Natural Language Processing

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Probability and Language

Quick Guide to Probability Theory

Log Probability

Basics of Information Theory

Probability and Language

Assign a probability to an input sequence

Given a URL: `choosespain.com`. What is this website about?

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choose spain	-8.35
chooses pain	-9.88
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The Goal

Find a good **scoring function** for input sequences.

Scoring Hypotheses

Acoustically Scored Hypotheses

Hypothesis	Score
the station signs are in deep in english	-14732
the stations signs are in deep in english	-14735
the station signs are in deep into english	-14739
the station 's signs are in deep in english	-14740
the station signs are in deep in the english	-14741
the station signs are indeed in english	-14757
the station 's signs are indeed in english	-14760
the station signs are indians in english	-14790
the station signs are indian in english	-14799
the stations signs are indians in english	-14807
the stations signs are indians and english	-14815

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- Use the model to **predict** the probability of new sequences.

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  - $\mathrm{P}(\mathrm{X} = \textit{choose}, \mathrm{Y} = \textit{spain}) == \mathrm{P}(\mathrm{Y} = \textit{spain}, \mathrm{X} = \textit{choose})$

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  - $P(\text{Kiki is a girl}) + P(\text{Kiki is fictional}) =$
    $P(\text{Kiki is a fictional girl})$,
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- Computing $P(f)$ from axioms:

$$P(f) = \sum_e P(e) \times P(f \mid e)$$

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- Use chain rule and simplify:

$$P(a, b, c, d \mid e) = P(d \mid e) \cdot P(c \mid d, e) \cdot P(b \mid c, e) \cdot P(a \mid b, e)$$

Probability: The Chain Rule

- $P(e_1, e_2, \ldots, e_n) = P(e_1) \times P(e_2 \mid e_1) \times P(e_3 \mid e_1, e_2) \ldots$

$$P(e_1, e_2, \ldots, e_n) = \prod_{i=1}^{n} P(e_i \mid e_{i-1}, e_{i-2}, \ldots, e_1)$$

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- y is an element of some implicit **event space**: $\mathcal{E}$

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- Finding the value that maximizes the probability value:

$$\hat{x} = \underset{x \in \mathcal{E}}{\arg\max} \; P(x)$$

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\begin{aligned}
\log(P(e)) &= \log(p_1 \times p_2 \times \ldots \times p_n) \\
&= \log(p_1) + \log(p_2) + \ldots + \log(p_n)
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- Also more efficient: addition instead of multiplication

Log Probability Arithmetic

p	$\log(p)$
0.0	$-\infty$
0.1	-3.32
0.2	-2.32
0.3	-1.74
0.4	-1.32
0.5	-1.00
0.6	-0.74
0.7	-0.51
0.8	-0.32
0.9	-0.15
1.0	0.00

Log Probability Arithmetic

- So: $(0.5 \times 0.5 \times \ldots 0.5) = (0.5)^n$ might get too small but $(-1 - 1 - 1 - 1) = -n$ is manageable

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- So: $(0.5 \times 0.5 \times \ldots 0.5) = (0.5)^n$ might get too small but $(-1 - 1 - 1 - 1) = -n$ is manageable
- Another useful fact when writing code ($\log_2$ is *log to the base 2*):

$$\log_2(x) = \frac{\log_{10}(x)}{\log_{10}(2)}$$

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- A more efficient soln, let *big* be a large constant e.g. 10^{30}:

function $logadd(x, y)$: # returns $\log(\exp(x) + \exp(y))$
if $(y - x) > \log(big)$ return y
elsif $(x - y) > \log(big)$ return x
else return
$$min(x, y) + \log(\exp(x - min(x, y)) + \exp(y - min(x, y)))$$
endif

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  endif
- There is a more efficient way of computing $\log(\exp(x - min(x, y)) + \exp(y - min(x, y)))$

Log Probability Arithmetic

function $logadd(x, y)$:
    if $(y - x) > \log(big)$ return y
    elsif $(x - y) > \log(big)$ return x
    elsif $(x \geq y)$ return $x + \log(1 + \exp(y - x))$
        $\#$ note that $max(x, y) = x$ and $y - x \leq 0$
    else return $y + \log(\exp(x - y) + 1)$
        $\#$ note that $max(x, y) = y$ and $x - y \leq 0$
    endif

Also, in ANSI C, `log1p` efficiently computes $\log(1 + x)$

`http://www.ling.ohio-state.edu/~jansche/src/logadd.c`

In Python, numpy.logaddexp2(x1,x2) for base 2

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- Information theory is the use of probability theory to quantify and measure “information”.
- Consider the task of efficiently sending a message. Sender Alice wants to send several messages to Receiver Bob. Alice wants to do this as efficiently as possible.
- Let’s say that Alice is sending a message where the entire message is just one character a, e.g. $aaaa\ldots$. In this case we can save space by simply sending the length of the message and the single character.

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- This formulation of entropy by Claude Shannon was adapted from thermodynamics, converting information into a quantity that can be measured.
- Information theory is built around this notion of message compression as a way to evaluate the amount of information.

Expectation

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- **Expectation** with respect to p is a weighted average:

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\begin{aligned}
E_p[x] &= \frac{x_1 \cdot p_1 + x_2 \cdot p_2 + \ldots + x_n p_n}{p_1 + p_2 + \ldots + p_n} \\
&= x_1 \cdot p_1 + x_2 \cdot p_2 + \ldots + x_n p_n \\
&= \sum_{x \in \mathcal{E}} x \cdot p(x)
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- Example: for a six-sided die the expectation is:

$$E_p[x] = 1 \cdot \frac{1}{6} + 2 \cdot \frac{1}{6} + \ldots + 6 \cdot \frac{1}{6} = 3.5$$

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- What is the *expected* number of bits with respect to p:

$$-E_p[\log_2 p(x)] = H(p)$$

- Entropy answers the question: *What is the expected number of bits needed to transmit messages from event space $\mathcal{E}$, where $p(x)$ defines the probability of observing x?*

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  (lower bound on message length $=$ 3 bits in this encoding scheme)

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  (lower bound on message length $=$ 3 bits in this encoding scheme)
- Can we do better?

Entropy

Horse 1	$\frac{1}{2}$	Horse 5	$\frac{1}{64}$
Horse 2	$\frac{1}{4}$	Horse 6	$\frac{1}{64}$
Horse 3	$\frac{1}{8}$	Horse 7	$\frac{1}{64}$
Horse 4	$\frac{1}{16}$	Horse 8	$\frac{1}{64}$

- If we know how likely we are to bet on each horse, say based on the horse's probability of winning, then we can do better.

Entropy

Horse 1	$\frac{1}{2}$	Horse 5	$\frac{1}{64}$
Horse 2	$\frac{1}{4}$	Horse 6	$\frac{1}{64}$
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- If we know how likely we are to bet on each horse, say based on the horse's probability of winning, then we can do better.
- Let p be the probability distribution given in the table above. The entropy of p is $H(p)$

Entropy

$$
\begin{aligned}
H(p) =& \\
=&\ -\sum_{i=1}^{8} p(i)\log_2\ p(i) \\
=&\ -\left(\frac{1}{2}\log_2\frac{1}{2}+\frac{1}{4}\log_2\frac{1}{4}+\frac{1}{8}\log_2\frac{1}{8}+\frac{1}{16}\log_2\frac{1}{16}+4(\frac{1}{64}\log_2\frac{1}{64})\right) \\
=&\ -\left(\frac{1}{2}\times -1+\frac{1}{4}\times -2+\frac{1}{8}\times -3+\frac{1}{16}\times -4+4(\frac{1}{64}\times -6)\right) \\
=&\ -\left(-\frac{1}{2}-\frac{1}{2}-\frac{3}{8}-\frac{1}{4}-\frac{3}{8}\right) \\
=&\ 2\ bits
\end{aligned}
$$

- What is the entropy when the horses are equally likely to win?

$$H(uniform\ distribution) = -8(\frac{1}{8}\times -3) = 3\ bits$$

Entropy

- e.g., most likely horse gets code 0, next most likely gets 10, and then $110, 1110, \ldots$
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- Number of bits: $160*1 + 80*2 + 40*3 + 20*4 + 5*5 = 545$
- Total number of bits per message (per race): $\frac{545}{320} \approx 1.7$ bits (always less than 2 bits)

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- Choosing between the biased horses from before (H=2) is $2^2 = 4$.

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- The relative entropy is also called the *Kullback-Leibler divergence*.

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- The **relative entropy** can be written as the sum of two terms:

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- The term $H_q(p)$ is called the **cross entropy**.

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- *Mutual Information* between two random variables X and Y:

$$I(X;Y) = D(p(x,y) \| p(x)p(y)) = \sum_{x} \sum_{y} p(x,y) \log_2 \frac{p(x,y)}{p(x)p(y)}$$

Acknowledgements

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