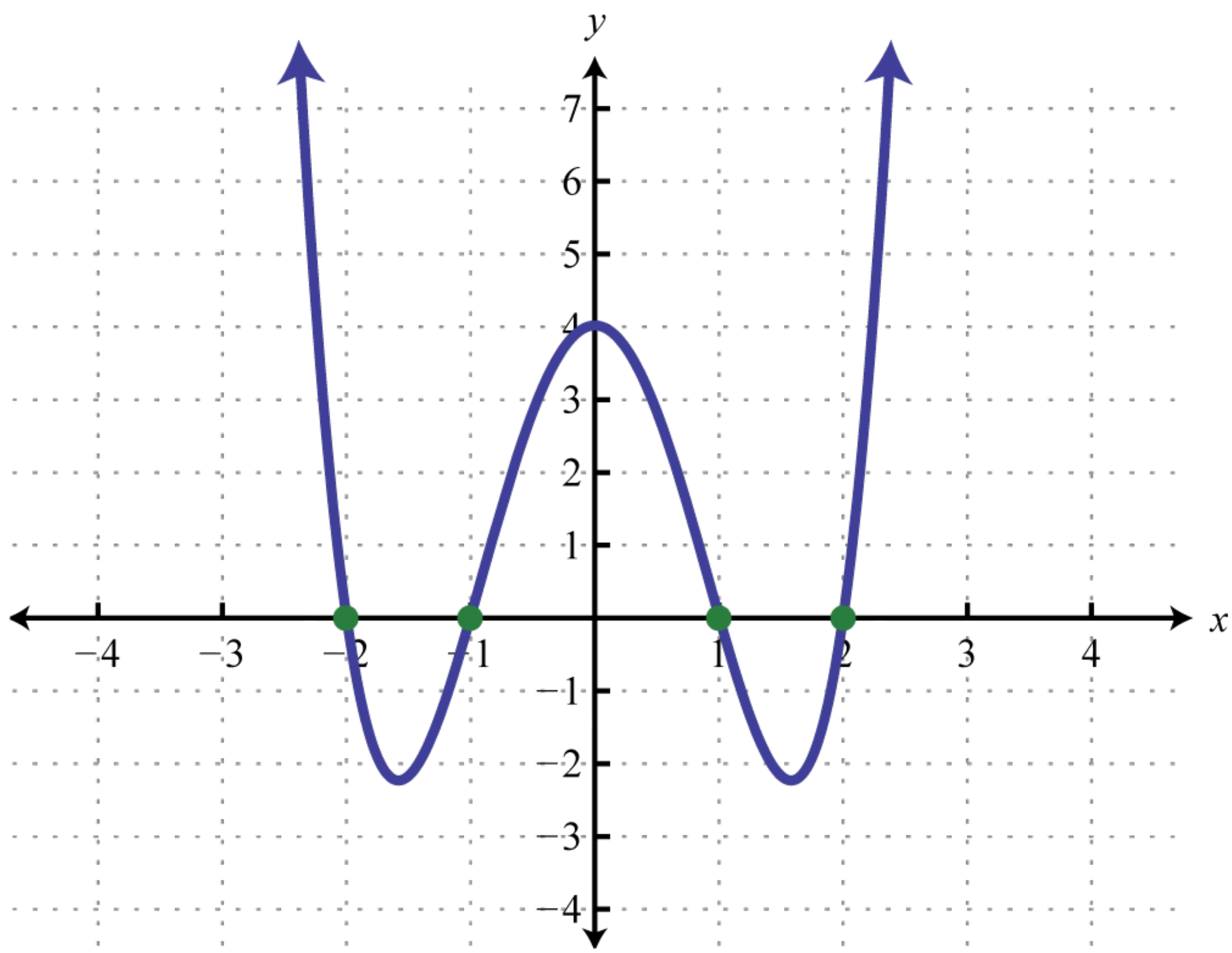
Function

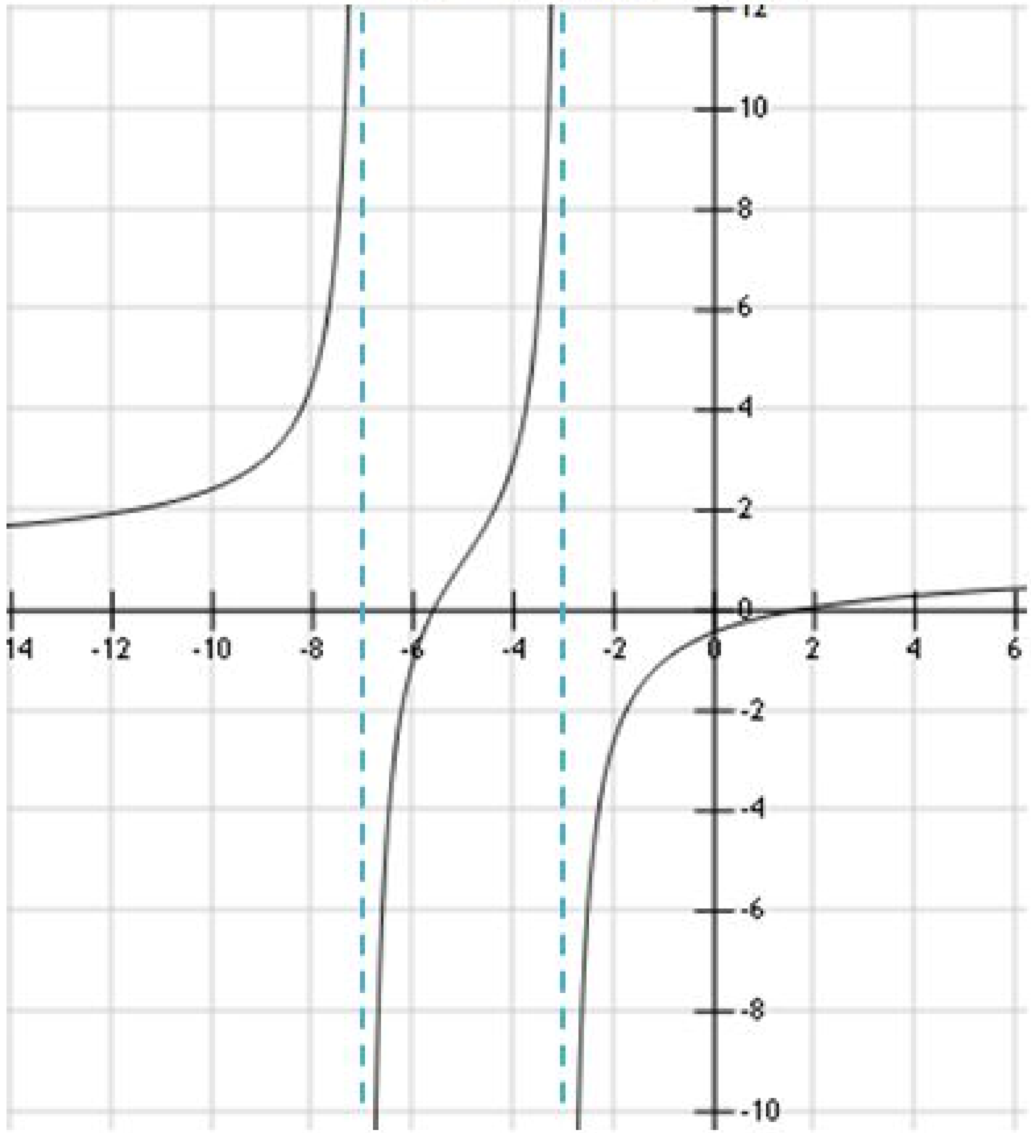
What is a function?

Functions, High-School Edition



$$f(x) = x^4 - 5x^2 + 4$$

$$f(x) = \frac{x^2 + 4x - 9}{x^2 + 10x + 21}$$



source: http://study.com/cimages/multimages/16/asymptote_1.JPG

Functions, High-School Edition

- In high school, functions are usually given as objects of the form

$$f(x) = \frac{x^3+3x^2+15x+7}{1-x^{137}}$$

- What does a function do?
  - It takes in as input a real number.
  - It outputs a real number
  - … except when there are vertical asymptotes or other discontinuities, in which case the function doesn't output anything.

Functions, CS Edition

```
int flipUntil(int n) {
  int numHeads = 0;
  int numTries = 0;

  while (numHeads < n) {
    if (randomBoolean()) numHeads++;

    numTries++;
  }

  return numTries;
}
```

Functions, CS Edition

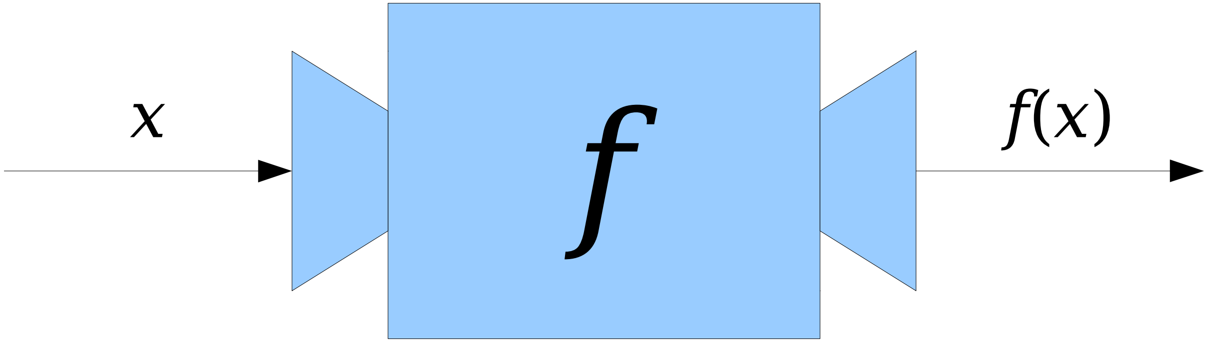
- In programming, functions
  - might take in inputs,
  - might return values,
  - might have side effects,
  - might never return anything,
  - might crash, and
  - might return different values when called multiple times.

What's Common?

- Although high-school math functions and CS functions are pretty different, they have two key aspects in common:
  - They take in inputs.
  - They produce outputs.
- In math, we like to keep things easy, so that's pretty much how we're going to define a function.

Rough Idea of a Function:

A function is an object f that takes in an input and produces exactly one output.



(This is not a complete definition – we'll revisit this in a bit.)

High School versus CS Functions

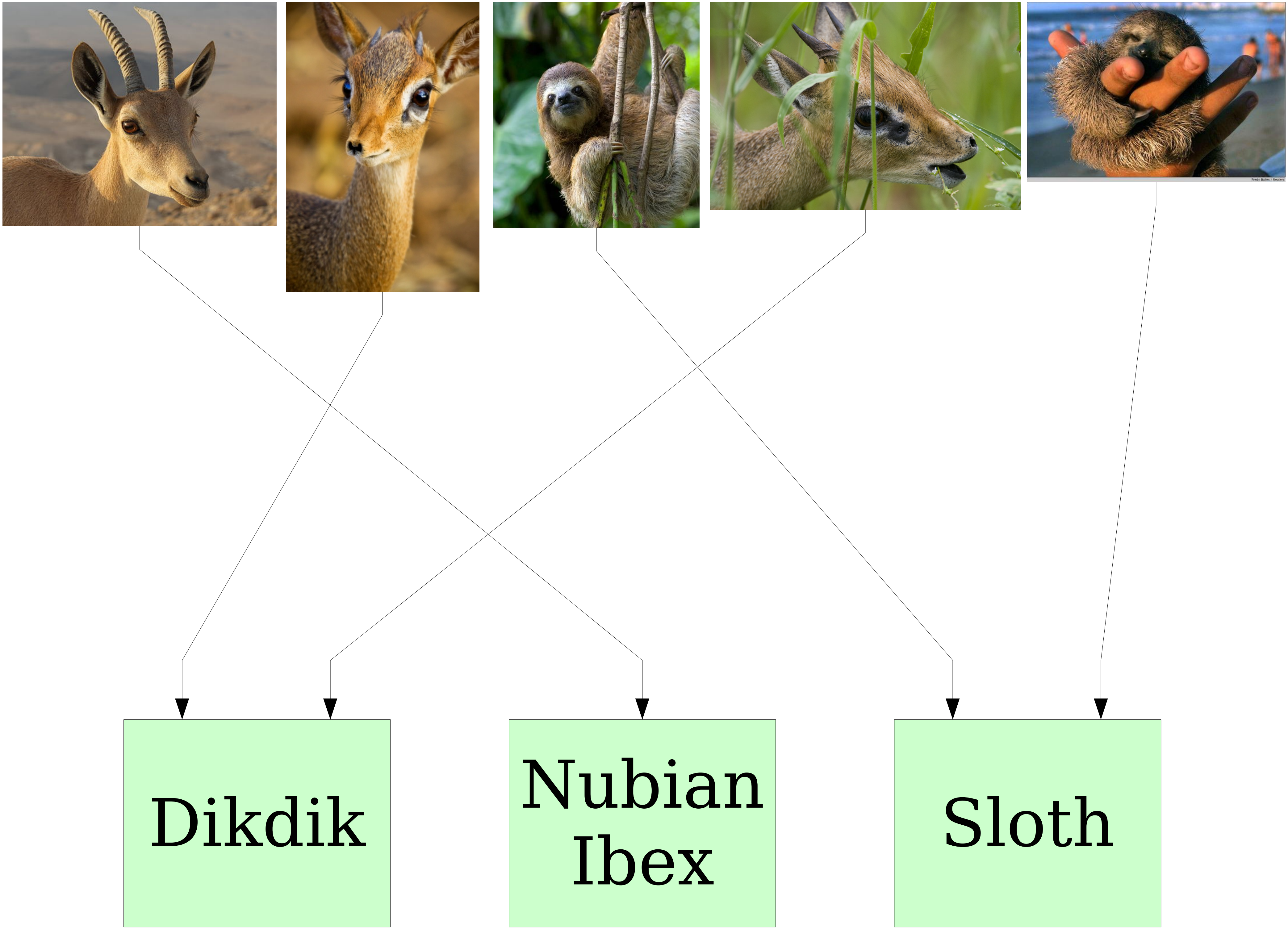
- In high school, functions usually were given by a rule:

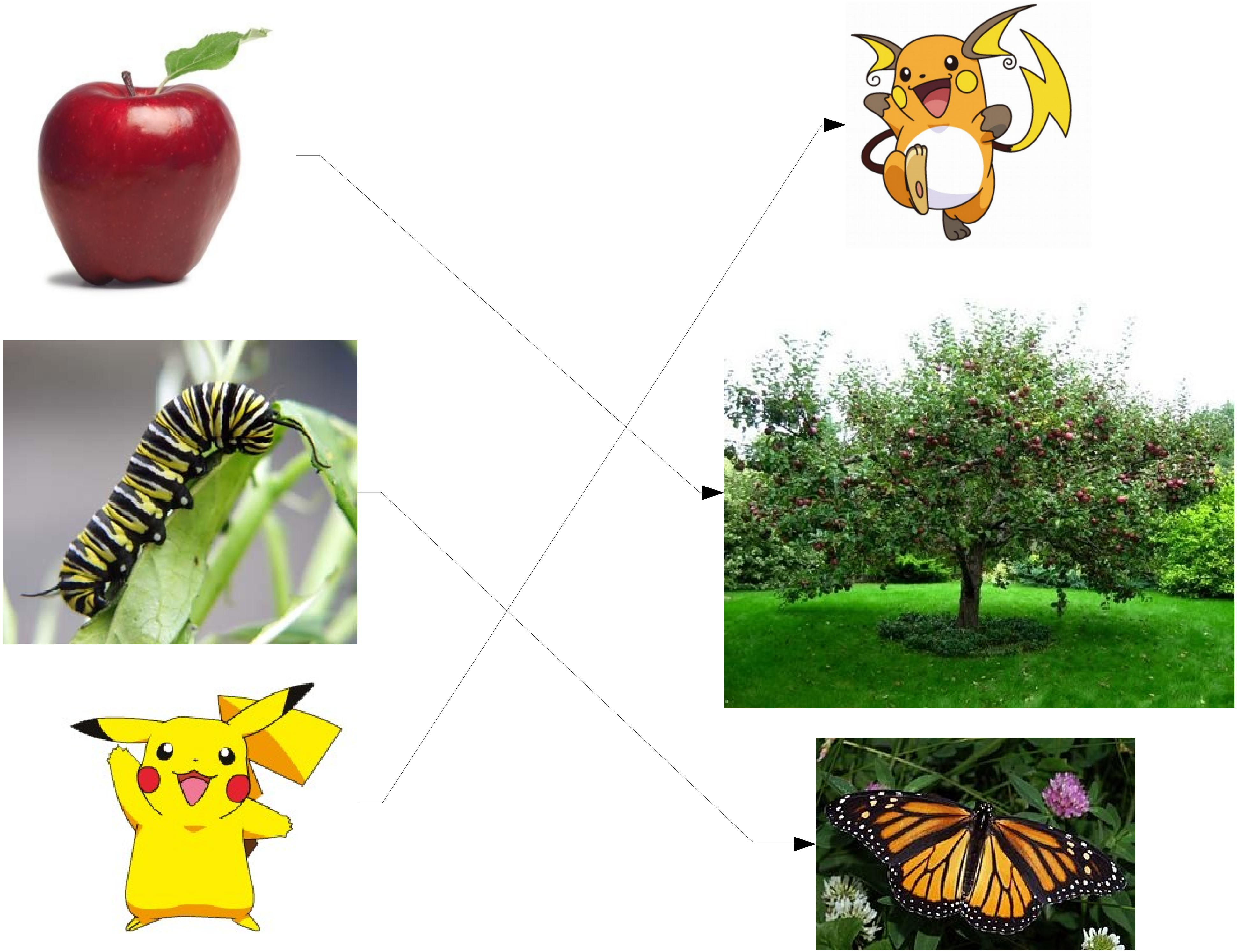
$$f(x) = 4x + 15$$

- In CS, functions are usually given by code:

```
int factorial(int n) {
    int result = 1;
    for (int i = 1; i <= n; i++) {
        result *= i;
    }
    return result;
}
```

- What sorts of functions are we going to allow from a mathematical perspective?


Dikdik
Nubian Ibex
Sloth



… but also …

$$f(x) = x^2 + 3x - 15$$

$$f(n) = \begin{cases} -n/2 & \text{if } n \text{ is even} \\ (n+1)/2 & \text{otherwise} \end{cases}$$

Functions like these are called ***piecewise functions***.

To define a function, you will typically either

- draw a picture, or
- give a rule for determining the output.

In mathematics, functions are ***deterministic***.

That is, given the same input, a function must always produce the same output.

The following is a perfectly valid piece of C++ code, but it's not a valid function under our definition:

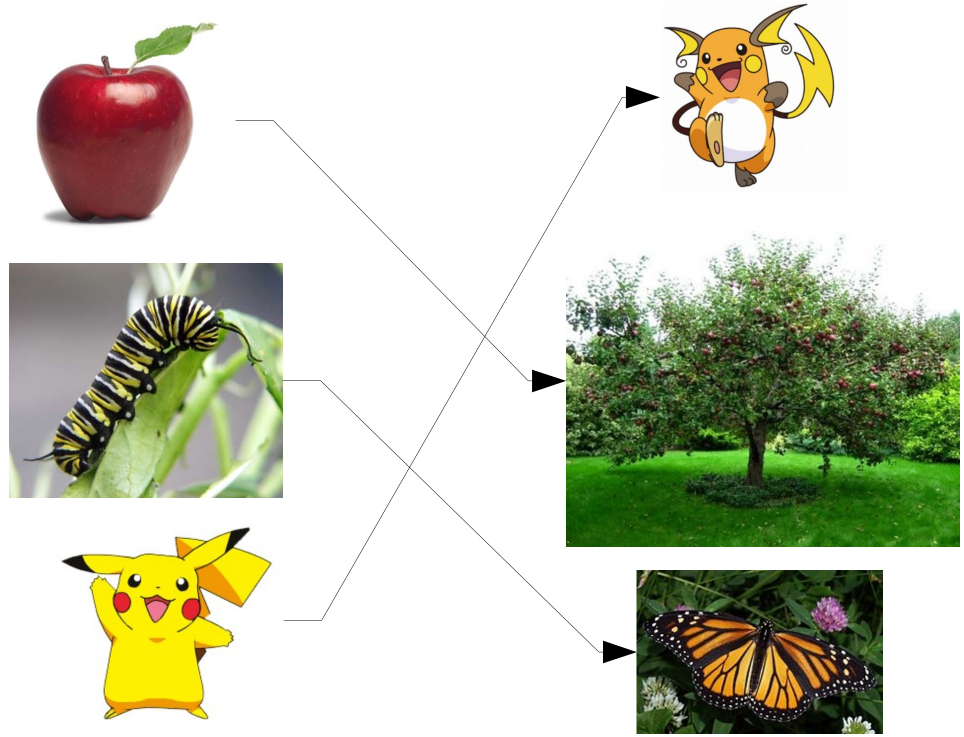
```
int randomNumber(int numOutcomes) {
    return rand() % numOutcomes;
}
```

One Challenge

$$f(x) = x^2 + 2x + 5$$

$$f(\ 3\ ) = 3^2 + 3 \cdot 2 + 5 = 20$$
$$f(\ 0\ ) = 0^2 + 0 \cdot 2 + 5 = 5$$

$f($  $) = \ldots\ ?$



$f($  $) =$ 

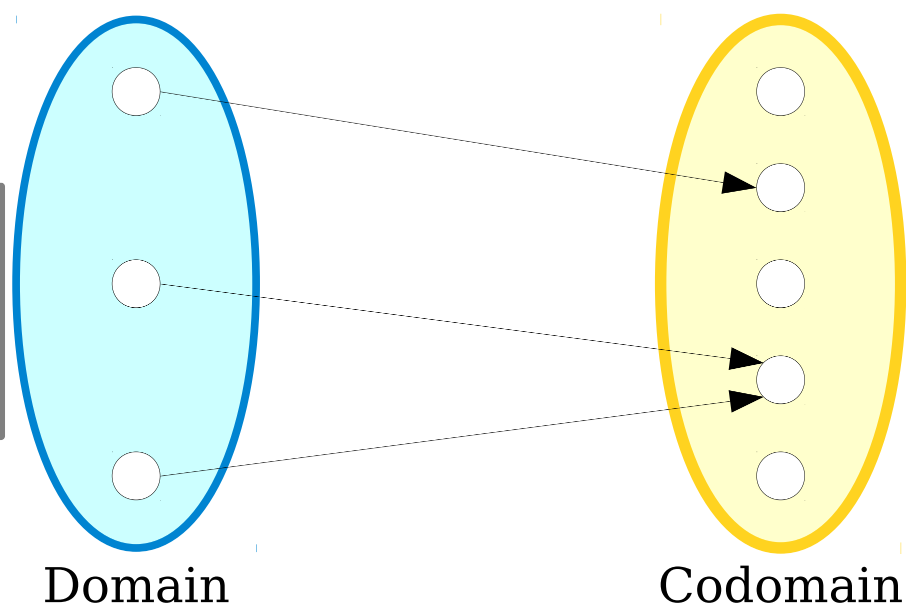
$$f(137) = \ldots?$$

We need to make sure we can't apply functions to meaningless inputs.

Domains and Codomains

- Every function *f* has two sets associated with it: its ***domain*** and its ***codomain***.
- A function *f* can only be applied to elements of its domain. For any *x* in the domain, *f*(*x*) belongs to the codomain.

The function must be defined for every element of the domain.



The output of the function must always be in the codomain, but not all elements of the codomain must be produced as outputs.

Domains and Codomains

- Every function f has two sets associated with it: its ***domain*** and its ***codomain***.
- A function f can only be applied to elements of its domain. For any x in the domain, $f(x)$ belongs to the codomain.

The codomain of this function is $\mathbb{R}$. Everything produced is a real number, but not all real numbers can be produced.

The domain of this function is $\mathbb{R}$. Any real number can be provided as input.

```
private double absoluteValueOf(double x) {
    if (x >= 0) {
        return x;
    } else {
        return -x;
    }
}
```

Domains and Codomains

- If f is a function whose domain is A and whose codomain is B, we write $\boldsymbol{f : A \to B}$.
- This notation just says what the domain and codomain of the function are. It doesn't say how the function is evaluated.
- Think of it like a "function prototype" in C or C++. The notation $f : ArgType \to RetType$ is like writing

  RetType *f*(*ArgType* `argument`);

  We know that f takes in an $ArgType$ and returns a $RetType$, but we don't know exactly which $RetType$ it's going to return for a given $ArgType$.

The Official Rules for Functions

- Formally speaking, we say that $f : A \to B$ if the following two rules hold.
- First, f must be obey its domain/codomain rules:

$$\forall a \in A.\ \exists b \in B.\ f(a) = b$$

(*"Every input in A maps to some output in B."*)

- Second, f must be deterministic:

$$\forall a_1 \in A.\ \forall a_2 \in A.\ (a_1 = a_2 \to f(a_1) = f(a_2))$$

(*"Equal inputs produce equal outputs."*)

- If you're ever curious about whether something is a function, look back at these rules and check! For example:
  - Can a function have an empty domain?
  - Can a function with a nonempty domain have an empty codomain?

Defining Functions

- Typically, we specify a function by describing a rule that maps every element of the domain to some element of the codomain.
- Examples:
  - $f(n) = n + 1$, where $f : \mathbb{Z} \to \mathbb{Z}$
  - $f(x) = \sin x$, where $f : \mathbb{R} \to \mathbb{R}$
  - $f(x) = \lceil x \rceil$, where $f : \mathbb{R} \to \mathbb{Z}$
- Notice that we're giving both a rule and the domain/codomain.

Defining Functions

Typically, we specify a function by describing a rule that maps every element of the domain to some

codomain.

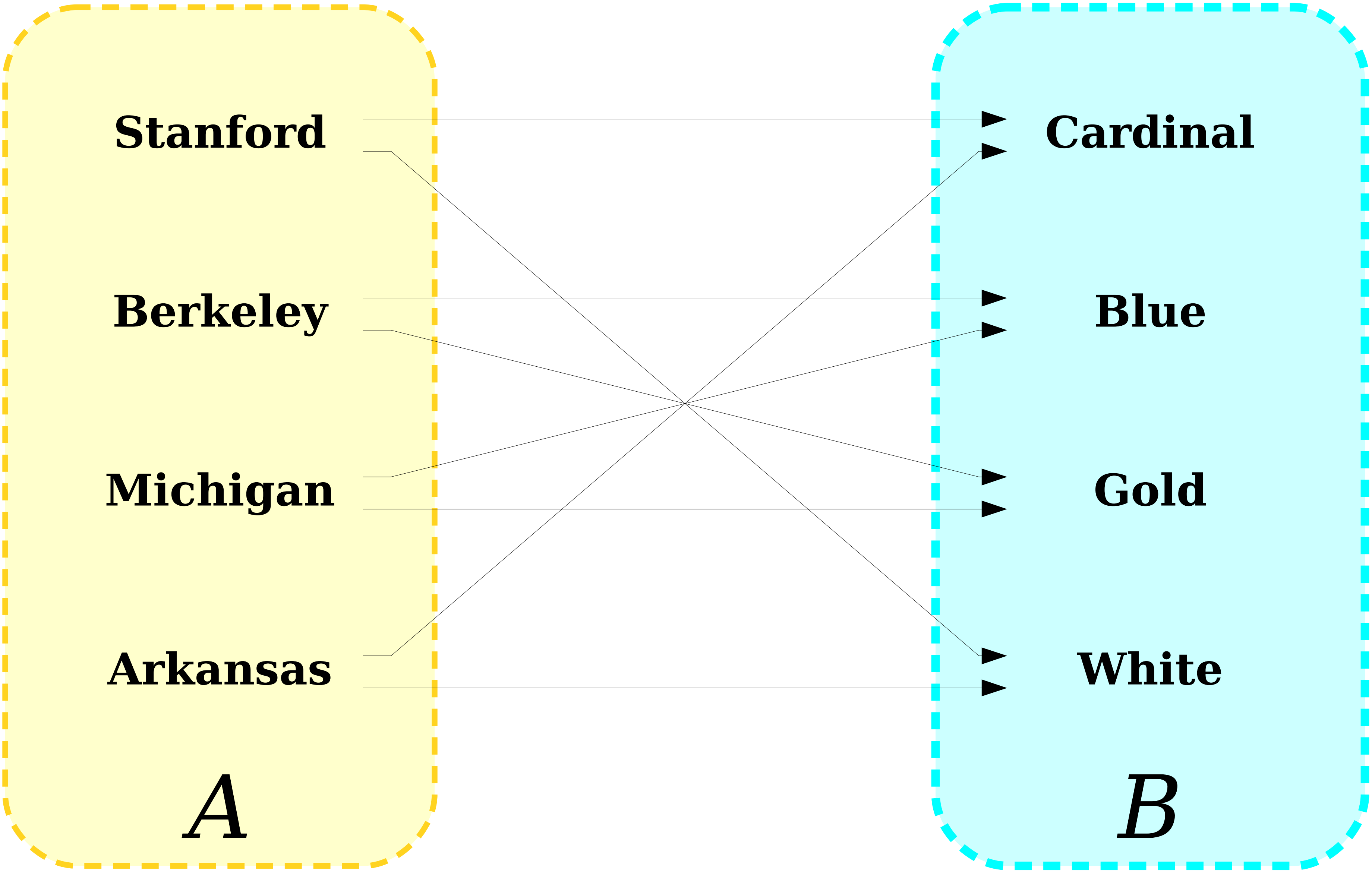
This is the ceiling function – the smallest integer greater than or equal to x. For example, $\lceil 1 \rceil = 1$, $\lceil 1.37 \rceil = 2$, and $\lceil \pi \rceil = 4$.

Examples:

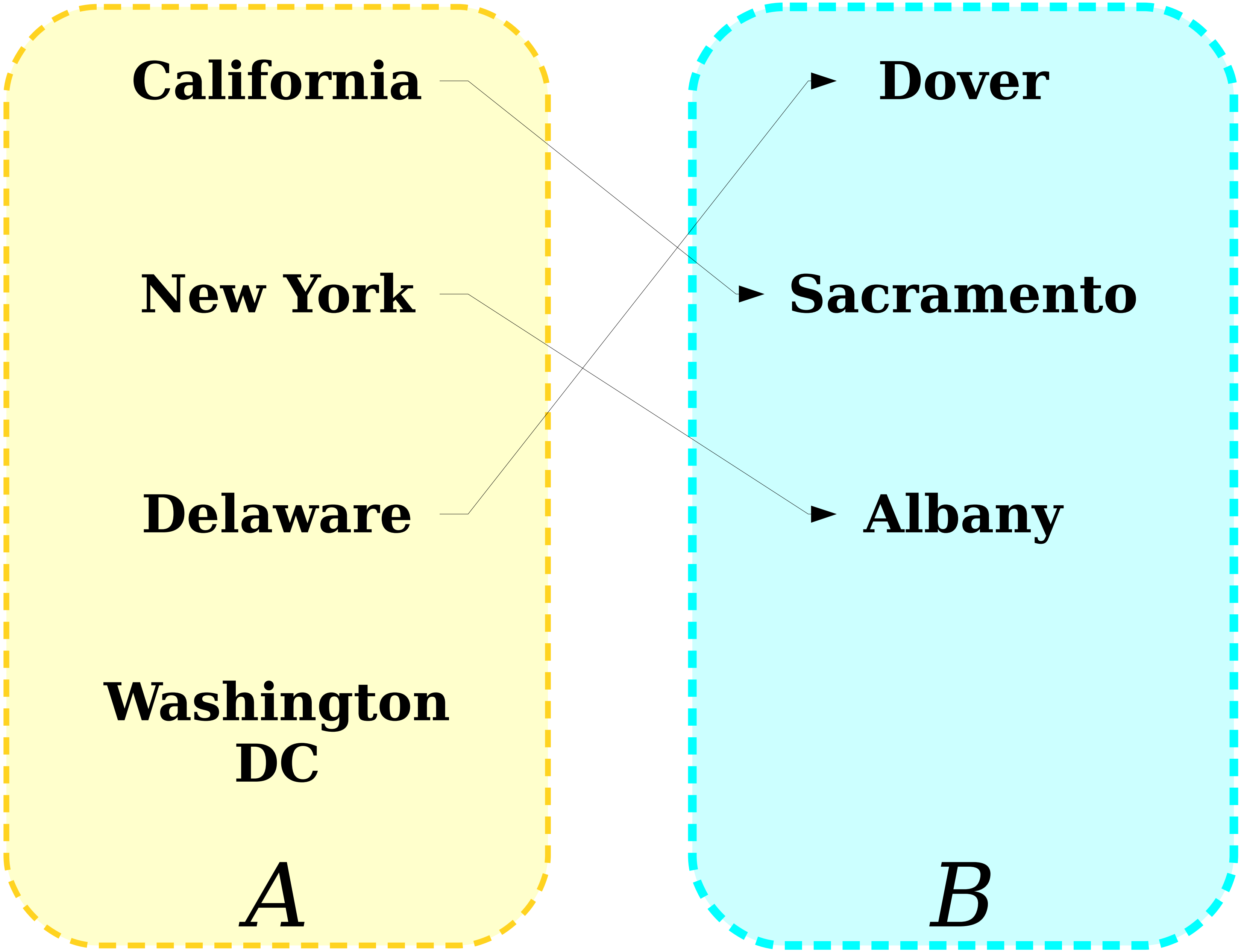
- $f(n) = n + 1$, where $f : \mathbb{Z}$
- $f(x) = \sin x$, where $f : \mathbb{R} \to \mathbb{R}$
- $f(x) = \lceil x \rceil$, where $f : \mathbb{R} \to \mathbb{Z}$

Notice that we're giving both a rule and the domain/codomain.

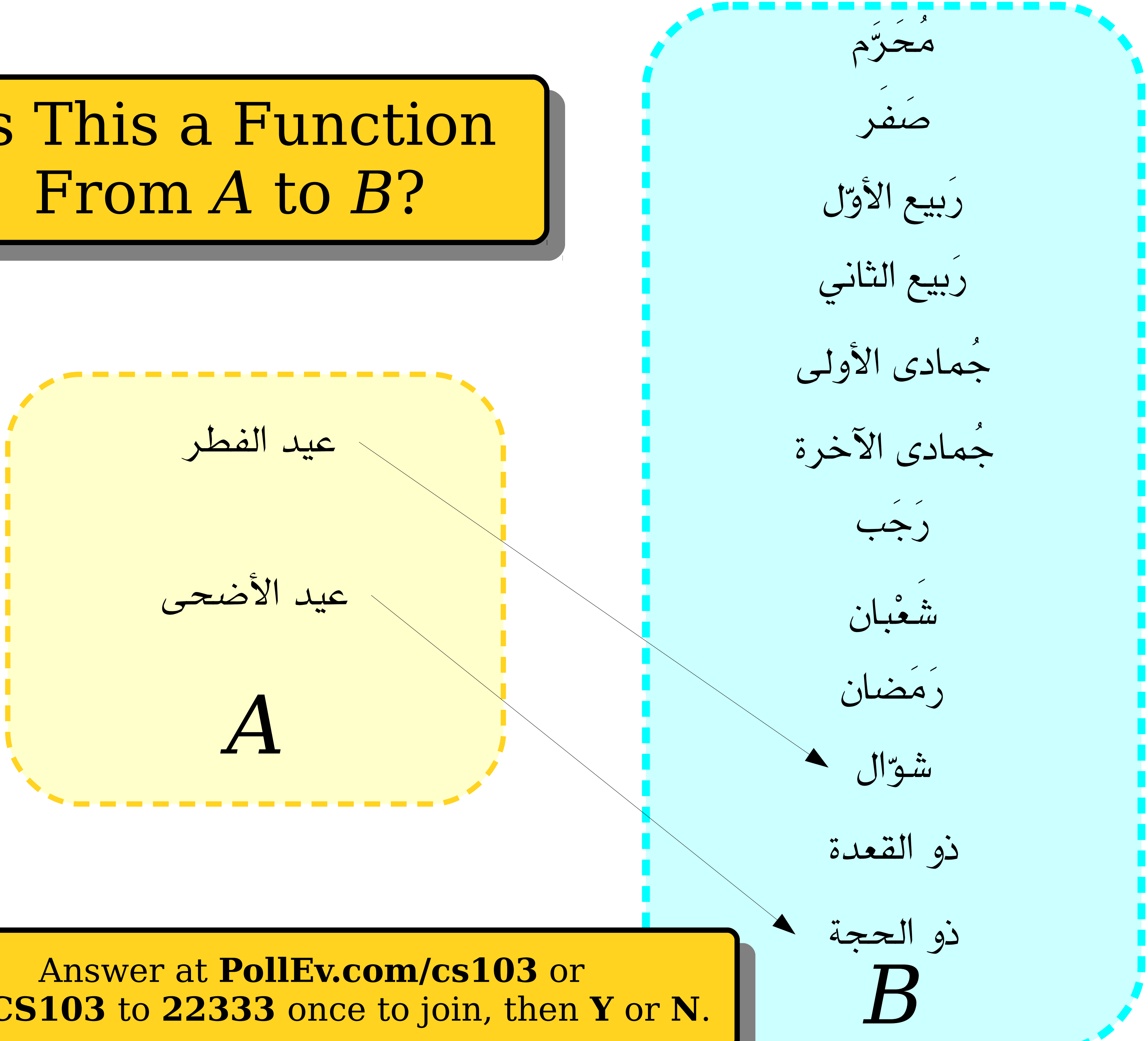
Is This a Function From A to B?



Is This a Function From A to B?

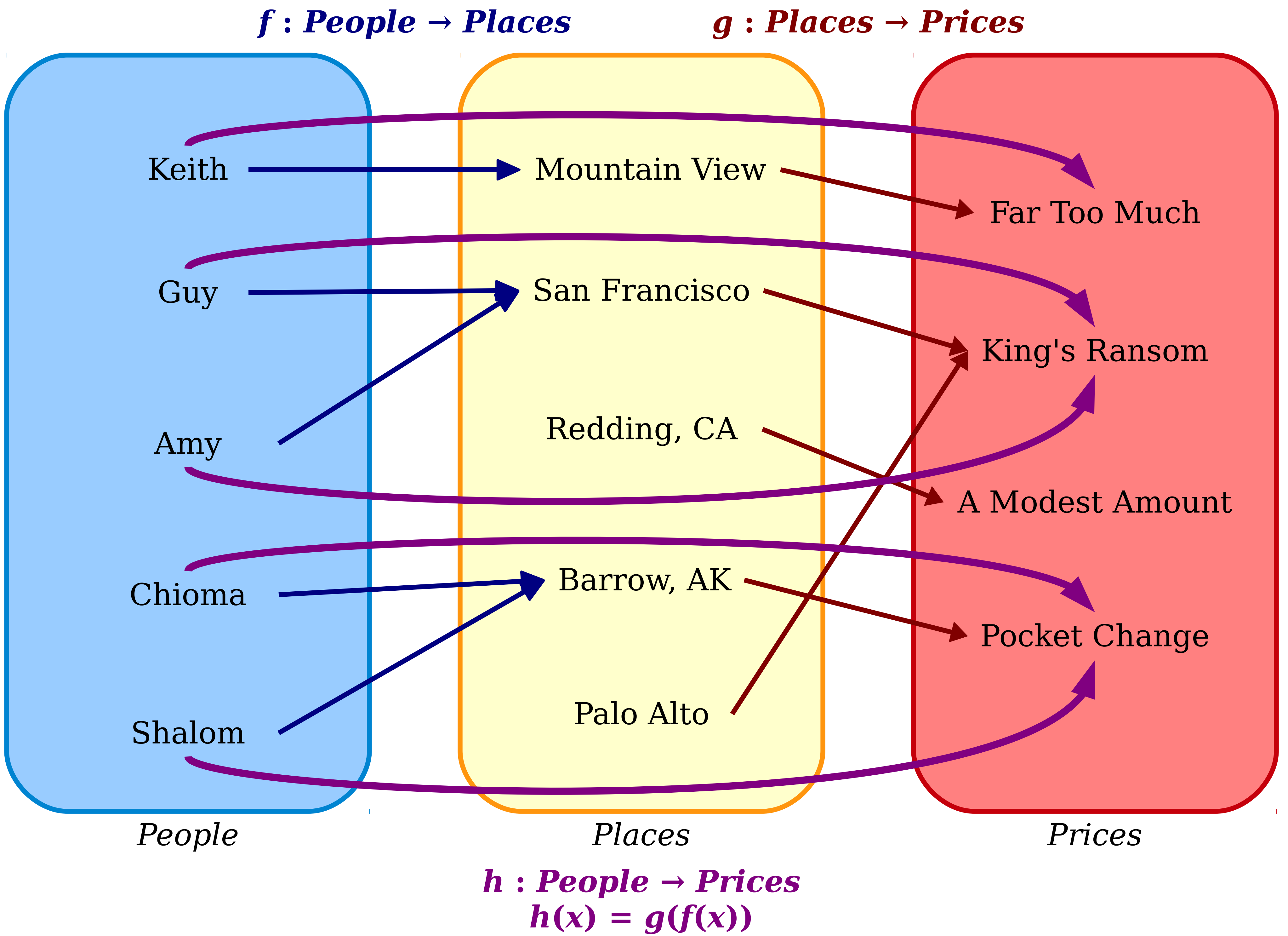


Is This a Function From A to B?



Answer at **PollEv.com/cs103** or
text **CS103** to **22333** once to join, then **Y** or **N**.

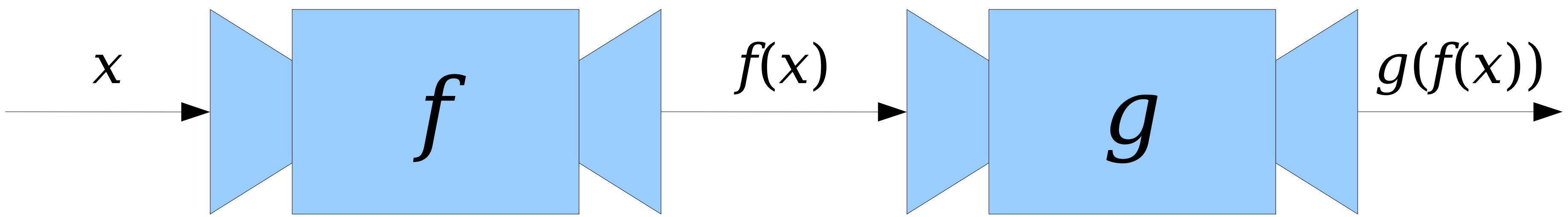
Combining Functions



f : People → Places
g : Places → Prices
Keith
Guy
Amy
Chioma
Shalom
Mountain View
San Francisco
Redding, CA
Barrow, AK
Palo Alto
Far Too Much
King's Ransom
A Modest Amount
Pocket Change
People
Places
Prices
h : People → Prices
h(x) = g(f(x))

Function Composition

- Suppose that we have two functions $f : A \to B$ and $g : B \to C$.
- Notice that the codomain of f is the domain of g. This means that we can use outputs from f as inputs to g.



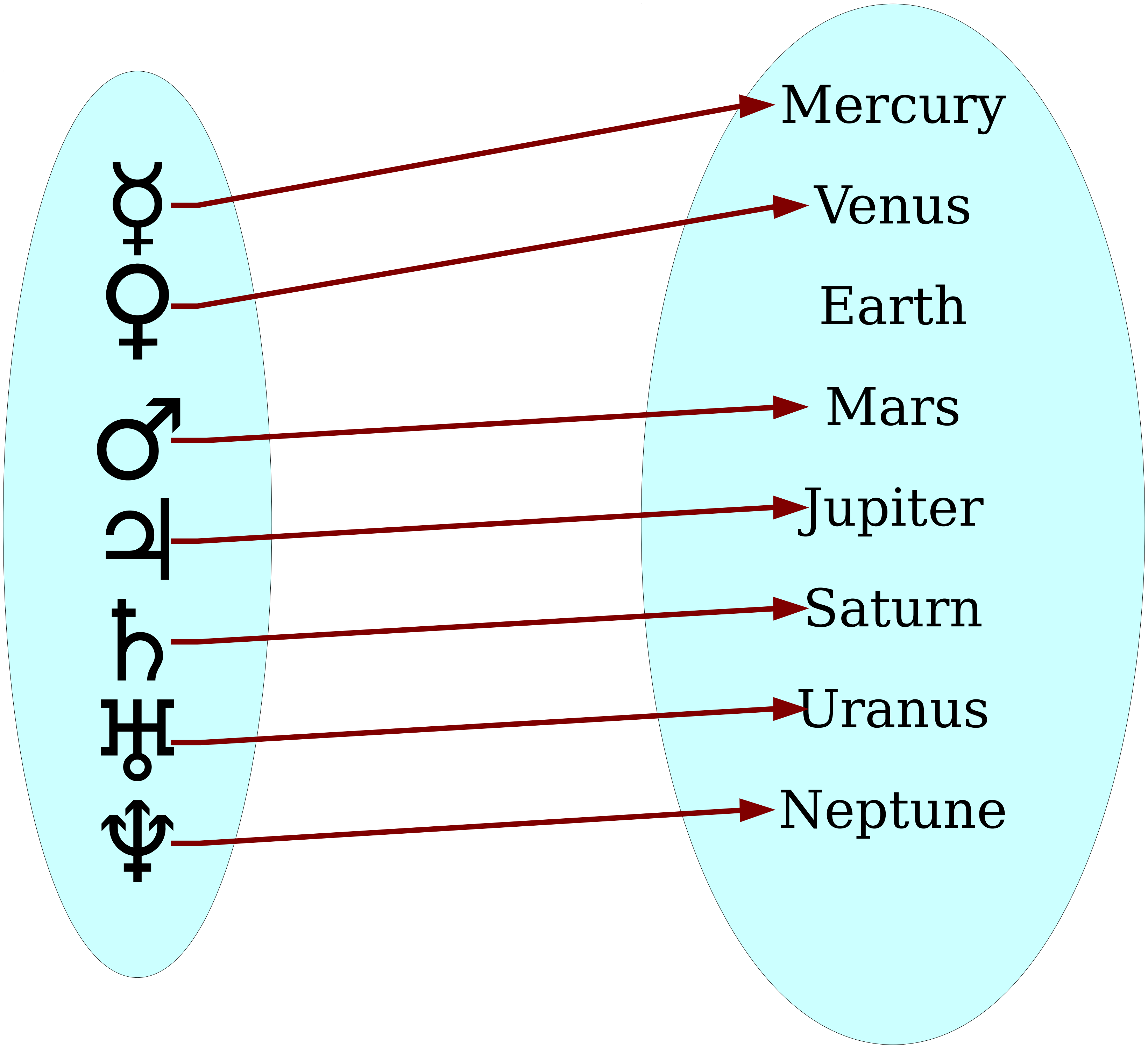
Function Composition

- Suppose that we have two functions $f : A \to B$ and $g : B \to C$.
- The ***composition of f and g***, denoted $\boldsymbol{g \circ f}$, is a function where
  - $g \circ f : A \to C$, and
  - $(g \circ f)(x) = g(f(x))$.

The name of the function is $g \circ f$. When we apply it to an input x, we write $(g \circ f)(x)$. I don't know why, but that's what we do.

- A few things to notice:
  - The domain of $g \circ f$ is the domain of f. Its codomain is the codomain of g.
  - Even though the composition is written $g \circ f$, when evaluating $(g \circ f)(x)$, the function f is evaluated first.

Special Types of Functions



☿
♀
♂
♃
♄
♅
♆
Mercury
Venus
Earth
Mars
Jupiter
Saturn
Uranus
Neptune

Injective Functions

- A function $f : A \to B$ is called ***injective*** (or ***one-to-one***) if the following statement is true about f:

$$\forall a_1 \in A.\ \forall a_2 \in A.\ (a_1 \neq a_2 \to f(a_1) \neq f(a_2))$$

(*"If the inputs are different, the outputs are different."*)

- The following first-order definition is equivalent and is often useful in proofs.

$$\forall a_1 \in A.\ \forall a_2 \in A.\ (f(a_1) = f(a_2) \to a_1 = a_2)$$

(*"If the outputs are the same, the inputs are the same."*)

- A function with this property is called an ***injection***.
- How does this compare to our second rule for functions?

Injective Functions

Theorem: Let $f : \mathbb{N} \to \mathbb{N}$ be defined as $f(n) = 2n + 7$. Then f is injective.

Proof:

How many of the following are correct ways of starting off this proof?

Consider any $n_1, n_2 \in \mathbb{N}$ where $n_1 = n_2$. We will prove that $f(n_1) = f(n_2)$.
Consider any $n_1, n_2 \in \mathbb{N}$ where $n_1 \neq n_2$. We will prove that $f(n_1) \neq f(n_2)$.
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Injective Functions

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Proof:

What does it mean for the function f to be injective?

$$\forall n_1 \in \mathbb{N}.\ \forall n_2 \in \mathbb{N}.\ (\ f(n_1) = f(n_2) \to n_1 = n_2\ )$$

$$\forall n_1 \in \mathbb{N}.\ \forall n_2 \in \mathbb{N}.\ (\ n_1 \neq n_2 \to f(n_1) \neq f(n_2)\ )$$

Therefore, we'll pick arbitrary $n_1, n_2 \in \mathbb{N}$ where $f(n_1) = f(n_2)$, then prove that $n_1 = n_2$.

Injective Functions

Theorem: Let $f : \mathbb{N} \to \mathbb{N}$ be defined as $f(n) = 2n + 7$. Then f is injective.

Proof: Consider any $n_1, n_2 \in \mathbb{N}$ where $f(n_1) = f(n_2)$. We will prove that $n_1 = n_2$.

Since $f(n_1) = f(n_2)$, we see that

$$2n_1 + 7 = 2n_2 + 7.$$

This in turn means that

$$2n_1 = 2n_2$$

so $n_1 = n_2$, as required. ■

Injective Functions

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Consider any $n_1, n_2 \in \mathbb{N}$ where $f(n_1) \neq f(n_2)$. We will prove that $n_1 \neq n_2$.

Good exercise: Repeat this proof using the other definition of injectivity!

Injective Functions

Theorem: Let $f : \mathbb{Z} \to \mathbb{N}$ be defined as $f(x) = x^4$. Then f is not injective.

Proof:

How many of the following are correct ways of starting off this proof?

Assume for the sake of contradiction that f is not injective.

Assume for the sake of contradiction that there are integers x_1 and x_2 where $f(x_1) = f(x_2)$ but $x_1 \neq x_2$.

Consider arbitrary integers x_1 and x_2 where $x_1 \neq x_2$. We will prove that $f(x_1) = f(x_2)$.

Consider arbitrary integers x_1 and x_2 where $f(x_1) = f(x_2)$. We will prove that $x_1 \neq x_2$.

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Injective Functions

Theorem: Let $f : \mathbb{Z} \to \mathbb{N}$ be defined as $f(x) = x^4$. Then f is not injective.

Proof:

What does it mean for f to be injective?

$$\forall x_1 \in \mathbb{Z}.\ \forall x_2 \in \mathbb{Z}.\ (x_1 \neq x_2 \to f(x_1) \neq f(x_2))$$

What is the negation of this statement?

$$\neg\forall x_1 \in \mathbb{Z}.\ \forall x_2 \in \mathbb{Z}.\ (x_1 \neq x_2 \to f(x_1) \neq f(x_2))$$
$$\exists x_1 \in \mathbb{Z}.\ \neg\forall x_2 \in \mathbb{Z}.\ (x_1 \neq x_2 \to f(x_1) \neq f(x_2))$$
$$\exists x_1 \in \mathbb{Z}.\ \exists x_2 \in \mathbb{Z}.\ \neg(x_1 \neq x_2 \to f(x_1) \neq f(x_2))$$
$$\exists x_1 \in \mathbb{Z}.\ \exists x_2 \in \mathbb{Z}.\ (x_1 \neq x_2 \land \neg(f(x_1) \neq f(x_2)))$$
$$\exists x_1 \in \mathbb{Z}.\ \exists x_2 \in \mathbb{Z}.\ (x_1 \neq x_2 \land f(x_1) = f(x_2))$$

Therefore, we need to find $x_1, x_2 \in \mathbb{Z}$ such that $x_1 \neq x_2$, but $f(x_1) = f(x_2)$.
Can we do that?

Injective Functions

Theorem: Let $f : \mathbb{Z} \to \mathbb{N}$ be defined as $f(x) = x^4$. Then f is not injective.

Proof: We will prove that there exist integers x_1 and x_2 such that $x_1 \neq x_2$, but $f(x_1) = f(x_2)$.

Let $x_1 = -1$ and $x_2 = +1$.

$$f(x_1) = f(-1) = (-1)^4 = 1$$

and

$$f(x_2) = f(1) = 1^4 = 1,$$

so $f(x_1) = f(x_2)$ even though $x_1 \neq x_2$, as required. ■

Injective Functions

Theorem: Let $f : \mathbb{Z} \to \mathbb{N}$ be defined as $f(x) = x^4$. Then f is not injective.

Proof: We will prove that there exist integers x_1 and x_2 such that $x_1 \neq x_2$, but $f(x_1) = f(x_2)$.

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Assume for the sake of contradiction that f is not injective.

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Consider arbitrary integers x_1 and x_2 where $x_1 \neq x_2$. We will prove that $f(x_1) = f(x_2)$.

Consider arbitrary integers x_1 and x_2 where $f(x_1) = f(x_2)$. We will prove that $x_1 \neq x_2$.

Injections and Composition

Injections and Composition

- ***Theorem:*** If $f : A \to B$ is an injection and $g : B \to C$ is an injection, then the function $g \circ f : A \to C$ is an injection.
- Our goal will be to prove this result. To do so, we're going to have to call back to the formal definitions of injectivity and function composition.

Theorem: If $f : A \to B$ is an injection and $g : B \to C$ is an injection, then the function $g \circ f : A \to C$ is also an injection.

Proof: Let $f : A \to B$ and $g : B \to C$ be arbitrary injections. We will prove that the function $g \circ f : A \to C$ is also injective.

There are two definitions of injectivity that we can use here:

$$\forall a_1 \in A.\ \forall a_2 \in A.\ ((g \circ f)(a_1) = (g \circ f)(a_2) \to a_1 = a_2)$$

$$\forall a_1 \in A.\ \forall a_2 \in A.\ (a_1 \neq a_2 \to (g \circ f)(a_1) \neq (g \circ f)(a_2))$$

Therefore, we'll choose an arbitrary $a_1, a_2 \in A$ where $a_1 \neq a_2$, then prove that $(g \circ f)(a_1) \neq (g \circ f)(a_2)$.

Theorem: If $f : A \to B$ is an injection and $g : B \to C$ is an injection, then the function $g \circ f : A \to C$ is also an injection.

Proof: Let $f : A \to B$ and $g : B \to C$ be arbitrary injections. We will prove that the function $g \circ f : A \to C$ is also injective. To do so, consider any $a_1, a_2 \in A$ where $a_1 \neq a_2$. We will prove that $(g \circ f)(a_1) \neq (g \circ f)(a_2)$. Equivalently, we need to show that $g(f(a_1)) \neq g(f(a_2))$.

How is $(g \circ f)(x)$ defined?

$$(g \circ f)(x) = g(f(x))$$

So we need to prove that $g(f(a_1)) \neq g(f(a_2))$.

Theorem: If $f : A \to B$ is an injection and $g : B \to C$ is an injection, then the function $g \circ f : A \to C$ is also an injection.

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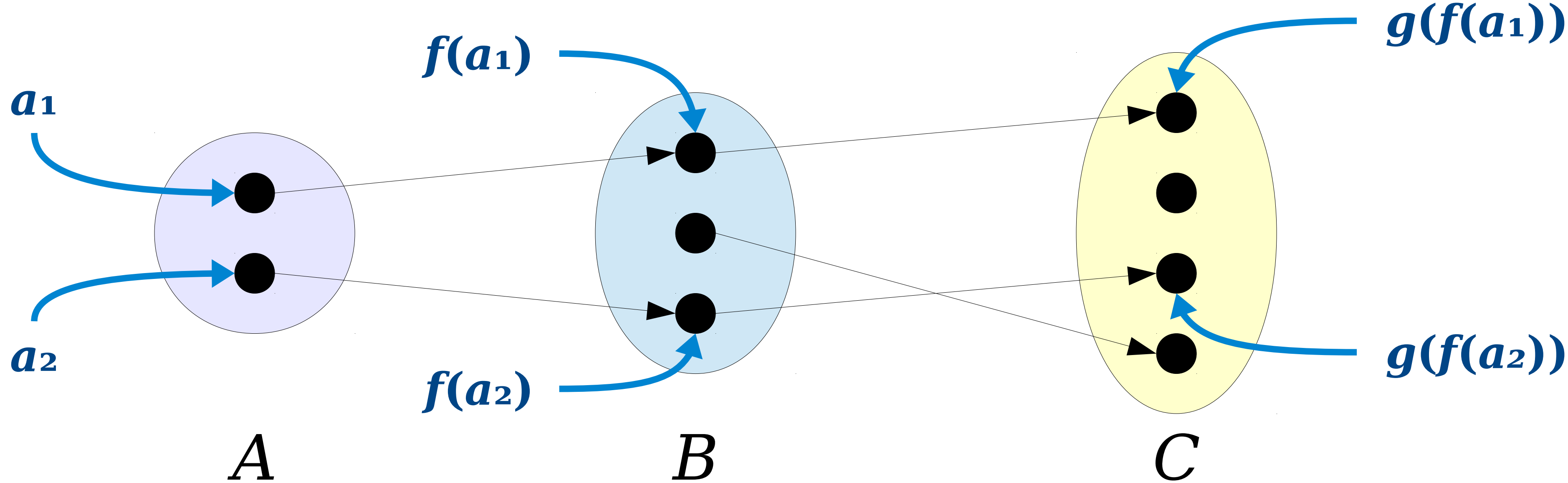
Since f is injective and $a_1 \neq a_2$, we see that $f(a_1) \neq f(a_2)$. Then, since g is injective and $f(a_1) \neq f(a_2)$, we see that $g(f(a_1)) \neq g(f(a_2))$, as required. ■



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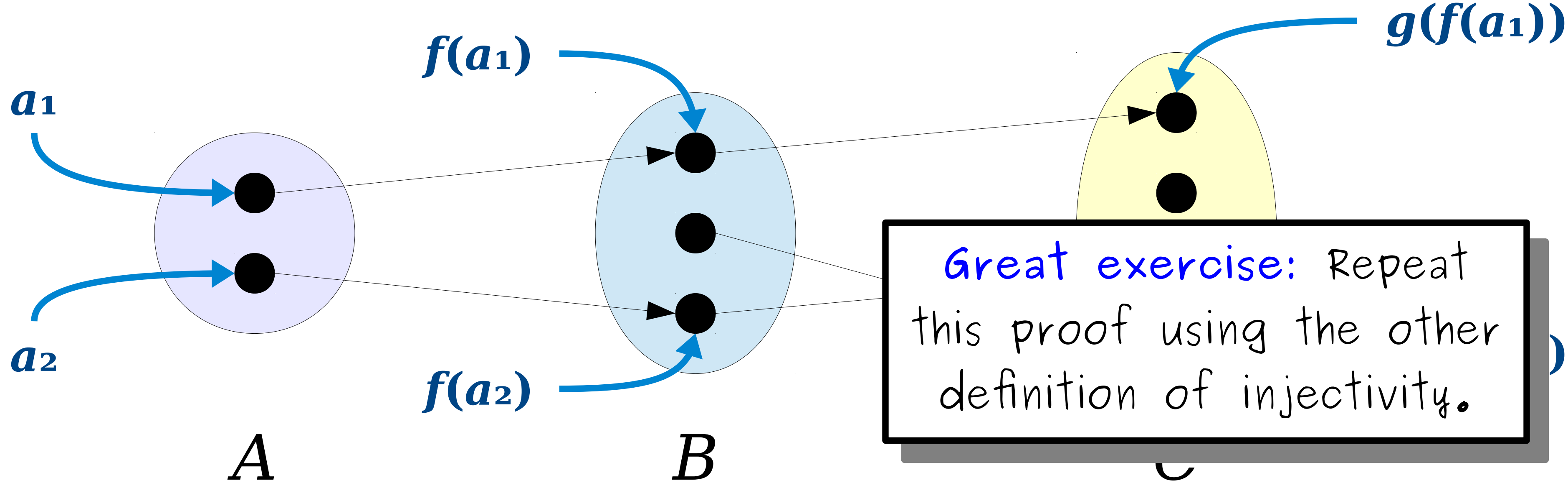
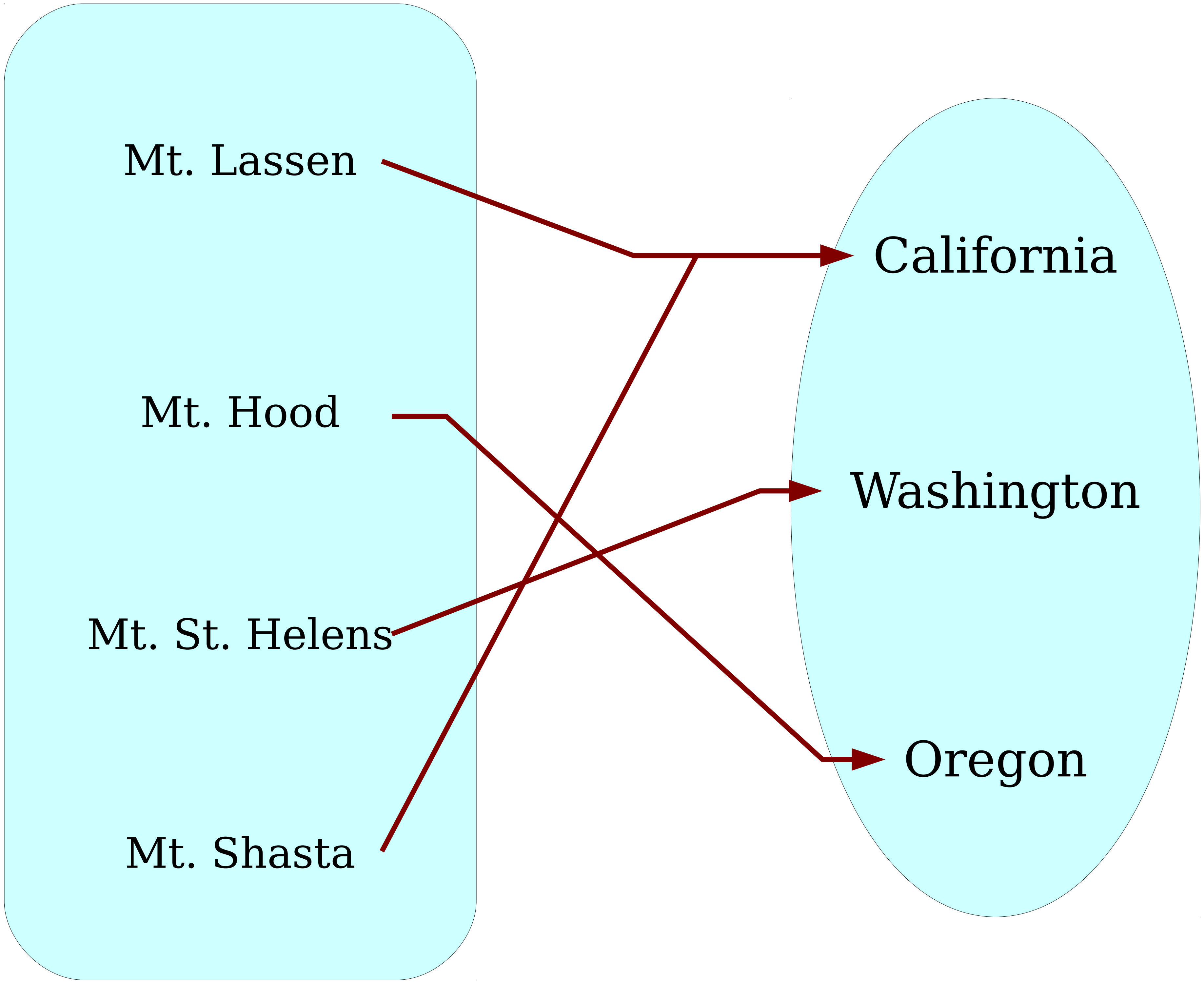
Since f is injective and $a_1 \neq a_2$, we see that $f(a_1) \neq f(a_2)$. Then, since g is injective and $f(a_1) \neq f(a_2)$, we see that $g(f(a_1)) \neq g(f(a_2))$, as required. ■



Great exercise: Repeat this proof using the other definition of injectivity.

Another Class of Functions


Mt. Lassen
Mt. Hood
Mt. St. Helens
Mt. Shasta
California
Washington
Oregon

Surjective Functions

- A function $f : A \to B$ is called ***surjective*** (or ***onto***) if this first-order logic statement is true about f:

$$\forall b \in B.\ \exists a \in A.\ f(a) = b$$

(*"For every possible output, there's at least one possible input that produces it"*)

- A function with this property is called a ***surjection***.
- How does this compare to our first rule of functions?

Surjective Functions

Theorem: Let $f : \mathbb{R} \to \mathbb{R}$ be defined as $f(x) = x / 2$. Then $f(x)$ is surjective.

Proof:

What does it mean for f to be surjective?

$$\forall y \in \mathbb{R}.\ \exists x \in \mathbb{R}.\ f(x) = y$$

Therefore, we'll choose an arbitrary $y \in \mathbb{R}$, then prove that there is some $x \in \mathbb{R}$ where $f(x) = y$.

Surjective Functions

Theorem: Let $f : \mathbb{R} \to \mathbb{R}$ be defined as $f(x) = x / 2$. Then $f(x)$ is surjective.

Proof: Consider any $y \in \mathbb{R}$. We will prove that there is a choice of $x \in \mathbb{R}$ such that $f(x) = y$.

Let $x = 2y$. Then we see that

$$f(x) = f(2y) = 2y / 2 = y.$$

So $f(x) = y$, as required. ■

Composing Surjections

Theorem: If $f : A \to B$ is surjective and $g : B \to C$ is surjective, then $g \circ f : A \to C$ is also surjective.

Proof: Let $f : A \to B$ and $g : B \to C$ be arbitrary surjections. We will prove that the function $g \circ f : A \to C$ is also surjective.

What does it mean for $g \circ f : A \to C$ to be surjective?

$$\forall c \in C.\ \exists a \in A.\ (g \circ f)(a) = c$$

Therefore, we'll choose arbitrary $c \in C$ and prove that there is some $a \in A$ such that $(g \circ f)(a) = c$.

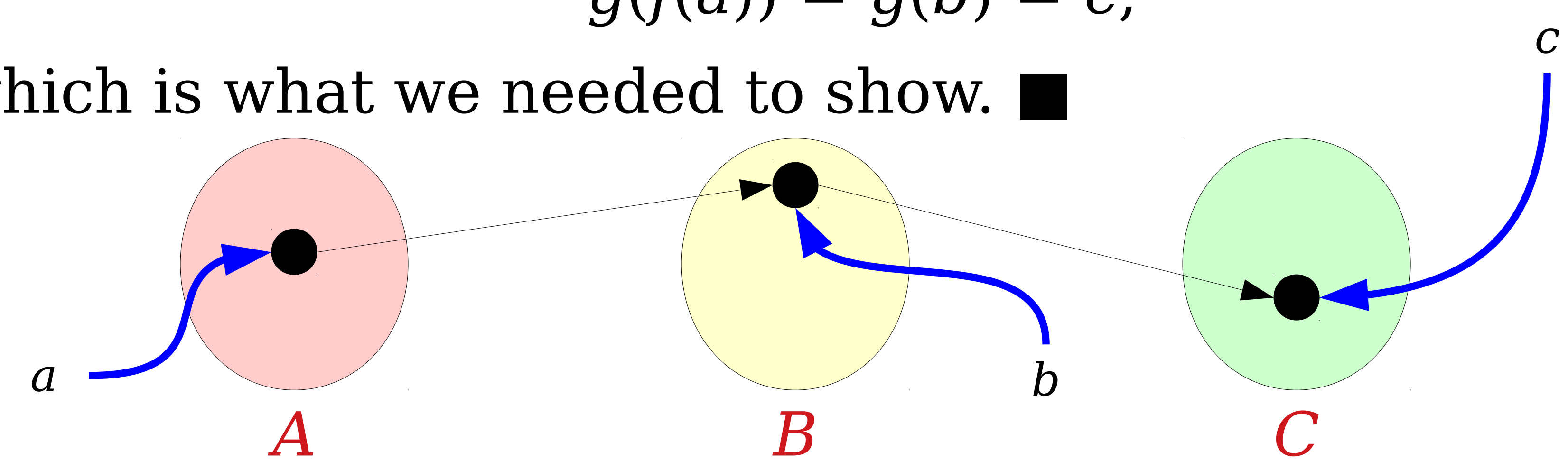
Theorem: If $f : A \to B$ is surjective and $g : B \to C$ is surjective, then $g \circ f : A \to C$ is also surjective.

Proof: Let $f : A \to B$ and $g : B \to C$ be arbitrary surjections. We will prove that the function $g \circ f : A \to C$ is also surjective. To do so, we will prove that for any $c \in C$, there is some $a \in A$ such that $(g \circ f)(a) = c$. Equivalently, we will prove that for any $c \in C$, there is some $a \in A$ such that $g(f(a)) = c$.

Consider any $c \in C$. Since $g : B \to C$ is surjective, there is some $b \in B$ such that $g(b) = c$. Similarly, since $f : A \to B$ is surjective, there is some $a \in A$ such that $f(a) = b$. This means that there is some $a \in A$ such that

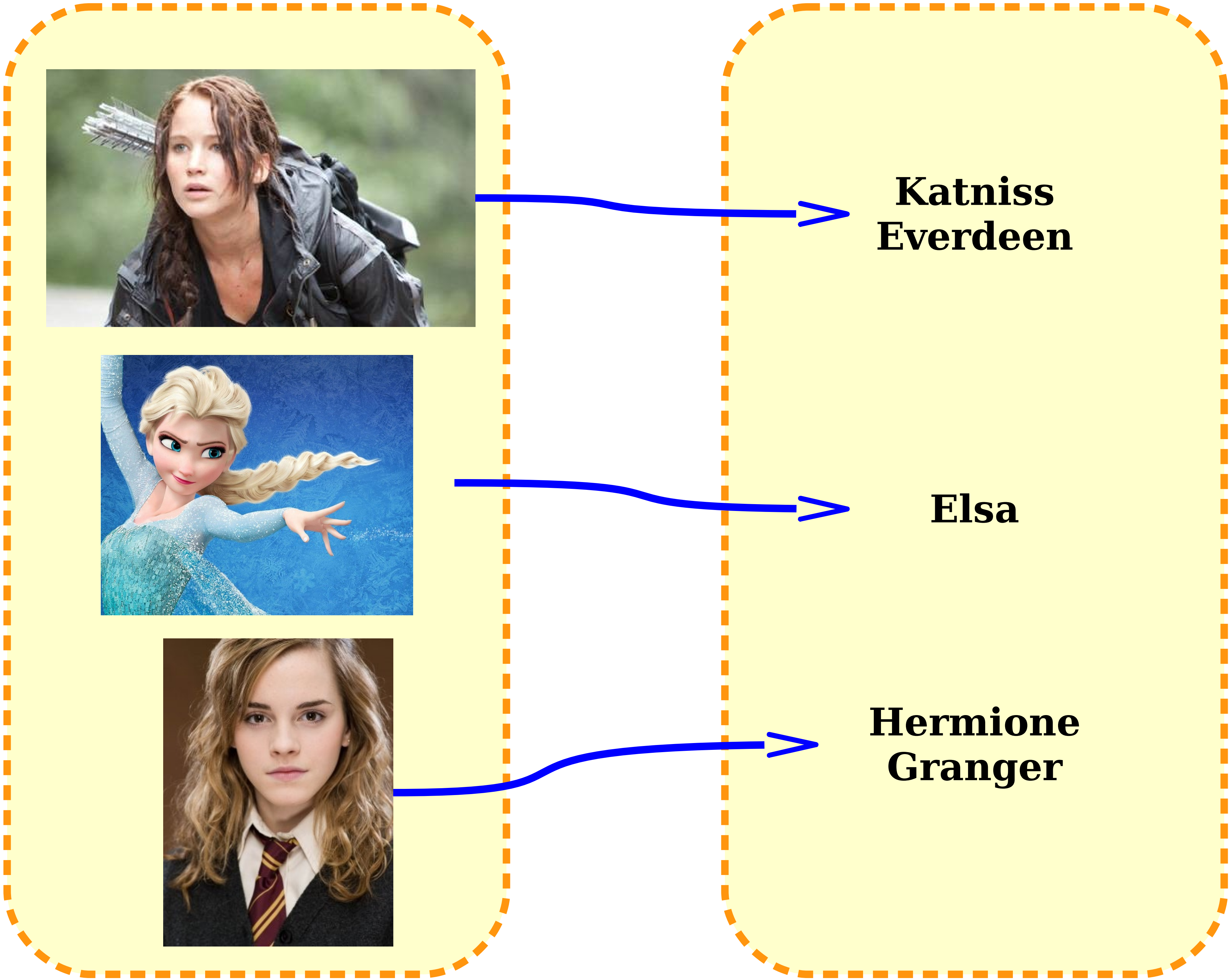
$$g(f(a)) = g(b) = c,$$

which is what we needed to show. ■



Injections and Surjections

- An injective function associates ***at most*** one element of the domain with each element of the codomain.
- A surjective function associates ***at least*** one element of the domain with each element of the codomain.
- What about functions that associate ***exactly one*** element of the domain with each element of the codomain?


Katniss
Everdeen
Elsa
Hermione
Granger

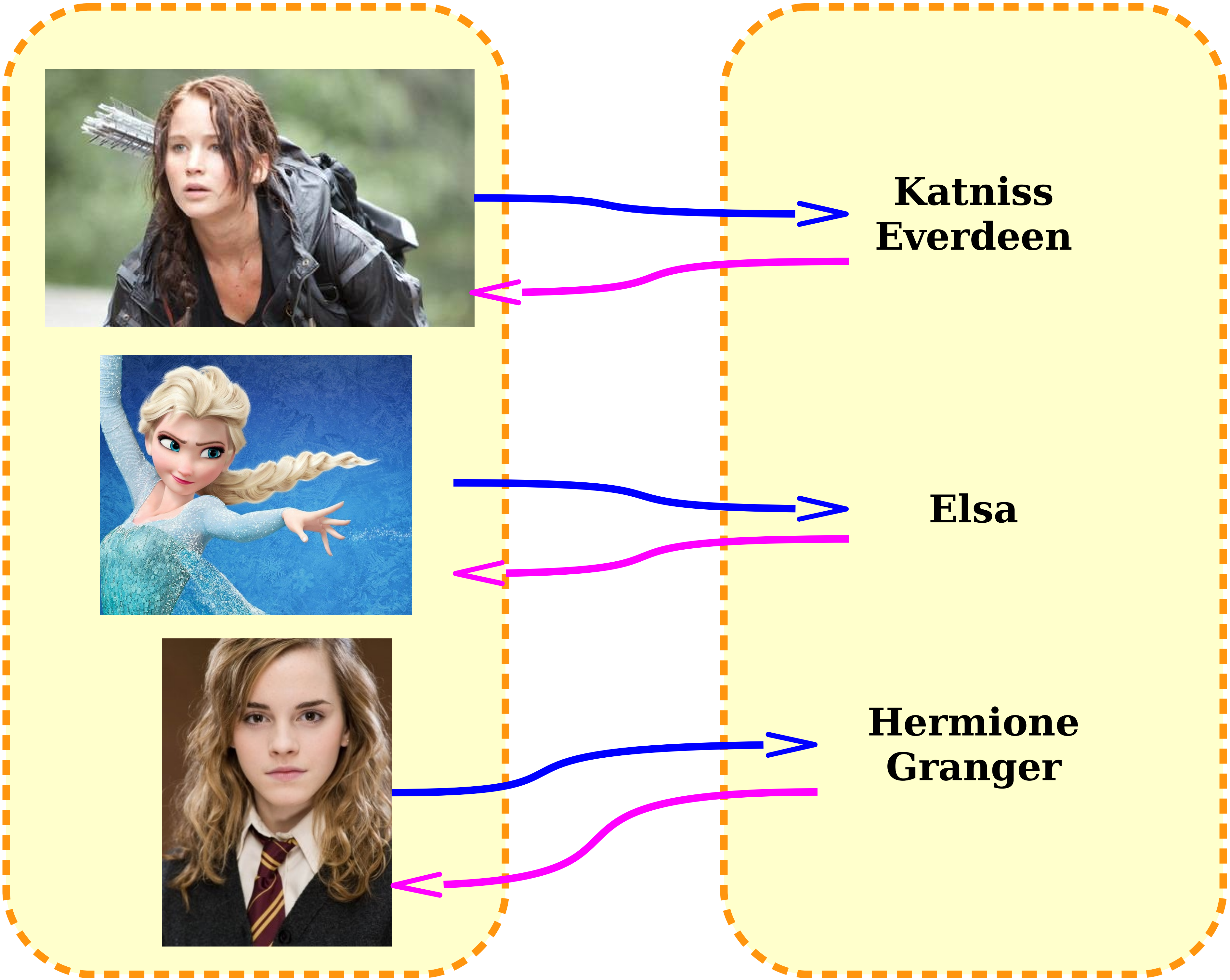
Bijections

- A function that associates each element of the codomain with a unique element of the domain is called ***bijective***.
  - Such a function is a ***bijection***.
- Formally, a bijection is a function that is both *injective* and *surjective*.
- Bijections are sometimes called ***one-to-one correspondences***.
  - Not to be confused with “one-to-one functions.”

Bijections and Composition

- Suppose that $f : A \to B$ and $g : B \to C$ are bijections.
- Is $g \circ f$ necessarily a bijection?
- ***Yes!***
  - Since both f and g are injective, we know that $g \circ f$ is injective.
  - Since both f and g are surjective, we know that $g \circ f$ is surjective.
  - Therefore, $g \circ f$ is a bijection.

Inverse Functions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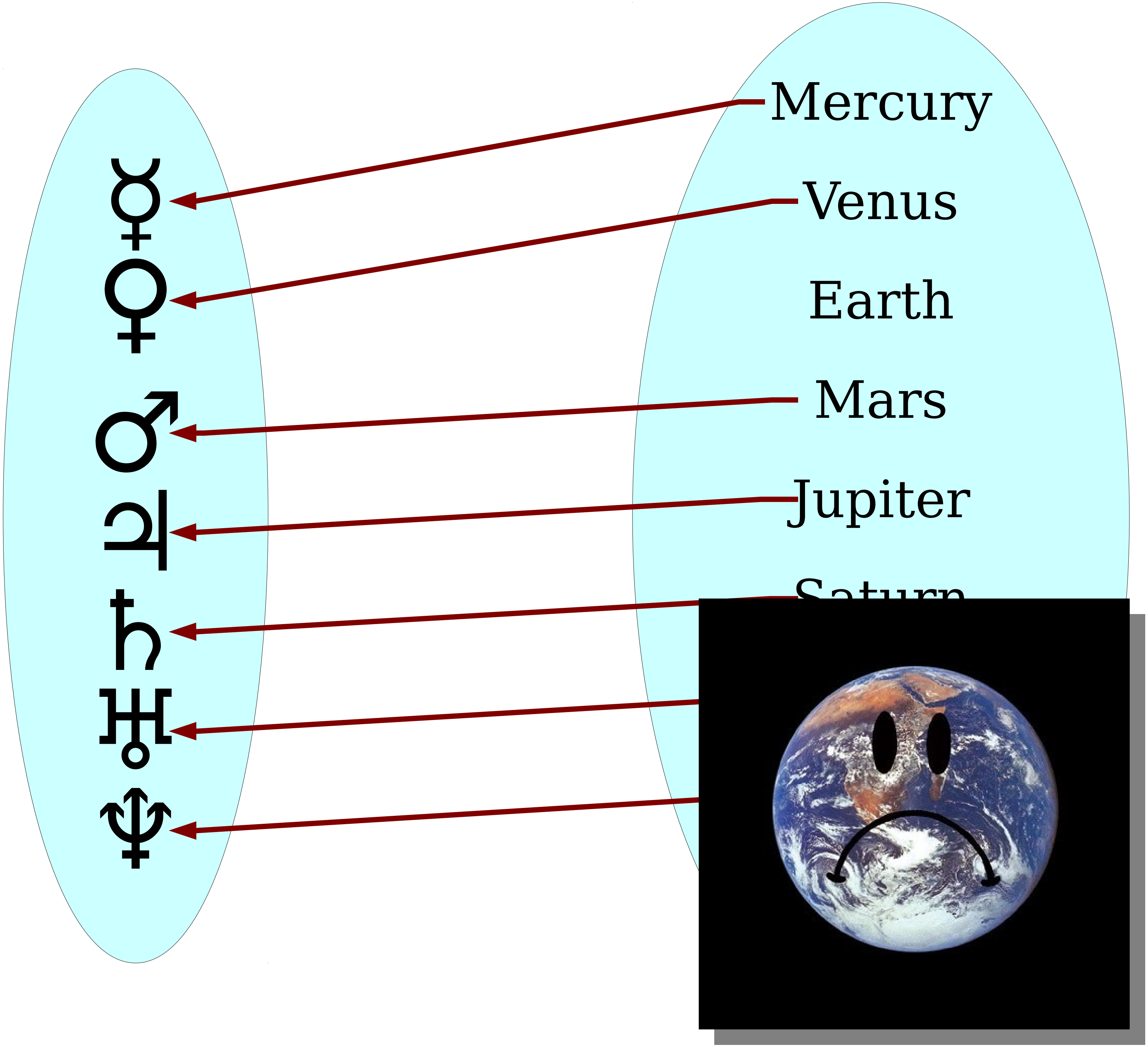
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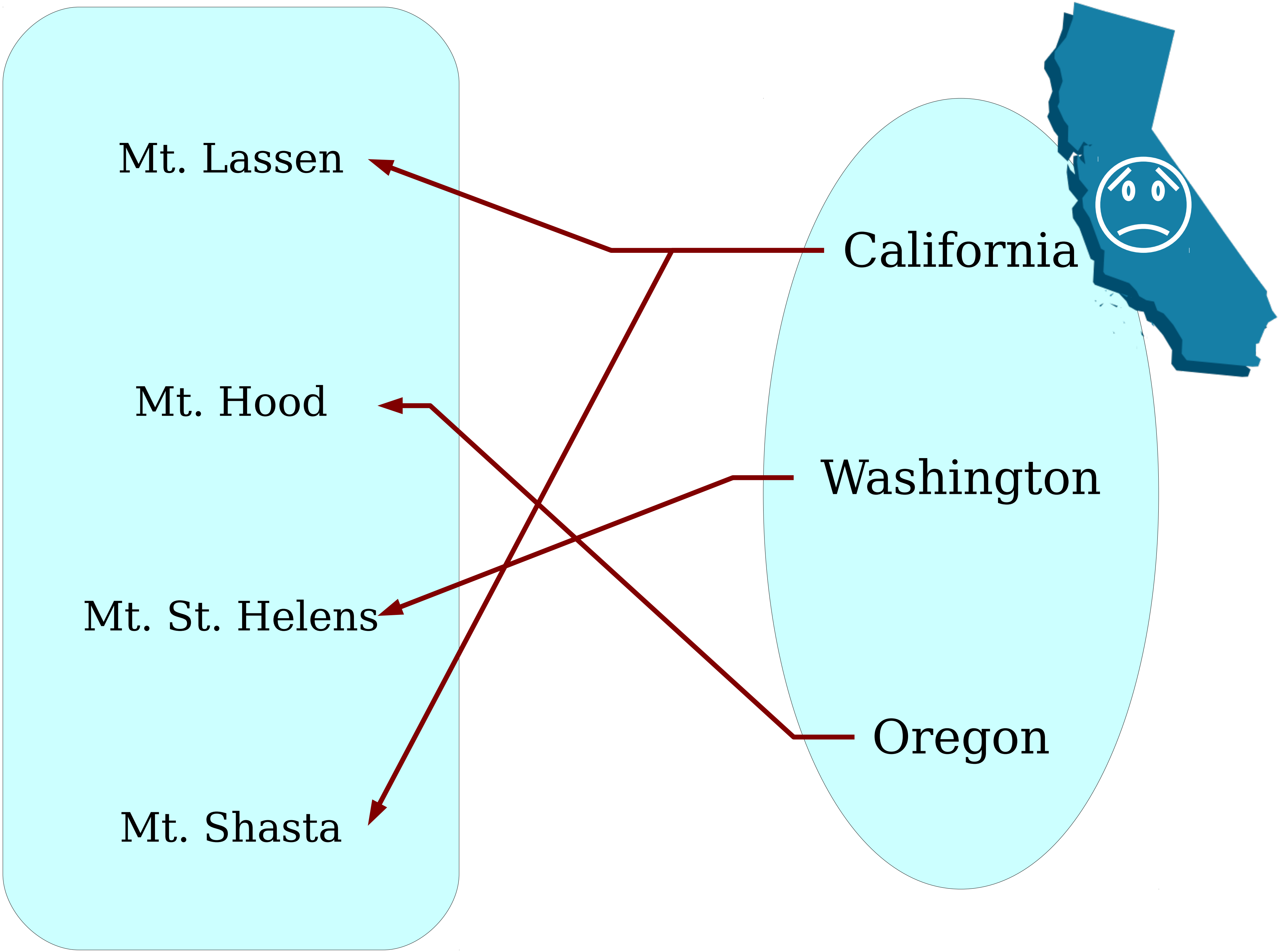
Earth
Mars
Jupiter


Mt. Lassen
Mt. Hood
Mt. St. Helens
Mt. Shasta
California
Washington
Oregon

Inverse Functions

- In some cases, it's possible to "turn a function around."
- Let $f : A \to B$ be a function. A function $f^{-1} : B \to A$ is called an ***inverse of f*** if the following first-order logic statements are true about f and f^{-1}

$$\forall a \in A. \ (f^{-1}(f(a)) = a) \qquad \forall b \in B. \ (f(f^{-1}(b)) = b)$$

- In other words, if f maps a to b, then f^{-1} maps b back to a and vice-versa.
- Not all functions have inverses (we just saw a few examples of functions with no inverses).
- If f is a function that has an inverse, then we say that f is ***invertible***.

Inverse Functions

- ***Theorem:*** Let $f : A \to B$. Then f is invertible if and only if f is a bijection.
- These proofs are in the course reader. Feel free to check them out if you'd like!
- ***Really cool observation:*** Look at the formal definition of a function. Look at the rules for injectivity and surjectivity. Do you see why this result makes sense?

Where We Are

- We now know
  - what an injection, surjection, and bijection are;
  - that the composition of two injections, surjections, or bijections is also an injection, surjection, or bijection, respectively; and
  - that bijections are invertible and invertible functions are bijections.
- You might wonder why this all matters. Well, there's a good reason…

Next Time

- ***Cardinality, Formally***
  - How do we rigorously define the idea that two sets have the same size?
- ***The Nature of Infinity***
  - It’s even weirder than you think!
- ***Cantor’s Theorem Revisited***
  - A formal proof of a major result!