Numerical Solution of the 2D Poisson Equation

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Outline

Problem Statement

- Solve the 2D Poisson equation:

$$-(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}) = f(x, y), \quad \text{on } \Omega = [0, 1] \times [0, 1].$$

- Boundary conditions:

$$u(x, y) = 0, \quad \text{on } \partial\Omega.$$

- Exact solution for validation:

$$u(x, y) = \sin^2(\pi x) \sin^2(\pi y).$$

Finite Difference Method

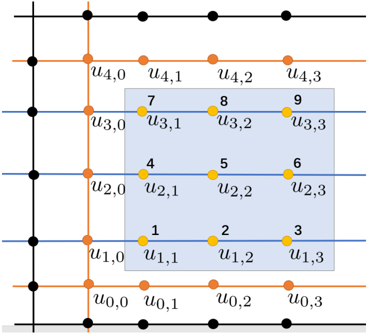
- Discretize the domain into a uniform grid with spacing $h = \frac{1}{N+1}$.
- Approximate derivatives using central differences:

$$\frac{\partial^2 u}{\partial x^2} \approx \frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h^2}, \quad \frac{\partial^2 u}{\partial y^2} \approx \frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{h^2}.$$

- Discretized Poisson equation:

$$-4u_{i,j} + u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1} = h^2 f_{i,j}.$$

- Diagram



Matrix Formulation

- Numbering the grid points in column-major order, we obtain a linear system,The equation is given as:

$$Du_{j-1} + Cu_j + Du_{j+1} = f_j, \quad 1 \leq j \leq n-1.$$

where

Matrix Formulation

$$C = \begin{pmatrix} 2\left(\frac{1}{h_1^2}+\frac{1}{h_2^2}\right) & -\frac{1}{h_2^2} & 0 & \cdots & 0 \\ -\frac{1}{h_2^2} & 2\left(\frac{1}{h_1^2}+\frac{1}{h_2^2}\right) & -\frac{1}{h_2^2} & \cdots & 0 \\ 0 & -\frac{1}{h_2^2} & 2\left(\frac{1}{h_1^2}+\frac{1}{h_2^2}\right) & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & -\frac{1}{h_2^2} \\ 0 & 0 & 0 & -\frac{1}{h_2^2} & 2\left(\frac{1}{h_1^2}+\frac{1}{h_2^2}\right) \end{pmatrix},$$

$$D = \begin{pmatrix} -\frac{1}{h_2^2} & 0 & 0 & \cdots & 0 \\ 0 & -\frac{1}{h_2^2} & 0 & \cdots & 0 \\ 0 & 0 & -\frac{1}{h_2^2} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ 0 & 0 & 0 & 0 & -\frac{1}{h_2^2} \end{pmatrix},$$

Matrix Formulation

$$f_j = f + b = \begin{pmatrix} f(x_1, y_j) + \frac{1}{h_1^2}\varphi(x_0, y_j) \\ f(x_2, y_j) \\ \vdots \\ f(x_{m-2}, y_j) \\ f(x_{m-1}, y_j) + \frac{1}{h_1^2}\varphi(x_m, y_j) \end{pmatrix}.$$

Equation can be further written as:

$$\begin{pmatrix} C & D & 0 & \cdots & 0 \\ D & C & D & \cdots & 0 \\ 0 & D & C & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & D \\ 0 & 0 & 0 & D & C \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_{n-2} \\ u_{n-1} \end{pmatrix} = \begin{pmatrix} f_1 - Du_0 \\ f_2 \\ \vdots \\ f_{n-2} \\ f_{n-1} - Du_n \end{pmatrix}.$$

Since $u(x,1) = u(x,0) = u(1,y) = u(0,y) = 0$, the vector b is a zero-vector the size of $(N_y - 1)^2$. For a procedure with a non-zero boundary, the vector resembles:

Matrix Formulation

$$A = \begin{pmatrix} C & D & 0 & \cdots & 0 \\ D & C & D & \cdots & 0 \\ 0 & D & C & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & D \\ 0 & 0 & 0 & D & C \end{pmatrix}$$

$$b + f = \begin{pmatrix} f_1 - Du_0 \\ f_2 \\ \vdots \\ f_{n-2} \\ f_{n-1} - Du_n \end{pmatrix}.$$

Validation of the Implementation

- The exact solution we are solving is given by:

$$u(x,y) = \sin^2(\pi x)\sin^2(\pi y).$$

- The corresponding Poisson equation:

$$-\Delta u(x,y) = f(x,y).$$

- Substituting the exact solution $u(x,y)$ into the Laplacian, the right-hand side $f(x,y)$ is derived as:

$$f(x,y) = 2\pi^2\left(\cos(2\pi x)\sin^2(\pi y) + \cos(2\pi y)\sin^2(\pi x)\right).$$

Validation of Boundary Conditions

- The exact solution is:

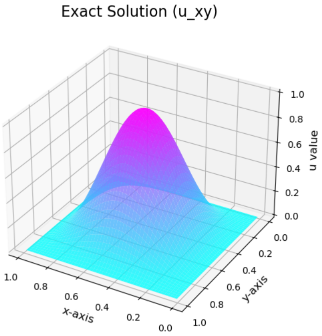
$$u_{ex}(x, y) = \sin^2(\pi x)\sin^2(\pi y).$$

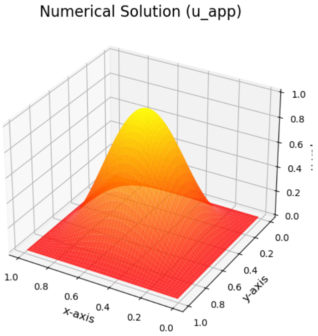
- Verify boundary conditions:
  1. At $x = 0$:
  $$u_{ex}(0, y) = \sin^2(\pi \cdot 0)\sin^2(\pi y) = 0.$$
  2. At $x = 1$:
  $$u_{ex}(1, y) = \sin^2(\pi \cdot 1)\sin^2(\pi y) = \sin^2(\pi)\sin^2(\pi y) = 0.$$
  3. At $y = 0$:
  $$u_{ex}(x, 0) = \sin^2(\pi x)\sin^2(\pi \cdot 0) = 0.$$
  4. At $y = 1$:
  $$u_{ex}(x, 1) = \sin^2(\pi x)\sin^2(\pi \cdot 1) = \sin^2(\pi x)\sin^2(\pi) = 0.$$

- Hence, $u_{ex}(x, y)$ satisfies:

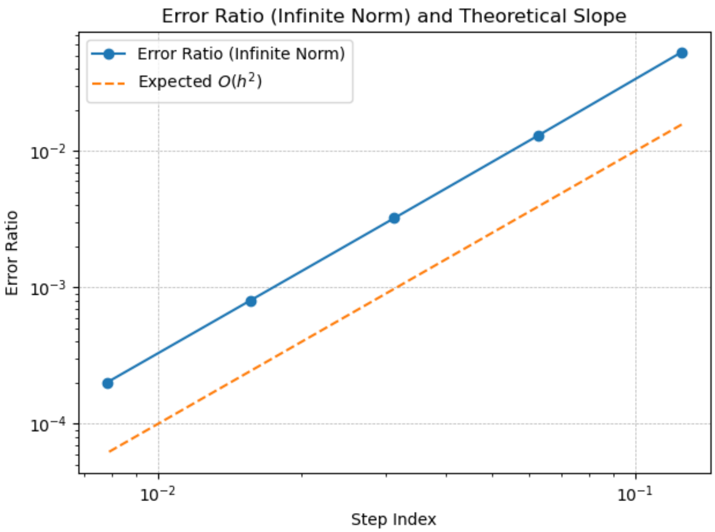
$$u(x, 0) = u(x, 1) = u(0, y) = u(1, y) = 0.$$

- result using numpy.linalg.solve:





Validation of the Implementation



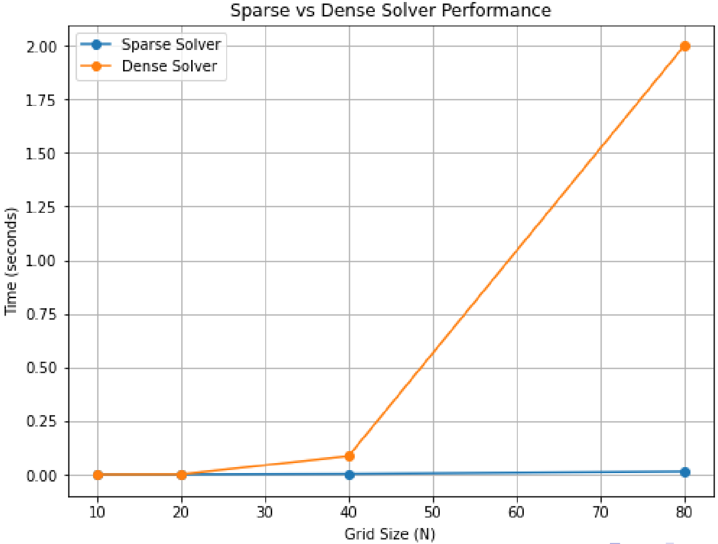
Sparse Matrix Solvers

- The coefficient matrix A has the form:

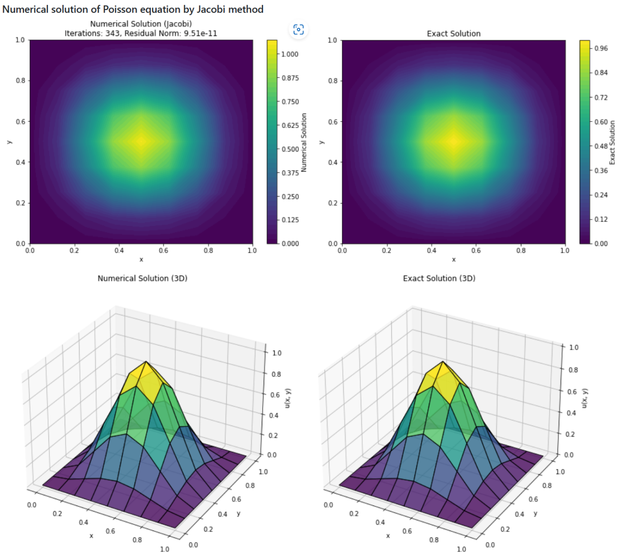
$$A = I_n \otimes T + T \otimes I_n,$$

where $\otimes$ is the Kronecker product. so we can use scipy.kron to create sparse matrix and use spsolve in scipy.sparse.linalg to solve the linear system.

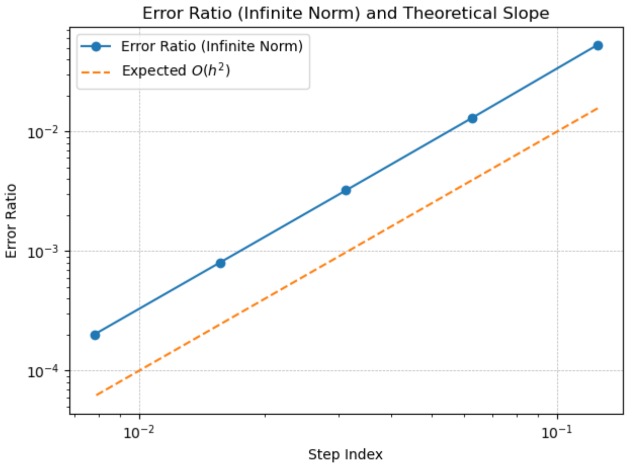
Computational Time Between The Dense and Sparse Matrix Solvers



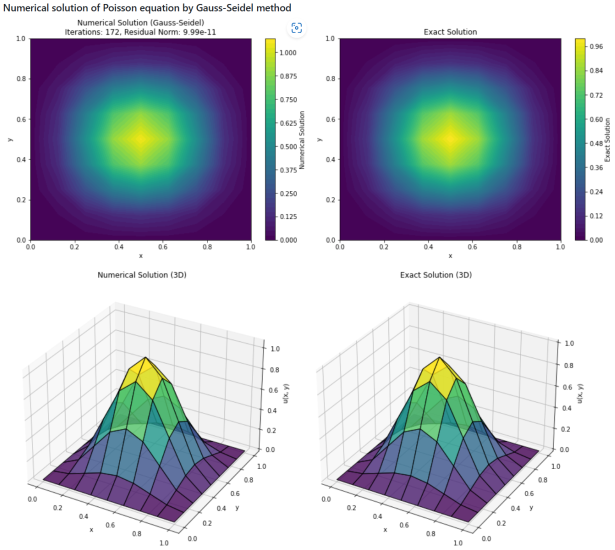
Jacobi Method



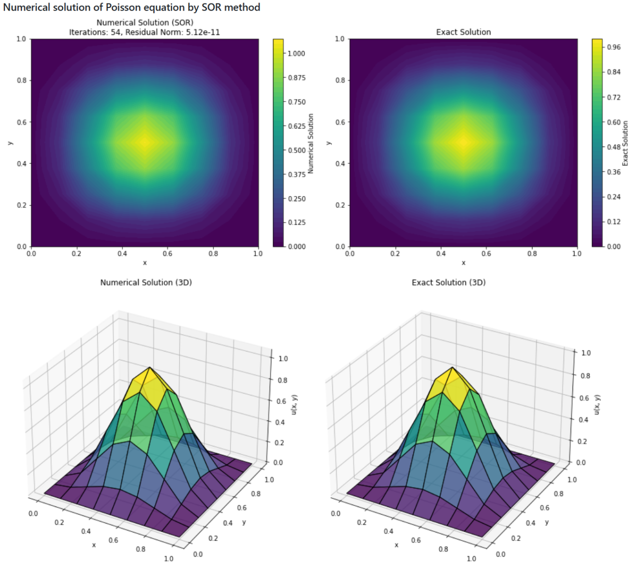
Jacobi Method



Gauss-Seidel method



SOR method



Numerical solution of Poisson equation by SOR method
Numerical Solution (SOR)
Iterations: 54, Residual Norm: 5.12e-11
Numerical Solution
Exact Solution
Numerical Solution (3D)
Exact Solution (3D)
u(x, y)
x
y

Successive Over-Relaxation (SOR) Method

- **Iteration matrix for SOR**:

$$T_{\mathrm{SOR}} = (D - \omega L)^{-1}[(1 - \omega)D + \omega U].$$

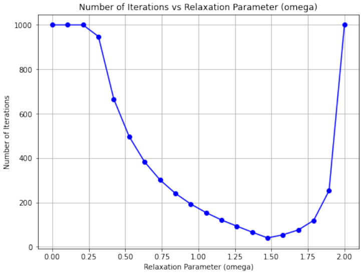
- Convergence rate depends on spectral radius $\rho(T_{\mathrm{SOR}})$.
- The smaller the spectral radius, the faster the convergence.

Optimal Relaxation Parameter ω_{opt}

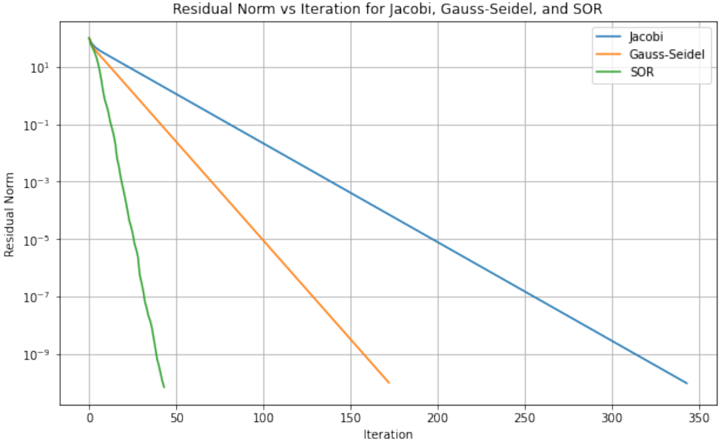
- For SOR, the optimal relaxation parameter ω_{opt} minimizes the spectral radius.
- It is given by:

$$\omega_{\text{opt}} = \frac{2}{1 + \sin\left(\frac{\pi}{n+1}\right)}.$$

- This value ensures faster convergence for the SOR method.
- finding the best omega by plot:



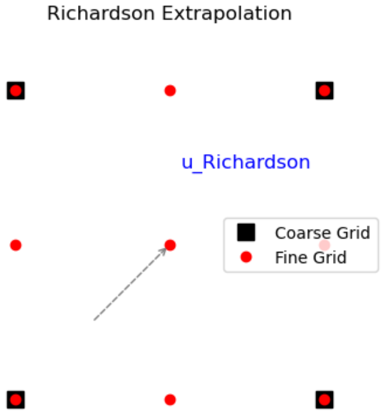
Comparison between the three methods



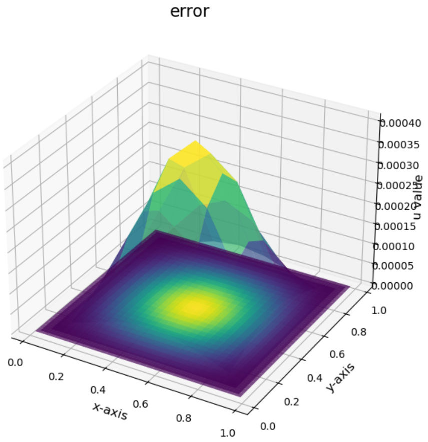
Richardson Extrapolation Formula

Higher Accuracy Approximation:

$$\frac{4}{3}u_{2i,2j}\left(\frac{h_1}{2},\frac{h_2}{2}\right)-\frac{1}{3}u_{ij}(h_1,h_2)=u(x_i,y_j)+O(h_1^4+h_2^4),\quad (i,j)\in\omega. \quad (1)$$



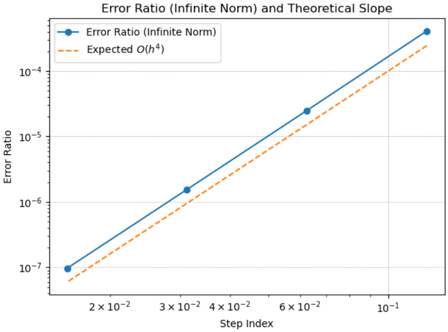
Richardson Extrapolation Formula



The error plot of Richardson:

Richardson Extrapolation Formula

Convergence order:



Construction of the Compact Difference Scheme

In this section, we establish a difference scheme with an accuracy of $O(h_1^4 + h_2^4)$ for solving the boundary value problems.
Let $v = \{v_{ij} \mid 0 \le i \le m, 0 \le j \le n\}$. Define the operators as follows:

$$(Av)_{ij} = \begin{cases} -\frac{1}{12}\left(v_{i-1,j} + 10v_{ij} + v_{i+1,j}\right), & 1 \le i \le m-1,\ 0 \le j \le n, \\ -v_{ij}, & i = 0,\ 0 \le j \le n, \end{cases} \tag{11}$$

$$(Bv)_{ij} = \begin{cases} -\frac{1}{12}\left(v_{i,j-1} + 10v_{ij} + v_{i,j+1}\right), & 1 \le j \le n-1,\ 0 \le i \le m, \\ -v_{ij}, & j = 0,\ 0 \le i \le m. \end{cases} \tag{12}$$

At the grid point (x_i, y_j), the differential equation is:

$$-\left(\frac{\partial^2 u}{\partial x^2}(x_i, y_j) + \frac{\partial^2 u}{\partial y^2}(x_i, y_j)\right) = f(x_i, y_j), \quad 0 \leq i \leq m,\ 0 \leq j \leq n. \tag{13}$$

Applying the operator AB gives:

$$-\left(AB\frac{\partial^2 u}{\partial x^2}(x_i, y_j) + AB\frac{\partial^2 u}{\partial y^2}(x_i, y_j)\right) = ABf(x_i, y_j), \quad 1 \leq i \leq m-1,\ 1 \leq j \leq n \tag{14}$$

This can be rewritten as:

$$-\left(B\left(A\frac{\partial^2 u}{\partial x^2}(x_i, y_j)\right) + A\left(B\frac{\partial^2 u}{\partial y^2}(x_i, y_j)\right)\right) = ABf(x_i, y_j), \quad 1 \leq i \leq m-1,\ 1 \tag{15}$$

By a Lemma, we have:

$$A\frac{\partial^2 u}{\partial x^2}(x_i, y_j) = \delta_x^2 u_{ij} + \frac{h_1^4}{240}\frac{\partial^6 u}{\partial x^6}(\xi_{ij}), \quad 1 \le i \le m-1,\ 0 \le j \le n, \quad (16)$$

$$B\frac{\partial^2 u}{\partial y^2}(x_i, y_j) = \delta_y^2 u_{ij} + \frac{h_2^4}{240}\frac{\partial^6 u}{\partial y^6}(\eta_{ij}), \quad 0 \le i \le m,\ 1 \le j \le n-1. \quad (17)$$

Here, $\xi_{ij} \in (x_{i-1}, x_{i+1})$, $\eta_{ij} \in (y_{j-1}, y_{j+1})$.
Define:

$$P_{ij} = \frac{h_1^4}{240}\frac{\partial^6 u}{\partial x^6}(\xi_{ij}), \quad 1 \le i \le m-1,\ 0 \le j \le n, \quad (18)$$

$$Q_{ij} = \frac{h_2^4}{240}\frac{\partial^6 u}{\partial y^6}(\eta_{ij}), \quad 0 \le i \le m,\ 1 \le j \le n-1. \quad (19)$$

From equations (16) and (17), we obtain:

$$A\frac{\partial^2 u}{\partial x^2}(x_i, y_j) = \delta_x^2 u_{ij} + P_{ij}, \quad 1 \le i \le m-1,\ 0 \le j \le n, \tag{20}$$

$$B\frac{\partial^2 u}{\partial y^2}(x_i, y_j) = \delta_y^2 u_{ij} + Q_{ij}, \quad 0 \le i \le m,\ 1 \le j \le n-1. \tag{21}$$

Substituting into (15):

$$-\left[B\left(\delta_x^2 u_{ij} + P_{ij}\right) + A\left(\delta_y^2 u_{ij} + Q_{ij}\right)\right] = ABf_{ij}, \quad (i,j) \in \omega. \tag{22}$$

Simplifies to:

$$-\left(B\delta_x^2 u_{ij} + A\delta_y^2 u_{ij}\right) = ABf_{ij} + R_{ij}, \quad (i,j) \in \omega, \tag{23}$$

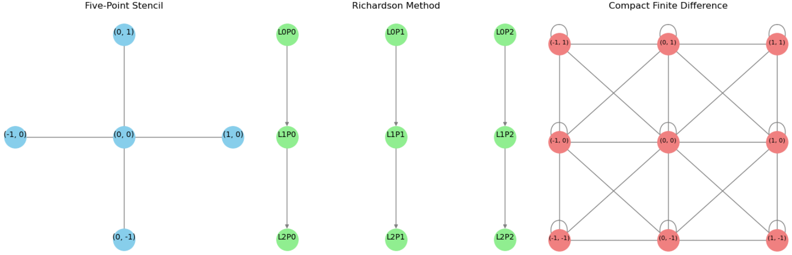
where:

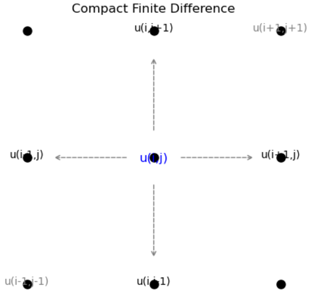
$$R_{ij} = BP_{ij} + AQ_{ij}, \quad (i,j) \in \omega. \tag{24}$$

Define:

$$M_6 = \max\left\{\max_{(x,y)\in\Omega}\left|\frac{\partial^6 u(x,y)}{\partial x^6}\right|, \max_{(x,y)\in\Omega}\left|\frac{\partial^6 u(x,y)}{\partial y^6}\right|\right\}. \tag{25}$$

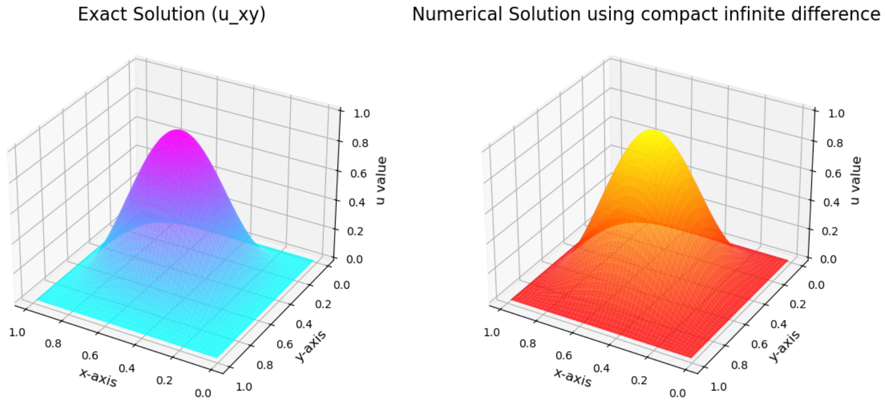
From equations (23), (20), and (21), we know:



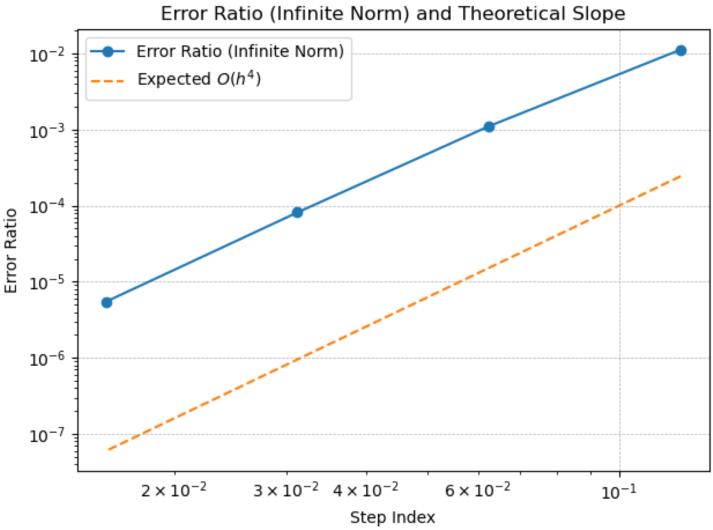


$$|R_{ij}| \leq \frac{1}{240} M_6 \left(h_1^4 + h_2^4\right), \quad (i,j) \in \omega. \tag{26}$$

Construction of the Compact Difference Scheme



Construction of the Compact Difference Scheme



Laplace equation with non-homogeneous boundary condition

- The Laplace equation is given by:

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0,$$

  for $(x, y) \in (0, 1) \times (0, 1)$.

- The boundary conditions are:

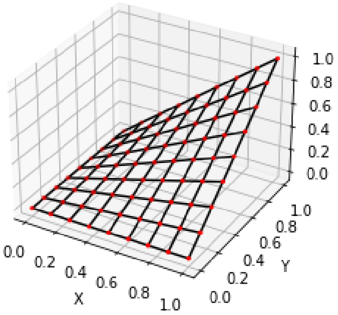
$$u(0, y) = 0, \quad \text{for } y \in [0, 1].$$

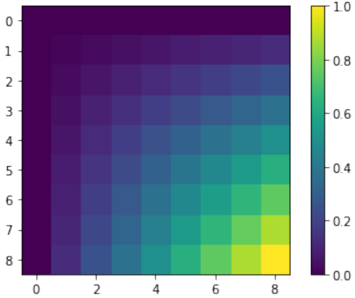
$$u(x, 0) = 0, \quad \text{for } x \in [0, 1].$$

$$u(1, y) = y, \quad \text{for } y \in [0, 1].$$

$$u(x, 1) = x, \quad \text{for } x \in [0, 1].$$

Laplace equation with non-homogeneous boundary condition





Conclusion

- Successfully solved the 2D Poisson equation using finite difference methods.
- Validated numerical results against exact solution.
- Explored iterative methods and their convergence properties.
- Constract higher order methods
- Implement our methods on Laplace eqution with non-homogeneous boundary condition.

Acknowledgment

Thank You!

- Thank you for your attention!
- Special thanks to Pro.Hadrien BERIOT, who taught us how to use the difference method and supported this work.
- Questions and discussions are welcome!