Ross Program 2021 Application Problems

This document is part of the application to the *Ross Mathematics Program*, and will remain posted at https://rossprogram.org/students/to-apply from January through March.

The Admissions Committee will start reading applications in March 2021. The deadline for applications is April 1, but for adequate consideration of your application, it is best to **submit your solutions some weeks before the end of March.**

Work independently on the problems below. We are interested in seeing how you approach unfamiliar math problems, not whether you can find answers by searching through web sites or books, or by asking other people.

Submit your own work on these problems.

For each problem, explore the situation (with calculations, tables, pictures, etc.), observe patterns, make some guesses, test the truth of those guesses, and write logical proofs when possible. Where were you led by your experimenting?

Include your thoughts (but not your scratch-paper) even if you might not have found a complete solution. If you've seen one of the problems before (e.g. in a class or online), please include a reference along with your solution.

We are not looking for quick answers written in minimal space. Instead, we hope to see evidence of your explorations, conjectures, and proofs written in a readable format.

> The quality of mathematical exposition, as well as the correctness and completeness of your solutions, are factors in admission decisions.

PDF format required.

Convert your problem solutions into a PDF file. You may type the solutions using LaTeX or a word processor, and convert the output to PDF format.
Alternatively, you may scan your solutions from a handwritten paper copy, and convert the output to PDF. (Use dark pencil or pen and write on only one side of the paper.) Submitting photos of your work is not recommended since file sizes of photos are often too large. (The Ross system cannot accept files much larger than 5 megabytes.) Rather than photographs, you might use a "scan" feature on your camera.

Note: Unlike the problems here, each Ross Program course concentrates deeply on one subject. These problems are intended to assess your general mathematical background and interests.

Problem 1

Suppose $A = (a_n) = (a_1, a_2, a_3, \dots)$ is an increasing sequence of positive integers. A number c is called *A-expressible* if c is the alternating sum of a finite subsequence of A. To form such a sum, choose a finite subset of the sequence A, list those numbers in increasing order (no repetitions allowed), and combine them with alternating plus and minus signs. We allow the trivial case of one-element subsequences, so that each a_n is A-expressible.

Definition. Sequence $A = (a_n)$ is an "alt-basis" if every positive integer is uniquely A-expressible. That is, for every integer $m > 0$, there is exactly one way to express m as an alternating sum of a finite subsequence of A.

Examples. Sequence $B = (2^{n-1}) = (1,\ 2,\ 4,\ 8,\ 16, \dots)$ is not an alt-basis because some numbers are B-expressible in more than one way. For instance $3 = -1 + 4 = 1 - 2 + 4$.

Sequence $C = (3^{n-1}) = (1,\ 3,\ 9,\ 27,\ 81, \dots)$ is not an alt-basis because some numbers (like 4 and 5) are not C-expressible.

(a) Let $D = (2^n - 1) = (1,\ 3,\ 7,\ 15,\ 31, \dots)$. Note that:

$$\mathbf{1} = 1, \quad \mathbf{2} = -1 + 3, \quad \mathbf{3} = 3, \quad \mathbf{4} = -3 + 7, \quad \mathbf{5} = 1 - 3 + 7,$$
$$\mathbf{6} = -1 + 7, \quad \mathbf{7} = 7, \quad \mathbf{8} = -7 + 15, \quad \mathbf{9} = 1 - 7 + 15, \quad \dots$$

Prove that D is an alt-basis.

(b) Can some $E = (4, 5, 7, \dots)$ be an alt-basis? That is, does there exist an alt-basis $E = (e_n)$ with $e_1 = 4,\ e_2 = 5$, and $e_3 = 7$? Justify your answer.

The first few values seem to work: $\mathbf{1} = -4 + 5, \quad \mathbf{2} = -5 + 7, \quad \mathbf{3} = -4 + 7.$

(c) Can $F = (1, 4, \dots)$ be an alt-basis? That is, does there exist an alt-basis $F = (f_n)$ with $f_1 = 1$ and $f_2 = 4$?

(d) Investigate some other examples. Is there some fairly simple test to determine whether a given sequence $A = (a_n)$ is an alt-basis?

Problem 2

A polynomial $f(x)$ has the *factor-square property* (or FSP) if $f(x)$ is a factor of $f(x^2)$. For instance, $g(x) = x - 1$ and $h(x) = x$ have FSP, but $k(x) = x + 2$ does not.

Reason: $x - 1$ is a factor of $x^2 - 1$, and x is a factor of x^2, but $x + 2$ is not a factor of $x^2 + 2$.

Multiplying by a nonzero constant "preserves" FSP, so we restrict attention to polynomials that are *monic* (i.e., have 1 as highest-degree coefficient).

What patterns do monic FSP polynomials satisfy?

To make progress on this topic, investigate the following questions and justify your answers.

(a) Are x and $x - 1$ the only monic FSP polynomials of degree 1?

(b) List all the monic FSP polynomials of degree 2.
To start, note that x^2, $x^2 - 1$, $x^2 - x$, and $x^2 + x + 1$ are on that list.
Some of them are products of FSP polynomials of smaller degree. For instance, x^2 and $x^2 - x$ arise from degree 1 cases. However, $x^2 - 1$ and $x^2 + x + 1$ are new, not expressible as a product of two smaller FSP polynomials.
Which terms in your list of degree 2 examples are new?

(c) List all the new monic FSP polynomials of degree 3.

Note: Some monic FSP polynomials of degree 3 have complex coefficients that are not real.

Can you make a similar list in degree 4 ?

(d) Are there monic FSP polynomials (of some degree) that have real number coefficients, but some of those coefficients are not integers?
Explain your reasoning.

Problem 3

Rossie is a simple robot in the plane, with Start position at the origin O, facing the positive x-axis.
An angle θ is entered into Rossie's memory. He can take only two actions:

> S: Rossie steps one meter in the direction he is facing.
> R: Rossie stays in place and rotates counterclockwise through angle θ.

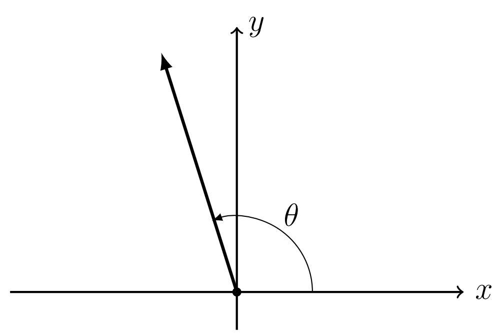
Notation: A string of symbols S and R (read from left to right) represents a sequence of Rossie's moves. For instance, $SRRSS$ indicates that Rossie steps one meter along the x-axis, rotates through angle 2θ, and then steps two meters in that new direction. In the questions below, we consider only those sequences of actions that include at least one S.

(a) For which θ can a sequence of actions result in Rossie's return to Start?
(Then by repeating that sequence of actions, Rossie will retrace the same path.)

For example, with $\theta = 2\pi/3 = 120°$, the actions $SRSRSR$ cause Rossie to trace an equilateral triangle and return to Start.

(b) Suppose θ is the angle pictured below, with $\cos(\theta) = -1/3$. Note that θ is approximately $109.47°$.

With this angle θ, explain why the actions
$SSSRSSRSSS$
cause Rossie to return to O.



Those moves return Rossie to O but he is not at Start:
He is *not* facing the positive x-axis.

With that θ, is there some sequence of actions that returns Rossie to Start? Justify your answer.

(c) Investigate the following question:
Which angles allow Rossie return to O? (Not necessarily facing the positive x-axis)

Provide more examples of such angles. Are there some angles θ that allow Rossie to return to O, but only after tracing some path more complicated than a triangle or a regular polygon?

(d) Are there some angles θ for which Rossie can never return to O?
Explain your reasoning.

Problem 4

Suppose $f = (f_1, f_2, f_3, \ldots)$ is a sequence of integers. For $0 \le k \le n$, define the "f-bin" numbers $\left[{n \atop k} \right]_f$ as follows: Define $\left[{n \atop 0} \right]_f = 1$, and for $k \ge 1$ let

$$\begin{bmatrix} n \\ k \end{bmatrix}_f = \frac{f_n f_{n-1} \cdots f_{n-k+1}}{f_k\, f_{k-1} \;\cdots\; f_1}.$$

If $I_n = n$, then $\left[{n \atop k} \right]_I = \binom{n}{k}$ is the usual binomial coefficient.*

Another example: Define $L = (1, 3, 4, 7, 11, 18, \ldots)$ by setting $L_1 = 1, L_2 = 3$ and
$L_n = L_{n-1} + L_{n-2}$ for $n > 2$.

Here are lists of some of the f-bin numbers $\left[{n \atop k} \right]_f$ for the sequences $f = I$ and $f = L$.

	0	1	2	3	4	5	6	
0	1							
1	1	1						
2	1	2	1					
3	1	3	3	1				
4	1	4	6	4	1			
5	1	5	10	10	5	1		
6	1	6	15	20	15	6	1	
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

I - bin numbers

	0	1	2	3	4	5	6	
0	1							
1	1	1						
2	1	3	1					
3	1	4	4	1				
4	1	7	$\frac{28}{3}$	7	1			
5	1	11	$\frac{77}{3}$	$\frac{77}{3}$	11	1		
6	1	18	66	$\frac{231}{2}$	66	18	1	
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

L - bin numbers

Definition. Sequence f is *binomid* if all the f-bin numbers $\left[{n \atop k} \right]_f$ are integers.

Equivalently: f is binomid when, for each $k \ge 1$:
Every product of k consecutive terms $f_n f_{n-1} \cdots f_{n-k+1}$ is an integer multiple of the product of the first k consecutive terms $f_k f_{k-1} \cdots f_1$.

Since every binomial coefficient $\binom{n}{k}$ is an integer, the sequence I is binomid. The table above shows that the sequence L is not binomid.

(a) Define sequences $P_n = 2^n = (2, 4, 8, \ldots)$, $Q_n = n^2 = (1, 4, 9, \ldots)$, and $D_n = 2n = (2, 4, 6, \ldots)$. In each case, find a simple formula for $\left[{n \atop k} \right]$, check that it is an integer, and conclude that P, Q and D are binomid.

(b) Is the sequence $M_n = 2^n - 1$ binomid? Justify your answer.

(c) Is the sequence $T_n = n(n+1)$ binomid?
As a first step, verify that $\left[{n \atop 2} \right]_T = \frac{T_n\, T_{n-1}}{T_2\, T_1} = \frac{n(n+1)}{6} \cdot \frac{(n-1)n}{2}$ is always an integer.

(d) Find some further examples of binomid sequences. Are there some interesting conditions on a sequence f that imply that f is binomid?

*Other notations for $\binom{n}{k}$ include ${}_nC_k$ and $C(n, k)$.