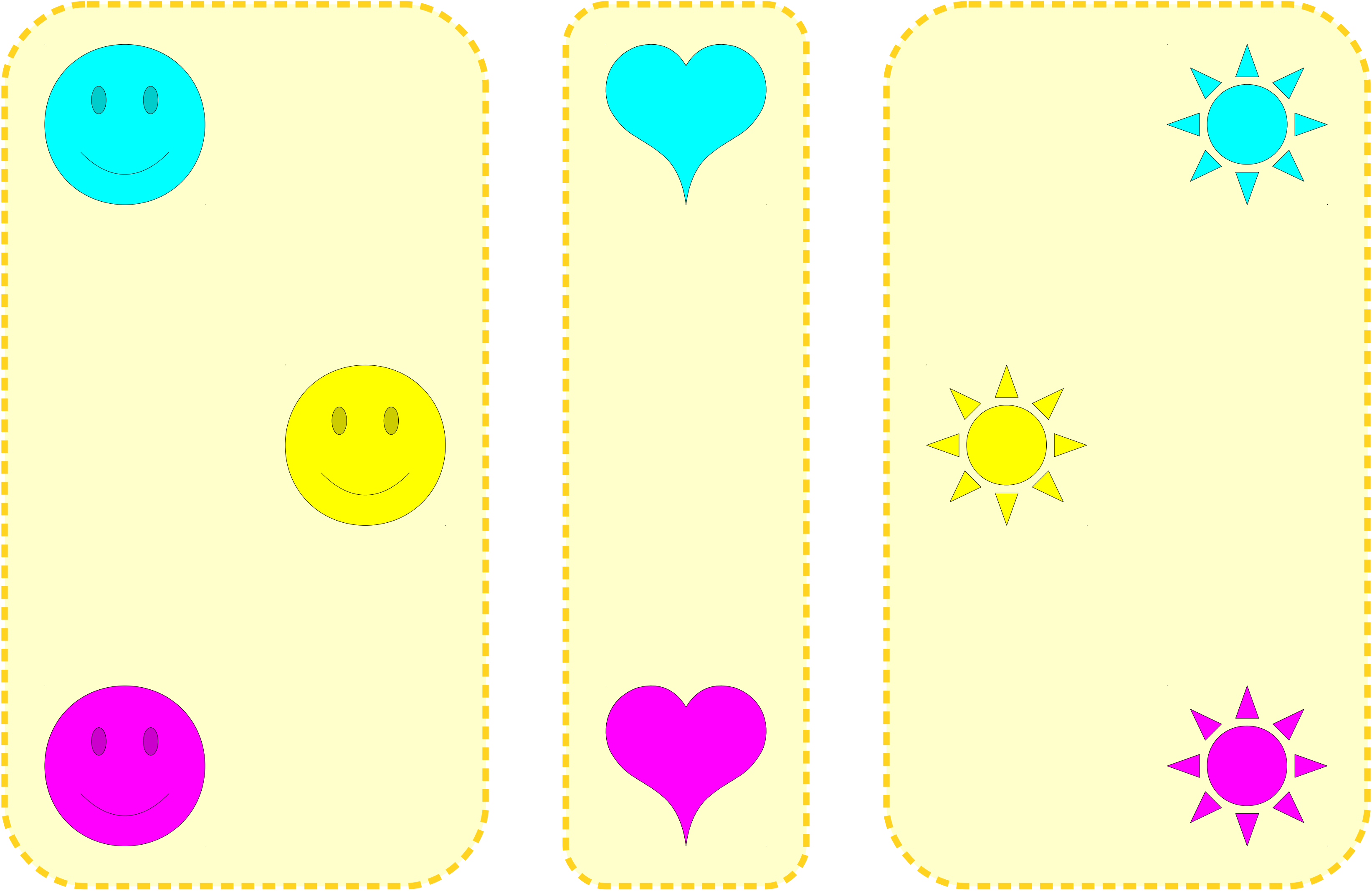
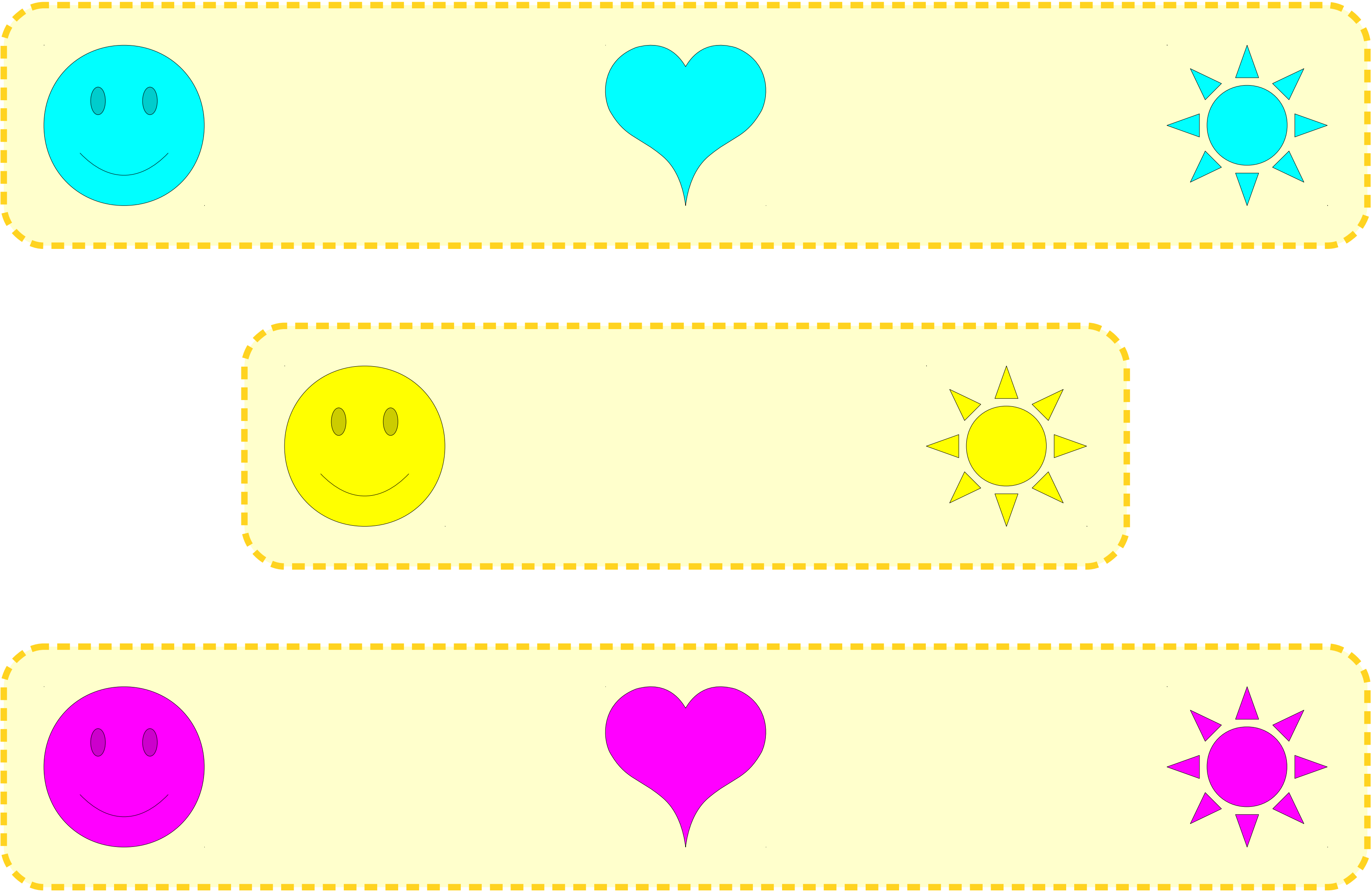
Properties of Equivalence Relations



xRy if x and y have the same shape



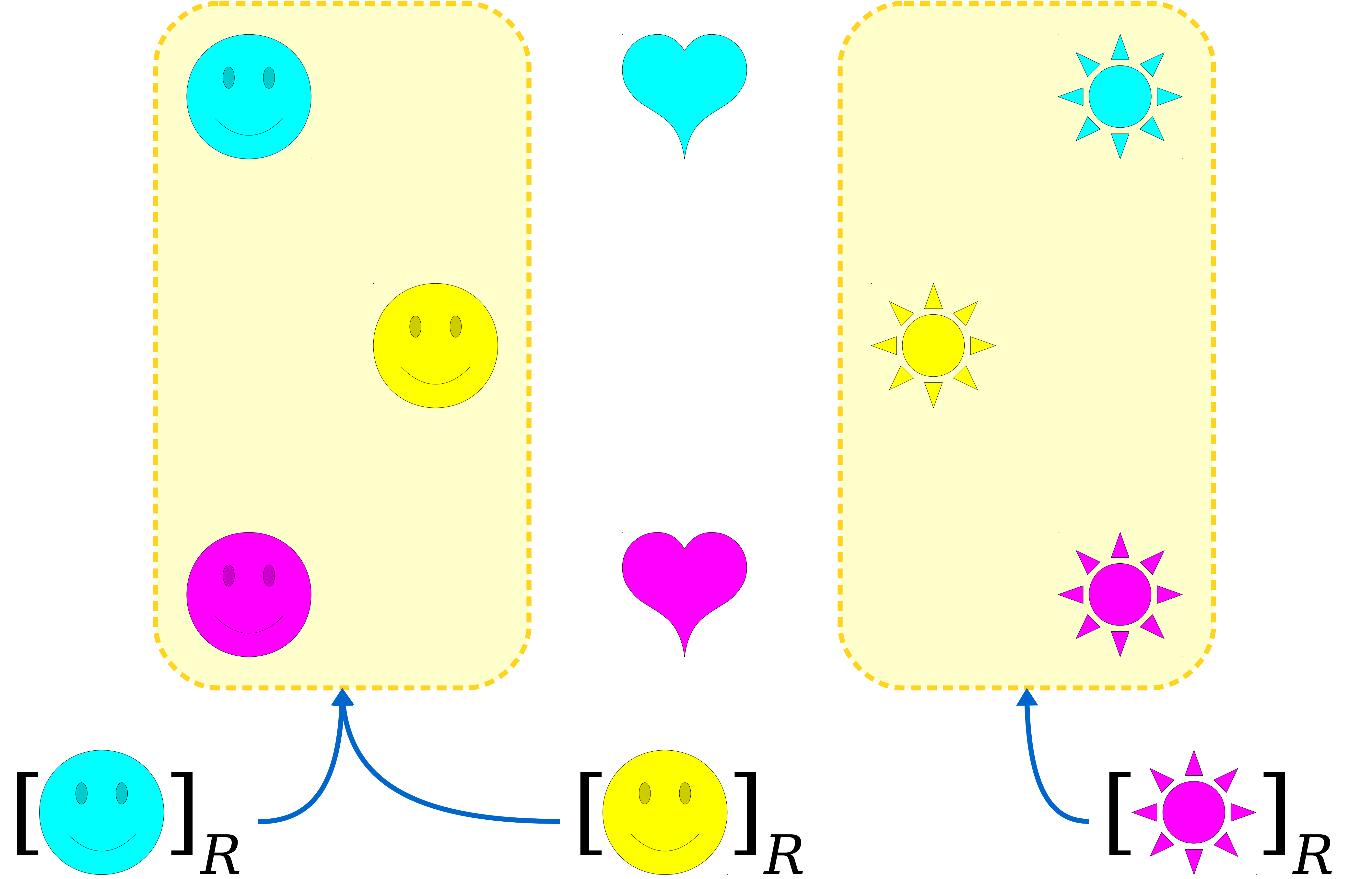
xTy if x and y have the same color

Equivalence Classes

- Given an equivalence relation R over a set A, for any $x \in A$, the ***equivalence class of x*** is the set

$$[x]_R = \{ \ y \in A \mid xRy \ \}$$

- Intuitively, the set $[x]_R$ contains all elements of A that are related to x by relation R.



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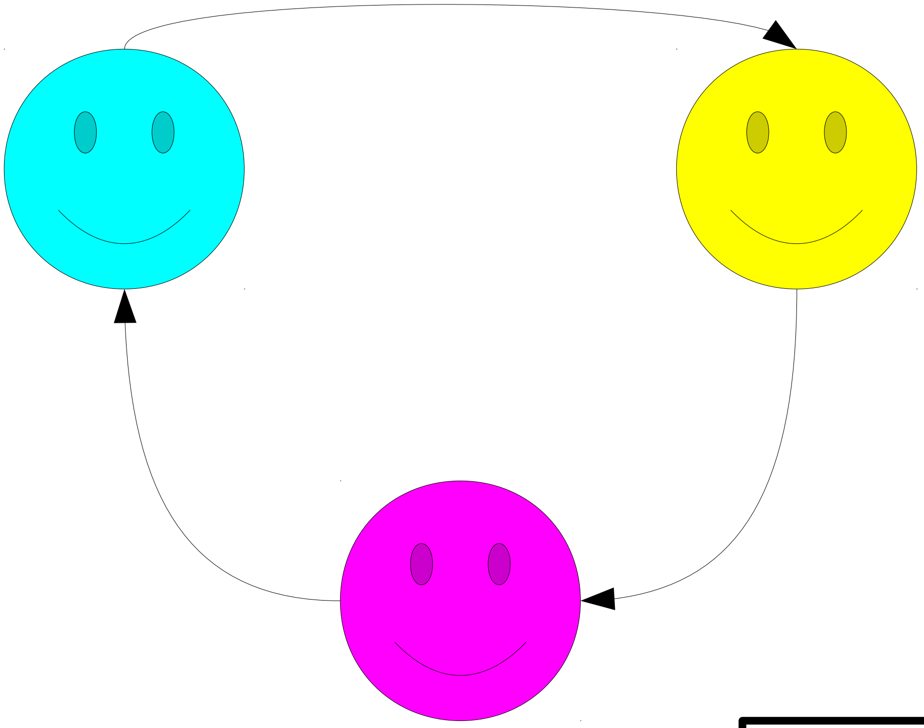
The Fundamental Theorem of Equivalence Relations: Let R be an equivalence relation over a set A. Then every element $a \in A$ belongs to exactly one equivalence class of R.

How’d We Get Here?

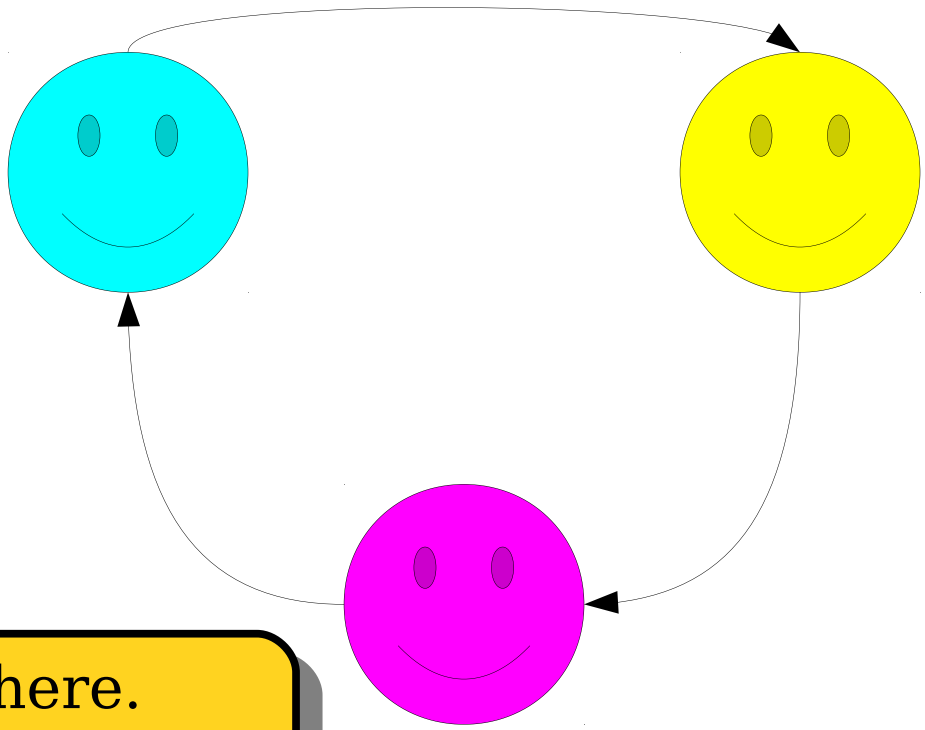
- We discovered equivalence relations by thinking about ***partitions*** of a set of elements.
- We saw that if we had a binary relation that tells us whether two elements are in the same group, it had to be reflexive, symmetric, and transitive.
- The FToER says that, in some sense, these rules precisely capture what it means to be a partition.
- ***Question:*** What’s so special about these three rules?

$$\forall a \in A.\ \forall b \in A.\ \forall c \in A.\ (aRb \land bRc \to cRa)$$



A binary relation
with this property
is called ***cyclic***.

$$\forall a \in A.\ \forall b \in A.\ \forall c \in A.\ (aRb \land bRc \to cRa)$$



Let R be the relation depicted here.
How many of the following claims are true?

R is reflexive.
R is symmetric.
R is transitive.
R is an equivalence relation.

Answer at **PollEv.com/cs103** or
text **CS103** to **22333** once to join, then **0**, **1**, **2**, **3**, or **4**.

$$\forall a \in A.\ \forall b \in A.\ \forall c \in A.\ (aRb \land bRc \rightarrow cRa)$$

Theorem: A binary relation R over a set A is an equivalence relation if and only if it is reflexive and cyclic.

Lemma 1: If R is an equivalence relation over a set A, then R is reflexive and cyclic.

Lemma 2: If R is a binary relation over a set A that is reflexive and cyclic, then R is an equivalence relation.

Lemma 1: If R is an equivalence relation over a set A, then R is reflexive and cyclic.

What We're Assuming

- R is an equivalence relation.
  - R is reflexive.
  - R is symmetric.
  - R is transitive.

What We Need To Show

- R is reflexive.
- R is cyclic.

Lemma 1: If R is an equivalence relation over a set A, then R is reflexive and cyclic.

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What We Need To Show

R is reflexive.

R is cyclic.

- If aRb and bRc, then cRa.

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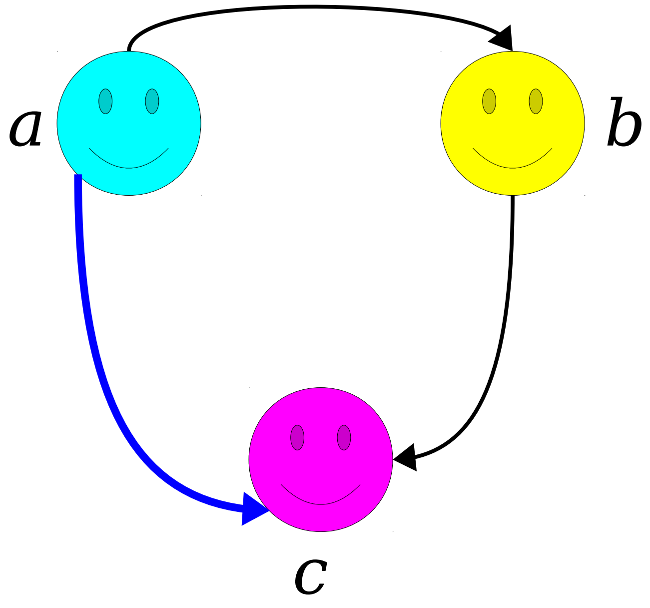

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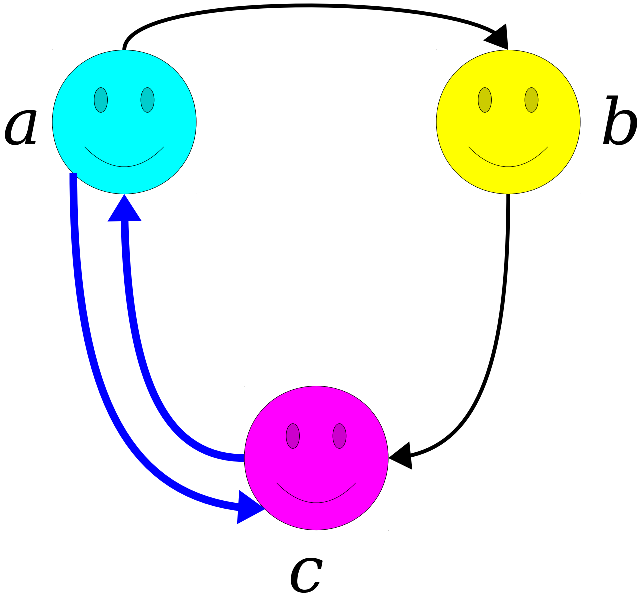

Lemma 1: If R is an equivalence relation over a set A, then R is reflexive and cyclic.

Proof: Let R be an arbitrary equivalence relation over some set A. We need to prove that R is reflexive and cyclic.

Since R is an equivalence relation, we know that R is reflexive, symmetric, and transitive. Consequently, we already know that R is reflexive, so we only need to show that R is cyclic.

To prove that R is cyclic, consider any arbitrary $a, b, c \in A$ where aRb and bRc. We need to prove that cRa holds. Since R is transitive, from aRb and bRc we see that aRc. Then, since R is symmetric, from aRc we see that cRa, which is what we needed to prove. ■

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Notice how the first few sentences of this proof mirror the structure of what needs to be proved. We're just following the templates from the first week of class!

Notice how this setup mirrors the first-order definition of cyclicity:

$$\forall a \in A.\ \forall b \in A.\ \forall c \in A.\ (aRb \land bRc \to cRa)$$

When writing proofs about terms with first-order definitions, it's critical to call back to those definitions!

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Although this proof is deeply informed by the first-order definitions, notice that there is no first-order logic notation anywhere in the proof. That's normal – it's actually quite rare to see first-order logic in written proofs.

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Lemma 2: If R is a binary relation over a set A that is reflexive and cyclic, then R is an equivalence relation.

What We're Assuming

- R is reflexive.
- R is cyclic.

What We Need To Show

- R is an equivalence relation.
  - R is reflexive.
  - R is symmetric.
  - R is transitive.

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R is reflexive.

- $\forall x \in A.\ xRx$

R is cyclic.

$xRy \land yRz \to zRx$

What We Need To Show

- R is symmetric.
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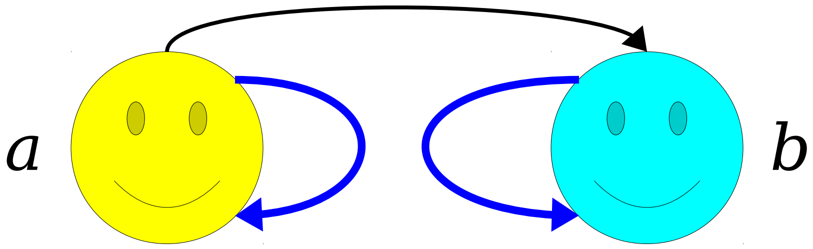

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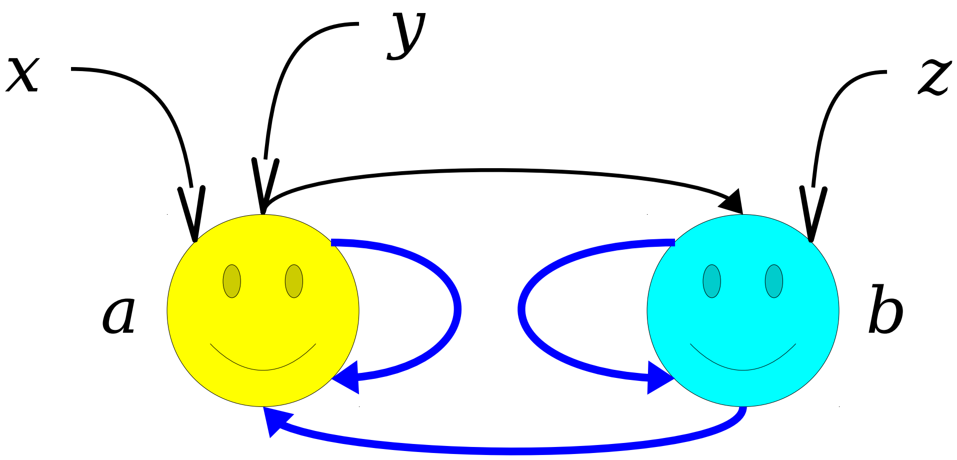

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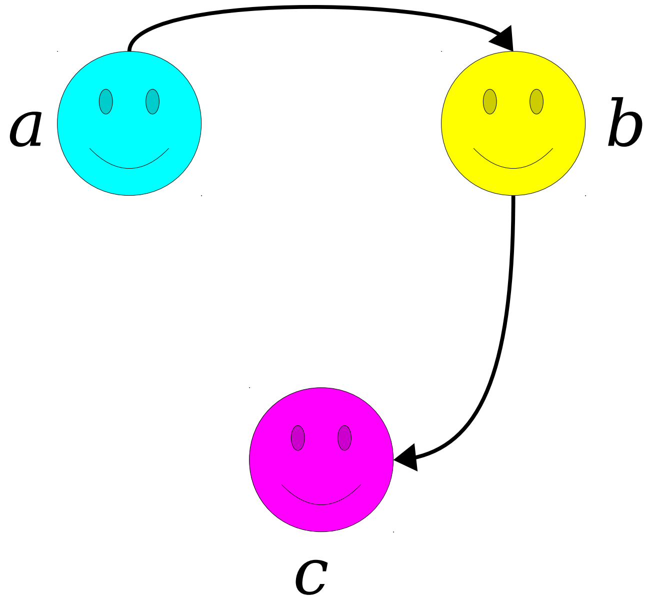

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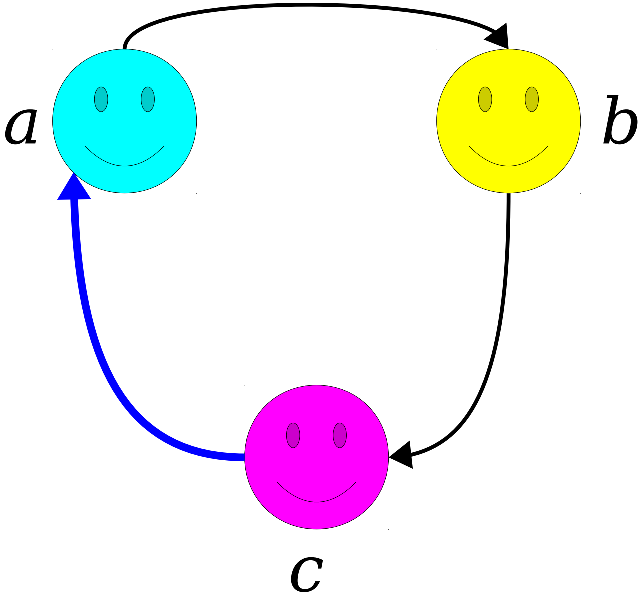

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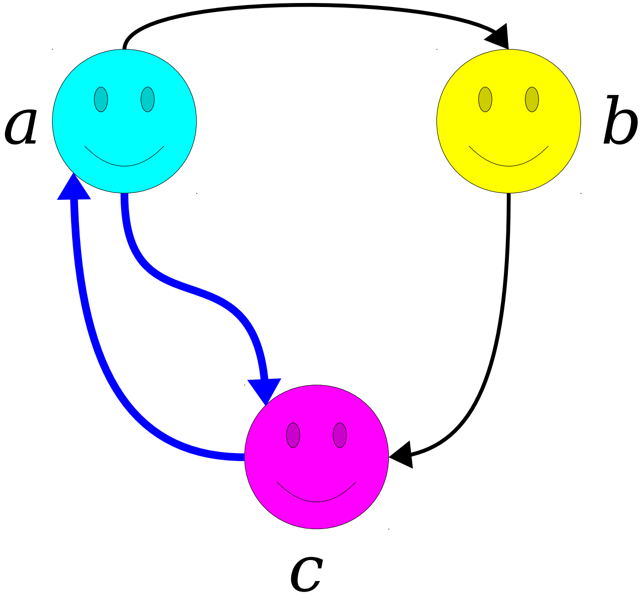

Lemma 2: If R is a binary relation over a set A that is cyclic and reflexive, then R is an equivalence relation.

Proof: Let R be an arbitrary binary relation over a set A that is cyclic and reflexive. We need to prove that R is an equivalence relation. To do so, we need to show that R is reflexive, symmetric, and transitive. Since we already know by assumption that R is reflexive, we just need to show that R is symmetric and transitive.

First, we'll prove that R is symmetric. To do so, pick any arbitrary $a, b \in A$ where aRb holds. We need to prove that bRa is true. Since R is reflexive, we know that aRa holds. Therefore, by cyclicity, since aRa and aRb, we learn that bRa, as required.

Next, we'll prove that R is transitive. Let a, b, and c be any elements of A where aRb and bRc. We need to prove that aRc. Since R is cyclic, from aRb and bRc we see that cRa. Earlier, we showed that R is symmetric. Therefore, from cRa we see that aRc is true, as required. ■

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Notice how this setup mirrors the first-order definition of symmetry:

$$\forall a \in A.\ \forall b \in A.\ (aRb \to bRa)$$

When writing proofs about terms with first-order definitions, it's critical to call back to those definitions!

First, we'll prove that *R* is symmetric. To do so, pick any arbitrary *a*, *b* ∈ *A* where *aRb* holds. We need to prove that *bRa* is true. Since *R* is reflexive, we know that *aRa* holds. Therefore, by cyclicity, since *aRa* and *aRb*, we learn that *bRa*, as required.

Next, we'll prove that *R* is transitive. Let *a*, *b*, and *c* be any elements of *A* where *aRb* and *bRc*. We need to prove that *aRc*. Since *R* is cyclic, from *aRb* and *bRc* we see that *cRa*. Earlier, we showed that *R* is symmetric. Therefore, from *cRa* we see that *aRc* is true, as required. ■

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Notice how this setup mirrors the first-order definition of transitivity:

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When writing proofs about terms with first-order definitions, it's critical to call back to those definitions!

Next, we'll prove that R is transitive. Let a, b, and c be any elements of A where aRb and bRc. We need to prove that aRc. Since R is cyclic, from aRb and bRc we see that cRa. Earlier, we showed that R is symmetric. Therefore, from cRa we see that aRc is true, as required. ■

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Refining Your Proofwriting

- When writing proofs about terms with formal definitions, you ***must*** call back to those definitions.
  - Use the first-order definition to see what you'll assume and what you'll need to prove.
- When writing proofs about terms with formal definitions, you ***must not*** include any first-order logic in your proofs.
  - Although you won't use any FOL *notation* in your proofs, your proof implicitly calls back to the FOL definitions.
- You'll get a lot of practice with this on Problem Set Three. If you have any questions about how to do this properly, please feel free to ask on Piazza or stop by office hours!