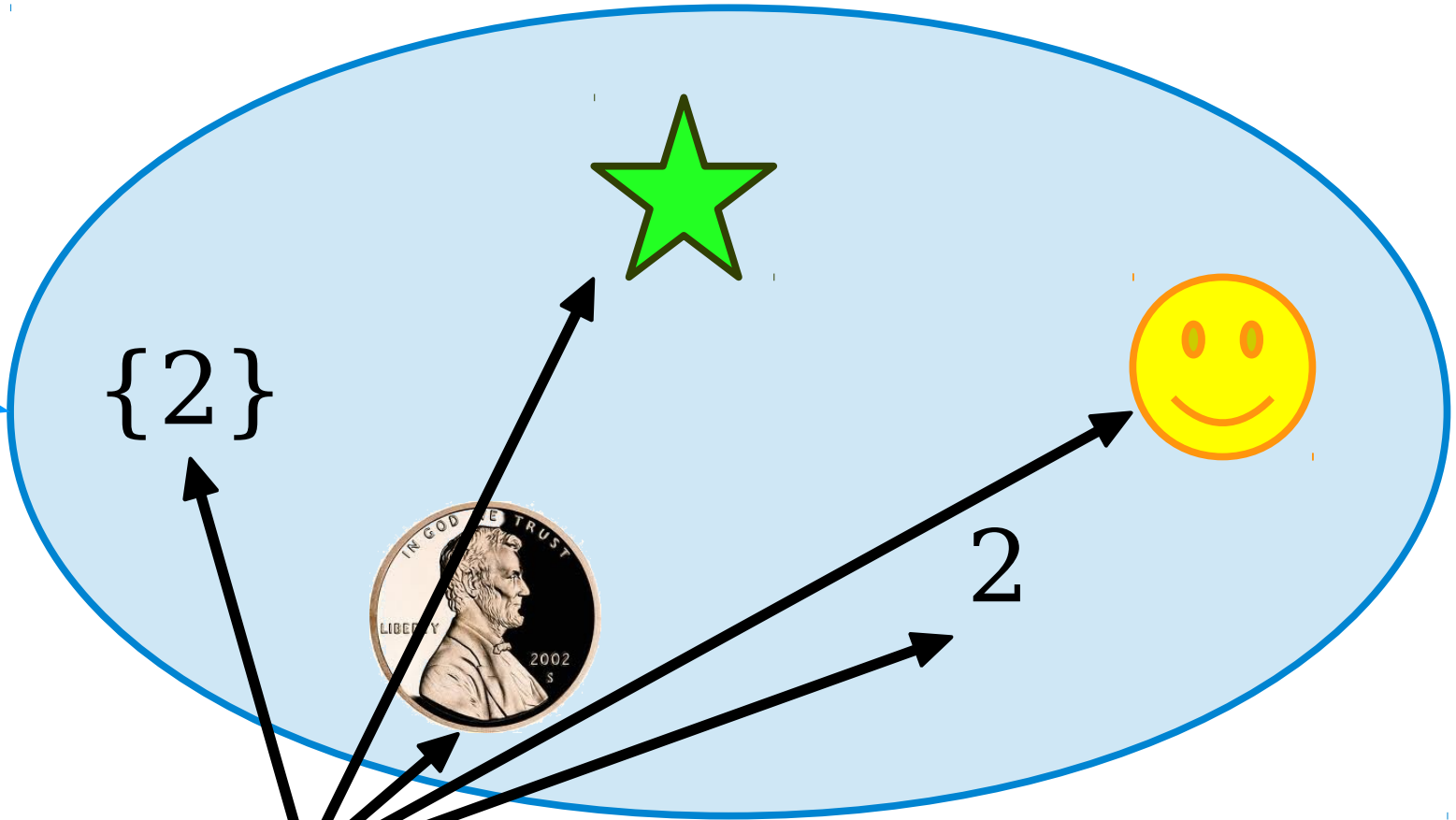
Set S



$$\left\{ \text{penny}, 2 \right\} \subseteq S$$

# Subsets and Elements

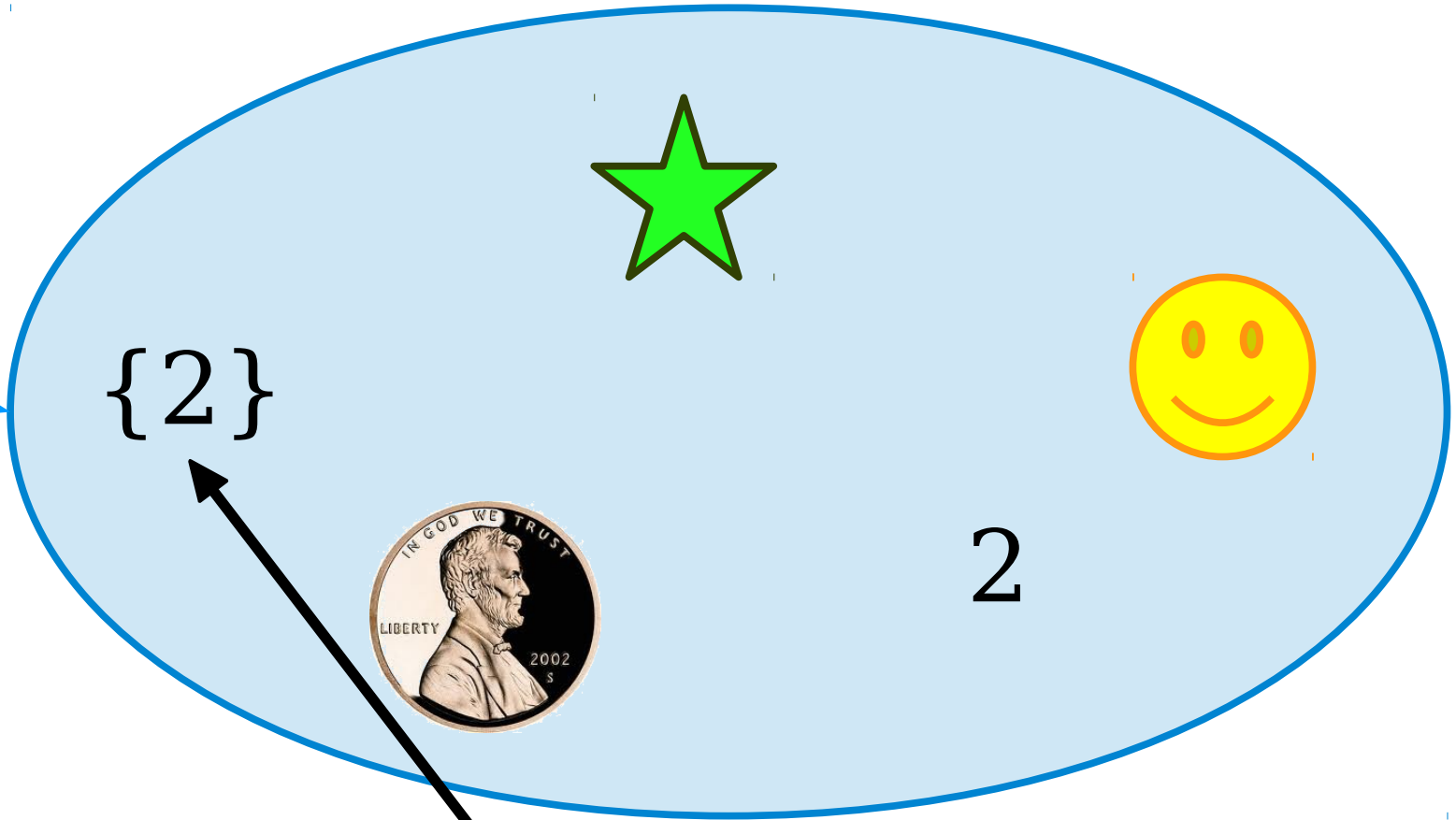
Set S



$\{ \text{Lincoln Penny}, 2 \} \notin S$

# Subsets and Elements

*Set S*

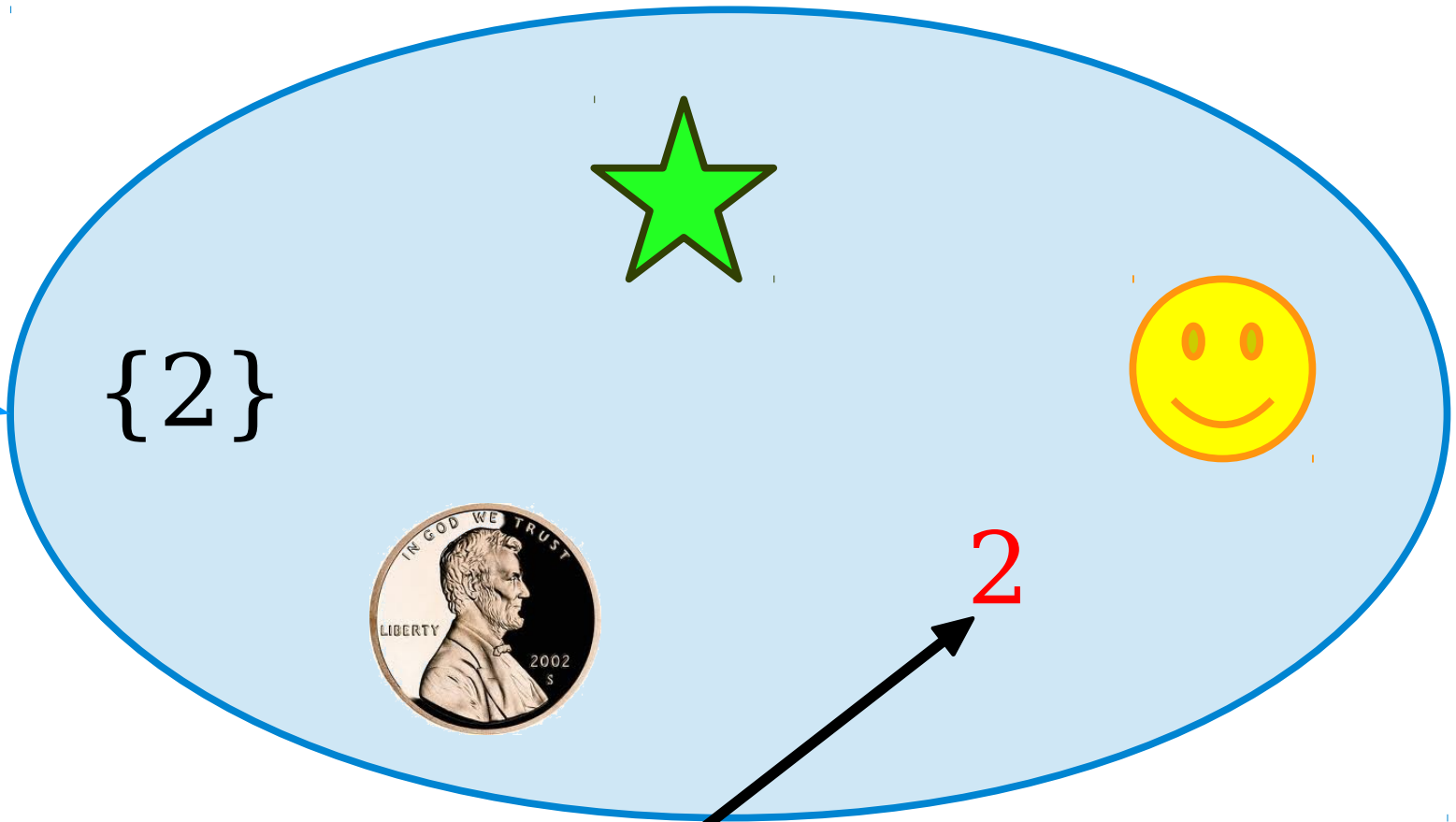


$\{2\} \in S$



# Subsets and Elements

*Set S*

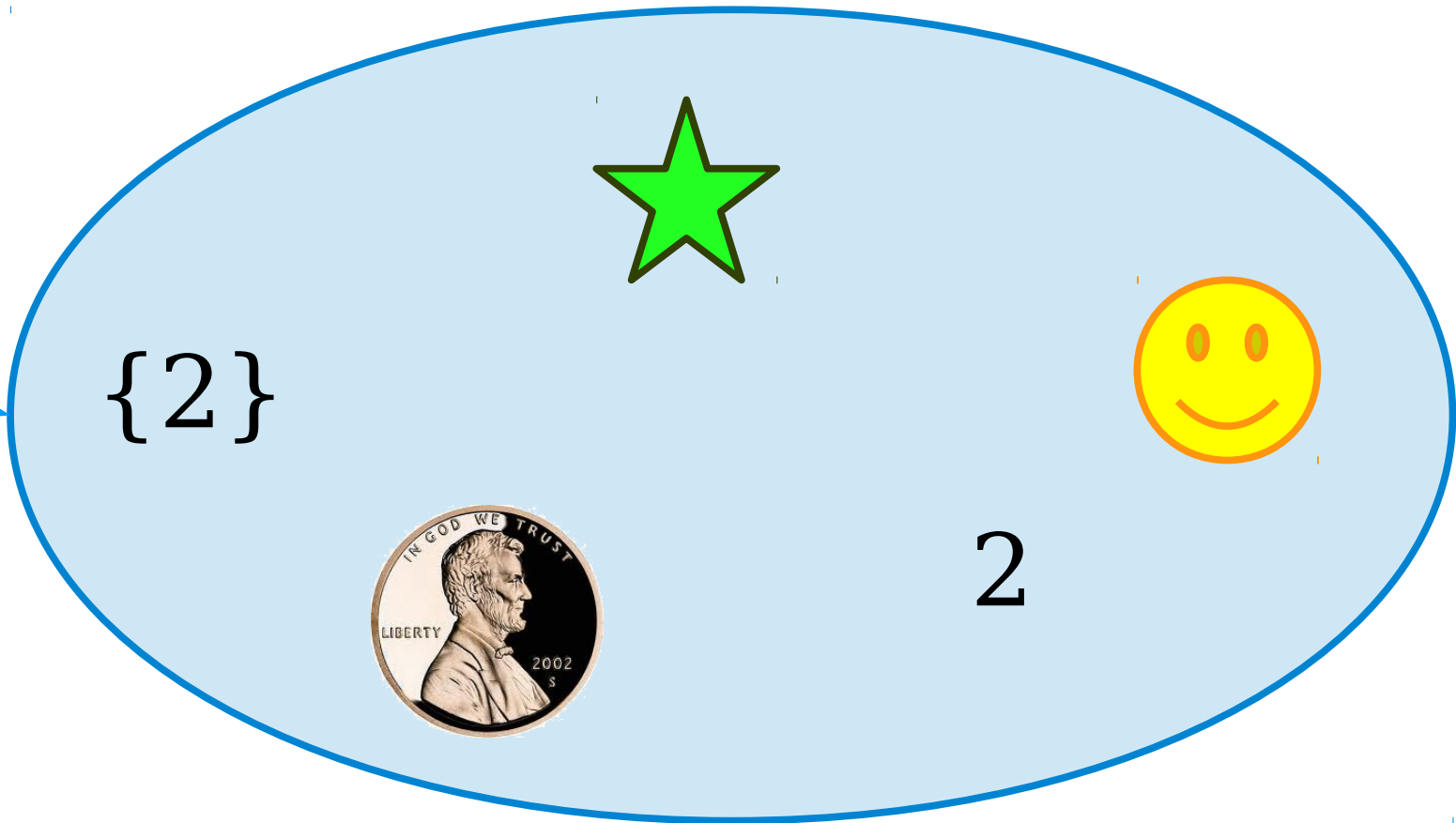


$$\{ \textcolor{red}{2} \} \subseteq S$$

A black arrow points from the red number 2 inside the curly braces of the equation  $\{ \textcolor{red}{2} \} \subseteq S$  to a red number 2 located inside the Set S oval.

# Subsets and Elements

Set S



$$2 \notin S$$

(Since 2  
isn't a set.)

# Subsets and Elements

- We say that  $S \in T$  if, among the elements of  $T$ , one of them is *exactly* the object  $S$ .
- We say that  $S \subseteq T$  if  $S$  is a set and every element of  $S$  is also an element of  $T$ . ( $S$  has to be a set for the statement  $S \subseteq T$  to be meaningful.)
- Although these concepts are similar, ***they are not the same!*** Not all elements of a set are subsets of that set and vice-versa.
- We have a resource on the course website, the Guide to Elements and Subsets, that explores this in more depth.

# What About the Empty Set?

- A set  $S$  is called a **subset** of a set  $T$  (denoted  **$S \subseteq T$** ) if all elements of  $S$  are also elements of  $T$ .
- Are there any sets  $S$  where  $\emptyset \subseteq S$ ?
- Equivalently, is there a set  $S$  where the following statement is true?

**“All elements of  $\emptyset$  are also elements of  $S$ ”**

- **Yes!** In fact, this statement is true for every choice of  $S$ !

# Vacuous Truth

- A statement of the form

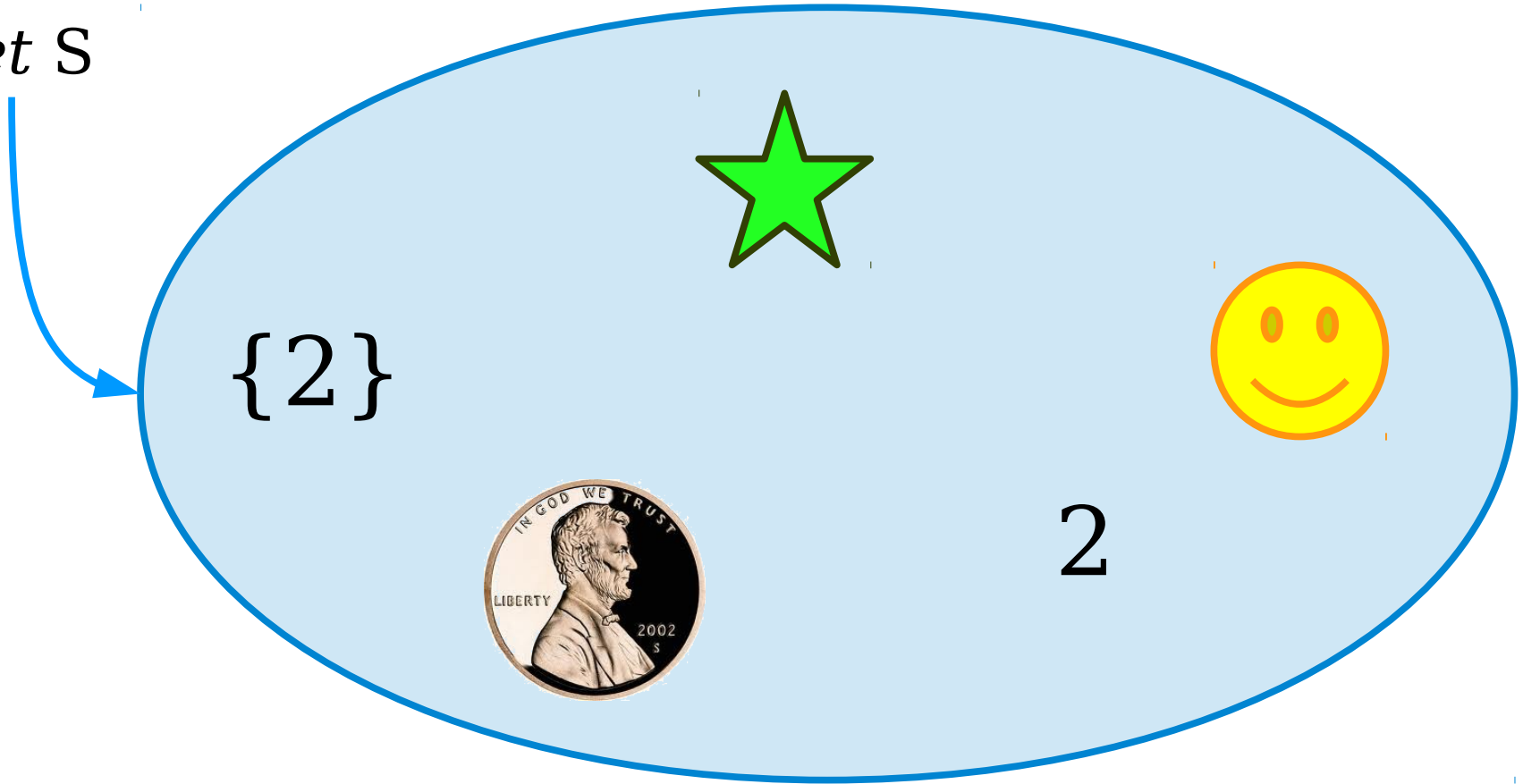
**“All objects of type  $P$   
are also of type  $Q$ ”**

is called ***vacuously true*** if there are no objects of type  $P$ .

- Vacuously true statements are true *by definition*. This is a convention used throughout mathematics.
- Some examples:
  - All unicorns are pink.
  - All unicorns are blue.
  - Every element of  $\emptyset$  is also an element of  $S$ .

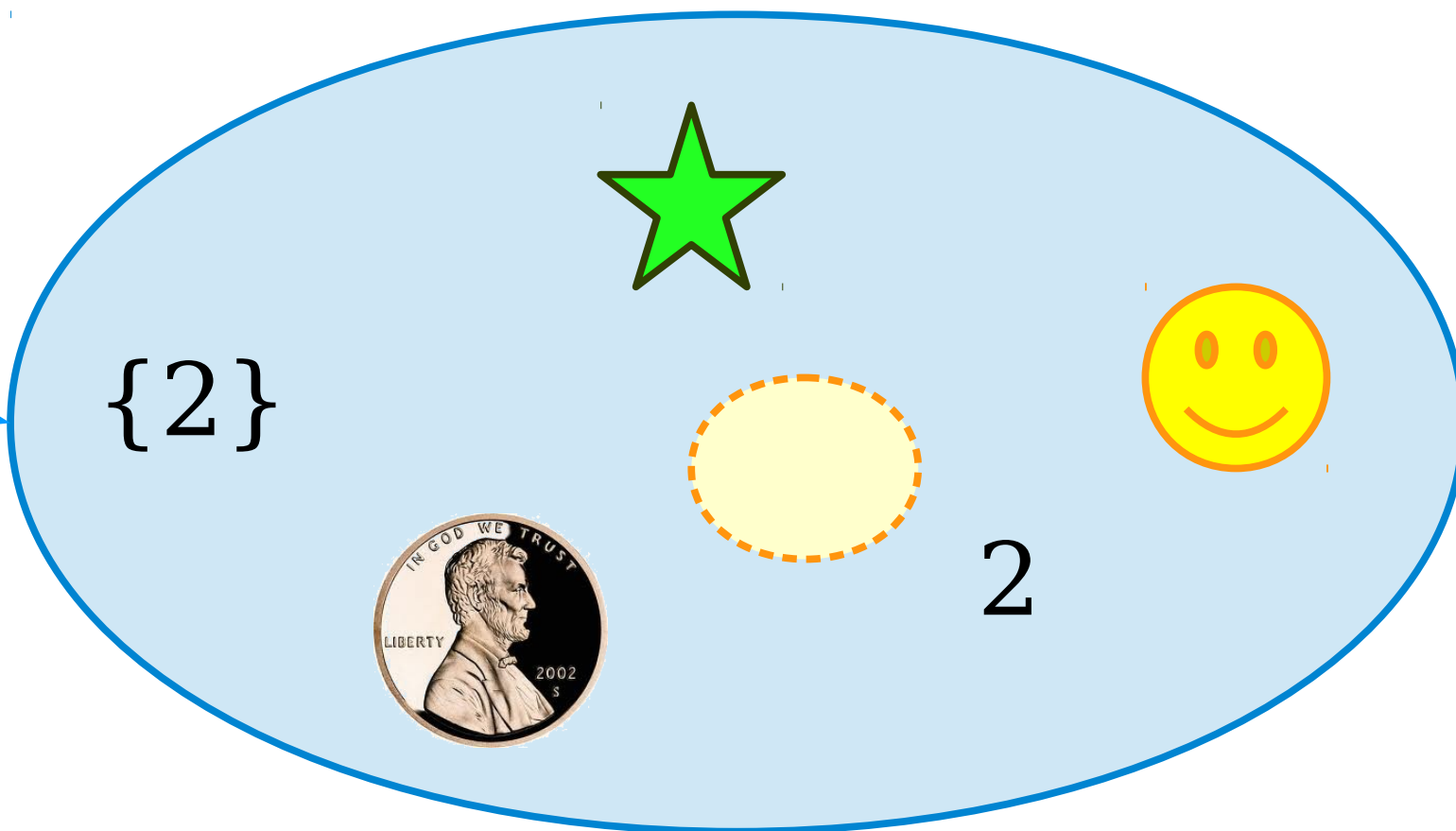
# Subsets and Elements

*Set S*



# Subsets and Elements

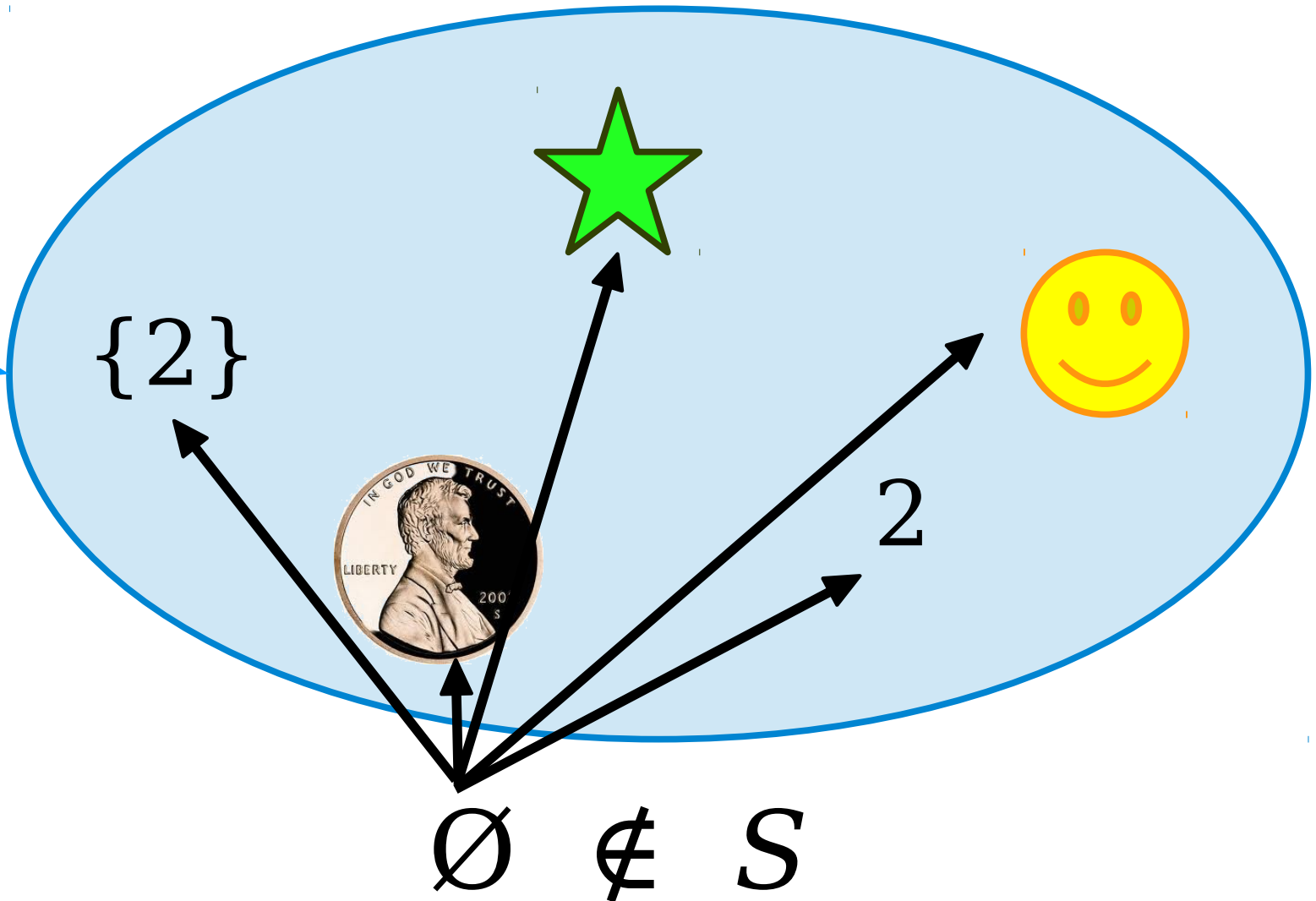
*Set S*



$$\emptyset \subseteq S$$

# Subsets and Elements

Set S





$$S = \left\{ \text{Lincoln Penny}, \text{Kennedy Half Dollar} \right\}$$

$$\wp(S) = \left\{ \emptyset, \left\{ \text{Kennedy Half Dollar} \right\}, \left\{ \text{Lincoln Penny} \right\}, \left\{ \text{Lincoln Penny}, \text{Kennedy Half Dollar} \right\} \right\}$$

The notation  $\wp(S)$  denotes the **power set** of  $S$  (the set of all subsets of  $S$ ).

$$\text{Formally, } \wp(S) = \{ T \mid T \subseteq S \}$$

What is  $\wp(\emptyset)$ ?

**Answer:**  $\{\emptyset\}$

*Remember that  $\emptyset \neq \{\emptyset\}$ !*

# Cardinality

Cardinality

# Cardinality

- The *cardinality* of a set is the number of elements it contains.
- If  $S$  is a set, we denote its cardinality by writing  $|S|$ .
- Examples:
  - $|\{a, b, c, d, e\}| = 5$
  - $|\{\{a, b\}, \{c, d, e, f, g\}, \{h\}\}| = 3$
  - $|\{1, 2, 3, 3, 3, 3, 3\}| = 3$
  - $|\{n \in \mathbb{N} \mid n < 137\}| = 137$

# The Cardinality of $\mathbb{N}$

- What is  $|\mathbb{N}|$ ?
  - There are infinitely many natural numbers.
  - $|\mathbb{N}|$  can't be a natural number, since it's infinitely large.
- We need to introduce a new term.
- Let's define  $\aleph_0 = |\mathbb{N}|$ .
  - $\aleph_0$  is pronounced “aleph-zero,” “aleph-nought,” or “aleph-null.”

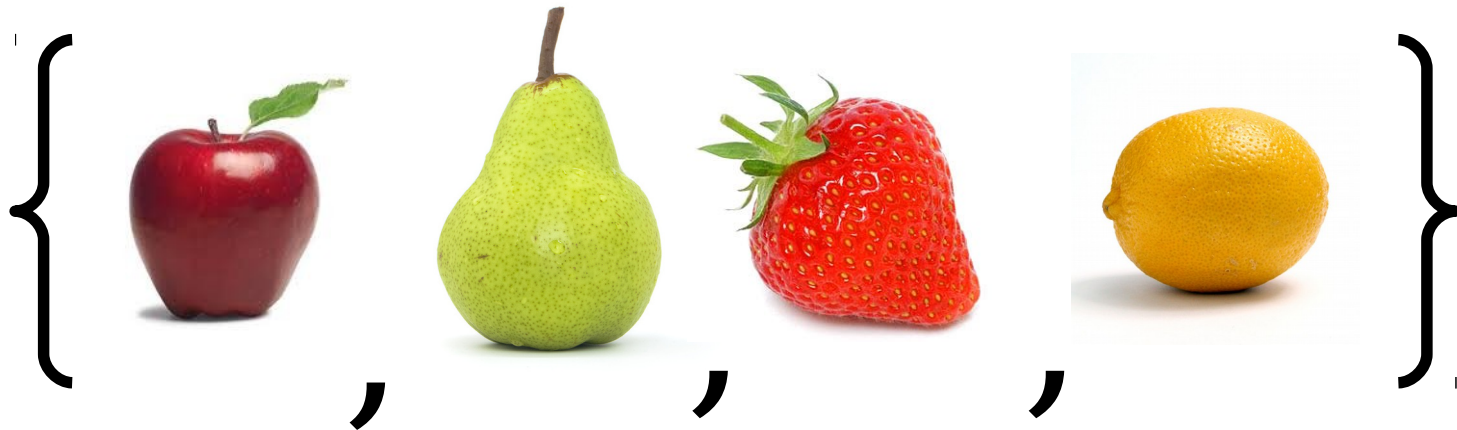
阿莱夫零

Consider the set

$$S = \{ n \mid n \in \mathbb{N} \text{ and } n \text{ is even} \}$$

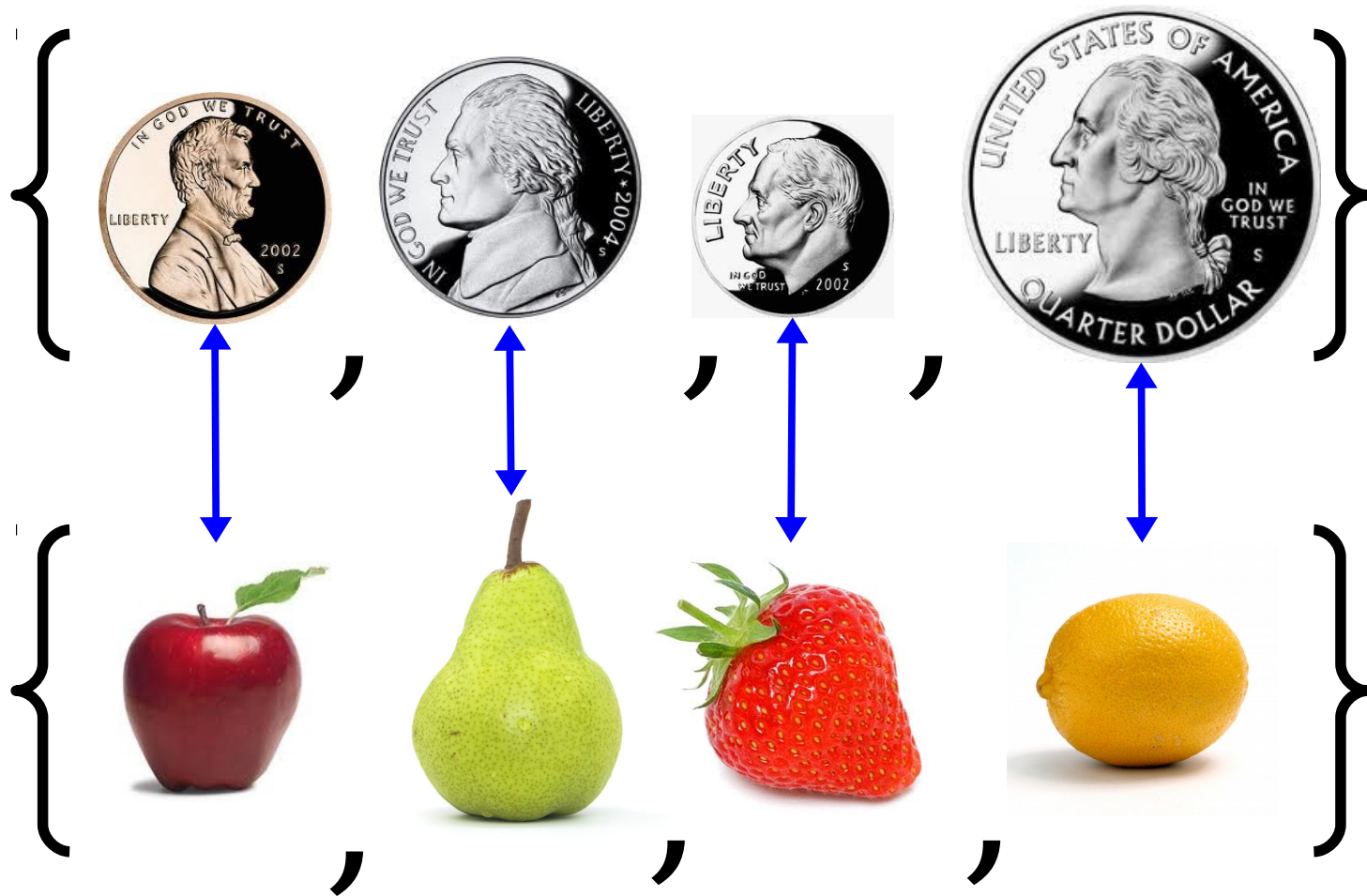
What is  $|S|$ ?

# How Big Are These Sets?



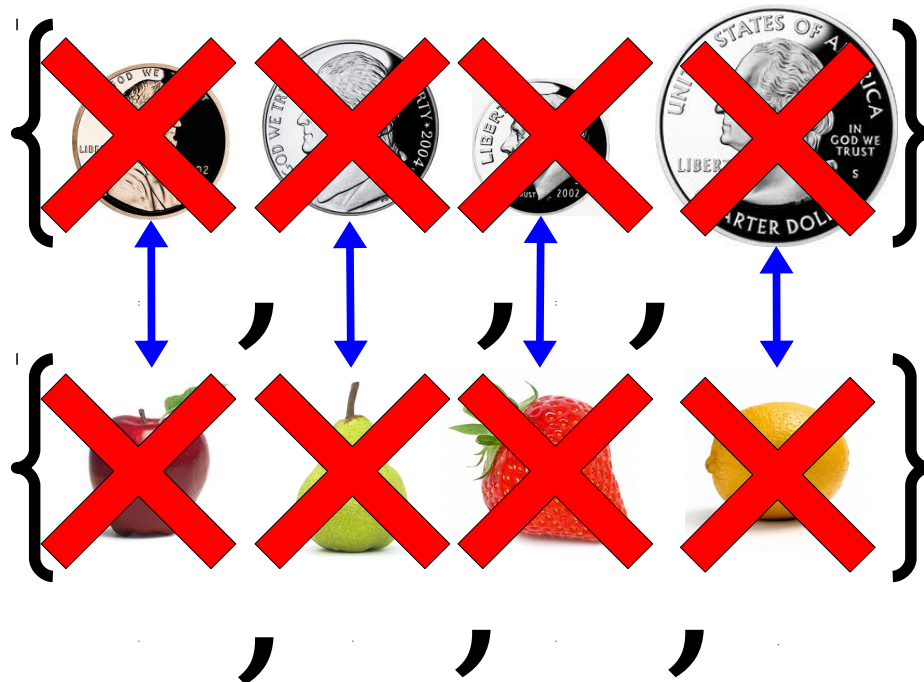


# How Big Are These Sets?



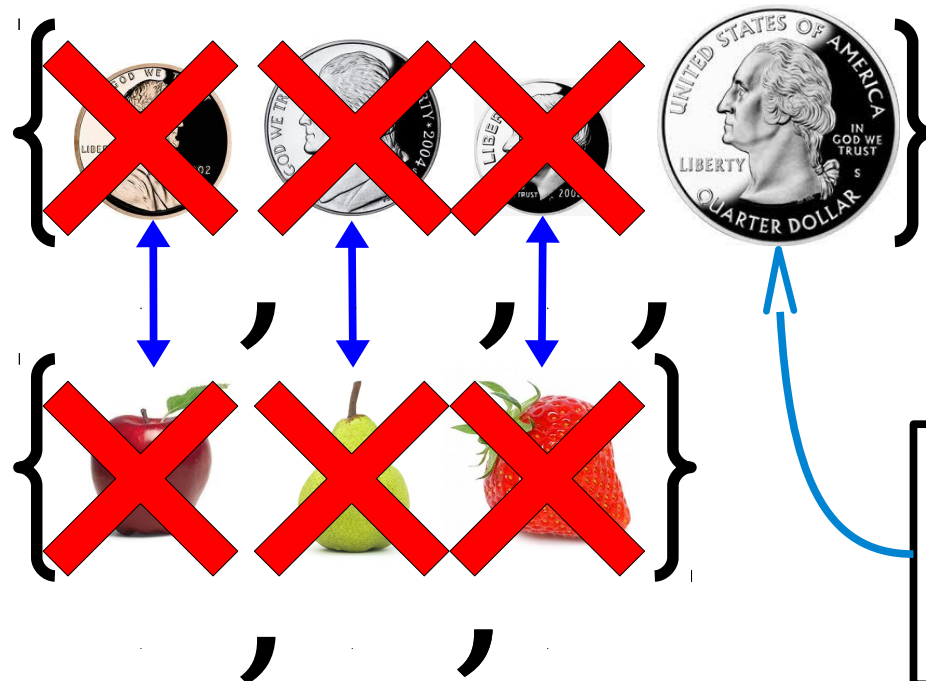
# Comparing Cardinalities

- *By definition*, two sets have the same size if there is a way to pair their elements off without leaving any elements uncovered.
- The intuition:



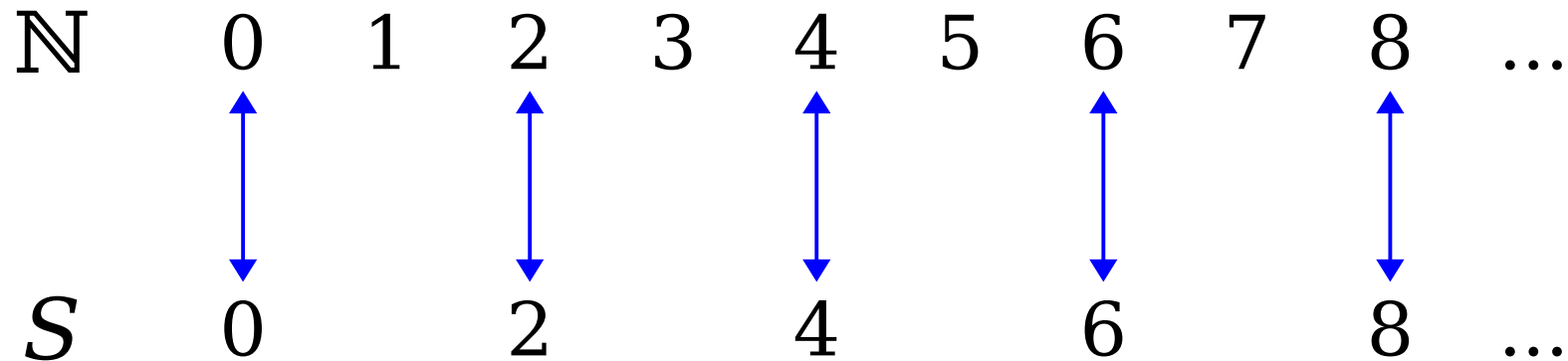
# Comparing Cardinalities

- *By definition*, two sets have the same size if there is a way to pair their elements off without leaving any elements uncovered.
- The intuition:



Everything has been paired up, and this one is all alone.

# Infinite Cardinalities



$$S = \{ n \mid n \in \mathbb{N} \text{ and } n \text{ is even} \}$$

Two sets have the same size if *there is a way* to pair their elements off without leaving any elements uncovered

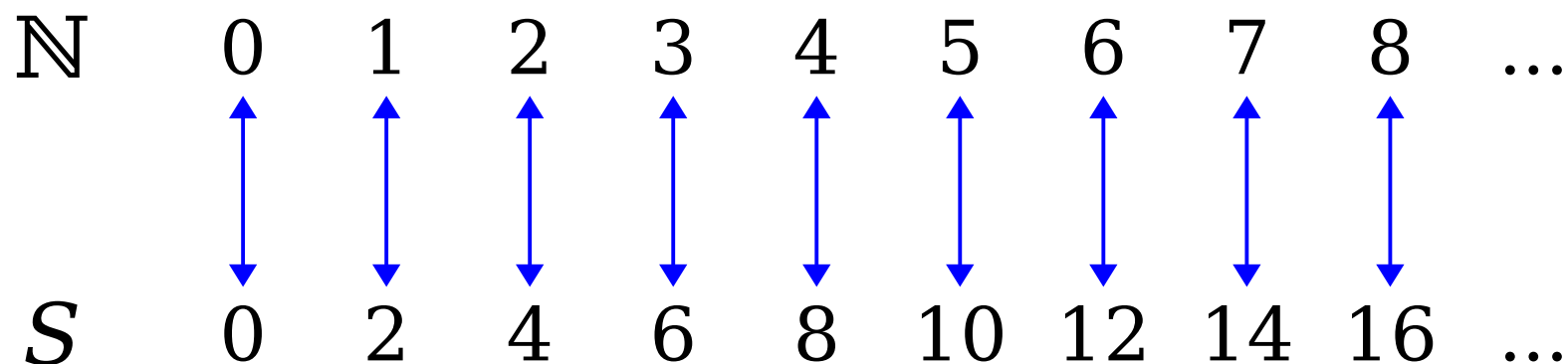
# Infinite Cardinalities

$\mathbb{N}$	0	1	2	3	4	5	6	7	8	...
--------------	---	---	---	---	---	---	---	---	---	-----

$S$	0		2		4		6		8	...
-----	---	--	---	--	---	--	---	--	---	-----

$$S = \{ n \mid n \in \mathbb{N} \text{ and } n \text{ is even} \}$$

# Infinite Cardinalities



$$n \leftrightarrow 2n$$

$$S = \{ n \mid n \in \mathbb{N} \text{ and } n \text{ is even} \}$$

$$|S| = |\mathbb{N}| = \aleph_0$$

# Infinite Cardinalities

$\mathbb{N}$     0    1    2    3    4    5    6    7    8    ...

$\mathbb{Z}$     ...    -3    -2    -1    0    1    2    3    4    ...

# Infinite Cardinalities

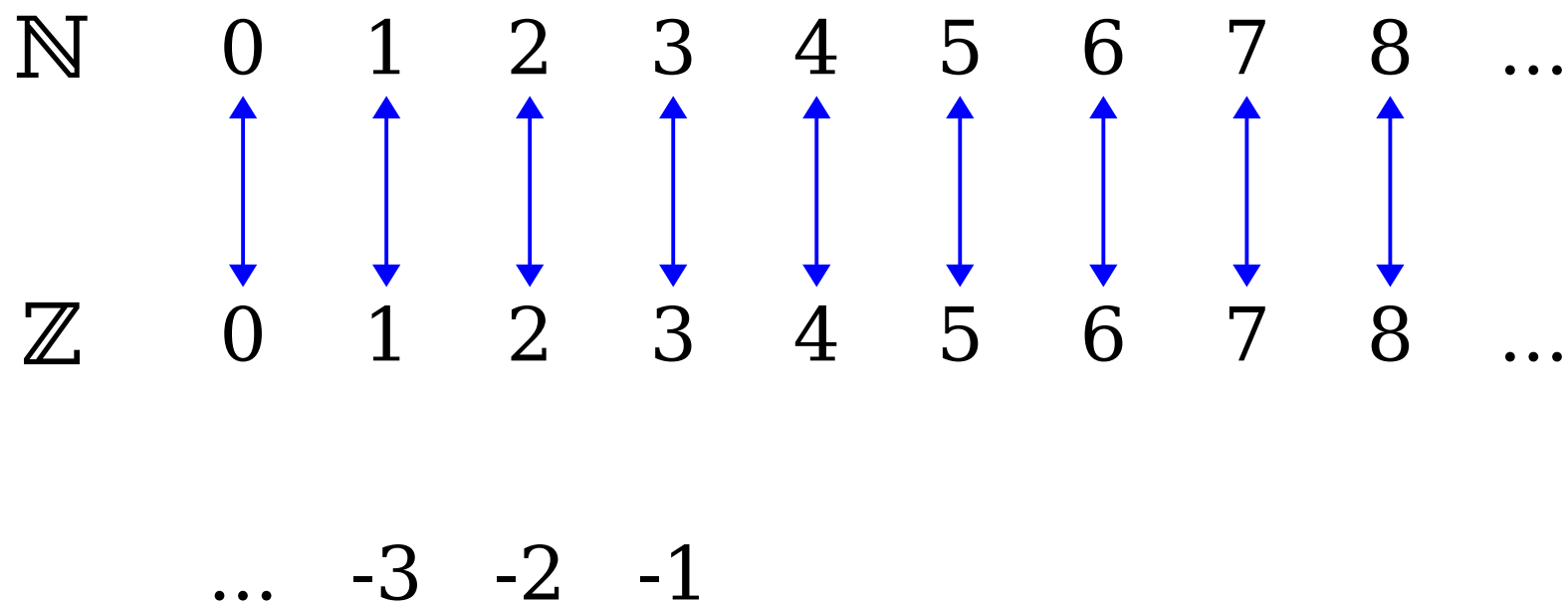
$\mathbb{N}$     0    1    2    3    4    5    6    7    8    ...

$\mathbb{Z}$                     0    1    2    3    4    ...

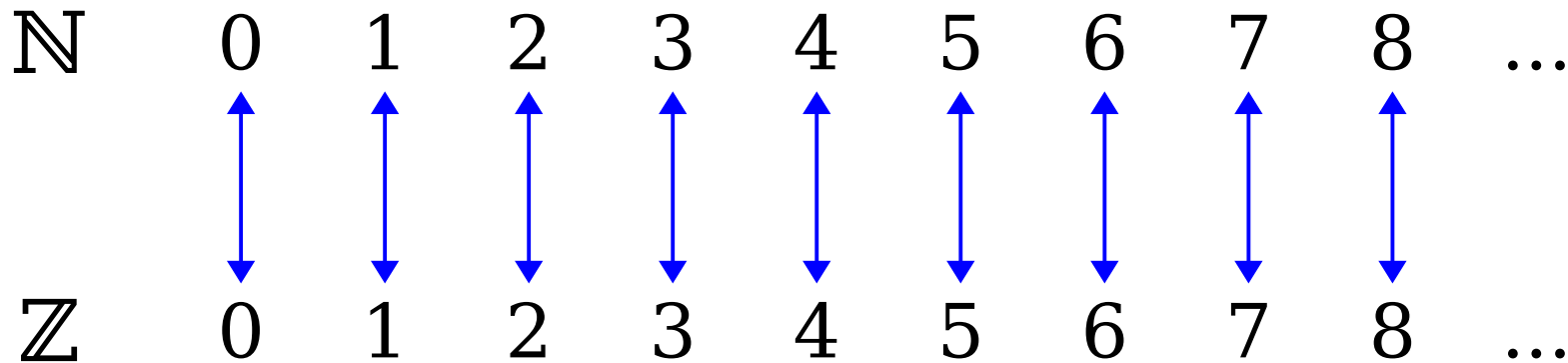
...   -3   -2   -1



# Infinite Cardinalities



# Infinite Cardinalities



... -3 -2 -1

Two sets have the same size if *there is a way* to pair their elements off without leaving any elements uncovered

# Infinite Cardinalities

$\mathbb{N}$     0    1    2    3    4    5    6    7    8    ...

$\mathbb{Z}$     ...    -3    -2    -1    0    1    2    3    4    ...

# Infinite Cardinalities

$\mathbb{N}$     0    1    2    3    4    5    6    7    8    ...

$\mathbb{Z}$

...   -3   -2   -1   0   1   2   3   4   ...

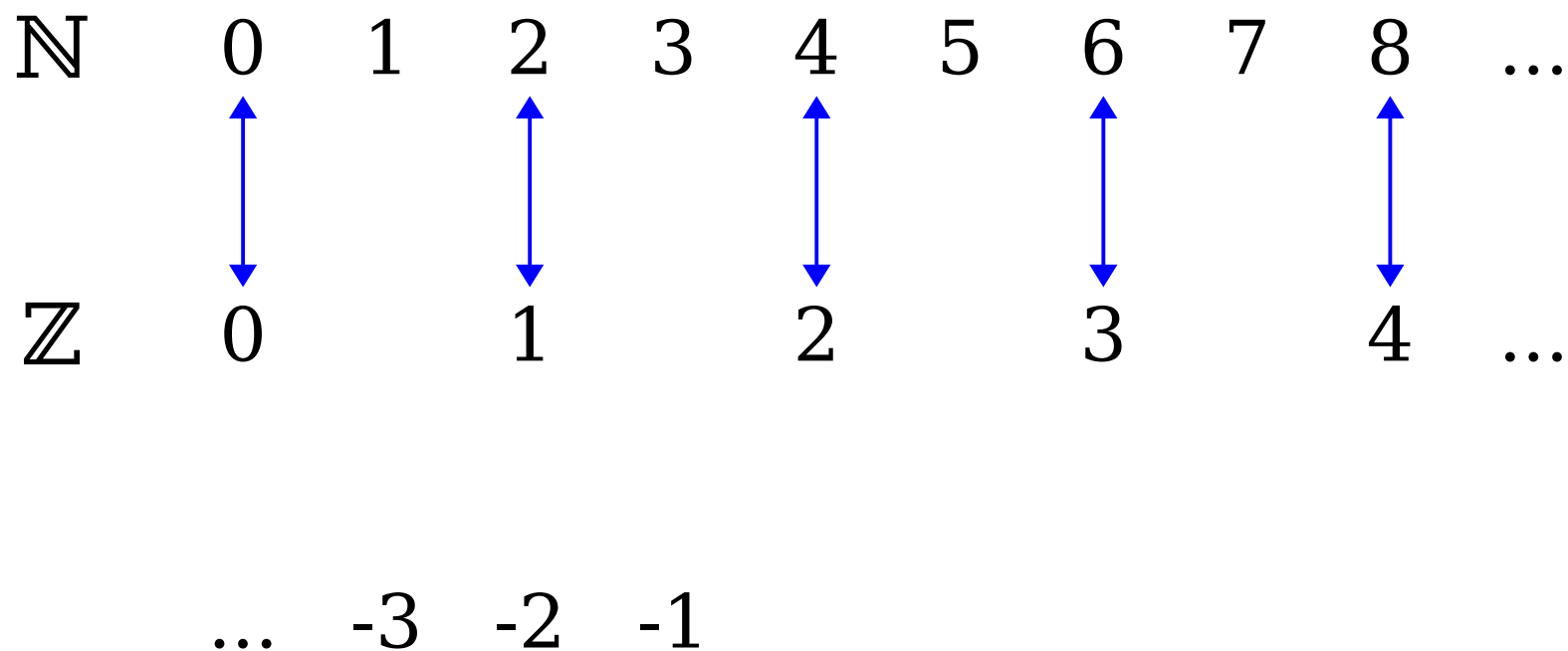
# Infinite Cardinalities

$\mathbb{N}$     0    1    2    3    4    5    6    7    8    ...

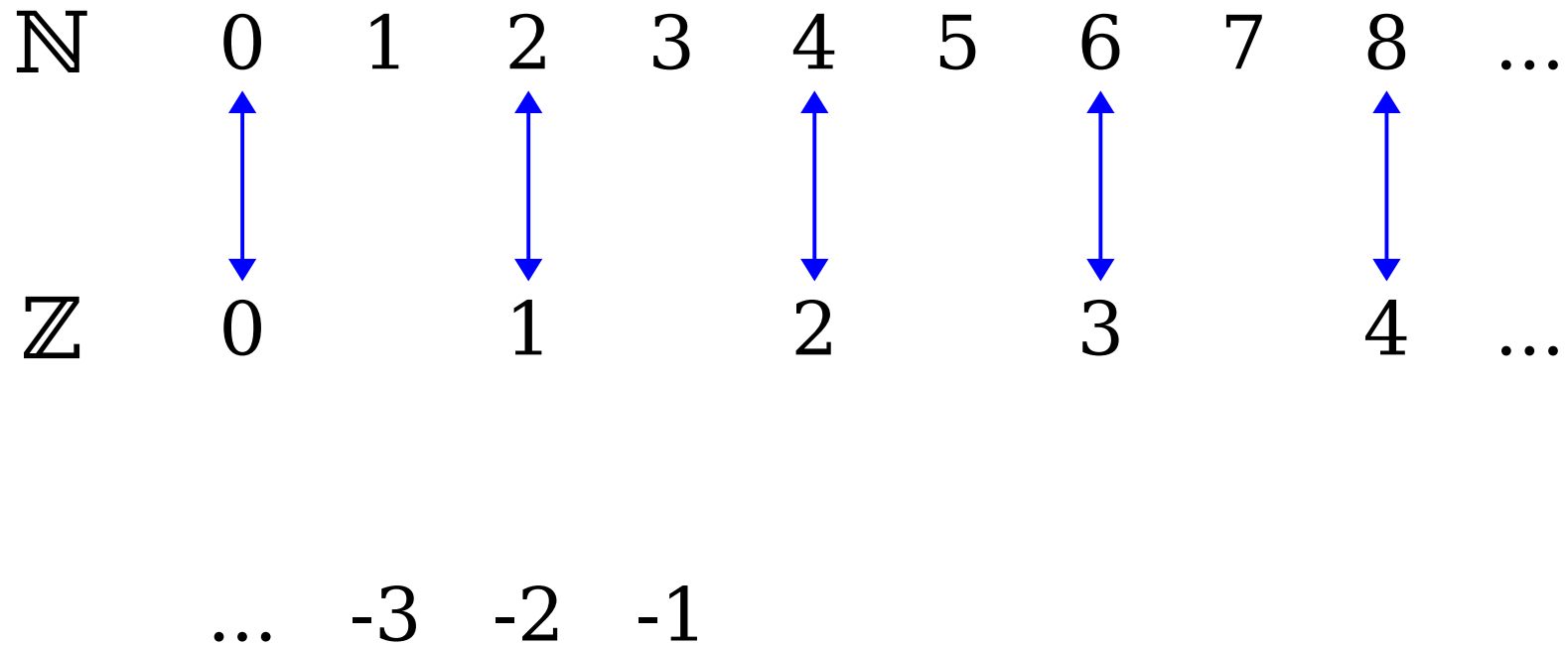
$\mathbb{Z}$     0            1            2            3            4    ...

...   -3   -2   -1

# Infinite Cardinalities

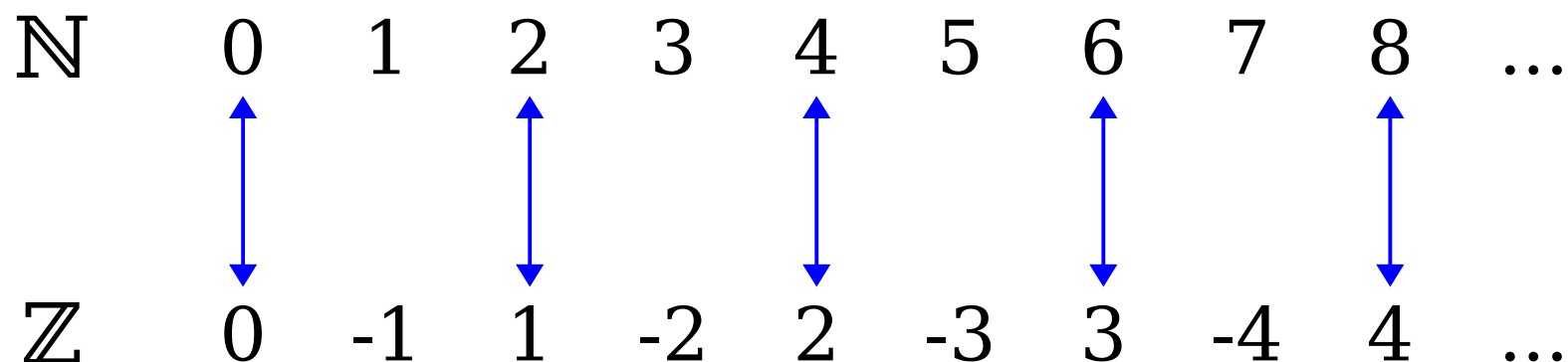


# Infinite Cardinalities



Pair nonnegative integers with even natural numbers.

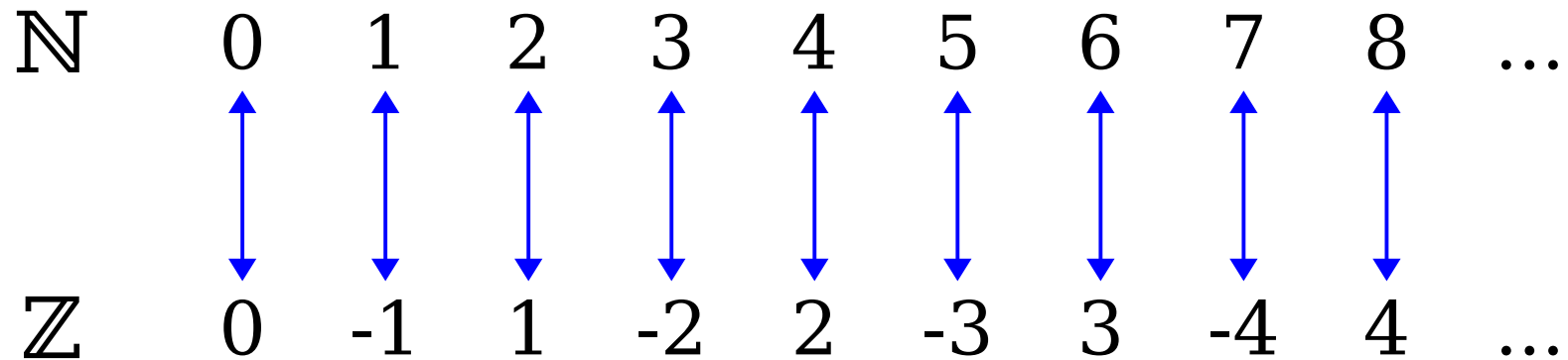
# Infinite Cardinalities



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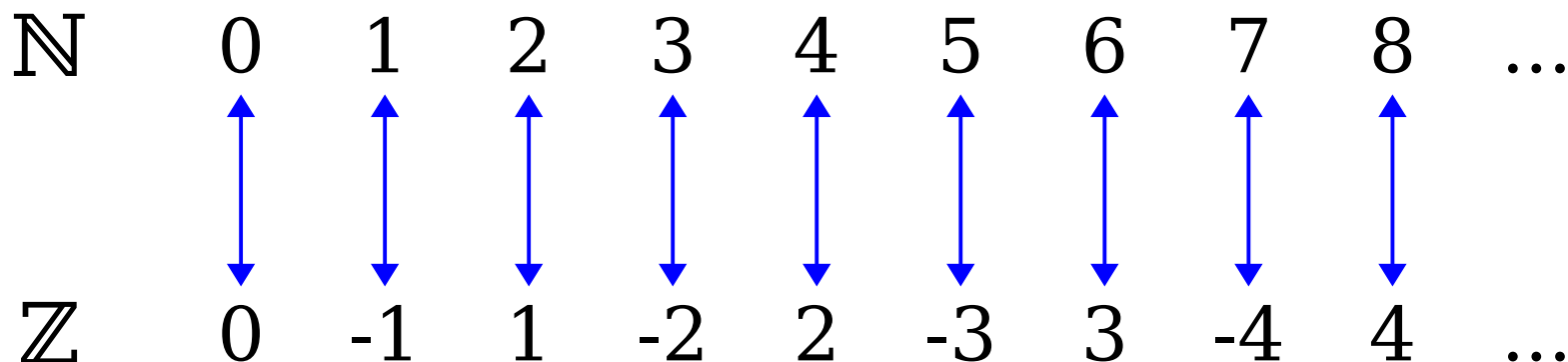


# Infinite Cardinalities



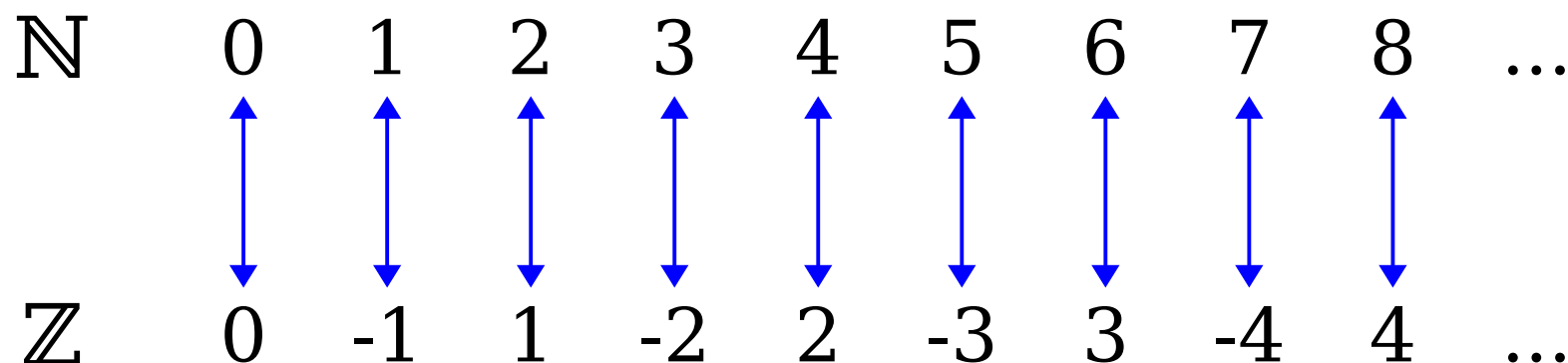
Pair nonnegative integers with even natural numbers.

# Infinite Cardinalities



Pair nonnegative integers with even natural numbers.  
Pair negative integers with odd natural numbers.

# Infinite Cardinalities



$$|\mathbb{N}| = |\mathbb{Z}| = \aleph_0$$

Pair nonnegative integers with even natural numbers.  
Pair negative integers with odd natural numbers.

## ***Important Question:***

Do all infinite sets have  
the same cardinality?

$$S = \left\{ \text{Lincoln Penny}, \text{Kennedy Half Dollar} \right\}$$

$$\wp(S) = \left\{ \emptyset, \left\{ \text{Kennedy Half Dollar} \right\}, \left\{ \text{Lincoln Penny} \right\}, \left\{ \text{Lincoln Penny}, \text{Kennedy Half Dollar} \right\} \right\}$$

$$|S| < |\wp(S)|$$

$$S = \left\{ \text{Lincoln Penny}, \text{Jefferson Nickel}, \text{Button} \right\}$$

$$\wp(S) = \left\{ \begin{array}{l} \emptyset, \{ \text{Lincoln Penny} \}, \{ \text{Jefferson Nickel} \}, \{ \text{Button} \}, \\ \{ \text{Lincoln Penny}, \text{Jefferson Nickel} \}, \{ \text{Lincoln Penny}, \text{Button} \}, \{ \text{Jefferson Nickel}, \text{Button} \}, \\ \{ \text{Lincoln Penny}, \text{Jefferson Nickel}, \text{Button} \} \end{array} \right\}$$

$$|S| < |\wp(S)|$$

$$S = \{a, b, c, d\}$$

$$\wp(S) = \{$$

$$\emptyset,$$

$$\{a\}, \{b\}, \{c\}, \{d\},$$

$$\{a, b\}, \{a, c\}, \{a, d\}, \{b, c\}, \{b, d\}, \{c, d\}$$

$$\{a, b, c\}, \{a, b, d\}, \{a, c, d\}, \{b, c, d\},$$

$$\{a, b, c, d\}$$

$$\}$$

$$|S| < |\wp(S)|$$

If  $|S|$  is infinite, what is the relation between  $|S|$  and  $|\wp(S)|$ ?

Does  $|S| = |\wp(S)|$ ?



If  $|S| = |\wp(S)|$ , we can pair up the elements of  $S$  and the elements of  $\wp(S)$  without leaving anything out.

If  $|S| = |\wp(S)|$ , we can pair up the elements of  $S$  and **the elements of  $\wp(S)$**  without leaving anything out.

If  $|S| = |\wp(S)|$ , we can pair up the elements of  $S$  and **the subsets of  $S$**  without leaving anything out.

If  $|S| = |\wp(S)|$ , we can pair up the elements of  $S$  and the subsets of  $S$  without leaving anything out.

If  $|S| = |\wp(S)|$ , we can pair up the elements of  $S$  and the subsets of  $S$  without leaving anything out.

What would that look like?

$x_0$

$x_1$

$x_2$

$x_3$

$x_4$

$x_5$

$\dots$

$$x_0 \longleftrightarrow \{ x_0, x_2, x_4, \dots \}$$

$$x_1 \longleftrightarrow \{ x_0, x_3, x_4, \dots \}$$

$$x_2 \longleftrightarrow \{ x_4, \dots \}$$

$$x_3 \longleftrightarrow \{ x_1, x_4, \dots \}$$

$$x_4 \longleftrightarrow \{ x_0, x_5, \dots \}$$

$$x_5 \longleftrightarrow \{ x_0, x_1, x_2, x_3, x_4, x_5, \dots \}$$

...

$x_0$	$x_1$	$x_2$	$x_3$	$x_4$	$x_5$	$\dots$
-------	-------	-------	-------	-------	-------	---------

$$x_0 \longleftrightarrow \{ x_0, x_2, x_4, \dots \}$$

$$x_1 \longleftrightarrow \{ x_0, x_3, x_4, \dots \}$$

$$x_2 \longleftrightarrow \{ x_4, \dots \}$$

$$x_3 \longleftrightarrow \{ x_1, x_4, \dots \}$$

$$x_4 \longleftrightarrow \{ x_0, x_5, \dots \}$$

$$x_5 \longleftrightarrow \{ x_0, x_1, x_2, x_3, x_4, x_5, \dots \}$$

$\dots$



	$x_0$	$x_1$	$x_2$	$x_3$	$x_4$	$x_5$	...
$x_0$	<b>Y</b>	<b>N</b>	<b>Y</b>	<b>N</b>	<b>Y</b>	<b>N</b>	...

$x_1 \longleftrightarrow \{ x_0, x_3, x_4, \dots \}$

$x_2 \longleftrightarrow \{ x_4, \dots \}$

$x_3 \longleftrightarrow \{ x_1, x_4, \dots \}$

$x_4 \longleftrightarrow \{ x_0, x_5, \dots \}$

$x_5 \longleftrightarrow \{ x_0, x_1, x_2, x_3, x_4, x_5, \dots \}$

...

		$x_0$	$x_1$	$x_2$	$x_3$	$x_4$	$x_5$	...
$x_0$	$\longleftrightarrow$	<b>Y</b>	<b>N</b>	<b>Y</b>	<b>N</b>	<b>Y</b>	<b>N</b>	...
$x_1$	$\longleftrightarrow$	<b>Y</b>	<b>N</b>	<b>N</b>	<b>Y</b>	<b>Y</b>	<b>N</b>	...

$x_2 \longleftrightarrow \{ x_4, \dots \}$

$x_3 \longleftrightarrow \{ x_1, x_4, \dots \}$

$x_4 \longleftrightarrow \{ x_0, x_5, \dots \}$

$x_5 \longleftrightarrow \{ x_0, x_1, x_2, x_3, x_4, x_5, \dots \}$

...





The diagram shows a sequence of variables  $X_0, X_1, X_2, X_3, X_4, X_5, \dots$  and a corresponding sequence of binary values  $Y, N, N, Y, Y, N, \dots$ . The variables  $X_0, X_1, X_2$  are connected to the first three rows of the table by blue double-headed arrows. The variables  $X_3, X_4, X_5$  are connected to the fourth, fifth, and sixth rows of the table by blue double-headed arrows. The table has columns for  $X_0, X_1, X_2, X_3, X_4, X_5, \dots$  and rows for  $Y, N, N, Y, Y, N, \dots$ .

	$X_0$	$X_1$	$X_2$	$X_3$	$X_4$	$X_5$	$\dots$
$X_0$	<b>Y</b>	<b>N</b>	<b>Y</b>	<b>N</b>	<b>Y</b>	<b>N</b>	$\dots$
$X_1$	<b>Y</b>	<b>N</b>	<b>N</b>	<b>Y</b>	<b>Y</b>	<b>N</b>	$\dots$
$X_2$	<b>N</b>	<b>N</b>	<b>N</b>	<b>N</b>	<b>Y</b>	<b>N</b>	$\dots$
$X_3$	$\{ \quad X_1, \quad \quad \quad X_4, \quad \quad \quad \dots \}$						
$X_4$	$\{ X_0, X_5, \dots \}$						
$X_5$	$\{ X_0, X_1, X_2, X_3, X_4, X_5, \dots \}$						
$\dots$							

		$x_0$	$x_1$	$x_2$	$x_3$	$x_4$	$x_5$	...
$x_0$	$\longleftrightarrow$	<b>Y</b>	<b>N</b>	<b>Y</b>	<b>N</b>	<b>Y</b>	<b>N</b>	...
$x_1$	$\longleftrightarrow$	<b>Y</b>	<b>N</b>	<b>N</b>	<b>Y</b>	<b>Y</b>	<b>N</b>	...
$x_2$	$\longleftrightarrow$	<b>N</b>	<b>N</b>	<b>N</b>	<b>N</b>	<b>Y</b>	<b>N</b>	...
$x_3$	$\longleftrightarrow$	<b>N</b>	<b>Y</b>	<b>N</b>	<b>N</b>	<b>Y</b>	<b>N</b>	...

$x_4 \longleftrightarrow \{ x_0, x_5, \dots \}$

$x_5 \longleftrightarrow \{ x_0, x_1, x_2, x_3, x_4, x_5, \dots \}$

...



		$x_0$	$x_1$	$x_2$	$x_3$	$x_4$	$x_5$	...
$x_0$	$\longleftrightarrow$	<b>Y</b>	<b>N</b>	<b>Y</b>	<b>N</b>	<b>Y</b>	<b>N</b>	...
$x_1$	$\longleftrightarrow$	<b>Y</b>	<b>N</b>	<b>N</b>	<b>Y</b>	<b>Y</b>	<b>N</b>	...
$x_2$	$\longleftrightarrow$	<b>N</b>	<b>N</b>	<b>N</b>	<b>N</b>	<b>Y</b>	<b>N</b>	...
$x_3$	$\longleftrightarrow$	<b>N</b>	<b>Y</b>	<b>N</b>	<b>N</b>	<b>Y</b>	<b>N</b>	...
$x_4$	$\longleftrightarrow$	<b>Y</b>	<b>N</b>	<b>N</b>	<b>N</b>	<b>N</b>	<b>Y</b>	...
$x_5$	$\longleftrightarrow$	<b>Y</b>	<b>Y</b>	<b>Y</b>	<b>Y</b>	<b>Y</b>	<b>Y</b>	...
...								



		$x_0$	$x_1$	$x_2$	$x_3$	$x_4$	$x_5$	...
$x_0$	$\longleftrightarrow$	<b>Y</b>	<b>N</b>	<b>Y</b>	<b>N</b>	<b>Y</b>	<b>N</b>	...
$x_1$	$\longleftrightarrow$	<b>Y</b>	<b>N</b>	<b>N</b>	<b>Y</b>	<b>Y</b>	<b>N</b>	...
$x_2$	$\longleftrightarrow$	<b>N</b>	<b>N</b>	<b>N</b>	<b>N</b>	<b>Y</b>	<b>N</b>	...
$x_3$	$\longleftrightarrow$	<b>N</b>	<b>Y</b>	<b>N</b>	<b>N</b>	<b>Y</b>	<b>N</b>	...
$x_4$	$\longleftrightarrow$	<b>Y</b>	<b>N</b>	<b>N</b>	<b>N</b>	<b>N</b>	<b>Y</b>	...
$x_5$	$\longleftrightarrow$	<b>Y</b>	<b>Y</b>	<b>Y</b>	<b>Y</b>	<b>Y</b>	<b>Y</b>	...
...		...	...	...	...	...	...	...

	$x_0$	$x_1$	$x_2$	$x_3$	$x_4$	$x_5$	...
$x_0$	<b>Y</b>	<b>N</b>	<b>Y</b>	<b>N</b>	<b>Y</b>	<b>N</b>	...
$x_1$	<b>Y</b>	<b>N</b>	<b>N</b>	<b>Y</b>	<b>Y</b>	<b>N</b>	...
$x_2$	<b>N</b>	<b>N</b>	<b>N</b>	<b>N</b>	<b>Y</b>	<b>N</b>	...
$x_3$	<b>N</b>	<b>Y</b>	<b>N</b>	<b>N</b>	<b>Y</b>	<b>N</b>	...
$x_4$	<b>Y</b>	<b>N</b>	<b>N</b>	<b>N</b>	<b>N</b>	<b>Y</b>	...
$x_5$	<b>Y</b>	<b>Y</b>	<b>Y</b>	<b>Y</b>	<b>Y</b>	<b>Y</b>	...
...	...	...	...	...	...	...	...

	$x_0$	$x_1$	$x_2$	$x_3$	$x_4$	$x_5$	...
$x_0$	<b>Y</b>	<b>N</b>	<b>Y</b>	<b>N</b>	<b>Y</b>	<b>N</b>	...
$x_1$	<b>Y</b>	<b>N</b>	<b>N</b>	<b>Y</b>	<b>Y</b>	<b>N</b>	...
$x_2$	<b>N</b>	<b>N</b>	<b>N</b>	<b>N</b>	<b>Y</b>	<b>N</b>	...
$x_3$	<b>N</b>	<b>Y</b>	<b>N</b>	<b>N</b>	<b>Y</b>	<b>N</b>	...
$x_4$	<b>Y</b>	<b>N</b>	<b>N</b>	<b>N</b>	<b>N</b>	<b>Y</b>	...
$x_5$	<b>Y</b>	<b>Y</b>	<b>Y</b>	<b>Y</b>	<b>Y</b>	<b>Y</b>	...
...	...	...	...	...	...	...	...

	$x_0$	$x_1$	$x_2$	$x_3$	$x_4$	$x_5$	$\dots$
$x_0$	<b>Y</b>	<b>N</b>	<b>Y</b>	<b>N</b>	<b>Y</b>	<b>N</b>	$\dots$
$x_1$	<b>Y</b>	<b>N</b>	<b>N</b>	<b>Y</b>	<b>Y</b>	<b>N</b>	$\dots$
$x_2$	<b>N</b>	<b>N</b>	<b>N</b>	<b>N</b>	<b>Y</b>	<b>N</b>	$\dots$
$x_3$	<b>N</b>	<b>Y</b>	<b>N</b>	<b>N</b>	<b>Y</b>	<b>N</b>	$\dots$
$x_4$	<b>Y</b>	<b>N</b>	<b>N</b>	<b>N</b>	<b>N</b>	<b>Y</b>	$\dots$
$x_5$	<b>Y</b>	<b>Y</b>	<b>Y</b>	<b>Y</b>	<b>Y</b>	<b>Y</b>	$\dots$
$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$

$\left\{ \begin{array}{c} x_0' \qquad \qquad \qquad x_5' \quad \dots \end{array} \right\}$

	$x_0$	$x_1$	$x_2$	$x_3$	$x_4$	$x_5$	...
$x_0$	Y	N	Y	N	Y	N	...
$x_1$	Y	N	N	Y	Y	N	...
$x_2$	N	N	N	N	Y	N	...
$x_3$	N	Y	N	N	Y	N	...
$x_4$	Y	N	N	N	N	Y	...
$x_5$	Y	Y	Y	Y	Y	Y	...
...	...	...	...	...	...	...	...

Y	N	N	N	N	Y	...
---	---	---	---	---	---	-----

Which row in the table is paired with this set?

	$x_0$	$x_1$	$x_2$	$x_3$	$x_4$	$x_5$	...
$x_0$	Y	N	Y	N	Y	N	...
$x_1$	Y	N	N	Y	Y	N	...
$x_2$	N	N	N	N	Y	N	...
$x_3$	N	Y	N	N	Y	N	...
$x_4$	Y	N	N	N	N	Y	...
$x_5$	Y	Y	Y	Y	Y	Y	...
...	...	...	...	...	...	...	...

Y	N	N	N	N	Y	...
---	---	---	---	---	---	-----

Which row in the table is paired with this set?

	$x_0$	$x_1$	$x_2$	$x_3$	$x_4$	$x_5$	...
$x_0$	Y	N	Y	N	Y	N	...
$x_1$	Y	N	N	Y	Y	N	...
$x_2$	N	N	N	N	Y	N	...
$x_3$	N	Y	N	N	Y	N	...
$x_4$	Y	N	N	N	N	Y	...
$x_5$	Y	Y	Y	Y	Y	Y	...
...	...	...	...	...	...	...	...

Y	N	N	N	N	Y	...
---	---	---	---	---	---	-----

	$x_0$	$x_1$	$x_2$	$x_3$	$x_4$	$x_5$	...
$x_0$	Y	N	Y	N	Y	N	...
$x_1$	Y	N	N	Y	Y	N	...
$x_2$	N	N	N	N	Y	N	...
$x_3$	N	Y	N	N	Y	N	...
$x_4$	Y	N	N	N	N	Y	...
$x_5$	Y	Y	Y	Y	Y	Y	...
...	...	...	...	...	...	...	...

Flip all Y's to N's and vice-versa to get a new set

N	Y	Y	Y	Y	N	...
---	---	---	---	---	---	-----



	$x_0$	$x_1$	$x_2$	$x_3$	$x_4$	$x_5$	...
$x_0$	Y	N	Y	N	Y	N	...
$x_1$	Y	N	N	Y	Y	N	...
$x_2$	N	N	N	N	Y	N	...
$x_3$	N	Y	N	N	Y	N	...
$x_4$	Y	N	N	N	N	Y	...
$x_5$	Y	Y	Y	Y	Y	Y	...
...	...	...	...	...	...	...	...

Flip all Y's to N's and vice-versa to get a new set

{  $x_1', x_2', x_3', x_4', \dots$  }

	$x_0$	$x_1$	$x_2$	$x_3$	$x_4$	$x_5$	...
$x_0$	Y	N	Y	N	Y	N	...
$x_1$	Y	N	N	Y	Y	N	...
$x_2$	N	N	N	N	Y	N	...
$x_3$	N	Y	N	N	Y	N	...
$x_4$	Y	N	N	N	N	Y	...
$x_5$	Y	Y	Y	Y	Y	Y	...
...	...	...	...	...	...	...	...

N	Y	Y	Y	Y	N	...
---	---	---	---	---	---	-----

Which row in the table is paired with this set?

	$x_0$	$x_1$	$x_2$	$x_3$	$x_4$	$x_5$	...
$x_0$	Y	N	Y	N	Y	N	...
$x_1$	Y	N	N	Y	Y	N	...
$x_2$	N	N	N	N	Y	N	...
$x_3$	N	Y	N	N	Y	N	...
$x_4$	Y	N	N	N	N	Y	...
$x_5$	Y	Y	Y	Y	Y	Y	...
...	...	...	...	...	...	...	...

N	Y	Y	Y	Y	N	...
---	---	---	---	---	---	-----

Which row in the table is paired with this set?

	$x_0$	$x_1$	$x_2$	$x_3$	$x_4$	$x_5$	...
$x_0$	Y	N	Y	N	Y	N	...
$x_1$	Y	N	N	Y	Y	N	...
$x_2$	N	N	N	N	Y	N	...
$x_3$	N	Y	N	N	Y	N	...
$x_4$	Y	N	N	N	N	Y	...
$x_5$	Y	Y	Y	Y	Y	Y	...
...	...	...	...	...	...	...	...

N	Y	Y	Y	Y	N	...
---	---	---	---	---	---	-----

Which row in the table is paired with this set?

	$x_0$	$x_1$	$x_2$	$x_3$	$x_4$	$x_5$	...
$x_0$	Y	N	Y	N	Y	N	...
$x_1$	Y	N	N	Y	Y	N	...
$x_2$	N	N	N	N	Y	N	...
$x_3$	N	Y	N	N	Y	N	...
$x_4$	Y	N	N	N	N	Y	...
$x_5$	Y	Y	Y	Y	Y	Y	...
...	...	...	...	...	...	...	...

N	Y	Y	Y	Y	N	...
---	---	---	---	---	---	-----

Which row in the table is paired with this set?

	$x_0$	$x_1$	$x_2$	$x_3$	$x_4$	$x_5$	...
$x_0$	Y	N	Y	N	Y	N	...
$x_1$	Y	N	N	Y	Y	N	...
$x_2$	N	N	N	N	Y	N	...
$x_3$	N	Y	N	N	Y	N	...
$x_4$	Y	N	N	N	N	Y	...
$x_5$	Y	Y	Y	Y	Y	Y	...
...	...	...	...	...	...	...	...

N	Y	Y	Y	Y	N	...
---	---	---	---	---	---	-----

Which row in the table is paired with this set?

	$x_0$	$x_1$	$x_2$	$x_3$	$x_4$	$x_5$	...
$x_0$	Y	N	Y	N	Y	N	...
$x_1$	Y	N	N	Y	Y	N	...
$x_2$	N	N	N	N	Y	N	...
$x_3$	N	Y	N	N	Y	N	...
$x_4$	Y	N	N	N	N	Y	...
$x_5$	Y	Y	Y	Y	Y	Y	...
...	...	...	...	...	...	...	...

N	Y	Y	Y	Y	N	...
---	---	---	---	---	---	-----

Which row in the table is paired with this set?

	$x_0$	$x_1$	$x_2$	$x_3$	$x_4$	$x_5$	...
$x_0$	Y	N	Y	N	Y	N	...
$x_1$	Y	N	N	Y	Y	N	...
$x_2$	N	N	N	N	Y	N	...
$x_3$	N	Y	N	N	Y	N	...
$x_4$	Y	N	N	N	N	Y	...
$x_5$	Y	Y	Y	Y	Y	Y	...
...	...	...	...	...	...	...	...

N	Y	Y	Y	Y	N	...
---	---	---	---	---	---	-----

Which row in the table is paired with this set?



	$x_0$	$x_1$	$x_2$	$x_3$	$x_4$	$x_5$	...
$x_0$	Y	N	Y	N	Y	N	...
$x_1$	Y	N	N	Y	Y	N	...
$x_2$	N	N	N	N	Y	N	...
$x_3$	N	Y	N	N	Y	N	...
$x_4$	Y	N	N	N	N	Y	...
$x_5$	Y	Y	Y	Y	Y	Y	...
...	...	...	...	...	...	...	...

N	Y	Y	Y	Y	N	...
---	---	---	---	---	---	-----

Which row in the table is paired with this set?

	$x_0$	$x_1$	$x_2$	$x_3$	$x_4$	$x_5$	...
$x_0$	Y	N	Y	N	Y	N	...
$x_1$	Y	N	N	Y	Y	N	...
$x_2$	N	N	N	N	Y	N	...
$x_3$	N	Y	N	N	Y	N	...
$x_4$	Y	N	N	N	N	Y	...
$x_5$	Y	Y	Y	Y	Y	Y	...
...	...	...	...	...	...	...	...

N	Y	Y	Y	Y	N	...
---	---	---	---	---	---	-----

Which row in the table is paired with this set?

	$x_0$	$x_1$	$x_2$	$x_3$	$x_4$	$x_5$	...
$x_0$	Y	N	Y	N	Y	N	...
$x_1$	Y	N	N	Y	Y	N	...
$x_2$	N	N	N	N	Y	N	...
$x_3$	N	Y	N	N	Y	N	...
$x_4$	Y	N	N	N	N	Y	...
$x_5$	Y	Y	Y	Y	Y	Y	...
...	...	...	...	...	...	...	...

N	Y	Y	Y	Y	N	...
---	---	---	---	---	---	-----

Which row in the table is paired with this set?

	$x_0$	$x_1$	$x_2$	$x_3$	$x_4$	$x_5$	...
$x_0$	<b>Y</b>	<b>N</b>	<b>Y</b>	<b>N</b>	<b>Y</b>	<b>N</b>	...
$x_1$	<b>Y</b>	<b>N</b>	<b>N</b>	<b>Y</b>	<b>Y</b>	<b>N</b>	...
$x_2$	<b>N</b>	<b>N</b>	<b>N</b>	<b>N</b>	<b>Y</b>	<b>N</b>	...
$x_3$	<b>N</b>	<b>Y</b>	<b>N</b>	<b>N</b>	<b>Y</b>	<b>N</b>	...
$x_4$	<b>Y</b>	<b>N</b>	<b>N</b>	<b>N</b>	<b>N</b>	<b>Y</b>	...
$x_5$	<b>Y</b>	<b>Y</b>	<b>Y</b>	<b>Y</b>	<b>Y</b>	<b>Y</b>	...
...	...	...	...	...	...	...	...

	$x_0$	$x_1$	$x_2$	$x_3$	$x_4$	$x_5$	...
$x_0$	<b>N</b>	<b>N</b>	<b>N</b>	<b>N</b>	<b>N</b>	<b>Y</b>	...
$x_1$	<b>Y</b>	<b>Y</b>	<b>Y</b>	<b>Y</b>	<b>Y</b>	<b>Y</b>	...
$x_2$	<b>N</b>	<b>Y</b>	<b>N</b>	<b>Y</b>	<b>Y</b>	<b>N</b>	...
$x_3$	<b>Y</b>	<b>N</b>	<b>Y</b>	<b>N</b>	<b>Y</b>	<b>N</b>	...
$x_4$	<b>N</b>	<b>N</b>	<b>Y</b>	<b>Y</b>	<b>N</b>	<b>Y</b>	...
$x_5$	<b>Y</b>	<b>N</b>	<b>N</b>	<b>N</b>	<b>N</b>	<b>Y</b>	...
...	...	...	...	...	...	...	...

	$x_0$	$x_1$	$x_2$	$x_3$	$x_4$	$x_5$	...
$x_0$	<b>N</b>	<b>N</b>	<b>N</b>	<b>N</b>	<b>N</b>	<b>Y</b>	...
$x_1$	<b>Y</b>	<b>Y</b>	<b>Y</b>	<b>Y</b>	<b>Y</b>	<b>Y</b>	...
$x_2$	<b>N</b>	<b>Y</b>	<b>N</b>	<b>Y</b>	<b>Y</b>	<b>N</b>	...
$x_3$	<b>Y</b>	<b>N</b>	<b>Y</b>	<b>N</b>	<b>Y</b>	<b>N</b>	...
$x_4$	<b>N</b>	<b>N</b>	<b>Y</b>	<b>Y</b>	<b>N</b>	<b>Y</b>	...
$x_5$	<b>Y</b>	<b>N</b>	<b>N</b>	<b>N</b>	<b>N</b>	<b>Y</b>	...
...	...	...	...	...	...	...	...

	$x_0$	$x_1$	$x_2$	$x_3$	$x_4$	$x_5$	...
$x_0$	<b>N</b>	<b>N</b>	<b>N</b>	<b>N</b>	<b>N</b>	<b>Y</b>	...
$x_1$	<b>Y</b>	<b>Y</b>	<b>Y</b>	<b>Y</b>	<b>Y</b>	<b>Y</b>	...
$x_2$	<b>N</b>	<b>Y</b>	<b>N</b>	<b>Y</b>	<b>Y</b>	<b>N</b>	...
$x_3$	<b>Y</b>	<b>N</b>	<b>Y</b>	<b>N</b>	<b>Y</b>	<b>N</b>	...
$x_4$	<b>N</b>	<b>N</b>	<b>Y</b>	<b>Y</b>	<b>N</b>	<b>Y</b>	...
$x_5$	<b>Y</b>	<b>N</b>	<b>N</b>	<b>N</b>	<b>N</b>	<b>Y</b>	...
...	...	...	...	...	...	...	...

<b>N</b>	<b>Y</b>	<b>N</b>	<b>N</b>	<b>N</b>	<b>Y</b>	...
----------	----------	----------	----------	----------	----------	-----

	$x_0$	$x_1$	$x_2$	$x_3$	$x_4$	$x_5$	...
$x_0$	<b>N</b>	<b>N</b>	<b>N</b>	<b>N</b>	<b>N</b>	<b>Y</b>	...
$x_1$	<b>Y</b>	<b>Y</b>	<b>Y</b>	<b>Y</b>	<b>Y</b>	<b>Y</b>	...
$x_2$	<b>N</b>	<b>Y</b>	<b>N</b>	<b>Y</b>	<b>Y</b>	<b>N</b>	...
$x_3$	<b>Y</b>	<b>N</b>	<b>Y</b>	<b>N</b>	<b>Y</b>	<b>N</b>	...
$x_4$	<b>N</b>	<b>N</b>	<b>Y</b>	<b>Y</b>	<b>N</b>	<b>Y</b>	...
$x_5$	<b>Y</b>	<b>N</b>	<b>N</b>	<b>N</b>	<b>N</b>	<b>Y</b>	...
...	...	...	...	...	...	...	...

<b>Y</b>	<b>N</b>	<b>Y</b>	<b>Y</b>	<b>Y</b>	<b>N</b>	...
----------	----------	----------	----------	----------	----------	-----



	$x_0$	$x_1$	$x_2$	$x_3$	$x_4$	$x_5$	...
$x_0$	<b>N</b>	<b>N</b>	<b>N</b>	<b>N</b>	<b>N</b>	<b>Y</b>	...
$x_1$	<b>Y</b>	<b>Y</b>	<b>Y</b>	<b>Y</b>	<b>Y</b>	<b>Y</b>	...
$x_2$	<b>N</b>	<b>Y</b>	<b>N</b>	<b>Y</b>	<b>Y</b>	<b>N</b>	...
$x_3$	<b>Y</b>	<b>N</b>	<b>Y</b>	<b>N</b>	<b>Y</b>	<b>N</b>	...
$x_4$	<b>N</b>	<b>N</b>	<b>Y</b>	<b>Y</b>	<b>N</b>	<b>Y</b>	...
$x_5$	<b>Y</b>	<b>N</b>	<b>N</b>	<b>N</b>	<b>N</b>	<b>Y</b>	...
...	...	...	...	...	...	...	...

<b>Y</b>	<b>N</b>	<b>Y</b>	<b>Y</b>	<b>Y</b>	<b>N</b>	...
----------	----------	----------	----------	----------	----------	-----

	$x_0$	$x_1$	$x_2$	$x_3$	$x_4$	$x_5$	...
$x_0$	N	N	N	N	N	Y	...
$x_1$	Y	Y	Y	Y	Y	Y	...
$x_2$	N	Y	N	Y	Y	N	...
$x_3$	Y	N	Y	N	Y	N	...
$x_4$	N	N	Y	Y	N	Y	...
$x_5$	Y	N	N	N	N	Y	...
...	...	...	...	...	...	...	...

Y	N	Y	Y	Y	N	...
---	---	---	---	---	---	-----

	$x_0$	$x_1$	$x_2$	$x_3$	$x_4$	$x_5$	...
$x_0$	N	N	N	N	N	Y	...
$x_1$	Y	Y	Y	Y	Y	Y	...
$x_2$	N	Y	N	Y	Y	N	...
$x_3$	Y	N	Y	N	Y	N	...
$x_4$	N	N	Y	Y	N	Y	...
$x_5$	Y	N	N	N	N	Y	...
...	...	...	...	...	...	...	...

Y	N	Y	Y	Y	N	...
---	---	---	---	---	---	-----

	$x_0$	$x_1$	$x_2$	$x_3$	$x_4$	$x_5$	...
$x_0$	N	N	N	N	N	Y	...
$x_1$	Y	Y	Y	Y	Y	Y	...
$x_2$	N	Y	N	Y	Y	N	...
$x_3$	Y	N	Y	N	Y	N	...
$x_4$	N	N	Y	Y	N	Y	...
$x_5$	Y	N	N	N	N	Y	...
...	...	...	...	...	...	...	...

Y	N	Y	Y	Y	N	...
---	---	---	---	---	---	-----

	$x_0$	$x_1$	$x_2$	$x_3$	$x_4$	$x_5$	...
$x_0$	<b>N</b>	<b>N</b>	<b>N</b>	<b>N</b>	<b>N</b>	<b>Y</b>	...
$x_1$	<b>Y</b>	<b>Y</b>	<b>Y</b>	<b>Y</b>	<b>Y</b>	<b>Y</b>	...
$x_2$	<b>N</b>	<b>Y</b>	<b>N</b>	<b>Y</b>	<b>Y</b>	<b>N</b>	...
$x_3$	<b>Y</b>	<b>N</b>	<b>Y</b>	<b>N</b>	<b>Y</b>	<b>N</b>	...
$x_4$	<b>N</b>	<b>N</b>	<b>Y</b>	<b>Y</b>	<b>N</b>	<b>Y</b>	...
$x_5$	<b>Y</b>	<b>N</b>	<b>N</b>	<b>N</b>	<b>N</b>	<b>Y</b>	...
...	...	...	...	...	...	...	...

<b>Y</b>	<b>N</b>	<b>Y</b>	<b>Y</b>	<b>Y</b>	<b>N</b>	...
----------	----------	----------	----------	----------	----------	-----

	$x_0$	$x_1$	$x_2$	$x_3$	$x_4$	$x_5$	...
$x_0$	N	N	N	N	N	Y	...
$x_1$	Y	Y	Y	Y	Y	Y	...
$x_2$	N	Y	N	Y	Y	N	...
$x_3$	Y	N	Y	N	Y	N	...
$x_4$	N	N	Y	Y	N	Y	...
$x_5$	Y	N	N	N	N	Y	...
...	...	...	...	...	...	...	...

Y	N	Y	Y	Y	N	...
---	---	---	---	---	---	-----

	$x_0$	$x_1$	$x_2$	$x_3$	$x_4$	$x_5$	...
$x_0$	N	N	N	N	N	Y	...
$x_1$	Y	Y	Y	Y	Y	Y	...
$x_2$	N	Y	N	Y	Y	N	...
$x_3$	Y	N	Y	N	Y	N	...
$x_4$	N	N	Y	Y	N	Y	...
$x_5$	Y	N	N	N	N	Y	...
...	...	...	...	...	...	...	...

Y	N	Y	Y	Y	N	...
---	---	---	---	---	---	-----

# The Diagonalization Proof

- No matter how we pair up elements of  $S$  and subsets of  $S$ , the complemented diagonal won't appear in the table.
  - In row  $n$ , the  $n$ th element must be wrong.
- No matter how we pair up elements of  $S$  and subsets of  $S$ , there is *always* at least one subset left over.
- This result is ***Cantor's theorem***: Every set is strictly smaller than its power set:

**If  $S$  is a set, then  $|S| < |\wp(S)|$ .**



# Infinite Cardinalities

- By Cantor's Theorem:

$$|\mathbb{N}| < |\wp(\mathbb{N})|$$

$$|\wp(\mathbb{N})| < |\wp(\wp(\mathbb{N}))|$$

$$|\wp(\wp(\mathbb{N}))| < |\wp(\wp(\wp(\mathbb{N})))|$$

$$|\wp(\wp(\wp(\mathbb{N})))| < |\wp(\wp(\wp(\wp(\mathbb{N}))))|$$

...

- ***Not all infinite sets have the same size!***
- ***There is no biggest infinity!***
- ***There are infinitely many infinities!***

What does this have to do  
with computation?

*“The set of all computer programs”*

*“The set of all problems to solve”*

# Where We're Going

- A *string* is a sequence of characters.
- We're going to prove the following results:
  - There are *at most* as many programs as there are strings.
  - There are *at least* as many problems as there are sets of strings.
- This leads to some *incredible* results – we'll see why in a minute!

# Where We're Going

A *string* is a sequence of characters.

We're going to prove the following results:

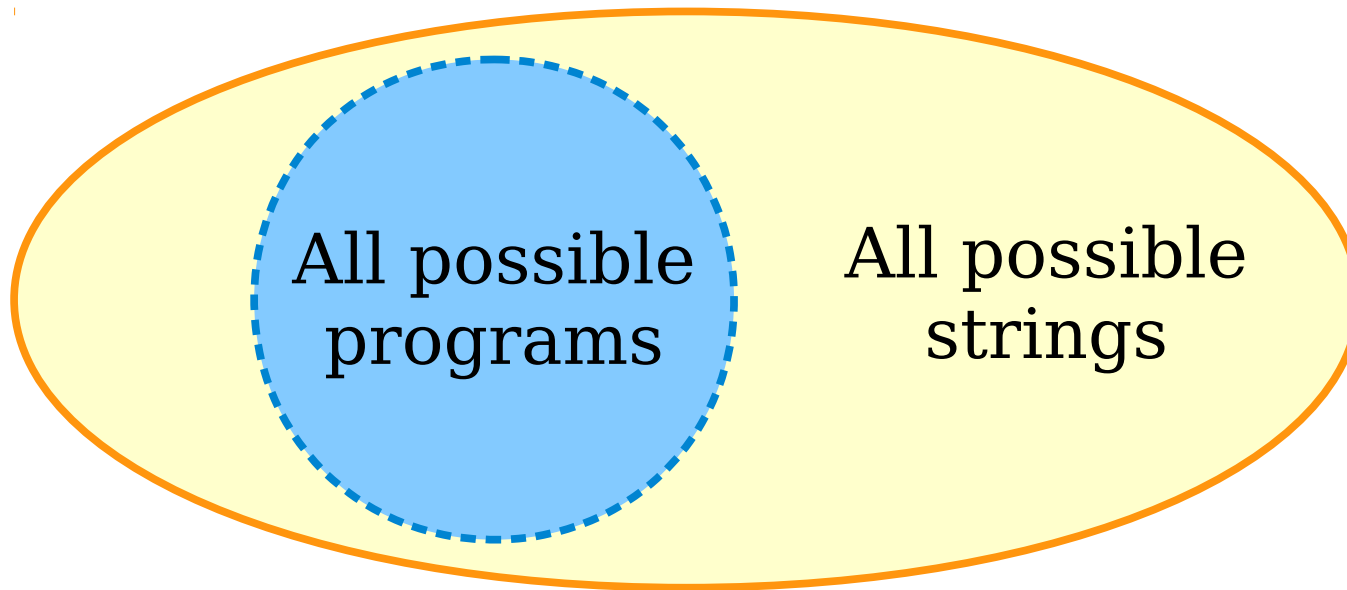
- There are *at most* as many programs as there are strings.

There are *at least* as many problems as there are sets of strings.

This leads to some *incredible* results – we'll see why in a minute!

# Strings and Programs

- The source code of a computer program is just a (long, structured, well-commented) string of text.
- All programs are strings, but not all strings are necessarily programs.



$$|\mathbf{Programs}| \leq |\mathbf{Strings}|$$

# Where We're Going

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- A **string** is a sequence of characters.
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# Where We're Going

A *string* is a sequence of characters.

We're going to prove the following results:

There are *at most* as many programs as there are strings. ✓

- There are *at least* as many problems as there are sets of strings.

This leads to some *incredible* results – we'll see why in a minute!

# Strings and Problems

- There is a connection between the number of sets of strings and the number of problems to solve.
- Let  $S$  be any set of strings. This set  $S$  gives rise to a problem to solve:

**Given a string  $w$ , determine whether  $w \in S$ .**

# Strings and Problems

**Given a string  $w$ , determine whether  $w \in S$ .**

- Suppose that  $S$  is the set

$$S = \{ "a", "b", "c", \dots, "z" \}$$

- From this set  $S$ , we get this problem:

**Given a string  $w$ , determine whether  $w$  is a single lower-case English letter.**

# Strings and Problems

**Given a string  $w$ , determine whether  $w \in S$ .**

- Suppose that  $S$  is the set
$$S = \{ "0", "1", "2", \dots, "9", "10", "11", \dots \}$$
- From this set  $S$ , we get this problem:

**Given a string  $w$ , determine whether  $w$  represents a natural number.**

# Strings and Problems

**Given a string  $w$ , determine whether  $w \in S$ .**

- Suppose that  $S$  is the set

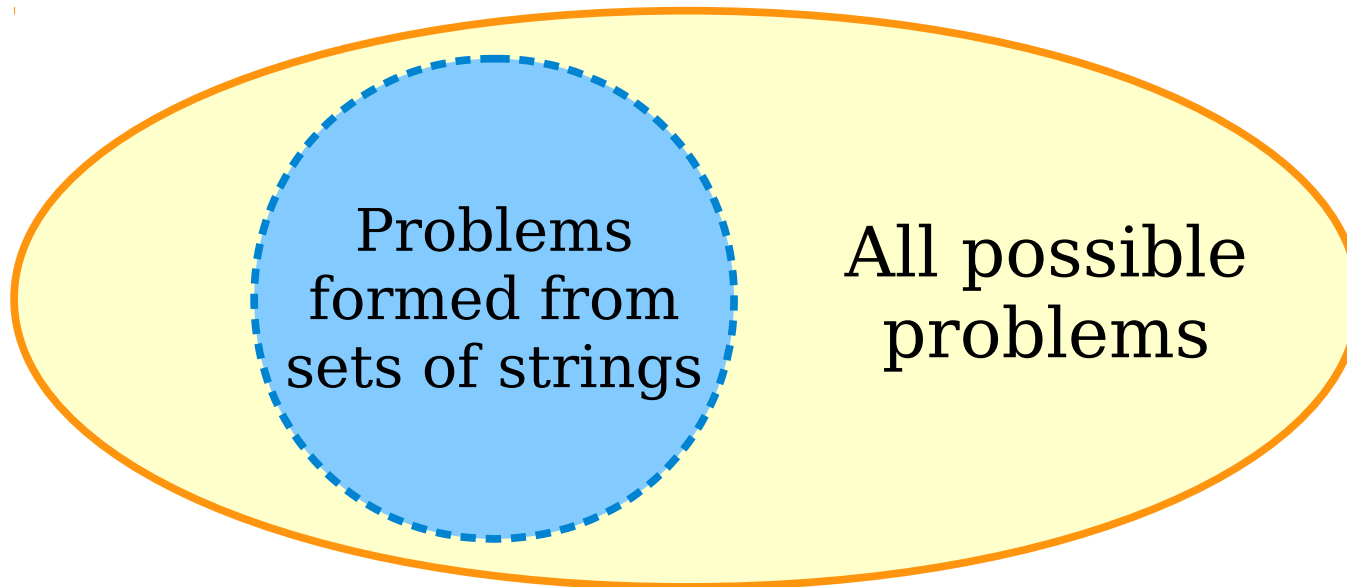
$$S = \{ p \mid p \text{ is a legal Java program} \}$$

- From this set  $S$ , we get this problem:

**Given a string  $w$ , determine whether  $w$  is a legal Java program.**

# Strings and Problems

- Every set of strings gives rise to a unique problem to solve.
- Other problems exist as well.



$$|\mathbf{Sets\ of\ Strings}| \leq |\mathbf{Problems}|$$

# Where We're Going

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There are *at most* as many programs as there are strings. ✓

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# Where We're Going

A *string* is a sequence of characters.

We're going to prove the following results:

There are *at most* as many programs as there are strings. ✓

There are *at least* as many problems as there are sets of strings. ✓

- This leads to some *incredible* results – we'll see why ~~in a minute!~~ *right now!*

Every computer program is a string.

So, the number of programs is at most the number of strings.

From Cantor's Theorem, we know that there are more sets of strings than strings.

There are at least as many problems as there are sets of strings.

$$|\mathbf{Programs}| \leq |\mathbf{Strings}| < |\mathbf{Sets\ of\ Strings}| \leq |\mathbf{Problems}|$$

Every computer program is a string.

So, the number of programs is at most the number of strings.

From Cantor's Theorem, we know that there are more sets of strings than strings.

There are at least as many problems as there are sets of strings.

$$|\mathbf{Programs}| < |\mathbf{Problems}|$$

*There are more problems to  
solve than there are programs  
to solve them.*

**|Programs| < |Problems|**

# It Gets Worse

- Using more advanced set theory, we can show that there are *infinitely more* problems than solutions.
- In fact, if you pick a totally random problem, the probability that you can solve it is *zero*.
- ***More troubling fact:*** We've just shown that *some* problems are impossible to solve with computers, but we don't know *which* problems those are!

We need to develop a more nuanced understanding of computation.

# Where We're Going

- ***What makes a problem impossible to solve with computers?***
  - Is there a deep reason why certain problems can't be solved with computers, or is it completely arbitrary?
  - How do you know when you're looking at an impossible problem?
  - Are these real-world problems, or are they highly contrived?
- ***How do we know that we're right?***
  - How can we back up our pictures with rigorous proofs?
  - How do we build a mathematical framework for studying computation?



# Next Time

- ***Mathematical Proof***
  - What is a mathematical proof?
  - How can we prove things with certainty?