Quaternions

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A *rigid motion* of the plane $\mathbb{R}^2$ is a transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ that "preserves distances"

$$\text{for every } a, b \in \mathbb{R}^2 : \ |T(a) - T(b)| = |a - b| \,.$$

Examples:

Translate by vector c:	$Trans_c$
Rotate about point A through angle θ:	$Rot_{A,\theta}$
Reflect across line ℓ:	$Refl_\ell$

Orientation.

A line-reflection reverses orientation. (Test an *L*, or a spiral.)
Translations and rotations preserve orientation.

Theorem. Every plane rigid motion is a composition of those three types.

- Compose reflections across different lines:

  Get a rotation?

- Compose two rotations: Get another rotation?

  E.g. Rotate 30° about $(0,0)$, then 60° about $(1,0)$.

  Is the result a rotation? What is the center?

- Compose reflections across 3 different lines?

  Is the result a reflection across some 4^{th} line?

The field $\mathbb{C}$ of complex numbers.

A point in the plane is a pair $(a_1, a_2) \in \mathbb{R}^2$. It's a vector. Can add vectors, and multiply them by scalars.

Want a way to *multiply* pairs, a product that includes useful geometric information.

Point (a_1, a_2) interpreted as a *complex number* $\alpha = a_1 + a_2\mathbf{i}$.
Define multiplication by the rule: $\boxed{\mathbf{i}^2 = -1.}$

If $\alpha = a_1 + a_2\mathbf{i}$ and $\beta = b_1 + b_2\mathbf{i}$, then

$$\alpha\beta = (a_1 b_1 - a_2 b_2) + (a_1 b_2 + a_2 b_1)\mathbf{i}.$$

Complex *conjugate* or "bar":

$$\text{If } \alpha = a_1 + a_2\mathbf{i}, \quad \text{define} \quad \overline{\alpha} := a_1 - a_2\mathbf{i}$$

Bar = Reflect across $\mathbb{R}$. Then: $\alpha \in \mathbb{R} \iff \overline{\alpha} = \alpha$.

Also: $\alpha\overline{\alpha} = (a_1 + a_2\mathbf{i})(a_1 - a_2\mathbf{i}) = a_1^2 + a_2^2 = |\alpha|^2$.

"bar" is additive: $\overline{\alpha + \beta} = \overline{\alpha} + \overline{\beta}$,

and multiplicative: $\overline{\alpha \cdot \beta} = \overline{\alpha} \cdot \overline{\beta}$.

Corollary. $|\alpha\beta| = |\alpha| \cdot |\beta|$.

Proof. $|\alpha\beta|^2 = \alpha\beta \cdot \overline{\alpha\beta} = \alpha\beta\overline{\alpha}\overline{\beta} = \alpha\overline{\alpha} \cdot \beta\overline{\beta} = |\alpha|^2 \cdot |\beta|^2$. □

That **"Law of Moduli"** connects algebra to geometry.

Define map $m_\alpha : \mathbb{C} \to \mathbb{C}$ by: $m_\alpha(z) = \alpha z$.

If $|\alpha| = 1$ (i.e. α is on the unit circle), then:

$$|m_\alpha(z)| = |z| \qquad \text{for every } z \in \mathbb{C}.$$

Therefore m_α preserves lengths. It's a rigid motion.

$$m_\alpha = \text{rotation about 0 through angle } \theta.$$

How about rotation about point A? For $Rot_{A,\theta}$:
translate A to 0, rotate about 0, translate 0 back to A.

In symbols: $Rot_{A,\theta}(z) \;=\; e^{\mathbf{i}\theta}(z - A) + A \;=\; e^{\mathbf{i}\theta} z + B,$

where $B = (1 - e^{\mathbf{i}\theta})A$.

Theorem. Every rotation and translation of $\mathbb{R}^2$ can be expressed algebraically as:

$$f(z) = az + b \quad \text{where } a, b \in \mathbb{C} \text{ and } |a| = 1.$$

Then the composition of rotations $Rot_{A,\theta}$ and $Rot_{B,\varphi}$ can be calculated within $\mathbb{C}$, without needing pictures.

Exercise. Express reflections similarly.

SUMMARY:

The algebra of complex numbers $\mathbb{C}$ provides tools for understanding rigid motions of the plane $\mathbb{R}^2$.

These ideas were developed in the early 1800s.

History. When did people start using complex numbers?

The quadratic formula is ancient, but getting something like $\sqrt{-5}$ just indicated "no solution."

1540s: Cubic formula in Cardano's book, *Ars Magna*.
Even negative numbers were suspect: separate chapters for $x^3 = ax + b$, and $x^3 + ax = b$.
Square roots of negatives became more of a puzzle.

In the 1700s came Euler with complex power series and $e^{i\theta}$. The "Fundamental Theorem" became a big topic before 1800:
Every nonconstant polynomial has roots in $\mathbb{C}$.

Early 1800s: People considered the "complex plane."
$\mathbb{C}$ is the multiplication lying beneath plane geometry.

What a wonderful idea, connecting algebra and geometry !

1840s: **William Rowan Hamilton** (1805 - 1865) was searching for a similar algebra on 3-space. Is there is a multiplication on $\mathbb{R}^3$ that satisfies that Law of Moduli:

$$|zw| = |z| \cdot |w| \, ?$$

Restrict attention to motions fixing the origin 0 (no translations).
Let $O(3)$ be the group of all those rigid motions of 3-space.

Types include:

Rotate about a line through some angle θ.
Reflect across a plane.
Any others?

Before continuing with W.R. Hamilton's story, let's look more closely at the group $O(3)$.

Let $SO(3)$ be the subgroup of *rotations*. Those are the rigid motions of 3-space that fix the origin 0 and preserve orientation.

How big is $SO(3)$?

Note: A rotation f is NOT the physical motion of twisting through many positions. It is the comparison of start position to end position for each point.

$f \in SO(3)$ is determined by the position of 3 perpendicular unit vectors.

The motion of those 3 vectors rotating in space is a **path** inside $SO(3)$.

Proposition. Each $f \in SO(3)$ is a line-rotation:
Rotate around an axis-line ℓ (through 0) through angle θ.

For instance, rotate around the z-axis line by 20°.
But which direction do we twist?

Better description: Use a *ray* ℓ, a half-line from 0.
The *right-hand rule* determines twist-direction.

Encode both choices with a single *vector* v.
The direction of v provides the ray ℓ.
The length $|v|$ is the number of degrees to twist.

Therefore: Each rotation in $SO(3)$ is represented as a point in the solid ball of radius 360.

There is duplication: Opposite rays are on the same line.

300° about the positive z-axis is the SAME as 60° about the negative z-axis.

Let's restrict to angles between 0° and 180°.
Then there are NO REPETITIONS except at the ends.

(180° turn about positive z-axis equals 180° turn about negative z-axis.)

$$\boxed{\left(\text{Solid ball of radius 180 in } \mathbb{R}^3\right) \longrightarrow SO(3).}$$

Conclusion: $SO(3)$ is a 3-dimensional object.

How big is $SO(4)$? (Answer: $\dim SO(4) = 6.$)

The algebra of rotations is another challenge.
Rotate 3-space 20° about the positive z-axis and then 60° about the positive x-axis, the result should be another line-rotation.
Which one? How to calculate its vector (ray and angle)?

(Before machinery of vector spaces & matrices was discovered.)

Imitating complex numbers, he tried units **i** and **j** in 3-space, with $\mathbf{i}^2 = -1$ and $\mathbf{j}^2 = -1$.

Multiply two numbers of the form $\alpha = a + b\mathbf{i} + c\mathbf{j}$?

For complex numbers, the Law of Moduli was the connection with geometry.

The length $|\alpha|$ should be related to this new product using a "bar" operation. That is:

$$|\alpha|^2 = \alpha\overline{\alpha}.$$

Imitating the complex number process, we hope that:

$$a^2 + b^2 + c^2 \;=\; (a + b\mathbf{i} + c\mathbf{j}) \cdot (a - b\mathbf{i} - c\mathbf{j})$$

Distribute and look: Squared terms are OK, and the *ab* and *ac* terms cancel. But what about the terms $(b\mathbf{i})(-c\mathbf{j}) + (c\mathbf{j})(-b\mathbf{i})$?

$$\mathbf{ij} + \mathbf{ji} = 0. \qquad \hookleftarrow\text{(Weird}$$

Hamilton was the first to seriously consider non-commutative products.

But if $\mathbf{ji} = -\mathbf{ij}$, could that product be in our 3-space?
Is it possible to express

$$\mathbf{ij} = x + y\mathbf{i} + z\mathbf{j} \quad \text{for real numbers } x, y, z \text{ ?}$$

(**IMPOSSIBLE.** The proof is left for you.)

The number $\mathbf{k} = \mathbf{ij}$ cannot be in the original 3 dimensions **!**

Hamilton had that flash of insight in 1843 as he was walking along the Royal Canal:

Use 4-space rather than 3-space **!**

He was so happy and excited that he scratched his "quaternion" formulas onto the cement of a nearby bridge:

$$\boxed{\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1}$$

Those formulas are still there, on a bronze plaque on Broom Bridge in Dublin Ireland.

From Hamilton's formulas, deduce the following rules:

$$\mathbf{ij} = \mathbf{k} \quad \text{and} \quad \mathbf{ji} = -\mathbf{k}.$$
$$\mathbf{jk} = \mathbf{i} \quad \text{and} \quad \mathbf{kj} = -\mathbf{i}.$$
$$\mathbf{ki} = \mathbf{j} \quad \text{and} \quad \mathbf{ik} = -\mathbf{j}.$$

A **quaternion** is a combination of those four units:

$$\alpha = a_0 + a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k} \quad \text{where each } a_n \in \mathbb{R}.$$

Hamilton's units lead to a general formulas:

$$\begin{aligned}(a_0 + a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}) \cdot (b_0 + b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}) \ = \\ (a_0b_0 - a_1b_1 - a_2b_2 - a_3b_3) + \\ (a_0b_1 + a_1b_0 + a_2b_3 - a_3b_2)\mathbf{i} + \\ (a_0b_2 - a_1b_3 + a_2b_0 + a_3b_1)\mathbf{j} + \\ (a_0b_3 + a_1b_2 - a_2b_1 + a_3b_0)\mathbf{k}.\end{aligned}$$

Define "bar" by

$$\overline{a_0 + a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}} = a_0 - a_1\mathbf{i} - a_2\mathbf{j} - a_3\mathbf{k}$$

The anti-commuting rules then imply:

Lemma. $\alpha\overline{\alpha} = |\alpha|^2$

This "bar" is almost multiplicative. It needs a twist:

Lemma. $\overline{\alpha \cdot \beta} = \overline{\beta} \cdot \overline{\alpha}$.

Lemma implies $\boxed{|\alpha\beta| = |\alpha| \cdot |\beta|}$ The **Law of Moduli.**

Proof. $|\alpha\beta|^2 = (\alpha\beta)(\overline{\alpha\beta}) = \alpha\beta \cdot \overline{\beta}\overline{\alpha} = \alpha \cdot |\beta|^2 \cdot \overline{\alpha} =$
$= \alpha\overline{\alpha}|\beta|^2 = |\alpha|^2 \cdot |\beta|^2$. □

This product makes $\mathbb{R}^4$ into the system $\mathbb{H}$ of quaternions.
It fits together, with amazing symmetries among **i**, **j**, **k**.
No wonder Hamilton was excited about it.

If $\alpha \in \mathbb{H}$ define $\lambda_\alpha : \mathbb{H} \to \mathbb{H}$ by: $\lambda(z) = \alpha z$.
If $|\alpha| = 1$ then λ_α is a rotation, so $\lambda_\alpha \in SO(4)$.

Define $\rho_\alpha : \mathbb{H} \to \mathbb{H}$ by: $\rho_\alpha(z) = z\alpha$. If $|\alpha| = 1$ then also $\rho_\alpha \in SO(4)$.

Lemma. The λ_α and ρ_α generate all rotations in 4-space.

But Hamilton's goal was to analyze 3-dimensions.

Let $\mathbb{H}_0 = \text{span}(\mathbf{i},\ \mathbf{j},\ \mathbf{k})$ be the space of *pure quaternions*.
Then $\mathbb{H}_0$ is 3-dimensional.

Suppose $\alpha \in \mathbb{H}$ is on the unit sphere: $|\alpha| = 1$.
Define $f_\alpha(z) = \alpha z \overline{\alpha}$. Then $|f_\alpha(z)| = |z|$ and f_α is a rotation.

Moreover, f_α preserves "purity":
If $z \in \mathbb{H}_0$, then $f_\alpha(z) \in \mathbb{H}_0$. (*Proof.* If $w = f_\alpha(z)$, show: $\overline{w} = -w$.)

Lemma. Every rotation on $\mathbb{H}_0$ equals f_α for some α.

This correspondence $\alpha \mapsto f_\alpha$ provides:
(Unit sphere in $\mathbb{H}$) $\longrightarrow$ $SO(3)$,

and the quaternion product on the left corresponds to composition of rotations on the right.

This is the algebraic tool Hamilton was looking for.

Composition of two rotations in $\mathbb{R}^3$.
Represent the rotations as unit quaternions, multiply those quaternions, express the answer as a rotation.

Hamilton had enthusiastic followers. They urged everyone to use quaternions for geometry and physics.
(Maxwell's first formulated his equations in quaternionic language !)

Hamilton wrote a book, and his followers (mostly in Great Britain) wrote more books. Between the 1860s and 1900, some enthusiasts became mystical, claiming quaternions to be the "science of pure time."

Rebellion against the tyranny of quaternions.
The quaternion product of two vectors in 3-space requires 4-dimensions. Theoretical physicists (led by Gibbs and Heaviside) wanted to keep things concrete, in $\mathbb{R}^3$.

Giiven two vectors (pure quaternions)

$$v = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k},$$
$$w = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k},$$

their quaternion product requires 4 components:

$$v \cdot w = -(a_1 b_1 + a_2 b_2 + a_3 b_3) + \\ + (a_2 b_3 - a_3 b_2)\mathbf{i} + (-a_1 b_3 + a_3 b_1)\mathbf{j} + (a_1 b_2 - a_2 b_1)\mathbf{k}.$$

Anti-quaternion reformers broke that product in two:
The "scalar product" $v \bullet w$ and the "vector product" $v \times w$.
Those are also called the dot and cross products.

People use them today without knowing that they arose from the 4-dimensional algebra of quaternions.

Quaternions were an amazing discovery and have an interesting history. I highly recommend Crowe's book

A History of Vector Analysis.

How have quaternions been generalized since the 1840s ?

Natural question: Which $\mathbb{R}^n$ admit a multiplication satisfying the Law of Moduli ?

In 1848, Graves and Cayley found such a multiplication on $\mathbb{R}^8$. That system of octonions is tricky since it is not associative!

Good products were known in $\mathbb{R}^n$ for dimensions $n = 1, 2, 4, 8$.

What about 16? Several people tried, but no one could find such a multiplication.

In 1898, Hurwitz proved his famous **"1, 2, 4, 8 Theorem."**

That's a topic for different lecture.

A lot of mathematical work was motivated by quaternions.
I will mention several topics and encourage you to read about them independently.

• Complex analysis is a beautiful subject: calculus is done over $\mathbb{C}$ rather than $\mathbb{R}$. Work on Quaternionic Analysis began with Hamilton's followers, but has not grown in importance since the late 1800s.

• A Clifford Algebra has generators $e_1, \ldots, e_n$ over $\mathbb{R}$ satisfying:

$$e_j^2 = -1 \quad \text{and} \quad e_i e_j = -e_j e_i \text{ whenever } i \neq j.$$

This is associative with dimension 2^n. (For $n = 1, 2$ get $\mathbb{C}$ and $\mathbb{H}$.) There is a growing theory of Clifford Analysis, where exterior products in integrals are replaced by Clifford algebra products.

• A multiplication on $\mathbb{R}^n$ is a division algebra if it is bilinear (i.e. distributive laws) and has the Zero-Product Property:

If $ab = 0$ then $a = 0$ or $b = 0$.

Exercise. Suppose a multiplication satisfies:

For every $a \neq 0$, there exists a' such that $aa' = a'a = 1$.

Must this be a division algebra?

(Answer: No. If $ab = 0$ we get $a'(ab) = 0$, but associativity is not assumed.)

A real algebra with the Law of Moduli is a division algebra, so we know examples when $n = 1, 2, 4$ and 8.

Are other dimensions possible for real division algebras?

Without assuming a rule like the Law of Moduli, this question is difficult. After many years of work, the "1, 2, 4, 8 Theorem" was finally proved in the 1950s, using sophisticated topological tools.

• For a field K and $a, b \in K$, build a generalized quaternion algebra with generators $\mathbf{i}$ and $\mathbf{j}$ satisfying:

$$\mathbf{i}^2 = a, \quad \mathbf{j}^2 = b, \quad \text{and} \quad \mathbf{ji} = -\mathbf{ij}.$$

This yields a 4-dimensional algebra $\mathbb{H}_K(a, b)$ over K.

When is it a division algebra?

These ideas led to interest in classifying division algebras over K, and to the study of central simple algebras and Brauer groups.

• Gaussian integers $\mathbb{Z}[\mathbf{i}]$ are a beautiful part of number theory, and were generalized to systems of integers in other number fields.

Integer quaternions were investigated by Hurwitz around 1919.

- What further generalizations of quaternions are still to come?

THANKS FOR LISTENING.

References.

Lecture notes on "Sums of Squares" posted at
https://u.osu.edu/shapiro.6/

Michael J. Crowe, *A History of Vector Analysis*,
University of Notre Dame Press, 1967. Dover paperback 1994.