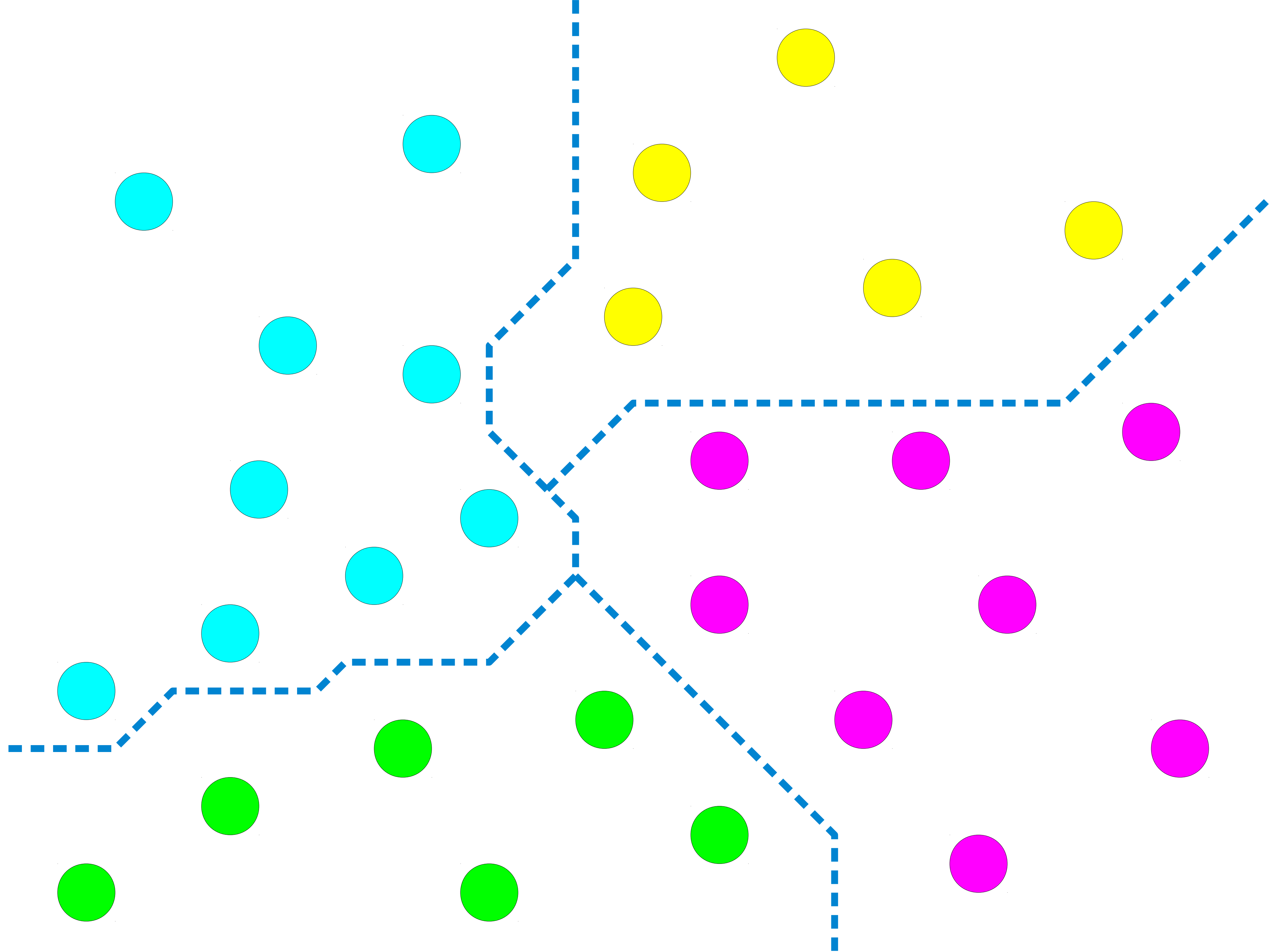
# Partitions



# Partitions

- A ***partition of a set*** is a way of splitting the set into disjoint, nonempty subsets so that every element belongs to exactly one subset.
  - Two sets are ***disjoint*** if their intersection is the empty set; formally, sets $S$ and $T$ are disjoint if $S \cap T = \emptyset$.
- Intuitively, a partition of a set breaks the set apart into smaller pieces.
- There doesn't have to be any rhyme or reason to what those pieces are, though often there is one.

# Partitions and Clustering

- If you have a set of data, you can often learn something from the data by finding a “good” partition of that data and inspecting the partitions.
  - Usually, the term ***clustering*** is used in data analysis rather than *partitioning*.
- Interested to learn more? Take CS161 or CS246!

What's the connection between partitions and binary relations?

$$\forall a \in A.\ aRa$$

$$\forall a \in A.\ \forall b \in A.\ (aRb \to bRa)$$

$$\forall a \in A.\ \forall b \in A.\ \forall c \in A.\ (aRb \land bRc \to aRc)$$

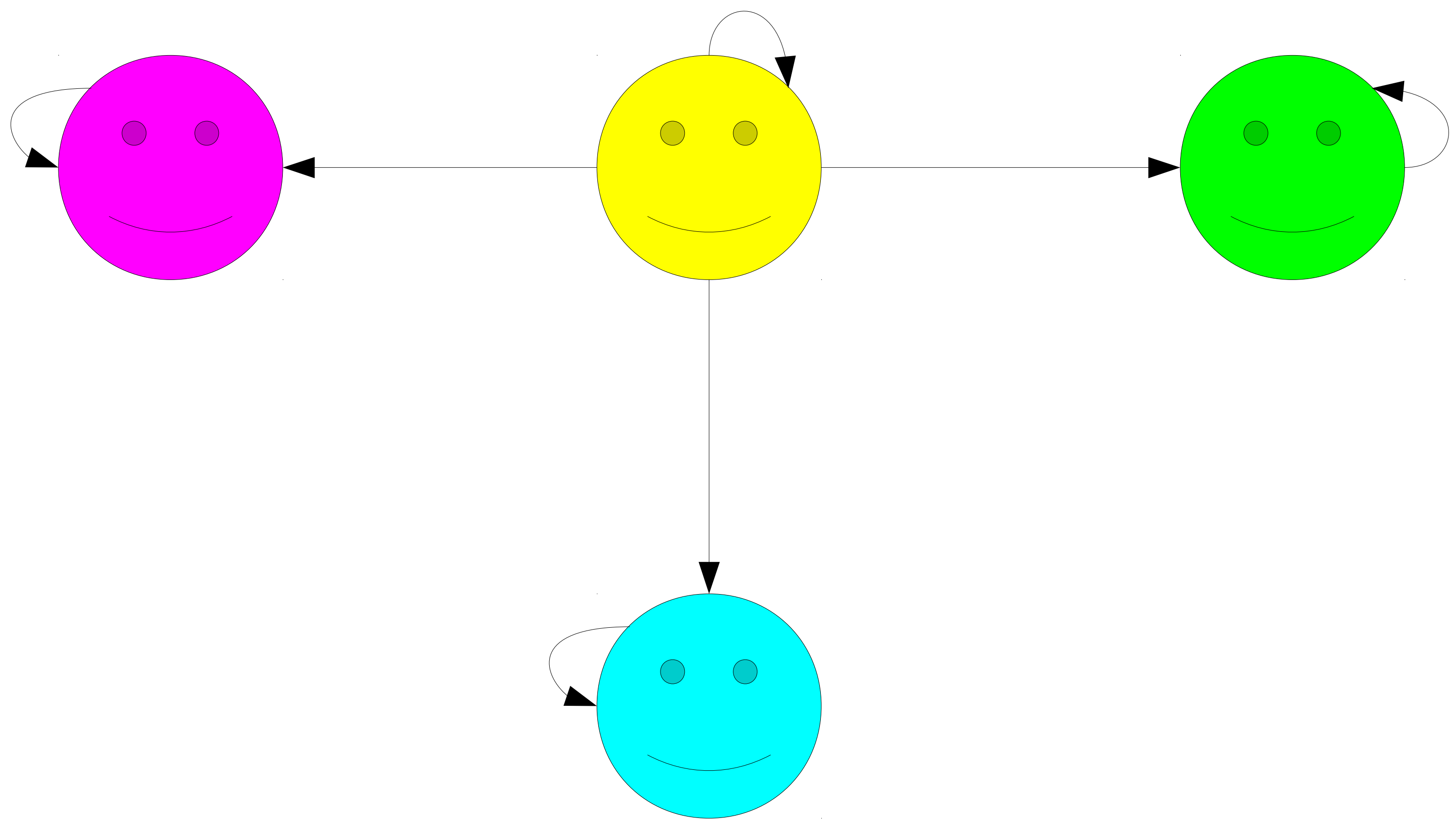
# Reflexivity

- Some relations always hold from any element to itself.
- Examples:
  - $x = x$ for any $x$.
  - $A \subseteq A$ for any set $A$.
  - $x \equiv_k x$ for any $x$.
- Relations of this sort are called ***reflexive***.
- Formally speaking, a binary relation $R$ over a set $A$ is reflexive if the following first-order statement is true:

$$\forall a \in A.\ aRa$$

(*"Every element is related to itself."*)

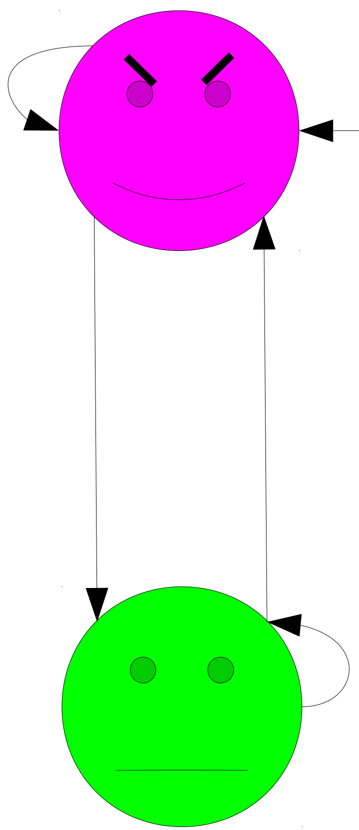
# Reflexivity Visualized



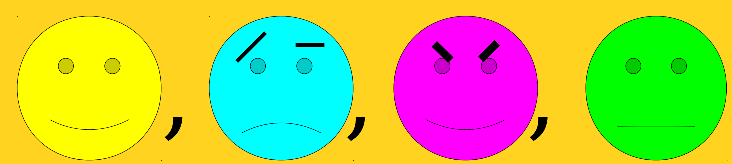
$\forall a \in A.\ aRa$

(*"Every element is related to itself."*)

Answer at **PollEv.com/cs103** or
text **CS103** to **22333** once to join, then **0**, **1**, **2**, **3**, **4**, or **5**.

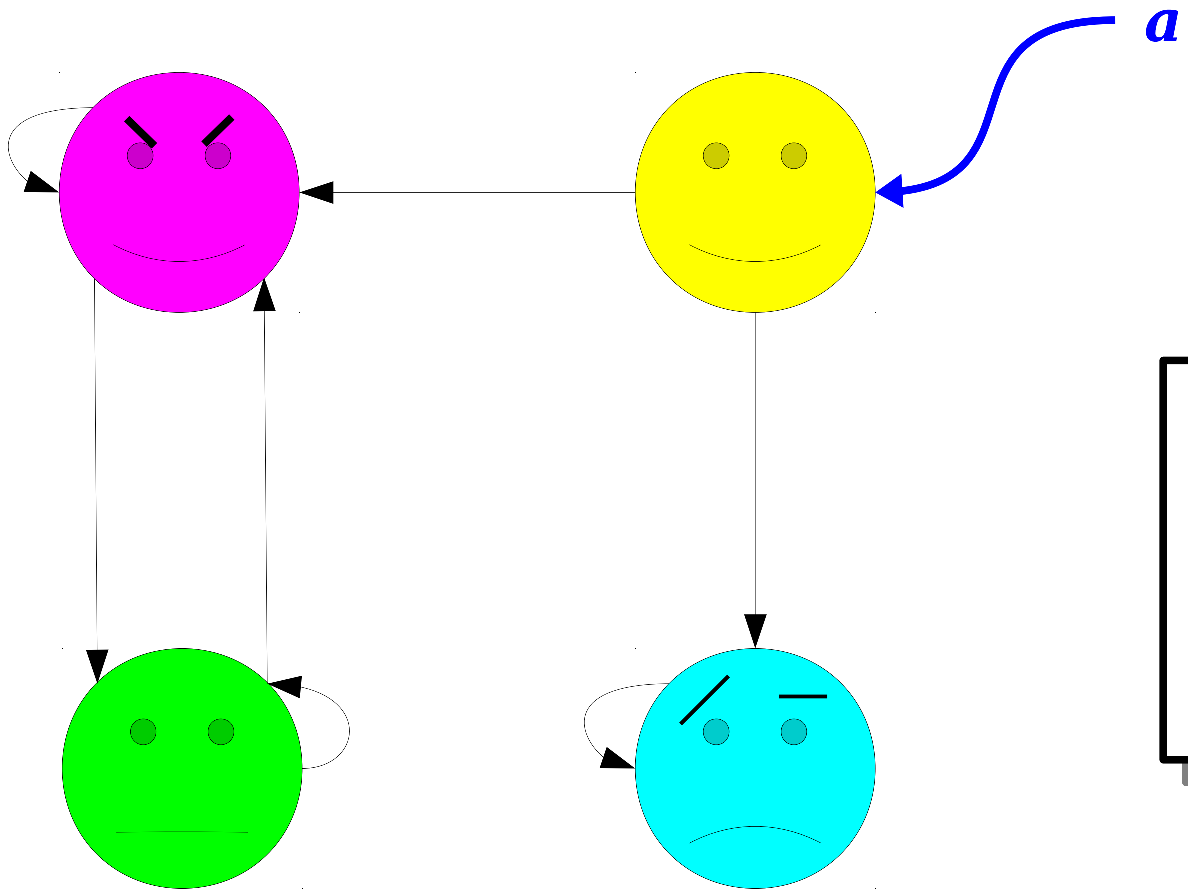


Let $R$ be the binary relation given by the drawing to the left. How many of the following objects are reflexive?

$R$, 

$$\forall a \in A.\ aRa$$

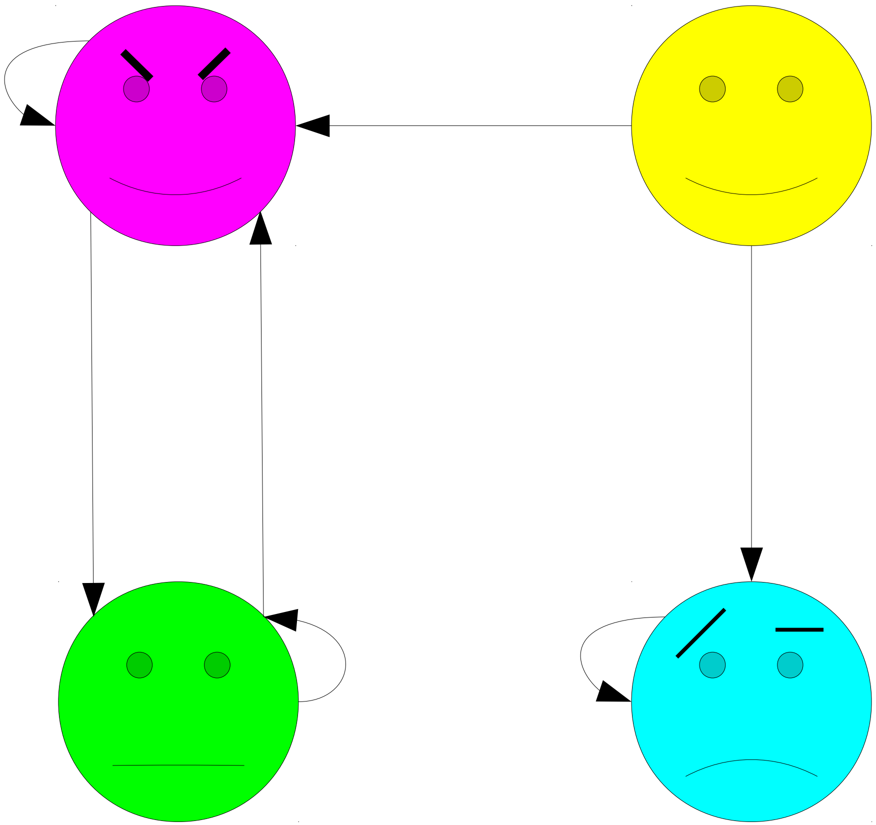
(*"Every element is related to itself."*)

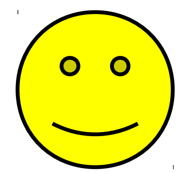



This means that $R$ is not reflexive, since the first-order logic statement given below is not true.

$$\forall a \in A.\ aRa$$

(*"Every element is related to itself."*)



Is  reflexive?

Reflexivity is a property of ***relations***, not ***individual objects***.

$\forall a \in ??.$ ***a***  ***a***

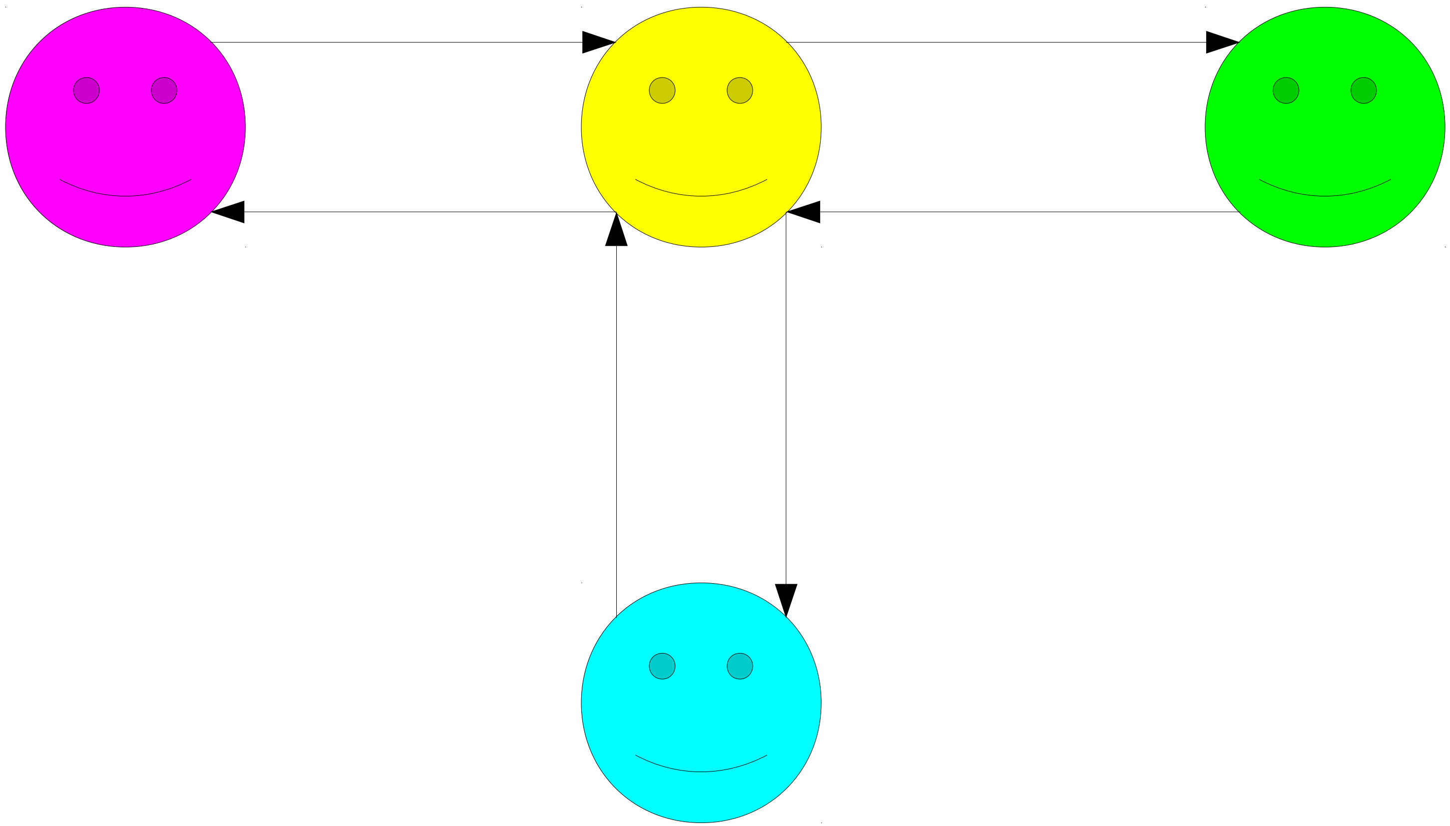
# Symmetry

- In some relations, the relative order of the objects doesn't matter.
- Examples:
  - If $x = y$, then $y = x$.
  - If $x \equiv_k y$, then $y \equiv_k x$.
- These relations are called ***symmetric***.
- Formally: a binary relation $R$ over a set $A$ is called *symmetric* if the following first-order statement is true about $R$:

$$\forall a \in A.\ \forall b \in A.\ (aRb \to bRa)$$

(*"If a is related to b, then b is related to a."*)

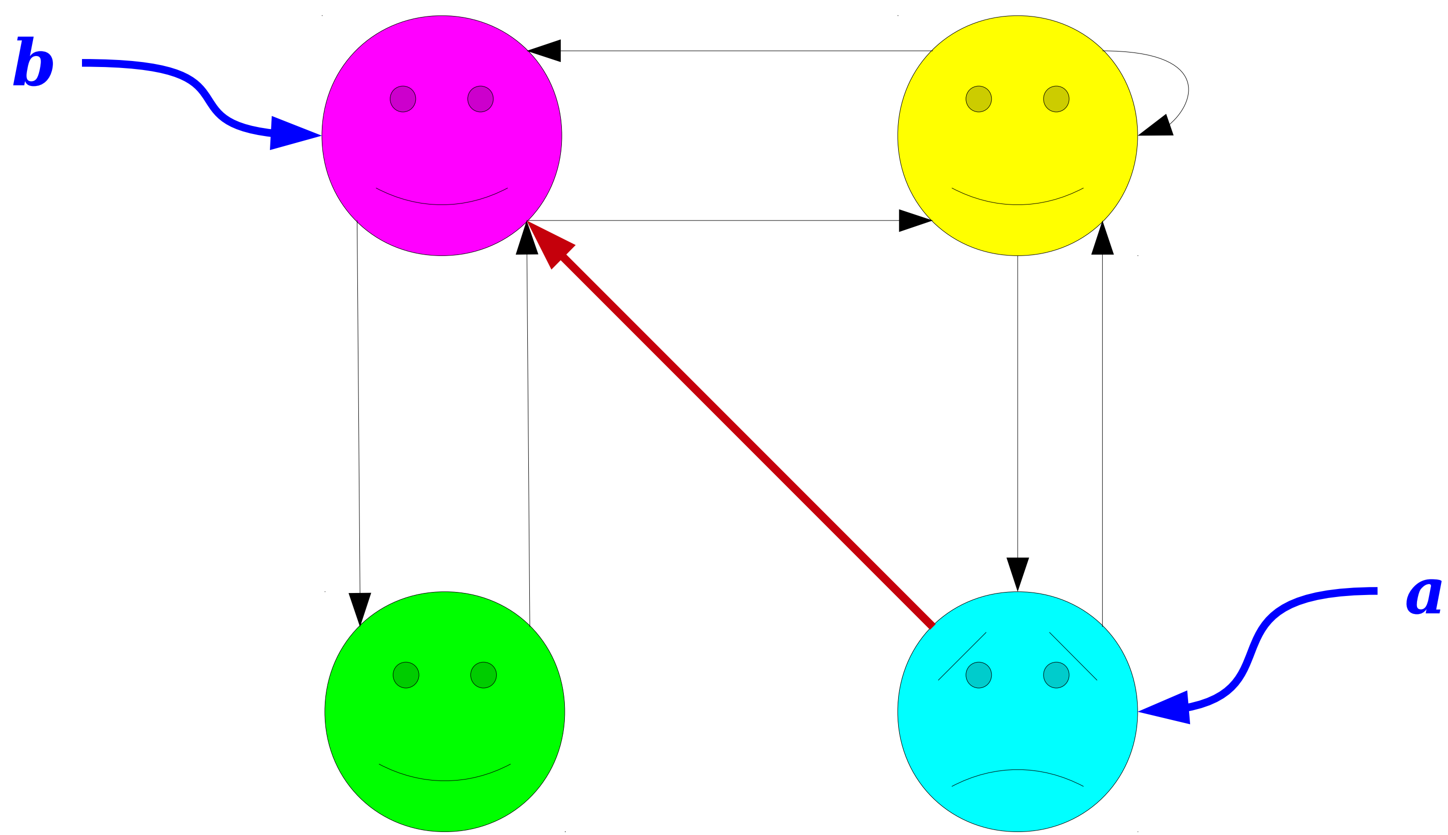
# Symmetry Visualized



$$\forall a \in A.\ \forall b \in A.\ (aRb \rightarrow bRa)$$

(*“If a is related to b, then b is related to a.”*)

# Is This Relation Symmetric?

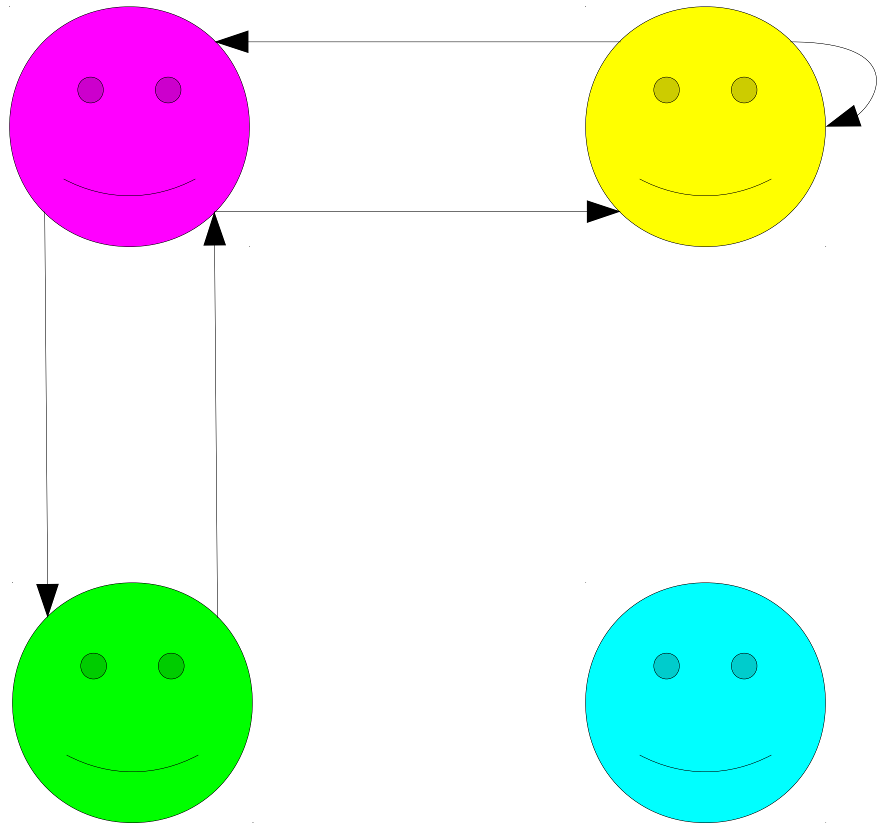



$\forall a \in A.\ \forall b \in A.\ (aRb \rightarrow bRa)$

(*"If a is related to b, then b is related to a."*)

Is this relation symmetric?

Answer at **PollEv.com/cs103** or
text **CS103** to **22333** once to join, then **Y** or **N**.



$$\forall a \in A.\ \forall b \in A.\ (aRb \rightarrow bRa)$$

(*"If a is related to b, then b is related to a."*)

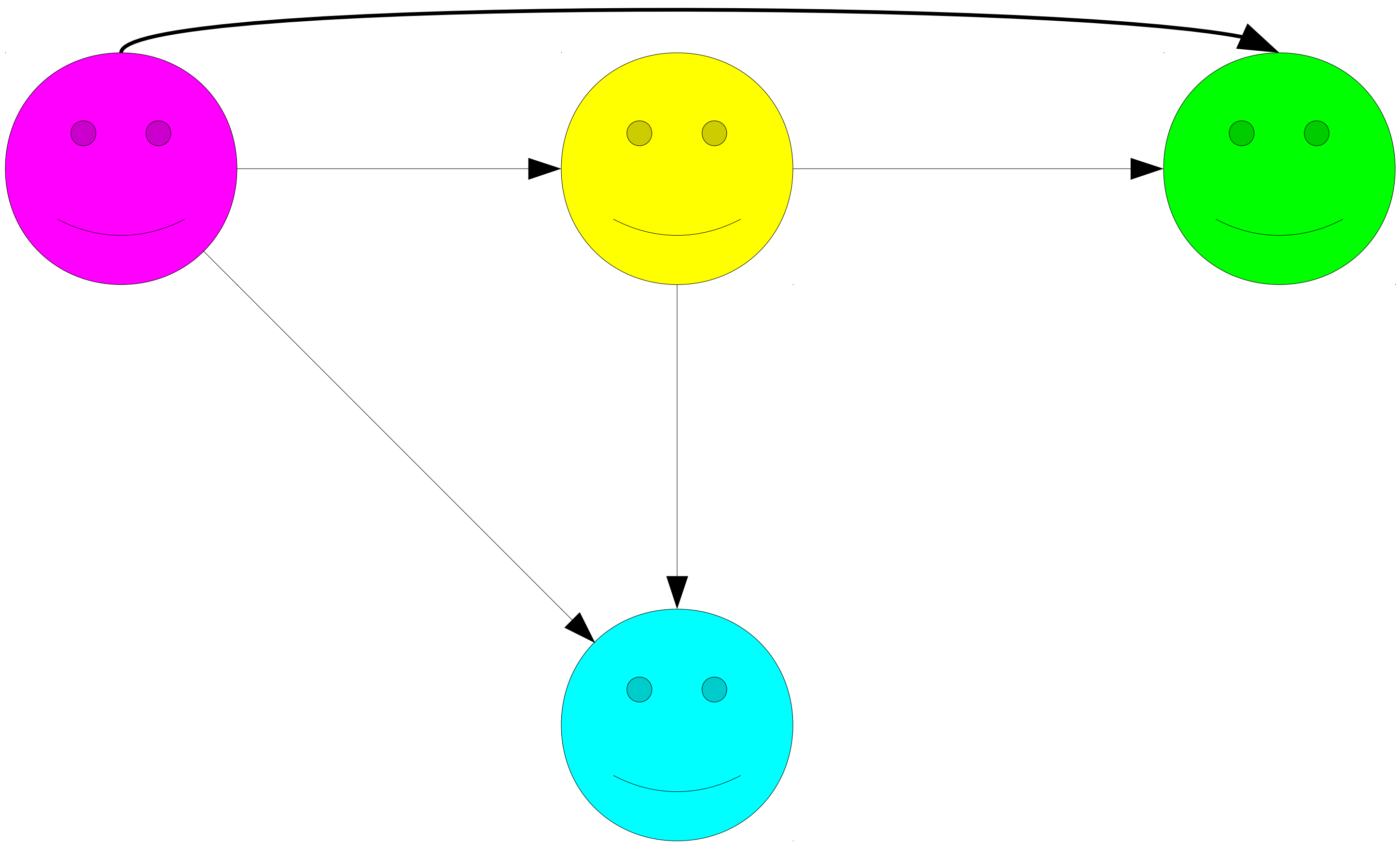
# Transitivity

- Many relations can be chained together.
- Examples:
  - If $x = y$ and $y = z$, then $x = z$.
  - If $R \subseteq S$ and $S \subseteq T$, then $R \subseteq T$.
  - If $x \equiv_k y$ and $y \equiv_k z$, then $x \equiv_k z$.
- These relations are called ***transitive***.
- A binary relation $R$ over a set $A$ is called *transitive* if the following first-order statement is true about $R$:

$$\forall a \in A.\ \forall b \in A.\ \forall c \in A.\ (aRb \land bRc \to aRc)$$

(*"Whenever a is related to b and b is related to c, we know a is related to c."*)

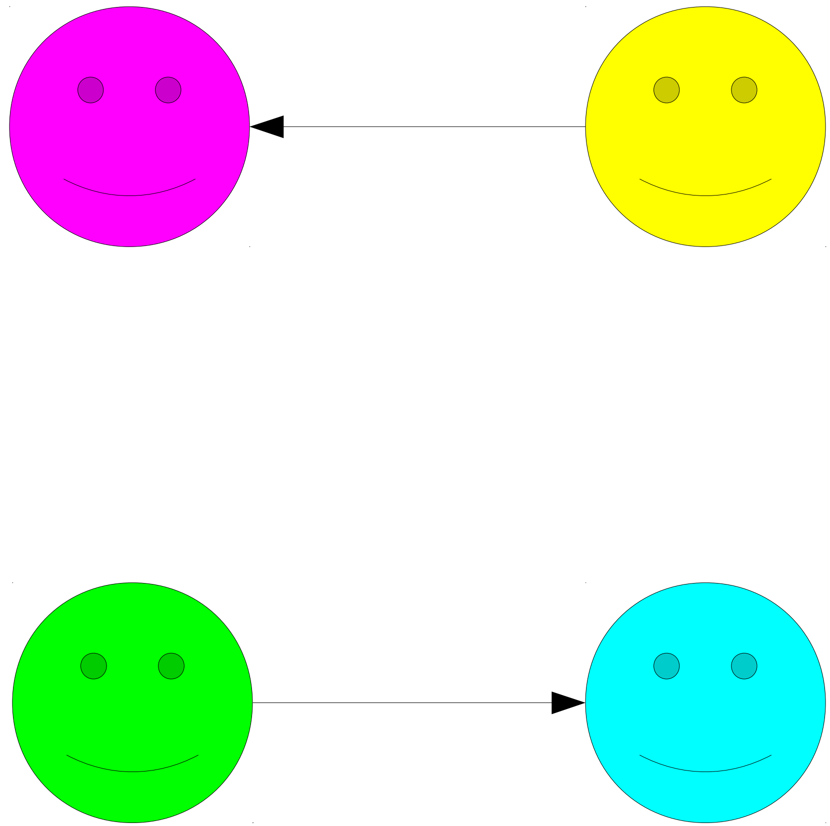
# Transitivity Visualized



$\forall a \in A. \forall b \in A. \forall c \in A. (aRb \land bRc \to aRc)$

*("Whenever a is related to b and b is related to c, we know a is related to c.)*

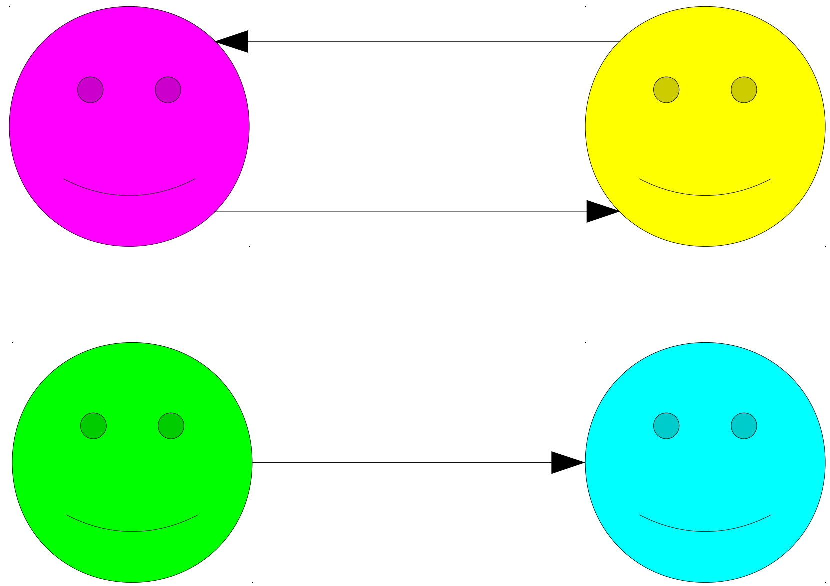
# Is This Relation Transitive?



$\forall a \in A.\ \forall b \in A.\ \forall c \in A.\ (aRb \land bRc \to aRc)$

(*"Whenever a is related to b and b is related to c, we know a is related to c.*)

Is this relation transitive?

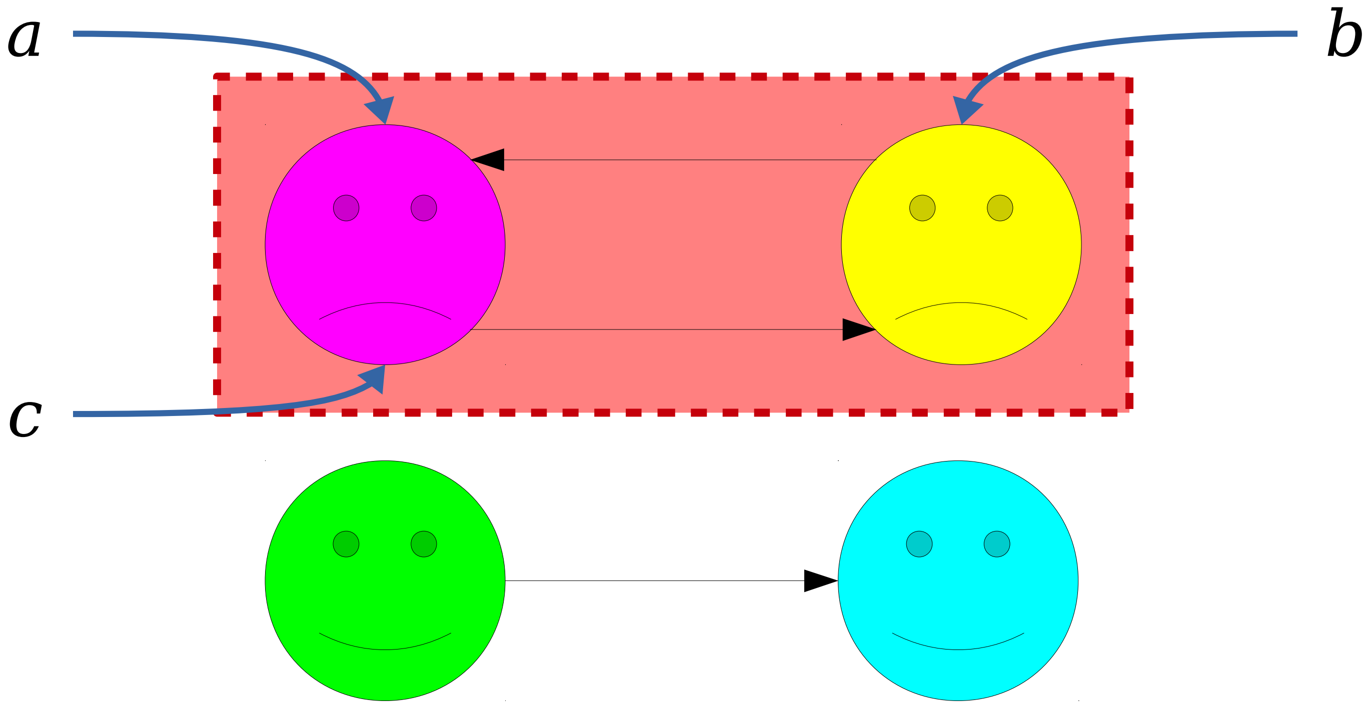


Answer at **PollEv.com/cs103** or
text **CS103** to **22333** once to join, then **Y** or **N**.

$\forall a \in A. \forall b \in A. \forall c \in A. (aRb \land bRc \to aRc)$

(*"Whenever a is related to b and b is related to c, we know a is related to c."*)

# Is This Relation Transitive?




$\forall a \in A.\ \forall b \in A.\ \forall c \in A.\ (\boldsymbol{aRb \land bRc \to aRc})$

(*"Whenever a is related to b and b is related to c, we know a is related to c.*)

# Equivalence Relations

- An ***equivalence relation*** is a relation that is reflexive, symmetric and transitive.
- Some examples:
  - $x = y$
  - $x \equiv_k y$
  - $x$ has the same color as $y$
  - $x$ has the same shape as $y$.

Binary relations give us a ***common language*** to describe ***common structures***.

# Equivalence Relations

- Most modern programming languages include some sort of hash table data structure.
  - Java: `HashMap`
  - C++: `std::unordered_map`
  - Python: `dict`
- If you insert a key/value pair and then try to look up a key, the implementation has to be able to tell whether two keys are equal.
- Although each language has a different mechanism for specifying this, many languages describe them in similar ways…

# Equivalence Relations

"The `equals` method implements an equivalence relation on non-null object references:

- It is *reflexive*: for any non-null reference value `x`, `x.equals(x)` should return true.
- It is *symmetric*: for any non-null reference values `x` and `y`, `x.equals(y)` should return true if and only if `y.equals(x)` returns true.
- It is *transitive*: for any non-null reference values `x`, `y`, and `z`, if `x.equals(y)` returns `true` and `y.equals(z)` returns true, then `x.equals(z)` should return `true`."

Java 8 Documentation

# Equivalence Relations

"Each unordered associative container is parameterized by `Key`, by a function object type `Hash` that meets the Hash requirements (17.6.3.4) and acts as a hash function for argument values of type `Key`, and by a binary predicate `Pred` that induces an equivalence relation on values of type `Key`. Additionally, `unordered_map` and `unordered_multimap` associate an arbitrary mapped type `T` with the `Key`."

C++14 ISO Spec, §23.2.5/3

# Equivalence Relation Proofs

- Let's suppose you've found a binary relation $R$ over a set $A$ and want to prove that it's an equivalence relation.
- How exactly would you go about doing this?

# An Example Relation

- Consider the binary relation ~ defined over the set $\mathbb{Z}$:

$$a \sim b \quad \textbf{if} \quad a+b \textbf{ is even}$$

- Some examples:

$$0 \sim 4 \qquad 1 \sim 9 \qquad 2 \sim 6 \qquad 5 \sim 5$$

- Turns out, this is an equivalence relation! Let's see how to prove it.

We can binary relations by giving a rule, like this:

***a~b*** **if** ***some property of a and b holds***

***This is the general template for defining a relation.*** Although we're using "if" rather than "iff" here, the two above statements are definitionally equivalent. For a variety of reasons, definitions are often introduced with "if" rather than "iff." Check the "Mathematical Vocabulary" handout for details.

What properties must ~ have to be an equivalence relation?

***Reflexivity***
***Symmetry***
***Transitivity***

Let's prove each property independently.

## $a \sim b$ if $a+b$ is even

***Lemma 1:*** The binary relation $\sim$ is reflexive.

***Proof:***

What is the formal definition of reflexivity?

$$\forall a \in \mathbb{Z}.\ a \sim a$$

Therefore, we'll choose an arbitrary integer $a$, then go prove that $a \sim a$.

# $a \sim b$ if $a+b$ is even

***Lemma 1:*** The binary relation $\sim$ is reflexive.

***Proof:*** Consider an arbitrary $a \in \mathbb{Z}$. We need to prove that $a \sim a$. From the definition of the $\sim$ relation, this means that we need to prove that $a+a$ is even.

To see this, notice that $a+a = 2a$, so the sum $a+a$ can be written as $2k$ for some integer $k$ (namely, $a$), so $a+a$ is even. Therefore, $a \sim a$ holds, as required. ■

# $a \sim b$ if $a + b$ is even

***Lemma 2:*** The binary relation $\sim$ is symmetric.

Which of the following works best as the opening of this proof?

A. Consider any integers $a$ and $b$. We will prove $a \sim b$ and $b \sim a$.
B. Pick $\forall a \in \mathbb{Z}$ and $\forall b \in \mathbb{Z}$. We will prove $a \sim b \to b \sim a$.
C. Consider any integers $a$ and $b$ where $a \sim b$ and $b \sim a$.
D. Consider any integer $a$ where $a \sim a$.
E. The relation $\sim$ is symmetric if for any $a, b \in \mathbb{Z}$, we have $a \sim b \to b \sim a$.
F. Consider any integers $a$ and $b$ where $a \sim b$. We will prove $b \sim a$.

Answer at **PollEv.com/cs103** or
text **CS103** to **22333** once to join, then **A**, **B**, **C**, **D**, **E**, or **F**.

## $a \sim b$ if $a+b$ is even

***Lemma 2:*** The binary relation ~ is symmetric.

***Proof:***

What is the formal definition of symmetry?

$$\forall a \in \mathbb{Z}.\ \forall b \in \mathbb{Z}.\ (a \sim b \to b \sim a)$$

Therefore, we'll choose arbitrary integers $a$ and $b$ where $a \sim b$, then prove that $b \sim a$.

## $a \sim b$ if $a+b$ is even

***Lemma 2:*** The binary relation $\sim$ is symmetric.

***Proof:*** Consider any integers $a$ and $b$ where $a \sim b$. We need to show that $b \sim a$.

Since $a \sim b$, we know that $a+b$ is even. Because $a+b = b+a$, this means that $b+a$ is even. Since $b+a$ is even, we know that $b \sim a$, as required. ■

$a \sim b$ **if** $a+b$ **is even**

***Lemma 3:*** The binary relation ~ is transitive.

***Proof:***

What is the formal definition of transitivity?

$$\forall a \in \mathbb{Z}.\ \forall b \in \mathbb{Z}.\ \forall c \in \mathbb{Z}.\ (a \sim b \land b \sim c \to a \sim c)$$

Therefore, we'll choose arbitrary integers $a$, $b$, and $c$ where $a \sim b$ and $b \sim c$, then prove that $a \sim c$.

## $a \sim b$ if $a+b$ is even

***Lemma 3:*** The binary relation $\sim$ is transitive.

***Proof:*** Consider arbitrary integers $a$, $b$ and $c$ where $a \sim b$ and $b \sim c$. We need to prove that $a \sim c$, meaning that we need to show that $a+c$ is even.

Since $a \sim b$ and $b \sim c$, we know that $a+b$ and $b+c$ are even. This means there are integers $k$ and $m$ where $a+b = 2k$ and $b+c = 2m$. Notice that

$$(a+b) + (b+c) = 2k + 2m.$$

Rearranging, we see that

$$a+c + 2b = 2k + 2m,$$

so

$$a+c = 2k + 2m - 2b = 2(k+m-b).$$

So there is an integer $r$, namely $k+m-b$, such that $a+c = 2r$. Thus $a+c$ is even, so $a \sim c$, as required. ■

# An Observation

# $a \sim b$ if $a+b$ is even

***Lemma 1:*** The binary relation $\sim$ is reflexive.

***Proof:*** Consider an arbitrary $a \in \mathbb{Z}$. We need to prove that $a \sim a$. From the definition of the $\sim$ relation, this means that we need to prove that $a+a$ is even.

To see this, notice that $a+a = 2a$, so the sum $a+a$ can be written as $2k$ for some integer $k$ (namely, $a$), so $a+a$ is even. Therefore, $a \sim a$ holds, as required. ■

The formal definition of reflexivity is given in first-order logic, but this proof does not contain any first-order logic symbols!

## $a \sim b$ if $a+b$ is even

***Lemma 2:*** The binary relation ~ is symmetric.

***Proof:*** Consider any integers $a$ and $b$ where $a \sim b$. We need to show that $b \sim a$.

Since $a \sim b$, we know that $a+b$ is even. Because $a+b = b+a$, this means that $b+a$ is even. Since $b+a$ is even, we know that $b \sim a$, as required. ■

The formal definition of symmetry is given in first-order logic, but this proof does not contain any first-order logic symbols!

## $a \sim b$ if $a+b$ is even

***Lemma 3:*** The binary relation $\sim$ is transitive.

***Proof:*** Consider arbitrary integers $a$, $b$ and $c$ where $a \sim b$ and $b \sim c$. We need to prove that $a \sim c$, meaning that we need to show that $a+c$ is even.

Since $a \sim b$ and $b \sim c$, we know that $a+b$ and $b+c$ are even. This means there are integers $k$ and $m$ where $a+b = 2k$ and $b+c = 2m$. Notice that

$$(a+b) + (b+c) = 2k + 2m.$$

Rearranging, we see that

$$a+c + 2b = 2k + 2m,$$

so

$$a+c = 2k +$$

So there is an integer $r$,

$a+c = 2r$. Thus $a+c$ is ev

The formal definition of transitivity is given in first-order logic, but this proof does not contain any first-order logic symbols!

# First-Order Logic and Proofs

- First-order logic is an excellent tool for giving formal definitions to key terms.
- While first-order logic *guides* the structure of proofs, it is *exceedingly rare* to see first-order logic in written proofs.
- Follow the example of these proofs:
  - Use the FOL definitions to determine what to assume and what to prove.
  - Write the proof in plain English using the conventions we set up in the first week of the class.
- ***Please, please, please, please, please internalize the contents of this slide!***