Output gap measures in Brazil

The output gap is a key concept for monetary policymakers, as it is a variable that seeks to capture whether the economic activity conditions are exerting an upward or downward pressure on inflation. It is the percentage difference between the real levels of effective output and potential output, the latter being defined as the economic activity level that does not generate inflationary or disinflationary pressures on the economy.

However, the potential output is an unobservable variable, so that the output gap estimation is subject to high uncertainty, with no consensus in the literature or among central banks about the best method to estimate it. Therefore, central banks usually estimate the output gap using different methodologies, a procedure that has also been adopted by the Banco Central do Brasil (BCB). This provides monetary policymakers with a better understanding about the degree of uncertainty involved in the output gap measurement. In fact, different statistical methodologies may indicate, via the estimated output gap, different states of the economy for an identical period. Furthermore, when using the same methodology, results may be reviewed should data be revised, or the sample size be changed. Information provided by the use of several methodologies, together with that coming from small-scale semi-structural models, is used by Copom to assess the output gap level. Moreover, it allows the construction of counterfactual scenarios using different output gap assessments from that used in the reference scenario, thus improving risk assessment.[1]

This box presents the output gap estimates derived from the following methodologies (for which a brief description is presented in Appendix 1)[2]:

Group I – Statistical univariate gaps

- I. Quadratic trend with breaks;
- II. Non-parametric trend;
- III. HP (Hodrick-Prescott) trend;
- IV. ℓ_1 trend;
- V. Modified HP trend;
- VI. Band-Pass filter – Christiano and Fitzgerald approximation;
- VII. Beveridge and Nelson – Kamber et al. (2018) variant.

Group II – Multivariate gaps

- I. Production function with a simple combination;
- II. Production function based on the Areosa (2008) approach;
- III. Production function based on the model used by the U. S. Congressional Budget Office (CBO);
- IV. Estimation based on the Jarocinski and Lenza (2018) model;
- V. Estimation based on principal components.

1/ For example, in the September 2022 IR, inflation projections were presented assuming a different output gap level from that used in the reference scenario.

2/ It is worth noting that, in New Keynesian structural models of general equilibrium, the potential output, typically in this framework, is the output that would prevail in a counterfactual situation with no nominal frictions and no monetary shocks and markups. Different models with different types of friction and shocks may be applied, thus making it difficult to reach a consensus on the potential output.

Statistical univariate gaps

Univariate methodologies perform the trend-cycle decomposition and generally share the assumption that trends are uncorrelated with the cycle, with the gap being the difference between the level of the observed activity variable and the trend. Seven traditional univariate methods for output gap decomposition are presented below as examples. It is noteworthy that this is a non-exhaustive set of the univariate methodologies available in the related literature.[3] The first methodology derives the trend by partial smoothing through sample plots, assuming that each part of the trend is deterministic, calculated through **quadratic trend regression with breaks**. The sample's breakpoints are determined by the statistical test of multiple structural breakpoints by Bai and Perron (2003). Thus, each additional piece of information may change substantially the trend adjustment after the last breakpoint, and may lead to changes in the previously selected breakpoints. The second methodology, "**non-parametric**", based on Cleveland (1979), derives the trend using local smoothing via locally weighted regressions, so that the trend change due to the addition of observations is local.

The third type of output gap measure is the **HP method** by Hodrick and Prescott (1997), whose trend is stochastic and smooth, obtained via Ridge regression with a smoothing parameter that is usual for quarterly data. The fourth methodology is the **ℓ_1 trend filter**, proposed by Kim et al (2009), which is a modification of the HP method, replacing the sum of squares used in this filter to penalize trend variations for a sum of absolute values (i.e., an ℓ_1 norm). The resulting trend is piecewise linear, and there is no need to specify, *a priori*, the number or location of the breakpoints. The fifth output gap estimation also introduces a **modification in the HP filter**, based on Andrle (2013), assuming that the long-term growth rate of the economy's productivity has a defined steady state.[4] The sixth output gap estimation is the **Band-Pass – "bp"**, using the approach of Christiano and Fitzgerald (2003), which typically represents frequencies between 8 and 32 quarters for the cyclical output component.

The seventh methodology introduces **a modification to the Beveridge-Nelson (BN) decomposition** carried out by Kamber et al. (2018), imposing a lower signal-to-noise ratio. The BN decomposition defines the GDP trend as the limit of conditional expectation over long forecast horizons. Following the approach of Kamber et al. (2018), an autoregressive model is used for the GDP growth rate conditional expectation, imposing long lags and restrictions on the coefficients to maximize the amplitude of the resulting gap.[5]

These methodologies are initially applied to the logarithm of the seasonally adjusted quarterly series of GDP at market prices, calculated by the Brazilian Institute of Geography and Statistics (IBGE). Figure 1A shows the estimates using a sample starting in 1996Q1 and ending in 2024Q1, the last released data.[6] There is a substantial dispersion of gap measurements, highlighting the high degree of uncertainty in the estimation of this variable. Different types of gaps may indicate different states of the economy for the same period such as, for instance, from 2010 to 2013, characterized by high GDP growth rates, immediately before the outbreak of the Covid-19 pandemic, or especially in 2024Q1, reflecting greater uncertainty at the end of the sample. This characteristic translates into an even more challenging conduct of monetary policy, since greater emphasis is given to the estimation of the current gap, as long as it not only reveals the current state of the economy but is also the starting point for projections of the output gap down the

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3/ Canova (2020) emphasizes that statistical decompositions between transitory and permanent output components might not recover the gap and potential output when assuming that the data-generating process comes from canonical New-Keynesian general equilibrium models. For example, in these canonical models that are a benchmark for economic policy analysis, the gap and potential output have correlated spectral characteristics – short and long frequency oscillations over time – while the basic assumption of several statistical filters is that the trend r is uncorrelated with the cycle. These types of spectral distortions of statistical gaps in relation to gaps in canonical general equilibrium models require caution by analysts when considering a specific output gap. That also corroborates the central motivation of this box.

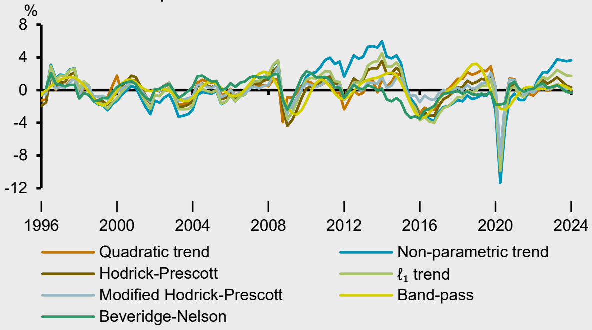
4/ This methodology is used in the Samba model. Further details in Fasolo et al. (2023). Andrle (2013) uses an AR(1) autoregressive term to characterize the cyclical component. To provide greater richness in the dynamics of the output gap, an AR(2) was introduced in the estimation for Brazil.

5/ As suggested by the authors, the average growth rate over the period is modeled as a constant. Furthermore, following Morley et al. (2023), the relative variance of shocks during the pandemic is calibrated.

6/ The selected breakpoints for the quadratic trend in this sample were 2000Q1, 2008Q4, 2013Q3, and 2020Q2. As for the ℓ_1, trend, the breakpoints were 2003Q3, 2012Q1, 2012Q4, and 2020Q3.

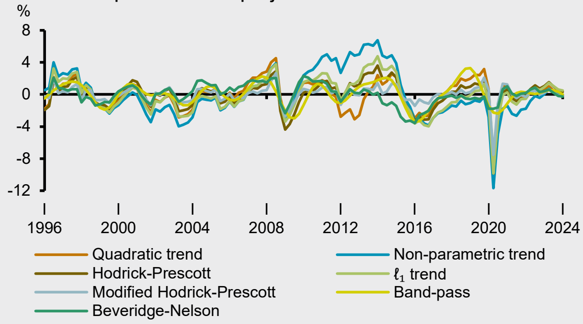
line. Conversely, despite differences observed in the output gap level, there is a high correlation among measures, i.e., all measures tend to move in the same direction.

Figure 1A – Univariate statistical output gaps
Observed data up to 2024Q1



Obs.: Data period: 1996Q1–2024Q1.

Figure 1B – Univariate statistical output gaps
Data filled up with Focus projections until 2028Q4



Obs.: Data period: 1996Q1–2024Q1.

To mitigate the end-of-sample problem, literature and experience have highlighted the benefits of extending the sample using some forecasting source before estimating the gap.[7] This reduces the usual end-of-sample problem, although results become, to some extent, dependent on the forecasts used. In this box, the GDP sample was extended to 2028Q4, using the median expectations from the Focus survey of June 14, 2024, with interpolation from 2026Q2 onwards to obtain quarterly values.[8] Results are shown in Figure 1B and a comparison for 2024Q1 is presented in Table 1. One may observe significant changes in this quarter's values in some specifications.[9]

Table 1 – Univariate statistical output gaps
Comparison between the gaps obtained for 2024Q1 using only observed data or filling up with Focus projections

%

Method	Output gap with observed data	Output gap with Focus projections	Difference (p.p.)
Quadratic trend	-0.31	-0.21	0.10
Non-parametric trend	3.67	-0.04	-3.71
Hodrick-Prescott	0.33	0.33	0.00
ℓ_1 trend	1.75	0.47	-1.27
Modified Hodrick-Prescott	0.12	-0.17	-0.30
Band-pass	0.03	0.02	-0.01
Beveridge-Nelson	-0.30	-0.28	0.02

Multivariate gaps

Multivariate gaps involve different methodologies and approaches. The output gap may result from a combination of observable variables or may be treated as an unobservable variable estimated through the

7/ Another approach is to use one-sided methodologies, whose gap estimate is not revised as the sample size increases, such as the one-sided HP-filter, the use of the YoY GDP change or of local projections. See Stock and Watson (1999) and Hamilton (2018). Of the univariate gaps presented, only the Beveridge-Nelson gap is a one-sided output gap method.
8/ For this sample, the selected breakpoints for the quadratic trend were 2012Q4 and 2020Q2. As for the ℓ_1 trend, the breakpoints were 2003Q2, 2012Q1, 2012Q4, and 2020Q3.
9/ Appendix 2 presents another exercise illustrating the end-of-sample problem, using pseudo real-time estimates.

Kalman filter. Macroeconomic relationships, particularly the Phillips curve, may be used to provide information on the output gap estimation.

The key feature of some multivariate estimates is the use of a production function that combines capital and labor through via a Cobb-Douglas technology. The main idea is to capture possible inflationary or disinflationary pressures based on estimated pressures on production factor markets. Estimates, however, may vary according to the techniques used to measure the degree of slack of production factors.

The first estimate uses a **production function** to make a linear combination of labor and capital gaps, obtained by the HP method applied to the employment rate from the quarterly IBGE'S Continuous National Household Sample Survey (PNAD Continuous) and the Level of Capacity Utilization (Nuci) from the Fundação Getulio Vargas (FGV), both seasonally adjusted.[10] The weights correspond to the estimated share of these factors in the national income.[11]

The second methodology, based on **Areosa** (2008), also combines two methods commonly used to estimate the potential output – the production function and the HP filter. Via a Cobb-Douglas production function, it is possible to express the output gap as a linear combination of two other gaps – the employment gap and the capacity utilization gap. This relationship shows that, when employment and level of capacity utilization deviate from their natural levels, the output deviates from its potential level. The methodology uses this relationship to create a filter that simultaneously estimates these three gaps by solving a single optimization problem that represents three HP filters interconnected by the constraint derived from the production function.

The third method is **based on the model used by the U. S. Congressional Budget Office** (CBO), presented in Shackleton (2018), but using aggregated data.[12] The estimation is based on the production function, whose potential levels are decomposed into three components: labor contribution, obtained by the potential employment level; capital contribution, obtained by the potential stock of capital; and the residual, which would represent the potential total factor productivity. Piecewise linear regressions are used to estimate these values, which also include terms for capturing the cyclical component. Then, to find the non-cyclical values, these terms are set to zero. Each part of the trend is built based on the cycle classification by the Brazilian Business Cycle Dating Committee (Codace). The series estimated by Souza Júnior and Cornelio (2020) was used for capital.

The following approach is an application of the model developed by **Jarocinski and Lenza** - JL (2018) (JL) for Brazil. This is a Bayesian dynamic factor model in which output gap estimates are consistent with inflation behavior. The output gap is an unobservable variable and a common factor to activity variables and inflation. Inflation is measured by the core inflation Ex-0 (a combination of services inflation and industrial goods inflation), modeled as a deviation from a trend and a function of its past values, the output gap, and imported inflation. In addition, the variance of shocks is allowed to change over time (stochastic volatility). The activity variables in this approach are the same used in the small-scale semi-structural model: GDP, IBGE's unemployment rate, stock of formal jobs measured by the New Caged of the Ministry of Labor and Employment, and Nuci calculated by the FGV.

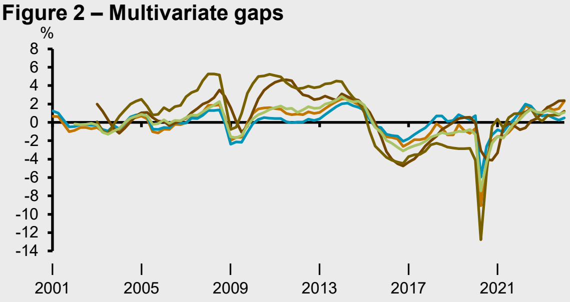
Finally, the method of **principal components** is used to obtain a common series that simultaneously explains the cyclical dynamics in economic activity and the labor market. The first estimated principal component summarizes 71.9% of the total variance of the database, which contains standardized GDP series, stock of formal jobs measured by the New Caged, Nuci calculated by the FGV, and the IBGE's employment rate, all of which have a positive correlation with market prices inflation.

10/ Further details in Alves and Correia (2013).
11/ Respective weights of 0.4 and 0.6 were used for capital and labor, obtained by estimating the average participation of these production factors in the GDP from 1999 to 2019.
12/ The U. S. Congressional Budget Office (CBO) uses disaggregated series for both labor and capital and thus manages to better capture differences arising from the heterogeneity in these factors.

Although multivariate models include more economic activity information and have theoretical references, output gaps are sensitive to specification, such as, the model's equations, the number of lags, the number and type of variables, the size of the sample, etc.[13]

Figure 2 presents output gaps estimated by multivariate methods. As in the case of univariate output gaps, there are high correlations among measures and significant level differences.

Figure 2 – Multivariate gaps

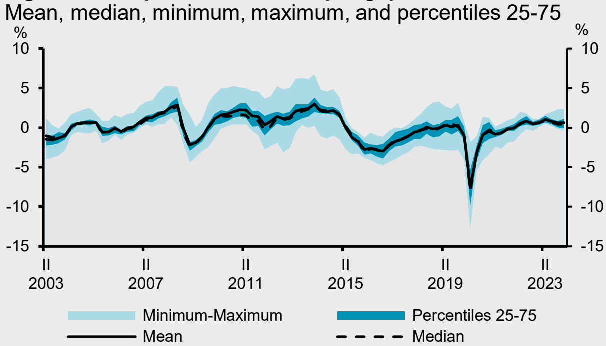


Obs.: Data period: 2001Q1–2024Q1.

Set of measures

Figure 3 shows the area covered by all univariate and multivariate output gaps and the curves with their simple average, median, the 25th and 75th percentiles, and minimum and maximum values. The amplitude of the area indicates the degree of uncertainty involved in these measurements. In the period between 2003Q2 and 2024Q1, the average difference between the most extreme measures was 4.25 p.p., and between the 25th and 75th percentiles, 1.31 p.p. Conversely, in general, the high correlation among the measures stands out (Table 2).

Figure 3 – Dispersion of the output gap measures
Mean, median, minimum, maximum, and percentiles 25-75



Obs.: Data period: 2003Q2–2024Q1.

Table 3 shows the output gap levels in 2023 and in 2024Q1 for all the discussed methodologies. Considering 2024Q1, the estimated range is between -0.28% and 2.37%, with an average of 0.63%, median of 0.40%, and 25th and 75th percentiles at -0.08% and 1.13%, respectively. Finally, the Pre-Copom Questionnaire (PCQ), sent to participants of the Market Expectations System before each Monetary Policy Committee (Copom) meeting, periodically includes a question about the participants' estimates of the output gap. In the June 2024 Copom's PCQ, the median estimate of the output gap for 2024Q1 was 0.3%, with 25th and 75th percentiles at -0.1% and 0.7%, respectively.

13/ Output gaps derived from these models are also not exempt from potential spectral distortions when using canonical general equilibrium models as a reference for the data-generating process.

Table 2 – Correlation of output gap measures
Sample 2003Q2–2024Q1

	Quadratic trend	Non-parametric trend	Hodrick-Prescott	ℓ_1 trend	Modified Hodrick-Prescott	Modified Hodrick-Prescott	Band-pass	Production function	Areosa	CBO	Beveridge-Nelson	Jarocinski & Lenza
Quadratic trend	1.00											
Non-parametric trend	0.53	1.00										
Hodrick-Prescott	0.81	0.81	1.00									
ℓ_1 trend	0.77	0.92	0.94	1.00								
Modified Hodrick-Prescott	0.85	0.65	0.91	0.82	1.00							
Band-pass	0.67	0.50	0.74	0.64	0.50	1.00						
Beveridge-Nelson	0.52	0.40	0.52	0.51	0.47	0.44	1.00					
Production function	0.60	0.85	0.83	0.89	0.70	0.52	0.42	1.00				
Areosa	0.73	0.72	0.90	0.85	0.77	0.71	0.38	0.90	1.00			
CBO	0.36	0.77	0.57	0.72	0.33	0.48	0.41	0.75	0.60	1.00		
Jarocinski & Lenza	0.48	0.83	0.69	0.80	0.56	0.40	0.68	0.85	0.65	0.78	1.00	
Principal components	0.55	0.89	0.82	0.89	0.63	0.54	0.47	0.96	0.86	0.83	0.89	1.00

Table 3 – Output gap levels from 2023Q1 to 2024Q1 by type of methodology

%

	2023				2024
	Q1	Q2	Q3	Q4	Q1
Statistical univariate gaps					
Quadratic trend	0.57	1.00	0.49	0.00	-0.21
Non-parametric trend	0.07	0.69	0.35	0.02	-0.04
Hodrick-Prescott	1.07	1.53	1.03	0.54	0.33
ℓ_1 trend	0.93	1.47	1.04	0.62	0.47
Modified Hodrick-Prescott	0.23	0.64	0.24	-0.09	-0.17
Band-pass	0.38	0.53	0.52	0.33	0.02
Beveridge-Nelson	0.41	0.63	0.22	-0.18	-0.28
Multivariate gaps					
Production Function	0.97	1.67	1.35	1.49	2.31
Areosa	0.70	0.79	0.47	0.24	0.48
CBO	1.07	1.65	1.96	2.33	2.37
Jarocinski & Lenza	0.15	0.74	0.79	0.82	1.22
Principal components	1.06	1.31	1.16	0.89	1.10
Summary					
Mean	0.63	1.06	0.80	0.58	0.63
Median	0.63	0.90	0.66	0.43	0.40
25th percentile	0.34	0.68	0.44	0.01	-0.08
75th percentile	1.00	1.49	1.07	0.84	1.13

Conclusion

This box presented a set of output gap measures, thus highlighting the high degree of uncertainty involved in the estimation of this variable. This box enhances the transparency of the BCB's decision-making processes.[14] The BCB monitors different output gap measures and has made efforts to improve the methodologies used.

14/ Regarding the analysis and projections system that supports the Copom's decision-making process, see box BCB's Analysis and Projections System of the March 2023 IR.

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Appendix 1 – Brief description of methodologies

1. Univariate gaps

a) Quadratic trend with breaks: deterministic and uncorrelated with the economic cycle $y_t = a + bt + ct^2 + \epsilon_t$; where y : observable variable; $\hat{\epsilon}_t$: cycle.

b) Non-parametric trend: smoothed by a locally weighted regression. For each t, smoothed trend y_t^s is the following weighted prediction: The subset to calculate y_t^s is made up of the indexes $t_- = \max(1, t-k)$ up to $t_+ = \min(t+k, T)$, where $k = \lfloor (\mathrm{T} \times \mathrm{bwidth} - 0.5)/2 \rfloor$; bwidth=0,4. The weights for each observation $j = t_-, \ldots, t_+$ follow the tricube:

$$w_j = \left\{1 - \left(\frac{|t_j - t_i|}{\triangle}\right)^3\right\}^3$$, where $\triangle = 1{,}0001 \ \max(t_+ - t, t - t_-)$; cycle $= y_t$ - y_t^s .

c) HP (Hodrick-Prescott) trend: stochastic and smooth – uncorrelated with the cycle: trend via Ridge estimator: $\tilde{y} = (H'H + \lambda Q'Q)^{-1} + H'y$; where: y : observable variable; $\tilde{y}$: trend; $H = (I_{t\times t} 0_{t\times 2})$; $Q_{t\times(t+2)}$; smoothing parameter (1600 – quarterly data); $\mathrm{y} - \tilde{y}$: cycle.

d) ℓ_1 trend (Kim et al. (2009)): piecewise linear trend

ℓ_1 trend is obtained by solving the following minimization problem:

$$\tilde{y} = argmin_{\mu \in \mathbb{R}^t} \left\{\sum_{i=1}^{t} (y_i - \mu_i)^2 + \lambda \sum_{i=3}^{t} |\Delta^2 \mu_i|\right\}$$; where: y : observable variable; $\tilde{y}$: trend;

λ : smoothing parameter; $\Delta^2 \mu_i = \Delta\mu_i - \Delta\mu_{i-1} = \mu_i - 2\mu_{i-1} + \mu_{i-2}$.

Unlike the HP filter, in which the trend converges to a linear trend when $\lambda \to \infty$, in the ℓ_1 filter the trend becomes linear without breakpoints when $\lambda \geq \lambda_{max}$, where $\lambda_{max} = \|(DD^T)^{-1} Dy\|_\infty$ and, with D being the second-order difference matrix:

$$D = \begin{bmatrix} 1 & -2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 & 0 & 0 \\ 0 & 0 & \ddots & \ddots & \ddots & 0 & 0 \\ 0 & 0 & 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & -2 & 1 \end{bmatrix}$$

For this box, the ℓ_1 trend was obtained using $\lambda = (1/2)^5 \times \lambda_{max}$, which resulted in four breakpoints for both samples presented in the "Statistical univariate gaps" section, as per footnotes 6 and 8.

e) HP trend modified by Andrle (2013): long-term growth rate of the economy productivity with a defined steady state. The estimation uses the following state-space form:

$$Y_t^{obs} = y_t + \log Z_t$$

$$y_t = \alpha_1 y_{t-1} + (\alpha_2 - \alpha_1) y_{t-2} + \sigma_y \epsilon_t^y$$

$$\log\left(\frac{Z_t}{Z_{t-1}}\right) = \log Z_t^z = \log Z_{ss}^z + \log Z_t^{zc} + \log Z_t^{zt}$$

$$\log\left(\frac{Z_t^{zc}}{Z_{ss}^z}\right) = \rho_z \log\left(\frac{Z_{t-1}^{zc}}{Z_{ss}^z}\right) + (1 - \rho_z^2)^{0.5} \sigma_{zc} \varepsilon_t^{zc}$$

$$\log Z_t^{zt} = (1 - \rho_z^2)^{0.5} \sigma_{zt} \varepsilon_t^{zt}$$

$$\epsilon_t^y \sim N(0,1) \quad \varepsilon_t^{zc} \sim N(0,1) \quad \varepsilon_t^{zt} \sim N(0,1)$$

where Y_t^{obs} is the (logarithm of) real GDP per capita; y_t is the (logarithm of) output cyclical component, modeled as an AR(2) where the restriction of the parameters, combined with the priors defined for the model's estimation, ensure stationarity of the output gap; $\log Z_t$ is the (logarithm of) the trend level. The third equation of the model characterizes the model's growth rate trend consisting of a cyclical component, $\log Z_t^{zc}$, which follows an AR(1) process, and a temporary component, $\log Z_t^{zt}$, besides the deterministic growth rate $\log Z_{ss}^{z}$. The exogenous shocks $\epsilon_t^{y}, \varepsilon_t^{zc}$ and ε_t^{zt} follow the Standard Normal distribution, with the coefficients σ_y, σ_{zc} and σ_{zt} defining, respectively, the standard deviation of each of the system's components.

f) Band-Pass filter (8-32 quarters – Christiano and Fitzgerald approach): quadratic function equals 1 for frequencies between (ω_1, ω_2) quarters and 0 outside this range. Low Pass: $\mathcal{B}_0^{lp} = {}^{\omega_1}/_{\pi}$; $\mathcal{B}_j^{lp} = {}^{\sin(j\omega_1)}/_{j\pi}$; $0 < j < \infty$, for some ω_1. High Pass: $\mathcal{B}_0^{hp} = 1 - \mathcal{B}_0^{lp}$; $\mathcal{B}_j^{hp} = -\mathcal{B}_j^{lp}$; $0 < j < \infty$. Band-Pass for the cycle: $\mathcal{B}_0^{hp} = \mathcal{B}_j^{lp}(\omega_2) - \mathcal{B}_j^{lp}(\omega_1)$; $0 < j < \infty$; $\omega_2 > \omega_1$. CF use non-stationary, asymmetric, and optimal approach (min. error).

g) Beveridge-Nelson modified by Kamber et al. (2018): the BN trend is defined by $\tau_t = \lim_{j \to \infty} E_t y_{t+j}$, without loss of generality, disregarding deterministic terms. Kamber et al. (2018) use an autoregressive $\phi(L)\Delta y_t = e_t$ model for quarterly GDP, imposing twelve quarters of lags and carrying out an exhaustive search in a grid to $\phi(1)$ so that maximizing the amplitude of the resulting gap $h_t = y_t - \tau_t$. Morley et al. (2023) also proposed correction for heteroscedasticity in e_t during the pandemic.

2. Multivariate gaps

a) Production function: output gap with Cobb-Douglas technology: $\frac{Y_t}{Y_t^n} = \left(\frac{C_t}{C_t^n}\right)^{1-\alpha} \left(\frac{1-U_t}{1-U_t^n}\right)^{\alpha}$; where: Y_t : output; Y_t^n: potential output; C_t: industry capacity utilization; C_t^n: Naicu; U_t: unemployment rate; U_t^n: Nairu; α: share of employment in output (0.6). Output gap in log: $\hat{y}_t = (1-\alpha)\hat{c}_t - \alpha(\hat{u}_t)$, where $\hat{y}_t$; $\hat{c}_t$ and $\hat{u}_t$ are the output, level of capacity utilization and unemployment rate gaps, respectively. The level of capacity utilization and the unemployment rate gaps are calculated using the HP filter.

b) Production function based on the Areosa approach (2008): The Kalman filter algorithm may be used to solve the proposed optimization problem. To do this is necessary to build a state-space model and impose restrictions on the variance-covariance matrix of the errors so that the resulting likelihood function, to be maximized by the Kalman filter, is the objective function of the proposed filter. Thus, the solution found by the Kalman filter will be the same as that of the HP filters restricted by the relationship extracted from the production function. The state-space representation, used in the Kalman filter, would then be given by:

$$\begin{bmatrix} x_{1,t} \\ x_{2,t} \\ x_{3,t} \\ x_{4,t} \\ x_{5,t} \\ x_{6,t} \end{bmatrix} = \begin{bmatrix} 2 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_{1,t-1} \\ x_{2,t-1} \\ x_{3,t-1} \\ x_{4,t-1} \\ x_{5,t-1} \\ x_{6,t-1} \end{bmatrix} + \begin{bmatrix} e_{1,t} \\ 0 \\ e_{2,t} \\ 0 \\ e_{3,t} \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} y_{1,t} \\ y_{2,t} \\ y_{3,t} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_{1,t} \\ x_{2,t} \\ x_{3,t} \\ x_{4,t} \\ x_{5,t} \\ x_{6,t} \end{bmatrix} + \begin{bmatrix} \varepsilon_{1,t} \\ \varepsilon_{2,t} \\ 0{,}6 \cdot \varepsilon_{1,t} + 0{,}4 \cdot \varepsilon_{2,t} \end{bmatrix}$$

Where the states $x_{1,t}$, $x_{3,t}$ and $x_{5,t}$ represent the potential unemployment series, the level of capacity utilization, and the output, while $y_{1,t}$, $y_{2,t}$ and $y_{3,t}$ are the observable series of these same three variables. In this case, the restriction on the variance-covariance matrix of the errors is given by:

$$\begin{bmatrix} V_{e_1} & 0 & 0 & 0 & 0 \\ 0 & V_{e_2} & 0 & 0 & 0 \\ 0 & 0 & V_{e_3} & 0 & 0 \\ 0 & 0 & 0 & V_{\varepsilon_1} & Cov(\varepsilon_1, \varepsilon_2) \\ 0 & 0 & 0 & Cov(\varepsilon_1, \varepsilon_2) & V_{\varepsilon_2} \end{bmatrix} =$$

$$\begin{bmatrix} {V_{e_3}}/{\beta_1} & 0 & 0 & 0 & 0 \\ 0 & {V_{e_3}}/{\beta_2} & 0 & 0 & 0 \\ 0 & 0 & V_{e_3} & 0 & 0 \\ 0 & 0 & 0 & {1600 \cdot V_{e_3} \cdot (\beta_2 + 0.4^2)}/{den} & {-1600 \cdot V_{e_3} \cdot 0.4 \cdot 0.6}/{den} \\ 0 & 0 & 0 & {-1600 \cdot V_{e_3} \cdot 0.4 \cdot 0.6}/{den} & {1600 \cdot V_{e_3} \cdot (\beta_1 + 0.6^2)}/{den} \end{bmatrix}$$

where den = $(\beta_1 \cdot \beta_2) + (\beta_2 \cdot 0.6^2) + (\beta_1 \cdot 0.4^2)$.[15]

c) Production function based on the model used by the U. S. Congressional Budget Office (CBO): The estimation is based on Shackleton (2018), used in the U.S. Congress potential output model, but with publicly available aggregate data for Brazil. It consists in a potential output version with conventional Cobb-Douglas production function.

In general terms, variations of Okun's Law are applied in piecewise linear regressions that associate production factors (total factors productivity – TFP, labor, and capital stock) with the employment gap and time dummies associated with the peak of economic cycles to extract the trends. The equations for extracting the trend for each production input are presented below.

- Labor (retropolated PNAD-C[16] – sample from 2002Q1):
  - Natural unemployment rate (U_t^*): uses Nairu as a *proxy*, as in the CBO potential output.[17]
  - Working Age Population (PEA):

$$\ln(PEA_t) = \alpha + \beta_1 Egap_t \times Covid + \beta_2 Egap_{t-1} \times Covid + \beta_3 T_{2002} + \beta_4 T_{2008} + \beta_5 T_{2014} + \beta_6 T_{2019} + \varepsilon_t$$

where time trends (*T*) correspond to a specific business cycle, defined by Codace, dated similarly to the original CBO potential output (demarcating cycles as defined by the NBER).[18] Potential PEA is the prediction of this regression by applying a zero value to the employment rate gap coefficients (cyclical

15/ The weights in this relation represent the labor elasticity and the labor force elasticity, usually estimated at 0.6 and 0.4 for Brazil.
16/ See Alves and Fasolo (2015).
17/ Estimated using quarterly data (four-quarter moving average) with a sample from 2006Q1 to 2019Q4. The GDP deflator (GDP data from the IBGE Quarterly National Accounts System) is regressed against a constant, four lag terms for the GDP deflator (each lag term is a third-degree distributed lag polynomial), a four-quarter lag term for the unemployment rate (demean), a quarter lag term for food-at-home and energy sub-index in the IPCA and a productivity deviation variable (difference between the growth rate of labor productivity (GDP/occupation) and the trend, which is measured as the potential working age population in the methodology. The regression is conditioned to the restrictions that the GDP deflator lags add up to 1 (one), in order to solve for Nairu, and the last term in the past is restricted to zero. – See "The Economic and Budget Outlook" (CBO, 1994) for details on the original implementation.
18/ Cycles dating is obtained on the basis of peak to peak values in each cycle. For example, the trend T_{2008} assumes zero values until the peak of the previous cycle, which T_{2002} is in 2002Q4, when it assumes a value of 25 and then a value of 25 is added every quarter until the peak of the trend cycle T_{2008}, which occurs in September 2008, repeating the peak value until the end of the sample. The same computing method is applied for all time trends.

variable). The trends of all other variables – GFCF, Nuci, and TFP – are extracted in this way.[19] For the PEA, a dummy was applied for the period following the outbreak of the pandemic, interacting with the terms of the employment rate gaps to differentiate the relatively disparate cyclical dynamics in this series before and after the pandemic.

- Employment rate (E_t), potential employment rate (E_t^*), employment rate gap ($Egap_t$) and potential employment ($OCUP_t^*$):

$$E_t = \left[1 - \left({}^{U_t}/_{100}\right)\right]$$

$$E_t^* = \left[1 - \left({}^{U_t^*}/_{100}\right)\right]$$

$$Egap_t = \left[\left({}^{E_t}/_{E_t^*}\right) - 1\right] \times 100$$

$$OCUP_t^* = E_t^* \times PEA_t^*$$

- Stock of capital:

  - Gross Fixed Capital Formation (GFCF): Obtained from the quarterly difference of the IPEA stock of capital series (K) released by the Institute of Applied Economic Research (Ipea) (see Souza Júnior and Cornelio, 2020). Since the capital stock is obtained using the perpetual stock methodology, the quarterly difference in the stock of capital includes the depreciation rate between quarters. Ipea's stock of capital stock series goes until 2023Q4. For the 2024Q1 GFCF values, the quarterly change in the volume of GFCF in seasonally adjusted real terms calculated by the IBGE on the last observation in the series (2023Q4) is applied. This output is added by a pro-rata of the average implicit depreciation rate for 2010-2017, estimated at 6.39% p.a. in Souza Júnior and Cornelio (2020).

$$FBKF_t \equiv K_t - K_{t-1}; \quad ;$$

$$\ln(FBKF_t) = \alpha + \beta_1 Egap_t + \beta_2 Egap_{t-1} + \beta_3 T_{2002} + \beta_4 T_{2008} + \beta_5 T_{2014} + \beta_6 T_{2019} + \varepsilon_t;$$

  - Potential GFCF ($FBKF_t^*$) is obtained by predicting this regression and applying zero to the employment rate gap coefficients.

  - Nuci (industry – FGV): Used together with GFCF to add cyclical variation to the stock of capital.

$$NUCI_t = \alpha + \beta_1 Egap_t + \beta_2 Egap_{t-1} + \beta_3 T_{2002} + \beta_4 T_{2008} + \beta_5 T_{2014} + \beta_6 T_{2019} + \varepsilon_t \, ;$$

  - Potential Nuci ($NUCI_t^*$) is obtained by predicting this regression and applying zero to the employment rate gap coefficients.

  - Stock of capital adjusted by Nuci:

$K_0^* \equiv K_0$; where the asterisk refers to the potential

$$K_1^* = K_0 + FBKF_t^* ;$$

$$K_t^* = (K_{t-1} + FBKF_t^*) \times NUCI_t^*; \; t > 0$$

19/ Nairu and, consequently, potential PEA and occupation were projected to correct for the more recent time trend since the outbreak of the pandemic.

- TFP (derivative of a production function with stock of capital adjusted by Nuci):

$$\ln(A_t) \equiv \ln(QPIB_t) - 0{,}6 \times \ln(IOCUP_t) \quad - 0{,}4 \times \ln(IKN_t);$$

  where: $IKN_t \equiv IK_t \times NUCI_t$; *I*: Index numbers are based on 2002Q4.

$$\ln(A_t) = \alpha + \beta_1 Egap_t + \beta_2 Egap_{t-1} + \beta_3 T_{2002} + \beta_4 T_{2008} + \beta_5 T_{2014} + \beta_6 T_{2019} + \varepsilon_t;$$

  Potential TFP using Nuci in the formulation is $\ln(A_t)^*$ and is obtained by predicting this regression and applying a zero value to the employment rate gap coefficients.

- Potential output and output gap:

$$\ln(Gap_t) \equiv \ln(QGDP_t) - \ln(QGDP_t)^* \quad ;$$

  Where: $\ln(QGDP_t)^* = \ln(A_t)^* + 0{,}6 \times \ln(IOCUP_t)^* - 0{,}4 \times \ln(IKN_t)^*$.

d) <u>Jarocinski and Lenza Model</u>:

Observable:

$$y_t^n = b^n(L)g_t + w_t^n + \varepsilon_t^n, for\ n = 1, \dots, 4$$

$$(\pi_t - z_t) = a_g(L)g_t + a_p(L)(\pi_{t-1} - z_{t-1}) + a_v(L)v + e^{\frac{1}{2}h_t}\epsilon_t^\pi$$

$$\pi_t^e = c_0 + c_1 z_t + \varepsilon_t^e$$

Laws of motion:

$$g_t = \phi_1 g_{t-1} + \phi_2 g_{t-2} + \eta_t^g$$

$$w_t^n = d^n + w_{t-1}^n + \eta_t^n, for\ n = 1, \dots, 4$$

$$z_t = z_{t-1} + e^{\frac{1}{2}f_t}\epsilon_t^z$$

where: $b^n(L)$: 1 lead, contemporaneous and 2 lags; $a_g(L)$: 1 lead, contemporaneous and 1 lag $a_p(L)$: 1 lag; $a_v(L)$: 2 lags.

The output gap (g_t) is a common factor of the GDP (y_t^1), Nuci (y_t^2), the unemployment rate (y_t^3), and the stock of formal jobs measured by the New Caged (y_t^4), which have specific trends w_t^1, w_t^2, w_t^3 and w_t^4 . g_t follows AR(2), w_t^1 follows a random walk with drift and w_t^2, a random walk without drift. $\varepsilon_t^1 = 0$ is imposed so that the output gap g_t so that it coincides with the cyclical GDP trend.

The Phillips curve describes the relationship between the deviation of core inflation Ex-0 from its trend (π_t-z_t) and the output gap, past values of Ex-0, and the IC-BR change (v_t) and considers stochastic volatility (h_t). The last equation of observable variables shows the relationship between the inflation trend (z_t) and 12-month inflation expectations starting in two years (π_t^e). Thus, medium-term inflation expectations and the inflation trend are linked, also following a random walk and presenting stochastic volatility (f_t).

e) Estimation by principal components:

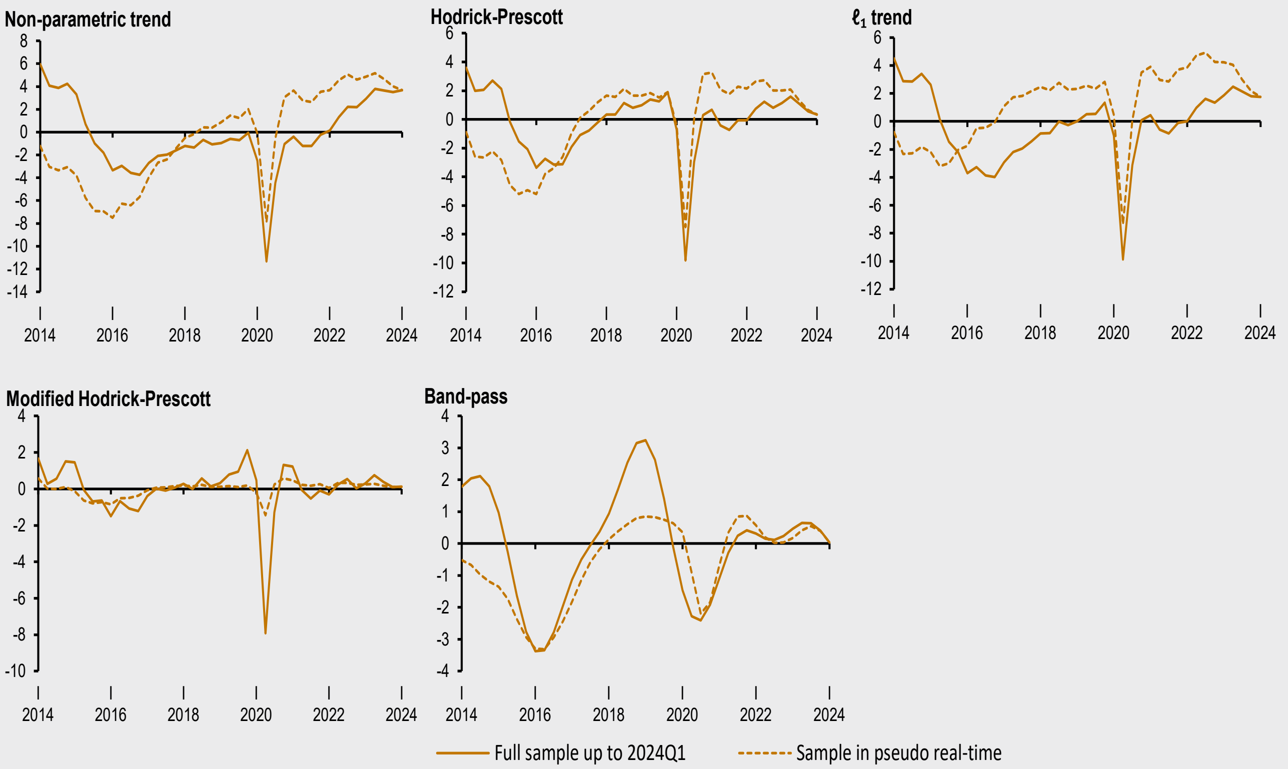
Principal components are used to summarize the relevant information in a data set and are obtained by solving the characteristic equation $\det[S - \lambda I]=0$, where S is the data covariance matrix. The principal components are the eigenvectors associated with the eigenvalues that solve the equation.

To obtain the common series that summarizes changes in activity and in the labor market, a database is used with the ex-trend of GDP series (IBGE), industrial capacity utilization (FGV); the PNAD unemployment rate (IBGE), with inverted signal; and the Caged employment series (MTE). The output and Caged employment gaps are calculated previously through an HP filter and the four stationary series are standardized before the components are estimated through a singular value decomposition.

Appendix 2 – Pseudo real-time estimation exercise

Bilateral methodologies are known to present end-of-sample problems, as each new observation at the end causes a general revision of the filtered series. Although useful for analyzing historical patterns, they are noisier indicators of real-time economic conditions.[20] Figure 4 presents estimates using the entire sample until 2024Q1 and the pseudo real-time estimate for the statistical univariate gaps.[21] The latest estimates are obtained by fixing the beginning of the sample (1996Q1), but varying its end for each period from 2014Q1 to 2024Q1. For example, the gap estimate in 2014Q1 uses the GDP series only up to that quarter.[22] The substantial difference between the two series in most estimates is noteworthy.

Figure 4 – Forecasting excercise in pseudo real-time



Obs.: Data period: 2014Q1–2024Q1.

20/ Further details in Orphanides and Norden (2002).
21/ For this exercise were not included the quadratic trend gap, since changes in the samples could lead to the selection of different structural breaks; and the Beveridge-Nelson gap, since it is a one-sided gap, for which the pseudo real-time exercise presents the same result as using the full sample.
22/ As the latest available GDP vintage is used in all cases, this is a pseudo real-time estimation. If the vintage available at that time was used, it would be a real-time estimation. For an example of the effects of real-time and pseudo real-time estimates for Brazil, see Cusinato, Minella, and Pôrto Júnior (2013).