SOLUZIONI PROVA INTERCORSO 31/5/2021

ESERCIZIO 1

X assume valori in $S = \{1, 2, 4, 8\}$

$$p(2k) = \frac{p(k)}{k} \quad \text{per } k = 1, 2, 4$$

$k=1 \Rightarrow p(2) = p(1)$

$k=2 \Rightarrow p(4) = \frac{p(2)}{2} = \frac{p(1)}{2}$

$k=4 \Rightarrow p(8) = \frac{p(4)}{4} = \frac{p(1)}{8}$

(i)

k	1	2	4	8
p(k)	$\frac{8}{21}$	$\frac{8}{21}$	$\frac{4}{21}$	$\frac{1}{21}$

$$\Uparrow$$

$$p(1) + p(2) + p(4) + p(8) = 1 \Rightarrow p(1) + p(1) + \frac{p(1)}{2} + \frac{p(1)}{8} = \frac{21}{8} \cdot p(1) = 1 \Rightarrow p(1) = \frac{8}{21}$$

(ii) $$E(X) = \sum_{k=1}^{8} k \cdot p(k) = 1 \cdot \frac{8}{21} + 2 \cdot \frac{8}{21} + 4 \cdot \frac{4}{21} + 8 \cdot \frac{1}{21} = \frac{48}{21} = \frac{16}{7} = 2{,}2857$$

$$E(X^2) = \sum_{k=1}^{8} k^2 \cdot p(k) = 1 \cdot \frac{8}{21} + 4 \cdot \frac{8}{21} + 16 \cdot \frac{4}{21} + 64 \cdot \frac{1}{21} = \frac{8 + 12 + 64 + 64}{21} = \frac{168}{21} = 8$$

$$\mathrm{Var}(X) = E(X^2) - (E(X))^2 = 8 - \left(\frac{16}{7}\right)^2 = 8 - \frac{256}{49} = \frac{392-256}{49} = \frac{163}{49} = 2,7755$$

(iii) $$E[X(\alpha - X)] = \alpha E(X) - E(X^2) = \alpha \cdot \frac{16}{7} - 8 = 0 \Rightarrow \alpha = 8 \cdot \frac{7}{16} = \frac{7}{2}$$

(iv) $$P(X=k \mid X>1) = \frac{P(X=k, X>1)}{P(X>1)} = \frac{P(X=k)}{P(X>1)} = \begin{cases} 8/13, & k=2 \\ 4/13, & k=4 \\ 1/13, & k=8 \end{cases}$$

$$P(X>1) = 1 - P(X=1) = 1 - \frac{8}{21} = \frac{13}{21}$$

$$E[X \mid X>1] = \sum_{k=2}^{8} k \cdot P(X=k \mid X>1) = 2 \cdot \frac{8}{13} + 4 \cdot \frac{4}{13} + 8 \cdot \frac{1}{13} = \frac{40}{13} = 3,0769$$

ESERCIZIO 2

(i) $\begin{cases} f(x) \geq 0 \\ \int_{-\infty}^{\infty} f(x)\,dx = 1 \end{cases} \Rightarrow \begin{cases} \dfrac{c}{(x+2)^2} \geq 0 \Rightarrow c \geq 0 \\ c\int_0^{\infty} \dfrac{1}{(x+2)^2}\,dx = c \cdot \dfrac{(-1)}{x+2}\Big|_{x=0}^{x\to\infty} = c\,\dfrac{1}{2} = 1 \Rightarrow c = 2 \end{cases}$

(ii) $F(x) = \int_{-\infty}^{x} f(t)\,dt = \begin{cases} 0 & \text{per } x<0 \\ \int_0^x \dfrac{2}{(t+2)^2}\,dt = \dfrac{(-2)}{t+2}\Big|_{t=0}^{t=x} = \dfrac{(-2)}{x+2} - \dfrac{(-2)}{2} = 1 - \dfrac{2}{x+2} = \dfrac{x}{x+2} & \text{per } x \geq 0 \end{cases}$

(iii) $F(m) = \dfrac{1}{2} \Rightarrow \dfrac{m}{m+2} = \dfrac{1}{2} \Rightarrow 2m = m+2 \Rightarrow m = 2$

(iv) $P(X>1) = 1 - F(1) = \dfrac{2}{3} \Rightarrow P(Y \geq 2) = 1 - P(Y=0) - P(Y=1) = 1 - \binom{5}{0}\left(\dfrac{2}{3}\right)^0\left(\dfrac{1}{3}\right)^5 - \binom{5}{1}\left(\dfrac{2}{3}\right)^1\left(\dfrac{1}{3}\right)^4$

essendo $Y \sim Bin(5, \frac{2}{3})$

$= 1 - \dfrac{1}{243} - \dfrac{10}{243} = \dfrac{232}{243} = 0,9547$

ESERCIZIO 3

(i)

ω	X	Y
0000	0	2
0001	1	1
0010	1	1
0011	2	0
0100	1	1
0101	2	2
0110	2	0
0111	3	1
1000	1	1
1001	2	0
1010	2	2
1011	3	1
1100	2	0
1101	3	1
1110	3	1
1111	4	2

$x \backslash y$	0	1	2	$p_X(x)$
0	0	0	$\frac{1}{16}$	$1/16$
1	0	$\frac{4}{16}$	0	$4/16$
2	$\frac{4}{16}$	0	$\frac{2}{16}$	$6/16$
3	0	$\frac{4}{16}$	0	$4/16$
4	0	0	$\frac{1}{16}$	$1/16$
$p_Y(y)$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$	1

(ii) $p(0,0) \neq p_X(0)\cdot p_Y(0) > 0 \Rightarrow$ X e Y non sono indipendenti

(iii) $X \sim Bin(4, \frac{1}{2})$, $Y \sim Bin(2, \frac{1}{2})$

$\Rightarrow E(X) = 4 \cdot \frac{1}{2} = 2 \qquad Var(X) = 4 \cdot \frac{1}{2} \cdot \frac{1}{2} = 1$

$E(Y) = 2 \cdot \frac{1}{2} = 1 \qquad Var(Y) = 2 \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{2}$

$E(X \cdot Y) = \sum_x \sum_y x \cdot y \cdot p(x,y) =$

$= \frac{4}{16} + \frac{8}{16} + \frac{12}{16} + \frac{8}{16} = \frac{32}{16} = 2$

$\Rightarrow Cov(X,Y) = E(X\cdot Y) - E(X)\cdot E(Y) = 0 \Rightarrow \rho(X,Y) = 0$

(iv) $P(X = k \mid Y = 2) = \begin{cases} 1/4 & \text{se } k=0,\ k=4 \\ 1/2 & \text{se } k=2 \end{cases}$

$E[X \mid Y=2] = 0 \cdot \frac{1}{4} + 2 \cdot \frac{1}{2} + 4 \cdot \frac{1}{4} = 2$

ESERCIZIO 4

(i) $\begin{cases} E(1-X) + Var\left(\frac{1}{2}X\right) = 0 \\ E(X) - \frac{1}{2} Var(X) = 0 \end{cases} \Rightarrow \begin{cases} 1 - E(X) + \frac{1}{4} Var(X) = 0 \\ E(X) - \frac{1}{2} Var(X) = 0 \end{cases} \Rightarrow \begin{cases} Var(X) = 4 = \sigma^2 \\ E(X) = 2 = \mu \end{cases}$

(ii) $P(X>0) = P\left(\frac{X-\mu}{\sigma} > -\frac{2}{2}\right) = P(Z>-1) = 1-\Phi(-1) = \Phi(1) = 0{,}8413$ con $Z \sim N(0,1)$

$P(X<1) = P\left(\frac{X-\mu}{\sigma} < \frac{1-2}{2}\right) = P(Z<-0{,}5) = \Phi(-0{,}5) = 1-\Phi(0{,}5) = 1-0{,}6915 = 0{,}3085$

$P(X<1 \mid X>0) = \frac{P(0<X<1)}{P(X>0)} = \frac{P(X<1)-P(X<0)}{P(X>0)} = \frac{P(X<1) - 1 + P(X>0)}{P(X>0)} =$

$= \frac{0{,}3085 - 1 + 0{,}8413}{0{,}8413} = \frac{0{,}1498}{0{,}8413} = 0{,}1781$

(iii) $E(X-Y) = E(X) - E(Y) = 2-2=0$

$Var(X-Y) = Var(X) + Var(Y) = 4+4 = 8$

$$E[(X+Y)^2] = E(X^2) + E(Y^2) + 2E(XY) = (E(X))^2 + Var(X) + (E(Y))^2 + Var(Y)$$

$$+ 2E(X) \cdot E(Y) = 4+4+4+4+2 \cdot 2 \cdot 2 = 24$$

(iv) $E(X-T) = E(X-Y) = 0$

$Var(X-T) = Var(X) + Var(T) - 2Cov(X,T) = 4+4-2 \cdot 1 = 6$

$$E[(X+T)^2] = E(X^2) + E(T^2) + 2E(X \cdot T) = 8+8+2 \cdot 5 = 26$$

$$Cov(X,T) = E(X \cdot T) - E(X) \cdot E(T) \Rightarrow E(XT) = Cov(X,T) + E(X)E(T) = 1+2 \cdot 2 = 5$$