



# Natural Language Processing

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September 13, 2016

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Part 1: Probability models of Language

# The Language Modeling problem

## Setup

- ▶ Assume a (finite) vocabulary of words:

$$\mathcal{V} = \{killer, crazy, clown\}$$

- ▶ Use  $\mathcal{V}$  to construct an infinite set of *sentences*

$$\mathcal{V}^+ = \left\{ \begin{array}{l} clown, killer clown, crazy clown, \\ crazy killer clown, killer crazy clown, \\ \dots \end{array} \right\}$$

- ▶ A *sentence* is **defined** as each  $s \in \mathcal{V}^+$

# The Language Modeling problem

## Data

Given a training data set of example sentences  $s \in \mathcal{V}^+$

## Language Modeling problem

Estimate a probability model:

$$\sum_{s \in \mathcal{V}^+} p(s) = 1.0$$

- ▶  $p(\text{clown}) = 1\text{e-}5$
- ▶  $p(\text{killer}) = 1\text{e-}6$
- ▶  $p(\text{killer clown}) = 1\text{e-}12$
- ▶  $p(\text{crazy killer clown}) = 1\text{e-}21$
- ▶  $p(\text{crazy killer clown killer}) = 1\text{e-}110$
- ▶  $p(\text{crazy clown killer killer}) = 1\text{e-}127$

Why do we want to do this?

# Scoring Hypotheses in Speech Recognition

## From acoustic signal to candidate transcriptions

Hypothesis	Score
the station signs are in deep in english	-14732
the stations signs are in deep in english	-14735
the station signs are in deep into english	-14739
the station 's signs are in deep in english	-14740
the station signs are in deep in the english	-14741
the station signs are indeed in english	-14757
the station 's signs are indeed in english	-14760
the station signs are indians in english	-14790
the station signs are indian in english	-14799
the stations signs are indians in english	-14807
the stations signs are indians and english	-14815

# Scoring Hypotheses in Machine Translation

From source language to target language candidates

Hypothesis	Score
we must also discuss a vision .	-29.63
we must also discuss on a vision .	-31.58
it is also discuss a vision .	-31.96
we must discuss on greater vision .	-36.09
⋮	⋮

# Scoring Hypotheses in Decryption

## Character substitutions on ciphertext to plaintext candidates

Hypothesis	Score
Heopaj, zk ukq swjp pk gjks w oaynap?	-93
Urbcnw, mx hxd fjwc cx twxf j bnanc?	-92
Wtdepy, oz jzf hlye ez vyzh l dpncpe?	-91
Mjtufo, ep zpv xbou up lopx b tfdsfu?	-89
Nkuvgp, fq aqw ycpv vq mpqy c ugetgv?	-87
Gdnozi, yj tjp rvio oj fijr v nzxmzo?	-86
Czjkve, uf pfl nrek kf befn r jvtivk?	-85
Yvfgra, qb lbh jnag gb xabj n frperg?	-84
Zwghsb, rc mci kobh hc ybck o gsqfsh?	-83
Byijud, te oek mqdj je adem q iushuj?	-77
Jgqrcl, bm wms uylr rm ilmu y qcapcr?	-76
Listen, do you want to know a secret?	-25

# Scoring Hypotheses in Spelling Correction

Substitute spelling variants to generate hypotheses

Hypothesis	Score
... stellar and versatile <b>acress</b> whose combination of sass and glamour has defined her ...	-18920
... stellar and versatile <b>acres</b> whose combination of sass and glamour has defined her ...	-10209
... stellar and versatile <b>actress</b> whose combination of sass and glamour has defined her ...	-9801



# Probability models of language

## Question

- ▶ Given a finite vocabulary set  $\mathcal{V}$
- ▶ We want to build a probability model  $P(s)$  for all  $s \in \mathcal{V}^+$
- ▶ **But** we want to consider sentences  $s$  of each length  $\ell$  separately.
- ▶ Write down a new model over  $\mathcal{V}^+$  such that  $P(s \mid \ell)$  is in the model
- ▶ **And** the model should be equal to  $\sum_{s \in \mathcal{V}^+} P(s)$ .
- ▶ Write down the model

$$\sum_{s \in \mathcal{V}^+} P(s) = \dots$$

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Part 2:  $n$ -grams for Language Modeling

## Language models

### $n$ -grams for Language Modeling

#### Smoothing $n$ -gram Models

##### Smoothing Counts

- Add-one Smoothing

- Additive Smoothing

- Good-Turing Smoothing

##### Smoothing by Interpolation

- Interpolation: Jelinek-Mercer Smoothing

##### Backoff Smoothing

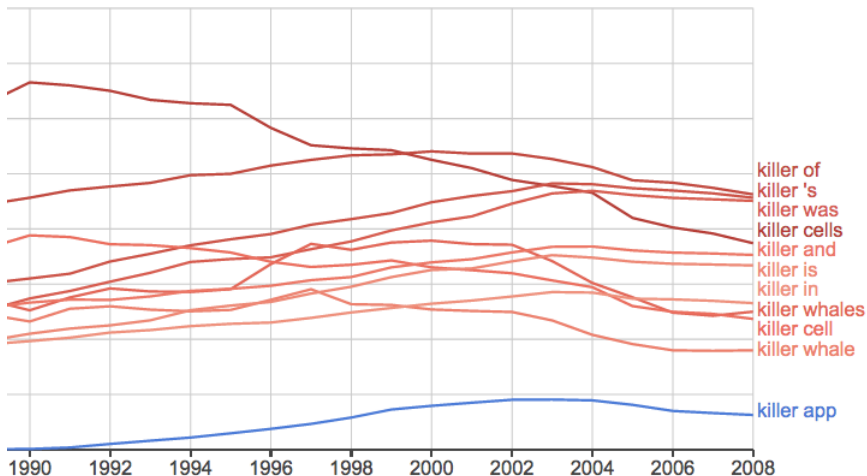
- Katz Backoff

- Backoff Smoothing with Discounting

## Evaluating Language Models

# $n$ -gram Models

Google  $n$ -gram viewer



# Learning Language Models

- ▶ Directly count using a training data set of sentences:  
 $w_1, \dots, w_n$ :

$$p(w_1, \dots, w_n) = \frac{n(w_1, \dots, w_n)}{N}$$

- ▶  $n$  is a function that counts how many times each sentence occurs
- ▶  $N$  is the sum over all possible  $n(\cdot)$  values
- ▶ Problem: does not generalize to new sentences unseen in the training data.
- ▶ What are the chances you will see a sentence: crazy killer clown crazy killer?
- ▶ In NLP applications we often need to assign non-zero probability to previously unseen sentences.

# Learning Language Models

Apply the Chain Rule: the unigram model

$$\begin{aligned} p(w_1, \dots, w_n) &\approx p(w_1)p(w_2) \dots p(w_n) \\ &= \prod_i p(w_i) \end{aligned}$$

Big problem with a unigram language model

$p(\text{the the the the the the the}) > p(\text{we must also discuss a vision .})$

# Learning Language Models

Apply the Chain Rule: the bigram model

$$\begin{aligned} p(w_1, \dots, w_n) &\approx p(w_1)p(w_2 \mid w_1) \dots p(w_n \mid w_{n-1}) \\ &= p(w_1) \prod_{i=2}^n p(w_i \mid w_{i-1}) \end{aligned}$$

Better than unigram

$p(\text{the the the the the the the}) < p(\text{we must also discuss a vision .})$

# Learning Language Models

Apply the Chain Rule: the trigram model

$$\begin{aligned} p(w_1, \dots, w_n) &\approx \\ &p(w_1)p(w_2 \mid w_1)p(w_3 \mid w_1, w_2) \dots p(w_n \mid w_{n-2}, w_{n-1}) \\ &p(w_1)p(w_2 \mid w_1) \prod_{i=3}^n p(w_i \mid w_{i-2}, w_{i-1}) \end{aligned}$$

Better than bigram, but ...

$p(\text{we must also discuss a vision .})$  might be zero because we have not seen  $p(\text{discuss} \mid \text{must also})$



# Maximum Likelihood Estimate

## Using training data to learn a trigram model

- ▶ Let  $c(u, v, w)$  be the count of the trigram  $u, v, w$ , e.g.  $c(\text{crazy}, \text{killer}, \text{clown})$
- ▶ Let  $c(u, v)$  be the count of the bigram  $u, v$ , e.g.  $c(\text{crazy}, \text{killer})$
- ▶ For any  $u, v, w$  we can compute the conditional probability of generating  $w$  given  $u, v$ :

$$p(w \mid u, v) = \frac{c(u, v, w)}{c(u, v)}$$

- ▶ For example:

$$p(\text{clown} \mid \text{crazy}, \text{killer}) = \frac{c(\text{crazy}, \text{killer}, \text{clown})}{c(\text{crazy}, \text{killer})}$$

# Number of Parameters

How many probabilities in each  $n$ -gram model

- ▶ Assume  $\mathcal{V} = \{killer, crazy, clown, UNK\}$

Question

How many unigram probabilities:  $P(x)$  for  $x \in \mathcal{V}$ ?

4

# Number of Parameters

How many probabilities in each  $n$ -gram model

- ▶ Assume  $\mathcal{V} = \{killer, crazy, clown, UNK\}$

## Question

How many bigram probabilities:  $P(y|x)$  for  $x, y \in \mathcal{V}$ ?

$$4^2 = 16$$

# Number of Parameters

How many probabilities in each  $n$ -gram model

- ▶ Assume  $\mathcal{V} = \{killer, crazy, clown, UNK\}$

## Question

How many trigram probabilities:  $P(z|x, y)$  for  $x, y, z \in \mathcal{V}$ ?

$$4^3 = 64$$

# Number of Parameters

## Question

- ▶ Assume  $|\mathcal{V}| = 50,000$  (a realistic vocabulary size for English)
- ▶ What is the minimum size of training data in tokens?
  - ▶ If you wanted to observe all unigrams at least once.
  - ▶ If you wanted to observe all trigrams at least once.

125,000,000,000,000 (125 Ttokens)

Some trigrams should be zero since they do not occur in the language,  $P(\textit{the} \mid \textit{the}, \textit{the})$ .

But others are simply unobserved in the training data,  $P(\textit{idea} \mid \textit{colourless}, \textit{green})$ .

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Part 3: Smoothing Probability Models

## Language models

### $n$ -grams for Language Modeling

#### Smoothing $n$ -gram Models

##### Smoothing Counts

- Add-one Smoothing

- Additive Smoothing

- Good-Turing Smoothing

##### Smoothing by Interpolation

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##### Backoff Smoothing

- Katz Backoff

- Backoff Smoothing with Discounting

### Evaluating Language Models

# Bigram Models

- In practice:

$$\begin{aligned}P(\text{Mork read a book}) = \\&P(\text{Mork} \mid \langle \text{start} \rangle) \times P(\text{read} \mid \text{Mork}) \times \\&P(\text{a} \mid \text{read}) \times P(\text{book} \mid \text{a}) \times \\&P(\langle \text{stop} \rangle \mid \text{book})\end{aligned}$$

- $P(w_i \mid w_{i-1}) = \frac{c(w_{i-1}, w_i)}{c(w_{i-1})}$   
On unseen data,  $c(w_{i-1}, w_i)$  or worse  $c(w_{i-1})$  could be zero

$$\sum_{w_i} \frac{c(w_{i-1}, w_i)}{c(w_{i-1})} = ?$$



# Smoothing

- ▶ **Smoothing** deals with events that have been observed zero times
- ▶ Smoothing algorithms also tend to improve the accuracy of the model

$$P(w_i \mid w_{i-1}) = \frac{c(w_{i-1}, w_i)}{c(w_{i-1})}$$

- ▶ Not just unobserved events: what about events observed once?

## Add-one Smoothing

$$P(w_i \mid w_{i-1}) = \frac{c(w_{i-1}, w_i)}{c(w_{i-1})}$$

- Add-one Smoothing:

$$P(w_i \mid w_{i-1}) = \frac{1 + c(w_{i-1}, w_i)}{V + c(w_{i-1})}$$

- Let  $V$  be the number of words in our vocabulary  
Assign count of 1 to unseen bigrams

## Add-one Smoothing

$$\begin{aligned} P(\text{Mindy read a book}) = & \\ & P(\text{Mindy} \mid \langle \text{start} \rangle) \times P(\text{read} \mid \text{Mindy}) \times \\ & P(\text{a} \mid \text{read}) \times P(\text{book} \mid \text{a}) \times \\ & P(\langle \text{stop} \rangle \mid \text{book}) \end{aligned}$$

- ▶ Without smoothing:

$$P(\text{read} \mid \text{Mindy}) = \frac{c(\text{Mindy, read})}{c(\text{Mindy})} = 0$$

- ▶ With add-one smoothing (assuming  $c(\text{Mindy}) = 1$  but  $c(\text{Mindy, read}) = 0$ ):

$$P(\text{read} \mid \text{Mindy}) = \frac{1}{V + 1}$$

## Additive Smoothing: (Lidstone 1920, Jeffreys 1948)

$$P(w_i \mid w_{i-1}) = \frac{c(w_{i-1}, w_i)}{c(w_{i-1})}$$

- ▶ Add-one smoothing works horribly in practice. Seems like 1 is too large a count for unobserved events.
- ▶ Additive Smoothing:

$$P(w_i \mid w_{i-1}) = \frac{\delta + c(w_{i-1}, w_i)}{(\delta \times V) + c(w_{i-1})}$$

- ▶  $0 < \delta \leq 1$   
Still works horribly in practice, but better than add-one smoothing.

## Good-Turing Smoothing: (Good, 1953)

$$P(w_i \mid w_{i-1}) = \frac{c(w_{i-1}, w_i)}{c(w_{i-1})}$$

- ▶ Imagine you're sitting at a sushi bar with a conveyor belt.
- ▶ You see going past you 10 plates of tuna, 3 plates of unagi, 2 plates of salmon, 1 plate of shrimp, 1 plate of octopus, and 1 plate of yellowtail
- ▶ Chance you will observe a new kind of seafood:  $\frac{3}{18}$
- ▶ How likely are you to see another plate of salmon: should be  $< \frac{2}{18}$

# Good-Turing Smoothing

- ▶ How many types of seafood (words) were seen once? Use this to predict probabilities for unseen events

Let  $n_1$  be the number of events that occurred once:  $p_0 = \frac{n_1}{N}$

- ▶ The Good-Turing estimate states that for any  $n$ -gram that occurs  $r$  times, we should pretend that it occurs  $r^*$  times

$$r^* = (r + 1) \frac{n_{r+1}}{n_r}$$

- ▶  $n_r$ : number of different objects seen  $r$  times

# Good-Turing Smoothing

- ▶ 10 tuna, 3 unagi, 2 salmon, 1 shrimp, 1 octopus, 1 yellowtail
- ▶ How likely is new data? Let  $n_1$  be the number of items occurring once, which is 3 in this case.  $N$  is the total, which is 18.

$$p_0 = \frac{n_1}{N} = \frac{3}{18} = 0.166$$

# Good-Turing Smoothing

- ▶ 10 tuna, 3 unagi, 2 salmon, 1 shrimp, 1 octopus, 1 yellowtail
- ▶ How likely is *octopus*? Since  $c(\text{octopus}) = 1$  The GT estimate is  $1^*$ .

$$r^* = (r + 1) \frac{n_{r+1}}{n_r}$$

$$p_{GT} = \frac{r^*}{N}$$

- ▶ To compute  $1^*$ , we need  $n_1 = 3$  and  $n_2 = 1$

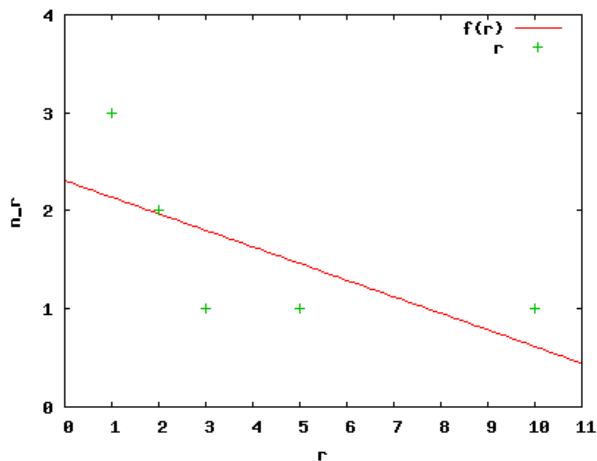
$$1^* = 2 \times \frac{1}{3} = \frac{2}{3}$$

$$p_1 = \frac{1^*}{18} = 0.037$$

- ▶ What happens when  $n_{r+1} = 0$ ? (smoothing before smoothing)



## Simple Good-Turing: linear interpolation for missing $n_{r+1}$



$$f(r) = a + b * r$$

$$a = 2.3$$

$$b = -0.17$$

$r$	$n_r = f(r)$
1	2.14
2	1.97
3	1.80
4	1.63
5	1.46
6	1.29
7	1.12
8	0.95
9	0.78
10	0.61
11	0.44

## Comparison between Add-one and Good-Turing

freq $r$	num with freq $r$ $n_r$	NS $p_r$	Add1 $p_r$	SGT $p_r$
0	0	0	0.0294	0.12
1	3	0.04	0.0588	0.03079
2	2	0.08	0.0882	0.06719
3	1	0.12	0.1176	0.1045
5	1	0.2	0.1764	0.1797
10	1	0.4	0.3235	0.3691

- ▶  $N = (1 * 3) + (2 * 2) + 3 + 5 + 10 = 25$
- ▶  $V = 1 + 3 + 2 + 1 + 1 + 1 = 9$
- ▶ Important: we added a new word type for unseen words. Let's call it UNK, the unknown word.
- ▶ Check that:  $1.0 == \sum_r n_r \times p_r$   
 $0.12 + (3 * 0.03079) + (2 * 0.06719) + 0.1045 + 0.1797 + 0.3691 = 1.0$

## Comparison between Add-one and Good-Turing

freq $r$	num with freq $r$ $n_r$	NS $p_r$	Add1 $p_r$	SGT $p_r$
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- ▶ NS = No smoothing:  $p_r = \frac{r}{N}$
- ▶ Add1 = Add-one smoothing:  $p_r = \frac{1+r}{V+N}$
- ▶ SGT = Simple Good-Turing:  $p_0 = \frac{n_1}{N}$ ,  $p_r = \frac{(r+1) \frac{n_{r+1}}{n_r}}{N}$   
with linear interpolation for missing values where  $n_{r+1} = 0$   
(Gale and Sampson, 1995) <http://www.grsampson.net/AGtf1.html>

## Using unigrams to smooth bigrams: incorrect version

$$P(w_i \mid w_{i-1}) = \frac{c(w_{i-1}, w_i)}{c(w_{i-1})}$$

- ▶ In add-one or Good-Turing:  
 $P(\text{the} \mid \text{string}) = P(\text{Fonz} \mid \text{string})$
- ▶ If  $c(w_{i-1}, w_i) = 0$ , then use  $P(w_i)$  (back off)
- ▶ Works for trigrams too: back off to bigrams and then unigrams
- ▶ Problem: probabilities get mixed up (unseen bigrams, for example will get higher probabilities than seen bigrams)

# Interpolation: Jelinek-Mercer Smoothing

$$P_{ML}(w_i \mid w_{i-1}) = \frac{c(w_{i-1}, w_i)}{c(w_{i-1})}$$

- ▶  $P_{JM}(w_i \mid w_{i-1}) = \lambda P_{ML}(w_i \mid w_{i-1}) + (1 - \lambda)P_{ML}(w_i)$   
where,  $0 \leq \lambda \leq 1$
- ▶ Notice that  $P_{JM}(\text{the} \mid \text{string}) > P_{JM}(\text{Fonz} \mid \text{string})$  as we wanted
- ▶ Jelinek-Mercer (1980) describe an elegant form of this **interpolation**:

$$P_{JM}(n\text{gram}) = \lambda P_{ML}(n\text{gram}) + (1 - \lambda)P_{JM}(n - 1\text{gram})$$

- ▶ What about  $P_{JM}(w_i)$ ?  
For missing unigrams:  $P_{JM}(w_i) = \lambda P_{ML}(w_i) + (1 - \lambda)\frac{\delta}{V}$

## Interpolation: Finding $\lambda$

$$P_{JM}(n\text{gram}) = \lambda P_{ML}(n\text{gram}) + (1 - \lambda)P_{JM}(n - 1\text{gram})$$

- ▶ Deleted Interpolation (Jelinek, Mercer)  
compute  $\lambda$  values to minimize cross-entropy on **held-out** data  
which is **deleted** from the initial set of training data
- ▶ Improved JM smoothing, a separate  $\lambda$  for each  $w_{i-1}$ :

$$P_{JM}(w_i \mid w_{i-1}) = \lambda(w_{i-1})P_{ML}(w_i \mid w_{i-1}) + (1 - \lambda(w_{i-1}))P_{ML}(w_i)$$

## Backoff Smoothing: Katz Backoff

- ▶ Use smoothing over counts for backoff smoothing.
- ▶ Also called discounting since we remove some probability from observed events.
- ▶ Katz Backoff (include Good-Turing with Backoff Smoothing)

$$P_{katz}(y \mid x) = \begin{cases} \frac{c^*(xy)}{c(x)} & \text{if } c(xy) > 0 \\ \alpha(x)P_{katz}(y) & \text{otherwise} \end{cases}$$

- ▶ where  $\alpha(x)$  is chosen to make sure that  $P_{katz}(y \mid x)$  is a proper probability

$$\alpha(x) = 1 - \sum_y \frac{c^*(xy)}{c(x)}$$

## Backoff Smoothing: Katz Backoff

$x$	$c(x)$	$c^*(x)$	$\frac{c^*(x)}{c(the)}$
the	48		
the,dog	15	14.5	14.5/48
the,woman	11	10.5	10.5/48
the,man	10	9.5	9.5/48
the,park	5	4.5	4.5/48
the,job	2	1.5	4.5/48
the,telescope	1	0.5	0.5/48
the>manual	1	0.5	0.5/48
the,afternoon	1	0.5	0.5/48
the,country	1	0.5	0.5/48
the,street	1	0.5	0.5/48
TOTAL			0.9479
the,UNK	0		0.052



# Backoff Smoothing with Discounting

- ▶ Witten-Bell discounting  
use the  $n - 1$  gram model when the  $n$  gram model has too few unique words **in the  $n$  gram context**
- ▶ Absolute discounting (Ney, Essen, Kneser)

$$P_{abs}(y \mid x) = \begin{cases} \frac{c(xy) - D}{c(x)} & \text{if } c(xy) > 0 \\ \alpha(x) P_{abs}(y) & \text{otherwise} \end{cases}$$

compute  $\alpha(x)$  as was done in Katz smoothing

## Language models

### $n$ -grams for Language Modeling

#### Smoothing $n$ -gram Models

##### Smoothing Counts

- Add-one Smoothing

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## Evaluating Language Models

# Evaluating Language Models

- ▶ So far we've seen the probability of a sentence:  $P(w_0, \dots, w_n)$
- ▶ What is the probability of a collection of sentences, that is what is the probability of an unseen test corpus  $T$
- ▶ Let  $T = s_0, \dots, s_m$  be a test corpus with sentences  $s_i$
- ▶  $T$  is assumed to be separate from the training data used to train our language model  $P(s)$
- ▶ What is  $P(T)$ ?

# Evaluating Language Models: Independence assumption

- ▶  $T = s_0, \dots, s_m$  is the text corpus with sentences  $s_0$  through  $s_m$
- ▶  $P(T) = P(s_0, s_1, s_2, \dots, s_m)$  – but each sentence is independent from the other sentences
- ▶  $P(T) = P(s_0) \cdot P(s_1) \cdot P(s_2) \cdot \dots \cdot P(s_m) = \prod_{i=0}^m P(s_i)$
- ▶  $P(s_i) = P(w_0^i, \dots, w_n^i)$  – which can be any  $n$ -gram language model
- ▶ A language model is better if the value of  $P(T)$  is higher for unseen sentences  $T$ , we want to maximize:

$$P(T) = \prod_{i=0}^m P(s_i)$$

# Evaluating Language Models: Computing the Average

- ▶ However,  $T$  can be any arbitrary size
- ▶  $P(T)$  will be lower if  $T$  is larger.
- ▶ Instead of the probability for a given  $T$  we can compute the *average* probability.
- ▶  $M$  is the total number of tokens in the test corpus  $T$ :

$$M = \sum_{i=1}^m \text{length}(s_i)$$

- ▶ The average *log* probability of the test corpus  $T$  is:

$$\frac{1}{M} \log_2 \prod_{i=1}^m P(s_i) = \frac{1}{M} \sum_{i=1}^m \log_2 P(s_i)$$

# Evaluating Language Models: Perplexity

- ▶ The average *log* probability of the test corpus  $T$  is:

$$\ell = \frac{1}{M} \sum_{i=1}^m \log_2 P(s_i)$$

- ▶ Note that  $\ell$  is a negative number
- ▶ We evaluate a language model using *Perplexity* which is  $2^{-\ell}$

# Evaluating Language Models

## Question

Show that:

$$2^{-\frac{1}{M} \log_2 \prod_{i=1}^m P(s_i)} = \frac{1}{\sqrt[M]{\prod_{i=1}^m P(s_i)}}$$

# Evaluating Language Models

## Question

What happens to  $2^{-\ell}$  if any  $n$ -gram probability for computing  $P(T)$  is zero?



# Evaluating Language Models: Typical Perplexity Values

From 'A Bit of Progress in Language Modeling' by Chen and Goodman

Model	Perplexity
unigram	955
bigram	137
trigram	74

# Natural Language Processing

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Part 4: Event space in Language Models

# Trigram Models

- ▶ The trigram model:

$$P(w_1, w_2, \dots, w_n) = \\ P(w_1) \times P(w_2 \mid w_1) \times P(w_3 \mid w_1, w_2) \times P(w_4 \mid w_2, w_3) \times \\ \dots P(w_i \mid w_{i-2}, w_{i-1}) \dots \times P(w_n \mid w_{n-2}, \dots, w_{n-1})$$

- ▶ Notice that the length of the sentence  $n$  is variable
- ▶ What is the event space?

# The stop symbol

- ▶ Let  $\mathcal{V} = \{a, b\}$  and the language  $L$  be  $\mathcal{V}^*$
- ▶ Consider a unigram model:  $P(a) = P(b) = 0.5$
- ▶ So strings in this language  $L$  are:

$a$ stop	$0.5$
$b$ stop	$0.5$
$aa$ stop	$0.5^2$
$bb$ stop	$0.5^2$
$\vdots$	

- ▶ The sum over all strings in  $L$  should be equal to 1:

$$\sum_{w \in L} P(w) = 1$$

- ▶ But  $P(a) + P(b) + P(aa) + P(bb) = 1.5$  !!

# The stop symbol

- ▶ What went wrong?  
We need to model variable length sequences
- ▶ Add an explicit probability for the stopsymbol:

$$P(a) = P(b) = 0.25$$

$$P(\text{stop}) = 0.5$$

- ▶  $P(\text{stop}) = 0.5$ ,  $P(a \text{ stop}) = P(b \text{ stop}) = 0.25 \times 0.5 = 0.125$ ,  
 $P(aa \text{ stop}) = 0.25^2 \times 0.5 = 0.03125$  (now the sum is no longer greater than one)

## The stop symbol

- ▶ With this new stop symbol we can show that  $\sum_w P(w) = 1$   
Notice that the probability of any sequence of length  $n$  is  $0.25^n \times 0.5$   
Also there are  $2^n$  sequences of length  $n$

$$\begin{aligned}\sum_w P(w) &= \\& \sum_{n=0}^{\infty} 2^n \times 0.25^n \times 0.5 \\& \sum_{n=0}^{\infty} 0.5^n \times 0.5 = \sum_{n=0}^{\infty} 0.5^{n+1} \\& \sum_{n=1}^{\infty} 0.5^n = 1\end{aligned}$$

## The stop symbol

- ▶ With this new stop symbol we can show that  $\sum_w P(w) = 1$   
Using  $p_s = P(\text{stop})$  the probability of any sequence of length  $n$  is  $p(n) = p(w_1, \dots, w_{n-1}) \times p_s(w_n)$

$$\begin{aligned}\sum_w P(w) &= \sum_{n=0}^{\infty} p(n) \sum_{w_1, \dots, w_n} p(w_1, \dots, w_n) \\ &= \sum_{n=0}^{\infty} p(n) \sum_{w_1, \dots, w_n} \prod_{i=1}^n p(w_i)\end{aligned}$$

$$\begin{aligned}\sum_{w_1, \dots, w_n} \prod_i p(w_i) &= \\ \sum_{w_1} \sum_{w_2} \dots \sum_{w_n} p(w_1)p(w_2) \dots p(w_n) &= 1\end{aligned}$$

## The stop symbol

$$\sum_{w_1} \sum_{w_2} \dots \sum_{w_n} p(w_1)p(w_2) \dots p(w_n) = 1$$

$$\begin{aligned} \sum_{n=0}^{\infty} p(n) &= \sum_{n=0}^{\infty} p_s(1 - p_s)^n \\ &= p_s \sum_{n=0}^{\infty} (1 - p_s)^n \\ &= p_s \frac{1}{1 - (1 - p_s)} = p_s \frac{1}{p_s} = 1 \end{aligned}$$



## Acknowledgements

Many slides borrowed or inspired from lecture notes by Michael Collins, Chris Dyer, Kevin Knight, Philipp Koehn, Adam Lopez, and Luke Zettlemoyer from their NLP course materials.

All mistakes are my own.