Equidistribution Course

Scribe: James Leng, Jiazhen Tan, and Daniel Shapiro

Ross 2020

Contents

Lecture #1

1.1 Uniform Distribution of Sequences

Recall that a subset A of $\mathbb{R}$ is *dense* if every $x \in \mathbb{R}$ is arbitrarily close to A. That is, for every $\epsilon > 0$, that x is ϵ-close to some element of A. More precisely:

$$\forall x \in \mathbb{R} \text{ and } \forall \epsilon > 0, \text{ there exists } a \in A \text{ such that } |x - a| < \epsilon.$$

Exercise 1. *Examples of dense subsets:*

1. *$\mathbb{Q}$ is dense in $\mathbb{R}$.*
2. *$\mathbb{Q}^2 = \mathbb{Q} \times \mathbb{Q}$ is dense in $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$.*

   That is: $\forall \epsilon > 0,\ \forall (x, y) \in \mathbb{R}^2,\ \exists (r, s) \in \mathbb{Q}^2$ such that $\ distance\big((x, y), (r, s)\big) < \epsilon$.

   Here we use the Euclidean distance.
3. *(optional) Let C be the classical middle-thirds Cantor set. Verify that the set of endpoints of all the removed intervals is a dense subset of C.*

If $x \in \mathbb{R}$ define its *fractional part* to be $\{x\} = x - \lfloor x \rfloor$, where $\lfloor \cdot \rfloor$ is the floor function. Then $\{x\} \in [0, 1)$. We sometimes write (x mod 1) in place of $\{x\}$.

MOTIVATING EXAMPLE.

Proposition 1.1. $\boxed{\textit{If } \alpha \in \mathbb{R} \textit{ is irrational, then the sequence } \{n\alpha\}_{n\in\mathbb{N}} \textit{ is dense in } [0, 1].}$

How do we define a dense "sequence"? We defined denseness above only for subsets.

Corollary 1.1.1. *(Easy exercise.) There are uncountably many dense sequences.*

We will prove Proposition 1.1 in several steps. Before starting the proof, we recall some terminology. Given $\epsilon > 0$, a sequence (a_n) in $[0, 1]$ is called ϵ-dense if:

$$\forall x \in [0, 1],\ \exists n \in \mathbb{N} \text{ such that } |a_n - x| < \epsilon.$$

This says: For any subinterval $J \subseteq [0, 1]$ of length 2ϵ, some terms of the sequence (a_n) are in J.

Check that a sequence (a_n) is dense in $[0, 1]$ if it is $\frac{1}{k}$-dense for every $k \in \mathbb{N}$.

To prove the Proposition, we need to show: $\forall k \in \mathbb{N}$ the sequence $\{na\}_{n\in\mathbb{N}}$ is $\frac{1}{k}$-dense.

Let's consider the case $k = 10$.

Claim 1.2. *If there exists m such that* $\{m\alpha\} < \frac{1}{10}$, *then the sequence* $\{n\alpha\}_{n\in\mathbb{N}}$ *is* $\frac{1}{10}$*-dense.*

Proof. Given such m, the sequence $\{m\alpha\}, \{2m\alpha\}, \{3m\alpha\}, \ldots$ must enter every subinterval of length $\frac{1}{10}$. □

Claim 1.3. *Among* $\{\alpha\}, \{2\alpha\}, \ldots, \{11\alpha\}$ *in* $[0, 1]$, *there are two terms with distance* $< 1/10$.

Proof. Pigeonhole with the subintervals $\left[0, \frac{1}{10}\right], \left[\frac{1}{10}, \frac{2}{10}\right], \cdots, \left[\frac{9}{10}, 1\right]$. □

Proof of Proposition 1.1. By Claim 1.3, there exist $i > j$ such that $\left|\{i\alpha\} - \{j\alpha\}\right| < \frac{1}{10}$. If $\{i\alpha\} > \{j\alpha\}$ in $[0, 1]$, then $m = i - j$ satisfies $\{m\alpha\} = \{i\alpha - j\alpha\} = \{i\alpha\} - \{j\alpha\} < \frac{1}{10}$, and Claim 1.2 applies.

What if $\{i\alpha\} < \{j\alpha\}$ in $[0, 1]$? Show that $\frac{9}{10} < \{m\alpha\} < 1$. Can we conclude that the sequence is $\frac{1}{10}$-dense i this case? *[Yes. Successive multiples of* $\{m\alpha\}$ *differ by less than* $\frac{1}{10}$, *so they also enter every length* $\frac{1}{10}$ *subintervals.]*

Finish the proof by replacing 10 with arbitrarily large integer k. □

Question 2. *Can you generalize the above for* $[0, \alpha)$ *where* $\alpha \neq 1$?

Question 3. *When is the sequence* $(\{n\alpha\}, \{n\beta\})_{n\in\mathbb{N}}$ *dense in* $[0, 1] \times [0, 1]$?

(Much easier variant) When is the sequence $\left(\{m\alpha\}, \{n\beta\}\right)_{m,n\in\mathbb{N}}$ *dense in* $[0, 1] \times [0, 1]$?

Question 4 (Daniel)**.** *If* $B \subset A \subset S$ *and A is dense in S, B is dense in A, is B dense in S?*

(Yes, by the $\epsilon/2$ *trick. For any x in S and any* $\epsilon > 0$, *approximate x in A with error at most* $\epsilon/2$. *Approximate this approximation again in B with error at most* $\epsilon/2$.*)*

Question 5. *Suppose* α *is irrational and* $A \subset \mathbb{N}$ *is infinite. Is* $\{n\alpha \mid n \in A\}$ *necessarily dense in* $[0, 1]$?

(No. See an example of A in Exercise 6.)

Exercise 6. *For* $0 < x_1 < x_2 < 1$, *the set* $A = \left\{n : \{n\alpha\} \in [x_1, x_2]\right\}$ *is infinite.*

[Deduce this as a corollary of the denseness of $\{A\alpha\}$*in* $[x_1, x_2]$.*]*

Clearly $\{n\alpha \mid n \in A\}$ is not dense for the A in the exercise above.

Question 7. *Is* $\{n\alpha \mid n \equiv 0 \bmod 17\}$ *dense in* $[0, 1]$?

(Write $n = 17k, k \in \mathbb{N}$. *Since* 17α *is irrational, apply Proposition 1.1 to* $S = \{k \cdot (17\alpha) \mid k \in \mathbb{N}\}$ *to show S is dense.)*

It is natural to ask:

Which sequences (a_n) satisfy: For every irrational α, the sequence $(\{a_n\alpha\})_{n\in\mathbb{N}}$ is dense in $[0, 1]$?

Question 8 (Pico)**.** *Is the set* $\left\{n^2\alpha\right\}_{n\in\mathbb{N}}$ *dense in* $[0, 1]$?

Theorem 1.4 (Misha and Aditya)**.** *If* α *is irrational, the sequence* $\left\{p_n\alpha\right\}_{n\in\mathbb{N}}$ *is dense in* $[0, 1]$. *Here* p_n *is the* n^{th} *prime.*

Claim 1.5. *The set* $\left\{n^2\alpha\right\}_{n\in\mathbb{N}}$ *is dense. For polynomial* $f(x) \in \mathbf{Z}[x]$ *and irrational* α, *the sequence* $\left\{f(n)\alpha\right\}_{n\in\mathbb{N}}$ *is dense.*

The following result is even stronger.

Theorem 1.6 (Barz and Weyl)**.** *If $f(t)$ is a real polynomial with at least one coefficient irrational, other than the constant term, then the sequence $\{f(n)\}_{n\in\mathbb{N}}$ is dense in $[0,1]$.*

(Misha Donchenko) In fact: $\{f(p_n)\}$ is also dense!

1.2 Next time

Please review Riemann integration of functions $f : [0,1] \to \mathbb{R}$.

Theorem 1.7 (Weierstrass Approximation Theorem)**.** *Any continuous function on a closed interval $[a,b]$ can be uniformly approximated by polynomials.*

Theorem 1.8 (Furstenberg-Sàrközy Theorem)**.** *if $S \subset \mathbb{N}$ satisfy that no two numbers in S differ by a square number, then the asymptotic density of S is zero.*

1.3 Solutions to Exercises

Exercise 1.

1. For every $r \in \mathbb{R}$, truncations of its decimal expression, or its continued fraction approximation, are rational numbers that approximate x to arbitrary precision.

2. This problem only asks about a finite product. In fact, this works for arbitrary products.

   Claim 1.9 (More generally)**.** *For arbitrary product spaces $\prod_{i\in I} A_i$ such that each A_i is dense in B_i, we have*

   $$\prod_{i\in I} A_i \text{ dense in } \prod_{i\in I} B_i$$

   when we take the product topology for $\prod_{i\in I} B_i$.

   Proof. Take an arbitrary $x \in \prod\limits_{i\in I} B_i$, and an arbitrary open neighborhood O of x. By the definition of a topology generated by a basis, there exists some basis element $\prod_{i\in I} U_i$ such that

   $$x \in \prod_{i\in I} U_i \subset O$$

   where each U_i is open in B_i. Take $p_i \in U_i \cap A_i$ for all i; the point with i^{th} coordinate p_i $\forall i$ is in O.

   Note that I don't need to split into cases for the finitely many $U_i \neq B_i$ and the remaining U_i that equals to the whole space; it's an open set in B_i either case and $U_i \cap A_i$ is always nonempty by denseness. □

3.
   - Using the base-3 decimal definition, successively longer truncations of decimals who only has digits 0, 2 still gives a decimal with only digits 0, 2, but is an endpoint.
   - Using the intersection of nested set definition, for $x \in C$ and $\epsilon > 0$, take $n \in \mathbb{N}$ large enough so that $\frac{1}{3^n} < \epsilon$. Consider the segment containing x in the n^{th} iteration. It has length less than ϵ, and both endpoints are boundary points of some removed open intervals. So x is within ϵ distance from endpoints of removed intervals for arbitrary positive ϵ.

Question 2.

Answer. There is a bijection betwen dense sequences $\{n\gamma \bmod 1\}$ and $\{n\lambda \bmod \alpha\}$ via the map $f : \gamma \mapsto \gamma\alpha$. Scale all intervals by a factor of α or $\frac{1}{\alpha}$ to show having an element in all intervals of one implies that for the other.

Question 3.

1. When $\alpha, \beta, \frac{\beta}{\alpha} \in \mathbb{R} \setminus \mathbb{Q}$.

   Proof. See Sophie's later proof of Theorem 3.7, the uniform distribution of $\{n\alpha\}$.

   Similarly to prove the U.D. of $\{n\alpha, n\beta\}$ we consider continuous $f : \mathbb{T}^2 \to \mathbb{R}$ with fourier series

   $$f = \sum_{j,k \in \mathbf{Z}} c_{j,k} e^{2\pi i(jx+ky)}.$$

   Since f can be uniformly approximated by certain truncation of its series (e.g. by Fejér kernel, the average of the first n truncations) We can change the order of summation for the corresponding finite sums and only consider $e^{2\pi i(jx+ky)}$ for arbitrary j, k when considering definition 2.4:

   $$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \sum_{j,k<M} c_{j,k} e^{2\pi i(jn\alpha+kn\beta)} \quad ? \quad \int_{\mathbb{T}^2} f(n\alpha, n\beta) d\mu$$

   □

   Consider when $n(j\alpha + k\beta) \neq 0$,

   $$\frac{1}{N}\sum_{n=1}^{N} e^{2\pi i(jn\alpha+kn\beta)} = \frac{1}{N}\sum_{n=1}^{N} e^{2\pi i n(j\alpha+k\beta)} = \frac{1}{N}\frac{e^{2\pi i(N+1)(j\alpha+k\beta)} - 1}{e^{2\pi i(j\alpha+k\beta)} - 1}.$$

   Consider the modulus of the right side; it goes to 0 as N gets large because the modulus of $e^{i\theta}$ is bounded by 1. When $j = k = 0$, this limit is

   $$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} 1 = 1.$$

   When $n(j\alpha + k\beta) \neq 0$ for all pairs $j, k \in \mathbf{Z}$ s.t. j, k aren't both zero, which is when $\alpha, \beta, \frac{\beta}{\alpha} \in \mathbb{R} \setminus \mathbb{Q}$, the limit of this sum $\frac{1}{N}\sum_{n=1}^{N} e^{2\pi i(jn\alpha+kn\beta)}$ equals to the integral

   $$\int e^{2\pi i(jx+ky)} d\mu \forall j, k \in \mathbf{Z}.$$

2. When $\alpha, \beta, \in \mathbb{R} \setminus \mathbb{Q}$.

   The proof is entirely analogous to the one dimensional case. In fact, this set is $\{n\alpha\}_{n\in\mathbb{N}} \times \{m\beta\}_{m\in\mathbb{N}}$, and both components are dense by Proposition 1.1. In Claim 1.9, I have shown this gives a dense set.

Exercise 6.

1. *By contradiction.* Suppose the set is finite. All finite subsets of a linearly ordered set, $[0, 1]$ in our case, has a least element, so it makes sense to talk about the smallest and the second smallest element of $A\alpha := \{n\alpha \mid n \in A\}$.

   The interval between the smallest $\{n\alpha\}$ in $[x_1, x_2]$ and the second smallest contains at least one $\{m\alpha\}$ for some $m \in \mathbb{N}$ due to the denseness of $\{n\alpha | n \in \mathbb{N}\}$ in $[0, 1]$. Contradiction. $\square$

2. *Direct Proof.* By denseness, there is at least one $\{m_1\alpha\} \in [x_1, x_2] := I_1$. Since $x_1 \neq x_2$, $\{m_1\alpha\}$ can't coincide with both endpoints of the interval, and WLOG assume $m_1 \neq x_2$. Take $I_2 = [m_1, x_2]$. Let S be the set of n where this process stops working. WOP on it to show $S = \emptyset$ and you now constructed an infinite sequence $\{m_i\alpha\}_{i=1}^{\infty}$. $\square$

Lecture #2

2.1 Questions from Last Time

Question 9 (Lev)**.** *Let $A \in M_{n\times n}(\mathbb{Z})$ integer-valued matrices. Take $A\vec{v} \pmod 1$. Are orbits dense?*

Example 10. *Take*

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$$

the "Arnold Cat map." We will get to this later.

Question 11. *What if the matrix A in the above question has the property that $A^n = \mathrm{Id}$? (All orbits end up being periodic).*

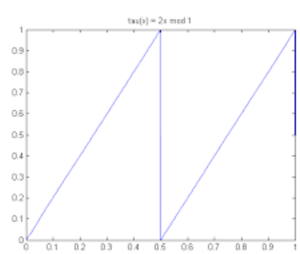
Question 12. *Let $A \in M_{n\times n}(\mathbb{Z})$ integer-valued matrices. Take $A\vec{v} \pmod 1$. Which points are periodic under the orbit? That is, for which $\vec{v}$ such that there exists n with $A^n\vec{v} = \vec{v}$?*

Let's understand the one dimensional case first.

Example 13. *What about the case if we let $n = 1$ in the above example and $A = (2)$? In other words, for which $x \in [0, 1)$ is $2^n x \equiv x \pmod 1$?*

Observe that (2) is not invertible as an 1×1 integer valued matrix.

We can see exponentiation as iterated application of the function $f(x) = 2x \bmod 1$ on $[0, 1]$.



Question 14. *What happens if our matrix A is invertible?*

Question 15. *For which x with $2^n x \pmod 1$ dense in $[0, 1]$?*

LEV'S QUOTE: This should occur if x is normal [in base 2].

Conjecture 16 (Pratyush)**.** This should occur if x is irrational.

This turns out to not be true.

Question 17. *Is it true that* $2^n x \pmod 1$*,* $k \in \mathbb{N}$*, is dense for uncountably many* x*?*

It turns out this is true. For $x \in [0, 1]$, write x in base two:

$$x = \sum_{i=1}^{\infty} \frac{\beta_i}{2^i} = 0.\beta_1\beta_2\beta_3\ldots_2 : \beta_i \in \{0, 1\}.$$

Then

$$2x = \sum_{i=1}^{\infty} \frac{\beta_i}{2^{i-1}} = \beta_1 + \sum_{i=1}^{\infty} \frac{\beta_{i+1}}{2^i} = \beta_1.\beta_2\beta_3\beta_4...$$

Taking mod 1, it's equivalent to $0.\beta_2\beta_3\beta_4....$

If we represent real numbers by the sequence of its binary digits then

$$\begin{aligned} x &\sim (\beta_1, \beta_2, \beta_3, \beta_4, \ldots) \\ 2x &\sim (\beta_2, \beta_3, \beta_4, \beta_5, \ldots) \end{aligned}$$

So $f(x) = 2x$ is the shift-one-place-left operation.

Question 18 (James)**.** *Some numbers don't have a unique decimal expansion. What do we do for these numbers? Is it a countable set?*

Answer. If a number has two expansions: one infinite and one finite, then we always pick the finite representation.

In fact, if we have two binary expansion of the same number

$$\sum_{i=1}^{\infty} \frac{\beta_i}{2^i} = \sum_{i=1}^{\infty} \frac{\gamma_i}{2^i},$$

and they first differ at the k^{th} digit, then WLOG let $b_k = 1, \gamma_k = 0$, and we see

$$\sum_{i=1}^{\infty} \frac{\beta_i}{2^i} \geq \left(\sum_{i=1}^{k-1} \frac{\beta_i}{2^i} \right) + \frac{1}{2^k} \geq \sum_{i=1}^{k-1} \frac{\gamma_i}{2^i} + \sum_{i=k+1}^{\infty} \frac{\gamma_i}{2^i}.$$

Equality is only obtained when $\beta_m = 0$ and $\gamma_m = 1$ for all $m > k$. Also, for a finite expansion we can always rewrite the last nonzero digit $\ldots 1$ as $\ldots 0\dot{1}$. So numbers with two expansions are exactly those with a finite expansion. So yes for countable and the above criterion suffices.

Theorem 2.1 (Srinath Mahankali)**.** $2^n x \pmod 1$ *is dense if and only if every finite length string of* 0*'s and* 1*'s appears in the binary expansion of* x*.*

2.2 Uniform Distribution

Definition 2.2. A sequence $(x_n) \in [0, 1]$ is *uniformly distributed* (U.D.) if for any subinterval $[a, b]$ one has

$$\lim_{n\to\infty} \frac{|\{x_n : 1 \leq n \leq N\} \cap [a, b]|}{n} \to b - a.$$

Here's another description:

Definition 2.3. (x_n) is U.D. if the probability of hitting $[a, b]$ is $b - a$.

It's not obvious that such a sequence exists. There are uncountably many intervals $[a, b]$ of a fixed length r and the sequence has to spend r of its lifetime in such an interval.

Example 19. *Take the sequence*

$$0, 1, 0, \frac{1}{2}, 1, 0, \frac{1}{4}, \frac{2}{4}, \frac{3}{4}, 1, \ldots .0, \frac{1}{2^k}, \frac{2}{2^k} \ldots .$$

Our intuition suggests that this is uniformly distributed.

Exercise 20 (Mandatory (Pico's) exercise)**.** *Prove that in the definition of U.D., it suffices to prove this for intervals* $[a, b]$ *with* $a, b \in \left\{ \frac{n}{2^d} \mid n, d \in \mathbb{N} \right\}$.

Here's an equivalent definition of uniform distribution:

Definition 2.4. A sequence $(x_n) \in [0, 1]$ is uniformly distributed if for all $f \in C^0([0, 1])$, i.e. continuous functions $f : [0, 1] \longrightarrow \mathbb{R}$,

$$\lim_{N\to\infty} \frac{1}{N} \sum_{i=1}^{N} f(x_i) = \int_0^1 f(x)dx. \tag{2.1}$$

If instead of f, we take $1_{[a,b]}$, then we end up with the other definition of uniform distribution. Here's our next equivalent form:

Definition 2.5. A sequence $(x_n) \in [0, 1]$ is uniformly distributed if for all Riemann integrable f, (2.1) holds:

$$\lim_{N\to\infty} \frac{1}{N} \sum_{i=1}^{N} f(x_i) = \int_0^1 f(x)dx.$$

Exercise 21. *All definitions are equivalent.*

Question 22 (Jacob)**.** *Would replacing any* f *in* 2.1 *with any Lebesgue measurable [integrable] functions work?*

2.3 Weierstrass Approximation Theorem

Theorem 2.6 (polynomial version)**.** *For any* $f \in C([0, 1])$ *and* $\epsilon > 0$ *there exists a polynomial* p *with*

$$|f(x) - p(x)| < \epsilon$$

for all x.

Theorem 2.7 (trignometric version)**.** *For any* $f \in C([0, 1])$ *and* $\epsilon > 0$ *there exists a trigonometric polynomial* τ *with*

$$|f(x) - \tau(x)| < \epsilon.$$

where a trigonometric polynomial is a finite sum

$$\sum_{i=1}^{n} a_i \sin(nx) + b_i \cos(nx).$$

2.4 Additional Exercises

Exercise 23. *Assume that for all $\epsilon > 0$ there exists n such that $n^2\alpha \pmod 1 < \epsilon$. Prove that $n^2\alpha$ is dense everywhere.*

2.5 Next time

Metric Spaces. Examples include $\mathbb{R}^n$ with the metric

$$\delta_p(x,y) = \sqrt[p]{\sum_{i=1}^{n} |x_i - y_i|^p} \quad \text{for } p \geq 1$$

In particular,

- $p = \infty$ corresponds to the max norm: $\delta(x,y) = \max\{|x_i - y_i|\}$.
- $p = 2$ corresponds to the Euclidean distance.

Cantor Sets. Construct them and show that they are homeomorphic to the space $\{0,1\}^{\mathbb{N}}$ under the metric

$$d(\vec{x},\vec{y}) = \sum_{i\in\mathbb{N}} \frac{|x_i - y_i|}{2^i}, \quad \vec{x},\vec{y} \in \{0,1\}^{\mathbb{N}}$$

2.6 Solution to Exercises

Pico's Exercise. Take arbitrary $[a,b] \in \mathbb{R}$. Take small $\epsilon > 0$ s.t. $\epsilon/4 < b - a$ (for simplicity). Take dyadic rationals m, k s.t. $m \in [a, a + \epsilon/4]$ and $k \in [b - \epsilon/4, b]$. Then

$$\lim_{N\to\infty} \frac{|\{x_n : 1 \leq n \leq N\} \cap [m,k]|}{N} \to k - m.$$

Now take large N so that the fraction inside the limit is within $\epsilon/2$ from $k - m$ for all $N' \geq N$. Note that

$$\frac{|\{x_n : 1 \leq n \leq N\} \cap [m,k]|}{n} \leq \frac{|\{x_n : 1 \leq n \leq N\} \cap [a,b]|}{n}$$

Similarly take $a - \frac{\epsilon}{4} < m' < a < b < k' < b + \frac{\epsilon}{4}$ to squeeze the RHS above between $b - a - \epsilon$ and $b - a + \epsilon$ for sufficient large N.

Exercise 21.

Lecture #3

3.1 Exploration: U.D. and Denseness

Question 24 (Michael Barz and Aditya Jambhale)**.** *Can we classify functions $f : [0,1] \to [0,1]$ such that if $\{a_n\}_{n\in\mathbb{N}}$ is U.D. then $\{f(a_n)\}_{n\in\mathbb{N}}$ is also U.D.?*

KEVIN DU: Must we have $f' = \pm 1$?

MICHAEL BARZ: If you assume f is nice, then $|f'| = 1$ can be shown, so yes.

But what if f is not differentiable?

PICO: How about decimal part of $2x$, why wouldn't that work?

Or $2x$ on $[0, \frac{1}{2}]$ and $2 - 2x$ on $[\frac{1}{2}, 1]$.

PICO: The requirement that you need isnt very strong, and for Riemann integrable functions all you need is the image of (a, b) to have "measure" $b - a$. Being Riemann integrable is much stronger than we need, but simplifies a lot of the ugly stuff.

Proposed Answer. Let $f : [0,1] \to [0,1]$ be a surjective continuous function then it preserves denseness. □

Question 25. *For which sequences $(n_k)_{k\in\mathbb{N}}$ satisfy: if (x_n) is uniformly distributed, then (x_{n_k}) is uniformly distributed?*

Definition 3.1. We call a sequence $(n_k)_{k\in\mathbb{N}}$ in Question 25 a universal sequence.

Question 26. *For which sequences n_k satisfy: if (x_n) is dense, then (x_{n_k}) is dense?*

Proposed Answer. Sequences $(n_k)_{k\in\mathbb{N}}$ that misses at most finitely many elements of $\mathbb{N}$.

Proof. NECESSITY. For any $(n_k)_{k\in\mathbb{N}}$ whose complement in $\mathbb{N}$ is also an infinite sequence $(m_k)_{k\in\mathbb{N}}$, take the sequence $\{x_i\}_{i\in\mathbb{N}}$ that such that

$$x_{n_k} = 1, x_{m_k} = r_k, \quad \forall k \in \mathbb{N}$$

where r_k is the k^{th} rational number under some enumeration.

SUFFICIENCY. For an interval $[a, b]$, a dense sequence must have infinitely many elements in it as you can divide it into N subintervals for any $N \in \mathbb{N}$. Thus, throwing finitely many terms out of a dense sequence does not affect its denseness because there are more terms between any two $[a, b]$ than what you throw away. □

Example 27. *$n\alpha$ for $\alpha \notin \mathbb{Q}$ is dense mod 1, and $n^2\alpha$ is dense mod 1.*

Question 28. *Which function preserves denseness?*

Here's a fact:

Theorem 3.2. *If $\alpha \notin \mathbb{Q}$, then $n^2\alpha$ is uniformly distributed.*

Question 29. *For which α are $2^n\alpha$ uniformly distributed?*

Review of the fact:

Theorem 3.3. *If in its expansion α contains all 0-1 strings, then $2^n\alpha$ is dense.*

We can generalize this:

Theorem 3.4. *Let k be a positive integer. Then $k^n\alpha$ is dense mod 1 if the base k expansion of α contains all words in $\{0, 1, 2, \ldots, k-1\}$.*

But the above only covers integers only. We can ask a question about more general rational or real numbers:

Question 30. *For what x is $\pi^n x$ dense mod 1? What about $\frac{3}{2}^n x$? (Base π-expansion? Base β expansion?)*

Theorem 3.5 (Open Problem)**.** *Is $(3/2)^n$ dense mod 1?*

Exercise 31. *Give an example of x such that x^n is dense mod 1.*

Theorem 3.6 (Furstenberg)**.** *Let α be irrational. Then the set*

$$\{2^n 3^m \alpha \pmod 1 : n, m \in \mathbb{N}\}$$

is dense in $(0, 1)$.

Remark. When we deal with denseness, the order of the sequence is immaterial. This is not true if we are interested in U.D. phenomenon!

For example, $\mathbb{Q}$ is dense, but it takes work to find a ordering that makes it U.D. On the other hand, $(n\alpha)_{n\in\mathbb{N}}$ has a natural ordering.

We can order the set

$$\{2^n 3^m \mid m, n \in \mathbb{N}\}$$

by size. Define the sequence

$$(a_n)_{n\in\mathbb{N}} := \{1, 2, 3, 4, 6, 8, 9, 12, 16, 18, \cdots\}.$$

Question 32 (**B**ig **B**onus **P**roblem.)**.** *Find α, such that $a_n\alpha$ is not U.D., a_n defined as above.*

Exercise 33. *There are uncountably many irrational α for which $2^n\alpha$ is not dense mod 1.*

3.2 Criterion of U.D.

Question 34 (Big Problem)**.** *Which sequences are uniformly distributed and why?*

Theorem 3.7. *$(n\alpha)$ is uniformly distributed mod 1.*

Sophie's solution (a sketch). Refer to definition 2.4. We show that for all $f \in C([0,1])$,

$$\frac{1}{n}\sum_{j=0}^{n-1} f(j\alpha) \to \int_0^1 f(x)dx.$$

Since f may be approximated by sums of trig polynomials $e^{2\pi ikx}$, it suffices to show this for $f(x) = e^{2\pi ikx}$. We have

$$\frac{1}{n}\sum_{j=0}^{n-1} e^{2\pi ijk\alpha} = \frac{1}{n}\frac{e^{2\pi ikn\alpha} - 1}{e^{2\pi ik\alpha} - 1} \to \delta_k$$

if $k \neq 0$ and

$$\int_0^1 e^{2\pi ikx} dx = \delta_k = \begin{cases} 0 & k \neq 0 \\ 1 & k = 0 \end{cases}.$$

$\square$

There is a subtle point: only f that satisfy $f(0) = f(1)$ can be approximated by trigonometric polynomials.

Theorem 3.8 (Weyl's Criterion)**.** *A sequence $(x_n) \subset [0,1]$ is uniformly distributed modulo 1 if and only if for any $h \in \mathbb{Z} \setminus \{0\}$ one has*

$$\frac{1}{N}\sum_{n=0}^{N-1} e^{2\pi ihx_n} \to 0.$$

Example 35. *More examples of U.D.*

- $\sqrt{n}$ mod 1.
- $\log^2 n$ mod 1.
- $n^2\alpha + \log^2 n$ mod 1.

Conjecture 36 (Kevin Du)**.** If $\lim_{n\to\infty} a_{n+1} - a_n \to 0$ and $\lim_{n\to\infty} a_n \to \infty$, is it true that $(a_n)_{n\in\mathbb{N}}$ is U.D.?

Fact 3.9. *The sequence* $\log n$ mod 1 *is not U.D., but* $\log^{1+\epsilon} n$ mod 1 *for any $\epsilon > 0$ is U.D.*

Exercise 37. *Kevin's criterion is enough for denseness!*

Exercise 38 (Pico)**.** *Does the harmonic series not being U.D. follow from* $\log(n)$ *not being U.D. ?*

Lecture #4

Recall conjecture last time:

Conjecture 39 (Kevin's Conjecture)**.** Assume $(a_n) \subset \mathbb{R}$ such that $a_n \to \infty$ monotonically, $a_{n+1} - a_n \to 0$, and $n(a_{n+1} - a_n) \to \infty$. Then $a_n \pmod 1$ is uniformly distributed.

Compare with the classical Fejer's theorem:

Theorem 4.1 (Fejer)**.** *Let $f : [0, \infty) \to [0, \infty)$ be a differentiable function such that $f(x) \to \infty$ as $x \to \infty$, $f'(x) \to 0$, and $xf'(x) \to \infty$. Then $f(n) \pmod 1$ is uniformly distributed.*

Question 40 (Aditya)**.** *Are there interesting functions $f : [0, 1] \to [0, 1]$ with $f^n(x_0)$ is uniformly distributed for some x_0?*

What about f continuous?

Example 41. $f(x) = 2x \pmod 1$. *Not strictly continuous, but can make a "tent function." In addition, can take S^1 instead of $[0, 1]$. Then the discontinuity at $\frac{1}{2}$ disappears.*

Example 42. n^c *for* $0 < c < 1$.

Example 43. $\log(n)^c$ *for* $c > 1$.

Exercise 44. $\sin(n)$ *is dense mod* 1.

Example 45. $\log(n) \log\log(n)$

Exercise 46. *How about n^c, $c > 0$, $c \in \mathbb{N}$? How about $n^c \log^b(n)$? For which parameters b, c uniformly distributed? Dense?*

Theorem 4.2. *Let $x \in [0, 1]$ be a base 2 normal number. Then $2^n x \pmod 1$ is uniformly distributed.*

Definition 4.3. A number $x \in [0, 1]$ is base 2 normal if any finite w $0-1$ word appears in the binary expansion of x with probability

$$\frac{1}{2^{|w|}}$$

where $|w|$ is the length of w.

Theorem 4.4. *Base 2 normal numbers in $0 - 1$ have full measure. In other words, the complement of normal numbers has measure 0.*

Corollary 4.4.1. *Almost all numbers in $[0, 1]$ are normal in every base.*

Example 47. *Champernone's constant:* $0.12345678910111213\ldots$ *is normal in base* 10*. Square concatenation is also normal:* $0.1491625\ldots$*. So is prime concatenation:* $0.2357111317\ldots$*.*

There are rather general theorems of this time: if $f : (0, \infty)$ is "nice", then $0.f(1)f(2)f(3)\ldots$ is normal in base 10.

Theorem 4.5. *Let* $x \in [0, 1]$*. Then* $2^n x \pmod 1$ *is uniformly distributed if and only if* x *is a base* 2 *normal number.*

Question 48. *How can you define* $p - q$ *normality? Where* 0 *appears with probability* p *and* 1 *appears with probability* $q = 1 - p$*.*

Theorem 4.6. *There exists a* $1 - 1$ *correspondence between subsets of* $\mathbb{N}$ *and* $\{0, 1\}^{\mathbb{N}}$

Proof. 1 if the element is in the subset and 0 otherwise. $\square$

Let us call a subset S of $\mathbb{N}$ normal if 1_S is a normal binary sequence.

Definition 4.7. Let $S \subset \mathbb{N}$. The density of $S \subset \mathbb{N}$ is defined by

$$d(S) = \lim_{N\to\infty} \frac{|S \cap \{1, 2, \ldots, N\}|}{N} = \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} 1_S(x).$$

Exercise 49. *If S is normal, then* $d(S) = \frac{1}{2}$*. Also,* $d(\mathbb{N} \setminus S) = \frac{1}{2}$*.*

Exercise 50. $d(S \cap S - n) = \frac{1}{4}$ *for all integers n.*

Exercise 51. *For any* $n_1, n_2, \ldots, n_k$ *not equal,* $d\big(S \cap (S - n_1) \cap (S - n_2) \cap \cdots \cap (S - n_k)\big) = \frac{1}{2^{k+1}}$*.*

Exercise 52. *If we replace the sets in the above equation with their complements, then the formula is still valid.*

Example 53. *Example of U.D. sequences based on Fejèr's theorem.*

1. $\sqrt{n}$*, or more generally,* n^c *for* $0 < c < 1$*.*
2. $\log^c n$*, for* $c > 1$*.*
3. $\sin n$*.*
4. *Candidates:*
   - $\log n \cdot \log\log n$
   - $\log n \cdot \log\log n \log\log\log n$
   - $\log n \cdot \log^c \log n$
   - $\log n \cdot \log\log^c \log n$
   - *How about* $n^c \bmod 1$*,* $c \notin \mathbb{N}$*,* $c > 0$*?*
   - *How about* $n^c \log^b n$ *for parameter* $c \notin \mathbb{N}, b > 1$ *- are they U.D.? Dense?*

Exercise 54. $\sin n \bmod 1$ *is dense.*

Lecture #5

Fact 5.1 (In response to Pico)**.** *After appropriately defining U.D.* mod 2 (*your exercise), you can show that* $\sqrt{n}$ *U.D.* mod2*.*

Exercise 55 (Additional)**.** *How about U.D. mod* $\sqrt{2}$

Is $n\sqrt{3}$ *or* $\frac{n}{\sqrt{3}}$ *U.D.* mod$\sqrt{2}$*?* mod2*?* mod3*?*

Conjecture 56 (Michael Barz and Aditya Jambhale)**.** Every dense sequence has a U.D. subsequence.

Theorem 5.2. *Any dense sequence has a U.D. rearrangement.*

Bergelson: This is proved by a familiar technique, by a famous guy, youngish at the time, in 20th century, who is already dead. Ex: guess who is it.

Definition 5.3 (Terminology)**.** We say a sequence $(x_n)_{n\in\mathbb{N}} \subset \mathbb{R}$ is U.D. if $(x_n \mod 1)$ is U.D.

When convevient, we'll identify it with the 1 dimensional torus $\mathbb{T} = [0, 1)$.

More generally, consider $\mathbb{T}^n = \mathbb{R}^n/\mathbf{Z}^n$.

Recall our equivalent definitions of U.D.:

1. The frequency that a sequence hit $[a, b]$ is proportional to the length of $[a, b]$, as $n \longrightarrow \infty$.
2. The average value of any continuous $f \in C[0, 1]$ on its first n terms, as we take $n \longrightarrow \infty$, converges to its integral on $[0, 1]$.
3. The average value of Riemann integrable functions on the first $n \longrightarrow \infty$ terms converges to its intergral.

We now add:

4. For any nonzero integer h,
$$\frac{1}{N}\sum_{n=1}^{N} e^{2\pi i h x_n} \longrightarrow 0.$$
5. The criterion using continuous functions can be restricted to the subset of $C[0, 1]$ s.t. $f(0) = f(1)$.

Theorem 5.4 (Weyl)**.** *For any sequence* (n_k) *that goes to infinity, and* $n_k \in \mathbb{N}$*, then then set* $x \in \mathbb{R}$ *for which* $n_k x$ *is uniformly distributed has full measure.*

Theorem 5.5 (Borel)**.** *A sequence* $2^n x \pmod 1$ *is uniformly distributed if and only if* x *is normal.*

These two theorems imply that normal numbers are of full measure. Both of these deal with measure 0 since the complement of something full measure is measure 0.

Definition 5.6. A sequence $f_n : [0,1] \to \mathbb{R}$ is almost everywhere convergent if

$$\lim_{n\to\infty} f_n(x)$$

exists for a set of x of full measure in $[0,1]$.

Definition 5.7. A sequence $f_n : [0,1] \to \mathbb{R}$ is uniformly convergent to f if for all $\epsilon > 0$, there exists N_0 such that for all $n \geq N_0$

$$\max_{x\in[0,1]} |f_n(x) - f(x)| < \epsilon$$

Definition 5.8. $f_n \to f$ if $\int_0^1 |f_n(x) - f(x)|dx = 0$ (L^1 convergence).

Exercise 57. *All these methods of convergence are different notions.*

Fact 5.9. *For any $\alpha \notin \mathbb{Q}$, $n^2\alpha$ is uniformly distributed mod 1. In particular, this sequence is dense mod 1.*

Proof. We use the Van Der Corput trick, which is as follows.

Lemma 5.10. *Let $(x_n) \subset \mathbb{R}$. Assume that for any $h \in \mathbb{N}$, the sequence $x_{n+h} - x_n \pmod 1$ for $n \in \mathbb{N}$ is uniformly distributed. Then $x_n \pmod 1$ is uniformly distributed.*

Proof. See `https://terrytao.wordpress.com/2008/06/14/the-van-der-corputs-trick-and-equidistribution-on-nilmanifolds/` lemma 1 and corollary 2. □

Let $x_n = n\alpha$. Then $x_{n+h} - x_n = h^2\alpha + 2nh\alpha$ which is a uniformly distributed sequence shifted by a constant amount. Hence, $n^2\alpha \pmod 1$ is dense. □

Definition 5.11. A set $E \subset \mathbb{N}$ is called a Van Der Corput set if if in order to apply the van der corput trick you only need to check for $h \in E$.

Exercise 58. *Show that E is not finite.*

Observation (Misha Donchenko): Any $k\mathbf{Z}$ is a VDC set because this means the subsequences sorted by remainder mod$k$ are each U.D. and merging U.D. sets gives you a U.D. set.

Example 59. *Here are examples of van der Corput sets:*
$\{n^2\}$, $\{17n\}$, $\cancel{n^2+1}$, n^2-1, $P-1$, $P+1$, $\cancel{P+17}$, $\cancel{P-17}$.
where P is the set of primes.
Primes don't work. The only prime shifts that work are $P-1$ and $P+1$.

Theorem 5.12. *For any unbounded sequence that goes to infinity, $(a_n) \subset \mathbb{N}$, $a_n - a_m$ is a van der Corput set.*

Question 60 (Jessie)**.** *Can we define a sense of U.D.* mod1 *for f in $C[0,1]$?*

PICO: We start by considering ~~all~~ rational coefficient polynomials on $[0,1]$. We don't want uncountably many things.

Definition 5.13. How would you define (and bring interesting examples) a notion of uniformly distribution for $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$?

Exercise 61. *Is there a version of van der Corput for denseness?*

Exercise 62. *Prove Weyl's theorem on U.D. of polynomials by VDC's trick.*