Mathematics OB1 | Take-home Exam | Pavel Ghazaryan
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MATH 19801
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SECTION B

1/ $x^2 - y^2 = 15$; $y = \pm x$ asymp.

a) $x^2 - y^2 = 15$
$x^2 + y^2 = 113$

sum each side $\Rightarrow x^2 + x^2 + y^2 - y^2 = 128$

$2x^2 = 128 \Rightarrow x = \pm 8 \Rightarrow$

$\Rightarrow y = \pm 7$ as $64 + y^2 = 113 \Rightarrow y = \sqrt{49} = \pm 7$

This shows four intersections $(8;7)$, $(8;-7)$, $(-8;7)$; $(-8;-7)$

b) $2x - 2y\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = \frac{2x}{2y} = \frac{x}{y}$ The gradient at $(4,1)$ is

$\frac{dy}{dx} = \frac{4}{1} = 4 \Rightarrow y = 4x + c$ but point $(4,1)$ is on $y = 4x + c \Rightarrow$

$\Rightarrow 1 = 16 + c \Rightarrow c = -15 \Rightarrow y = 4x - 15$

c) $y = 4x - 15$
$y = \pm x$
$\Rightarrow$

$x = 4x - 15$
$-3x = -15$
$x = 5 \Rightarrow (5;5)$

$-x = 4x - 15$
$-5x = -15$
$x = 3 \Rightarrow (3;-3)$

two points $(5;5)$ and $(3;-3)$

d) Tangent intersects y axis at $y = -15$ and $x = 0$ while asymptote $y = x$ at $(0;0)$. As tangent as higher slope (4) than asymptote (1), it goes closer to Y axis than $y = x$ meaning that Left bottom part of hyperbole cannot intesect with tangent because it is ove the other side of asymptote $y = x$. Tangent continues to the first quadrant as it is tangent to point $(4;1)$ meaning there is no way for it to meet the upper left side of hyperbole! Shown

2/

a) $y = x + \frac{4}{x^2} - 3$

$x^2 \neq 0$

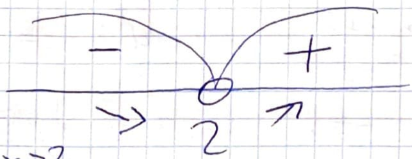
$x \neq 0 \Rightarrow$ there is no intersection with Y axis.

$y' = 1 - 8x^{-3} = 0$

$8x^{-3} = 1$

$x^{-3} = \frac{1}{8}$

$x^3 = 8 \Rightarrow x = 2$



for $x > 2$

$f'(3) = 1 - \frac{8}{27} > 0 \Rightarrow +$

this shows that point (2;0) is a local minimum.

Find intersections with x-axis: $x + \frac{4}{x^2} - 3 = 0$

$\frac{x^3 + 4 - 3x^2}{x^2} = 0 \Rightarrow x^3 - 3x^2 + 4 = 0$

$x \cdot (x^2 - 3x) = -4$

-2	2	$x = -2$ $x^2 - 3x \neq 2$ wrong
2	-2	$x = 2$ $x^2 - 3x = -2$ ✓
4	1	$x = 4$ wrong
1	-4	wrong
-1	4	$x = -1$ $x^2 - 3x = 4$ true ✓

$\Rightarrow$ Meaning intersection with x axis: $(2;0)$; $(-1;0)$

Asymptote:

$$\begin{array}{l|l} x^3 + 4 - 3x^2 & x^2 \\ \hline x^3 & x - 3 \\ \hline 4 - 3x^2 & \\ -3x^2 & \\ \hline 4 & \end{array}$$

$\Rightarrow$ this shows that $y = x - 3$ is an asymptote.

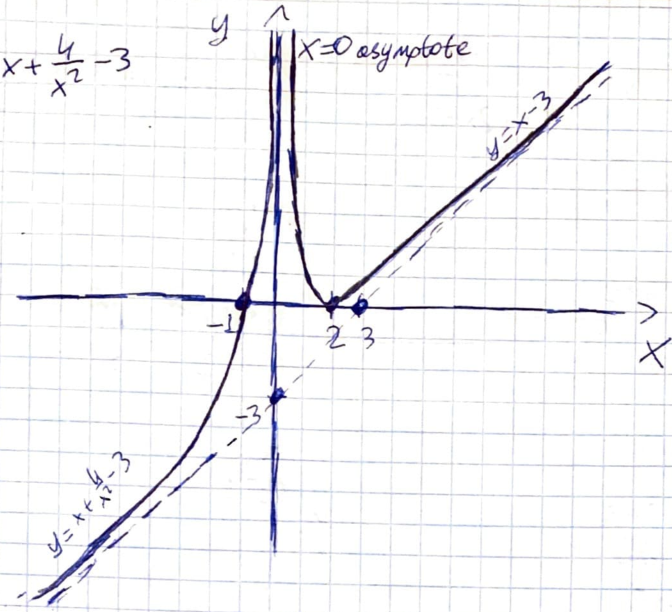
for $\lim\limits_{x \to +\infty} x + \frac{4}{x^2} - 3 = \lim\limits_{x \to +\infty} x - 3 \Rightarrow$ meaning y also goes to $+\infty$

for $\lim\limits_{x \to -\infty} x + \frac{4}{x^2} - 3 = \lim\limits_{x \to -\infty} x - 3 \Rightarrow$ y also goes to $-\infty$

graph on next page

$\Longrightarrow$

$f(x) = x + \frac{4}{x^2} - 3$



b) $y = 2x + \frac{4}{(2x-2)^2} - 5$ if $f(x) = x + \frac{4}{x^2} - 3$ than the given equation is just $f(2x-2) = 2x - 2 + \frac{4}{(2x-2)^2} - 3 =$

$= 2x + \frac{4}{(2x-2)^2} - 5$. This means that it is a transformation of $f(x)$ 2 towards right and 2 times closer in the middle:

$f(2x-2) = 2x + \frac{4}{(2x-2)^2} - 5$

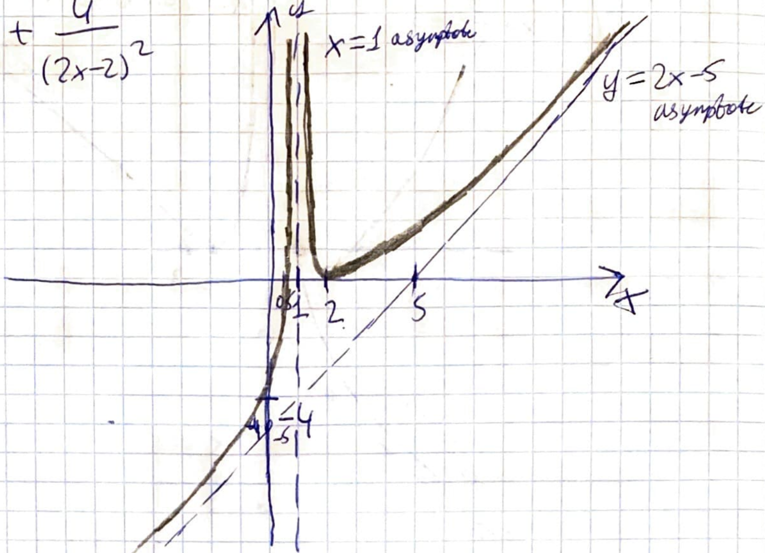
please see on next page

$$f(2x-2) = 2x-5+\frac{4}{(2x-2)^2}$$

$$x_1 = \frac{x_0+2}{2}$$

all new x_1 are $= \frac{x_0+2}{2}$

$y=2x-5$ is new asyptote because instead of x we have $2x-2$ so $y=x-3$ changed to $y=2x-5$



3/ a)(i) $y=-2x+5$ $\quad y=x+2$ $\quad y=2x-3$

$-2x+5=2x-3$

$4x=8$

$x=2 \Rightarrow y=1 \Rightarrow \underline{(2;1)}$

$-2x+5=x+2$

$-3x=-3$

$x=1 \Rightarrow y=3 \Rightarrow \underline{(1;3)}$

$x+2=2x-3$

$x=5 \Rightarrow y=7$

$\Downarrow$

$\underline{(5;7)}$

Three points: $(2;1)$, $(1;3)$, $(5;7)$

(ii) $y=x+2$ is above from other two lines which can be seen from obtained points $\Rightarrow$

$$\Rightarrow \text{Area} = \int_1^5 (x+2)\,dx - \int_1^2 (-2x+5)\,dx - \int_2^5 (2x-3)\,dx$$

$$\int_1^5 (x+2)\,dx = \left[\frac{x^2}{2}+2x\right]_1^5 = 20$$

$$\int_1^2 (-2x+5)\,dx = \left[-x^2+5x\right]_1^2 = 2$$

$$\int_2^5 (2x-3)\,dx = \left[x^2-3x\right]_2^5 = 12$$

$\Rightarrow$

$\Rightarrow \text{Area} = \int_1^5 (x+2)\,dx - \int_1^2 (-2x+5)\,dx - \int_2^5 (2x-3)\,dx =$

$= 20 - 2 - 12 = \underline{\underline{6}}$ Answer: $\underline{6}$

b)(i) $\int \frac{1}{x} \ln x \, dx =$

$u = \ln x$

$\frac{du}{dx} = \frac{1}{x}$

$du = \frac{1}{x} dx \Rightarrow \int \frac{1}{x} \ln x \, dx = \int u \, du = \frac{u^2}{2} =$

$= \frac{\ln^2 x}{2} + c = \boxed{\frac{(\ln x)^2}{2} + C}$

(ii) $\int x^2 \ln x \, dx = \frac{x^3}{3} \ln x - \int \frac{x^3}{3} \cdot \frac{1}{x} \, dx =$

$= \frac{x^3}{3} \ln x - \frac{1}{3} \cdot \frac{x^3}{3} + c = \boxed{\frac{3x^3 \ln x - x^3}{9} + C}$

4/ a) $u_1 = 2x+4$; $u_2 = -2x$; $u_3 = 3x-4$.

$u_2 = (2x+4) \cdot r = -2x \Rightarrow 2xr + 4r = -2x$

$u_3 = (2x+4) \cdot r^2 = 3x-4 \qquad 2xr^2 + 4r^2 = 3x-4$

$r = \frac{-2x}{2x+4} = -\frac{x}{x+2}$

$-2x \cdot \left(-\frac{x}{x+2}\right) = 3x-4$

$2x^2 = (3x-4)\cdot(x+2)$

$2x^2 = 3x^2 + 6x - 4x - 8$

$x^2 + 2x - 8 = 0 \quad D = 4 + 32 = 36 \Rightarrow$

$x_1 = \frac{-2+6}{2} = 2$

$x_2 = -4$ for $x_2 = -4$, $r = -2$ which is wrong becaus $r > -1$ and $r < 1$

$\Rightarrow \boxed{x = 2, \; r = -\frac{1}{2}}$

$$S_\infty = \frac{u_1}{1-r} = \frac{2x+4}{1-(-\frac{1}{2})} = \frac{8}{\frac{3}{2}} = \frac{8\cdot 2}{3} = \boxed{\frac{16}{3}}$$

$$\underline{\underline{\frac{16}{3} = S_\infty}} \quad \text{and} \quad \underline{\underline{x=2}}$$

b) $f(x) = e^x + \cos x$

$f'(x) = e^x - \sin x$ $\qquad f(0) = 2$

$f''(x) = e^x - \cos x$ $\qquad f'(0) = 1$

$\qquad f''(0) = 0$

$f'''(x) = e^x + \sin x$ $\qquad f'''(0) = 1$

$f^{IV}(x) = e^x + \cos x$ $\qquad f^{IV}(0) = 2.$

Maclaurin Series: $f(x) \approx 2 + \frac{x}{1!}\cdot 1 + \frac{x^2}{2!}\cdot 0 +$

$+ \frac{x^3}{3!}\cdot 1 \; \ldots + \frac{x^4}{4!}\cdot 2$

So first three non-zero terms are:

$$\boxed{2 \; ; \; x \; ; \; \frac{x^3}{6}}$$

The fourth is $\frac{x^4}{4!}\cdot 2 = \frac{x^4}{12}.$

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Sorry for bad handwritings