Proof by Contradiction

“When you have eliminated all which is impossible, then whatever remains, however improbable, must be the truth.”

- Sir Arthur Conan Doyle, *The Adventure of the Blanched Soldier*

Proof by Contradiction

- A ***proof by contradiction*** is a proof that works as follows:
  - To prove that P is true, assume that P is *not* true.
  - Beginning with this assumption, use logical reasoning to conclude something that is clearly impossible.
    - For example, that $1 = 0$, that $x \in S$ and $x \notin S$, etc.
  - This means that if P is false, something that cannot possibly happen, happens!
  - Therefore, P can't be false, so it must be true.

An Example: ***Set Cardinalities***

Set Cardinalities

- We've seen sets of many different cardinalities:
  - $|\emptyset| = 0$
  - $|\{1, 2, 3\}| = 3$
  - $|\{ n \in \mathbb{N} \mid n < 137\}| = 137$
  - $|\mathbb{N}| = \aleph_0$.
- These span from the finite up through the infinite.
- ***Question:*** Is there a "largest" set? That is, is there a set that's bigger than every other set?

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One option: "there *is* a largest set."

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Notice that we're announcing

1. that this is a proof by contradiction, and
2. what, specifically, we're assuming.

This helps the reader understand where we're going. Remember – proofs are meant to be read by other people!

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The three key pieces:

1. Say that the proof is by contradiction.
2. Say what you are assuming is the negation of the statement to prove.
3. Say you have reached a contradiction and what the contradiction means.

In CS103, please include all these steps in your proofs!

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