Quantum Measurements Homework 1

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1 Rabi oscillations of two-level atom

a)

For the two-level basis $\{|e\rangle, |g\rangle\}$ we choose the ordering $|e\rangle = \begin{pmatrix}1\\0\end{pmatrix}$, $|g\rangle = \begin{pmatrix}0\\1\end{pmatrix}$. The Pauli matrices according to these definitions are:

$$\sigma_x = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} \qquad \sigma_y = \begin{pmatrix}0 & i\\ -i & 0\end{pmatrix} \qquad \sigma_z = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} \tag{1}$$

b)

From the definitions of the Pauli matrices (1) we can derive the following cyclic commutator relations:

$$[\sigma_x, \sigma_y] = 2i\sigma_z \qquad [\sigma_y, \sigma_z] = 2i\sigma_x \qquad [\sigma_z, \sigma_x] = 2i\sigma_y \tag{2}$$

These can be expressed more concisely with the Levi-Civita symbol ϵ_{ijk}:

$$[\sigma_i, \sigma_j] = 2i\sigma_k\epsilon_{ijk} \quad \text{Where} \quad \varepsilon_{ijk} = \begin{cases} +1 \text{ if } (i,j,k) \text{ is } (x,y,z), (y,z,x), \text{ or } (z,x,y), \\ -1 \text{ if } (i,j,k) \text{ is } (z,y,x), (x,z,y), \text{ or } (y,x,z), \\ \ \ 0 \text{ if } i=j, \text{ or } j=k, \text{ or } k=i \end{cases} \tag{3}$$

c)

We identify $\hat{X} = -i\frac{\omega_0 t}{2}\sigma_z$, $\hat{Y} = \sigma_x$. We start with the expression and expand it using the given expansion theorem:

$$e^{-i\frac{\omega_0 t}{2}\sigma_z}\sigma_x e^{i\frac{\omega_0 t}{2}\sigma_z} = \sigma_x + \frac{1}{1!}[-i\frac{\omega_0 t}{2}\sigma_z, \sigma_x] + \frac{1}{2!}[-i\frac{\omega_0 t}{2}\sigma_z, [-i\frac{\omega_0 t}{2}\sigma_z, \sigma_x]] \tag{4}$$

$$+ \frac{1}{3!}[-i\frac{\omega_0 t}{2}\sigma_z, [-i\frac{\omega_0 t}{2}\sigma_z, [-i\frac{\omega_0 t}{2}\sigma_z, \sigma_x]]] + ... \tag{5}$$

We evaluate the commutators:

$$[-i\frac{\omega_0 t}{2}\sigma_z, \sigma_x] = -i\frac{\omega_0 t}{2}[\sigma_z, \sigma_x] = -i\frac{\omega_0 t}{2}(2i\sigma_y) = \omega_0 t\sigma_y \tag{6}$$

$$[-i\frac{\omega_0 t}{2}\sigma_z, (\omega_0 t)\sigma_y] = -(\omega_0 t)^2[\frac{i}{2}\sigma_z, \sigma_y] = -(\omega_0 t)^2(\frac{i}{2}(-2i\sigma_x)) = -(\omega_0 t)^2\sigma_x \tag{7}$$

$$[-i\frac{\omega_0 t}{2}\sigma_z,\ -(\omega_0 t)^2\sigma_x] = -(\omega_0 t)^2[-i\frac{\omega_0 t}{2}\sigma_z, \sigma_x] = -(\omega_0 t)^2(\omega_0 t\sigma_y) = -(\omega_0 t)^3\sigma_y \tag{8}$$

$$[-i\frac{\omega_0 t}{2}\sigma_z,\ -(\omega_0 t)^3\sigma_y] = -(\omega_0 t)^3[-i\frac{\omega_0 t}{2}\sigma_z, \sigma_y] = -(\omega_0 t)^3(-\omega_0 t\sigma_x) = (\omega_0 t)^4\sigma_x \tag{9}$$

$$[-i\frac{\omega_0 t}{2}\sigma_z,\ (\omega_0 t)^4\sigma_x] = ... \tag{10}$$

Inserting these back into (5) gives us:

$$e^{-i\frac{\omega_0 t}{2}\sigma_z}\sigma_x e^{i\frac{\omega_0 t}{2}\sigma_z} = \sigma_x + \frac{1}{1!}(\omega_0 t)\sigma_y - \frac{1}{2!}(\omega_0 t)^2\sigma_x - \frac{1}{3!}(\omega_0 t)^3\sigma_y \tag{11}$$
$$+ \frac{1}{4!}(\omega_0 t)^4\sigma_x + \frac{1}{5!}(\omega_0 t)^5\sigma_y - ...$$

$$= \sigma_x(1 - \frac{(\omega_0 t)^2}{2!} + \frac{(\omega_0 t)^4}{4!} - ...) \tag{12}$$
$$+ \sigma_y(\frac{(\omega_0 t)^1}{1!} - \frac{(\omega_0 t)^3}{3!} + \frac{(\omega_0 t)^5}{5!} - ...)$$

$$= \sigma_x \sum_{n=0}^{\infty}(-1)^n \frac{(\omega_0 t)^{2n}}{(2n)!} + \sigma_y \sum_{n=0}^{\infty}(-1)^n \frac{(\omega_0 t)^{2n+1}}{(2n+1)!} \tag{13}$$

$$= \sigma_x \cos(\omega_0 t) + \sigma_y \sin(\omega_0 t) \tag{14}$$

Which is the relation we set out to show.

d)

$$e^{\hat{u}}|m\rangle = \sum_{n=0}^{\infty}\frac{1}{n!}\hat{u}^n|m\rangle \tag{15}$$
$$= \sum_{n=0}^{\infty}\frac{1}{n!}\mu^n|m\rangle \tag{16}$$
$$= e^{\mu}|m\rangle \tag{17}$$

e)

Consider the initial state $|\psi(0)\rangle = \psi_e|e\rangle + \psi_g|g\rangle$, we time evolve it according to $U = e^{-iH_b t/\hbar}$:

$$|\psi(t)\rangle = \psi_e e^{-iH_b t/\hbar}|e\rangle + \psi_g e^{-iH_b t/\hbar}|g\rangle \tag{18}$$
$$= \psi_e e^{-i\omega_0/2t}|e\rangle + \psi_g e^{i\omega_0/2t}|g\rangle \tag{19}$$
$$= \psi_e e^{-i\omega_0 t}|e\rangle + \psi_g|g\rangle \tag{20}$$

Where $\pm\hbar\omega_0/2$ are the eigenenergies of the ground and excited states. We note that the populations of the two levels are constant in time:

$$P_e(t) = |\psi_e e^{-i\omega_0 t}|^2 = |\psi_e|^2 \tag{21}$$
$$P_g(t) = |\psi_g|^2 \tag{22}$$

f)

We now add a Rabi drive interation Hamiltonian $H_d = \hbar\omega_1\cos(\omega t)\sigma_x$, where ω_1 and ω are the Rabi frequency and drive frequency respectively. Writing out the full interaction:

$$H = H_b + H_d = \hbar\frac{\omega}{2}\sigma_z + \hbar\omega_1\cos(\omega t)\sigma_x \tag{23}$$

We define the detuning $\delta = \omega_0 - \omega$, which allows us to identify $H_{rot} = \hbar\frac{\omega}{2}\sigma_z$

$$H = H_{rot} + \hbar\frac{\delta}{2}\sigma_z + H_d \tag{24}$$

We transform into an interation frame via $U_{rot} = e^{-iH_{rot}t/\hbar}$

$$H_{d,I} = U_{rot}H_d U_{rot}^\dagger = \hbar\omega_1\cos(\omega t)e^{-i\frac{\omega}{2}\sigma_z}\sigma_x e^{i\frac{\omega}{2}\sigma_z} \tag{25}$$

Here we invoke the result (14) from section **c)**:

$$\begin{aligned}
H_{d,I} &= \hbar\omega_1 \cos(\omega t)\left(\sigma_x \cos(-\omega t) + \sigma_y \sin(-\omega t)\right) && (26)\\
&= \hbar\omega_1 \cos(\omega t)\left(\sigma_x \cos(\omega t) - \sigma_y \sin(\omega t)\right) && (27)\\
&= \hbar\omega_1 \cos^2(\omega t)\sigma_x - \hbar\omega_1 \cos(\omega t)\sin(\omega t)\sigma_y && (28)\\
&= \hbar\frac{\omega_1}{4}\left(e^{2i\omega t} + e^{-2i\omega t} + 2\right)\sigma_x - \hbar\frac{\omega_1}{4i}\left(e^{2i\omega t} - e^{-2i\omega t}\right)\sigma_y && (29)
\end{aligned}$$

Here we notice that there are extremely quickly rotating terms $\sim e^{\pm 2i\omega t}$. The typical timescale of these fast oscillations is much faster than the time evolution of the atomic populations (we will see later that the atomic populations oscillate at $\sim \omega_1$ near resonance), hence we neglect these terms by time averaging our $H_{d,I}$ over the timescale $T = 1/(2\omega)$.

$$H_{d,I} \approx \hbar\frac{\omega_1}{2}\sigma_x \tag{30}$$

g)

We now wish to determine the correct Hamiltonian which dictates the time evolution of states $|\psi\rangle_I = U_{rot}|\psi\rangle$ in the interaction frame. We start from the Schrödinger equation in the lab frame, from which we make $|\psi\rangle_I$ appear on the RHS.

$$i\hbar\partial_t|\psi\rangle = H|\psi\rangle = HU^\dagger_{rot}|\psi\rangle_I \tag{31}$$

Next, we examine the LHS of the Schrödinger equation:

$$\begin{aligned}
i\hbar\partial_t|\psi\rangle &= i\hbar\partial_t(U^\dagger_{rot}|\psi\rangle_I) && (32)\\
&= i\hbar(\partial_t U^\dagger_{rot})|\psi\rangle_I + i\hbar U^\dagger_{rot}\partial_t|\psi\rangle_I && (33)\\
&= H_{rot}U^\dagger_{rot}|\psi\rangle_I + i\hbar U^\dagger_{rot}\partial_t|\psi\rangle_I && (34)
\end{aligned}$$

Where in the last line we used the result $\partial_t U^\dagger_{rot} = -\frac{i}{\hbar}H_{rot}U^\dagger_{rot}$. We then proceed by combining (31) and (34):

$$i\hbar U^\dagger_{rot}|\psi_I\rangle = HU^\dagger_{rot}|\psi\rangle_-\, H_{rot}U^\dagger_{rot}|\psi\rangle_I \tag{35}$$

Applying U_{rot} from the left

$$\begin{aligned}
i\hbar|\psi\rangle_I &= U_{rot}HU^\dagger_{rot}|\psi\rangle_I - U_{rot}H_{rot}U^\dagger_{rot}|\psi\rangle_I && (36)\\
&= U_{rot}(H - H_{rot})U^\dagger_{rot}|\psi\rangle_I && (37)\\
&= U_{rot}\left(\hbar\frac{\delta}{2}\sigma_z + H_d\right)U^\dagger_{rot}|\psi\rangle_I && (38)\\
&= \left(\hbar\frac{\delta}{2}\sigma_z + H_{d,I}\right)|\psi\rangle_I && (39)
\end{aligned}$$

Where in the last step we indentified the definition of $H_{d,I}$ (25) as well as the fact that σ_z commutes with U_{rot}. Hence we have found the correct Hamiltonian V_I for our system, inserting in the expression (30) found through the rotating wave approximation we obtain:

$$V_I = \hbar\frac{\delta}{2}\sigma_z + \hbar\frac{\omega_1}{2}\sigma_x \tag{40}$$

h)

We are now faced with the eigenvalue problem of V_I, we start by obtaining the characteristic polynomial,

$$E^2 - \frac{\hbar^2}{4}\delta^2 - \frac{\hbar^2}{4}\omega_1^2 = 0 \tag{41}$$

We find the two eigenenergies

$$E_\pm = \pm\frac{\hbar}{2}\sqrt{\delta^2 + \omega_1^2} = \pm\frac{\hbar\Omega}{2} \tag{42}$$

Again working with the ordering $|e\rangle = \begin{pmatrix}1\\0\end{pmatrix}$, $|g\rangle = \begin{pmatrix}0\\1\end{pmatrix}$ we find the following eigenstates and promptly normalise them with mixing angle θ:

$$|-\rangle = a_- \begin{pmatrix} \frac{\delta-\Omega}{\omega_1} \\ 1 \end{pmatrix} = \begin{pmatrix} \sin(\theta) \\ \cos(\theta) \end{pmatrix} \qquad |+\rangle = b_+ \begin{pmatrix} 1 \\ \frac{\Omega-\delta}{\omega_1} \end{pmatrix} = \begin{pmatrix} \cos(\theta) \\ -\sin(\theta) \end{pmatrix} \tag{43}$$

Solving for a_- and b_+ which normalise their respective eigenstates, we obtain:

$$b_+ = -a_- = \sqrt{\frac{\Omega-\delta}{2\Omega}} = -\sin(\theta) \tag{44}$$

Which is indeed in line with the hint $\sin^2(\theta) = \frac{\Omega-\delta}{2\Omega}$.

i)

Consider an atom initially in the ground state $|\psi(0)\rangle_I = |g\rangle$

Since V_I is not diagonal in the $\{|e\rangle, |g\rangle\}$ basis, calculations would be more convenient if we expressed our states in the $\{|+\rangle, |-\rangle\}$.

$$|g\rangle = \cos(\theta)|-\rangle - \sin(\theta)|+\rangle \qquad |e\rangle = \sin(\theta)|-\rangle + \cos(\theta)|+\rangle \tag{45}$$

Thus,

$$\begin{aligned} |\psi(t)\rangle_I &= e^{-iV_I t/\hbar}|\psi(0)\rangle_0 & (46)\\ &= \sum_{n=0}^{\infty}\left(\frac{-it}{\hbar}\right)^n [\cos(\theta)V_I^n|-\rangle - \sin(\theta)V_I^n|+\rangle] & (47)\\ &= \cos(\theta)\sum_{n=0}^{\infty}\frac{1}{n!}(i\Omega t/2)^n|-\rangle - \sin(\theta)\sum_{n=0}^{\infty}\frac{1}{n!}(-i\Omega t/2)^n|+\rangle & (48)\\ &= \cos(\theta)e^{i\frac{\Omega}{2}t}|-\rangle - \sin(\theta)e^{-i\frac{\Omega}{2}t}|+\rangle & (49) \end{aligned}$$

j)

The probability of measuring the atom in the excited state is given at time t is given by:

$$P_e(t) = |\langle e|\psi(t)\rangle_I|^2 \tag{50}$$

Before proceeding, we will first work this inner product out

$$\begin{aligned} \langle e|\psi(t)\rangle_I &= \cos(\theta)e^{i\frac{\Omega}{2}t}\langle e|-\rangle - \sin(\theta)e^{-i\frac{\Omega}{2}t}\langle e|+\rangle & (51)\\ &= \cos(\theta)\sin(\theta)e^{i\frac{\Omega}{2}t} - \cos(\theta)\sin(\theta)e^{-i\frac{\Omega}{2}t} & (52)\\ &= 2i\cos(\theta)\sin(\theta)\sin(\Omega t/2) & (53) \end{aligned}$$

Where $\sin^2(\theta) = \frac{\Omega-\delta}{2\Omega}$ and $\cos^2(\theta) = \frac{\omega_1^2}{2\Omega(\Omega-\delta)}$. Therefore,

$$\begin{aligned} P_e(t) &= 4\cos^2(\theta)\sin^2(\theta)\sin^2(i\Omega t/2) & (54)\\ &= \frac{\omega_1^2}{\omega_1^2+\delta^2}\sin^2(\Omega t/2) & (55)\\ &= \frac{1}{2}\frac{\omega_1^2}{\omega_1^2+\delta^2}[1-\cos(\Omega t)] & (56) \end{aligned}$$

Thus we see that for a resonant drive ($\delta = 0$) the atomic population oscillates between the ground and excited state at the Rabi frequency ω_1 (note: $\Omega = \omega_1$ when $\delta = 0$), full population inversion being achieved when $\omega_1 t = \pi$. For non-zero detuning, full population transfer does not occur and the oscillations happen at the generalised Rabi frequency Ω.

2 Quantum jumps and quantum Zeno effect

a)

We start with the Hamiltonian:

$$H_I = |{\downarrow}\rangle\langle{\downarrow}|\, \hbar\Omega\sigma_x + |{\uparrow}\rangle\langle{\uparrow}|\, (\hbar g\sigma_z + \hbar\Omega\sigma_x) \tag{57}$$

Where $|{\downarrow}\rangle\langle{\downarrow}|$ and $|{\uparrow}\rangle\langle{\uparrow}|$ act on the ancillia system and $|{\downarrow}\rangle$, $|{\uparrow}\rangle$ are the ground and excited states of the two-level ancillia respectively. The Pauli operators act on our two-level atomic system. Time evolution after an interaction time Δt is given by the propagator:

$$\begin{aligned}
U &= \exp[-iH_I\Delta t/\hbar] && (58)\\
&= \exp[-i\,[\Delta t\Omega\, |{\downarrow}\rangle\langle{\downarrow}|\, \sigma_x + |{\uparrow}\rangle\langle{\uparrow}|\, (\Delta t g\sigma_z + \Delta t\Omega\sigma_x)]] && (59)\\
&= \exp[-i\,[\rho\, |{\downarrow}\rangle\langle{\downarrow}|\, \sigma_x + |{\uparrow}\rangle\langle{\uparrow}|\, (\theta\sigma_z + \rho\sigma_x)]] && (60)\\
&= \sum_{n=0}^{\infty} \frac{1}{n!}(-i)^n \left[\rho\, |{\downarrow}\rangle\langle{\downarrow}|\, \sigma_x + |{\uparrow}\rangle\langle{\uparrow}|\, (\theta\sigma_z + \rho\sigma_x)\right]^n && (61)\\
&= \sum_{n=0}^{\infty} \frac{1}{n!}(-i)^n \left[\rho^n\, |{\downarrow}\rangle\langle{\downarrow}|\, \sigma_x^n + |{\uparrow}\rangle\langle{\uparrow}|\, (\theta\sigma_z + \rho\sigma_x)^n\right] && (62)\\
&= |{\downarrow}\rangle\langle{\downarrow}|\, e^{-i\rho\sigma_x} + |{\uparrow}\rangle\langle{\uparrow}|\, e^{-i(\theta\sigma_z + \rho\sigma_x)} && (63)
\end{aligned}$$

Where $\rho = \Delta t\Omega$ and $\theta = \Delta t g$. Note that this θ has nothing to do with the mixing angle introduced in the previous section. In the last step, we used the orthonormality of $\{|{\uparrow}\rangle, |{\downarrow}\rangle\}$.

b)

If the initial state of the ancillia and atom are $|{\rightarrow_x}\rangle = \frac{1}{\sqrt{2}}(|{\downarrow}\rangle + |{\uparrow}\rangle)$ and $\psi_e\,|e\rangle + \psi_g\,|g\rangle$ respectively, then the initial state of the joint atom+ancillia system is given by:

$$\begin{aligned}
\left|\tilde{\psi}(0)\right\rangle &= |{\rightarrow_x}\rangle \otimes (\psi_e\,|e\rangle + \psi_g\,|g\rangle) && (64)\\
&= \frac{1}{\sqrt{2}}\,|{\downarrow}\rangle\,(\psi_e\,|e\rangle + \psi_g\,|g\rangle) + \frac{1}{\sqrt{2}}\,|{\uparrow}\rangle\,(\psi_e\,|e\rangle + \psi_g\,|g\rangle) && (65)
\end{aligned}$$

c)

The joint wavefunction after an interaction time of Δt

$$\begin{aligned}
\left|\tilde{\psi}(\Delta t)\right\rangle &= U\left|\tilde{\psi}(0)\right\rangle && (66)\\
&= \frac{1}{\sqrt{2}}\,|{\downarrow}\rangle\, e^{-i\rho\sigma_x}(\psi_e\,|e\rangle + \psi_g\,|g\rangle) + \frac{1}{\sqrt{2}}\,|{\uparrow}\rangle\, e^{-i(\theta\sigma_z+\rho\sigma_x)}(\psi_e\,|e\rangle + \psi_g\,|g\rangle) && (67)
\end{aligned}$$

For small time steps Δt(such that $\rho, \theta \ll 1$) we use the following approximation $e^{-i(\theta\sigma_z+\rho\sigma_x)} \approx e^{-i\rho\sigma_x}e^{-i\theta\sigma_z}$. This is not true in general as σ_x and σ_z do not commute. Since $|e\rangle$, $|g\rangle$ are eigenstates of σ_x, we can directly make use of (17) to get rid of all occurences of $\exp(-i\theta\sigma_z)$:

$$\begin{aligned}
\left|\tilde{\psi}(\Delta t)\right\rangle &\approx \frac{1}{\sqrt{2}}\,|{\downarrow}\rangle\, e^{-i\rho\sigma_x}(\psi_e\,|e\rangle + \psi_g\,|g\rangle) + \frac{1}{\sqrt{2}}\,|{\uparrow}\rangle\, e^{-i\rho\sigma_x}e^{\theta\sigma_z}(\psi_e\,|e\rangle + \psi_g\,|g\rangle) && (68)\\
&= e^{-i\rho\sigma_x}\left[\frac{1}{\sqrt{2}}\,|{\downarrow}\rangle\,(\psi_e\,|e\rangle + \psi_g\,|g\rangle) + \frac{1}{\sqrt{2}}\,|{\uparrow}\rangle\,(\psi_e e^{-i\theta}\,|e\rangle + \psi_g e^{i\theta}\,|g\rangle)\right] && (69)
\end{aligned}$$

d)

We detect our ancillia in the basis $\{|{\leftarrow_y}\rangle, |{\rightarrow_y}\rangle\}$, the decomposition of $\{|{\uparrow}\rangle, |{\downarrow}\rangle\}$ in this basis is:

$$|{\uparrow}\rangle = \frac{1}{\sqrt{2}}\left[|{\rightarrow_y}\rangle + |{\leftarrow_y}\rangle\right] \qquad |{\downarrow}\rangle = \frac{i}{\sqrt{2}}\left[|{\rightarrow_y}\rangle - |{\leftarrow_y}\rangle\right] \tag{70}$$

Inserting (70) into (69) we obtain:

$$\left|\tilde{\psi}(\Delta t)\right\rangle = \left|\leftarrow_y\right\rangle e^{-i\rho\sigma_x}\left(\frac{e^{-i\theta}+i}{2}\psi_e\left|e\right\rangle + \frac{e^{i\theta}+i}{2}\psi_g\left|g\right\rangle\right) + \left|\rightarrow_y\right\rangle e^{-i\rho\sigma_x}\left(\frac{e^{-i\theta}-i}{2}\psi_e\left|e\right\rangle + \frac{e^{i\theta}-i}{2}\psi_g\left|g\right\rangle\right) \tag{71}$$

$$= \left|\leftarrow_y\right\rangle e^{-i\rho\sigma_x}\left|\psi'_\leftarrow\right\rangle + \left|\rightarrow_y\right\rangle e^{-i\rho\sigma_x}\left|\psi'_\rightarrow\right\rangle \tag{72}$$

Where we denote $|\psi'_\leftarrow\rangle$ and $|\psi'_\rightarrow\rangle$ as the ancillia dependent atomic states before applying the $e^{-i\rho\sigma_x}$ operator. We note for further reference that these states are not normalised, but that they do satisfy:

$$\langle\psi'_\leftarrow|\psi'_\leftarrow\rangle + \langle\psi'_\rightarrow|\psi'_\rightarrow\rangle = 1 \tag{73}$$

In other words, the total probability of measuring the atom in any of the two states is 1. For future brevity, we also introduce the following coefficients:

$$a_\leftarrow = \frac{e^{-i\theta}+i}{2} \qquad b_\leftarrow = \frac{e^{i\theta}+i}{2} \tag{74}$$

$$a_\rightarrow = \frac{e^{-i\theta}-i}{2} \qquad b_\rightarrow = \frac{e^{i\theta}-i}{2} \tag{75}$$

As a consequence of (73) these coefficients obey the following equalities:

$$|a_\leftarrow|^2 + |a_\rightarrow|^2 = 1 \qquad |b_\leftarrow|^2 + |b_\rightarrow|^2 = 1 \tag{76}$$

e)

We now evaluate the action $e^{-i\rho\sigma_x}$, for small Δt we expand it to linear order such that $e^{-i\rho\sigma_x} \approx I - i\rho\sigma_x$, where I is the identity operator. We have to be very careful applying this linearised operator as it is no longer unitary, so we have to include a normalising factor N when applying it in order to conserve the norm of the state it is applied on. We define:

$$\begin{aligned}|\psi_\leftarrow\rangle &= e^{-i\rho\sigma_x}\left|\psi'_\leftarrow\right\rangle &(77)\\ &\approx N(I - i\rho\sigma_x)\left(a_\leftarrow\psi_e\left|e\right\rangle + b_\leftarrow\psi_g\left|g\right\rangle\right) &(78)\\ &= N\left(\left[a_\leftarrow\psi_e - i\rho b_\leftarrow\psi_g\right]\left|e\right\rangle + \left[b_\leftarrow\psi_e - i\rho a_\leftarrow\psi_g\right]\left|e\right\rangle\right) &(79)\end{aligned}$$

Doing the same for $|\psi'_\rightarrow\rangle$:

$$|\psi_\rightarrow\rangle = N\left(\left[a_\rightarrow\psi_e - i\rho b_\rightarrow\psi_g\right]\left|e\right\rangle + \left[b_\rightarrow\psi_e - i\rho a_\rightarrow\psi_g\right]\left|e\right\rangle\right) \tag{80}$$

Imposing the condition $\langle\psi_\leftarrow|\psi_\leftarrow\rangle + \langle\psi_\rightarrow|\psi_\rightarrow\rangle = 1$ allows us to determine $N = (1+\rho^2)^{-\frac{1}{2}}$.

f)

Now we calculate the probability of detecting the ancillia either in the $|\rightarrow_y\rangle$ ($P(+1)$) or the $|\leftarrow_y\rangle$ ($P(-1)$) state.

$$\begin{aligned}P(+1) &= \langle\psi_\rightarrow|\psi_\rightarrow\rangle &(81)\\ &= N^2(1+\rho^2)\left(|a_\rightarrow|^2|\psi_e|^2 + |b_\rightarrow|^2|\psi_g|^2\right) &(82)\\ &= \frac{|\psi_e|^2}{4}(1 + ie^{i\theta} - ie^{-i\theta} + 1) + \frac{|\psi_g|^2}{4}(1 + ie^{-i\theta} - ie^{i\theta} + 1) &(83)\\ &= \frac{|\psi_e|^2}{2}(1 - \sin(\theta)) + \frac{|\psi_g|^2}{2}(1 + \sin(\theta)) &(84)\end{aligned}$$

Similiary, we find for $|\leftarrow_y\rangle$:

$$\begin{aligned}P(-1) &= \langle\psi_\leftarrow|\psi_\leftarrow\rangle &(85)\\ &= \frac{|\psi_e|^2}{2}(1 + \sin(\theta)) + \frac{|\psi_g|^2}{2}(1 - \sin(\theta)) &(86)\end{aligned}$$

These probabilities do indeed at up to one as $P(+1) + P(-1) = |\psi_e|^2 + |\psi_g|^2 = 1$. It is quite interesting to see that these probabilites do not at all depend on the Rabi drive ρ.

g)

First, we want to verify that the simulation works; for this, it makes sense to look at a few basic test cases.[1]

Case 1: $[\Delta t = 0, g = 0, \Omega = 0]$ If we have no interaction between the particle and either the ancilla or the Rabi drive we expect constant probabilities and a random walk.

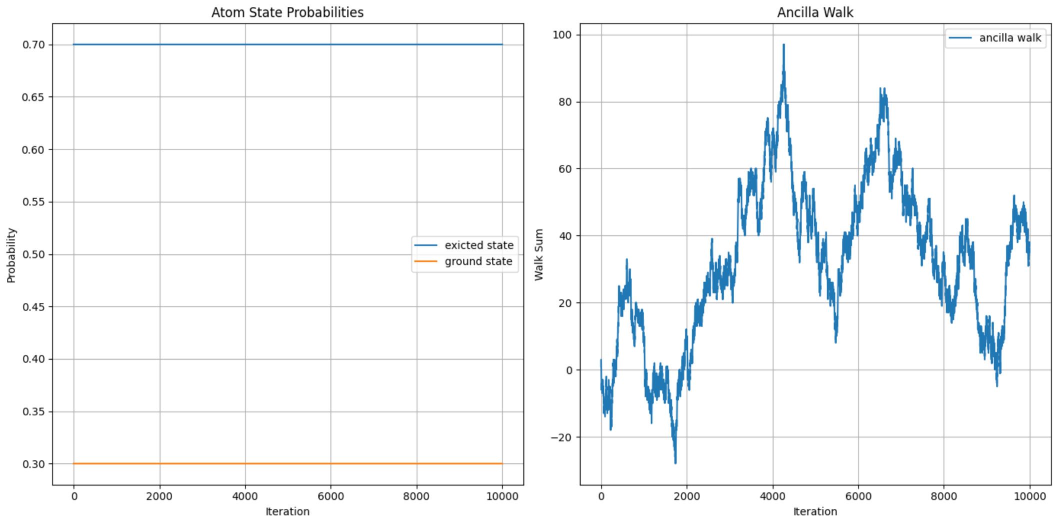

Figure 1: Simulation of the system with parameters $\rho = 0, \theta = 0$ and initial conditions $|\psi_g|^2 = 0.3, |\psi_e|^2 = 0.7$.

Case 2: $[\Delta t > 0, g > 0, \Omega = 0]$ If only the ancilla interacts with the atom then we expect the atom to converge to one of the states after sufficient interactions. We further expect a biased walk towards the final state. (The speed of the convergence is dependent on the measurement strength $\theta = g\Delta t$.)

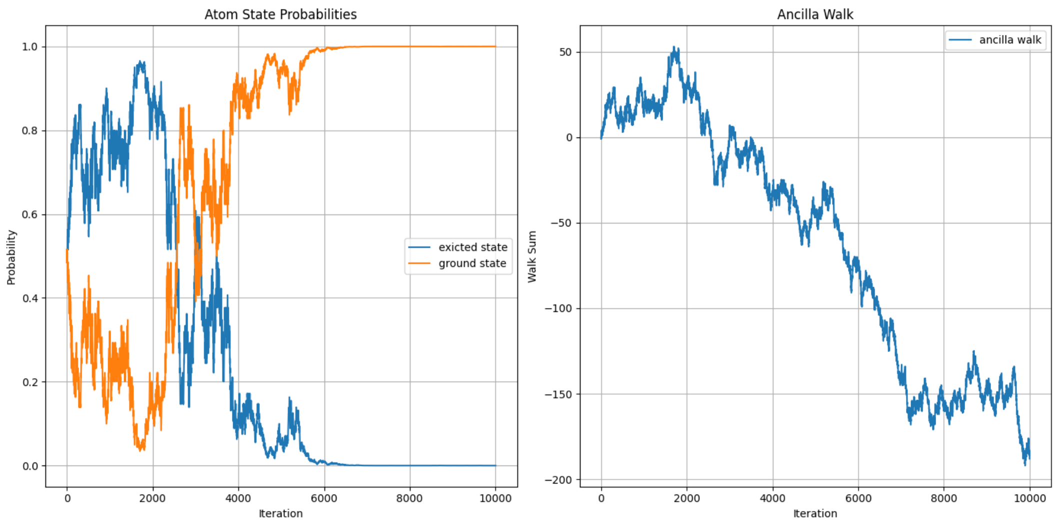

Figure 2: Simulation of the system with parameters $\rho = 0, \theta = 0.01$ and initial conditions $|\psi_g|^2 = |\psi_e|^2 = 0.5$.

Case 3: $[\Delta t > 0, g = 0, \Omega > 0]$ If only the Rabi drive interacts with the ancilla then we expect oscillating states and a random walk. The offset of the oscillations depends on the initial $|\psi_g|^2$ and $|\psi_e|^2$.

As can be seen in figures 1, 2, and 3, the simulation passes the basic test cases. From here on we denote the Rabi drive strength again as $\rho = \Omega\Delta t$ and the measurement strength as $\theta = g\Delta t$.

[1]The simulation was implemented in Python and can be found on GitHub:
https://github.com/Fzurbonsen/QuantumMeasurementsAndOptomechanicsHS2025.git
under `/ex1/code/quantum_jump_sim.py`.

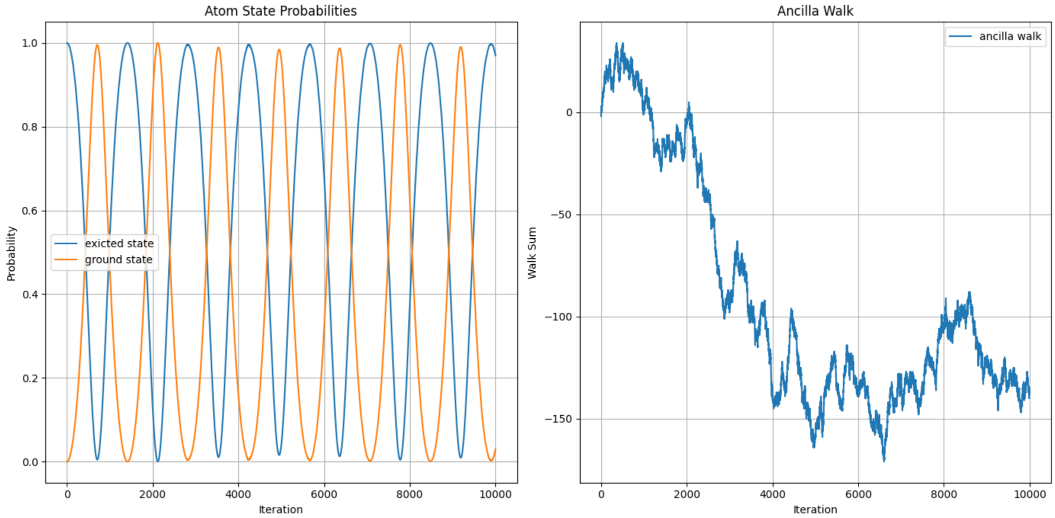

Figure 3: Simulation of the system with parameters $\rho = 0.01, \theta = 0$ and initial conditions $|\psi_g|^2 = 0, |\psi_e|^2 = 1$.

Let us now investigate some interesting parameter configurations.
The first configuration we investigate is $\theta = \rho$. We can see in figure 4 that in this case we get roughly half height oscillations.

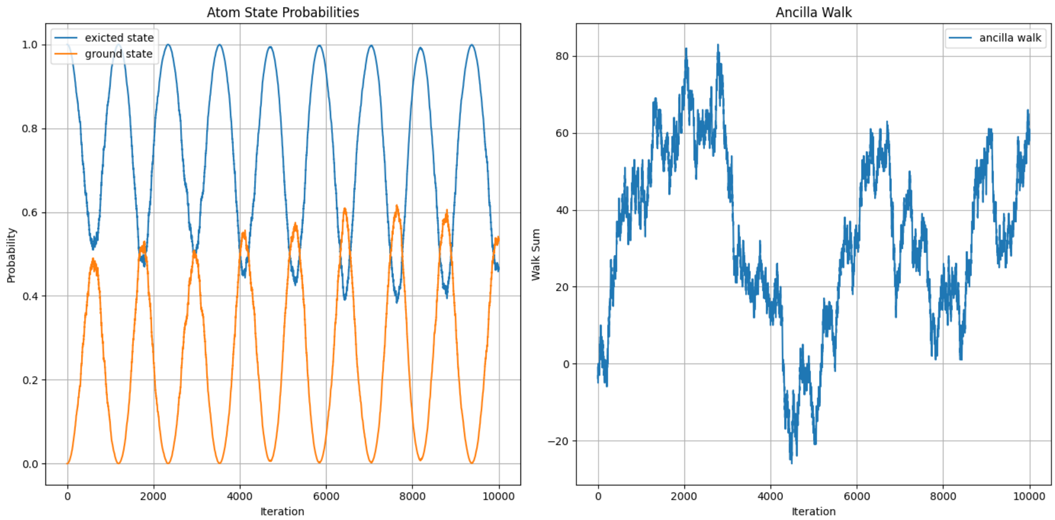

Figure 4: Simulation of the system with parameters $\rho = 0.01, \theta = 0.01$ and initial conditions $|\psi_g|^2 = 0, |\psi_e|^2 = 1$.

h)

Now we look at the configuration $\rho << \theta$ we see in figure 5 that for this configuration we can observe the quantum Zeno effect.

Last in figure 6 we can see that for the configuration $\rho = 0.01, \theta = 0.1$ we observe quantum jumps.

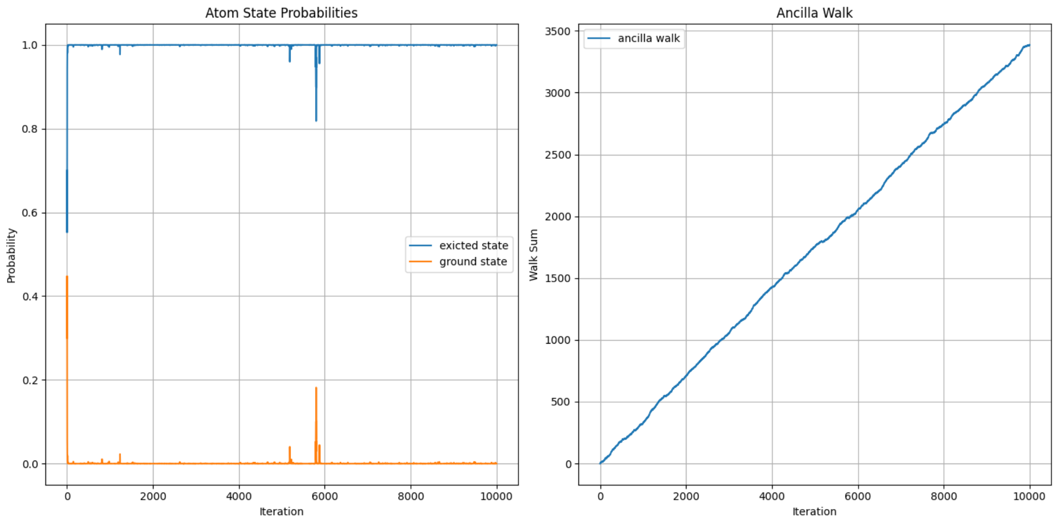

Figure 5: Simulation of the system with parameters $\rho = 0.001, \theta = 0.1$ and initial conditions $|\psi_g|^2 = 0.3, |\psi_e|^2 = 0.7$.

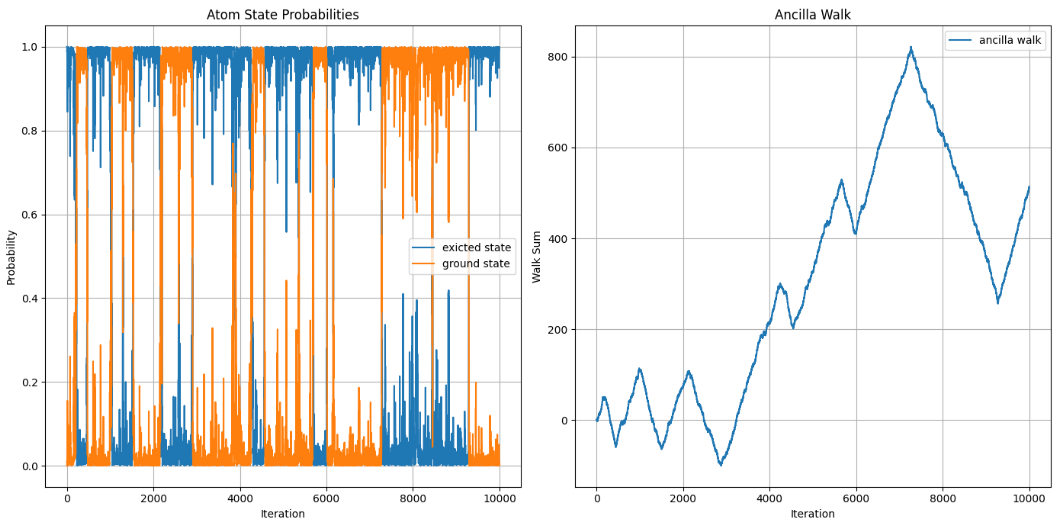

Figure 6: Simulation of the system with parameters $\rho = 0.01, \theta = 0.1$ and initial conditions $|\psi_g|^2 = 1, |\psi_e|^2 = 0$.