Natural Language Processing

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Part 1: Probability and Language

Probability and Language

Quick guide to probability theory

Entropy and Information Theory

Probability and Language

Assign a probability to an input sequence

Given a URL: `choosespain.com`. What is this website about?

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chooses pain	-9.88
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The Goal

Find a good **scoring function** for input sequences.

Scoring Hypotheses in Speech Recognition

From acoustic signal to candidate transcriptions

Hypothesis	Score
the station signs are in deep in english	-14732
the stations signs are in deep in english	-14735
the station signs are in deep into english	-14739
the station 's signs are in deep in english	-14740
the station signs are in deep in the english	-14741
the station signs are indeed in english	-14757
the station 's signs are indeed in english	-14760
the station signs are indians in english	-14790
the station signs are indian in english	-14799
the stations signs are indians in english	-14807
the stations signs are indians and english	-14815

Scoring Hypotheses in Machine Translation

From source language to target language candidates

Hypothesis	Score
we must also discuss a vision .	-29.63
we must also discuss on a vision .	-31.58
it is also discuss a vision .	-31.96
we must discuss on greater vision .	-36.09
⋮	⋮

Scoring Hypotheses in Decryption

Character substitutions on ciphertext to plaintext candidates

Hypothesis	Score
Heopaj, zk ukq swjp pk gjks w oaynap?	-93
Urbcnw, mx hxd fjwc cx twxf j bnlanc?	-92
Wtdepy, oz jzf hlye ez vyzh l dpncpe?	-91
Mjtufo, ep zpv xbou up lopx b tfdsfu?	-89
Nkuvgp, fq aqw ycpv vq mpqy c ugetgv?	-87
Gdnozi, yj tjp rvio oj fijr v nzxmzo?	-86
Czjkve, uf pfl nrek kf befn r jvtivk?	-85
Yvfgra, qb lbh jnag gb xabj n frperg?	-84
Zwghsb, rc mci kobh hc ybck o gsqfsh?	-83
Byijud, te oek mqdj je adem q iushuj?	-77
Jgqrcl, bm wms uylr rm ilmu y qcapcr?	-76
Listen, do you want to know a secret?	-25

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- **Learn** the parameters of the model from data.
- Use the model to **predict** the probability of new sequences.

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Part 2: Quick guide to probability theory

Probability and Language

Quick guide to probability theory

Entropy and Information Theory

Probability: The Basics

- Sample space

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- Sample space
- Event space
- Random variable

Probability distributions

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  - $P(X = \text{killer}) = 1.05\text{e-}05$
  - $P(X = \text{app}) = 1.19\text{e-}05$

Joint probability

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Joint Probability: P(X=value AND Y=value)

- Since X=value AND Y=value, the order does not matter
- P(X = killer, Y = app) $\Leftrightarrow$ P(Y = app, X = killer)
- In both cases it is P(X,Y) = P(Y,X) = P('killer app')
- In NLP, we often use numerical indices to express this: P(W_{i-1} = killer, W_i = app)

Joint probability

Joint probability table

W_{i-1}	$W_i = \text{app}$	$\mathrm{P}(W_{i-1}, W_i)$
$\langle S \rangle$	app	1.16e-19
an	app	1.76e-08
killer	app	1.24e-10
the	app	2.68e-07
this	app	3.74e-08
your	app	2.39e-08

There will be a similar table for each choice of W_i.

Get $\mathrm{P}(W_i)$ from $\mathrm{P}(W_{i-1}, W_i)$

$$P(W_i = \text{app}) = \sum_x P(W_{i-1} = x, W_i = \text{app}) = 1.19e-05$$

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Conditional probability from Joint probability

$$P(W_i \mid W_{i-1}) = \frac{P(W_{i-1}, W_i)}{P(W_{i-1})}$$

- P(killer) = 1.05e-05
- P(killer, app) = 1.24e-10
- P(app | killer) = 0.0096
- P(the | killer) = 1.82e-05

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- Computing $P(f)$ from axioms:

$$P(f) = \sum_e P(e) \times P(f \mid e)$$

Probability: The Chain Rule

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  - $\frac{p(a,b,c,d,e)}{p(e)} = \frac{p(d,e)}{p(e)} \cdot \frac{p(c,d,e)}{p(d,e)} \cdot \frac{p(b,c,d,e)}{p(c,d,e)} \cdot \frac{p(a,b,c,d,e)}{p(b,c,d,e)}$
- Use chain rule and simplify:

$$P(a, b, c, d \mid e) = P(d \mid e) \cdot P(c \mid d, e) \cdot P(b \mid c, e) \cdot P(a \mid b, e)$$

Probability: The Chain Rule

- $P(e_1, e_2, \ldots, e_n) = P(e_1) \times P(e_2 \mid e_1) \times P(e_3 \mid e_1, e_2) \ldots$

$$P(e_1, e_2, \ldots, e_n) = \prod_{i=1}^{n} P(e_i \mid e_{i-1}, e_{i-2}, \ldots, e_1)$$

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- Shorthand for value assigned to a random variable Y, e.g. $Y = y$
- y is an element of some implicit **event space**: $\mathcal{E}$

Probability: Random Variables and Events

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- Finding the value that maximizes the probability value:

$$\hat{x} = \arg\max_{x \in \mathcal{E}} P(x)$$

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\begin{aligned}
\log(P(e)) &= \log(p_1 \times p_2 \times \ldots \times p_n) \\
&= \log(p_1) + \log(p_2) + \ldots + \log(p_n)
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- Note that:

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- Also more efficient: addition instead of multiplication

Log Probability Arithmetic

p	$\log(p)$
0.0	$-\infty$
0.1	-3.32
0.2	-2.32
0.3	-1.74
0.4	-1.32
0.5	-1.00
0.6	-0.74
0.7	-0.51
0.8	-0.32
0.9	-0.15
1.0	0.00

Log Probability Arithmetic

- So: $(0.5 \times 0.5 \times \ldots 0.5) = (0.5)^n$ might get too small but $(-1 - 1 - 1 - 1) = -n$ is manageable

Log Probability Arithmetic

- So: $(0.5 \times 0.5 \times \ldots 0.5) = (0.5)^n$ might get too small but $(-1 - 1 - 1 - 1) = -n$ is manageable
- Another useful fact when writing code ($\log_2$ is *log to the base 2*):

$$\log_2(x) = \frac{\log_{10}(x)}{\log_{10}(2)}$$

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Part 3: Entropy and Information Theory

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Information Theory

- Information theory is the use of probability theory to quantify and measure "information".
- Consider the task of efficiently sending a message. Sender Alice wants to send several messages to Receiver Bob. Alice wants to do this as efficiently as possible.
- Let's say that Alice is sending a message where the entire message is just one character a, e.g. $aaaa\ldots$. In this case we can save space by simply sending the length of the message and the single character.

Information Theory

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- This formulation of entropy by Claude Shannon was adapted from thermodynamics, converting information into a quantity that can be measured.
- Information theory is built around this notion of message compression as a way to evaluate the amount of information.

Expectation

- For a probability distribution p

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- **Expectation** with respect to p is a weighted average:

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\begin{aligned}
E_p[x] &= \frac{x_1 \cdot p_1 + x_2 \cdot p_2 + \ldots + x_n p_n}{p_1 + p_2 + \ldots + p_n} \\
&= x_1 \cdot p_1 + x_2 \cdot p_2 + \ldots + x_n p_n \\
&= \sum_{x \in \mathcal{E}} x \cdot p(x)
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- Example: for a six-sided die the expectation is:

$$E_p[x] = 1 \cdot \frac{1}{6} + 2 \cdot \frac{1}{6} + \ldots + 6 \cdot \frac{1}{6} = 3.5$$

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- Entropy answers the question: *What is the expected number of bits needed to transmit messages from event space $\mathcal{E}$, where $p(x)$ defines the probability of observing x?*

Entropy

- Alice wants to bet on a horse race. She has to send a message to her bookie Bob to tell him which horse to bet on.

Entropy

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- Can we do better?

Entropy

Horse 1	$\frac{1}{2}$	Horse 5	$\frac{1}{64}$
Horse 2	$\frac{1}{4}$	Horse 6	$\frac{1}{64}$
Horse 3	$\frac{1}{8}$	Horse 7	$\frac{1}{64}$
Horse 4	$\frac{1}{16}$	Horse 8	$\frac{1}{64}$

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- If we know how likely we are to bet on each horse, say based on the horse's probability of winning, then we can do better.
- Let p be the probability distribution given in the table above. The entropy of p is $H(p)$

Entropy

$$
\begin{aligned}
H(p) = & \\
= & \ -\sum_{i=1}^{8} p(i) \log_2 \ p(i) \\
= & \ -\left(\frac{1}{2}\log_2\frac{1}{2} + \frac{1}{4}\log_2\frac{1}{4} + \frac{1}{8}\log_2\frac{1}{8} + \frac{1}{16}\log_2\frac{1}{16} + 4(\frac{1}{64}\log_2\frac{1}{64})\right) \\
= & \ -\left(\frac{1}{2}\times -1 + \frac{1}{4}\times -2 + \frac{1}{8}\times -3 + \frac{1}{16}\times -4 + 4(\frac{1}{64}\times -6)\right) \\
= & \ -\left({\color{red}-\frac{1}{2} - \frac{1}{2}} - {\color{blue}\frac{3}{8} - \frac{1}{4} - \frac{3}{8}}\right) \\
= & \ 2 \ bits
\end{aligned}
$$

- What is the entropy when the horses are equally likely to win?

$$H(uniform\ distribution) = -8(\frac{1}{8} \times -3) = 3\ bits$$

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- Total number of bits per message (per race): $\frac{545}{320} \approx 1.7$ bits (always less than 2 bits)

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- Choosing between the biased horses from before (H=2) is $2^2 = 4$.

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- The relative entropy is also called the *Kullback-Leibler divergence*.

Cross Entropy and Relative Entropy

- The **relative entropy** can be written as the sum of two terms:

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- The term $H_q(p)$ is called the **cross entropy**.

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- $D(q \| p)$ is not a true distance:
  - It is asymmetric: $D(q \| p) \neq D(p \| q)$,
  - It does not obey the triangle inequality: $D(p \| r) \nleq D(p \| q) + D(q \| r)$

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- *Entropy* of a random variable X:

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- *Mutual Information* between two random variables X and Y:

$$I(X;Y) = D(p(x,y) \| p(x)p(y)) = \sum_{x} \sum_{y} p(x,y) \log_2 \frac{p(x,y)}{p(x)p(y)}$$

Log Probability Arithmetic

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- A more efficient soln, let *big* be a large constant e.g. 10^{30}:

function $logadd(x, y)$: # returns $\log(\exp(x) + \exp(y))$
if $(y - x) > \log(big)$ return y
elsif $(x - y) > \log(big)$ return x
else return
$$min(x, y) + \log(\exp(x - min(x, y)) + \exp(y - min(x, y)))$$
endif

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  endif
- There is a more efficient way of computing
  $\log(\exp(x - min(x, y)) + \exp(y - min(x, y)))$

Log Probability Arithmetic

function $logadd(x, y)$:
    if $(y - x) > \log(big)$ return y
    elsif $(x - y) > \log(big)$ return x
    elsif $(x \geq y)$ return $x + \log(1 + \exp(y - x))$
        # note that $max(x, y) = x$ and $y - x \leq 0$
    else return $y + \log(\exp(x - y) + 1)$
        # note that $max(x, y) = y$ and $x - y \leq 0$
    endif

Also, in ANSI C, `log1p` efficiently computes $\log(1 + x)$

`http://www.ling.ohio-state.edu/~jansche/src/logadd.c`

In Python, numpy.logaddexp2(x1,x2) for base 2

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All mistakes are my own.