Direct Proofs

Two Quick Definitions

- An integer n is ***even*** if there is some integer k such that $n = 2k$.
  - This means that 0 is even.
- An integer n is ***odd*** if there is some integer k such that $n = 2k + 1$.
  - This means that 0 is not odd.
- We'll assume the following for now:
  - Every integer is either even or odd.
  - No integer is both even and odd.

Our First Direct Proof

Theorem: If n is an even integer, then n^2 is even.
Proof: Let n be an even integer.

Since n is even, there is some integer k such that $n = 2k$.

This means that $n^2 = (2k)^2 = 4k^2 = 2(2k^2)$.

From this, we see that there is an integer m (namely, $2k^2$) where $n^2 = 2m$.

Therefore, n^2 is even. ■

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Notice how we use the value of ***k*** that we obtained above. Giving names to quantities, even if we aren't fully sure what they are, allows us to manipulate them. This is similar to variables in programs.

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Our ultimate goal is to prove that $\boldsymbol{n^2}$ is even. This means that we need to find some $\boldsymbol{m}$ such that $\boldsymbol{n^2 = 2m}$. Here, we're explicitly showing how we can do that.

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Hey, that's what we were trying to show! We're done now.

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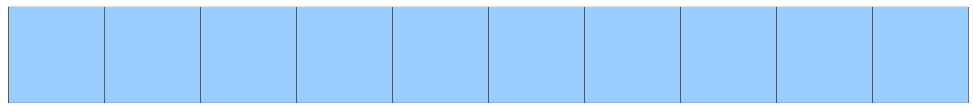
A Visual Intuition



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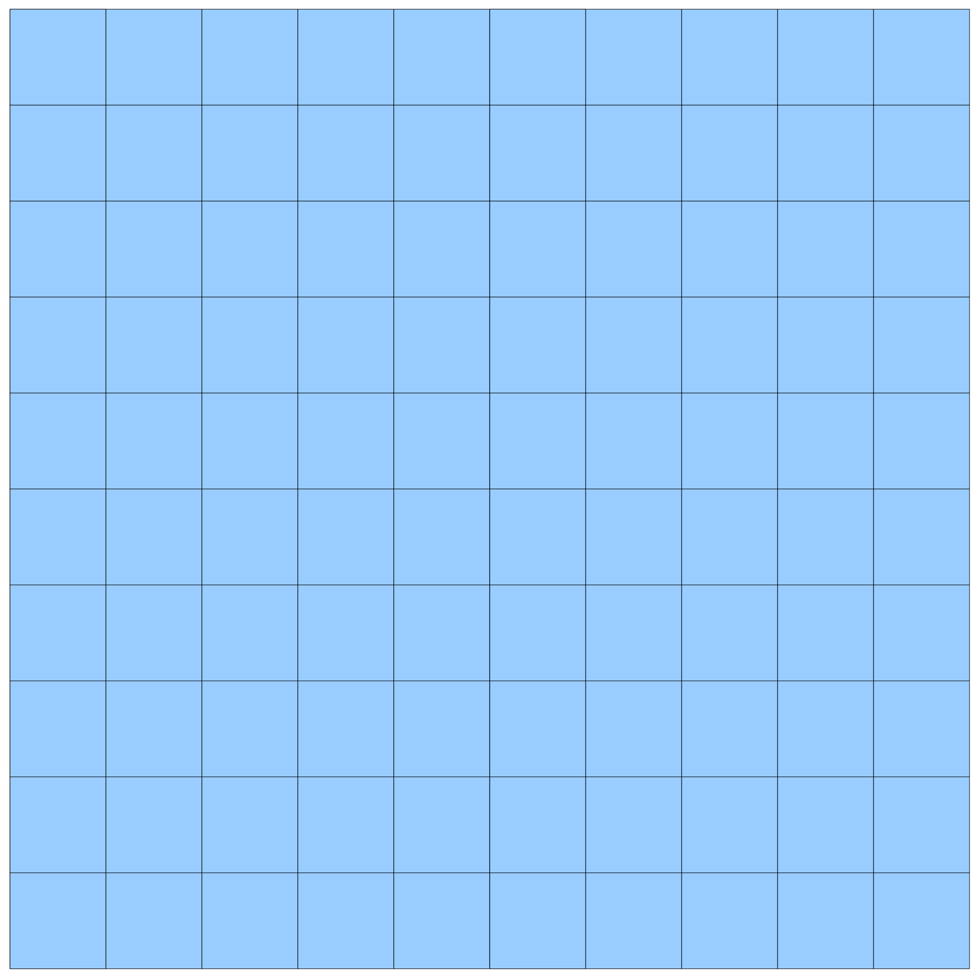

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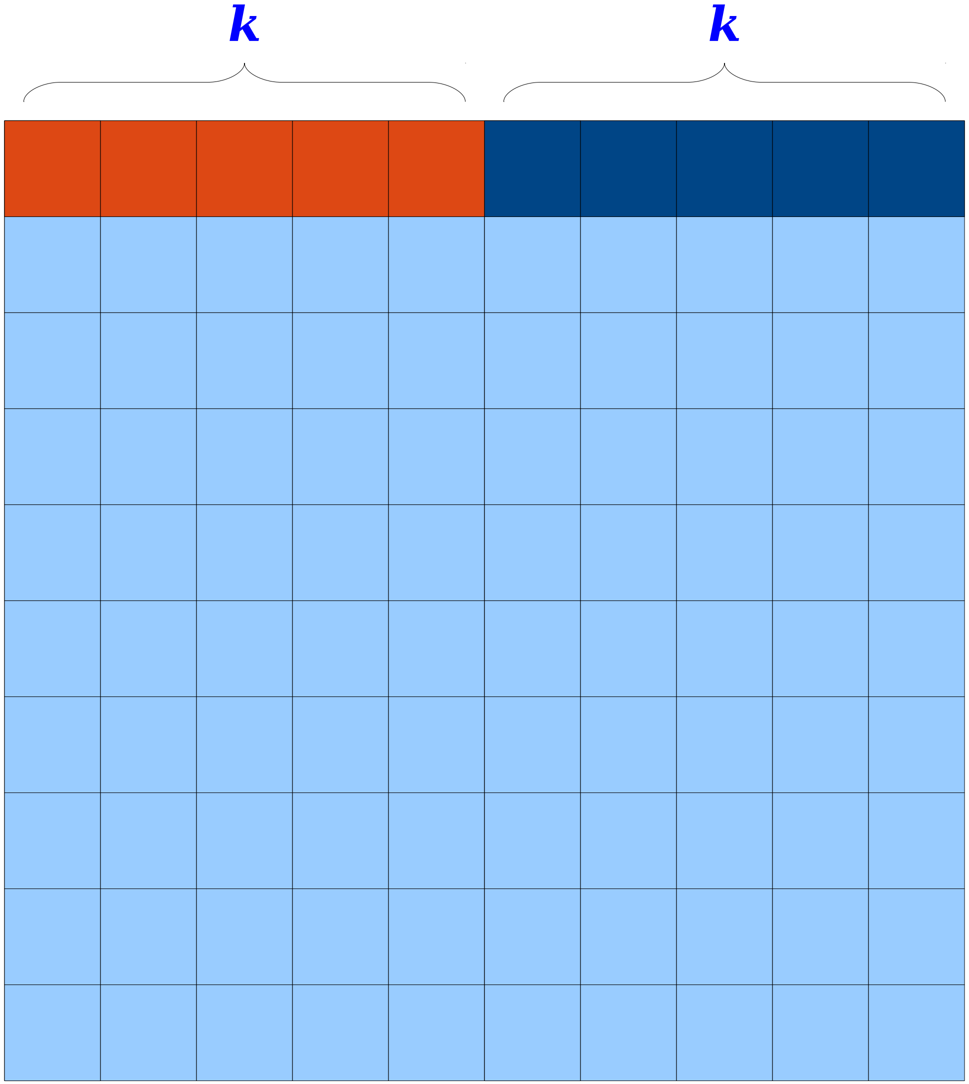

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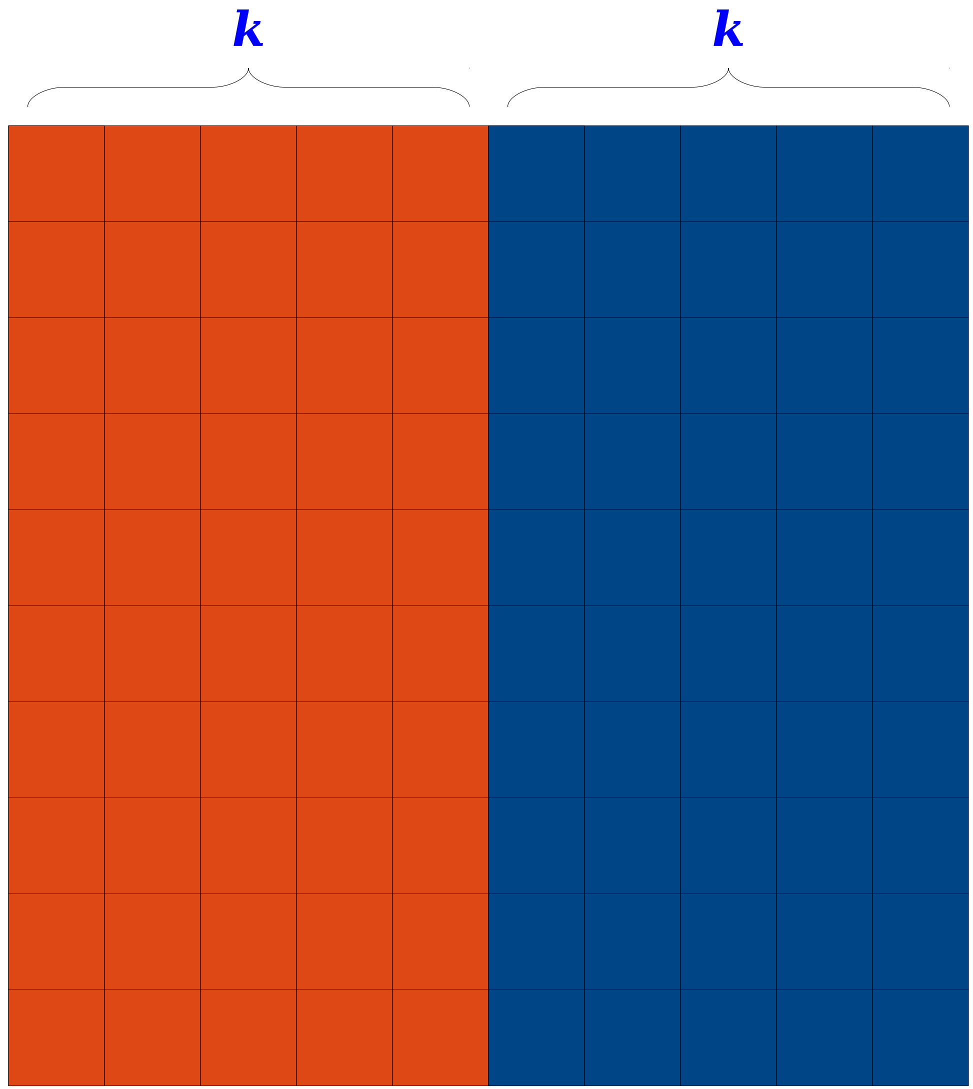

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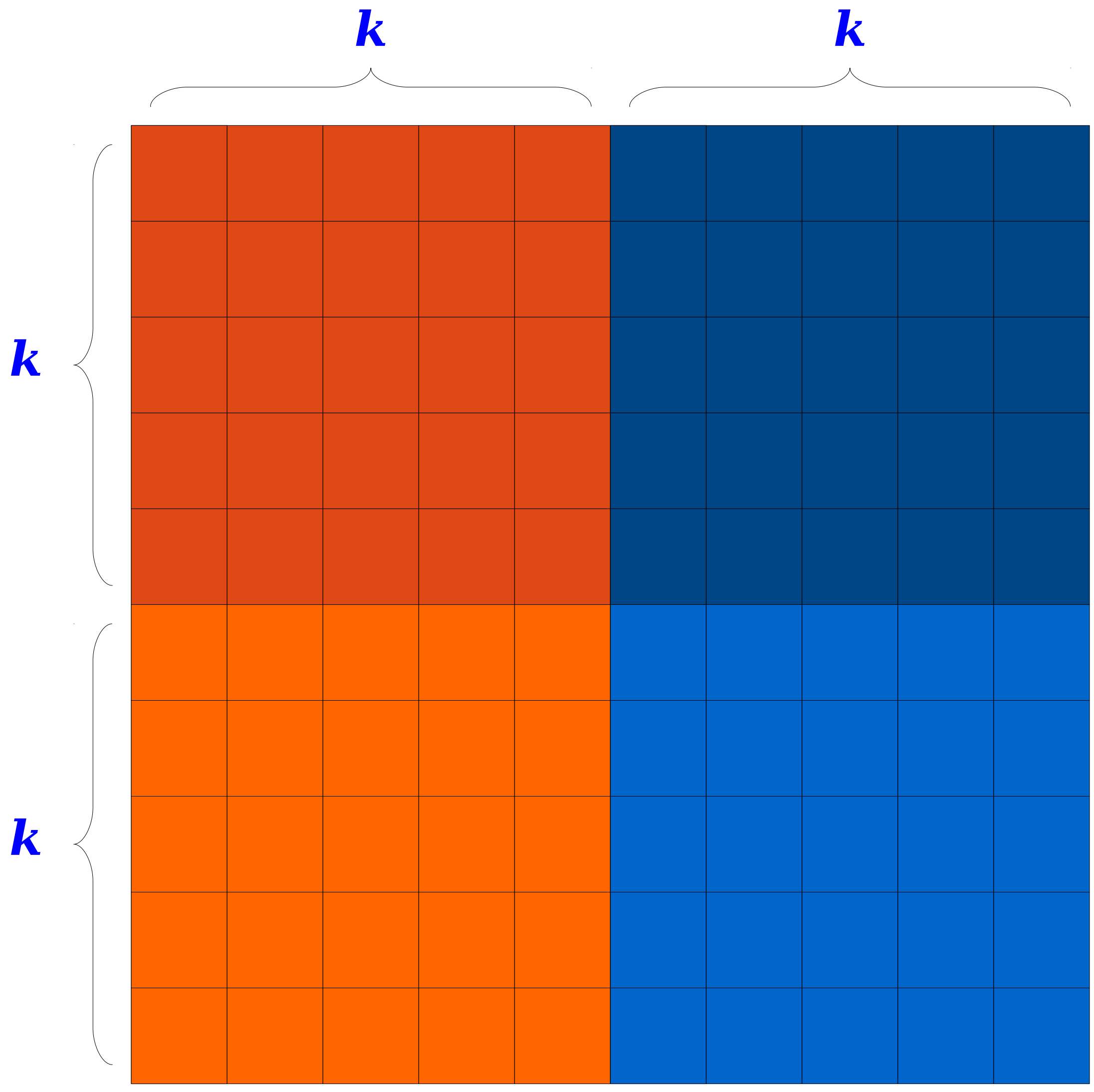
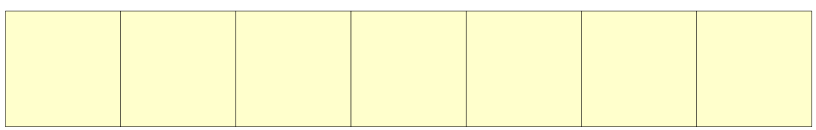
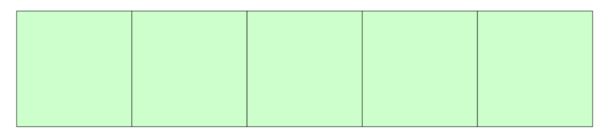


That wasn't so bad! Let's do another one.

Theorem: For any integers m and n, if m and n are odd, then $m + n$ is even.





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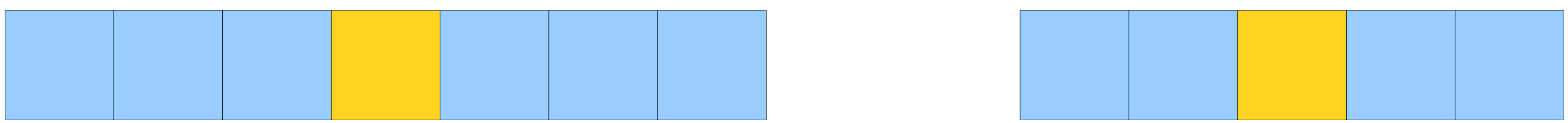

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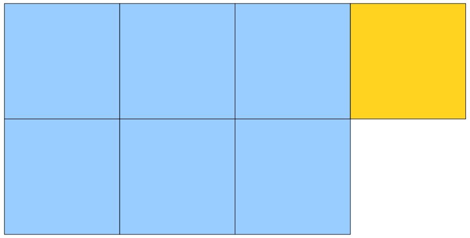

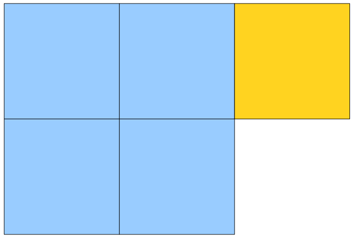

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How do we prove that this is true for *any* integers?

Proving Something Always Holds

- Many statements have the form

For any *x*, `[some-property]` holds of *x*.

- Examples:

For all integers n, if n is even, n^2 is even.

For any sets A, B, and C, if $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$.

For all sets S, $|S| < |\wp(S)|$.

Everything that drowns me makes me wanna fly.

- How do we prove these statements when there are (potentially) infinitely many cases to check?

Arbitrary Choices

- To prove that some property holds true for all possible x, show that no matter what choice of x you make, that property must be true.
- Start the proof by choosing x ***arbitrarily***:
  - "Let x be an arbitrary even integer."
  - "Let x be any set containing 137."
  - "Consider any x."
  - "Pick an odd integer x."
- Demonstrate that the property holds true for this choice of x.

ar·bi·trar·y
adjective /ˈärbiˌtrerē/

…not this one!

1. Based on random choice or personal whim, rather than any reason or system - *"his mealtimes were entirely arbitrary"*

2. *(of power or a ruling body)* Unrestrained and autocratic in the use of authority - *"arbitrary rule by King and bishops has been made impossible"*

3. *(of a constant or other quantity)* Of unspecified value

Use this definition…

Source: Google

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By picking ***m*** and ***n*** arbitrarily, anything we prove about ***m*** and ***n*** will generalize to all possible choices we could have made.

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Numbering these equalities lets us refer back to them later on, making the flow of the proof a bit easier to understand.

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Notice that we use k in the first equality and r in the second equality. That's because we know that n is twice something plus one, but we can't say for sure that it's k specifically.

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This is a grammatically correct and complete sentence! Proofs are expected to be written in complete sentences, so you'll often use punctuation at the end of formulas.

We recommend using the "mugga mugga" test – if you read a proof and replace all the mathematical notation with "mugga mugga," what comes back should be a valid sentence.

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Trace through this proof if $m = 7$ and $n = 9$. What is the resulting value of s?

A. 3
B. 8
C. 17

Answer at **PollEv.com/cs103** or text **CS103** to **22333** once to join, then **A**, **B**, or **C**.