Set Equality

Set Equality

- As we mentioned on Monday, two sets A and B are equal when they have exactly the same elements.
- Here's a little theorem that's very useful for showing that two sets are equal:

  Theorem: If A and B are sets where $A \subseteq B$ and $B \subseteq A$, then $A = B$.

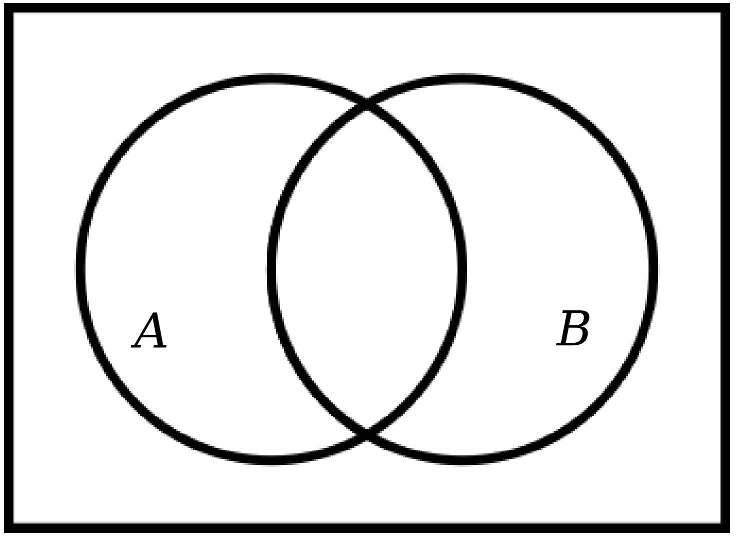
- We've included a proof of this result as an appendix to this slide deck. You should read over it on your own time.

A Trickier Theorem

- Our last theorem for today is this one, which comes to us from the annals of set theory:

Theorem: If A and B are sets and $A \cup B \subseteq A \cap B$, then $A = B$.

- Unlike our previous theorem, this one is a lot harder to see using Venn diagrams alone.



Tackling our Theorem

Theorem: If A and B are sets and $A \cup B \subseteq A \cap B$, then $A = B$.

- Before we Flail and Panic, let's see if we can tease out some info about what this proof might look like.

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- We're going to assume $A \cup B \subseteq A \cap B$.
- We're going to prove that $A = B$.

Reasonable guess: let's try proving that $A \subseteq B$ and that $B \subseteq A$.

Lemma: If S and T are sets and $S \cup T \subseteq S \cap T$, then $S \subseteq T$.

A *lemma* is a smaller proof that's designed to build into a larger one. Think of it like program decomposition, except for proofs!

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What We've Covered

- ***What is a mathematical proof?***
  - An argument – mostly written in English – outlining a mathematical argument.
- ***What is a direct proof?***
  - It's a proof where you begin from some initial assumptions and reason your way to the conclusion.
- ***What are universal and existential statements?***
  - Universal statements make a claim about all objects of one type. Existential statements make claims about at least one object of some type.
- ***How do we write proofs about set theory?***
  - By calling back to definitions! Definitions are key.

Appendix: Set Equality

Set Equality

- If A and B are sets, we say that $A = B$ precisely when the following statement is true:

For any object *x*, $x \in A$ if and only if $x \in B$.

- (This is called the ***axiom of extensionality***.)
- In practice, this definition is tricky to work with.
- It's often easier to use the following result to show that two sets are equal:

For any sets *A* and *B*,
if $A \subseteq B$ and $B \subseteq A$, then $A = B$.

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First, we'll prove that if $x \in A$, then $x \in B$. To do so, take any $x \in A$. Since $A \subseteq B$ and $x \in A$, we see that $x \in B$, as required.

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Since we've proven both directions of implication, we see that $A = B$.

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