Universal and Existential Statements

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Proof:

Which of the following should be the next sentence of this proof?

A. "Pick any odd integer, $n = 137$."
B. "Pick any odd integer n."
C. "Pick any odd integer n and arbitrary integers r and s where $r^2 - s^2 = n$."

Answer at **PollEv.com/cs103** or text **CS103** to **22333** once to join, then **A**, **B**, or **C**.

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This is a very different sort of request than what we've seen in the past. How on earth do we go about proving something like this?

Universal vs. Existential Statements

- A ***universal statement*** is a statement of the form

For all *x*, `[some-property]` holds for *x*.

- We've seen how to prove these statements.
- An ***existential statement*** is a statement of the form

There is some *x* where `[some-property]` holds for *x*.

- How do you prove an existential statement?

Proving an Existential Statement

- Over the course of the quarter, we will see several different ways to prove an existential statement of the form

  There is an *x* where `[some-property]` holds for *x*.

- ***Simplest approach:*** Search far and wide, find an *x* that has the right property, then show why your choice is correct.

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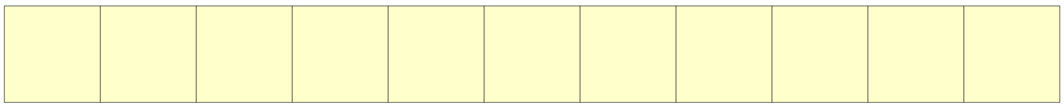

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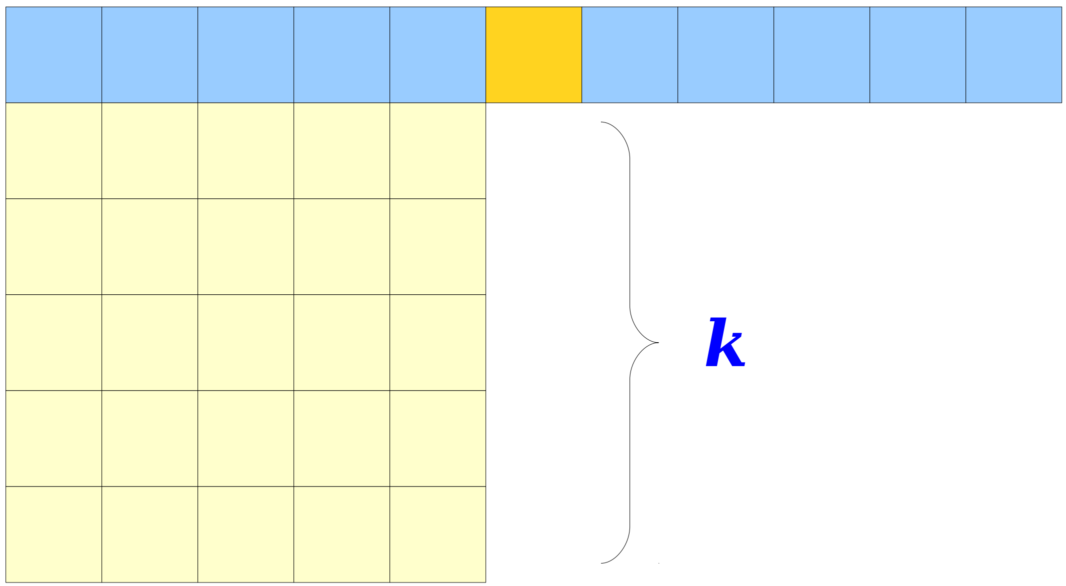

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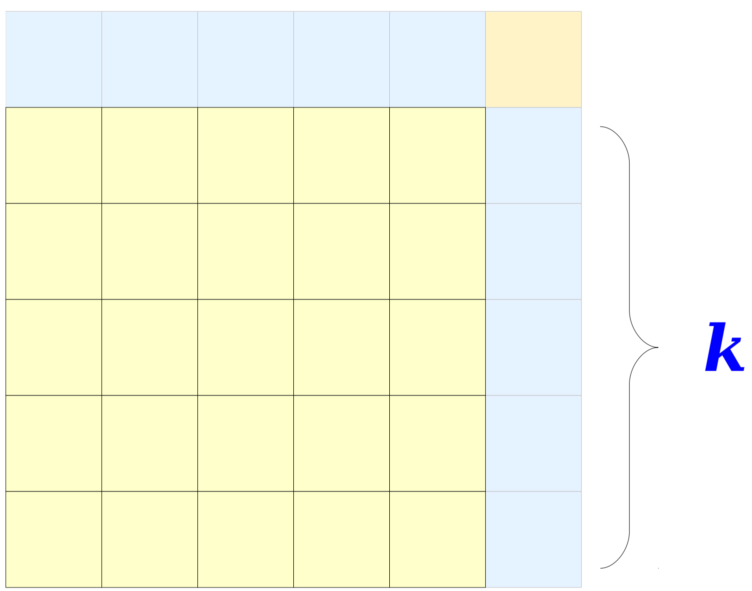

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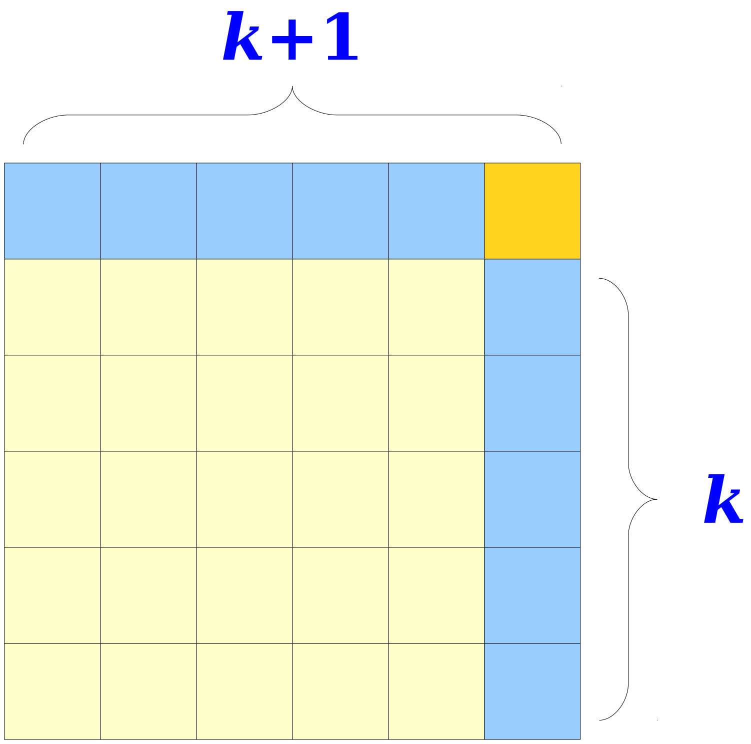
Our guess:

$(k+1)^2 - k^2 = n$



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Proof: Pick any odd integer n. Since n is odd, we know there is some integer k where $n = 2k + 1$.

Now, let $r = k+1$ and $s = k$.

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Proof: Pick any odd integer n. Since n is odd, we know there is some integer k where $n = 2k + 1$.

Now, let $r = k+1$ and $s = k$. Then we see that

$$r^2 - s^2 = (k+1)^2 - k^2$$

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Now, let $r = k+1$ and $s = k$. Then we see that

$$\begin{aligned} r^2 - s^2 &= (k+1)^2 - k^2 \\ &= k^2 + 2k + 1 - k^2 \end{aligned}$$

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Follow-Up Question: There are some integers that can't be written as $r^2 - s^2$ for any integers r and s.

Can you prove that every integer can be formed by adding and subtracting some combination of at most *three* perfect squares?