Natural Language Processing

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Part 1: Introducing Hidden Markov Models

Modelling pairs of sequences

Input: sequence of words; Output: sequence of labels

Input	British	left	waffles	on	Falkland	Islands
Output1	N	N	V	P	N	N
Output2	N	V	N	P	N	N

⋮

N Noun, e.g. islands

V Verb, e.g. leave, left

P Preposition, e.g. on

Modelling pairs of sequences

Input: sequence of words; Output: sequence of labels

Input	British	left	waffles	on	Falkland	Islands
Output1	N	N	V	P	N	N
Output2	N	V	N	P	N	N

$\vdots$

- 3 states: $\mathcal{S} = \{N, V, P\}$
- Input sequence: $x_1, x_2, \ldots, x_n$
- Output sequence: $t_1, t_2, \ldots, t_n$ where $t_i \in \mathcal{S}$
- How many output sequences?

$\mid \mathcal{S} \mid^n$

Modelling pairs of sequences

Input: sequence of characters; Output: sequence of labels

Input	北京大学生比赛	7 chars
Output1	BIBIIBI	7 labels
Output2	BIIIBBI	7 labels
⋮		7 labels

B Begin word

I Inside word

BIBIIBI 北京—大学生—比赛 (Beijing student competition)

BIIIBBI 北京大学—生—比赛 (Peking University Health Competition)

Hidden Markov Models

- Input: x
- Output space: $\mathcal{Y}(x)$
- Output: $y \in \mathcal{Y}(x)$
- We want to learn a function f such that $f(x) = y$

Hidden Markov Models

Conditional model

- Construct function f using a conditional probability:
$$f(x) = \arg\max_{y \in \mathcal{Y}(x)} p(y \mid x)$$
- We can construct this function f using two principles:
  - Discriminative learning: find the best output y given input x
  - Generative modelling: model the joint probability $p(x, y)$ to find $p(y \mid x)$

Hidden Markov Models

Generative Model

- Start from the joint probability $p(x, y)$:
$$p(x, y) = p(y)p(x \mid y)$$
- Also:
$$p(x, y) = p(x)p(y \mid x)$$

Bayes Rule:

$$p(y \mid x) = \frac{p(y)p(x \mid y)}{p(x)}$$

Hidden Markov Models

Generative Model

- Bayes Rule:

$$p(y \mid x) = \frac{p(y)p(x \mid y)}{p(x)}$$

- where:

$$p(x) = \sum_{y \in \mathcal{Y}(x)} p(x, y) = \sum_{y \in \mathcal{Y}(x)} p(y)p(x \mid y)$$

- So using a generative model, we can find the best output y using:

$$p(y \mid x) = \frac{p(y)p(x \mid y)}{\sum_{y \in \mathcal{Y}(x)} p(y)p(x \mid y)}$$

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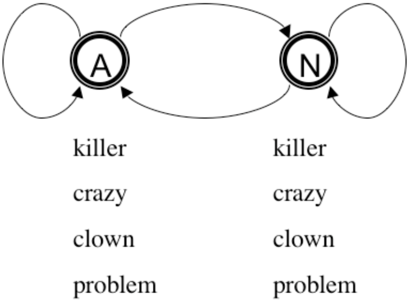
Part 2: Algorithms for Hidden Markov Models

Hidden Markov Model

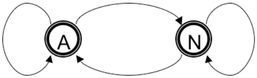
$$\text{Model } \theta = \begin{cases} \pi_i & \text{probability of starting at state } i \\ a_{i,j} & \text{probability of transition from state } i \text{ to state } j \\ b_i(o) & \text{probability of output } o \text{ at state } i \end{cases}$$



Hidden Markov Model Algorithms

- HMM as parser: compute the best sequence of states for a given observation sequence.
- HMM as language model: compute probability of given observation sequence.
- HMM as learner: given a corpus of observation sequences, learn its distribution, i.e. learn the parameters of the HMM from the corpus.
  - Learning from a set of observations with the sequence of states provided (states are not hidden) [Supervised Learning]
  - Learning from a set of observations without any state information. [Unsupervised Learning]

HMM as Parser



$\pi =$

A	0.25
N	0.75

$a =$

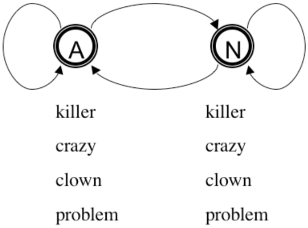
$a_{i,j}$	A	N
A	0.0	1.0
N	0.5	0.5

$b =$

$b_i(o)$	*clown*	*killer*	*problem*	*crazy*
A	0	0	0	1
N	0.4	0.3	0.3	0

The task: for a given observation sequence find the most likely state sequence.

HMM as Parser



- Find most likely sequence of states for *killer clown*
- Score every possible sequence of states: AA, AN, NN, NA
  - P(killer clown, AA) $= \pi_A \cdot b_A(killer) \cdot a_{A,A} \cdot b_A(clown) = 0.0$
  - P(killer clown, AN) $= \pi_A \cdot b_A(killer) \cdot a_{A,N} \cdot b_N(clown) = 0.0$
  - P(killer clown, NN) $= \pi_N \cdot b_N(killer) \cdot a_{N,N} \cdot b_N(clown) = 0.75 \cdot 0.3 \cdot 0.5 \cdot 0.4 = 0.045$
  - P(killer clown, NA) $= \pi_N \cdot b_N(killer) \cdot a_{N,A} \cdot b_A(clown) = 0.0$
- Pick the state sequence with highest probability (NN=0.045).

HMM as Parser

- As we have seen, for input of length 2, and a HMM with 2 states there are 2^2 possible state sequences.
- In general, if we have q states and input of length T there are q^T possible state sequences.
- Using our example HMM, for input *killer crazy clown problem* we will have 2^4 possible state sequences to score.
- Our naive algorithm takes exponential time to find the best state sequence for a given input.
- The **Viterbi algorithm** uses dynamic programming to provide the best state sequence with a time complexity of $q^2 \cdot T$

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Part 3: Viterbi Algorithm for HMMs

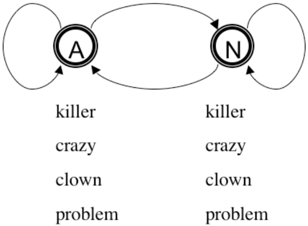
Viterbi Algorithm for HMMs

- For input of length T: $o_1, \ldots, o_T$, we want to find the sequence of states $s_1, \ldots, s_T$
- Each s_t in this sequence is one of the states in the HMM.
- So the task is to find the most likely sequence of states:

$$\arg\max_{s_1,\ldots,s_T} P(o_1, \ldots, o_T, s_1, \ldots, s_T)$$

- The Viterbi algorithm solves this by creating a table $V[s, t]$ where s is one of the states, and t is an index between $1, \ldots, T$.

Viterbi Algorithm for HMMs



- Consider the input *killer crazy clown problem*
- So the task is to find the most likely sequence of states:

$$\arg \max_{s_1, s_2, s_3, s_4} P(\textit{killer crazy clown problem}, s_1, s_2, s_3, s_4)$$

- A sub-problem is to find the most likely sequence of states for *killer crazy clown*:

$$\arg \max_{s_1, s_2, s_3} P(\textit{killer crazy clown}, s_1, s_2, s_3)$$

Viterbi Algorithm for HMMs

- In our example there are two possible values for s_4:

$$\max_{s_1,\ldots,s_4} P(\textit{killer crazy clown problem}, s_1, s_2, s_3, s_4) =$$

$$\max\left\{ \max_{s_1,s_2,s_3} P(\textit{killer crazy clown problem}, s_1, s_2, s_3, N), \right.$$

$$\left. \max_{s_1,s_2,s_3} P(\textit{killer crazy clown problem}, s_1, s_2, s_3, A) \right\}$$

- Similarly:

$$\max_{s_1,\ldots,s_3} P(\textit{killer crazy clown}, s_1, s_2, s_3) =$$

$$\max\left\{ \max_{s_1,s_2} P(\textit{killer crazy clown}, s_1, s_2, N), \right.$$

$$\left. \max_{s_1,s_2} P(\textit{killer crazy clown}, s_1, s_2, A) \right\}$$

Viterbi Algorithm for HMMs

- Putting them together:

$$P(\textit{killer crazy clown problem}, s_1, s_2, s_3, N) =$$
$$\max\{P(\textit{killer crazy clown}, s_1, s_2, N) \cdot a_{N,N} \cdot b_N(\textit{problem}),$$
$$P(\textit{killer crazy clown}, s_1, s_2, A) \cdot a_{A,N} \cdot b_N(\textit{problem})\}$$

$$P(\textit{killer crazy clown problem}, s_1, s_2, s_3, A) =$$
$$\max\{P(\textit{killer crazy clown}, s_1, s_2, N) \cdot a_{N,A} \cdot b_A(\textit{problem}),$$
$$P(\textit{killer crazy clown}, s_1, s_2, A) \cdot a_{A,A} \cdot b_A(\textit{problem})\}$$

- The best score is given by:

$$\max_{s_1,\ldots,s_4} P(\textit{killer crazy clown problem}, s_1, s_2, s_3, s_4) =$$
$$\max_{N,A}\left\{\max_{s_1,s_2,s_3} P(\textit{killer crazy clown problem}, s_1, s_2, s_3, N),\right.$$
$$\left.\max_{s_1,s_2,s_3} P(\textit{killer crazy clown problem}, s_1, s_2, s_3, A)\right\}$$

Viterbi Algorithm for HMMs

- Provide an index for each input symbol:
  1:killer 2:crazy 3:clown 4:problem

$$
\begin{aligned}
V[N,3] &= \max_{s_1,s_2} P(\textit{killer crazy clown}, s_1, s_2, N) \\
V[N,4] &= \max_{s_1,s_2,s_3} P(\textit{killer crazy clown problem}, s_1, s_2, s_3, N)
\end{aligned}
$$

- Putting them together:

$$
\begin{aligned}
V[N,4] &= \max\{V[N,3] \cdot a_{N,N} \cdot b_N(\textit{problem}), \\
&\qquad V[A,3] \cdot a_{A,N} \cdot b_N(\textit{problem})\} \\[1em]
V[A,4] &= \max\{V[N,3] \cdot a_{N,A} \cdot b_A(\textit{problem}), \\
&\qquad V[A,3] \cdot a_{A,A} \cdot b_A(\textit{problem})\}
\end{aligned}
$$

- The best score for the input is given by:
  $\max\{V[N,4], V[A,4]\}$
- To extract the best sequence of states we backtrack (same trick as obtaining alignments from minimum edit distance)

Viterbi Algorithm for HMMs

- For input of length T: $o_1, \ldots, o_T$, we want to find the sequence of states $s_1, \ldots, s_T$
- Each s_t in this sequence is one of the states in the HMM.
- For each state q we initialize our table: $V[q, 1] = \pi_q \cdot b_q(o_1)$
- Then compute for $t = 1 \ldots T - 1$ for each state q:

$$V[q, t+1] = \max_{q'} \left\{ V[q', t] \cdot a_{q',q} \cdot b_q(o_{t+1}) \right\}$$

- After the loop terminates, the best score is $\max_q V[q, T]$

Learning from Fully Observed Data

$\pi =$

A	0.25
N	0.75

$a =$

$a_{i,j}$	A	N
A	0.0	1.0
N	0.5	0.5

$b =$

$b_i(o)$	*clown*	*killer*	*problem*	*crazy*
A	0	0	0	1
N	0.4	0.3	0.3	0

Viterbi algorithm:

V	killer:1	crazy:2	clown:3	problem:4
A				
N				

Learning from Fully Observed Data

$\pi =$

A	0.25
N	0.75

$a =$

$a_{i,j}$	A	N
A	0.0	1.0
N	0.5	0.5

$b =$

$b_i(o)$	*clown*	*killer*	*problem*	*crazy*
A	0	0	0	1
N	0.4	0.3	0.3	0

Viterbi algorithm:

V	killer:1	crazy:2	clown:3	problem:4
A	0	0.1125	0	0
N	0.225	0	0.045	0.00675

Probability models of language

Question

$\pi =$

V	0.25
N	0.75

$a =$

$a_{i,j}$	V	N
V	0.5	0.5
N	0.5	0.5

$b =$

$b_i(o)$	*time*	*flies*	*can*
V	0.1	0.1	0.8
N	0.5	0.4	0.1

What is the best sequence of tags for each string below:

1. *time*
2. *time flies*
3. *time flies can*

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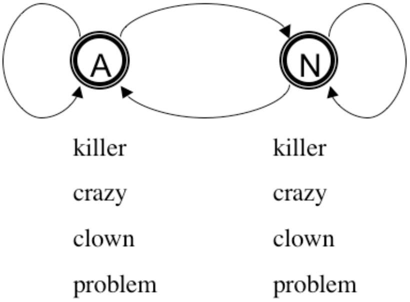
Part 4: HMM as a Language Model

Hidden Markov Model

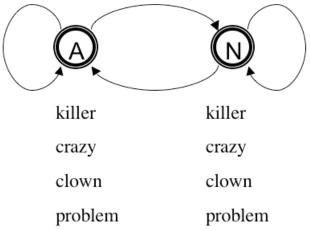
$$\text{Model } \theta = \begin{cases} \pi_i & \text{probability of starting at state } i \\ a_{i,j} & \text{probability of transition from state } i \text{ to state } j \\ b_i(o) & \text{probability of output } o \text{ at state } i \end{cases}$$



Hidden Markov Model Algorithms

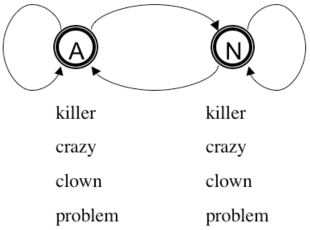
- HMM as parser: compute the best sequence of states for a given observation sequence.
- HMM as language model: compute probability of given observation sequence.
- HMM as learner: given a corpus of observation sequences, learn its distribution, i.e. learn the parameters of the HMM from the corpus.
  - Learning from a set of observations with the sequence of states provided (states are not hidden) [Supervised Learning]
  - Learning from a set of observations without any state information. [Unsupervised Learning]

HMM as a Language Model



- Find $P(\textit{killer clown}) = \sum_y P(y, \textit{killer clown})$
- $P(\textit{killer clown}) = P(AA, \textit{killer clown}) + P(AN, \textit{killer clown}) + P(NN, \textit{killer clown}) + P(NA, \textit{killer clown})$

HMM as a Language Model



- Consider the input *killer crazy clown problem*
- So the task is to find the sum over all sequences of states:

$$\sum_{s_1,s_2,s_3,s_4} P(\textit{killer crazy clown problem}, s_1, s_2, s_3, s_4)$$

- A sub-problem is to find the most likely sequence of states for *killer crazy clown*:

$$\sum_{s_1,s_2,s_3} P(\textit{killer crazy clown}, s_1, s_2, s_3)$$

HMM as a Language Model

- In our example there are two possible values for s_4:

$$\sum_{s_1,\ldots,s_4} P(\mathit{killer\ crazy\ clown\ problem}, s_1, s_2, s_3, s_4) =$$

$$\sum_{s_1,s_2,s_3} P(\mathit{killer\ crazy\ clown\ problem}, s_1, s_2, s_3, N) +$$

$$\sum_{s_1,s_2,s_3} P(\mathit{killer\ crazy\ clown\ problem}, s_1, s_2, s_3, A)$$

- Very similar to the Viterbi algorithm. Sum instead of max, and that's the only difference!

HMM as a Language Model

- Provide an index for each input symbol:
  1:killer 2:crazy 3:clown 4:problem

$$V[N,3] = \sum_{s_1,s_2} P(\textit{killer crazy clown}, s_1, s_2, N)$$

$$V[N,4] = \sum_{s_1,s_2,s_3} P(\textit{killer crazy clown problem}, s_1, s_2, s_3, N)$$

- Putting them together:

$$\begin{aligned} V[N,4] &= V[N,3] \cdot a_{N,N} \cdot b_N(\textit{problem}) + \\ &\quad V[A,3] \cdot a_{A,N} \cdot b_N(\textit{problem}) \end{aligned}$$

$$\begin{aligned} V[A,4] &= V[N,3] \cdot a_{N,A} \cdot b_A(\textit{problem}) + \\ &\quad V[A,3] \cdot a_{A,A} \cdot b_A(\textit{problem}) \end{aligned}$$

- The best score for the input is given by: $V[N,4] + V[A,4]$

HMM as a Language Model

- For input of length T: $o_1, \ldots, o_T$, we want to find $P(o_1, \ldots, o_T) = \sum_{y_1,\ldots,y_T} P(y_1, \ldots, y_T, o_1, \ldots, o_T)$
- Each y_t in this sequence is one of the states in the HMM.
- For each state q we initialize our table: $V[q, 1] = \pi_q \cdot b_q(o_1)$
- Then compute recursively for $t = 1 \ldots T - 1$ for each state q:

$$V[q, t+1] = \sum_{q'} \left\{ V[q', t] \cdot a_{q',q} \cdot b_q(o_{t+1}) \right\}$$

- After the loop terminates, the best score is $\sum_q V[q, T]$
- So: Viterbi with sum instead of max gives us an algorithm for HMM as a language model.
- This algorithm is sometimes called the *forward algorithm*.

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Part 5: Supervised Learning for HMMs

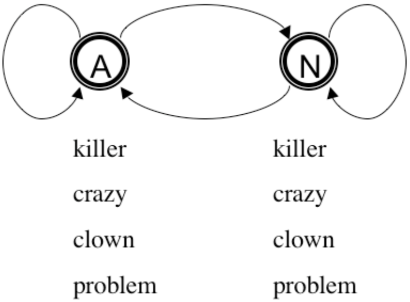
Hidden Markov Model Algorithms

- HMM as parser: compute the best sequence of states for a given observation sequence.
- HMM as language model: compute probability of given observation sequence.
- HMM as learner: given a corpus of observation sequences, learn its distribution, i.e. learn the parameters of the HMM from the corpus.
  - Learning from a set of observations with the sequence of states provided (states are not hidden) [Supervised Learning]
  - Learning from a set of observations without any state information. [Unsupervised Learning]

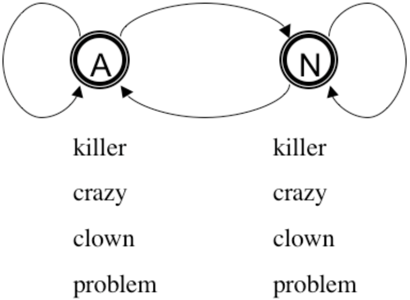
Hidden Markov Model

$$\text{Model } \theta = \begin{cases} \pi_i & \text{probability of starting at state } i \\ a_{i,j} & \text{probability of transition from state } i \text{ to state } j \\ b_i(o) & \text{probability of output } o \text{ at state } i \end{cases}$$

$$\text{Constraints}: \sum_i \pi_i = 1, \sum_j a_{i,j} = 1, \sum_o b_i(o) = 1$$



HMM Learning from Labeled Data



- The task: to find the values for the parameters of the HMM:
  - π_A, π_N
  - $a_{A,A}, a_{A,N}, a_{N,N}, a_{N,A}$
  - $b_A(killer), b_A(crazy), b_A(clown), b_A(problem)$
  - $b_N(killer), b_N(crazy), b_N(clown), b_N(problem)$

Learning from Fully Observed Data

Labeled Data L

```
x1,y1: killer/N clown/N      (x1 = killer,clown; y1 = N,N)
x2,y2: killer/N problem/N    (x2 = killer,problem; y2 = N,N)
x3,y3: crazy/A problem/N     ...
x4,y4: crazy/A clown/N
x5,y5: problem/N crazy/A clown/N
x6,y6: clown/N crazy/A killer/N
```

Learning from Fully Observed Data

- Let's say we have m labeled examples:
  $L = (x_1, y_1), \ldots, (x_m, y_m)$
- Each $(x_\ell, y_\ell) = \{o_1, \ldots, o_T, s_1, \ldots, s_T\}$
- For each (x_ℓ, y_ℓ) we can compute the probability using the HMM:
  - $(x_1 = \textit{killer}, \textit{clown}; y_1 = N, N)$:
    $P(x_1, y_1) = \pi_N \cdot b_N(\textit{killer}) \cdot a_{N,N} \cdot b_N(\textit{clown})$
  - $(x_2 = \textit{killer}, \textit{problem}; y_2 = N, N)$:
    $P(x_2, y_2) = \pi_N \cdot b_N(\textit{killer}) \cdot a_{N,N} \cdot b_N(\textit{problem})$
  - $(x_3 = \textit{crazy}, \textit{problem}; y_3 = A, N)$:
    $P(x_3, y_3) = \pi_A \cdot b_A(\textit{crazy}) \cdot a_{A,N} \cdot b_N(\textit{problem})$
  - $(x_4 = \textit{crazy}, \textit{clown}; y_4 = A, N)$:
    $P(x_4, y_4) = \pi_A \cdot b_A(\textit{crazy}) \cdot a_{A,N} \cdot b_N(\textit{clown})$
  - $(x_5 = \textit{problem}, \textit{crazy}, \textit{clown}; y_5 = N, A, N)$:
    $P(x_5, y_5) = \pi_N \cdot b_N(\textit{problem}) \cdot a_{N,A} \cdot b_A(\textit{crazy}) \cdot a_{A,N} \cdot b_N(\textit{clown})$
  - $(x_6 = \textit{clown}, \textit{crazy}, \textit{killer}; y_6 = N, A, N)$:
    $P(x_6, y_6) = \pi_N \cdot b_N(\textit{clown}) \cdot a_{N,A} \cdot b_A(\textit{crazy}) \cdot a_{A,N} \cdot b_N(\textit{killer})$
- $\prod_\ell P(x_\ell, y_\ell) = {\pi_N}^4 \cdot {\pi_A}^2 \cdot {a_{N,N}}^2 \cdot {a_{N,A}}^2 \cdot {a_{A,N}}^4 \cdot {a_{A,A}}^0 \cdot b_N(\textit{killer})^3 \cdot b_N(\textit{clown})^4 \cdot b_N(\textit{problem})^3 \cdot b_A(\textit{crazy})^4$

Learning from Fully Observed Data

- We can easily collect frequency of observing a word with a state (tag)
  - $f(i, x, y)$ = number of times i is the initial state in (x, y)
  - $f(i, j, x, y)$ = number of times j follows i in (x, y)
  - $f(i, o, x, y)$ = number of times i is paired with observation o
- Then according to our HMM the probability of x, y is:

$$P(x, y) = \prod_i \pi_i^{f(i,x,y)} \cdot \prod_{i,j} a_{i,j}^{f(i,j,x,y)} \cdot \prod_{i,o} b_i(o)^{f(i,o,x,y)}$$

Learning from Fully Observed Data

- According to our HMM the probability of x, y is:

$$P(x,y) = \prod_i \pi_i^{f(i,x,y)} \cdot \prod_{i,j} a_{i,j}^{f(i,j,x,y)} \cdot \prod_{i,o} b_i(o)^{f(i,o,x,y)}$$

- For the labeled data $L = (x_1, y_1), \ldots, (x_\ell, y_\ell), \ldots, (x_m, y_m)$

$$\begin{aligned} P(L) &= \prod_{\ell=1}^{m} P(x_\ell, y_\ell) \\ &= \prod_{\ell=1}^{m} \left(\prod_i \pi_i^{f(i,x_\ell,y_\ell)} \cdot \prod_{i,j} a_{i,j}^{f(i,j,x_\ell,y_\ell)} \cdot \prod_{i,o} b_i(o)^{f(i,o,x_\ell,y_\ell)} \right) \end{aligned}$$

Learning from Fully Observed Data

- According to our HMM the probability of x, y is:

$$P(L) = \prod_{\ell=1}^{m} \left(\prod_{i} \pi_i^{f(i,x_\ell,y_\ell)} \cdot \prod_{i,j} a_{i,j}^{f(i,j,x_\ell,y_\ell)} \cdot \prod_{i,o} b_i(o)^{f(i,o,x_\ell,y_\ell)} \right)$$

- The log probability of the labeled data $(x_1, y_1), \ldots, (x_m, y_m)$ according to HMM with parameters θ is:

$$\begin{aligned} L(\theta) &= \sum_{\ell=1}^{m} \log P(x_\ell, y_\ell) \\ &= \sum_{\ell=1}^{m} \sum_{i} f(i, x_\ell, y_\ell) \log \pi_i + \\ &\quad \sum_{i,j} f(i, j, x_\ell, y_\ell) \log a_{i,j} + \\ &\quad \sum_{i,o} f(i, o, x_\ell, y_\ell) \log b_i(o) \end{aligned}$$

Learning from Fully Observed Data

$$L(\theta) = \sum_{\ell=1}^{m}$$

$$\sum_{i} f(i, x_\ell, y_\ell) \log \pi_i + \sum_{i,j} f(i, j, x_\ell, y_\ell) \log a_{i,j} + \sum_{i,o} f(i, o, x_\ell, y_\ell) \log b_i(o)$$

- $\theta = (\pi, a, b)$
- $L(\theta)$ is the log probability of the labeled data $(x_1, y_1), \ldots, (x_m, y_m)$
- We want to find a θ that will give us the maximum value of $L(\theta)$
- Find the θ such that $\frac{dL(\theta)}{d\theta} = 0$

Learning from Fully Observed Data

$$L(\theta) = \sum_{\ell=1}^{m}$$

$$\sum_i f(i, x_\ell, y_\ell) \log \pi_i + \sum_{i,j} f(i, j, x_\ell, y_\ell) \log a_{i,j} + \sum_{i,o} f(i, o, x_\ell, y_\ell) \log b_i(o)$$

- The values of $\pi_i, a_{i,j}, b_i(o)$ that maximize $L(\theta)$ are:

$$\begin{aligned}
\pi_i &= \frac{\sum_\ell f(i, x_\ell, y_\ell)}{\sum_\ell \sum_k f(k, x_\ell, y_\ell)} \\
a_{i,j} &= \frac{\sum_\ell f(i, j, x_\ell, y_\ell)}{\sum_\ell \sum_k f(i, k, x_\ell, y_\ell)} \\
b_i(o) &= \frac{\sum_\ell f(i, o, x_\ell, y_\ell)}{\sum_\ell \sum_{o' \in V} f(i, o', x_\ell, y_\ell)}
\end{aligned}$$

Learning from Fully Observed Data

Labeled Data:

```
x1,y1: killer/N clown/N
x2,y2: killer/N problem/N
x3,y3: crazy/A problem/N
x4,y4: crazy/A clown/N
x5,y5: problem/N crazy/A clown/N
x6,y6: clown/N crazy/A killer/N
```

Learning from Fully Observed Data

- The values of π_i that maximize $L(\theta)$ are:

$$\pi_i = \frac{\sum_\ell f(i, x_\ell, y_\ell)}{\sum_\ell \sum_k f(k, x_\ell, y_\ell)}$$

- $\pi_N = \frac{2}{3}$ and $\pi_A = \frac{1}{3}$ because:

$$\sum_\ell f(N, x_\ell, y_\ell) = 4$$
$$\sum_\ell f(A, x_\ell, y_\ell) = 2$$

Learning from Fully Observed Data

- The values of $a_{i,j}$ that maximize $L(\theta)$ are:

$$a_{i,j} \quad = \quad \frac{\sum_{\ell} f(i,j,x_\ell,y_\ell)}{\sum_{\ell}\sum_{k} f(i,k,x_\ell,y_\ell)}$$

- $a_{N,N} = \frac{1}{2}$; $a_{N,A} = \frac{1}{2}$; $a_{A,N} = 1$ and $a_{A,A} = 0$ because:

$$\sum_{\ell} f(N,N,x_\ell,y_\ell) \quad = \quad 2 \qquad\qquad \sum_{\ell} f(A,N,x_\ell,y_\ell) \quad = \quad 4$$

$$\sum_{\ell} f(N,A,x_\ell,y_\ell) \quad = \quad 2 \qquad\qquad \sum_{\ell} f(A,A,x_\ell,y_\ell) \quad = \quad 0$$

Learning from Fully Observed Data

- The values of $b_i(o)$ that maximize $L(\theta)$ are:

$$b_i(o) = \frac{\sum_\ell f(i, o, x_\ell, y_\ell)}{\sum_\ell \sum_{o' \in V} f(i, o', x_\ell, y_\ell)}$$

- $b_N(killer) = \frac{3}{10}$; $b_N(clown) = \frac{4}{10}$; $b_N(problem) = \frac{3}{10}$ and $b_A(crazy) = 1$ because:

$$\begin{aligned} \sum_\ell f(N, killer, x_\ell, y_\ell) &= 3 \\ \sum_\ell f(N, clown, x_\ell, y_\ell) &= 4 \\ \sum_\ell f(N, crazy, x_\ell, y_\ell) &= 0 \\ \sum_\ell f(N, problem, x_\ell, y_\ell) &= 3 \end{aligned}$$

$$\begin{aligned} \sum_\ell f(A, killer, x_\ell, y_\ell) &= 0 \\ \sum_\ell f(A, clown, x_\ell, y_\ell) &= 0 \\ \sum_\ell f(A, crazy, x_\ell, y_\ell) &= 4 \\ \sum_\ell f(A, problem, x_\ell, y_\ell) &= 0 \end{aligned}$$

Learning from Fully Observed Data

```
x1,y1: killer/N clown/N
x2,y2: killer/N problem/N
x3,y3: crazy/A problem/N
x4,y4: crazy/A clown/N
x5,y5: problem/N crazy/A clown/N
x6,y6: clown/N crazy/A killer/N
```

$\pi =$

A	0.25
N	0.75

$a =$

$a_{i,j}$	A	N
A	0.0	1.0
N	0.5	0.5

$b =$

$b_i(o)$	*clown*	*killer*	*problem*	*crazy*
A	0	0	0	1
N	0.4	0.3	0.3	0

Natural Language Processing

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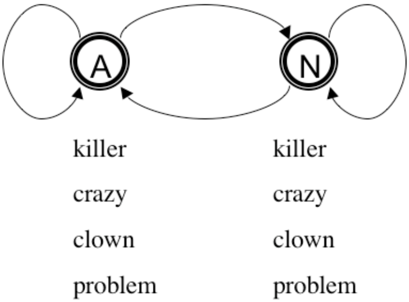
Part 6: Lagrange Multipliers

Hidden Markov Model

$$\text{Model } \theta = \begin{cases} \pi_i & \text{probability of starting at state } i \\ a_{i,j} & \text{probability of transition from state } i \text{ to state } j \\ b_i(o) & \text{probability of output } o \text{ at state } i \end{cases}$$

$$\text{Constraints}: \sum_i \pi_i = 1, \sum_j a_{i,j} = 1, \sum_o b_i(o) = 1$$



Learning from Fully Observed Data

$$L(\theta) = \sum_{\ell=1}^{m} \sum_{i} f(i, x_\ell, y_\ell) \log \pi_i + \sum_{i,j} f(i, j, x_\ell, y_\ell) \log a_{i,j} + \sum_{i,o} f(i, o, x_\ell, y_\ell) \log b_i(o)$$

- $\theta = (\pi, a, b)$
- $L(\theta)$ is the log probability of the labeled data $(x_1, y_1), \ldots, (x_m, y_m)$
- We want to find a θ that will give us the maximum value of $L(\theta)$
- Find the θ such that $\frac{dL(\theta)}{d\theta} = 0$

Learning from Fully Observed Data

$$L(\theta) = \sum_{\ell=1}^{m} \sum_{i} f(i, x_\ell, y_\ell) \log \pi_i + \sum_{i,j} f(i, j, x_\ell, y_\ell) \log a_{i,j} + \sum_{i,o} f(i, o, x_\ell, y_\ell) \log b_i(o)$$

- Find the θ such that $\frac{dL(\theta)}{d\theta} = 0$ and $\theta = (\pi, a, b)$
- Split up $L(\theta)$ into $L(\pi), L(a), L(b)$
- Let $\nabla L = \forall i, j, o : \frac{\partial L(\pi)}{\partial \pi_i}, \frac{\partial L(a)}{\partial a_{i,j}}, \frac{\partial L(b)}{\partial b_i(o)}$
- We must also obey constraints:
  $\sum_k \pi_k = 1, \sum_k a_{i,k} = 1, \sum_o b_i(o) = 1$

Learning from Fully Observed Data

$$L(\pi) = \sum_{\ell=1}^{m} \sum_{i} f(i, x_\ell, y_\ell) \log \pi_i$$

- Let us focus on $\nabla L(\pi)$ (the other two: a and b are similar)
- For the constraint $\sum_k \pi_k = 1$ we introduce a new variable into our search for a maximum:

$$L(\pi, \lambda) = L(\pi) + \lambda(1 - \sum_k \pi_k)$$

- λ is called the Lagrange multiplier
- λ penalizes any solution that does not obey the constraint
- The constraint ensures that π is a probability distribution

Learning from Fully Observed Data

$$\frac{\partial L(\pi)}{\partial \pi_i} = \frac{\partial}{\partial \pi_i} \underbrace{\sum_{\ell=1}^{m} f(i, x_\ell, y_\ell) \log \pi_i}_{\text{the only part with variable } \pi_i} + \underbrace{\sum_{\ell=1}^{m} \sum_{j:j \neq i} f(j, x_\ell, y_\ell) \log \pi_j}_{\text{no } \pi_i \text{ so derivative is } 0}$$

- We want a value of π_i such that $\frac{\partial L(\pi,\lambda)}{\partial \pi_i} = 0$

$$\frac{\partial}{\partial \pi_i} \sum_{\ell=1}^{m} \left(f(i, x_\ell, y_\ell) \log \pi_i + \lambda(1 - \sum_k \pi_k) \right) = 0$$

$$\frac{\partial}{\partial \pi_i} \sum_{\ell=1}^{m} \left(\underbrace{f(i, x_\ell, y_\ell) \log \pi_i}_{\frac{\partial}{\partial \pi_i} = \frac{f(i, x_\ell, y_\ell)}{\pi_i}} + \lambda - \underbrace{\lambda \pi_i}_{\frac{\partial}{\partial \pi_i} = \lambda} - \lambda \sum_{j:j \neq i} \pi_j) \right) = 0$$

Learning from Fully Observed Data

$$\frac{\partial L(\pi)}{\partial \pi_i} = \frac{\partial}{\partial \pi_i} \underbrace{\sum_{\ell=1}^{m} f(i, x_\ell, y_\ell) \log \pi_i}_{\text{the only part with variable } \pi_i} + \underbrace{\sum_{\ell=1}^{m} \sum_{j:j \neq i} f(j, x_\ell, y_\ell) \log \pi_j}_{\text{no } \pi_i \text{ so derivative is } 0}$$

- We can obtain a value of π_i wrt λ:

$$\frac{\partial L(\pi, \lambda)}{\partial \pi_i} = \underbrace{\sum_{\ell=1}^{m} \frac{f(i, x_\ell, y_\ell)}{\pi_i} - \lambda}_{\text{see previous slide}} = 0$$

$$\pi_i = \frac{\sum_{\ell=1}^{m} f(i, x_\ell, y_\ell)}{\lambda} \tag{1}$$

- Combine π_is from Eqn (1) with constraint $\sum_k \pi_k = 1$

$$\lambda = \sum_k \sum_{\ell=1}^{m} f(k, x_\ell, y_\ell)$$

Learning from Fully Observed Data

$$\frac{\partial L(\pi)}{\partial \pi_i} = \frac{\partial}{\partial \pi_i} \quad \underbrace{\sum_{\ell=1}^{m} f(i, x_\ell, y_\ell) \log \pi_i}_{\text{the only part with variable } \pi_i} \quad + \underbrace{\sum_{\ell=1}^{m} \sum_{j:j \neq i} f(j, x_\ell, y_\ell) \log \pi_j}_{\text{no } \pi_i \text{ so derivative is } 0}$$

- The value of π_i for which $\frac{\partial L(\pi,\lambda)}{\partial \pi_i} = 0$ is Eqn (2) which can be combined with the value of λ from Eqn (3).

$$\pi_i = \frac{\sum_{\ell=1}^{m} f(i, x_\ell, y_\ell)}{\lambda} \tag{2}$$

$$\lambda = \sum_{k} \sum_{\ell=1}^{m} f(k, x_\ell, y_\ell) \tag{3}$$

$$\pi_i = \frac{\sum_{\ell=1}^{m} f(i, x_\ell, y_\ell)}{\sum_k \sum_{\ell=1}^{m} f(k, x_\ell, y_\ell)}$$

Learning from Fully Observed Data

$$L(\theta) = \sum_{\ell=1}^{m}$$
$$\sum_i f(i, x_\ell, y_\ell) \log \pi_i + \sum_{i,j} f(i, j, x_\ell, y_\ell) \log a_{i,j} + \sum_{i,o} f(i, o, x_\ell, y_\ell) \log b_i(o)$$

- The values of $\pi_i, a_{i,j}, b_i(o)$ that maximize $L(\theta)$ are:

$$\begin{aligned}
\pi_i &= \frac{\sum_\ell f(i, x_\ell, y_\ell)}{\sum_\ell \sum_k f(k, x_\ell, y_\ell)} \\
a_{i,j} &= \frac{\sum_\ell f(i, j, x_\ell, y_\ell)}{\sum_\ell \sum_k f(i, k, x_\ell, y_\ell)} \\
b_i(o) &= \frac{\sum_\ell f(i, o, x_\ell, y_\ell)}{\sum_\ell \sum_{o' \in V} f(i, o', x_\ell, y_\ell)}
\end{aligned}$$

Natural Language Processing

Anoop Sarkar
anoopsarkar.github.io/nlp-class

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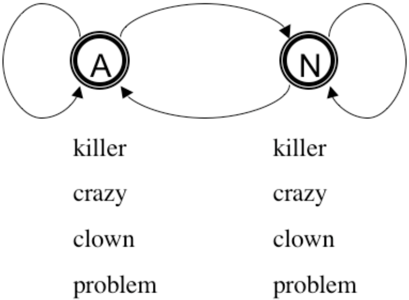
Part 7: Unsupervised Learning for HMMs

Hidden Markov Model

$$\text{Model } \theta = \begin{cases} \pi_i & \text{probability of starting at state } i \\ a_{i,j} & \text{probability of transition from state } i \text{ to state } j \\ b_i(o) & \text{probability of output } o \text{ at state } i \end{cases}$$



Hidden Markov Model Algorithms

- HMM as parser: compute the best sequence of states for a given observation sequence.
- HMM as language model: compute probability of given observation sequence.
- HMM as learner: given a corpus of observation sequences, learn its distribution, i.e. learn the parameters of the HMM from the corpus.
  - Learning from a set of observations with the sequence of states provided (states are not hidden) [Supervised Learning]
  - Learning from a set of observations without any state information. [Unsupervised Learning]

Learning from Unlabeled Data

Unlabeled Data $U = x_1, \ldots, x_m$:

```
x1: killer clown
x2: killer problem
x3: crazy problem
x4: crazy clown
```

- `y1, y2, y3, y4` are unknown.
- But we can enumerate all possible values for `y1, y2, y3, y4`
- For example, for `x1: killer clown`

`x1,y1,1:`	`killer/A clown/A`	$p_1 = \pi_A \cdot b_A(killer) \cdot a_{A,A} \cdot b_A(clown)$
`x1,y1,2:`	`killer/A clown/N`	$p_2 = \pi_A \cdot b_A(killer) \cdot a_{A,N} \cdot b_N(clown)$
`x1,y1,3:`	`killer/N clown/N`	$p_3 = \pi_N \cdot b_N(killer) \cdot a_{N,N} \cdot b_N(clown)$
`x1,y1,4:`	`killer/N clown/A`	$p_4 = \pi_N \cdot b_N(killer) \cdot a_{N,A} \cdot b_A(clown)$

Learning from Unlabeled Data

- Assume some values for $\theta = \pi, a, b$
- We can compute $P(y \mid x_\ell, \theta)$ for any y for a given x_ℓ

$$P(y \mid x_\ell, \theta) = \frac{P(x, y \mid \theta)}{\sum_{y'} P(x, y' \mid \theta)}$$

- For example, we can compute $P(\texttt{NN} \mid \texttt{killer clown}, \theta)$ as follows:

$$\frac{\pi_N \cdot b_N(killer) \cdot a_{N,N} \cdot b_N(clown)}{\sum_{i,j} \pi_i \cdot b_i(killer) \cdot a_{i,j} \cdot b_j(clown)}$$

- $P(y \mid x_\ell, \theta)$ is called the *posterior probability*

Learning from Unlabeled Data

- Compute the posterior for all possible outputs for each example in training:
- For `x1: killer clown`
  - `x1,y1,1: killer/A clown/A` $P(\texttt{AA} \mid \texttt{killer clown}, \theta)$
  - `x1,y1,2: killer/A clown/N` $P(\texttt{AN} \mid \texttt{killer clown}, \theta)$
  - `x1,y1,3: killer/N clown/N` $P(\texttt{NN} \mid \texttt{killer clown}, \theta)$
  - `x1,y1,4: killer/N clown/A` $P(\texttt{NA} \mid \texttt{killer clown}, \theta)$
- For `x2: killer problem`
  - `x2,y2,1: killer/A problem/A` $P(\texttt{AA} \mid \texttt{killer problem}, \theta)$
  - `x2,y2,2: killer/A problem/N` $P(\texttt{AN} \mid \texttt{killer problem}, \theta)$
  - `x2,y2,3: killer/N problem/N` $P(\texttt{NN} \mid \texttt{killer problem}, \theta)$
  - `x2,y2,4: killer/N problem/A` $P(\texttt{NA} \mid \texttt{killer problem}, \theta)$
- Similarly for `x3: crazy problem`
- And `x4: crazy clown`

Learning from Unlabeled Data

- For unlabeled data, the log probability of the data given θ is:

$$
\begin{aligned}
L(\theta) &= \sum_{\ell=1}^{m} \log \sum_{y} P(x_\ell, y \mid \theta) \\
&= \sum_{\ell=1}^{m} \log \sum_{y} P(y \mid x_\ell, \theta) \cdot P(x_\ell \mid \theta)
\end{aligned}
$$

- Unlike the fully observed case there is no simple solution to finding θ to maximize $L(\theta)$
- We instead initialize θ to some values, and then iteratively find better values of θ: $\theta^0, \theta^1, \ldots$ using the following formula:

$$
\begin{aligned}
\theta^t &= \arg\max_{\theta} Q(\theta, \theta^{t-1}) \\
&= \sum_{\ell=1}^{m} \sum_{y} P(y \mid x_\ell, \theta^{t-1}) \cdot \log P(x_\ell, y \mid \theta)
\end{aligned}
$$

Learning from Unlabeled Data

$$\begin{aligned}
\theta^t &= \arg\max_{\theta} Q(\theta, \theta^{t-1}) \\
Q(\theta, \theta^{t-1}) &= \sum_{\ell=1}^{m} \sum_{y} P(y \mid x_\ell, \theta^{t-1}) \cdot \log P(x_\ell, y \mid \theta) \\
&= \sum_{\ell=1}^{m} \sum_{y} P(y \mid x_\ell, \theta^{t-1}) \cdot \\
&\quad \left(\sum_{i} f(i, x_\ell, y) \cdot \log \pi_i \right. \\
&\quad + \sum_{i,j} f(i, j, x_\ell, y) \cdot \log a_{i,j} \\
&\quad \left. + \sum_{i,o} f(i, o, x_\ell, y) \cdot \log b_i(o) \right)
\end{aligned}$$

Learning from Unlabeled Data

$$
\begin{aligned}
g(i, x_\ell) &= \sum_y P(y \mid x_\ell, \theta^{t-1}) \cdot f(i, x_\ell, y) \\
g(i, j, x_\ell) &= \sum_y P(y \mid x_\ell, \theta^{t-1}) \cdot f(i, j, x_\ell, y) \\
g(i, o, x_\ell) &= \sum_y P(y \mid x_\ell, \theta^{t-1}) \cdot f(i, o, x_\ell, y)
\end{aligned}
$$

$$
\begin{aligned}
\theta^t &= \arg\max_{\pi, a, b} \sum_{\ell=1}^{m} \sum_i g(i, x_\ell) \cdot \log \pi_i \\
&\quad + \sum_{i,j} g(i, j, x_\ell) \cdot \log a_{i,j} \\
&\quad + \sum_{i,o} g(i, o, x_\ell) \cdot \log b_j(o)
\end{aligned}
$$

Learning from Unlabeled Data

$$Q(\theta, \theta^{t-1}) = \sum_{\ell=1}^{m}$$
$$\sum_i g(i, x_\ell) \log \pi_i + \sum_{i,j} g(i, j, x_\ell) \log a_{i,j} + \sum_{i,o} g(i, o, x_\ell) \log b_i(o)$$

- The values of $\pi_i, a_{i,j}, b_i(o)$ that maximize $L(\theta)$ are:

$$\begin{aligned}
\pi_i &= \frac{\sum_\ell g(i, x_\ell)}{\sum_\ell \sum_k g(k, x_\ell)} \\
a_{i,j} &= \frac{\sum_\ell g(i, j, x_\ell)}{\sum_\ell \sum_k g(i, k, x_\ell)} \\
b_i(o) &= \frac{\sum_\ell g(i, o, x_\ell)}{\sum_\ell \sum_{o' \in V} g(i, o', x_\ell)}
\end{aligned}$$

EM Algorithm for Learning HMMs

- Initialize θ^0 at random. Let $t = 0$.
- The EM Algorithm:
  - E-step: compute expected values of y, $P(y \mid x, \theta)$ and calculate $g(i, x), g(i, j, x), g(i, o, x)$
  - M-step: compute $\theta^t = \arg\max_\theta Q(\theta, \theta^{t-1})$
  - Stop if $L(\theta^t)$ did not change much since last iteration. Else continue.
- The above algorithm is guaranteed to improve likelihood of the unlabeled data.
- In other words, $L(\theta^t) \geq L(\theta^{t-1})$
- *But* it all depends on θ^0!

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All mistakes are my own.