PROBLEMS ON "HIDDEN" INDEPENDENCE AND UNIFORMITY

All the problems below when looked at the right way, can be solved by elegant arguments avoiding induction, recurrence relations, complicated sums, etc. They all have a vague theme in common, related to certain probabilities being either uniform or independent. However, it is not necessary to look at a problem from this point of view in order to find the elegant solution. If you solve a problem in a complicated way, the answer might suggest to you a simpler method.

1. Slips of paper with the numbers from 1 to 99 are placed in a hat. Five numbers are randomly drawn out of the hat one at a time (without replacement). What is the probability that the numbers are chosen in increasing order?

2. In how many ways can a positive integer n be written as a sum of positive integers, taking order into account? For instance, 4 can be written as a sum in the eight ways $4 = 3+1 = 1+3 = 2+2 = 2+1+1 = 1+2+1 = 1+1+2 = 1+1+1+1$.

3. How many 8×8 matrices of 0's and 1's are there, such that every row and column contains an odd number of 1's?

4. Let $f(n)$ be the number of ways to take an n-element set S, and, if S has more than one element, to partition S into two disjoint nonempty subsets S_1 and S_2, then to take one of the sets S_1, S_2 with more than one element and partition it into two disjoint nonempty subsets S_3 and S_4, then to take one of the sets with more than one element not yet partitioned and partition it into two disjoint nonempty subsets, etc., always taking a set with more than one element that is not yet partitioned and partitioning it into two nonempty disjoint subsets, until only one-element subsets remain. For example, we could start with 12345678 (short for $\{1,2,3,4,5,6,7,8\}$), then partition it into 126 and 34578, then partition 34578 into 4 and 3578, then 126 into 6 and 12, then 3578 into 37 and 58, then 58 into 5 and 8, then 12 into 1 and 2, and finally 37 into 3 and 7. (The order we partition the sets is important; for instance, partitioning 1234 into 12 and 34, then 12 into 1 and 2, and then 34 into 3 and 4, is different from partitioning 1234 into 12 and 34, then 34 into 3 and 4, and then 12 into 1 and 2. However, partitioning 1234 into 12 and 34 is the same as partitioning it into 34 and 12.) Find a simple formula for $f(n)$. For instance, $f(1) = 1$, $f(2) = 1$, $f(3) = 3$, and $f(4) = 18$.

5. Fix positive integers n and k. Find the number of k-tuples $(S_1, S_2, \dots, S_k)$ of subsets S_i of $\{1, 2, \dots, n\}$ subject to each of the following conditions:

   (a) $S_1 \subseteq S_2 \subseteq \cdots \subseteq S_k$

   (b) The S_i's are pairwise disjoint.

   (c) $S_1 \cap S_2 \cap \cdots \cap S_k = \emptyset$

   (d) $S_1 \subseteq S_2 \supseteq S_3 \subseteq S_4 \supseteq S_5 \subseteq \cdots S_k$ (The symbols $\subseteq$ and $\supseteq$ alternate.)

6. Let p be a prime number and $1 \le k \le p-1$. How many k-element subsets $\{a_1, \dots, a_k\}$ of $\{1, 2 \dots, p\}$ are there such that $a_1 + \cdots + a_k \equiv 0 \pmod{p}$?

7. Let π be a random permutation of $1, 2, \dots, n$. Fix a positive integer $1 \le k \le n$. What is the probability that in the disjoint cycle decomposition of π, the length of the cycle containing 1 is k? In other words, what is the probability that k is the least positive integer for which $\pi^k(1) = 1$?

8. Let π be a random permutation of $1, 2, \ldots, n$. What is the probability that 1 and 2 are in the same cycle of π?

9. Choose n real numbers $x_1, \ldots, x_n$ uniformly and independently from the interval $[0, 1]$. What is the expected value of $\min_i x_i$, the minimum of $x_1, \ldots, x_n$?

10. (a) Let m and n be nonnegative integers. Evaluate the integral

$$B(m, n) = \int_0^1 x^m (1-x)^n dx,$$

by interpreting the integral as a probability.

(b) Let R be the region consisting of all triples (x, y, z) of nonnegative real numbers satisfying $x + y + z \leq 1$. Let $w = 1 - x - y - z$. Express the value of the triple integral (taken over the region R)

$$\iiint x^1 y^9 z^8 w^4 \, dx \, dy \, dz$$

in the form $a!\, b!\, c!\, d!/n!$, where $a,\ b,\ c,\ d$, and n are positive integers.

11. (a) Choose n points at random (uniformly and independently) on the circumference of a circle. Find the probability p_n that all the points lie on a semicircle. (For instance, $p_1 = p_2 = 1$.)

(b) Fix $\theta < 2\pi$ and find the probability that the n points lie on an arc subtending an angle θ .

(c) Choose four points at random on the surface of a sphere. Find the probability that the center of the sphere is contained within the convex hull of the four points.

(d) (more difficult—I don't know an elegant proof) Choose n points uniformly at random in a square (or more generally, a parallelogram). Show that the probability that the points are in convex position (i.e., each is a vertex of their convex hull) is given by

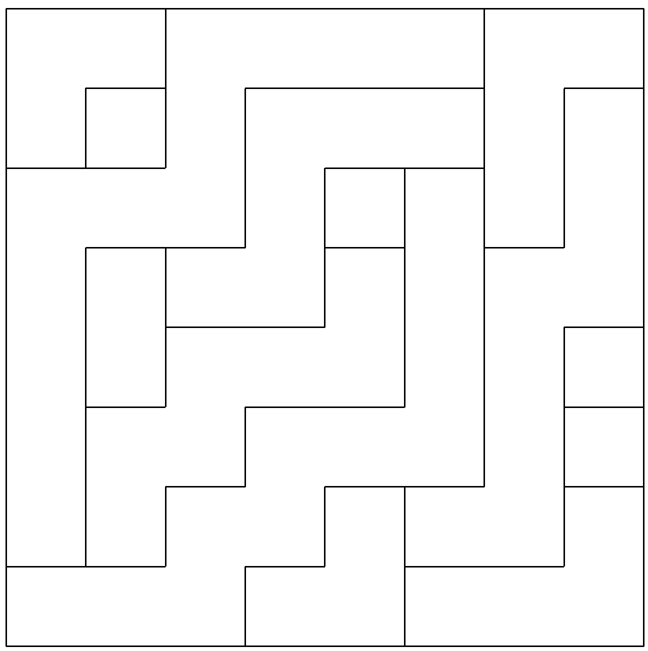
$$P_n = \left[\frac{1}{n!} \binom{2n-2}{n-1} \right]^2 .$$

12. Passengers $P_1, \ldots, P_n$ enter a plane with n seats. Each passenger has a different assigned seat. The first passenger sits in the wrong seat. Thereafter, each passenger either sits in their seat if unoccupied or otherwise sits in a random unoccupied seat. What is the probability that the last passenger sits in his or her own seat?

13. Let $x_1, x_2, \ldots, x_n$ be n points (in that order) on the circumference of a circle. A person starts at the point x_1 and walks to one of the two neighboring points with probability $1/2$ for each. The person continues to walk in this way, always moving from the present point to one of the two neighboring points with probability $1/2$ for each. Find the probability p_i that the point x_i is the last of the n points to be visited for the first time. In other words, find the probability that when x_i is visited for the first time, all the other points will have already been visited. For instance, $p_1 = 0$ (when $n > 1$), since x_1 is the *first* of the n points to be visited.

14. There are n parking spaces $1, 2, \ldots, n$ (in that order) on a one-way street. Cars $C_1, \ldots, C_n$ enter the street in that order and try to park. Each car C_i has a preferred space a_i. A car will drive to its preferred space and try to park there. If the space is already occupied, the car will park in the next available space. If the car must leave the street without parking, then the process fails. If $\alpha = (a_1, \ldots, a_n)$ is a sequence of preferences that allows every car to park, then we call α a *parking function*. For instance, there are 16 parking functions of length 3, given by (abbreviating $(1, 1, 1)$ as 111, etc.) 111, 112, 121, 211, 113, 131, 311, 122, 212, 221, 123, 132, 213, 231, 312, 321. Show that the number of parking functions of length n is equal to $(n+1)^{n-1}$.

15. A *snake* on the 8×8 chessboard is a nonempty subset S of the squares of the board obtained as follows: Start at one of the squares and continue walking one step up or to the right, stopping at any time. The squares visited are the squares of the snake. Here is an example of the 8×8 chessboard covered with disjoint snakes.



Find the total number of ways to cover an 8×8 chessboard with disjoint snakes. Generalize to an $m \times n$ chessboard.

16. Let n balls be thrown uniformly and independently into n bins.
    (a) Find the probability that bin 1 is empty.
    (b) Find the expected number of empty bins.

17. Consider $2m$ persons forming m couples who live together at a given time. Suppose that at some later time, the probability of each person being alive is p, independently of other persons. At that later time, let A be the number of persons that are alive and let S be the number of couples in which both partners are alive. For any number of total surviving persons a, find expected number of surviving couples, i.e., $\mathbb{E}[S|A = a]$.

18. Let X_1, X_2 and X_3 be three independent, continuous random variables with the same distribution. Given that X_2 is smaller than X_3, what is the conditional probability that X_1 is smaller than X_2?

19. In game show each contestant i spins an infinitely calibrated "fair" wheel of fortune, which assigns the contestant a real number X_i between 1 and 100.

   (a) Find $\mathbb{P}(X_1 < X_2)$. Explain your answer.

   (b) Find $\mathbb{P}(X_1 < X_2,\, X_1 < X_3)$, i.e., find the probability that the first contestant will have the smallest value of the first three contestants.

   (c) Consider a new random variable N, which is integer valued. N is the index of the first contestant who is assigned a smaller number than contestant 1. As an illustration, if contestant 1 has a smaller value than contestants 2, 3, and 4, but contestant 5 has a smaller value than contestant 1 ($X_5 < X_1$), then $N = 5$. Find $\mathbb{P}(N > n)$ as a function of n.

20. The adventures of Ant Alice. On a meter-long rod sit 25 ants placed uniformly at random, each facing a uniformly chosen direction (east or west). They proceed to march forward at 1 cm/sec; whenever two ants collide, they reverse directions. Ants fall off the rod when they reach either endpoint. Alice is the ant initially positioned 13th from the west.

   (a) How long does it take before we can be certain that Alice is off the rod?

   (b) What is the probability that Alice falls off the rod facing the same direction as her initial direction?

   (c) What is the probability that Alice is the last ant to fall off the rod?

   (d) What is the expected number of collisions that Alice has?

   (e) What is the probability that Alice has more collisions than any other ant?

   (f) Alice has a cold, which is transmitted from ant to ant instantly upon collision. How is the expected number of ants infected in the process?

PROBLEMS ON SUMS AND INTEGRALS

Contents

1 Swapping sums

1.1 Finite sums

Here is an example that many of you might already know from high school math contests.

Example 1. Let n be a positive integer. Prove that

$$\sum_{k\geq 1} \varphi(k) \left\lfloor \frac{n}{k} \right\rfloor = \frac{1}{2}n(n+1).$$

Proof. The key idea is to rewrite the floor as a sum involving divisors:

$$\sum_{k\geq 1} \varphi(k) \left\lfloor \frac{n}{k} \right\rfloor = \sum_{k\geq 1} \varphi(k) \sum_{\substack{k\mid m \\ k\leq n}} 1 = \sum_{k\geq 1} \sum_{\substack{k\mid m \\ m\leq n}} \varphi(k).$$

Thus we're computing the sum of $\varphi(k)$ over several pairs of integers (k,m) for which $k \mid m$, $m \leq n$. For example, if $n = 6$, the possible pairs (k,m) are given be the following table:

$$(k,m) \in \left\{ \begin{array}{cccccc} (1,1) & (1,2) & (1,3) & (1,4) & (1,5) & (1,6) \\ & (2,2) & & (2,4) & & (2,6) \\ & & (3,3) & & & (3,6) \\ & & & (4,4) & & \\ & & & & (5,5) & \\ & & & & & (6,6) \end{array} \right\}$$

Nominally, we're supposed to be summing by the rows of this table (i.e. fix k and run the sum over corresponding m). However, by interchanging the order of summation we can instead consider this as a sum over the rows: if we instead pick the value of m first, we see that

$$\sum_{k\geq 1} \sum_{\substack{k|m \\ m\leq n}} \varphi(k) = \sum_{m=1}^{n} \sum_{k|m} \varphi(k).$$

Using the famous fact $\sum_{d|n} \varphi(d) = n$, we conclude

$$\sum_{m=1}^{n} \sum_{k|m} \varphi(k) = \sum_{m=1}^{n} m = \frac{1}{2}n(n+1).$$

□

Here one has the idea that one can "swap the order of summation": even though there is a single $\sum$ initially, by rewriting it as a double $\sum$ and then swapping the order, we are able to solve the problem.

The goal of this lecture is to try and push this idea to allow us to do similar calculations over both infinite sums and integrals. Because of the introduction of infinity, things become a little more complicated and some more care is necessary. So, in the first part of the lecture we will address conditions on which rearranging the order of summation or integration is permissible. After that we will see several applications.

1.2 Absolute and conditional convergence

Let $\sum_n a_n$ be an infinite series of complex numbers; then its limit is defined as

$$\sum_n a_n := \lim_{N\to\infty} \left(\sum_{n=1}^{N} a_n \right).$$

Note that this depends on the order of the terms: if we permute the sequence, the limit might change! This is weird and bad since we would want "infinite addition" to be commutative, so we want a way to avoid this behavior. This is accomplished by using the so-called notion of absolute convergence.

Definition 2. If $\sum a_k$ converges, we say it *converges absolutely* if $\sum |a_k| < \infty$, and *converges conditionally* otherwise.

Theorem 3 (Rearrangement okay iff absolutely convergent). *Let $\sum a_n$ be a convergent series of complex numbers.*

(a) If $\sum a_n$ is absolutely convergent, it is invariant under permutation of the terms (the sum will still converge, and the limit remains the same).

(b) If $\sum a_n$ is conditionally convergent and a_n are real numbers, then there exists a permutation of the terms for which the sum converges to 2018.

Thus, any time before you try to rearrange the series, you must check first that it's absolutely convergent. With two $\sum$ signs the statement reads:

Theorem 4 (Fubini for doubly-indexed infinite sums)**.** *Let $a_{m,n} \in \mathbb{C}$. If any of the three quantities*

$$\sum_{(m,n)\in\mathbb{N}^2} |a_{m,n}|, \qquad \sum_m \left(\sum_n |a_{m,n}|\right), \qquad \sum_n \left(\sum_m |a_{m,n}|\right)$$

are convergent, then

$$\sum_{(m,n)\in\mathbb{N}^2} a_{m,n} = \sum_m \left(\sum_n a_{m,n}\right) = \sum_n \left(\sum_m a_{m,n}\right)$$

and all three series are convergent.

Corollary 5 (Tonelli for doubly-indexed infinite sums)**.** *Let $a_{m,n} \in \mathbb{R}_{\geq 0}$. Then*

$$\sum_{(m,n)\in\mathbb{N}^2} a_{m,n} = \sum_m \left(\sum_n a_{m,n}\right) = \sum_n \left(\sum_m a_{m,n}\right)$$

where we allow the possibility all three diverge.

Here is the classic example.

Example 6 (Putnam 2016 B6). Evaluate

$$\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} \sum_{n=0}^{\infty} \frac{1}{k2^n+1}.$$

Proof. Before anything else, the sum is absolutely convergent since we have

$$\sum_{\substack{k\geq 1\\ n\geq 0}} \frac{1}{k(k\cdot 2^n+1)} < \sum_{k\geq 1} k^{-2} \sum_{n\geq 0} \frac{1}{2^n} = \frac{\pi^2}{6}\cdot 2 < \infty.$$

Thus we may swap the order of summation freely.

We use $d = k\cdot 2^n + 1 \geq 2$ as the summation variable, so that the sum in question is

$$\sum_{d\geq 2} \frac{1}{d} \sum_{\substack{k\\ \exists n: d-1=k\cdot 2^n}} \frac{(-1)^{k-1}}{k}.$$

Now we claim that the inner sum is exactly $\frac{1}{d-1}$. Indeed, if $d-1 = 2^r m$ with m odd, then the sum is

$$\begin{aligned}\frac{(-1)^{m-1}}{m} + \frac{(-1)^{2m-1}}{2m} + \dots + \frac{(-1)^{2^r m-1}}{2^r m} &= \frac{1}{m}\left(\frac{1}{1} - \frac{1}{2} - \dots - \frac{1}{2^r}\right)\\ &= \frac{1}{2^r m}\\ &= \frac{1}{d-1}.\end{aligned}$$

Consequently, the final answer is

$$\sum_{d\geq 2} \frac{1}{d(d-1)} = \sum_{d\geq 2}\left(\frac{1}{d-1} - \frac{1}{d}\right) = 1.$$

□

2 Riemann integral

So far all of this is fair-game on high school. We'll now move into the realm of calculus.

Definition 7. A *tagged partition* P of $[a,b]$ consists of a partition of $[a,b]$ into n intervals, with a point ξ_i in the nth interval, denoted

$$a \le x_0 < x_1 < x_2 < \cdots < x_n \le b \qquad \text{and} \qquad \xi_i \in [x_{i-1}, x_i] \quad \forall\, 1 \le i \le n.$$

The *mesh* of P is the width of the longest interval, i.e. $\max_i(x_i - x_{i-1})$.

Theorem 8 (Riemann integral)**.** *Let $f\colon [a,b] \to \mathbb{C}$ be continuous. Then the definition*

$$\int_a^b f(x)\,dx = \lim_{\substack{P \text{ tagged partition} \\ \text{mesh } P \to 0}} \left(\sum_{i=1}^n f(\xi_i)(x_i - x_{i-1}) \right)$$

is well-defined (and finite).

There are a bunch of remarks I want to make about this result.

2.1 Compactness and improper integrals

We won't prove the definition of the Riemann integral works out, but we will mention that its proof hinges crucially on:

Fact 9. The interval $[a,b]$ is *compact*, so continuous functions $f\colon [a,b] \to \mathbb{C}$ behave well. In particular, f is bounded, and "uniformly continuous".

This fact is false for open (or unbounded) intervals: consider the function $1/\sqrt{x}$ on $(0,1)$, for example. This gives rise to the notion of "improper integrals", such as

$$\int_0^1 \frac{1}{\sqrt{x}}\,dx.$$

As written, this does not officially make sense as a Riemann integral, since $f(x) = \frac{1}{\sqrt{x}}$ is not a function on $[0,1]$. Rather, we implicitly mean

$$\lim_{\varepsilon \to 0^+} \int_\varepsilon^1 \frac{1}{\sqrt{x}}\,dx$$

since $f(x)$ *is* well-defined on $[\varepsilon, 1]$. In this case, there is no guarantee the limit exists; for example $\int_0^1 x^{-1}\,dx = \infty$.

Similarly, it's possible to set endpoints at ∞ by e.g.

$$\int_{-\infty}^{\infty} f(x) \coloneqq \lim_{B \to \infty} \int_{-B}^{B} f(x)$$

for example.

2.2 Mesh sums

Sometimes, you will find that a sum can be written in such a way that it corresponds to the mesh of a Riemann integral. In that case, one is very happy, because then it turns the entire sum into a single integral!

Example 10. Evaluate

$$\lim_{n\to\infty}\left(\frac{1}{n+1}+\frac{1}{n+2}+\cdots+\frac{1}{2n}\right).$$

Proof. Write as

$$\lim_{n\to\infty}\left(\frac{1}{n}\sum_{k=1}^{n}\frac{1}{1+\frac{k}{n}}\right).$$

Then, this is a mesh sum for $f(x)=\frac{1}{1+x}$ over $[0,1]$. Thus by definition it approaches $\int_0^1 \frac{1}{1+x}\,dx = \log 2$. $\square$

2.3 Discretization and inequalities

If asked to prove an identity or inequality about integrals, it is often possible to revert back to a discrete sum, a technique called *discretization*. For example, suppose one wishes to prove the Cauchy-Schwarz inequality in the form

$$\left(\int_a^b f(x)g(x)\,dx\right)^2 \le \left(\int_a^b f(x)^2\,dx\right)\left(\int_a^b g(x)^2\,dx\right)$$

for continuous functions $f,g\colon[a,b]\to\mathbb{R}$. By taking meshes, it is sufficient to prove

$$\left(\frac{1}{n}\sum_i f(a_i)g(a_i)\,dx\right)^2 \le \left(\frac{1}{n}\sum_i f(a_i)^2\,dx\right)\left(\frac{1}{n}\sum_i g(a_i)^2\,dx\right).$$

which is of course just the classical Cauchy-Schwarz from high school.

In practice, aside from discretization, most integral inequalities on competitions will really just use the following fact:

Lemma 11 (The obvious inequality)**.** *Let $f,g\colon[a,b]\to\mathbb{R}$ be continuous functions. If $f(x)\le g(x)$ for all $x\in[a,b]$ then*

$$\int_a^b f(x)\,dx \le \int_a^b g(x)\,dx.$$

3 Lebesgue integrals

3.1 Advantages of Lebesgue integrals

Unfortunately, Riemann integrals are terrible. In order to properly state theorems about interchanging order of summation, it'll be much more convenient to proceed with the **Lebesgue integral**, which I will generally denote by $\int_X$ to distinguish it from the Riemann integral $\int_a^b$.

Defining the Lebesgue integral is much more involved, because it involves a bunch of measure theory, so I *won't* define what it is (but those of you taking 18.175 will find out really soon). However, I'll at least mention the following reasons it's appreciated.

- **Better theorems about swapping limits and sums**. For example, for the Riemann integral, swapping $\sum_n \int_a^b f_n$ and $\int_a^b \sum_n f_n$ requires *uniform convergence*, which is a pretty strong condition (although it'll be true for Taylor series, which is a frequent use case for us).

- **Improper integrals can be handled natively**. You can write $\int_{(0,1)} \frac{1}{\sqrt{x}}\, dx$ and $\int_{\mathbb{R}} \exp(-x^2)\, dx$ and it makes sense, unlike for the Riemann case where one has to use an improper integral.

- **More versatile**. Although we won't encounter any, some functions that were previously not Riemann integrable can now be assigned values. The classic example is $\int_{[0,1]} \mathbf{1}_{\mathbb{Q}} = 0$.

3.2 Riemann integrals and Lebesgue integrals

Of course, it'd be really silly if there wasn't some guarantee that the Riemann integrals and Lebesgue integrals agree.

The rules for converting a Riemann and Lebesgue integral are as follows:

- For continuous functions $f\colon [a,b] \to \mathbb{C}$, the Riemann integral and Lebesgue integrals coincide. So **proper Riemann integrals work out of the box**.

- For continuous nonnegative functions $f\colon (a,b) \to \mathbb{R}_{\ge 0}$ on an open (or half-open) interval where one needs improper integrals, the improper Riemann integral and Lebesgue integrals coincide (where we allow the possibility that the integrals are both $+\infty$). Here, $a = -\infty$ and $b = +\infty$ are allowed too.

- For general $f\colon (a,b) \to \mathbb{C}$, if the partial integrals $\int_c^d |f|\, dx$ are bounded for any $[c,d] \subset (a,b)$ then we can also swap as above.

On the other hand, if your signs are all over the place, then there isn't hope in general of converting improper Riemann integrals to Lebesgue ones. A famous textbook example is $\int_0^\infty \frac{\sin x}{x}\, dx$ which in fact is *not* covered by Lebesgue integration.

3.3 Swapping double integrals

I'll state this in the full generality, though we'll only use it in the cases where the "σ-finite measure spaces" are $\mathbb{N}$ (corresponding to infinite sums) or sub-intervals of $\mathbb{R}$ (corresponding to Riemann integrals).

Theorem 12 (Fubini)**.** *Let X and Y be "σ-finite measure spaces". Let $f\colon X \times Y \to \mathbb{C}$ be continuous (or just "measurable"). If any of $\int_X \left(\int_Y |f(x,y)|\, dy\right)\, dx$, $\int_Y \left(\int_X |f(x,y)|\, dx\right)\, dy$, $\int_{X\times Y} |f(x,y)|\, d(x,y)$ are finite, then we have*

$$\int_X \left(\int_Y f(x,y)\, dy\right)\, dx = \int_Y \left(\int_X f(x,y)\, dx\right)\, dy = \int_{X\times Y} f(x,y)\, d(x,y).$$

Corollary 13 (Tonelli)**.** *Let X and Y be "σ-finite measure spaces". Let $f\colon X \times Y \to \mathbb{R}_{\ge 0}$ be continuous (or just "measurable") and nonnegative. Then*

$$\int_X \left(\int_Y f(x,y)\, dy\right)\, dx = \int_Y \left(\int_X f(x,y)\, dx\right)\, dy = \int_{X\times Y} f(x,y)\, d(x,y)$$

where we allow the possibility that all three are $+\infty$.

Remark 14.

- If $X = \mathbb{N}$ and $Y = \mathbb{N}$, then this corresponds to the double sums we stated earlier.
- If $X = \mathbb{N}$ and $Y \subset \mathbb{R}$ is an interval, then this states that $\sum \int$ and $\int \sum$ can be swapped.
- Note that if X and Y are finite closed intervals and $f\colon X \times Y \to \mathbb{C}$ is continuous, then hypotheses of Fubini are automatically satisfied, since $X \times Y$ is compact. The situation where X and Y are open/infinite is more slippery, although in most cases we'll have nonnegativity and then Tonelli will save us.

Tonelli's theorem (together with the result that even improper Riemann integrals are okay with nonnegative functions) means that whenever you have nonnegative functions, you can proceed *no holds barred* — everything works beautifully. In other words **nonnegative $\implies$ euphoria**.

3.4 Interchanging limits and Lebesgue integrals

You can read this off of the results on sums, but we'll state them here since they have names.

Theorem 15 (Dominated convergence theorem). *Let $f_n\colon I \to \mathbb{C}$ be a sequence of continuous functions on an interval $I \subseteq \mathbb{R}$. Assume that $|f_n(x)| \le g(x)$ for all x, where $\int_I g(x) < \infty$ (i.e. g is integrable). Then $\lim_n f_n(x)$ is integrable and*

$$\lim_{n\to\infty} \int_I f_n(x)\, dx = \int_I \lim_{n\to\infty} f_n(x)\, dx.$$

Theorem 16 (Monotone convergence theorem). *Suppose that $f_n\colon I \to \mathbb{R}_{\ge 0}$ is a sequence of continuous functions on an interval $I \subseteq \mathbb{R}$ which are also nonnegative. Assume further that $f_n(x) \le f_{n+1}(x)$ for $n \in \mathbb{N}$, $x \in I$. Then*

$$\lim_{n\to\infty} \int_I f_n(x)\, dx = \int_I \lim_{n\to\infty} f_n(x)\, dx$$

where the value of any of these integrals is allowed to be infinite.

4 Techniques for introducing more sums

4.1 Taylor series

Some common ones:

$$\begin{aligned}
\exp(x) &= \sum_{n\ge0} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots && \forall\, x \in \mathbb{R} \\
\log(1-x) &= -\sum_{n\ge1} \frac{x^n}{n} = -x - \frac{x^2}{2} - \frac{x^3}{3} - \dots && \forall\, |x| < 1 \\
\frac{1}{1-x} &= \sum_{n\ge0} x^n = 1 + x + x^2 + \dots && \forall\, |x| < 1 \\
\arctan(x) &= \sum_{n\ge0} \frac{(-1)^n x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \dots && \forall\, |x| < 1.
\end{aligned}$$

There is a nice theorem about Taylor series in general:

Theorem 17 (Convergence of Taylor series)**.** *Let f be an analytic function. Within its radius of convergence, the Taylor series for f will*

- converge absolutely *for any x as a series of complex numbers, and*
- converge uniformly *on any compact sub-interval, as a series of functions (i.e. it is compactly convergent).*

I mention uniform convergence here since it's actually strong enough to allow swapping integration even for the Riemann integral. Here's the definition:

Definition 18. A sequence of functions $F_n\colon [a,b] \to \mathbb{C}$ is said to *converge uniformly* to the function $F\colon [a,b] \to \mathbb{C}$ if

$$\lim_{n\to\infty} \sup_{x\in[a,b]} |F_n(x) - F(x)| = 0.$$

A series $\sum_n f_n$ converges uniformly if its partial sums $F_n = \sum_{k=1}^n f_k$ do.

But we'll be mostly using Lebesgue integrals anyways.

So, whenever you have an analytic function on a closed interval, all the summation results work fine! Here is a very famous example.

Example 19. Compute

$$\int_0^1 \log x \log(1-x)\ dx.$$

There is some subtlety here since this integral looks like it might be improper! Fortunately, it's not quite, since $\lim_{x\to 0^+} \log(x)\log(1-x) = 0$, and in this way we can actually regarding $\log(x)\log(1-x)$ as a proper integral on $[0,1]$.

Proof. Switch to Lebesgue integration. The integral is then

$$\begin{aligned}
I &= -\int_{[0,1]} \log x \sum_{n\ge1} \frac{x^n}{n}\ dx \\
&= -\sum_{n\ge1} \frac{1}{n} \int_0^1 x^n \log x\ dx && \text{(by Tonelli)} \\
&= -\sum_{n\ge1} \frac{1}{n} \left[x^{n+1} \cdot \frac{(n+1)\log x - 1}{(n+1)^2} \right]_{x=0}^{x=1} \text{(integration by parts)} \\
&= -\sum_{n\ge1} \frac{1}{n} \left[x^{n+1} \cdot \frac{(n+1)\log x - 1}{(n+1)^2} \right]_{x=0}^{x=1} \\
&= \sum_{n\ge1} \frac{1}{n(n+1)^2} \\
&= \sum_{n\ge1} \left[\frac{1}{n} - \frac{1}{n+1} - \frac{1}{(n+1)^2} \right].
\end{aligned}$$

The Nth partial sum of this is equal to $1 - \frac{1}{N+1} - \sum_{n=1}^N \frac{1}{(n+1)^2}$ which gives $2 - \frac{\pi^2}{6}$ as $N \to \infty$. $\square$

Remark 20 (An application of Feynman's trick). In my original notes, I had obtained the identity $\int_0^1 x^n \log x\ dx = -\frac{1}{(n+1)^2}$ using integration by parts. In class it was pointed out that *Feynman's* *trick*, more descriptively called "differentiating under the integral sign", gives a shorter way to prove this. Start by writing

$$\int_0^1 x^n \, dx = \frac{1}{n+1}$$

and then treat $n \in \mathbb{R}$ as a parameter. This allows one to differentiate both sides with respect to n, yielding

$$\begin{aligned} \int_0^1 \frac{d}{dn} x^n \, dx &= \frac{d}{dn} \frac{1}{n+1} \\ \implies \int_0^1 x^n \log x \, dx &= -\frac{1}{(n+1)^2}. \end{aligned}$$

See `http://www.math.uconn.edu/~kconrad/blurbs/analysis/diffunderint.pdf` for more details on this trick.

4.2 Eliminating fractions

The following seemingly obvious statement is surprisingly useful.

Lemma 21 (Denominator $\to$ integral)**.** *For any real number $s > -1$ we have*

$$\frac{1}{s+1} = \int_{(0,1)} t^s \, ds.$$

As a simple use case, let's suppose we were given $\sum_n \frac{x^n}{n}$ for some $|x| < 1$ and wanted to figure out what function it was (without knowing anything about log in advance). We can write

$$\begin{aligned} \sum_n \frac{x^n}{n} &= \sum_n x^n \int_{[0,1]} t^{n-1} \, dt \\ &= \sum_n \int_{[0,1]} x(xt)^{n-1} \, dt \\ &= \int_{[0,1]} \sum_n x(xt)^{n-1} \, dt \\ &= \int_{[0,1]} \frac{x}{1-xt} \, dt \\ &= [-\log(1-xt)]_{t=0}^{1} \\ &= [-\log(1-xt)]_{t=0}^{1} = -\log(1-x). \end{aligned}$$

Let's also see a solution to the earlier double sum.

Example 22 (Putnam 2016 B6). Evaluate

$$\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} \sum_{n=0}^{\infty} \frac{1}{k2^n + 1}.$$

Proof. Check conditional convergence of the double sum in the same way as before. Thus we apply Fubini freely:

$$\begin{aligned}
\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} \sum_{n=0}^{\infty} \frac{1}{k2^n+1} &= \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} \sum_{n=0}^{\infty} \int_{[0,1]} t^{k2^n} \, dt \\
&= \int_{[0,1]} \left(- \sum_{n=0}^{\infty} \sum_{k=1}^{\infty} \frac{(-t^{2^n})^k}{k} \right) \, dt \\
&= \int_{[0,1]} \sum_{n=0}^{\infty} \log(1+t^{2^n}) \, dt = \int_{[0,1]} \log \left(\prod_{n=0}^{\infty} (1+t^{2^n}) \right) \, dt \\
&= \int_{[0,1]} \log \left(\frac{1}{1-t} \right) \, dt = \int_{[0,1]} -\log(1-t) \, dt = 1.
\end{aligned}$$

□

4.3 Fourier series

If $f\colon \mathbb{R} \to \mathbb{C}$ is continuous with period 1, then

$$f(x) = \lim_{N\to\infty} \sum_{m=-N}^{N} a_m \exp(2\pi i m x).$$

The Fourier coefficients a_m are given by

$$a_m = \int_0^1 f(x) \exp(-2\pi i m x) \, dx.$$

We again have convergence results:

Theorem 23. *Let $f\colon [0,1] \to \mathbb{C}$ be periodic.*

(a) The Fourier series converges uniformly provided f is continuously differentiable (this can be weakened to "absolutely continuous", but we won't need that level of generality).

(b) The Fourier series converges absolutely as long as $\sum_{m\in\mathbb{Z}} |a_m| < \infty$.

Example 24. If $f\colon \mathbb{R} \to \mathbb{C}$ is continuously differentiable with period 1, and α is an irrational number, then

$$\lim_{n\to\infty} \frac{f(\alpha)+\cdots+f(n\alpha)}{n} = \int_0^1 f(x) \, dx.$$

Proof. Just write $f(x) = \sum_m a_m \exp(2\pi i m x)$. Then note that for $m \neq 0$, if we let $z = \exp(2\pi i m\alpha)$ then

$$\frac{z^1+z^2+\cdots+z^n}{n} = \frac{z(1-z^n)}{n(1-z)} \to 0$$

as long as $z \neq 1$, which holds since z is not a root of unity. This leaves just the contribution form $a_0 = \int_0^1 f(x) \, dx$. □

In general, Fourier-type sums are good things to keep an eye out for, even if they don't explicitly come from Fourier series. For example, given a complex polynomial $p(z)$ (or even a series):

- The discrete sum $\sum_{k=0}^{n-1} p\left(e^{\frac{2\pi i k}{n}}\right)$ extracts the coefficients with indices divisible by n,

- the integral $\int_{t=0}^{2\pi} p(e^{it})\, dt = 2\pi \cdot p(0)$ extracts the constant term of the polynomial,

and so on. This is related to complex analysis, in which it turns complex differentiable functions $\mathbb{C} \to \mathbb{C}$ *are* exactly the same as complex analytic functions, which means you can go nuts with all sorts of beautiful results such as Cauchy's theorem.

5 Problems

1. Evaluate the improper integral
$$\int_0^1 \frac{\log(1-x)}{x}\, dx.$$

2. Determine the value of the improper integral
$$\int_0^\infty \frac{x}{e^x - 1}\, dx.$$

3. (a) Show that that $\min(a, b) = \int_0^\infty \mathbf{1}_{\le a}(t)\mathbf{1}_{\le b}(t)\, dt$ for any nonnegative real numbers $a, b \ge 0$. (What do you think $\mathbf{1}_{\le c}(t)$ means?)

   (b) Show that if $r_1, \dots, r_n$ are nonnegative reals and $x_1, \dots, x_n$ are real numbers then
$$\sum_{i=1}^{n}\sum_{j=1}^{n} \min(r_i, r_j) x_i x_j \ge 0.$$

4. For each continuous function $f\colon [0,1] \to \mathbb{R}$ let $I(f) = \int_0^1 x^2 f(x)\, dx$ and $J(f) = \int_0^1 x f(x)^2\, dx$. Find the maximum value of $I(f) - J(f)$ over all such functions f.

5. Compute
$$\lim_{n\to\infty} \left[\frac{1}{\sqrt{4n^2 - 1^2}} + \frac{1}{\sqrt{4n^2 - 2^2}} + \cdots + \frac{1}{\sqrt{4n^2 - n^2}} \right].$$

6. Let a and b be real numbers with $a < b$, and let f and g be continuous functions from $[a, b]$ to $(0, \infty)$ such that $\int_a^b f(x)\, dx = \int_a^b g(x)\, dx$ but $f \neq g$. For every positive integer n, define
$$I_n = \int_a^b \frac{(f(x))^{n+1}}{(g(x))^n}\, dx.$$
   Show that $I_1, I_2, I_3, \dots$ is an increasing sequence with $\lim_{n\to\infty} I_n = \infty$.

7. Let $a_0, a_1, \dots, a_n, x$ be real numbers, where $0 < x < 1$, satisfying
$$a_0 + \frac{a_1}{1+x} + \frac{a_2}{1+x+x^2} + \cdots + \frac{a_n}{1+x+x^2+\cdots+x^n} = 0.$$
   Prove that for some $0 < y < 1$ we have
$$a_0 + a_1 y + a_2 y^2 + \cdots + a_n y^n = 0.$$

8. Find

$$\lim_{n\to\infty} \frac{1}{n} \sum_{a=1}^{n} \sum_{b=1}^{n} \frac{a}{a^2+b^2}.$$

9. Evaluate the following:

$$\int_0^\infty \left(x - \frac{x^3}{2} + \frac{x^5}{2\cdot 4} - \frac{x^7}{2\cdot 4\cdot 6} + \cdots \right) \left(1 + \frac{x^2}{2^2} + \frac{x^4}{2^2\cdot 4^2} + \frac{x^6}{2^2\cdot 4^2\cdot 6^2} + \cdots \right) dx.$$

10. Show that

$$\int_0^1 x^{-x}\, dx = \sum_{n\geq 1} n^{-n}.$$

11. Suppose that f is a function on the interval $[1,3]$ such that $-1 \leq f(x) \leq 1$ for all x and $\int_1^3 f(x)\, dx = 0$. Determine the largest possible value of

$$\int_1^3 \frac{f(x)}{x}\, dx.$$

12. Let $f\colon \mathbb{R} \to \mathbb{R}$ be continuous and satisfy $f(x) \geq 1$ for all x. Suppose that

$$f(x)f(2x)\dots f(nx) \leq 2018n^{2019}$$

for every positive integer n and $x \in \mathbb{R}$. Must f be constant?

13. Show that $\int_{-\infty}^{\infty} e^{-x^2}\, dx = \sqrt{\pi}$.

14. A rectangle in $\mathbb{R}^2$ is called *great* if either its width or height is an integer. Prove that if a rectangle X can be dissected into great rectangles, then the rectangle X is itself great.

15. Compute

$$\sum_{k\geq 0} \frac{2^k}{5^{2^k}+1}.$$

16. Determine the value of

$$\lim_{n\to\infty} \left[\frac{2^{1/n}}{n+1} + \frac{2^{2/n}}{n+\frac{1}{2}} + \cdots + \frac{2^{n/n}}{n+\frac{1}{n}} \right].$$

17. For any continuous function $f\colon [0,1] \to \mathbb{R}$ let

$$\mu(f) = \int_0^1 f(x)\, dx, \quad \mathrm{Var}(f) = \int_0^1 (f(x)-\mu(f))^2\, dx, \quad M(f) = \max_{0\leq x\leq 1} |f(x)|.$$

Show that if $f, g\colon [0,1] \to \mathbb{R}$ are continuous functions then

$$\mathrm{Var}(fg) \leq 2\,\mathrm{Var}(f)M(g)^2 + 2\,\mathrm{Var}(g)M(f)^2.$$

18. For $m \geq 3$, a list of $\binom{m}{3}$ real numbers a_{ijk} (where $1 \leq i < j < k \leq m$) is said to be area definite for $\mathbb{R}^n$ if the inequality

$$\sum_{1 \leq i<j<k \leq m} a_{ijk} \cdot \text{Area}(\triangle A_i A_j A_k) \geq 0$$

holds for every choice of m points $A_1, \ldots, A_m$ in $\mathbb{R}^n$. For example, the list of four numbers $a_{123} = a_{124} = a_{134} = 1$, $a_{234} = -1$ is area definite for $\mathbb{R}^2$. Prove that if a list of $\binom{m}{3}$ numbers is area definite for $\mathbb{R}^2$, then it is area definite for $\mathbb{R}^3$.

19. Prove that

$$\lim_{n\to\infty} \left(\prod_{k=0}^{n} \binom{n}{k} \right)^{\frac{1}{n(n+1)}} = \sqrt{e}.$$

20. Let $f\colon [0,1] \to \mathbb{R}$ be continuous. Show that

$$\int_0^1 \int_0^1 |f(x) + f(y)| \, dx \, dy \geq \int_0^1 |f(x)| \, dx.$$

21. Let $f\colon \mathbb{R}_{\geq 0} \to \mathbb{R}$ be a strictly decreasing continuous function such that $\lim_{x\to\infty} f(x) = 0$. Prove that

$$\int_0^\infty \frac{f(x) - f(x+1)}{f(x)} \, dx$$

diverges.

22. A rectangular prism X is contained within a rectangular prism Y.

    (a) Is it possible the surface area of X exceeds that of Y?

    (b) Is it possible the sum of the 12 side lengths of X exceeds that of Y?

23. For $a, b, c > 0$ prove that

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{4}{a+b} + \frac{4}{b+c} + \frac{4}{c+a} \geq \frac{12}{3a+b} + \frac{12}{3b+c} + \frac{12}{3c+a}.$$

24. Define a function $w\colon \mathbb{Z} \to \mathbb{Z}$ as follows. For $|a|, |b| \leq 2$, let $w(a,b)$ be as in the table shown; otherwise, let $w(a,b) = 0$.

	$w(a,b)$	$b = -2$	-1	0	1	2
a	-2	-1	-2	2	-2	-1
	-1	-2	4	-4	4	-2
	0	2	-4	12	-4	2
	1	-2	4	-4	4	-2
	2	-1	-2	2	-2	-1

For every finite nonempty subset S of $\mathbb{Z} \times \mathbb{Z}$, prove that

$$A(S) := \sum_{(\mathbf{s},\mathbf{s}') \in S \times S} w(\mathbf{s} - \mathbf{s}') > 0.$$

25. Evaluate

$$\lim_{x\to 1^-} \prod_{n\geq 0} \left(\frac{1+x^{n+1}}{1+x^n}\right)^{x^n}.$$

26. Suppose that $f\colon [0,1]^2 \to \mathbb{R}$ is continuous. Show that

$$\begin{aligned}&\int_0^1 \left(\int_0^1 f(x,y)dx\right)^2 dy + \int_0^1 \left(\int_0^1 f(x,y)dy\right)^2 dx \\ &\leq \left(\int_0^1 \int_0^1 f(x,y)\, dx\, dy\right)^2 + \int_0^1 \int_0^1 [f(x,y)]^2\, dx\, dy.\end{aligned}$$

27. For each positive integer k, let $A(k)$ be the number of odd divisors of k in the interval $\left[1, \sqrt{2k}\right)$. Evaluate:

$$\sum_{k=1}^{\infty} (-1)^{k-1} \frac{A(k)}{k}.$$

PROBLEMS ON ANALYSIS

We use the notation $f(x) \sim g(x)$ to mean $\lim_{x\to\infty} \frac{f(x)}{g(x)} = 1$. One says that $f(x)$ is *asymptotic* to $g(x)$.

1. Show that $\int_0^\infty \frac{\cos(ax)}{1+x^2} dx$ exists for $a \in \mathbb{R}$ and compute its value.
2. Find a simple function $f(x)$ for which $x^{1/x} - 1 \sim f(x)$ as $x \to \infty$.
3. For what pairs (a, b) of positive real numbers does the improper integral
$$\int_b^\infty \left(\sqrt{\sqrt{x+a} - \sqrt{x}} - \sqrt{\sqrt{x} - \sqrt{x-b}} \right) dx$$
converge?
4. Let a_n be the unique positive root of $x^n + x = 1$. Find a simple function $f(n)$ for which $1 - a_n \sim f(n)$ as $n \to \infty$.
5. For each continuous function $f : [0, 1] \to \mathbb{R}$, let $I(f) = \int_0^1 x^2 f(x) dx$ and $J(f) = \int_0^1 x f(x)^2 dx$. Find the maximum value of $I(f) - J(f)$ over all such functions f.
6. For a positive real number a, calculate $\int_0^\infty t^{-1/2} e^{-a(t+t^{-1})} dt$.
7. Let f be a function on $[0, \infty)$, differentiable and satisfying
$$f'(x) = -3f(x) + 6f(2x)$$
for $x > 0$. Assume that $|f(x)| \le e^{-\sqrt{x}}$ for $x \ge 0$ (so that $f(x)$ tends rapidly to ∞ as x increases). For n a nonnegative integer, define
$$\mu_n = \int_0^\infty x^n f(x) dx$$
(the nth moment of f).
   (a) Express μ_n in terms of μ_0.
   (b) Prove that the sequence $\{\mu_n \cdot 3^n/n!\}$ always converges, and that the limit is 0 only if $\mu_0 = 0$.
8. Suppose f and g are non-constant, differentiable, real-valued functions defined on $(-\infty, \infty)$. Furthermore, suppose that for each pair of real numbers x and y,
$$\begin{aligned} f(x+y) &= f(x)f(y) - g(x)g(y), \\ g(x+y) &= f(x)g(y) + g(x)f(y). \end{aligned}$$
If $f'(0) = 0$, prove that $(f(x))^2 + (g(x))^2 = 1$ for all x.
9. Let a and b be positive numbers. Find the largest number c, in terms of a and b, such that
$$a^x b^{1-x} \le a \frac{\sinh ux}{\sinh u} + b \frac{\sinh u(1-x)}{\sinh u}$$
for all u with $0 < |u| \le c$ and for all x, $0 < x < 1$. (Note: $\sinh u = (e^u - e^{-u})/2$.)

10. The function $K(x,y)$ is positive and continuous for $0 \le x \le 1, 0 \le y \le 1$, and the functions $f(x)$ and $g(x)$ are positive and continuous for $0 \le x \le 1$. Suppose that for all x, $0 \le x \le 1$,

$$\int_0^1 f(y)K(x,y)\,dy = g(x)$$

and

$$\int_0^1 g(y)K(x,y)\,dy = f(x).$$

Show that $f(x) = g(x)$ for $0 \le x \le 1$.

11. Evaluate

$$\int_0^\infty \left(x - \frac{x^3}{2} + \frac{x^5}{2\cdot 4} - \frac{x^7}{2\cdot 4\cdot 6} + \cdots\right)\left(1 + \frac{x^2}{2^2} + \frac{x^4}{2^2\cdot 4^2} + \frac{x^6}{2^2\cdot 4^2\cdot 6^2} + \cdots\right) dx.$$

12. Let f be a twice-differentiable real-valued function satisfying

$$f(x) + f''(x) = -xg(x)f'(x),$$

where $g(x) \ge 0$ for all real x. Prove that $|f(x)|$ is bounded.

13. Prove that there is a constant C such that, if $p(x)$ is a polynomial of degree 1999, then

$$|p(0)| \le C\int_{-1}^1 |p(x)|\,dx.$$

14. Find a real number c and a positive number L for which

$$\lim_{r\to\infty} \frac{r^c \int_0^{\pi/2} x^r \sin x dx}{\int_0^{\pi/2} x^r \cos x dx} = L.$$

15. Let $(a_1, b_1), (a_2, b_2), \ldots, (a_n, b_n)$ be the vertices of a convex polygon which contains the origin in its interior. Prove that there exist positive real numbers x and y such that

$$(a_1, b_1)x^{a_1}y^{b_1} + (a_2, b_2)x^{a_2}y^{b_2} + \cdots + (a_n, b_n)x^{a_n}y^{b_n} = (0, 0).$$

16. Show that all solutions of the differential equation $y'' + e^x y = 0$ remain bounded as $x \to \infty$.

17. Let f be a real-valued function having partial derivatives and which is defined for $x^2 + y^2 \le 1$ and is such that $|f(x,y)| \le 1$. Show that there exists a point (x_0, y_0) in the interior of the unit circle for which

$$\left(\frac{\partial f}{\partial x}(x_0, y_0)\right)^2 + \left(\frac{\partial f}{\partial y}(x_0, y_0)\right)^2 \le 16.$$

18. (a) On $[0,1]$, let f have a continuous derivative satisfying $0 < f'(x) \le 1$. Also, suppose that $f(0) = 0$. Prove that

$$\left(\int_0^1 f(x)dx\right)^2 \ge \int_0^1 f(x)^3 dx.$$

(b) Find an example where equality occurs.

19. Let $P(t)$ be a nonconstant polynomial with real coefficients. Prove that the system of simultaneous equations

$$0 = \int_0^x P(t) \sin t dt = \int_0^x P(t) \cos t dt$$

has only finitely many real solutions x.

20. Let C be the class of all real valued continuously differentiable functions f on the interval $0 \leq x \leq 1$ with $f(0) = 0$ and $f(1) = 1$. Determine the largest real number u such that

$$u \leq \int_0^1 |f'(x) - f(x)| dx$$

for all $f \in C$.

21. Given a convergent series $\sum a_n$ of positive terms, prove that the series $\sum \sqrt[n]{a_1 a_2 \cdots a_n}$ must also be convergent.

22. Given that $f(x) + f'(x) \to 0$ as $x \to \infty$, prove that both $f(x) \to 0$ and $f'(x) \to 0$.

23. Suppose that $f''(x)$ is continuous on $\mathbb{R}$, and that $|f(x)| \leq a$ on $\mathbb{R}$, and $|f''(x)| \leq b$ on $\mathbb{R}$. Find the best possible bound $|f'(x)| \leq c$ on $\mathbb{R}$.

24. Let f be a real function with a continuous third derivative such that $f(x), f'(x), f''(x), f'''(x)$ are positive for all x. Suppose that $f'''(x) \leq f(x)$ for all x. Show that $f'(x) < 2f(x)$ for all x. (Note that we cannot replace 2 by 1 because of the function $f(x) = e^x$.)

25. Show that the improper integral

$$\lim_{B \to \infty} \int_0^B \sin(x) \, \sin(x^2) \, dx$$

converges.

26. Fix an integer $b \geq 2$. Let $f(1) = 1$, $f(2) = 2$, and for each $n \geq 3$, define $f(n) = nf(d)$, where d is the number of base-b digits of n. For which values of b does

$$\sum_{n=1}^{\infty} \frac{1}{f(n)}$$

converge?

27. Evaluate

$$\lim_{x \to 1^-} \prod_{n=0}^{\infty} \left(\frac{1 + x^{n+1}}{1 + x^n} \right)^{x^n}.$$

28. Find all differentiable functions $f : (0, \infty) \to (0, \infty)$ for which there is a positive real number a such that

$$f'\left(\frac{a}{x}\right) = \frac{x}{f(x)}$$

for all $x > 0$.

29. Let k be an integer greater than 1. Suppose $a_0 > 0$, and define

$$a_{n+1} = a_n + \frac{1}{\sqrt[k]{a_n}}$$

for $n > 0$. Evaluate

$$\lim_{n\to\infty} \frac{a_n^{k+1}}{n^k}.$$

30. Define $f\colon \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} x & \text{if } x \le e \\ x f(\ln x) & \text{if } x > e. \end{cases}$$

Does $\sum_{n=1}^{\infty} \frac{1}{f(n)}$ converge?

31. Find all continuously differentiable functions $f : \mathbb{R} \to \mathbb{R}$ such that for every rational number q, the number $f(q)$ is rational and has the same denominator as q. (The denominator of a rational number q is the unique positive integer b such that $q = a/b$ for some integer a with $\gcd(a, b) = 1$.) (Note: gcd means greatest common divisor.)

32. Functions f, g, h are differentiable on some open interval around 0 and satisfy the equations and initial conditions

$$f' = 2f^2gh + \frac{1}{gh}, \quad f(0) = 1,$$
$$g' = fg^2h + \frac{4}{fh}, \quad g(0) = 1,$$
$$h' = 3fgh^2 + \frac{1}{fg}, \quad h(0) = 1.$$

Find an explicit formula for $f(x)$, valid in some open interval around 0.

33. Let $f : [0, 1]^2 \to \mathbb{R}$ be a continuous function on the closed unit square such that $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ exist and are continuous on the interior $(0, 1)^2$. Let $a = \int_0^1 f(0, y)\, dy$, $b = \int_0^1 f(1, y)\, dy$, $c = \int_0^1 f(x, 0)\, dx$, $d = \int_0^1 f(x, 1)\, dx$. Prove or disprove: There must be a point (x_0, y_0) in $(0, 1)^2$ such that

$$\frac{\partial f}{\partial x}(x_0, y_0) = b - a \quad \text{and} \quad \frac{\partial f}{\partial y}(x_0, y_0) = d - c.$$

34. Let $f : (1, \infty) \to \mathbb{R}$ be a differentiable function such that

$$f'(x) = \frac{x^2 - (f(x))^2}{x^2((f(x))^2 + 1)} \qquad \text{for all } x > 1.$$

Prove that $\lim_{x\to\infty} f(x) = \infty$.

35. Find all differentiable functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f'(x) = \frac{f(x + n) - f(x)}{n}$$

for all real numbers x and all positive integers n.

36. Suppose that the function $h : \mathbb{R}^2 \to \mathbb{R}$ has continuous partial derivatives and satisfies the equation

$$h(x, y) = a\frac{\partial h}{\partial x}(x, y) + b\frac{\partial h}{\partial y}(x, y)$$

for some constants a, b. Prove that if there is a constant M such that $|h(x, y)| \le M$ for all $(x, y) \in \mathbb{R}^2$, then h is identically zero.

37. Let $f : [0, \infty) \to \mathbb{R}$ be a strictly decreasing continuous function such that $\lim_{x\to\infty} f(x) = 0$. Prove that $\int_0^\infty \frac{f(x)-f(x+1)}{f(x)}\, dx$ diverges.

38. Is there a strictly increasing function $f : \mathbb{R} \to \mathbb{R}$ such that $f'(x) = f(f(x))$ for all x?

PROBLEMS ON RECURRENCES

1. Let $T_0 = 2, T_1 = 3, T_2 = 6$, and for $n \geq 3$,

$$T_n = (n+4)T_{n-1} - 4nT_{n-2} + (4n-8)T_{n-3}.$$

The first few terms are: 2, 3, 6, 14, 40, 152, 784, 5168, 40576, 363392. Find, with proof, a formula for T_n of the form $T_n = A_n + B_n$, where $\{A_n\}$ and $\{B_n\}$ are well-known sequences.

2. For which real numbers a does the sequence defined by the initial condition $u_0 = a$ and the recursion $u_{n+1} = 2u_n - n^2$ have $u_n > 0$ for all $n \geq 0$? (Express the answer in simplest form.)

3. Prove or disprove that there exists a positive real number u such that $[u^n] - n$ is an even integer for all positive integers n. (Here, $[x]$ is the greatest integer $\leq x$.)

4. Define u_n by $u_0 = 0, u_1 = 4$, and $u_{n+2} = \frac{6}{5}u_{n+1} - u_n$. Show that $|u_n| \leq 5$ for all n. (In fact, $|u_n| < 5$ for all n. Can you show this?)

5. Show that the next integer above $(\sqrt{3}+1)^{2n}$ is divisible by 2^{n+1}.

6. Let $a_0 = 0, a_1 = 1$, and for $n \geq 2$ let $a_n = 17a_{n-1} - 70a_{n-2}$. For $n > 6$, show that the first (most significant) digit of a_n (when written in base 10) is a 3.

7. Let a, b, c denote the (real) roots of the polynomial $P(t) = t^3 - 3t^2 - t + 1$. If $u_n = a^n + b^n + c^n$, what linear recursion is satisfied by $\{u_n\}$? If a is the largest of the three roots, what is the closest integer to a^5?

8. Solve the first order recursion given by $x_0 = 1$ and $x_n = 1 + (1/x_{n-1})$. Does $\{x_n\}$ approach a limiting value as n increases?

9. If $u_0 = 0, u_1 = 1$, and $u_{n+2} = 4(u_{n+1} - u_n)$, find u_{16}.

10. Let $a_0 = 1$, $a_1 = 2$, and $a_n = 4a_{n-1} - a_{n-2}$ for $n \geq 2$. Find an odd prime factor of a_{2015}.

11. Let $a_0 = 5/2$ and $a_k = a_{k-1}^2 - 2$ for $k \geq 1$. Compute

$$\prod_{i=0}^{\infty} \left(1 - \frac{1}{a_k}\right)$$

in closed form.

12. (a) Define $u_0 = 1, u_1 = 1$, and for $n \geq 1$,

$$2u_{n+1} = \sum_{k=0}^{n} \binom{n}{k} u_k u_{n-k}.$$

Find a simple expression for $F(x) = \sum_{n\geq 0} u_n \frac{x^n}{n!}$. Express your answer in the form $G(x) + H(x)$, where $G(x)$ is even (i.e., $G(-x) = G(x)$) and $H(x)$ is odd (i.e., $H(-x) = -H(x)$).

(b) Define $u_0 = 1$ and for $n \geq 0$,

$$2u_{n+1} = \sum_{k=0}^{n} \binom{n}{k} u_k u_{n-k}.$$

Find a simple expression for u_n.

13. For a positive integer n and any real number c , define x_k recursively by $x_0 = 0$, $x_1 = 1$, and for $k \geq 0$,

$$x_{k+2} = \frac{cx_{k+1} - (n-k)x_k}{k+1}.$$

Fix n and then take c to be the largest value for which $x_{n+1} = 0$. Find x_k in terms of n and k, $1 \leq k \leq n$.

14. Let $f(x)$ be a polynomial with integer coefficients. Define a sequence $a_0, a_1, \ldots$ of integers such that $a_0 = 0$ and $a_{n+1} = f(a_n)$ for all $n \geq 0$. Prove that if there exists a positive integer m for which $a_m = 0$ then either $a_1 = 0$ or $a_2 = 0$.

15. Define a sequence by $a_0 = 1$, together with the rules $a_{2n+1} = a_n$ and $a_{2n+2} = a_n + a_{n+1}$ for each integer $n \geq 0$. Prove that every positive rational number appears in the set

$$\left\{ \frac{a_{n-1}}{a_n} : n \geq 1 \right\} = \left\{ \frac{1}{1}, \frac{1}{2}, \frac{2}{1}, \frac{1}{3}, \frac{3}{2}, \cdots \right\}.$$

16. Let $1, 2, 3, \ldots, 2005, 2006, 2007, 2009, 2012, 2016, \ldots$ be a sequence defined by $x_k = k$ for $k = 1, 2, \ldots, 2006$ and $x_{k+1} = x_k + x_{k-2005}$ for $k \geq 2006$. Show that the sequence has 2005 consecutive terms each divisible by 2006.

17. Let $a_1 < a_2$ be two given integers. For any integer $n \geq 3$, let a_n be the smallest integer which is larger than a_{n-1} and can be uniquely represented as $a_i + a_j$, where $1 \leq i < j \leq n-1$. Given that there are only a finite number of even numbers in $\{a_n\}$, prove that the sequence $\{a_{n+1} - a_n\}$ is eventually periodic, i.e. that there exist positive integers T, N such that for all integers $n > N$, we have

$$a_{T+n+1} - a_{T+n} = a_{n+1} - a_n.$$

18. Let k be an integer greater than 1. Suppose that $a_0 > 0$, and define

$$a_{n+1} = a_n + \frac{1}{\sqrt[k]{a_n}}$$

for $n > 0$. Evaluate

$$\lim_{n\to\infty} \frac{a_n^{k+1}}{n^k}.$$

19. Let $x_0 = 1$ and for $n \geq 0$, let $x_{n+1} = 3x_n + \lfloor x_n\sqrt{5} \rfloor$. In particular, $x_1 = 5$, $x_2 = 26$, $x_3 = 136$, $x_4 = 712$. Find a closed-form expression for x_{2007}. ($\lfloor a \rfloor$ means the largest integer $\leq a$.)

20. (a) Let $a_0, \dots, a_{k-1}$ be real numbers, and define

$$a_n = \frac{1}{k}(a_{n-1} + a_{n-2} + \cdots + a_{n-k}), \quad n \geq k.$$

Find $\lim_{n\to\infty} a_n$ (in terms of $a_0, a_1, \dots, a_{k-1}$).

(b) Somewhat more generally, let $u_1, \dots, u_k \geq 0$ with $\sum u_i = 1$ and $u_k \neq 0$. Assume that the polynomial $x^k - u_1x^{k-1} - u_2x^{k-2} - \cdots - u_k$ cannot be written in the form $P(x^d)$ for some polynomial P and some $d > 1$. Now define

$$a_n = u_1a_{n-1} + u_2a_{n-2} + \cdots + u_ka_{n-k}, \quad n \geq k.$$

Again find $\lim_{n\to\infty} a_n$. (Part (a) is the case $u_1 = \cdots = u_k = 1/k$.)

21. (a) (repeats Congruence and Divisibility Problem #22) Define u_n recursively by $u_0 = u_1 = u_2 = u_3 = 1$ and

$$u_nu_{n-4} = u_{n-1}u_{n-3} + u_{n-2}^2, \quad n \geq 4.$$

Show that u_n is an integer.

(b) Do the same for $u_0 = u_1 = u_2 = u_3 = u_4 = 1$ and

$$u_nu_{n-5} = u_{n-1}u_{n-4} + u_{n-2}u_{n-3}, \quad n \geq 5.$$

(c) (much harder) Do the same for $u_0 = u_1 = u_2 = u_3 = u_4 = u_5 = 1$ and

$$u_nu_{n-6} = u_{n-1}u_{n-5} + u_{n-2}u_{n-4} + u_{n-3}^2, \quad n \geq 6,$$

and for $u_0 = u_1 = u_2 = u_3 = u_4 = u_5 = u_6 = 1$ and

$$u_nu_{n-7} = u_{n-1}u_{n-6} + u_{n-2}u_{n-5} + u_{n-3}u_{n-4}, \quad n \geq 7.$$

(d) What about $u_0 = u_1 = u_2 = u_3 = u_4 = u_5 = u_6 = u_7 = 1$ and

$$u_nu_{n-8} = u_{n-1}u_{n-7} + u_{n-2}u_{n-6} + u_{n-3}u_{n-5} + u_{n-4}^2, \quad n \geq 8?$$

22. (*very* difficult) Let $a_0, a_1, \dots$ satisfy a homogeneous linear recurrence (of finite degree) with constant coefficients. I.e., for some complex (or real, if you prefer) numbers $\nu_1, \dots, \nu_k$ we have

$$a_n = \nu_1a_{n-1} + \cdots + \nu_ka_{n-k}$$

for all $n \geq k$. Define

$$b_n = \begin{cases} 1, & a_n \neq 0 \\ 0, & a_n = 0. \end{cases}$$

Show that b_n is eventually periodic, i.e., there exists $p > 0$ such that $b_n = b_{n+p}$ for all n sufficiently large.

PROBLEMS ON INEQUALITIES

1. For $p > 1$ and $a_1, a_2, \ldots, a_n$ positive, show that

$$\sum_{k=1}^{n} \left(\frac{a_1 + a_2 + \cdots + a_k}{k} \right)^p < \left(\frac{p}{p-1} \right)^p \sum_{k=1}^{n} a_k^p.$$

2. If $a_n > 0$ for $n = 1, 2, \ldots$, show that

$$\sum_{n=1}^{\infty} \sqrt[n]{a_1 a_2 \cdots a_n} \leq e \sum_{n=1}^{\infty} a_n,$$

provided that $\sum_{n=1}^{\infty} a_n$ converges.

3. For $n = 1, 2, 3, \ldots$ let

$$x_n = \frac{1000^n}{n!}.$$

Find the largest term of the sequence.

4. Suppose that $a_1, a_2, \ldots, a_n$ with $n \geq 2$ are real numbers greater than -1, and all the numbers a_j have the same sign. Show that

$$(1 + a_1)(1 + a_2) \cdots (1 + a_n) > 1 + a_1 + a_2 + \cdots + a_n.$$

5. If $a_1, \ldots, a_{n+1}$ are positive real numbers with $a_1 = a_{n+1}$, show that

$$\sum_{i=1}^{n} \left(\frac{a_i}{a_{i+1}} \right)^n \geq \sum_{i=1}^{n} \frac{a_{i+1}}{a_i}.$$

6. Show that for any real numbers $a_1, a_2, \ldots, a_n$,

$$\left(\sum_{i=1}^{n} \frac{a_i}{i} \right)^2 \leq \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{a_i a_j}{i + j - 1}.$$

7. Let $y = f(x)$ be a continuous, strictly increasing function of x for $x \geq 0$, with $f(0) = 0$, and let f^{-1} denote the inverse function to f. If a and b are nonnegative constants, then show that

$$ab \leq \int_0^a f(x)dx + \int_0^b f^{-1}(y)dy.$$

8. Let $a_1, a_2, \ldots, a_n$ be real numbers. Show that

$$\min_{i<j} (a_i - a_j)^2 \leq M^2 \left(a_1^2 + \cdots + a_n^2 \right),$$

where

$$M^2 = \frac{12}{n(n^2 - 1)}.$$

9. Let f be a continuous function on the interval $[0, 1]$ such that $0 < m \leq f(x) \leq M$ for all x in $[0, 1]$. Show that

$$\left(\int_0^1 \frac{dx}{f(x)} \right) \left(\int_0^1 f(x)dx \right) \leq \frac{(m + M)^2}{4mM}.$$

8. Let $a_1, a_2, \ldots, a_n$ be real numbers. Show that

$$\min_{i<j}(a_i - a_j)^2 \leq M^2 \left(a_1^2 + \cdots + a_n^2\right),$$

where

$$M^2 = \frac{12}{n(n^2-1)}.$$

9. Let f be a continuous function on the interval $[0,1]$ such that $0 < m \leq f(x) \leq M$ for all x in $[0,1]$. Show that

$$\left(\int_0^1 \frac{dx}{f(x)}\right)\left(\int_0^1 f(x)dx\right) \leq \frac{(m+M)^2}{4mM}.$$

10. Consider any sequence $a_1, a_2, \ldots$ of real numbers. Show that

$$\sum_{n=1}^{\infty} a_n \leq \frac{2}{\sqrt{3}} \sum_{n=1}^{\infty} \left(\frac{r_n}{n}\right)^{1/2}$$

where

$$r_n = \sum_{k=n}^{\infty} a_k^2.$$

(If the left-hand side of the inequality is ∞, then so is the right-hand side.)

11. Show that

$$\frac{1}{(n-1)!}\int_n^{\infty} w(t)e^{-t}dt < \frac{1}{(e-1)^n},$$

where t is real, n is a positive integer, and

$$w(t) = (t-1)(t-2)\cdots(t-n+1).$$

12. Let $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ be two sequences of positive numbers. Show that the following statements are equivalent:

   - There is a sequence $(c_n)_{n=1}^{\infty}$ of positive numbers such that $\sum_{n=1}^{\infty} \frac{a_n}{c_n}$ and $\sum_{n=1}^{\infty} \frac{c_n}{b_n}$ both converge.
   - $\sum_{n=1}^{\infty} \sqrt{\frac{a_n}{b_n}}$

13. Suppose that a, b, c are real numbers in the interval $[-1,1]$ such that $1 + 2abc \geq a^2 + b^2 + c^2$. Prove that $1 + 2(abc)^n \geq a^{2n} + b^{2n} + c^{2n}$ for all positive integers n.

14. Suppose $f : \mathbb{R} \to \mathbb{R}$ is a two times differentiable function satisfying $f(0) = 1, f'(0) = 0$ and for all $x \in [0,\infty)$, it satisfies

$$f''(x) - 5f'(x) + 6f(x) \geq 0$$

Prove that, for all $x \in [0,\infty)$,

$$f(x) \geq 3e^{2x} - 2e^{3x}$$

15. Let $f : [0,1] \to \mathbb{R}$ be a continuous function satisfying $xf(y) + yf(x) \leq 1$ for every $x, y \in [0,1]$.
(a) Show that $\int_0^1 f(x)dx \leq \frac{\pi}{4}$.
(b) Find such a function for which equality occurs.

16. For what pairs of positive real numbers (a, b) does the improper integral shown converge?

$$\int_b^\infty \left(\sqrt{\sqrt{x+a} - \sqrt{x}} - \sqrt{\sqrt{x} - \sqrt{x-b}} \right) dx$$

17. Let A be a positive real number. What are the possible values of $\sum_{j=0}^{\infty} x_j^2$, given that $x_0, x_1, \cdots$ are positive numbers for which $\sum_{j=0}^{\infty} x_j = A$?

18. Let $f(x)$ be a continuous real-valued function de?ned on the interval $[0, 1]$. Show that

$$\int_0^1 \int_0^1 |f(x) + f(y)| dx\, dy \geq \int_0^1 |f(x)| dx$$

19. For each continuous function $f : [0,1] \to \mathbb{R}$, let $I(f) = \int_0^1 x^2 f(x)\, dx$ and $J(f) = \int_0^1 x\,(f(x))^2\, dx$. Find the maximum value of $I(f) - J(f)$ over all such functions f.

20. Suppose that $f : [0,1] \to \mathbb{R}$ has a continuous derivative and that $\int_0^1 f(x)\, dx = 0$. Prove that for every $\alpha \in (0,1)$,

$$\left| \int_0^\alpha f(x)\, dx \right| \leq \frac{1}{8} \max_{0 \leq x \leq 1} |f'(x)|$$

21. For $m \geq 3$, a list of $\binom{m}{3}$ real numbers a_{ijk} $(1 \leq i < j < k \leq m)$ is said to be area definite for $\mathbb{R}^n$ if the inequality

$$\sum_{1 \leq i < j < k \leq m} a_{ijk} \cdot \text{Area}(\triangle A_i A_j A_k) \geq 0$$

holds for every choice of m points $A_1, \ldots, A_m$ in $\mathbb{R}^n$. For example, the list of four numbers $a_{123} = a_{124} = a_{134} = 1, a_{234} = -1$ is area definite for $\mathbb{R}^2$. Prove that if a list of $\binom{m}{3}$ numbers is area definite for $\mathbb{R}^2$, then it is area definite for $\mathbb{R}^3$.

22. Let $X_1, X_2, ..$ be independent random variables with the same distribution, and let $S_n = X_1 + X_2 + ... + X_n, n = 1, 2, ...$ For what real numbers c is the following statement true:

$$\mathbb{P}\left(\left| \frac{S_{2n}}{2n} - c \right| \leq \left| \frac{S_n}{n} - c \right| \right) \geq \frac{1}{2}.$$

PROBLEMS ON PROBABILITY

1. Three closed boxes lie on a table. One box (you don't know which) contains a \$1000 bill. The others are empty. After paying an entry fee, you play the following game with the owner of the boxes: you point to a box but do not open it; the owner then opens one of the two remaining boxes and shows you that it is empty; you may now open either the box you first pointed to or else the other unopened box, but not both. If you find the \$1000, you get to keep it. Does it make any difference which box you choose? What is a fair entry fee for this game?

2. You are dealt two cards face down from a shuffled deck of 8 cards consisting of the four queens and four kings from a standard bridge deck. The dealer looks at both of your two cards (without showing them to you) and tells you (truthfully) that at least one card is a queen. What is the probability that you have been given two queens? What is this probability if the dealer tells you instead that at least one card is a red queen? What is this probability if the dealer tells you instead that at least one card (or exactly one card) is the queen of hearts?

3. An unfair coin (probability p of showing heads) is tossed n times. What is the probability that the number of heads will be even?

4. Two persons agreed to meet in a definite place between noon and one o'clock. If either person arrives while the other is not present, he or she will wait for up to 15 minutes. Calculate the probability that the meeting will occur, assuming that the arrival times are independent and uniformly distributed between noon and one o'clock.

5. Real numbers are chosen at random from the interval $[0, 1]$. If after choosing the nth number the sum of the numbers so chosen first exceeds 1, show that the expected or average value for n is e.

6. Let α and β be given positive real numbers with $\alpha < \beta$. If two points are selected at random from a straight line segment of length β, what is the probability that the distance between them is at least α?

7. Two real numbers x and y are chosen at random in the interval $(0, 1)$ with respect to the uniform distribution. What is the probability that the closest integer to x/y is even? Express the answer in the form $r + s\pi$, where r and s are rational numbers.

8. Four points are chosen at random on the surface of a sphere. What is the probability that the center of the sphere lies inside the tetrahedron whose vertices are at the four points? (It is understood that each point is independently chosen relative to a uniform distribution on the sphere.)

9. Four points are chosen uniformly and independently at random in the interior of a given circle. Find the probability that they are the vertices of a convex quadrilateral.

10. Let k be a positive integer. Suppose that the integers $1, 2, 3, \ldots, 3k+1$ are written down in random order. What is the probability that at no time during this process, the sum of the integers that have been written up to that time is a positive integer divisible by 3? Your answer should be in closed form, but may include factorials.

11. Let $(x_1, x_2, \ldots, x_n)$ be a point chosen at random from the n-dimensional region defined by $0 < x_1 < x_2 < \cdots < x_n < 1$. Let f be a continuous function on $[0, 1]$ with $f(1) = 0$. Set $x_0 = 0$ and $x_{n+1} = 1$. Show that the expected value of the Riemann sum

$$\sum_{i=0}^{n} (x_{i+1} - x_i) f(x_{i+1})$$

is $\int_0^1 f(t)\, P(t)\, dt$, where P is a polynomial of degree n, independent of f, with $0 \leq P(t) \leq 1$ for $0 \leq t \leq 1$.

12. Choose n points $x_1, \ldots, x_n$ at random from the unit interval $[0, 1]$. Let p_n be the probability that $x_i + x_{i+1} \leq 1$ for all $1 \leq i \leq n-1$. Find a simple expression for $\sum_{n \geq 0} p_n x^n = 1 + x + \frac{1}{2}x^2 + \frac{1}{3}x^3 + \cdots$.

13. A dart, thrown at random, hits a square target. Assuming any two parts of the target of equal area are equally likely to be hit, find the probability that the point hit is nearer to the center than to any edge. Express your answer in the form $(a\sqrt{b} + c)/d$, where a, b, c, d are integers.

14. If α is an irrational number, $0 < \alpha < 1$, is there a finite game with an honest coin such that the probability of one player winning the game is α? (An honest coin is one for which the probability of heads and the probability of tails are both $1/2$. A game is finite if, with probability 1, it must end in a finite number of moves.)

15. Let C be the unit circle $x^2 + y^2 = 1$. A point p is chosen randomly on the circumference C and another point q is chosen randomly from the interior of C (these points are chosen independently and uniformly over their domains). Let R be the rectangle with sides parallel to the x- and y-axes with diagonal pq. What is the probability that no point of R lies outside of C?

16. Let p_n be the probability that $c + d$ is a perfect square when the integers c and d are selected independently at random from the set $\{1, 2, \ldots, n\}$. Show that $\lim_{n\to\infty}(p_n\sqrt{n})$ exists, and express this limit in the form $r(\sqrt{s} - t)$ where s and t are integers and r is a rational number.

17. The points $1, 2, \ldots, 1000$ are paired up at random to form 500 intervals $[i, j]$. What is the probability that among these intervals is one which intersects all the others?

18. The temperatures in Chicago and Detroit are x° and y°, respectively. These temperatures are not assumed to be independent; namely, we are given:

(i) $P(x^\circ = 70^\circ)$, the probability that the temperature in Chicago is 70°,

(ii) $P(y^\circ = 70^\circ)$, and

(iii) $P(\max(x^\circ, y^\circ) = 70^\circ)$.

Determine $P(\min(x^\circ, y^\circ) = 70^\circ)$.

19. In the Massachusetts MEGABUCKS lottery, six distinct integers from 1 to 36 are selected each week. Great care is exercised to insure that the selection is completely random. If $N_{\max}$ denotes the largest of the six numbers, find the expected value for $N_{\max}$.

20. (a) A fair die is tossed repeatedly. Let p_n be the probability that after some number of tosses the sum of the numbers that have appeared is n. (For instance, $p_1 = 1/6$ and $p_2 = 7/36$.) Find $\lim_{n\to\infty} p_n$.

    (b) More generally, suppose that a "die" has infinitely many faces, marked $1, 2, \ldots$. When the die is thrown, the probability is a_i that face i appears (so $\sum_{i=1}^{\infty} a_i = 1$). Let p_n be as in (a), and find $\lim_{n\to\infty} p_n$. Assume that there does not exist $k > 1$ such that if $a_i \neq 0$, then $k|i$ (otherwise it is easy to see that $\lim p_n$ doesn't exist).

21. Suppose that each of n people write down the numbers $1, 2, 3$ in random order in one column of a $3 \times n$ matrix, with all orders equally likely and with the orders for different columns independent of each other. Let the row sums a, b, c of the resulting matrix by rearranged (if necessary) so that $a \leq b \leq c$. Show that for some $n \geq 1995$, it is at least four times as likely that both $b = a + 1$ and $c = a + 2$ as that $a = b = c$.

22. At time $t = 1$ choose two numbers x_1, y_1 uniformly and independently from $[0, 1]$. At time $t = 2$ choose two further numbers x_2, y_2, etc. What is the expected time n at which $\sum_{i=1}^{n}(x_i^2 + y_i^2) > 1$ for the first time?

    Note. It may seem more natural to choose just *one* number at a time, but then the answer is not as elegant.

23. A fair coin is flipped until the number of heads exceeds the number of tails. What is the expected number of flips?

PROBLEMS ON PROBABILITY GAMES

24. An integer n, unknown to you, has been randomly chosen in the interval $[1, 2002]$ with uniform probability. Your objective is to select n in an **odd** number of guesses. After each incorrect guess, you are informed whether n is higher of lower, and you **must** guess an integer on your next turn among the numbers that are still feasibly correct. Show that you have a strategy so that the chance of winning is greater than $2/3$.

25. A deck of cards (with 26 red cards and 26 black cards) is shuffled, and the cards are turned face up one at a time. At any point during this process before the last card is

turned up, you can stay "stop." If the next card is red, you win \$1; if it is black, you win nothing. What is your best strategy? In particular, is there a strategy which gives you an expectation of better than 50 cents?

26. In the previous problem suppose that you start with \$1 and after each card is shown you can bet (at even odds) on any outcome you choose (red or black) an amount equal to any fraction of your current worth. You can certainly guarantee that you end up with \$2 — just wait until one card remains before you bet. Can you *guarantee* that you will end up with more than \$2? If so, what is the maximum amount you can be sure of winning?

27. Alice takes two slips of paper and writes a different integer on each. Bob then chooses one of the slips and looks at the integer written on it. He can then keep this slip of paper or exchange it for the other slip. If he ends up with the larger integer, he wins. Is there a strategy for Bob which gives him a probability of more than 50% of winning?

28. Suppose in the previous problem that two real numbers in the interval $[0, 1]$ are chosen uniformly at random. Alice looks at the two numbers and then decides which one to show Bob. Now if Alice chooses optimally can Bob do better than break even? What are the optimal strategies of Bob and Alice?

PROBLEMS ON LINEAR ALGEBRA

1 Basic Linear Algebra

1. Let M_n be the $(2n+1) \times (2n+1)$ for which

$$(M_n)_{ij} = \begin{cases} 0, & i = j \\ 1, & i - j \equiv 1, \dots, n \pmod{2n+1} \\ -1, & i - j \equiv n+1, \dots, 2n \pmod{2n+1}. \end{cases}$$

Find the rank of M_n.

2. Let a_{ij} $(i, j = 1, 2, 3)$ be real numbers such that $a_{ij} > 0$ for $i = j$, and $a_{ij} < 0$ for $i \neq j$. Prove that there exist positive real numbers c_1, c_2, c_3 such that the quantities $a_{i1}c_1 + a_{i2}c_2 + a_{i3}c_3$ for $i = 1, 2, 3$ are either all positive, all negative, or all zero.

3. Let x, y, z be positive real numbers, not all equal, and define

$$a = x^2 - yz, \qquad b = y^2 - zx, \qquad c = z^2 - xy.$$

Express x, y, z in terms of a, b, c. (Hint: can you find a linear algebra interpretation of a, b, c, by making a certain matrix involving x, y, z?)

4. If A and B are square matrices of the same size such that $ABAB = 0$, does it follow that $BABA = 0$?

5. Suppose that A, B, C, D are $n \times n$ matrices (with entries in some field), such that AB^T and CD^T are symmetric, and $AD^T - BC^T = I$. Prove that $A^T D - C^T B = I$. (Hint: find a more "matricial" interpretation of the condition $AD^T - BC^T = I$.)

6. Let $x_1, \dots, x_N$ be distinct unit vectors in $\mathbb{R}^n$ forming a regular simplex in projective space $\mathbb{RP}^{n-1}$, i.e. $|\langle x_i, x_j \rangle| = \alpha$ for some $0 \leq \alpha < 1$. Show that $N \leq n(n+1)/2$.

2 Determinants

7. Let D_n denote the value of the $(n-1) \times (n-1)$ determinant

$$\begin{vmatrix} 3 & 1 & 1 & 1 & \cdots & 1 \\ 1 & 4 & 1 & 1 & \cdots & 1 \\ 1 & 1 & 5 & 1 & \cdots & 1 \\ 1 & 1 & 1 & 6 & \cdots & 1 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & 1 & \cdots & n+1 \end{vmatrix}.$$

Is the set $\{D_n/n!\}_{n \geq 2}$ bounded?

8. Let p be a prime number. Prove that the determinant of the matrix

$$\begin{bmatrix} x & y & z \\ x^p & y^p & z^p \\ x^{p^2} & y^{p^2} & z^{p^2} \end{bmatrix}.$$

is congruent modulo p to a product of polynomials of the form $ax + by + cz$, where a, b, c are integers.

9. Let A be a $2n \times 2n$ skew-symmetric matrix (i.e., a matrix in which $A_{ij} = -A_{ji}$) with integer entries. Prove that the determinant of A is a perfect square. (Hint: prove a polynomial identity.)

10. Let x_i, $i = 1, \dots, n$ and y_j, $j = 1, \dots, n$ be $2n$ distinct real numbers. Calculate the determinant of the matrix whose (i, j) entry is $1/(x_i - y_j)$. Using this, show that the matrix whose entries are $1/(i + j - 1)$ is invertible and that its inverse has integer entries.

11. Let A be the $n \times n$ matrix with $A_{jk} = \cos(2\pi(j + k)/n)$. Find the determinant of $I + A$.

12. Let A be a $2n \times 2n$ matrix, with entries chosen independently at random. Every entry is chosen to be 0 or 1, each with probability 1/2. Find the expected value of $\det(A - A^t)$ (as a function of n), where A^t is the transpose of A.

3 Eigenvalues and Related Things

13. Let $x_1, x_2, \dots, x_n$ be differentiable (real-valued) functions of a single variable t which satisfy

$$\begin{aligned} \frac{dx_1}{dt} &= a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ \frac{dx_2}{dt} &= a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\ &\vdots \\ \frac{dx_n}{dt} &= a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n \end{aligned}$$

for some constants $a_{ij} > 0$. Suppose that for all i, $x_i(t) \to 0$ as $t \to \infty$. Are the functions $x_1, x_2, \dots, x_n$ necessarily linearly dependent?

14. Let G be a finite set of real $n \times n$ matrices $\{M_i\}$, $1 \le i \le r$, which form a group under matrix multiplication. Suppose that $\sum_{i=1}^{r} \mathsf{tr}(M_i) = 0$, where $\mathsf{tr}(A)$ denotes the trace of the matrix A. Prove that $\sum_{i=1}^{r} M_i$ is the $n \times n$ zero matrix.

15. Let A be an $n \times n$ real symmetric matrix and B an $n \times n$ positive definite matrix. (A square matrix over $\mathbb{R}$ is *positive definite* if it is symmetric and all its eigenvalues are positive.) Show that all eigenvalues of AB are real. Hint. Use the following two facts from linear algebra: (a) all eigenvalues of a real symmetric matrix are real, and (b) a positive definite matrix has a positive definite square root.

16. Let A be an $n \times n$ matrix of real numbers for some $n \ge 1$. For each positive integer k, let $A^{[k]}$ be the matrix obtained by raising each entry to the kth power. Show that if $A^k = A^{[k]}$ for $k = 1, 2, \cdots, n + 1$, then $A^k = A^{[k]}$ for all $k \ge 1$.

17. Let n be a positive integer. Suppose that A, B, and M are $n \times n$ matrices with real entries such that $AM = MB$, and such that A and B have the same characteristic polynomial. Prove that $\det(A - MX) = \det(B - XM)$ for every $n \times n$ matrix X with real entries.

4 Combinatorics

18. A mansion has n rooms. Each room has a lamp and a switch connected to its lamp. However, switches may also be connected to lamps in other rooms, subject to the following condition: if the switch in room a is connected to the lamp in room b, then the switch in room b is also connected to the lamp in room a. Each switch, when flipped, changes the state (from on to off or vice versa) of each lamp connected to it. Suppose at some points the lamps are all off. Prove that no matter how the switches are wired, it is possible to flip some of the switches to turn all of the lamps on. (Hint: interpret as a linear algebra problem over the field of two elements.)

19. Let n and k be positive integers. Say that a permutation σ of $\{1, 2, \ldots, n\}$ is *k-limited* if $|\sigma(i) - i| \leq k$ for all i. Prove that the number of k-limited permutations of $\{1, 2, \ldots, n\}$ is odd if and only if $n \equiv 0$ or $1 \pmod{2k+1}$.

20. Consider bipartite graphs where the vertices on each side are labeled $\{1, 2, \ldots, n\}$. Find the number of such graphs for which there are an odd number of perfect matchings.

21. Show that the edges of the complete graph on n vertices cannot be partitioned into fewer than $n - 1$ edge-disjoint complete bipartite subgraphs.

5 Miscellaneous

22. Let p be a prime, and let $A = (a_{ij})_{i,j=0}^{p-1}$ be the $p \times p$ matrix defined by

$$a_{ij} = \binom{i+j}{i}, \quad 0 \leq i, j \leq p-1.$$

Show that $A^3 \equiv I \pmod{p}$, where I denotes the identity matrix. In other words, every entry of $A^3 - I$, evaluated over $\mathbb{Z}$, is divisible by p.

23. Let A be an $n \times n$ real matrix with every row and column sum equal to 0. Let $A[i, j]$ denote A with row i and column j removed. Show that $(-1)^{i+j} \det A[i, j]$ is independent of i and j. Can you express this determinant in terms of the eigenvalues of A?

24. Find the unique sequence $a_0, a_1, \ldots$ of real numbers such that for all $n \geq 0$ we have

$$\det \begin{bmatrix} a_0 & a_1 & \cdots & a_n \\ a_1 & a_2 & \cdots & a_{n+1} \\ \vdots & \vdots & & \vdots \\ a_n & a_{n+1} & \cdots & a_{2n} \end{bmatrix} = \det \begin{bmatrix} a_1 & a_2 & \cdots & a_n \\ a_2 & a_3 & \cdots & a_{n+1} \\ \vdots & \vdots & & \vdots \\ a_n & a_{n+1} & \cdots & a_{2n-1} \end{bmatrix} = 1.$$

(When $n = 0$ the second matrix is empty and by convention has determinant one.)

25. Let $A = A(n)$ be the $n \times n$ real matrix given by

$$A_{ij} = \begin{cases} 1, & j = i+1 \ (1 \leq i \leq n-1) \\ 1, & j = i-1 \ (2 \leq i \leq n) \\ 0, & \text{otherwise.} \end{cases}$$

Let $V_n(x) = \det(xI - A)$, so $V_0(x) = 1$, $V_1(x) = x$, $V_2(x) = x^2 - 1$, $V_3(x) = x^3 - 2x$. Show that $V_{n+1}(x) = xV_n(x) - V_{n-1}(x)$, $n \geq 1$.

26. Show that
$$V_n(2\cos\theta) = \frac{\sin((n+1)\theta)}{\sin(\theta)}.$$
Deduce that the eigenvalues of $A(n)$ are $2\cos(j\pi/(n+1))$, $1 \leq j \leq n$.

27. Given $v = (v_1, \ldots, v_n)$ where each $v_i = 0$ or 1, let $f(v)$ be the number of even numbers among the n numbers
$$v_1 + v_2 + v_3, v_2 + v_3 + v_4, \ldots, v_{n-2} + v_{n-1} + v_n, v_{n-1} + v_n + v_1, v_n + v_1 + v_2.$$
For which positive integers n is the following true: for all $0 \leq k \leq n$, exactly $\binom{n}{k}$ vectors of the 2^n vectors $v \in \{0,1\}^n$ satisfy $f(v) = k$?

28. Let $M(n)$ denote the space of all real $n \times n$ matrices. Thus $M(n)$ is a real vector space of dimension n^2. Let $f(n)$ denote the maximum dimension of a subspace V of $M(n)$ such that every nonzero element of V is invertible.
    (a) (easy) Show that $f(n) \leq n$.
    (b) (fairly easy) Show that if n is odd, then $f(n) = 1$.
    (c) (extremely difficult) For what n does $f(n) = n$?
    (d) (even more difficult) Find a formula for $f(n)$ for all n.

29. (a) Let n be a positive integer. Prove that the matrix
$$\left(\frac{1}{(i+j)^2}\right)_{i,j=1}^{n}$$
is positive definite.
    (b) Let $0 < a < b$. Prove that for any continuous function $f\colon [a,b] \to \mathbb{R}$,
$$\int_a^b \int_a^b \frac{f(x)f(y)}{(x+y)^2}\, dxdy \geq 0.$$

Remark. Feel free to use any of equivalent characterizations of what is meant by saying "a symmetric matrix $A = (a_{ij})_{i,j=1}^n$ is positive definite":

- All its eigenvalues are positive.
- For any $x \in \mathbb{R}^n, x \neq 0$, we have $x^T Ax > 0$.
- (Sylvester's criterion) All its principal minors are positive. Principal minors are the values $\det(a_{ij})_{i,j\in I}$, for $I \subseteq \{1, \ldots, n\}$.

30. Define $A_0 = \begin{pmatrix} 0 \end{pmatrix}$, $A_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$,
$$A_{n+1} = \begin{pmatrix} A_n & I_{2^n} \\ I_{2^n} & A_n \end{pmatrix},$$
where I_m is the $m \times m$ identity matrix. Prove that A_n has $n+1$ distinct eigenvalues with multiplicities $\binom{n}{0}, \binom{n}{1}, \ldots, \binom{n}{n}$.

31. Let $f(z) = z^{n-1} + c_{n-2}z^{n-2} + \ldots + c_1 z + c_0$ be a polynomial with complex coefficients, such that $c_0 c_{n-2} = c_1 c_{n-3} = \ldots = 0$. Prove that $f(z)$ and $z^n - 1$ have at most $n - \sqrt{n}$ common roots.

PROBLEMS ON ABSTRACT ALGEBRA

1 (Putnam 1972 A2)**.** Let S be a set and let $*$ be a binary operation on S satisfying the laws

$$\begin{aligned} x*(x*y) &= y \qquad \text{for all } x,y \text{ in } S, \\ (y*x)*x &= y \qquad \text{for all } x,y \text{ in } S. \end{aligned}$$

Show that $*$ is commutative but not necessarily associative.

2 (Putnam 1972 B3)**.** Let A and B be two elements in a group such that $ABA = BA^2B$, $A^3 = 1$ and $B^{2n-1} = 1$ for some positive integer n. Prove $B = 1$.

3 (Putnam 2007 A5)**.** Suppose that a finite group has exactly n elements of order p, where p is a prime. Prove that either $n = 0$ or p divides $n + 1$.

4 (Putnam 2011 A6)**.** Let G be an abelian group with n elements, and let $\{g_1 = e, g_2, \ldots, g_k\} \subsetneq G$ be a (not necessarily minimal) set of distinct generators of G. A special die, which randomly selects one of the elements $g_1, g_2, \ldots, g_k$ with equal probability, is rolled m times and the selected elements are multiplied to produce an element $g \in G$. Prove that there exists a real number $b \in (0,1)$ such that

$$\lim_{m\to\infty} \frac{1}{b^{2m}} \sum_{x\in G} \left(\mathrm{Prob}(g = x) - \frac{1}{n}\right)^2$$

is positive and finite.

5 (Putnam 1990 B4)**.** Let G be a finite group of order n generated by a and b. Prove or disprove: there is a sequence

$$g_1, g_2, g_3, \ldots, g_{2n}$$

such that

(a) every element of G occurs exactly twice, and

(b) g_{i+1} equals $g_i a$ or $g_i b$ for $i = 1, 2, \ldots, 2n$. (Interpret g_{2n+1} as g_1.)

6 (Putnam 2016 A5)**.** Suppose that G is a finite group generated by the two elements g and h, where the order of g is odd. Show that every element of G can be written in the form

$$g^{m_1} h^{n_1} g^{m_2} h^{n_2} \cdots g^{m_r} h^{n_r}$$

with $1 \le r \le |G|$ and $m_n, n_1, m_2, n_2, \ldots, m_r, n_r \in \{1, -1\}$. (Here $|G|$ is the number of elements of G.)

7 (Putnam 1977 B6)**.** Let H be a subgroup with h elements in a group G. Suppose that G has an element a such that for all x in H, $(xa)^3 = 1$, the identity. In G, let P be the subset of all products $x_1 a x_2 a \cdots x_n a$, with n a positive integer and the x_i's in H.

(a) Show that P is a finite set.

(b) Show that, in fact, P has no more than $3h^2$ elements.

8 (Putnam 1984 B3)**.** Prove or disprove the following statement: If F is a finite set with two or more elements, then there exists a binary operation $*$ on F such that for all x, y, z in F,

(i) $x * z = y * z$ implies $x = y$ (right cancellation holds), and

(ii) $x * (y * z) \neq (x * y) * z$ (*no* case of associativity holds).

9 (Putnam 1987 B6). Let F be the field of p^2 elements where p is an odd prime. Suppose S is a set of $(p^2 - 1)/2$ distinct nonzero elements of F with the property that for each $a \neq 0$ in F, exactly one of a and $-a$ is in S. Let N be the number of elements in the intersection $S \cap \{2a : a \in S\}$. Prove that N is even.

10 (Putnam 1989 B2). Let S be a nonempty set with an associative operation that is left and right cancellative ($xy = xz$ implies $y = z$, and $yx = zx$ implies $y = z$). Assume that for every a in S the set $\{a^n : n = 1, 2, 3, \dots\}$ is finite. Must S be a group?

11 (Putnam 1992 B6). Let $\mathcal{M}$ be a set of real $n \times n$ matrices such that

(i) $I \in \mathcal{M}$, where I is the $n \times n$ identity matrix;

(ii) if $A \in \mathcal{M}$ and $B \in \mathcal{M}$, then either $AB \in \mathcal{M}$ or $-AB \in \mathcal{M}$, but not both;

(iii) if $A \in \mathcal{M}$ and $B \in \mathcal{M}$, then either $AB = BA$ or $AB = -BA$;

(iv) if $A \in \mathcal{M}$ and $A \notin I$, there is at least one $B \in \mathcal{M}$ such that $AB = -BA$.

Prove that $\mathcal{M}$ contains at most n^2 matrices.

12 (Putnam 1996 A4). Let S be a set of ordered triples (a, b, c) of distinct elements of a finite set A. Suppose that

(1) $(a, b, c) \in S$ if and only if $(b, c, a) \in S$;

(2) $(a, b, c) \in S$ if and only if $(c, b, a) \notin S$ [for a, b, c distinct];

(3) (a, b, c) and (c, d, a) are both in S if and only if (b, c, d) and (d, a, b) are both in S.

Prove that there exists a one-to-one function g from A to $\mathbb{R}$ such that $g(a) < g(b) < g(c)$ implies $(a, b, c) \in S$.

13 (Putnam 2008 A6). Prove that there exists a constant $c > 0$ such that in every nontrivial finite group G there exists a sequence of length at most $c \ln |G|$ with the property that each element of G equals the product of some subsequence. (The elements of G in the sequence are not required to be distinct. A *subsequence* of a sequence is obtained by selecting some of the terms, not necessarily consecutive, without reordering them; for example, $4, 4, 2$ is a subsequence of $2, 4, 6, 4, 2$, but $2, 2, 4$ is not.)

14 (Putnam 2009 A5). Is there a finite abelian group G such that the product of the orders of all its elements is 2^{2009}?

15 (Putname 2010 A5). Let G be a group, with operation $*$. Suppose that

1. G is a subset of $\mathbb{R}^3$ (but $*$ need not be related to addition of vectors);
2. For each $\mathbf{a}, \mathbf{b} \in G$, either $\mathbf{a} \times \mathbf{b} = \mathbf{a} * \mathbf{b}$ or $\mathbf{a} \times \mathbf{b} = \mathbf{0}$ (or both), where $\times$ is the usual cross product in $\mathbb{R}^3$.

Prove that $\mathbf{a} \times \mathbf{b} = \mathbf{0}$ for all $\mathbf{a}, \mathbf{b} \in G$.

16. Let R be a *noncommutative* ring with identity. Suppose that x, y are elements of R such that $1 - xy$ and $1 - yx$ are invertible. (By the previous problem it suffice to assume that only $1 - xy$ is invertible, but this is irrelevant.) Show that

$$(1 + x)(1 - yx)^{-1}(1 + y) = (1 + y)(1 - xy)^{-1}(1 + x). \tag{1}$$

This problem illustrates that "noncommutative high school algebra" is a lot harder than ordinary (commutative) high school algebra.

Note. Formally we have

$$(1 - yx)^{-1} = 1 + yx + yxyx + yxyxyx + \cdots$$

and similarly for $(1-xy)^{-1}$. Thus both sides of (1) are formally equal to the sum of all "alternating words" (products of x's and y's with no two x's or y's appearing consecutively). This makes the identity (1) plausible, but our formal argument is not a proof.

17. Let G be a group of order $4n + 2$, $n \geq 1$. Prove that G is not a simple group, i.e., G has a proper normal subgroup.

18. Let R satisfy all the axioms of a ring except commutativity of addition. Show that $ax + by = by + ax$ for all $a, b, x, y \in R$.

19. Let G denote the set of all infinite sequences $(a_1, a_2, \dots)$ of integers a_i. We can add elements of G coordinate-wise, i.e.,

$$(a_1, a_2, \dots) + (b_1, b_2, \dots) = (a_1 + b_1, a_2 + b_2, \dots).$$

Let $\mathbb{Z}$ denote the set of integers. Suppose $f\colon \to \mathbb{Z}$ is a function satisfying $f(x + y) = f(x) + f(y)$ for all $x, y \in G$. Let e_i be the element of G with a 1 in position i and 0's elsewhere.

(a) Suppose that $f(e_i) = 0$ for all i. Show that $f(x) = 0$ for all $x \in G$.

(b) Show that $f(e_i) = 0$ for all but finitely many i.

20. Let G be a finite group, and set $f(G) = \#\{(u, v) \in G \times G : uv = vu\}$. Find a formula for $f(G)$ in terms of the order of G and the number $k(G)$ of conjugacy classes of G. (Two elements $x, y \in G$ are *conjugate* if $y = axa^{-1}$ for some $a \in G$. Conjugacy is an equivalence relation whose equivalence classes are called *conjugacy classes*.)

21 (difficult)**.** Let n be an odd positive integer. Show that the number of ways to write the identity permutation ι of $1, 2, \dots, n$ as a product $uvw = \iota$ of three n-cycles is $2(n-1)!^2/(n+1)$.

22. Let G be any finite group, and let $w \in G$. Find the number of pairs $(u, v) \in G \times G$ satisfying $w = uvu^2vuv$.

23. Show that the number of ways to write the cycle $(1, 2, \dots, n)$ as a product of $n-1$ transpositions is n^{n-2}. For instance, when $n = 3$ we have (multiplying permutations left-to-right) three ways:

$$(1, 2, 3) = (1, 3)(2, 3) = (1, 2)(1, 3) = (2, 3)(1, 2).$$

24 (difficult)**.** Let $s_i = (i, i+1) \in S_n$, i.e., s_i is the permutation of $1, 2, \dots, n$ that transposes i and $i+1$ and fixes all other j. Let $f(n)$ be the number of ways to write the permutation $n, n-1, \dots, 1$ in the form $s_{i_1} s_{i_2} \cdots s_{i_p}$, where $p = \binom{n}{2}$. For instance, $321 = s_1 s_2 s_1 = s_2 s_1 s_2$, so $f(3) = 2$. Moreover, $f(4) = 16$. Show that $f(n)$ is the number of sequences $a_1, \dots, a_p$ of $n-1$ 1's, $n-2$ 2's, ..., one $n-1$, such that in any prefix $a_1, a_2, \dots, a_k$, the number of $i+1$'s does not exceed the number of i's. For instance, when $n = 3$ there are the two sequences 112 and 121.

Note. An explicit formula is known for $f(n)$, but this is irrelevant here.

25 (difficult)**.** In the notation of the previous problem, show that

$$\sum_{i_1,i_2,\ldots,i_p} i_1 i_2 \cdots i_p = p!,$$

where the sum is over all sequences $i_1, \ldots, i_p$ for which $n, n-1, \ldots, 1 = s_{i_1} s_{i_2} \cdots s_{i_p}$. For instance, when $n = 3$ we get $1 \cdot 2 \cdot 1 + 2 \cdot 1 \cdot 2 = 3!$.

Note. The only known proofs are algebraic. It would be interesting to give a combinatorial proof.

PROBLEMS ON CONGRUENCES AND DIVISIBILITY

1. Let $n_1, n_2, \ldots, n_s$ be distinct integers such that

$$(n_1 + k)(n_2 + k) \cdots (n_s + k)$$

is an integral multiple of $n_1 n_2 \cdots n_s$ for every integer k. For each of the following assertions, give a proof or a counterexample:

- $|n_i| = 1$ for some i.
- If further all n_i are positive, then

$$\{n_1, n_2, \ldots, n_s\} = \{1, 2, \ldots, s\}.$$

2. How many coefficients of the polynomial

$$P_n(x_1, \ldots, x_n) = \prod_{1 \leq i < j \leq n} (x_i + x_j)$$

are odd?

3. If p is a prime number greater than 3 and $k = \lfloor 2p/3 \rfloor$, prove that the sum

$$\binom{p}{1} + \binom{p}{2} + \cdots + \binom{p}{k}$$

of binomial coefficients is divisible by p^2.

4. Do there exist positive integers a and b with $b - a > 1$ such for every $a < k < b$, either $\gcd(a, k) > 1$ or $\gcd(b, k) > 1$?

5. Suppose that $f(x)$ and $g(x)$ are polynomials (with $f(x)$ not identically 0) taking integers to integers such that for all $n \in \mathbb{Z}$, either $f(n) = 0$ or $f(n)|g(n)$. Show that $f(x)|g(x)$, i.e., there is a polynomial $h(x)$ with rational coefficients such that $g(x) = f(x)h(x)$.

6. Let q be an odd positive integer, and let N_q denote the number of integers a such that $0 < a < q/4$ and $\gcd(a, q) = 1$. Show that N_q is odd if and only if q is of the form p^k with k a positive integer and p a prime congruent to 5 or 7 modulo 8.

7. Let p be in the set $\{3, 5, 7, 11, \ldots\}$ of odd primes, and let

$$F(n) = 1 + 2n + 3n^2 + \cdots + (p-1)n^{p-2}.$$

Prove that if a and b are distinct integers in $\{0, 1, 2, \ldots, p-1\}$ then $F(a)$ and $F(b)$ are not congruent modulo p, that is, $F(a) - F(b)$ is not exactly divisible by p.

8. Do there exist 1,000,000 consecutive integers each of which contains a repeated prime factor?

9. A positive integer n is *powerful* if for every prime p dividing n, we have that p^2 divides n. Show that for any $k \geq 1$ there exist k consecutive integers, none of which is powerful.

10. Show that for any $k \geq 1$ there exist k consecutive positive integers, none of which is a sum of two squares. (You may use the fact that a positive integer n is a sum of two squares if and only if for every prime $p \equiv 3 \pmod 4$, the largest power of p dividing n is an even power of p.)

11. Prove that every positive integer has a multiple whose decimal representation involves all ten digits.

12. Prove that among any ten consecutive integers at least one is relatively prime to each of the others.

13. Find the length of the longest sequence of equal nonzero digits in which an integral square can terminate (in base 10), and find the smallest square which terminates in such a sequence.

14. Show that if n is an integer greater than 1, then n does not divide $2^n - 1$.

15. Show that if n is an odd integer greater than 1, then n does not divide $2^n + 2$.

16. Define a sequence $\{a_i\}$ by $a_1 = 3$ and $a_{i+1} = 3^{a_i}$ for $i \geq 1$. Which integers between 00 and 99 inclusive occur as the last two digits in the decimal expansion of infinitely many a_i?

17. What is the units (i.e., rightmost) digit of

$$\left[\frac{10^{20000}}{10^{100}+3}\right]?$$

Here $[x]$ is the greatest integer $\leq x$.

18. Suppose p is an odd prime. Prove that

$$\sum_{j=0}^{p} \binom{p}{j}\binom{p+j}{j} \equiv 2^p + 1 \pmod{p^2}.$$

19. Prove that for $n \geq 2$,

$$\overbrace{2^{2^{\cdots^{2}}}}^{n \text{ terms}} \equiv \overbrace{2^{2^{\cdots^{2}}}}^{n-1 \text{ terms}} \pmod{n}.$$

20. The sequence $(a_n)_{n\geq 1}$ is defined by $a_1 = 1$, $a_2 = 2$, $a_3 = 24$, and, for $n \geq 4$,

$$a_n = \frac{6a_{n-1}^2 a_{n-3} - 8a_{n-1}a_{n-2}^2}{a_{n-2}a_{n-3}}.$$

Show that, for all n, a_n is an integer multiple of n.

21. Prove that the expression

$$\frac{\gcd(m,n)}{n}\binom{n}{m}$$

is an integer for all pairs of integers $n \geq m \geq 1$.

22. Show that for each positive integer n,

$$n! = \prod_{i=1}^{n} \operatorname{lcm}\{1, 2, \ldots, \lfloor n/i \rfloor\}.$$

(Here lcm denotes the least common multiple, and $\lfloor x \rfloor$ denotes the greatest integer $\leq x$.)

23. Define a sequence $\{u_n\}_{n=0}^{\infty}$ by $u_0 = u_1 = u_2 = 1$, and thereafter by the condition that

$$\det \begin{pmatrix} u_n & u_{n+1} \\ u_{n+2} & u_{n+3} \end{pmatrix} = n!$$

for all $n \geq 0$. Show that u_n is an integer for all n. (By convention, $0! = 1$.)

24. Let p be a prime number. Let $h(x)$ be a polynomial with integer coefficients such that $h(0), h(1), \ldots, h(p^2-1)$ are distinct modulo p^2. Show that $h(0), h(1), \ldots, h(p^3-1)$ are distinct modulo p^3.

25. Define $a_0 = a_1 = 1$ and

$$a_n = \frac{1}{n-1} \sum_{i=0}^{n-1} a_i^2, \quad n > 1.$$

Is a_n an integer for all $n \geq 0$?

26. Let $f(x) = a_0 + a_1 x + \cdots$ be a power series with integer coefficients, with $a_0 \neq 0$. Suppose that the power series expansion of $f'(x)/f(x)$ at $x = 0$ also has integer coefficients. Prove or disprove that $a_0 | a_n$ for all $n \geq 0$.

27. Let S be a set of rational numbers such that

    (a) $0 \in S$;

    (b) If $x \in S$ then $x + 1 \in S$ and $x - 1 \in S$; and

    (c) If $x \in S$ and $x \notin \{0, 1\}$, then $1/(x(x-1)) \in S$.

    Must S contain all rational numbers?

28. Prove that for each positive integer n, the number $10^{10^{10^n}} + 10^{10^n} + 10^n - 1$ is not prime.

29. Let p be an odd prime. Show that for at least $(p+1)/2$ values of n in $\{0, 1, 2, \ldots, p-1\}$, $\sum_{k=0}^{p-1} k! n^k$ is not divisible by p.

30. Let a and b be distinct rational numbers such that $a^n - b^n$ is an integer for all positive integers n. Prove or disprove that a and b must themselves be integers.

31. Find the smallest integer $n \geq 2$ for which there exists an integer m with the following property: for each $i \in \{1, \ldots, n\}$, there exists $j \in \{1, \ldots, n\}$ different from i such that $\gcd(m+i, m+j) > 1$.

32. Let p be an odd prime number such that $p \equiv 2 \pmod 3$. Define a permutation π of the residue classes modulo p by $\pi(x) \equiv x^3 \pmod p$. Show that π is an even permutation if and only if $p \equiv 3 \pmod 4$.

33. Suppose that a positive integer N can be expressed as the sum of k consecutive positive integers

$$N = a + (a+1) + (a+2) + \cdots + (a+k-1)$$

for $k = 2017$ but for no other values of $k > 1$. Considering all positive integers N with this property, what is the smallest positive integer a that occurs in any of these expressions?

PROBLEMS ON COMBINATORIAL CONFIGURATIONS

These problems aim to capture the flavors of combinatorial questions which have been common in recent years on the Putnam. Difficulty is very roughly increasing, with the first several being more exercise-like.

1. Can a 2018×2018 grid be tiled with rotations and reflections of the L-tetromino?
2. Each edge of a complete graph on $2k$ vertices is removed with probability $\frac{1}{2}$. Prove that with probability greater than $\frac{1}{4^k}$, the maximum degree of the remaining graph is at most $k-1$.
3. Let n be a fixed positive integer. How many ways are there to write n as a sum of positive integers,
$$n = a_1 + a_2 + \ldots + a_k,$$
with k an arbitrary positive integer and $a_1 \leq a_2 \leq \ldots \leq a_k \leq a_1 + 1$?
4. How many n-digit numbers whose digits are in the set $\{2, 3, 7, 9\}$ are divisible by 3?
5. A 6×6 grid is tiled by dominoes. Prove that there exists some line which cuts the board into two nonempty parts without cutting through any domino.
6. On each face of a regular icosahedron is written a nonnegative integer such that the sum of all 20 integers is 39. Show that there are two faces that share a vertex and have the same integer written on them.
7. Given a positive integer n, what is the largest k such that the numbers $1, 2, \ldots, n$ can be put into k boxes such that the sum of the numbers in each box is the same?
8. Callie and Marie play a game in which they take turns filling entries of an initially empty 2008×2008 array. Callie plays first. At each turn, a player chooses a real number and places it in a vacant entry. The game ends when all the entries are filled. Callie wins if the determinant of the resulting matrix is nonzero; Marie wins if it is zero. Which player has a winning strategy?
9. Consider a $(2m-1) \times (2n-1)$ rectangular region, where m, n are integers such that $m, n \geq 4$. This region is to be tiled without overlap using rotations and reflections of the bent triomino and the S-tetromino. What is the fewest number of these tiles which can be used?
10. Let B be a set of more than $2^{n+1}/n$ distinct points with coordinates of the form $\{\pm 1, \ldots, \pm 1\}$ in n-dimensional space with $n \geq 3$. Show that there are three distinct points in B which are the vertices of an equilateral triangle.
11. A set of n points is given in the plane such that the distance between any two of them is greater than 1. Prove that one can choose at least $\frac{n}{7}$ of these points such that the distance between any two of them is greater than $\sqrt{3}$.
12. A round-robin tournament of $2n$ teams lasted for $2n-1$ days, as follows. On each day, every team played one game against another team, with one team winning and one team losing in each of the n games. Over the course of the tournament, each team played every other team exactly once. Can one necessarily choose one winning team from each day without choosing any team more than once?

13. Callie and Marie play a game with r red and g green stones. At every step, one can remove an amount k of stones of either color, where k divides the number of stones remaining of the other color. She who moves last wins, and Callie starts. For which (r, g) does Callie win?

14. Consider a polyhedron with at least five faces such that exactly three edges emerge from each of its vertices, and no two edges share more than one face. Two players play the following game:

    Each player, in turn, signs his or her name on a previously unsigned face. The winner is the player who first succeeds in signing three faces that share a common vertex.

    Show that the player who signs first will always win by playing as well as possible.

15. Two hundred students participated in a mathematical contest. They had six problems to solve. It is known that each problem was correctly solved by at least 120 participants. Prove that there must be two participants such that every problem was solved by at least one of these two students.

16. Let $n \geq 2$ be an integer and T_n be the number of nonempty subsets S of $\{1, 2, 3, \ldots, n\}$ with the property that the average of the elements in S is an integer. Prove that $T_n - n$ is always even.

17. Prove that it is not possible to color the squares of a 11×11 grid using three colors such that no four squares whose centers form the vertices of a rectangle with sides parallel to the sides of the grid, have the same color.

18. Callie and Marie play a game in which they take turns choosing an element from the group of invertible $n \times n$ matrices with entries in the field $\mathbb{Z}/p\mathbb{Z}$ of integers modulo p, where n is fixed positive integer and p is a fixed prime number, The rules of the game are:

    - A player cannot choose an element that has been chosen by either player on any previous turn.
    - A player can only choose an element that commutes with all previously chosen elements.
    - A player who cannot choose an element on his/her turn loses the game.

    Callie takes the first turn. For which (n, p) does Callie have a winning strategy?

19. Suppose a finite number of integer arithmetic progressions are given, such that every positive integer belongs to exactly one of them. Prove that two of these progressions have the same difference.

20. Suppose you are given a binary string which is not entirely zeroes. Prove that you may insert $+$ between some of the digits so that the resulting summation, when carried out in base 2, yields a power of 2. Example: for 101111 the split $1 + 0 + 1111$ is possible.

21. An alphabet consists of three letters. Some of the words of length 2 or more are prohibited, and all of the prohibited sequences have different lengths. A *word* is a sequence of letters of any length. A *correct* word does not contain any prohibited sequence. Prove that there are correct words of any length.

22. There are 2007 senators in a senate. Each senator has enemies within the senate. Prove that there is a non-empty subset K of senators such that, for every senator in the senate, the number of enemies of that senator in the set K is an even number.

23. Call a subset S of $\{1, 2, \ldots, n\}$ mediocre if it has the following property: Whenever a, b are elements of S whose average is an integer, that average is also an element of S. Let $A(n)$ be the number of mediocre subsets of $\{1, 2, \ldots, n\}$. Find all positive integers n such that $A(n+2) + A(n) = 2A(n+1) + 1$.

24. Let $S_1, S_2, \ldots, S_m$ be distinct subsets of $\{1, 2, \ldots, n\}$ such that $|S_i \cap S_j| = 1$ for all $i \neq j$. Prove that $m \leq n$.

25. For a set S of nonnegative integers, let $r_S(n)$ denote the number of ordered pairs (s_1, s_2) such that $s_1 \in S, s_2 \in S, s_1 \neq s_2, s_1 + s_2 = n$. Is it possible to partition the nonnegative integers into two sets A, B in such a way that $r_A(n) = r_B(n)$ for all n?

26. The 30 edges of a regular icosahedron are distinguished by labeling them $1, 2, \ldots, 30$. How many different ways are there to paint each edge red, white, or blue such that each of the 20 triangular faces of the icosahedron has two edges of the same color and a third edge of a different color?

27. There are 2010 boxes labeled $B_1, B_2, \ldots, B_{2010}$ and $2010n$ balls have been distributed among them, for some positive integer n. You may redistribute the balls by a sequence of moves, each of which consists of choosing an i and moving exactly i balls from box B_i into any one other box. For which values of n is it possible to reach the distribution with exactly n balls in each box, regardless of the initial distribution of balls?

PROBLEMS ON GENERATING FUNCTIONS

NOTE. All the problems below can be done using generating functions. Many of them can also be done by other methods. However, you should hand in only solutions which use generating functions. **No credit** for solving a problem without using generating functions!

1. Let $f(m,1) = f(1,n) = 1$ for $m \geq 1$, $n \geq 1$, and let

$$f(m,n) = f(m-1,n) + f(m,n-1) + f(m-1,n-1) \text{ for } m > 1 \text{ and } n > 1.$$

Also let

$$S(n) = \sum_{a+b=n} f(a,b), \; a \geq 1 \text{ and } b \geq 1.$$

Prove that

$$S(n+2) = S(n) + 2S(n+1) \text{ for } n \geq 2.$$

2. Let $x^{(n)} = x(x-1)\cdots(x-n+1)$ for n a positive integer, and let $x^{(0)} = 1$. Prove that

$$(x+y)^{(n)} = \sum_{k=0}^{n} \binom{n}{k} x^{(k)} y^{(n-k)}.$$

NOTE: $\displaystyle\binom{n}{k} = \frac{n(n-1)\cdots(n-k+1)}{1\cdot 2\cdots k}$.

3. For a set with n elements, how many subsets are there whose cardinality (the number of elements in the subset) is respectively $\equiv 0 (\text{mod } 3)$, $\equiv 1 (\text{mod } 3)$, $\equiv 2 (\text{mod } 3)$? In other words, calculate

$$s_{i,n} = \sum_{k \equiv i \pmod 3} \binom{n}{k} \text{ for } i = 0,1,2.$$

Your result should be strong enough to permit direct evaluation of the numbers $s_{i,n}$ and to show clearly the relationship of $s_{0,n}$ and $s_{1,n}$ and $s_{2,n}$ to each other for all positive integers n. In particular, show the relationships among these three sums for $n = 1000$. [An illustration of the definition of $s_{i,n}$ is $s_{0,6} = \binom{6}{0} + \binom{6}{3} + \binom{6}{6} = 22$.]

4. Given the power series

$$a_0 + a_1 x + a_2 x^2 + \cdots$$

in which

$$a_n = (n^2+1)3^n,$$

show that there is a relationship of the form

$$a_n + pa_{n+1} + qa_{n+2} + ra_{n+3} = 0,$$

in which p, q, r are constants independent of n. Find these constants and the sum of the power series.

5. Show that

$$x + \frac{2}{3}x^3 + \frac{2}{3}\frac{4}{5}x^5 + \frac{2}{3}\frac{4}{5}\frac{6}{7}x^7 + \cdots = \frac{\text{arc sin } x}{\sqrt{1-x^2}}.$$

NOTE (not on Putnam Exam): arc $\sin x$ is the same as $\sin^{-1} x$.

6. Let k be a positive integer and let $m = 6k - 1$. Let

$$S(m) = \sum_{j=1}^{2k-1} (-1)^{j+1} \binom{m}{3j-1}.$$

For example with $k = 3$,

$$S(17) = \binom{17}{2} - \binom{17}{5} + \binom{17}{8} - \binom{17}{11} + \binom{17}{14}.$$

Prove that $S(m)$ is never zero. [As usual, $\binom{m}{r} = \frac{m!}{r!(m-r)!}$.]

7. For nonnegative integers n and k, define $Q(n,k)$ to be the coefficient of x^k in the expansion of $(1 + x + x^2 + x^3)^n$. Prove that

$$Q(n,k) = \sum_{j=0}^{n} \binom{n}{j} \binom{n}{k-2j},$$

where $\binom{a}{b}$ is the standard binomial coefficient. (Reminder: For integers a and b with $a \geq 0$, $\binom{a}{b} = \frac{a!}{b!(a-b)!}$ for $0 \leq b \leq a$, and $\binom{a}{b} = 0$ otherwise.)

8. Let $a_{m,n}$ denote the coefficient of x^n in the expansion of $(1 + x + x^2)^m$. Prove that for all $k \geq 0$,

$$0 \leq \sum_{i=0}^{\lfloor 2k/3 \rfloor} (-1)^i a_{k-i,i} \leq 1.$$

9. Consider the power series expansion

$$\frac{1}{1 - 2x - x^2} = \sum_{n=0}^{\infty} a_n x^n.$$

Prove that, for each integer $n \geq 0$, there is an integer m such that

$$a_n^2 + a_{n+1}^2 = a_m.$$

10. Let $A = \{(x,y) : 0 \leq x, y \leq 1\}$. For $(x,y) \in A$, let

$$S(x,y) = \sum_{\frac{1}{2} \leq \frac{m}{n} \leq 2} x^m y^n,$$

where the sum ranges over all pairs (m,n) of positive integers satisfying the indicated inequalities. Evaluate

$$\lim_{(x,y) \to (1,1),\ (x,y) \in A} (1 - xy^2)(1 - x^2 y) S(x,y).$$

11. For a set S of nonnegative integers, let $r_S(n)$ denote the number of ordered pairs (s_1, s_2) such that $s_1 \in S$, $s_2 \in S$, $s_1 \neq s_2$, and $s_1 + s_2 = n$. Is it possible to partition the nonnegative integers into two sets A and B in such a way that $r_A(n) = r_B(n)$ for all n?

12. For positive integers m and n, let $f(m,n)$ denote the number of n-tuples $(x_1, x_2, \ldots, x_n)$ of integers such that $|x_1| + |x_2| + \cdots + |x_n| \leq m$. Show that $f(m,n) = f(n,m)$.

13. Let S_n denote the set of all permutations of the numbers $1, 2, \ldots, n$. For $\pi \in S_n$, let $\sigma(\pi) = 1$ if π is an even permutation and $\sigma(\pi) = -1$ if π is an odd permutation. Also, let $\nu(\pi)$ denote the number of fixed points of π. Show that

$$\sum_{\pi \in S_n} \frac{\sigma(\pi)}{\nu(\pi)+1} = (-1)^{n+1} \frac{n}{n+1}.$$

14. Given $a_0 = 1$ and $a_{n+1} = (n+1)a_n - \binom{n}{2} a_{n-2}$ for $n \geq 0$, compute $y = \sum_{n \geq 0} a_n \frac{x^n}{n!}$.

15. Solve the recurrence

$$(n+1)(n+2)a_{n+2} - 3(n+1)a_{n+1} + 2a_n = 0,$$

with the initial conditions $a_0 = 2,\ a_1 = 3$.

16. Find the coefficients of the power series $y = 1 + 3x + 15x^2 + 184x^3 + 495x^4 + \cdots$ satisfying

$$(27x - 4)y^3 + 3y + 1 = 0.$$

17. Find the unique power series $y = 1 + x - \frac{1}{2}x^2 + \frac{2}{3}x^3 + \cdots$ such that the constant term is 1, the coefficient of x is 1, and for all $n \geq 2$ the coefficient of x^n in y^n is 0. (Give a simple formula for the coefficients of y, not for y itself.)

18. Let $f(m, 0) = f(0, n) = 1$ and $f(m, n) = f(m-1, n) + f(m, n-1) + f(m-1, n-1)$ for $m, n > 0$. Show that

$$\sum_{n=0}^{\infty} f(n, n)x^n = \frac{1}{\sqrt{1 - 6x + x^2}}.$$

19. Suppose that $\mathbb{Z}$ is written as a disjoint union of $n < \infty$ arithmetic progressions, with differences $d_1 \geq d_2 \geq \cdots \geq d_n \geq 1$. Show that $d_1 = d_2$.

20. Solve the following equation for the power series $F(x, y) = \sum_{m,n \geq 0} a_{mn} x^m y^n$, where $a_{mn} \in \mathbb{R}$:

$$(xy^2 + x - y)F(x, y) = xF(x, 0) - y.$$

The point is to make sure that your solution has a power series expansion at $(0, 0)$.

21. Find a simple description of the coefficients $a_n \in \mathbb{Z}$ of the power series $F(x) = x + a_2 x^2 + a_3 x^3 + \cdots$ satisfying the functional equation

$$F(x) = (1 + x)F(x^2) + \frac{x}{1 - x^2}.$$

18.A34 PROBLEMS #1

Problems are marked by the following difficulty ratings.

[1] Easy. Most students should be able to solve it.

[2] Somewhat difficult or tricky. Many students should be able to solve it.

[3] Difficult. Only a few students should be able to solve it.

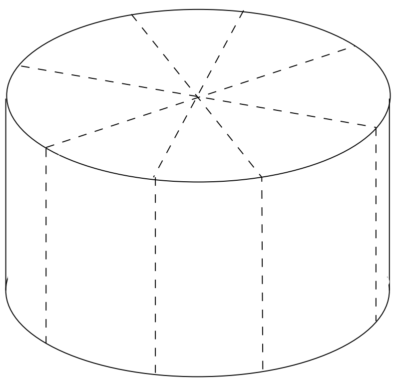
[4] Horrendously difficult. We don't really expect anyone to solve it, but those who like a challenge might want to give it a try.

[5] Unsolved.

Further gradations are indicated by + and −. Thus [1–] denotes an utterly trivial problem, and [5–] denotes an unsolved problem that has received little attention and may not be too difficult. A few students may be capable of solving a [3–] problem, while almost none could solve a [3] in a reasonable period of time. Of course these ratings are subjective, so you shouldn't take them *too* seriously.

1. [1] A single elimination tennis tournament is held among 215 players. A player is eliminated as soon as (s)he loses a match. Thus on the first round there are 107 matches and one player receives a bye (waits until the next round). On the second round there are 54 matches with no byes, etc. How many matches are played in all? What if there are 10^{100} players?

2. (a) [1] It's easy to see that a cylinder of cheese can be cut into eight identical pieces with four straight cuts.



Can this be done with only three straight cuts?

(b) [2+] What about a torus (doughnut)? What is the most number of pieces into which a solid torus can be cut by three straight cuts, or more generally by n straight cuts (without rearranging the pieces)?

(c) [3–] What is the most number of pieces into which a solid torus can be cut by three straight cuts, if one is allowed to rearrange the pieces after each cut? So far as I know, the answer is not known for $n \geq 4$ cuts.

3. [2] Can a cube of cheese three inches on a side be cut into 27 one-inch cubes with five straight cuts? What if one can move the pieces prior to cutting?

4. (a) [1] In how many zeros does 10000! end?

   (b) [3] What is the last nonzero digit of 10000!? (No fair using a computer to actually calculate 10000!.)

5. [1] In this problem, "knights" always tell the truth and "knaves" always lie. In (a)-(c), all persons are either knights or knaves.

   (a) There are two persons, A and B. A says, "At least one of us is a knave." What are A and B?

   (b) A says, "Either I am a knave or B is a knight." What are A and B?

   (c) Now we have three persons, A, B, and C. A says, "All of us are knaves." B says, "Exactly one of us is a knight." What are A, B, and C?

   (d) Now we have a third type of person, called "normal," who sometimes lies and sometimes tells the truth. A says, "I am normal." B says, "That is true." C says, "I am not normal." Exactly one of A, B, C is a knight, one is a knave, and one is normal. What are A, B, and C?

6. [1] Without using calculus, find the minimum value of $x + \frac{1}{x}$ for $x > 0$. What about $x + \frac{3}{x}$?

7. [3] Let n and k be nonnegative integers, with $n \geq k$. The *binomial coefficient* $\binom{n}{k}$ is defined by $\binom{n}{k} = n!/k!(n-k)!$. (Recall that $0! = 1$. If $n < k$, then it is convenient to define $\binom{n}{k} = 0$.)

(a) For what values of n and k is $\binom{n}{k}$ odd? Find as simple and elegant a criterion as possible.

(b) More generally, given a prime p, find a simple and elegant description of the largest power of p dividing $\binom{n}{k}$.

8. [3] (a) Let P be a convex polygon in the plane with a prime number p of sides, all angles equal, and all sides of rational length. Show that P is regular (i.e., all sides also have equal length).

   (b) (1990 Olympiad) Show that there exists an equiangular polygon with side lengths $1^2, 2^2, \ldots, 1990^2$ (in some order).

9. [2] What positive integers can be expressed as the sum of two or more consecutive positive integers? (The first three are $3 = 1 + 2$, $5 = 2 + 3$, $6 = 1 + 2 + 3$.)

10. [2] (a) Find the maximum value of $x^{1/x}$ for $x > 0$.

    (b) Without doing any numerical calculations, decide which is bigger, π^e or e^π.

11. [3] Does there exist an infinite sequence $a_0 a_1 a_2 \cdots$ of 1's, 2's, and 3's, such that no two consecutive blocks are identical? (In other words, for no $1 \le i < j$ do we have $a_i = a_j, a_{i+1} = a_{j+1}, \ldots, a_{j-1} = a_{2j-i-1}$.) For instance, if we begin our sequence with 1213121, then we're stuck.

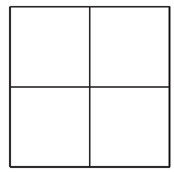
12. [2] A *lattice point* in the plane is a point with integer coordinates. For instance, $(3, 5)$ and $(0, -2)$ are lattice points, but $(3/2, 1)$ and $(\sqrt{2}, \sqrt{2})$ are not. Show that no three lattice points in the plane can be the vertices of an equilateral triangle. What about in three dimensions?

13. [3.5] True or false? Let n be a positive integer. Then

$$\left\lceil \frac{2}{2^{1/n} - 1} \right\rceil = \left\lfloor \frac{2n}{\log 2} \right\rfloor .$$

18.A34 PROBLEMS #2

14. [1] Here is a square divided into four congruent pieces:



Can a square be divided into *five* congruent pieces?

15. [1] Choose any 1000 points in the plane. Does there always exist a straight line which divides the points exactly in two, i.e., such that exactly 500 of the points lie on one side of the line and 500 on the other side?

16. [1] The *Fibonacci numbers* are defined by

$$F_{n+2} = F_{n+1} + F_n, \qquad F_1 = 1, F_2 = 1.$$

Thus $F_3 = 2$, $F_4 = 3$, $F_5 = 5$, $F_6 = 8$, etc.

(a) Compute $F_n^2 - F_{n-1}F_{n+1}$.

(b) Express $F_1 + F_2 + \cdots + F_n$ in terms of a single Fibonacci number.

17. [1] In how many ways can the positive integer n be written as a sum of positive integers, taking order into account? (E.g., for $n = 3$, there are the four ways $1+1+1$, $1+2$, $2+1$, 3.)

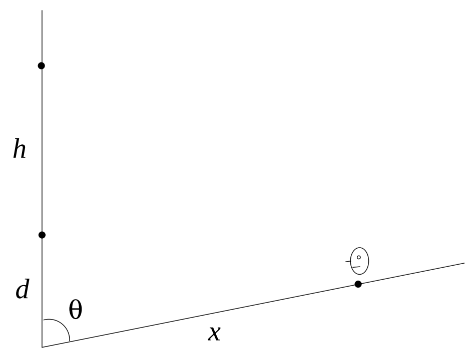
18. [1] What is the sum of all the digits used in writing down the numbers from one to a billion?

19. [2] (due to Oswald Jacoby; broadcast on National Public Radio, September 8, 1991) A hat contains a certain number N of blue balls and red balls. Five balls are picked randomly out of the hat (without replacement). The probability is exactly $1/2$ that all five balls are blue. What is the smallest possible value of N? Try to give a simple argument avoiding factorials, binomial coefficients, etc.

20. [2] (appeared in a letter from Martin Gardner to Marilyn Vos Savant, *Parade*, September 8, 1991) The Greens and Blacks are playing bridge. After a deal, Mr. Brown, an onlooker, asks Mrs. Black: "Do you have an ace in your hand?" She nods. There is a certain probability that her hand holds at least one other ace. After the next deal, he asks her: "Do you have the ace of spades?" She nods. Again, there is a certain probability that her hand holds at least another ace. Which probability is greater? Or are they both the same?

21. [2.5] Let d be any divisor of an integer of the form n^2+1. Prove that $d-3$ is not divisible by 4.

22. [2] A movie screen of height h is mounted on a vertical wall at a distance d from a floor which makes an angle of θ with the wall. How far from the wall should a customer sit to get the best possible view? (Here you will have to make some reasonable assumption about what is meant by "best possible." Try to avoid the use of trigonometry and/or calculus.)



23. [2] Let F_n be a Fibonacci number, as defined above.

    (a) Show that $F_n = \frac{1}{\sqrt{5}}(\tau^n - \bar{\tau}^n)$, where $\tau = \frac{1}{2}(1+\sqrt{5}) = 1.61803398\cdots$, $\bar{\tau} = \frac{1}{2}(1-\sqrt{5}) = -.61803398\cdots$. Deduce that F_n is the nearest integer to $\tau^n/\sqrt{5}$.

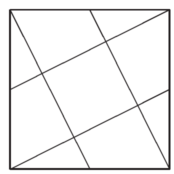
    (b) Show that

$$F_1^2 + F_2^2 + \cdots + F_n^2 = \frac{1}{5}(F_{2n+3} - F_{2n-1} - (-1)^n).$$

24. [3.5] Do there exist relatively prime positive integers a, b with the following property? Define $u_1 = a, u_2 = b$, and $u_n = u_{n-1} + u_{n-2}$ for $n \geq 3$. Then all the numbers u_n, $n \geq 1$, are composite.

25. [3] Define a sequence $a_1 < a_2 < \cdots$ of positive integers as follows. Pick $a_1 = 1$. Once $a_1, \ldots, a_n$ have been chosen, let a_{n+1} be the least positive integer not already chosen and not of the form $a_i + i$ for $1 \leq i \leq n$. Thus $a_1 + 1 = 2$ is not allowed, so $a_2 = 3$ and $a_3 = 4$. Now $a_2 + 2 = 5$ is also not allowed, so $a_4 = 6$, etc. The sequence begins $1, 3, 4, 6, 8, 9, 11, 12, 14, 16, 17, 19, \ldots$. Find a simple formula for a_n. Your formula should allow you, for instance, to compute $a_{1000000}$ quickly using a hand calculator. (Hint: The answer involves $\tau = \frac{1}{2}(1 + \sqrt{5})$.)

26. [2] Lines are drawn from the vertices of a square to the midpoints of the sides as shown below.



What is the ratio of the area of the original square to the area of the center square S? Can you solve the problem without making any arithmetic or algebraic calculations?

27. [2.5] Let θ be *any* positive real number, with $0 < \theta < 2\pi$. We are given an ordinary cylindrical cake with frosting on the top. Cut out a piece in the usual manner which makes an angle of θ (radians), turn it upside-down, and put it back in the cake. Then cut out another piece adjacent to the first piece (say in a clockwise direction) of the same size, and again turn it upside-down and put it back in the cake. Continue. Each piece is clockwise adjacent to the previous piece. After a while our pieces might consist of several subsets of previously cut pieces, but we still follow the same procedure. Prove that after finitely many steps, all the frosting will be back on top!

18.A34 PROBLEMS #3

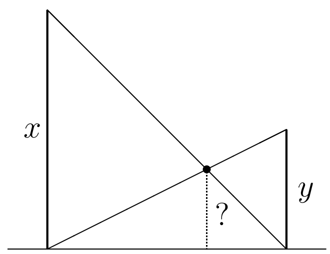
28. [1] Let $x, y > 0$. The *harmonic mean* of x and y is defined to be $2xy/(x+y)$. The *geometric mean* is $\sqrt{xy}$. The *arithmetic mean* (or *average*) is $(x+y)/2$. Show that

$$\frac{2xy}{x+y} \leq \sqrt{xy} \leq \frac{x+y}{2},$$

with equality if and only if $x = y$.

29. [1] A car travels one mile at a speed of x mi/hr and another mile at y mi/hr. What is the average speed? What kind of mean of x and y is this?

30. [1] Consider two telephone poles of heights x and y. Connect the top of each pole to the bottom of the other with a rope. What is the height of the point where the ropes cross? What kind of mean is this related to?



31. [1] Given two line segments of lengths x and y, describe a simple geometric construction for constructing a segment of length $\sqrt{xy}$.

32. [1] Suppose x and y are real numbers such that $x^2 + y^2 = x + y$. What is the largest possible value of x?

33. [2.5] (a) Let $x, y > 0$ and $p \neq 0$. The *p-th power mean* of x and y is defined to be

$$M_p(x,y) = \left(\frac{x^p + y^p}{2}\right)^{1/p}.$$

Note that $M_{-1}(x,y)$ is the harmonic mean and $M_1(x,y)$ is the arithmetic mean. If $p < q$, then show that

$$M_p(x,y) \leq M_q(x,y),$$

with equality if and only if $x = y$.

(b) Compute $\lim_{p\to\infty} M_p(x,y)$, $\lim_{p\to 0} M_p(x,y)$, $\lim_{p\to -\infty} M_p(x,y)$.

34. [3.5] Let $x, y > 0$. Define two sequences $x_1, x_2, \ldots$ and $y_1, y_2, \ldots$ as follows:

$$\begin{aligned} x_1 &= x & y_1 &= y \\ x_{n+1} &= \frac{x_n + y_n}{2} & y_{n+1} &= \sqrt{x_n y_n}, \quad n > 1. \end{aligned}$$

It's not hard to see that $\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n$. This limit is denoted $AG(x,y)$ and is called the *arithmetic-geometric mean* of x and y. Show that

$$AG(x,y) = \frac{\pi}{\displaystyle\int_0^{\pi} \frac{d\theta}{\sqrt{x^2\sin^2\theta + y^2\cos^2\theta}}}.$$

35. Let $x > 0$. Define

$$f(x) = x^{x^{x^{\cdot^{\cdot^{\cdot}}}}}.$$

More precisely, let $x_1 = x$ and $x_{n+1} = x^{x_n}$ if $n > 1$, and define $f(x) = \lim_{n\to\infty} x_n$.

(a) [1] Compute $f(\sqrt{2})$.

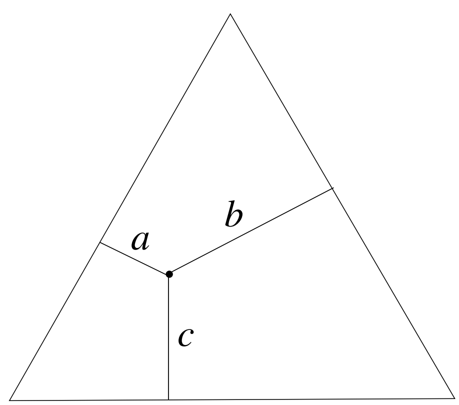
(b) [3.5] For what values of x does $f(x)$ exist?

(c) [3.5] Let

$$\begin{aligned} f(x+1) &= \sum_{n\geq 0} a_n \frac{x^n}{n!} \\ &= 1 + x + 2\frac{x^2}{2!} + 9\frac{x^3}{3!} + 56\frac{x^4}{4!} + 480\frac{x^5}{5!} + 5094\frac{x^6}{6!} + \cdots. \end{aligned}$$

Show that a_n is a positive integer.

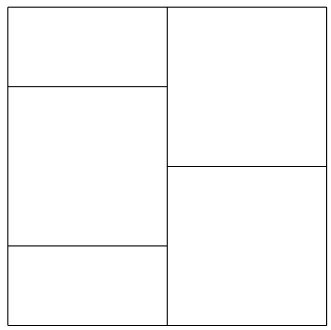
36. [2] Let T be an equilateral triangle. Find all points x in T that minimize the sum $a+b+c$ of the distances a, b, c of x from the three sides of T.



37. [2.5] Alice and Bob play the following game. They begin with a sequence $(a_1, \ldots, a_{2n})$ of positive integers. The players alternate turns, with Alice moving first. When it is someone's turn to move, that person can remove either the first or last remaining term of the sequence. A player's score at the end is the sum of his/her chosen numbers. Show that Alice has a strategy that guarantees a score at least as large as Bob's.

38. [3] Let F_n denote the nth Fibonacci number. Let p be a prime not equal to 5. Show that either F_{p-1} or F_{p+1} is divisible by p. Which?

39. Fix an integer $n > 0$. Let $f(n)$ be the most number of rectangles into which a square can be divided so that every line which is parallel to one of the sides of the square intersects the interiors of at most n of the rectangles. For instance, in the following figure there are five rectangles, and every horizontal or vertical line intersects the interior of at most 3 of them. This is not best possible, since we can obviously do the same with nine rectangles.



(a) [2] It is obvious that $f(n) \geq n^2$. Show that in fact $f(n) > n^2$ for $n \geq 3$.

(b) [3] Show that $f(n) < \infty$, i.e., for any fixed n we cannot divide a square into *arbitrarily many* rectangles with the desired property. In fact, one can show $f(n) \leq n^n$.

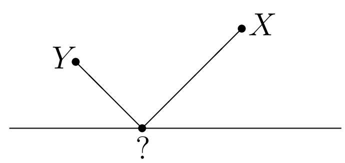
(c) [5] Find the actual value of $f(n)$. The best lower bound known is $f(n) \geq 3 \cdot 2^{n-1} - 2$.

18.A34 PROBLEMS #4

40. [1] Three students A, B, C compete in a series of tests. For coming in first in a test, a student is awarded x points; for coming second, y points; for coming third, z points. Here x, y, z are positive integers with $x > y > z$. There were no ties in any of the tests. Altogether A accumulated 20 points, B 10 points, and C 9 points. Student A came in second in the algebra test. Who came in second in the geometry test?

41. [1] Remove the upper-left and lower-right corner squares from an 8×8 chessboard. Show that the resulting board cannot be covered by 31 dominoes. (A domino consists of two squares with an edge in common.)

42. [1] Mr. X brings some laundry from his house to a nearby river. After washing it in the river, he delivers it to Ms. Y who lives on the same side of the river.



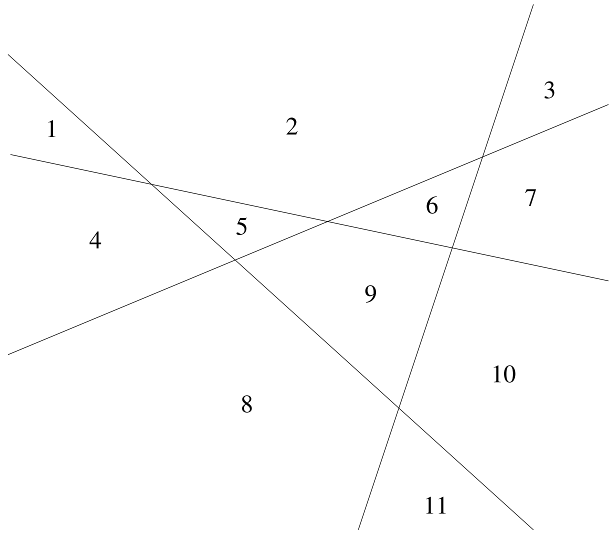
At what point on the river should Mr. X bring the laundry in order to travel the least possible distance? Try to do this problem without using calculus. (Assume of course that the river is a straight line.)

43. [2] An *antimagic square* is an $n \times n$ matrix whose entries are the distinct integers $1, 2, \ldots, n^2$ such that any set of n entries, no two in the same row or column, have the same sum of their elements. For instance,

$$\begin{bmatrix} 14 & 8 & 16 & 6 \\ 9 & 3 & 11 & 1 \\ 10 & 4 & 12 & 2 \\ 13 & 7 & 15 & 5 \end{bmatrix}$$

For what values of n do there exist $n \times n$ antimagic squares?

44. [1] Let $f(n)$ be the number of regions which are formed by n lines in the plane, where no two lines are parallel and no three meet in a point. E.g., $f(4) = 11$.



Find a simple formula for $f(n)$.

45. [2.5] Define a sequence $a_0, a_1, a_2, \ldots$ of integers as follows: $a_0 = 0$, and given $a_0, a_1, \ldots, a_n$, then a_{n+1} is the least integer greater than a_n such that no three distinct terms (not necessarily consecutive) of $a_0, a_1, \ldots, a_{n+1}$ are in arithmetic progression. (This means that for no $0 \leq i < j < k \leq n+1$ do we have $a_j - a_i = a_k - a_j$.) Find a simple rule for determining a_n. For instance, what is $a_{1000000}$? The sequence begins $0, 1, 3, 4, 9, 10, 12, \ldots$.

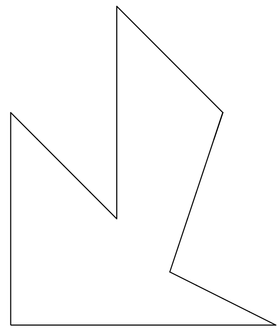
46. (a) [1] Let a, b, m, n be positive integers. Suppose that an $m \times n$ checkerboard can be tiled with $a \times b$ boards (in any orientation), i.e., the $a \times b$ boards can be placed on the $m \times n$ board to cover it completely, with no overlapping of the interiors of the $a \times b$ boards. Show that mn is divisible by ab.

    (b) [2.5] Assuming the condition of (a), show in fact that at least one of m and n is divisible by a. (Thus by symmetry, at least one of m and n is divisible by b.) For instance, a 6×30 board cannot be tiled with 4×3 boards.

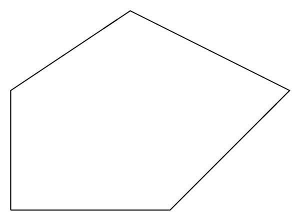
    (c) [2.5] Generalize (b) to any number of dimensions.

47. [2.5] Let R be a rectangle whose sides can have any positive real lengths. Show that if R can be tiled with finitely many rectangles all with at least one side of integer length, then R has at least one side of integer length.

48. [3] A *polygon* is a plane region enclosed by non-intersecting straight line segments, such as



A polygon P is *convex* if any straight line segment whose endpoints lie in P lies entirely in P. For instance, the above polygon is not convex, but



is convex. Can a convex polygon be dissected into non-convex quadrilaterals? (A quadrilateral is a four-sided polygon. The non-convex quadrilaterals in the above question may be of any size and shape, provided none are convex.) This problem was formulated and solved by a Berkeley undergraduate; none of the mathematics professors to whom he showed it were able to solve it.

49. (a) [3] Let $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$. Assume that f is a polynomial in each variable separately, i.e., for all $a \in \mathbb{R}$, the functions $f(a, x)$ and $f(x, a)$ are polynomials in x. Prove that $f(x, y)$ is a polynomial in x and y.

    (b) [2.5] Show that (a) is false if $\mathbb{R}$ is replaced by $\mathbb{Q}$ (the rational numbers).

50. [3.5] Does there exist a polynomial $f(x)$ with real coefficients such that $f(x)^2$ has fewer nonzero coefficients than $f(x)$?

18.A34 PROBLEMS #5

51. [1] A person buys a 30-year $\$100,000$ mortgage at an annual rate of 8%. What is his or her monthly payment?

52. (a) [1] Person A chooses an integer between 0 and $2^{11} - 1$, inclusive. Person B tries to guess A's number by asking yes-no questions. What is the minimum number of questions needed to guarantee that B finds A's number? Can the questions all be chosen in advance in an elegant way?

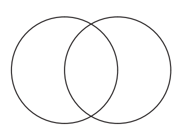
    (b) [2.5] What if A is allowed to lie at most once?

53. [1] Let M be an $n \times n$ *symmetric* matrix such that each row and column is a permutation of $1, 2, \ldots, n$. ("Symmetric" means that the entry in row i and column j is the same as the entry in row j and column i.) If n is odd, then show that every number $1, 2, \ldots, n$ appears exactly once on the main diagonal. For instance,

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 4 & 1 & 5 & 3 \\ 3 & 1 & 5 & 2 & 4 \\ 4 & 5 & 2 & 3 & 1 \\ 5 & 3 & 4 & 1 & 2 \end{bmatrix}$$

54. [1] Find all 10 digit numbers $a_0 a_1 \cdots a_9$ such that a_i is the number of digits equal to i, for all $0 \le i \le 9$.

55. [1] Two circles of radius one pass through each other's centers. What is the area of their intersection?



56. (a) [2] Given any 1000 points in the plane, show that there is a circle which contains exactly 500 of the points in its interior, and none on its circumference.

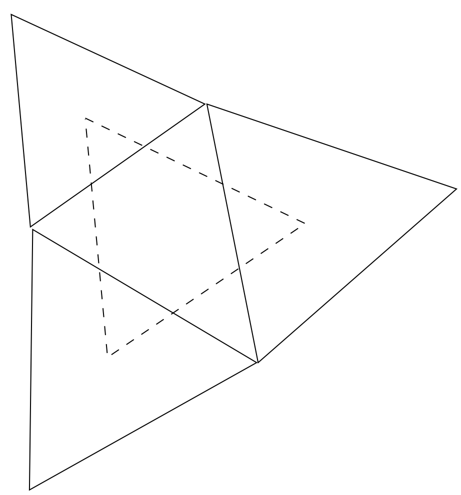
(b) [3] Given 1001 points in the plane, no three collinear and no four concyclic (i.e., no four on a circle), show that that there are exactly 250,000 circles with three of the points on the circumference, 499 points inside, and 499 points outside.

57. (a) [2.5] Let n be an integer, and suppose that $n^4+n^3+n^2+n+1$ is divisible by k. Show that either k or $k-1$ is divisible by 5. HINT. First show that one may assume that k is prime. Use *Fermat's theorem* for the prime k, which states that if m is not divisible by k, then $m^{k-1}-1$ is divisible by k. Try to avoid more sophisticated tools.

(b) [2] Deduce that there are infinitely many primes of the form $5j+1$.

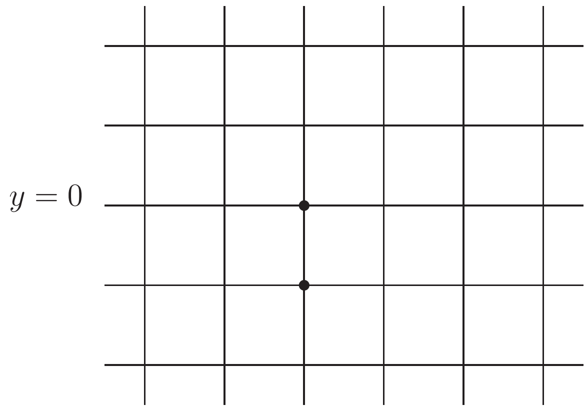
58. [2] A cylindrical hole is drilled straight through and all the way through the center of a sphere. After the hole is drilled, its length is six inches. What is the volume that remains?

59. [2.5] Let T be a triangle. Erect an equilateral triangle on each side of T (facing outwards). Show that the centers of these equilateral triangles form the vertices of an equilateral triangle.

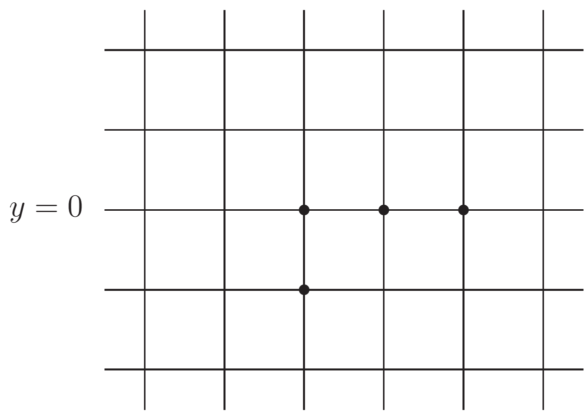


60. [5] Define a sequence $X_0, X_1, \ldots$ of rational numbers by $X_0 = 2$ and $X_{n+1} = X_n - \frac{1}{X_n}$ for $n \geq 0$. Is the sequence bounded?

61. Let $B = \mathbb{Z} \times \mathbb{Z}$, regarded as an infinite chessboard. (Here $\mathbb{Z}$ denotes the set of integers.) Suppose that counters are placed on some subset of the points of B. A counter can jump over another counter one step vertically or horizontally to an empty point, and then remove the counter that was jumped over. Given $n > 0$, let $f(n)$ denote the least number of counters that can be placed on B such that all their y-coordinates are ≤ 0, and such that by some sequence of jumps it is possible for a counter to reach a point with y-coordinate equal to n. For instance, $f(1) = 2$, as shown by the following diagram.



Similarly $f(2) = 4$, as shown by:



(a) [2] Show that $f(3) = 8$ (or at least that $f(3) \leq 8$ by constructing a suitable example).

(b) [2.5] Show that $f(4) = 20$ (or at least that $f(4) \leq 20$).

(c) [3] Find an upper bound for $f(5)$.

62. [3.5] Generalize Problem 12 to n dimensions as follows. Show that there exist $n+1$ lattice points (i.e., points with integer coordinates) in $\mathbb{R}^n$ such that any two of them are the same distance apart if and only if n satisfies the following conditions:

(a) If n is even, then $n+1$ is a square.

(b) If $n \equiv 3 \pmod 4$, then it is always possible.

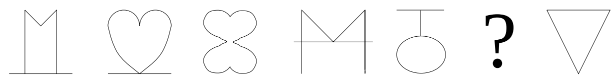
(c) If $n \equiv 1 \pmod 4$, then $n+1$ is a sum of two squares (of nonnegative integers). The well-known condition for this is that if $n+1 = p_1^{a_1} \cdots p_r^{a_r}$ is the factorization of $n+1$ into prime powers, then a_i is even whenever $p_i \equiv 3 (\mathrm{mod}\ 4)$.

63. [5+] Let $H_n = \sum_{j=1}^{n} 1/j$. Show that for all $n \geq 1$,

$$\sum_{d|n} d \leq H_n + (\log H_n)e^{H_n}.$$

18.A34 PROBLEMS #6

64. [1] Find the missing term:



65. [1] (a) A drugstore received a shipment of ten bottles of a certain drug. Each bottle contains one thousand pills. The drugstore received a telegram from the drug company saying that the pills in one bottle each weigh 10 milligrams too much and should be returned immediately. How can the faulty bottle be found with only one weighing?

(b) The next time the druggist received a shipment of ten bottles of the same drug, he again received a telegram from the drug company, this time saying that any any number of the bottles might contain pills each of which was 10 milligrams too heavy. Can all the faulty bottles still be determined with only one weighing?

66. (a) [2] A person has twelve coins, one of which is counterfeit and is either lighter or heavier than the other eleven. How few weighings are necessary on a balance to determine the counterfeit coin and to decide whether it's heavy or light?

(b) [2.5] Fix a positive integer k. Suppose one is given n coins, one of which is counterfeit and is either lighter or heavier than the other $n-1$. What is the largest possible value of n such that in k weighings on a balance it is possible to determine the counterfeit coin and whether it is heavy or light?

67. (a) [1] Show that for any integer x, $x^2 - x$ is divisible by 2.

(b) [1] Show that for any integer x, $x^3 - x$ is divisible by 6.

(c) [2.5] Let $f(n)$ be the largest integer k such that $x^n - x$ is divisible by k for all integers x. For instance, (a) and (b) assert that $f(2) \geq 2$ and $f(3) \geq 6$. Find a "nice" number-theoretic description of $f(n)$ not directly involving $x^n - x$. Show for instance that $f(2) = 2$, $f(3) = 6$, $f(4) = 2$, and $f(5) = 30$.

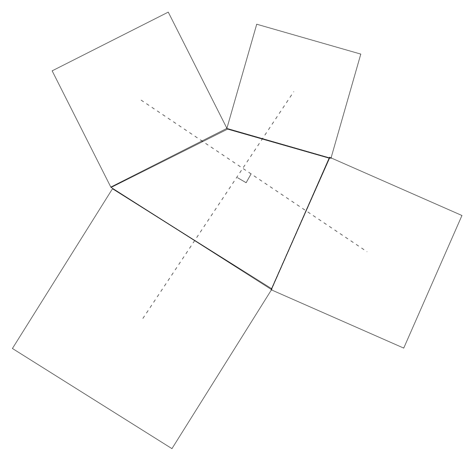
68. [1] (a) A person starts at the point $x = 1$ ft. at time $t = 0$ sec. and moves along the x-axis so that his velocity in feet per second is equal to his distance in feet from $x = 0$. (So in particular at time $t = 0$ his velocity is $1\ \frac{\text{ft}}{\text{sec}}$.) Where will he be in one second?

(b) What if his velocity is $x^2\ \frac{\text{ft}}{\text{sec}}$ when his/her distance from $x = 0$ is x ft?

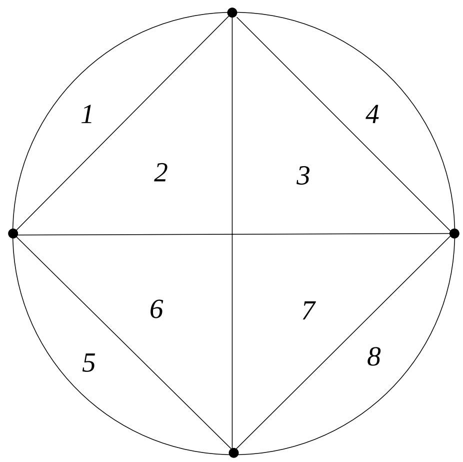
69. [2] Let $P(x)$ be a polynomial of degree n satisfying $P(i) = 2^i$ if $0 \le i \le n$. What is $P(n+1)$?

70. [2] Let n be an integer greater than one. Show that $n^4 + 4^n$ is not prime.

71. [2] Erect a square (facing outwards) on each side of a convex quadrilateral. Join the centers of each pair of opposite squares by a straight line. Show that these two straight lines intersect in right angles.



72. [2] Let $h(n)$ be the number of regions formed inside a circle by choosing n points on the circumference and drawing a straight line segment between every two of the points. Assume the points have been chosen so that no three of these chords intersect in a single point in the interior of the circle. For instance, $h(4) = 8$:



Find a simple formula for $h(n)$.

73. (a) [3] Let V be a finite set of vertices in the plane, no three colinear, and let E be a set of straight line segments (edges) joining two of the vertices in V. Suppose that any two edges in E either have a vertex in common or properly cross (i.e., intersect in their interiors). Show that $\#E \leq \#V$.

    (b) [5] Suppose that the edges in E joining two vertices need not be straight lines. We now assume that every pair of edges intersects exactly once, either at a common vertex or at a proper crossing. Show that the conclusion still holds.

74. [2] A man and a fly both start out at the point $x = 0$ at time $t = 0$. The man walks 4 mi/hr in the positive x-direction. The fly flies at a rate of 10 mi/hr. It continually flies between the man and the point $x = 0$. Where will the fly be after one hour? (Do not confuse this problem with a similar one where a fly flies between two trains moving toward each other.)

75. [5] Show that 462 is the largest integer that cannot be written in the form $ab + ac + bc$, where a, b, c are positive integers. (It is known that there is at most one such integer $n > 462$. If it exists, then it satisfies $n > 2 \cdot 10^{11}$.)

76. [3] Does there exist an infinite binary word $w = a_1 a_2 \cdots$ ($a_i = 0$ or 1) that is not eventually periodic, such that for all n sufficiently large,

the prefix $a_1 a_2 \cdots a_n$ ends in a (nonempty) square, i.e., we can write $a_1 a_2 \cdots a_n = xyy$ (concatenation of words) with y nonempty?

18.A34 PROBLEMS #7

77. [1] (a) What is the least number of weights necessary to weigh any integral number of pounds from 1 lb. to 63 lb. inclusive, if the weights must be placed on only one of the scale-pans of a balance? Generalize to any number of pounds.

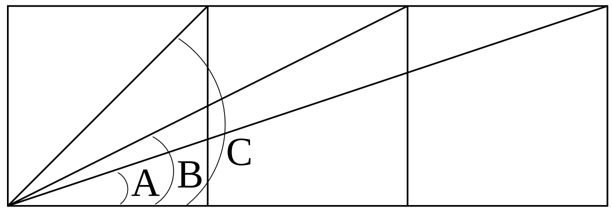
    (b) Same as (a), but from 1 lb. to 40 lb. if the weights can be placed in either of the scale-pans. Generalize.

    (c) A gold chain contains 23 links. What is the least number of links which need to be cut so a jeweler can sell any number of links from 1 to 23, inclusive? Generalize.

78. [2.5] A *perfect partition* of the positive integer n is a finite sequence $a_1 \geq a_2 \geq \cdots \geq a_k$ of positive integers a_i, such that each integer $1 \leq m \leq n$ can be written *uniquely* (regarding equal a_i's as indistinguishable) as a sum of a_i's. For instance, there are three perfect partitions of 5, viz., 11111, 221, and 311, since we have the unique representations 1, $1+1$, $1+1+1$, $1+1+1+1$, $1+1+1+1+1$ in the first case; 1, 2, $2+1$, $2+2$, $2+2+1$ in the second; and 1, $1+1$, 3, $3+1$, $3+1+1$ in the third. Show that the number of perfect partitions of n is equal to the number of *ordered factorizations* of $n+1$ into parts greater than one. For instance, the ordered factorizations of 12 are 12, $6 \cdot 2$, $2 \cdot 6$, $4 \cdot 3$, $3 \cdot 4$, $2 \cdot 2 \cdot 3$, $2 \cdot 3 \cdot 2$, and $3 \cdot 2 \cdot 2$, so there are eight perfect partitions of 11.

79. [1] Here is a proof by induction that all people have the same height. We prove that for any positive integer n, any group of n people all have the same height. This is clearly true for $n = 1$. Now assume it for n, and suppose we have a group of $n+1$ persons, say $P_1, P_2, \ldots, P_{n+1}$. By the induction hypothesis, the n people $P_1, P_2, \ldots, P_n$ all have the same height. Similarly the n people $P_2, P_3, \ldots, P_{n+1}$ all have the same height. Both groups of people contain $P_2, P_3, \ldots, P_n$, so P_1 and P_{n+1} have the same height as $P_2, P_3, \ldots, P_n$. Thus all of $P_1, P_2, \ldots, P_{n+1}$ have the same height. Hence by induction, for any n any group of n people have the same height. Letting n be the total number of people in the world, we conclude that all people have the same height. Is there a flaw in this argument?

80. [1] The following figure consists of three equal squares lined up together, with three diagonals as shown.



Show that angle C is the sum of angles A and B.

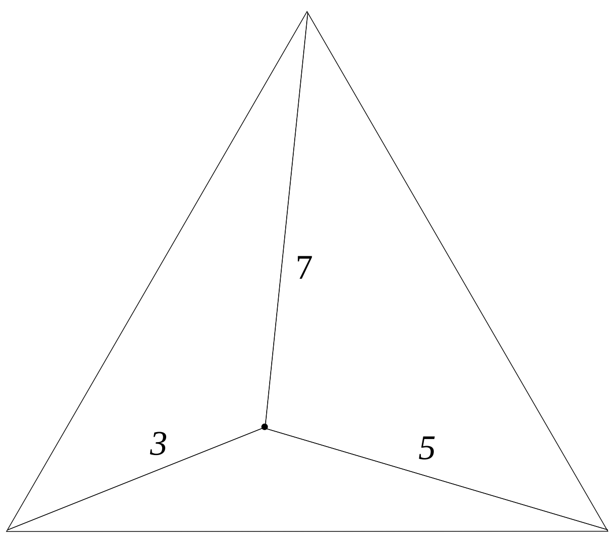
81. [1] Let n be a positive integer.

    (a) Show that if $2^n - 1$ is prime, then n is prime.

    (b) Show that if $2^n + 1$ is prime, then n is a power of two.

    Hint: The simplest way to show that a number is not prime is to factor it explicitly.

82. (a) [2.5] A point P in the interior of an equilateral triangle T is at a distance of 3, 5, and 7 units from the three vertices of T. What is the length of a side of T?



(b) [2.5] More generally, let the point P be at distances a, b, c from the vertices A, B, C of an equilateral triangle of side length d. Find a (nonzero) polynomial equation $f(a, b, c, d) = 0$, symmetric in a, b, c, d.

(c) [2.5] The symmetry of f in a, b, c is obvious, but why also the "hidden symmetry" in d? Find a noncomputational proof.

(d) [2.8] Generalize to n dimensions, i.e., find a (nonzero) polynomial equation $f(a_0, \ldots, a_n, d) = 0$, symmetric in all $n+2$ variables, satisfied by the distances $a_0, \ldots, a_n$ from a point to the vertices of a regular simplex of side length d.

(e) [5] Give a noncomputational explanation of the hidden symmetry of the variable d.

83. [3] Into how few pieces can an equilateral triangle be cut and reassembled to form a square?

84. [2] Let n be an integer greater than one. Show that $1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$ is not an integer.

85. [2.5]

(a) Persons X and Y have nonnegative integers painted on their foreheads which only the other can see. They are told that the sum of the two numbers is either 100 or 101. A third person P asks X if he knows the number on his forehead. If X says "no," then P asks Y. If Y says "no," then P asks X again, etc. Assume both X and Y are perfect logicians. Show that eventually one of them will answer "yes." (This may seem paradoxical. For instance, if X and Y both have 50 then Y knows that X will answer "no" to the first question, since from Y's viewpoint X will see either 50 or 51, and in either case cannot deduce his number. So how does either person gain information?)

(b) Generalize to more than two persons.

86. [2.5] Let $a(n)$ be the exponent of the largest power of 2 dividing the numerator of $\sum_{i=1}^{n} \frac{2^i}{i}$ (when written as a fraction in lowest terms). For instance, $a(1) = 1$, $a(2) = 2$, $a(3) = 2$, $a(4) = 5$. Show that $\lim_{n \to \infty} a(n) = \infty$.

87. [3–] Write the permutation $n, n-1, \ldots, 1$ as a product of $\binom{n}{2}$ (the minimum possible) adjacent transpositions $s_i = (i, i+1)$, $1 \le i \le n-1$. For instance, $321 = s_1 s_2 s_1$ (or $s_2 s_1 s_2$). What is the least number of s_i's we need to remove from this product in order to get a product equal to the identity permutation $1, 2, \ldots, n$? For instance, if we remove s_2 from $s_1 s_2 s_1$ then we get $s_1^2 = 123$ (clearly the minimum possible for $n = 3$). Does the answer depend on the way in which we write $n, n-1, \ldots, 1$ as a product of s_i's?

18.A34 PROBLEMS #8

89. [1] Pat beat Stacy in a set of tennis, winning six games to Stacy's three. Five games were won by the player who did not serve. Who served first?

90. [1] Given a seven-minute hourglass and an eleven-minute hourglass, what is the quickest way to time the boiling of an egg for two minutes? Assume that an hourglass cannot be turned upside-down unless it is inactive, i.e., all the sand is at the bottom.

91. [1] We all know

$$\int_{-1}^{1} \frac{dx}{1+x^2} = \frac{\pi}{2}.$$

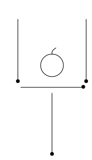
Let $x = 1/y$, so $dx = -dy/y^2$. When $x = 1$ and -1, then also $y = 1$ and -1. Thus

$$\frac{\pi}{2} = \int_{-1}^{1} \frac{-dy/y^2}{1+(1/y)^2} = -\int_{-1}^{1} \frac{dy}{1+y^2} = -\frac{\pi}{2}.$$

Hence $\pi = 0$. What (if anything) is wrong?

92. [1] Three men play a game with the understanding that the loser is to double the money of the other two. After three games, each has lost just once; and each has \$24. How much did each have to start?

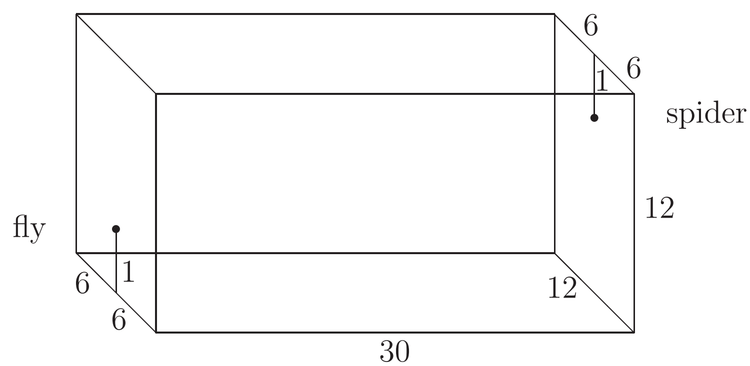
93. [1] Move the smallest number of matches to get the cherry outside the glass. The glass may have any orientation at the finish, but of course the cherry cannot be moved.



94. [2] Let n be a positive integer. How many subsets of the set $\{1, 2, \ldots, n\}$ do not contain two consecutive integers? For instance, when $n = 3$ there are the five subsets $\emptyset$, $\{1\}$, $\{2\}$, $\{3\}$, $\{1, 3\}$.

95. [2] What positive integers can be expressed as the di erence between two squares? The first six are $1 = 1^2 - 0^2$, $3 = 2^2 - 1^2$, $4 = 2^2 - 0^2$, $5 = 3^2 - 2^2$, $7 = 4^2 - 3^2$, $8 = 3^2 - 1^2$.

96. [2] In a room 30 feet long, 12 feet wide, and 12 feet high, a fly stands on an end wall one foot above the floor and 6 feet from each side wall. A spider stands on the opposite end wall one foor below the ceiling and 6 feet from each side wall. What is the minimum distance the spider needs to walk along the walls, floor, and ceiling in order to reach the fly?



97. (a) [2.5] Can an obtuse triangle be dissected into finitely many acute triangles?

    (b) [3.5] Can a 3-dimensional cube be dissected into finitely many acute simplices? A (3-dimensional) *simplex* is any polyhedron with four vertices, six edges, and four triangular faces. A simplex is *acute* if the angle between any two faces is less than $\pi/2$.

98. [2.5] Two baseball teams A and B play against each other for the first time. All that is known about the teams is that A has won a fraction p of its games and that B has won a fraction q, playing against comparable opponents. What is the most reasonable probability $f(p, q)$ that A will beat B? For instance, a little thought will convince you that $f(p, 1/2) = p$, $f(p, 1) = 0$ if $p \neq 1$, $f(p, q) + f(q, p) = 1$, etc. But it's not so obvious, for instance, what $f(.7, .4)$ should be.

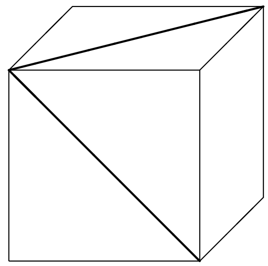
99. [2.5] n coins are placed in a circle, either heads or tails up. Player A wants to turn all the coins heads up but is never aware of the status

of the coins. When it is his turn, he can specify any subset of the n *positions* occupied by the coins. Player B can then rotate the coins around the circle as many spaces as she wants. She then turns over all the coins in the positions specified by A. For what values of n can A win, i.e., in a finite amount of time guarantee that all coins are heads? For instance, A can win when $n = 2$, by first choosing positions $\{1, 2\}$, then $\{1\}$, and then $\{1, 2\}$. It is easy to check that no matter what the initial configuration of the coins, at some stage both coins will be heads.

100. [3] Let M be the $2^n \times n$ matrix whose rows consist of all 2^n distinct vectors of ± 1's (in some order) of length n. Change any subset of the entries to 0's. Show that some nonempty subset of the rows of the resulting matrix sums to the vector $[0, 0, \ldots, 0]$.

18.A34 PROBLEMS #9

101. [1] Find the size of the planar angle formed by two face diagonals of a cube with a common vertex. (Try to find an elegant, noncomputational solution.)



102. [1.5] Find the missing term:

$$10, 11, 12, 13, 14, 15, 16, 17, 20, 22, 24, 31, 100, \_\_\_\_, 10000.$$

103. [1.5] Explain the rule which generates the following sequence:

$$2, 3, 10, 12, 13, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30,$$
$$31, 32, 33, 34, 35, 36, 37, 38, 39, 200, 201, 202, \dots$$

HINT: *Don't* think mathematically!

104. [1] (a) Two players play the following game. They start with a pile of 101 stones. The players take turns removing either 1, 2, 3, or 4 stones from the pile. The player who takes the last stone wins. Assuming both players play perfectly, will the first or second player win?

(b) What if the person who takes the last stone loses?

105. [1.5] Why does a mirror reverse left and right but not up and down? (This is not a frivolous question.)

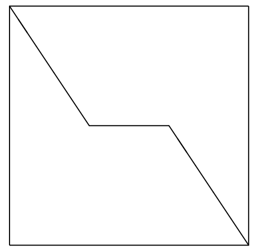
106. [1] Solve the recurrence

$$f(n+1) = nf(n) + (n-1)f(n-1) + \cdots + 2f(2) + f(1) + 1, \quad f(0) = 1.$$

107. [2] From a 100×100 chessboard remove any white square and any black square. Show that the remaining board can be covered with 4999 non-overlapping dominoes. (Each domino covers two adjacent squares.)

108. A collection of line segments inside or on the boundary of a square of side one is said to be *opaque* if every (infinite) straight line which crosses the square makes contact with at least one of the segments. For example, the two diagonals are opaque of total length $2\sqrt{2} \approx 2.82$.

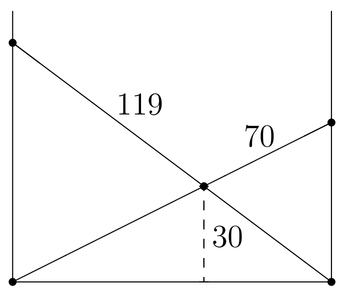
(a) [2] The following symmetric pattern is opaque. (The sides of the square are not part of the pattern.)



Show that its minimum total length is $1+\sqrt{3}\approx 2.73$.

(b) [2.5] Can you find a shorter opaque set? So far as I know, the smallest known opaque set has length $\sqrt{2}+\frac{1}{2}\sqrt{6}\approx 2.64$, and it is not known whether a smaller one exists.

109. [2.5] Two ladders of length 119 feet and 70 feet lean between two vertical walls so that they cross 30 feet above the ground. How far apart are the walls?



110. [5] Let $\left(\frac{n}{7}\right)$ denote the Legendre symbol. Specifically,

$$\left(\frac{n}{7}\right)=\begin{cases} 0, & n\equiv 0 \pmod 7 \\ 1, & n\equiv 1,2,4 \pmod 7 \\ -1, & n\equiv 3,5,6 \pmod 7). \end{cases}$$

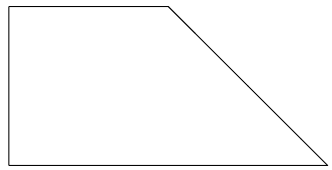
Show that

$$\frac{24}{7\sqrt{7}}\int_{\pi/3}^{\pi/2}\log\left|\frac{\tan t+\sqrt{7}}{\tan t-\sqrt{7}}\right|dt=\sum_{n\geq 1}\left(\frac{n}{7}\right)\frac{1}{n^2}.$$

The main point of this exercise is to give an example of an explicit identity that can be computed to any degree of accuracy but which is only conjecturally true.

18.A34 PROBLEMS #10

111. [1] The following shape consists of a square and half of another square of the same size, divided diagonally.



Cut the shape into four congruent pieces.

112. [1.5] Take five right triangles with legs of length one and two, cut one of them, and put the resulting six pieces together to form a square.

113. [1.5] Find an integer n whose first digit is three, such that $3n/2$ is obtained by removing the 3 at the beginning and putting it at the end.

114. (a) [1] Show that for any real x, $e^x > x$.
   (b) [1.5] Find the largest real number α for which it is *false* that $\alpha^x > x$ for all real x.

115. (a) [1] What amounts of postage cannot be obtained using only 5 cent stamps and 7 cent stamps? (For instance, 9 cents cannot be obtained, but $17 = 5 + 5 + 7$ cents can be.)
   (b) [2.5] Let a and b be relatively prime positive integers. For how many positive integers c is it impossible to obtain postage of c cents using only a cent and b cent stamps? What is the largest value of c with this property?

116. [2.5] Two players A and B play the following game. Fix a positive real number x. A and B each choose the number 1 or 2. A gives B one dollar if the numbers are different. B gives A x dollars times the sum of their numbers. For instance, if A chooses 1 and B chooses 2, then A gives B one dollar and B gives A $3x$ dollars. Both players are playing their best possible strategy. What value of x makes the game fair, i.e., in the long run both players should break even?

117. [3] Given positive integers n and b, define the *total b-ary expansion* $T_b(n)$ as follows: Write n as a sum of powers of b, with no power occurring more than $b-1$ times. (This is just the usual base b expansion of n.) For instance, if $n = 357948$ and $b = 3$, then we get

$$3^{11} + 3^{11} + 3^7 + 3^6 + 3^6 + 3^2.$$

Now do the same for each exponent, giving

$$3^{3^2+1+1} + 3^{3^2+1+1} + 3^{3+3+1} + 3^{3+3} + 3^{3+3} + 3^{1+1}.$$

Continue doing the same for every exponent not already a b or 1, until finally only b's and 1's appear. In the present case we get that $T_3(357948)$ is the array

$$3^{3^{1+1}+1+1} + 3^{3^{1+1}+1+1} + 3^{3+3+1} + 3^{3+3} + 3^{3+3} + 3^{1+1}.$$

Now define a sequence $a_0, a_1, \dots$ as follows. Choose a_0 to be any positive integer, and choose a base $b_0 > 1$. To get a_1, write the total b_0-ary expansion $T_{b_0}(a_0 - 1)$ of $a_0 - 1$, choose a base $b_1 > b_0$, and replace every appearance of b_0 in $T_{b_0}(a_0 - 1)$ by b_1. This gives the total b_1-ary expansion of the next term a_1. To get a_2, write the total b_1-ary expansion $T_{b_1}(a_1 - 1)$ of $a_1 - 1$, choose a base $b_2 > b_1$, and replace every appearance of b_1 in $T_{b_1}(a_1 - 1)$ by b_2. This gives the total b_2-ary expansion of the next term a_2. Continue in this way to obtain $a_3, a_4, \dots$. In other words, given a_n and the previously chosen base b_n, o get a_{n+1}, write the total b_n-ary expansion $T_{b_n}(a_n - 1)$ of $a_n - 1$, choose a base $b_{n+1} > b_n$, and replace every appearance of b_n in $T_{b_n}(a_n - 1)$ by b_{n+1}. This gives the total b_{n+1}-ary expansion of the next term a_{n+1}.

Example. Choose $a_0 = 357948$ and $b_0 = 3$ as above. Then

$$a_0 - 1 = 357947 = 3^{3^{1+1}+1+1} + 3^{3^{1+1}+1+1} + 3^{3+3+1} + 3^{3+3} + 3^{3+3} + 3 + 3 + 1 + 1.$$

Choose $b_1 = 10$. Then

$$\begin{aligned} a_1 &= 10^{10^{1+1}+1+1} + 10^{10^{1+1}+1+1} + 10^{10+10+1} + 10^{10+10} + 10^{10+10} + 10 + 10 + 1 + 1 \\ &= 10^{102} + 10^{102} + 10^{21} + 10^{20} + 10^{20} + 22. \end{aligned}$$

Then

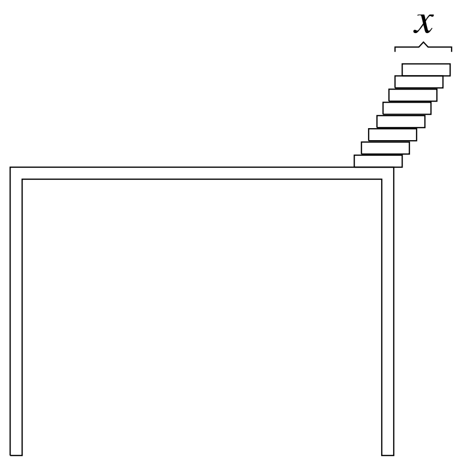
$$a_1 - 1 = 10^{10^{1+1}+1+1} + 10^{10^{1+1}+1+1} + 10^{10+10+1} + 10^{10+10} + 10^{10+10} + 10 + 10 + 1.$$

Choose $b_2 = 766$. Then

$$a_2 = 766^{766^{1+1}+1+1} + 766^{766^{1+1}+1+1} + 766^{766+766+1} + 766^{766+766} + 766^{766+766} + 766 + 766 + 1,$$

etc. Prove that for some n we have $a_n = a_{n+1} = \cdots = 0$. (Note how counterintuitive this seems. How could we not force $a_n \to \infty$ by choosing the b_n's sufficiently large?)

118. [2.5] What is the longest possible overhang x that can be obtained by stacking dominos of unit length over the edge of a table, as illustrated below? (The condition for the dominos not to fall is that the center of mass of all the dominos above any domino D lies directly above D.)

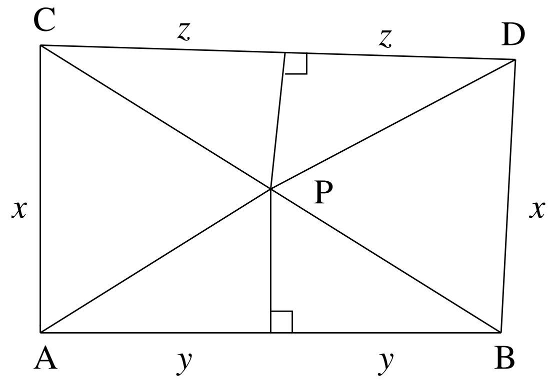


119. [3] Let G be a simple (i.e., no loops or multiple edges) finite graph and v a vertex of G. The *neighborhood* $N(v)$ of v consists of v and all adjacent vertices. Show that there exists a subset S of the vertex set of G such that $\#(S \cap N(v))$ is odd for all vertices v of G.

120. [2.5] Let G be a simple graph with an odd number of vertices. Show that there exists a nonempty subset S of the vertices such that every vertex of G is adjacent to an even number of elements of S. (A vertex is *not* considered to be adjacent to itself.)

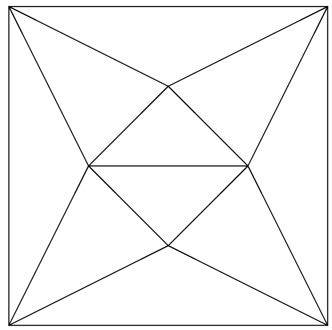
121. [2.5] An ant is constrained to walk on the walls, floor, and ceiling of a $1 \times 1 \times 2$ room. The ant stands in a corner of the room. From its perspective, what point(s) in the room is the farthest away, and what is the distance of this point from the ant? Hint. The farthest point is *not* the opposite corner!

122. [2] Draw a line segment AB in the plane. Let AC be perpendicular to AB. Draw BD so that D is on the same side of AB as C, angle ABD is 91°, and AC and BD have the same length. Let the perpendicular bisectors of AB and CD meet in P. Angles PAB and PBA are equal since P is on the perpendicular bisector of AB. For the same reason AP and BP have the same length. Similarly CP and DP have the same length. Thus triangles ACP and BDP are congruent, so the angles CAP and DBP are equal. It follows that the angles CAB and DBA are equal, i.e., $90° = 91°$. Is there a flaw in this argument, or is this the end of mathematics?


C
z
z
D
x
P
x
A
y
y
B

18.A34 PROBLEMS #11

123. [1] Can the following figure be drawn without lifting one's pencil from the paper or retracing any lines?



124. A *covering congruence* is a finite set $(a_1, m_1), \ldots, (a_k, m_k)$ of ordered pairs of integers, where $0 < m_1 < \cdots < m_k$, such that every integer can be written in at least one of the forms $m_i x + a_i$, where x is an integer.

(a) [1] Find a covering congruence with $m_1 > 1$.

(b) [4] Does there exist a covering congruence with m_1 arbitrarily large? (There exists one with $m_1 = 40$ and $k > 10^{50}$.)

(c) [5] Does there exist a covering congruence with each m_i odd?

(d) [1.5] Show that in a covering congruence,

$$\frac{1}{m_1} + \cdots + \frac{1}{m_k} \geq 1.$$

(e) [2.5] Show that in a covering congruence,

$$\frac{1}{m_1} + \cdots + \frac{1}{m_k} > 1.$$

This is equivalent [why?] to saying that there does not exist a covering congruence where every integer can be written *uniquely* in the form $m_i x + a_i$. In other words, the set $\mathbb{Z}$ of integers cannot be written as a *disjoint* union of arithmetic progressions, all with different differences.

125. (a) [2] Evaluate the infinite product

$$\prod_{n=0}^{\infty} \left(1 + x^{2^n}\right), \qquad |x| < 1.$$

(b) [3.5] Consider the Taylor series expansion

$$\prod_{n=0}^{\infty}\left(1+x^{2^n}+x^{2^{n+1}}\right)=a_0+a_1x+\cdots.$$

The sequence $(a_0, a_1, \ldots)$ begins

$$(1,1,2,1,3,2,3,1,4,3,5,2,5,3,4,1,5,4,7,3,8,\ldots).$$

Show that every positive rational number occurs exactly once among the fractions a_{i+1}/a_i, and moreover these fractions are in lowest terms.

(c) [3] Let $f(x) = 1/(1+2\lfloor x\rfloor - x)$, and define a sequence $x_0, x_1, \ldots$ by $x_0 = 1$ and $x_{n+1} = f(x_n)$ for $n \geq 0$. Show that $x_n = a_{n+1}/a_n$.

126. [1]–[5] Color the (x, y)-plane in two colors, i.e., color every point in the plane either red or blue, say. Will there always exist two points at distance one which are the same color? What about three colors? Four colors? Etc.

127. [2.5] Let $f(n)$ be a nonconstant polynomial with integer coefficients. Show that $f(n)$ is composite for infinitely many values of n. (NOTE: It is not known whether there exists a polynomial $g(n)$ of degree greater than one with integer coefficients such that $g(n)$ is prime for infinitely many values of n. However, many polynomials, such as $n^2 + 1$, are conjectured to have this property.)

128. [2.5] What positive real numbers a have the following property? For every continuous function $f(x)$ on the closed interval $[0, 1]$ satisfying $f(0) = f(1) = 0$, there is some value of x for which $f(x) = f(x + a)$.

129. [2.5] Person A writes down one positive real number on each of two slips of paper, so that one of the numbers is twice the other. Person A then turns the slips face down and asks person B to choose a slip. A agrees to pay B the amount on the slip (in dollars) which B has chosen. After B picks his slip and turns it over, showing a number x, A then says that B if he wants can choose the other slip instead. B reasons as follows: There is a 50% chance that the number on the other slip is $2x$, and a 50% chance it is $x/2$. Therefore the expected amount of money B will receive if he chooses the other slip is $.5(2x) + .5(x/2) = 5x/4 > x$. Thus B should always choose the other slip. Is there a flaw in this reasoning?

130. [3.5] Let x be a positive real number. Show that a square can be tiled with finitely many rectangles (in any orientation) similar to a $1 \times x$ rectangle if and only if x is algebraic (satisfies a polynomial equation with integer coefficients) and all its conjugates (the other roots of the polynomial of least degree satisfied by x) have positive real part. For instance, $x = \sqrt{2}$ and $x = 1 + \sqrt{2}$ cannot be done, since the conjugates are $-\sqrt{2} < 0$ and $1 - \sqrt{2} < 0$. However, $x = \frac{3}{2} + \sqrt{2}$ is possible since $\frac{3}{2} - \sqrt{2} > 0$.

131. [3] Let $F(x) = (1-x)(1-x^2)(1-x^3)(1-x^5)(1-x^8)\cdots$, where the exponents are Fibonacci numbers. Show that every coefficient of $F(x)$ (when expanded as a power series in x) is 0, 1, or -1.

132. [3] Suppose that $f : \mathbb{R} \to \mathbb{R}$ is a function such that f^2 and f^3 are infinitely differentiable on all of $\mathbb{R}$. Is the same true for f?

133. [2.5] Let S be an n-element set and $\mathcal{A}$ a collection of subsets of S such that (a) every $T \in \mathcal{A}$ has an odd number of elements, and (b) the intersection of any two (distinct) subsets in $\mathcal{A}$ has an even number of elements. Show that $\#\mathcal{A} \leq n$.

18.S34 (FALL, 2007)

PIGEONHOLE PROBLEMS

NOTE: Notation such as (78P) means a problem from the 1978 Putnam Exam.

1. (78P) Let A be any set of 20 distinct integers chosen from the arithmetic progression $1, 4, 7, \ldots, 100$. Prove that there must be two distinct integers in A whose sum is 104. [Actually, 20 can be replaced by 19.]

2. Five points are situated inside an equilateral triangle whose side has length one unit. Show that two of them may be chosen which are less than one half unit apart. What if the equilateral triangle is replaced by a square whose side has length of one unit?

3. (71P) Let there be given nine lattice points (points with integral coordinates) in three dimensional Euclidean space. Show that there is a lattice point on the interior of one of the line segments joining two of these points.) [To test your understanding, how many lattice points does one need in *four* dimensions to reach the same conclusion?]

4. (72IMO) Prove that from a set of ten distinct two-digit numbers (in the decimal system), it is possible to select two disjoint subsets whose members have the same sum. [Though not stated in the problem, one should assume that not both the subsets are empty, or even that neither of the subsets is empty.]

5. (80P) Let $A_1, A_2, \ldots, A_{1066}$ be subsets of a finite set X such that $|A_i| > \frac{1}{2}|X|$ for $1 \leq i \leq 1066$. Prove that there exist ten elements $x_1, \ldots, x_{10}$ of X such that every A_i contains at least one of $x_1, \ldots, x_{10}$. (Here $|S|$ means the number of elements in the set S.)

6. Given any $n+2$ integers, show that there exist two of them whose sum, or else whose difference, is divisible by $2n$.

7. Given any $n+1$ distinct integers between 1 and $2n$, show that two of them are relatively prime. Is this result best possible, i.e., is the conclusion still true for n integers between 1 and $2n$?

8. Given any $n+1$ integers between 1 and $2n$, show that one of them is divisible by another. Is this best possible, i.e., is the conclusion still true for n integers between 1 and $2n$?

9. Given any $2n-1$ integers, show that there are n of them whose sum is divisible by n. (Though superficially similar to some other pigeonhole problems, this problem is much more difficult and does not really involve the pigeonhole principle.)

10. Is it possible to cut an 8×8 chessboard with 13 straight lines (none passing through the midpoint of a square) such that every piece contains at most one midpoint of a square?

11. Let each of nine lines cut a square into two quadrilaterals whose areas are in the proportion $2:3$. Prove that at least three of the lines pass through the same point.

12. Let $a_1 < \cdots < a_n$, $b_1 > \cdots > b_n$, and $\{a_1, \ldots, a_n, b_1, \ldots, b_n\} = \{1, 2, \ldots, 2n\}$. Show that

$$\sum_{i=1}^{n} |a_i - b_i| = n^2.$$

13. Let u be an irrational real number. Let S be the set of all real numbers of the form $a + bu$, where a and b are integers. Show that S is dense in the real numbers, i.e., for any real number x and any $\epsilon > 0$, there is a element $y \in S$ such that $|x - y| < \epsilon$. (HINT. First let $x = 0$.)

14. Two disks, one smaller than the other, are each divided into 200 congruent sectors. In the larger disk 100 of the sectors are chosen arbitrarily and painted red; the other 100 sectors are painted blue. In the smaller disk each sector is painted either red or blue with no stipulation on the number of red and blue sectors. The small disk is then placed on the larger disk so that their centers coincide. Show that it is possible to align the two disks so that the number of sectors of the small disk whose color matches the corresponding sector of the large disk is at least 100.

15. A collection of subsets of $\{1, 2, \ldots, n\}$ has the property that each pair of subsets has at least one element in common. Prove that there are at most 2^{n-1} subsets in the collection.

16. (95P) For a partition π of $\{1,2,3,4,5,6,7,8,9\}$, let $\pi(x)$ be the number of elements in the part containing x. Prove that for any two partitions π and π', there are two distinct numbers x and y in $\{1,2,3,4,5,6,7,8,9\}$ such that $\pi(x)=\pi(y)$ and $\pi'(x)=\pi'(y)$.

    [A *partition* of a set S is a collection of (nonempty) disjoint subsets (parts) whose union is S.]

17. (a) Let p be a prime of the form $8k+1$. Prove that there exist positive integers a,b,m with $m<2p$ such that $mp=a^4+b^4$.

    HINT. Show that there is an integer x such that x^4+1 is divisible by p, and consider the numbers $u+vx$, where $0 \le u \le \lfloor\sqrt{p}\rfloor$ and $0 \le v \le \lfloor\sqrt{p}\rfloor$.

    NOTE. This is a minor variation of a standard application of the pigeonhole principle going back to Fermat. Do not hand in this problem if you've seen it or something similar before.

    (b) Improve the bound $m<2p$. In particular, find a constant $c<2$ such that one can take $m<cp$ for p large. (The best possible value of c requires some sophisticated number theory not involving the pigeonhole principle.)

18. $\mathbb{N}$ is the set of nonnegative integers. For any subset S of $\mathbb{N}$, let $P(S)$ be the set of all *pairs* of members of S. (A *pair* is a set (unordered) with two distinct members.) Partition $P(\mathbb{N})$, arbitrarily, into two sets (of pairs) P_1 and P_2. Prove that $\mathbb{N}$ must contain an infinite subset S such that either $P(S)$ is contained in P_1 or $P(S)$ is contained in P_2.

18.S34 (FALL, 2007)

GREATEST INTEGER PROBLEMS

NOTE: We use the notation $\lfloor x \rfloor$ for the greatest integer $\leq x$, even if the original source used the older notation $[x]$.

1. (48P) If n is a positive integer, prove that
$$\left\lfloor \sqrt{n} + \sqrt{n+1} \right\rfloor = \left\lfloor \sqrt{4n+2} \right\rfloor.$$

2. (a) Let p denote a prime number, and let m be any positive integer. Show that the exponent of the highest power of p which divides $m!$ is
$$\left\lfloor \frac{m}{p} \right\rfloor + \left\lfloor \frac{m}{p^2} \right\rfloor + \cdots + \left\lfloor \frac{m}{p^s} \right\rfloor,$$
where $p^{s+1} > m$.

   (b) In how many zeros does the number 1000! end, when written in base 10?

3. (a) Prove that the exponent of the highest power of p which divides $\binom{n}{m}$ is equal to the number of carries that occur when n and $m-n$ are added in base p (Kummer's theorem).

   (b) For $n > 1$ a composite integer, prove that not all of
$$\binom{n}{1}, \ldots, \binom{n}{n-1}$$
can be divisible by n.

4. Prove that for any positive integers i, j, k,
$$\frac{(3i)!(3j)!(3k)!}{i!j!k!(i+j)!(j+k)!(k+i)!}$$
is an integer.

5. Prove that for any integers $n_1, \ldots n_k$, the product
$$\prod_{1 \leq i < j \leq k} \frac{n_j - n_i}{j - i}$$
is an integer.

6. (68IMO) For every natural number n, evaluate the sum

$$\sum_{k=0}^{\infty} \left\lfloor \frac{n+2^k}{2^{k+1}} \right\rfloor = \left\lfloor \frac{n+1}{2} \right\rfloor + \left\lfloor \frac{n+2}{4} \right\rfloor + \cdots + \left\lfloor \frac{n+2^k}{2^{k+1}} \right\rfloor + \cdots.$$

7. A sequence of real numbers is defined by the *nonlinear* first order recurrence

$$u_{n+1} = u_n(u_n^2 - 3).$$

   (a) If $u_0 = 5/2$, give a simple formula for u_n.

   (b) If $u_0 = 4$, how many digits (in base ten) does $\lfloor u_{10} \rfloor$ have?

8. Define a sequence $a_1 < a_2 < \cdots$ of positive integers as follows. Pick $a_1 = 1$. Once $a_1, \ldots, a_n$ have been chosen, let a_{n+1} be the least positive integer not already chosen and not of the form $a_i + i$ for $1 \leq i \leq n$. Thus $a_1 + 1 = 2$ is not allowed, so $a_2 = 3$. Now $a_2 + 2 = 5$ is also not allowed, so $a_3 = 4$. Then $a_3 + 3 = 7$ is not allowed, so $a_4 = 6$, etc. The sequence begins:

$$1, 3, 4, 6, 8, 9, 11, 12, 14, 16, 17, 19, \ldots.$$

   Find a simple formula for a_n. Your formula should enable you, for instance, to compute $a_{1,000,000}$.

9. (a) (Problem A6, 93P; no contestant solved it.) The infinite sequence of 2's and 3's

$$2, 3, 3, 2, 3, 3, 3, 2, 3, 3, 3, 2, 3, 3, 2, 3, 3, 3, 2, 3, 3, 3, 2, 3, 3, 3, 2, 3, 3, 2, 3, 3, 3, 2, \ldots$$

   has the property that, if one forms a second sequence that records the number of 3's between successive 2's, the result is identical to the first sequence. Show that there exists a real number r such that, for any n, the nth term of the sequence is 2 if and only if $n = 1 + \lfloor rm \rfloor$ for some nonnegative integer m.

   (b) (similar in flavor to (a), though not involving the greatest integer function) Let $a_1, a_2, \ldots$ be the sequence

$$1, 2, 2, 3, 3, 4, 4, 4, 5, 5, 5, 6, 6, 6, 6, 7, 7, 7, 7, 8, 8, 8, 8, 9, 9, 9, 9, 9, \ldots$$

of integers a_n defined as follows: $a_1 = 1$, $a_1 \le a_2 \le a_3 \le \cdots$, and a_n is the number of n's appearing in the sequence. Find real numbers $\alpha, c > 0$ such that

$$\lim_{n\to\infty} \frac{a_n}{n^\alpha} = c.$$

10. (Problem B6, 95P; five of the top 204 contestants received at least 9 points (out of 10), and no one received 3–8 points.) For a positive real number α, define

$$S(\alpha) = \{\lfloor n\alpha \rfloor \,:\, n = 1, 2, 3, \ldots\}.$$

Prove that $\{1, 2, 3, \ldots\}$ cannot be expressed as the disjoint union of three sets $S(\alpha)$, $S(\beta)$, and $S(\gamma)$.

11. Let m be a positive integer and k any integer. Define a sequence $a_m, a_{m+1}, \ldots$ as follows:

$$\begin{aligned} a_m &= k \\ a_{n+1} &= \left\lfloor \frac{n+2}{n} a_n \right\rfloor, \quad n \ge m. \end{aligned}$$

Show that there exists a positive integer N and polynomials $P_0(n)$,$P_1(n)$, $\ldots$, $P_{N-1}(n)$ such that for all $0 \le i \le N-1$ and all integers t for which $tN + i \ge m$, we have

$$a_{tN+i} = P_i(t).$$

12. (Problem B1, 97P; 171 of the top 205 contestants received 10 points, and 14 others received 8–9 points.) Let $\{x\}$ denote the distance between the real number x and the nearest integer. For each positive integer n, evaluate

$$F_n = \sum_{m=1}^{6n-1} \min\left(\left\{\frac{m}{6n}\right\}, \left\{\frac{m}{3n}\right\}\right).$$

(Here $\min(a, b)$ denotes the minimum of a and b.)

13. (Problem B4, 98P; 73 of the top 199 contestants received at least 8 points.) Find necessary and sufficient conditions on positive integers m and n so that

$$\sum_{i=0}^{mn-1} (-1)^{\lfloor i/m \rfloor + \lfloor i/n \rfloor} = 0.$$

14. (Problem B3, 01P; 92 of the top 200 contestants received at least 8 points.) For any positive integer n, let $\langle n\rangle$ denote the closest integer to $\sqrt{n}$. Evaluate

$$\sum_{n=1}^{\infty} \frac{2^{\langle n\rangle}+2^{-\langle n\rangle}}{2^n}.$$

15. (Problem B3, 03P; 152 of the top 201 contestants received at least 8 points.) Show that for each positive integer n,

$$n! = \prod_{i=1}^{n} \mathrm{lcm}\{1,2,\dots,\lfloor n/i\rfloor\}.$$

(Here lcm denotes the least common multiple.)

16. Define $a_1 = 1$ and

$$a_{n+1} = \lfloor\sqrt{2a_n(a_n+1)}\rfloor, \quad n \geq 1.$$

Thus $(a_1,\dots,a_{10}) = (1,2,3,4,6,9,13,19,27,38)$. Show that $a_{2n+1} - a_{2n} = 2^{n-1}$, and find a simple description of $a_{2n+1} - 2a_{2n-1}$.

17. Prove that for all positive integers m, n,

$$\gcd(m,n) = m + n - mn + 2\sum_{k=0}^{m-1} \left\lfloor \frac{kn}{m} \right\rfloor.$$

18. Let a, b, c, d be real numbers such that $\lfloor na\rfloor + \lfloor nb\rfloor = \lfloor nc\rfloor + \lfloor nd\rfloor$ for all positive integers n. Prove that at least one of $a+b, a-c, a-d$ is an integer.

19. Let p be a prime congruent to 1 modulo 4. Prove that

$$\sum_{i=1}^{(p-1)/4} \lfloor\sqrt{ip}\rfloor = \frac{p^2-1}{12}.$$

20. Which positive integers can be written in the form $n + \lfloor\sqrt{n} + \frac{1}{2}\rfloor$ for some positive integer n?

21. For n a positive integer, let x_n be the last digit in the decimal representation of $\lfloor 2^{n/2}\rfloor$. Is the sequence $x_1, x_2, \dots$ periodic?

18.S34 (FALL 2007)

PROBLEMS ON ROOTS OF POLYNOMIALS

NOTE. The terms "root" and "zero" of a polynomial are synonyms. Those problems which appeared on the Putnam Exam are stated as they appeared verbatim (except for one minor correction and one clarification).

1. (39P) Find the cubic equation whose roots are the cubes of the roots of
$$x^3 + ax^2 + bx + c = 0.$$

2. (a) (40P) Determine all rational values for which a, b, c are the roots of
$$x^3 + ax^2 + bx + c = 0.$$
   (b) (not on Putnam Exam) Show that the only real polynomials $\prod_{i=0}^{n-1}(x-a_i) = x^n + a_{n-1}x^{n-1} + \cdots + a_0$ in addition to those given by (a) are $x^n, x^2 + x - 2$, and exactly two others, which are approximately equal to
$$x^3 + .56519772x^2 - 1.76929234x + .63889690$$
   and
$$x^4 + x^3 - 1.7548782x^2 - .5698401x + .3247183.$$

3. (51P) Assuming that all the roots of the cubic equation x^3+ax^2+bx+c are real, show that the difference between the greatest and the least roots is not less than $\sqrt{a^2 - 3b}$ nor greater than $2\sqrt{(a^2 - 3b)/3}$.

4. (56P) The nonconstant polynomials $P(z)$ and $Q(z)$ with complex coefficients have the same set of numbers for their zeros but possibly different multiplicities. The same is true of the polynomials $P(z) + 1$ and $Q(z)+1$. Prove that $P(z) = Q(z)$. (On the original Exam, the assumption that $P(z)$ and $Q(z)$ are nonconstant was inadvertently omitted.)

5. (58P) If $a_0, a_1, \ldots, a_n$ are real numbers satisfying
$$\frac{a_0}{1} + \frac{a_1}{2} + \cdots + \frac{a_n}{n+1} = 0,$$
   show that the equation $a_0 + a_1x + a_2x^2 + \cdots + a_nx^n = 0$ has at least one real root.

6. (68P) Determine all polynomials of the form

$$\sum_{0}^{n} a_i x^{n-i} \quad \text{with } a_i = \pm 1$$

$(0 \leq i \leq n,\ \ 1 \leq n < \infty)$ such that each has only real zeros.

7. (81P) Let $P(x)$ be a polynomial with real coefficients and form the polynomial

$$Q(x) = (x^2+1)P(x)P^{'}(x) + x(P(x)^2 + P^{'}(x)^2).$$

Given that the equation $P(x) = 0$ has n distinct real roots exceeding 1, prove or disprove that the equation $Q(x) = 0$ has at least $2n-1$ distinct real roots.

8. (89P) Prove that if

$$11z^{10} + 10iz^9 + 10iz - 11 = 0,$$

then $|z| = 1$. (Here z is a complex number and $i^2 = -1$.)

9. (90P) Is there an infinite sequence $a_0, a_1, a_2, \ldots$ of nonzero real numbers such that for each $n = 1, 2, 3, \ldots$ the polynomial

$$p_n(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$$

has exactly n distinct real roots?

10. (91P) Find all real polynomials $p(x)$ of degree $n \geq 2$ for which there exist real numbers $r_1 < r_2 < \cdots < r_n$ such that

(i) $p(r_i) = 0, \quad i = 1, 2, \ldots, n,$

and

(ii) $p'\left(\frac{r_i + r_{i+1}}{2}\right) = 0, \quad i = 1, 2, \ldots, n-1,$

where $p'(x)$ denotes the derivative of $p(x)$.

11. (a) (85P) (relatively easy) Let k be the smallest positive integer with the following property:

There are distinct integers m_1, m_2, m_3, m_4, m_5 such that the polynomial $p(x) = (x-m_1)(x-m_2)(x-m_3)(x-m_4)(x-m_5)$ has exactly k nonzero coefficients.

Find, with proof, a set of integers m_1, m_2, m_3, m_4, m_5 for which this minimum k is achieved.

(b) (considerably more difficult) Let $P(x) = x^{11} + a_{10}x^{10} + \cdots + a_0$ be a monic polynomial of degree eleven with real coefficients a_i, with $a_0 \neq 0$. Suppose that all the zeros of $P(x)$ are real, i.e., if α is a complex number such that $P(\alpha) = 0$, then α is real. Find (with proof) the least possible number of nonzero coefficients of $P(x)$ (including the coefficient 1 of x^{11}).

12. (99P) Let $P(x)$ be a polynomial of degree n such that $P(x) = Q(x)P''(x)$, where $Q(x)$ is a quadratic polynomial and $P''(x)$ is the second derivative of $P(x)$. Show that if $P(x)$ has at least two distinct roots then it must have n distinct roots.

13. (a) (05P) Let $p(z)$ be a polynomial of degree n, all of whose zeros have absolute value 1 in the complex plane. Put $g(z) = p(z)/z^{n/2}$. Show that all zeros of $g'(z) = 0$ have absolute value 1.

    (b) (00P) Let $f(t) = \sum_{j=1}^{N} a_j \sin(2\pi jt)$, where each a_j is real and a_N is not equal to 0. Let N_k denote the number of zeros (including multiplicities) of $\frac{d^k f}{dt^k}$ in the half-open interval $[0, 1)$. Prove that

$$N_0 \leq N_1 \leq N_2 \leq \cdots \quad \text{and} \quad \lim_{k\to\infty} N_k = 2N.$$

    (On the original Exam, it was not stated that the zeros should be taken in $[0, 1)$.)

14. Let $ax^3 + bx^2 + cx + d$ be a polynomial with three distinct real roots. How many real roots are there of the equation

$$4(ax^3 + bx^2 + cx + d)(3ax + b) = (3ax^2 + 2bx + c)^2?$$

15. Does there exist a finite set M of nonzero real numbers, such that for any positive integer n, there exists a polynomial of degree at least n with all coefficients in M, all of whose roots are real and belong to M?

16. Suppose that the polynomial $ax^2+(c-b)x+(e-d)$ has two real roots, both greater than 1. Prove that $ax^4+bx^3+cx^2+dx+e$ has at least one real root.

17. Suppose that $a, b, c \in \mathbb{C}$ are such that the roots of the polynomial z^3+az^2+bz+c all satisfy $|z|=1$. Prove that the roots of $x^3+|a|x^2+|b|x+|c|$ all satisfy $|x|=1$.

18. Let $P(x)=x^n+a_{n-1}x^{n-1}+\cdots+a_0$ be a monic polynomial of degree n with complex coefficients a_i. Suppose that the roots of $P(x)$ are $x_1, x_2, \cdots, x_n$, i.e., we have $P(x)=(x-x_1)(x-x_2)\cdots(x-x_n)$. The *discriminant* $\Delta(P(x))$ is defined by

$$\Delta(P(x))=\prod_{1\leq i<j\leq n}(x_i-x_j)^2.$$

Show that

$$\Delta(x^n+ax+b)=(-1)^{\binom{n}{2}}\left(n^nb^{n-1}+(-1)^{n-1}(n-1)^{n-1}a^n\right).$$

HINT. First note that

$$P^{'}(x)=P(x)\left(\frac{1}{x-x_1}+\cdots+\frac{1}{x-x_n}\right).$$

Use this formula to establish a connection between $\Delta(P(x))$ and the values $P^{'}(x_i)$, $1\leq i\leq n$.

19. Let $P_n(x)=(x+n)(x+n-1)\cdots(x+1)-(x-1)(x-2)\cdots(x-n)$. Show that all the zeros of $P_n(x)$ are purely imaginary, i.e., have real part 0.

20. Let $P(x)$ be a polynomial with complex coefficients such that every root has real part a. Let $z\in\mathbb{C}$ with $|z|=1$. Show that every root of the polynomial $R(x)=P(x-1)-zP(x)$ has real part $a+\frac{1}{2}$.

21. Let $d\geq 1$. It is not hard to see that there exists a polynomial $A_d(x)$ of degree d such that

$$F_d(x):=\sum_{n\geq 0}n^dx^n=\frac{A_d(x)}{(1-x)^{d+1}}. \tag{1}$$

For instance, $A_1(x) = x$, $A_2(x) = x + x^2$, $A_3(x) = x + 4x^2 + x^3$. Show that every root of $A_d(x)$ is real. HINT. First obtain a recurrence for $A_d(x)$ by differentiating (1).

22. Let $P(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_0$ be a monic polynomial with complex coefficients. Choose $j \in \{0, \ldots, n\}$ so that the roots of P can be labeled $\alpha_1, \ldots, \alpha_n$ with

$$|\alpha_1|, \ldots, |\alpha_j| > 1, \qquad |\alpha_{j+1}|, \ldots, |\alpha_n| \leq 1.$$

Prove that

$$\prod_{i=1}^{j} |\alpha_i| \leq \sqrt{|a_0|^2 + \cdots + |a_{n-1}|^2 + 1}.$$

HINT. One approach is to deduce this from an identity involving the polynomials $(z - \alpha_1) \cdots (z - \alpha_j)$ and $(\alpha_{j+1}z - 1) \cdots (\alpha_n z - 1)$.

23. Let $Q(x)$ be any monic polynomial of degree n with real coefficients. Prove that

$$\sup_{x \in [-2,2]} |Q(x)| \geq 2.$$

HINT. Let $P_n(x)$ be the monic polynomial satisfying

$$P_n(2\cos\theta) = 2\cos(n\theta) \qquad (\theta \in \mathbb{R}),$$

and examine the values of $P_n(x) - Q(x)$ at points where $|P_n(x)| = 2$.

OPTIONAL. Prove that equality only holds for $Q = P_n$.

24. Let $P(x), Q(x)$ be two polynomials with all real roots $r_1 \leq r_2 \leq \cdots \leq r_n$ and $s_1 \leq s_2 \leq \cdots \leq s_{n-1}$, respectively. We say that $P(x)$ and $Q(x)$ are *interlaced* if

$$r_1 \leq s_1 \leq r_2 \leq s_2 \leq \cdots \leq s_{n-1} \leq r_n.$$

Prove that $P(x)$ and $Q(x)$ are interlaced if and only if the polynomial $P + tQ$ has all real roots for all $t \in \mathbb{R}$.

25. Let $P(x)$ be a polynomial with real coefficients. For $t \in \mathbb{R}$, let $V(P, t)$ denote the number of sign changes in the sequence

$$P(t), P'(t), P''(t), \ldots.$$

(A *sign change* in a sequence is a pair of terms, one positive and one negative, with only zeros in between.) Prove that for any $a, b \in \mathbb{R}$, the number of roots of P in the half-open interval $(a, b]$, counted with multiplicities, is equal to $V(P, a) - V(P, b)$ minus a nonnegative even integer. Then deduce Descartes's rule of signs as a corollary.

26. Let $P(x)$ be a squarefree polynomial with real coefficients. Define the sequence of polynomials $P_0, P_1, \ldots$ by setting $P_0 = P$, $P_1 = P'$, and

$$P_{i+2} = -\text{rem}(P_i, P_{i+1}),$$

where $\text{rem}(A, B)$ means the remainder upon Euclidean division of A by B; upon arriving at a nonzero constant polynomial P_r, stop. Prove that for any $a, b \in \mathbb{R}$, the number of zeros of P in $(a, b]$ is $\sigma(a) - \sigma(b)$, where $\sigma(t)$ is the number of sign changes in the sequence

$$P_0(t), P_1(t), \ldots, P_r(t).$$

18.S34 (FALL 2007)
LIMIT PROBLEMS

1. Let a and b be positive real numbers. Prove that

$$\lim_{n\to\infty} (a^n + b^n)^{1/n}$$

equals the larger of a and b. What happens when $a = b$?

2. Show that $\lim_{n\to\infty}\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}-\log(n)\right)$ exists and lies between $\frac{1}{2}$ and 1.

   NOTE. This number, known as *Euler's constant* and denoted γ, is probably the third most important constant in the theory of complex variables, after π and e. Numerically we have

$$\gamma = 0.5772156649015328606065120900824024310421593359 3992\cdots.$$

   It is a famous unsolved problem to decide whether γ is irrational.

3. (47P) If (a_n) is a sequence of numbers such that, for $n \geq 1$,

$$(2 - a_n)a_{n+1} = 1,$$

   prove that $\lim_{n\to\infty} a_n$ exists and equals 1.

4. Let K be a positive real number. Take an arbitrary positive real number x_0 and form the sequence

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{K}{x_n}\right).$$

   Show that $\lim_{n\to\infty} x_n = \sqrt{K}$. (REMARK. this is how most calculators determine $\sqrt{K}$.)

5. (70P) Given a sequence (x_n) such that $\lim_{n\to\infty}(x_n - x_{n-2}) = 0$, prove that

$$\lim_{n\to\infty} \frac{x_n - x_{n-1}}{n} = 0.$$

6. Let $x_{n+1} = x_n^2 - 6x_n + 10$. For what values of x_0 is $\{x_n\}$ convergent, and how does the value of the limit depend on x_0?

7. (90P) Is $\sqrt{2}$ the limit of a sequence of numbers of the form $\sqrt[3]{n} - \sqrt[3]{m}$, $(n, m = 0, 1, 2, \ldots)$? Justify your answer.

8. Let $x_0 = 1$ and $x_{n+1} = x_n + 10^{-10x_n}$. Does $\lim_{n\to\infty} x_n$ exist? Explain.

9. (00P) Show that the improper integral

$$\lim_{B\to\infty} \int_0^B \sin(x)\, \sin(x^2)\, dx$$

converges.

10. Let $x > 0$. Define $a_1 = x$ and $a_{n+1} = x^{a_n}$ for $n \geq 1$. For which x does $\lim_{n\to\infty} a_n$ exist (and is finite)?

PART II

LIMITS. Two useful techniques are:

(a) *L'Hôpital's rule.* If $\lim_{x\to 0} f(x) = \lim_{x\to 0} g(x) = 0$, then

$$\lim_{x\to 0} \frac{f(x)}{g(x)} = \frac{f'(0)}{g'(0)},$$

provided the derivatives in question exist. Some limits can be converted to this form by first taking logarithms, or by substituting $1/x$ for x, etc.

(b) If $f(x)$ is reasonably well-behaved (e.g., continuous) on the closed interval $[a, b]$, then

$$\lim \sum_{i=1}^{n} f(x_i)(x_i - x_{i-1}) = \int_a^b f(x)dx,$$

where the limit is over any sequence of "partitions of $[a, b]$" $a = x_0 < x_1 < \cdots < x_n = b$ such that the maximum value of $x_i - x_{i-1}$ approaches 0. In particular, taking $a = 0$, $b = 1$, $x_i = i/n$, then

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} f(i/n) = \int_0^1 f(x)dx.$$

Sometimes a limit of products can be converted to this form by taking logarithms.

The next problems are all from the Putnam Exam.

11. Let $a > 0$, $a \neq 1$. Find

$$\lim_{x\to\infty} \left(\frac{1}{x}\frac{a^x - 1}{a - 1}\right)^{1/x}$$

12. Find

$$\lim_{n\to\infty} \left[\frac{1}{n^4} \prod_{i=1}^{2n} (n^2 + i^2)^{1/n}\right]$$

13. Let $0 < a < b$. Evaluate

$$\lim_{t\to 0} \left[\int_0^1 (bx + a(1-x))^t dx\right]^{1/t}$$

14. Evaluate

$$\lim_{x\to 0} \frac{1}{x} \int_0^x (1 + \sin(2t))^{1/t} dt$$

15. Evaluate

$$\lim_{n\to\infty} \sum_{j=1}^{n^2} \frac{n}{n^2 + j^2}$$

16. Evaluate

$$\prod_{n=2}^{\infty} \frac{n^3 - 1}{n^3 + 1}.$$

17. Evaluate

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \left(\left\lfloor \frac{2n}{k} \right\rfloor - 2\left\lfloor \frac{n}{k} \right\rfloor\right).$$

Express your answer in the form $\log(a) - b$, where a and b are positive integers.

18. Evaluate

$$\sqrt[8]{2207 - \frac{1}{2207 - \frac{1}{2207 - \cdots}}}.$$

Express your answer in the form $\frac{a+b\sqrt{c}}{d}$, where a, b, c, d are integers.

19. Assume that $(a_n)_{n \geq 1}$ is an increasing sequence of positive real numbers such that $\lim a_n/n = 0$. Must there exist infinitely many positive integers n such that $a_{n-i} + a_{n+i} < 2a_n$ for $i = 1, 2, \ldots, n-1$?

20. Evaluate

$$\lim_{x \to 1^-} \prod_{n=0}^{\infty} \left(\frac{1 + x^{n+1}}{1 + x^n} \right)^{x^n}.$$

21. Let k be an integer greater than 1. Suppose $a_0 > 0$, and define

$$a_{n+1} = a_n + \frac{1}{\sqrt[k]{a_n}}$$

for $n > 0$. Evaluate

$$\lim_{n \to \infty} \frac{a_n^{k+1}}{n^k}.$$