Proof by Exhaustion

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This is called a proof by cases (alternatively, a proof by exhaustion) and works by showing that the theorem is true regardless of what specific outcome arises.

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After splitting into cases, it's a good idea to summarize what you just did so that the reader knows what to take away from it.

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Some Little Exercises

- Here's a list of other theorems that are true about odd and even numbers:
  - ***Theorem:*** The sum and difference of any two even numbers is even.
  - ***Theorem:*** The sum and difference of an odd number and an even number is odd.
  - ***Theorem:*** The product of any integer and an even number is even.
  - ***Theorem:*** The product of any two odd numbers is odd.
- Feel free to use these results going forward.
- If you'd like to practice the techniques from today, try your hand at proving some of these results!