

Q5

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0.1 a)

The PCA is conducted by obtaining new parameters z by mapping them with the old parameters x in a linear relation.

The correlation of z and x is linked by a loading weight w , saying how much weighting of each old parameter is considered by the new parameter z .

$$z_1 = w_1^T x$$

To find the new z , we try to maximize the variance of the new z , such that it can capture the most information from the old x into the new z , hence reducing the dimension of the parameter space.

$$\begin{aligned} \text{Var}(z_1) &= \text{Var}(w_1^T x) \\ &= E((w_1^T x - w_1^T \mu)^2) \\ &= w_1^T E((x - \mu)(x - \mu)^T) w_1 \\ &= w_1^T \Sigma w_1 \end{aligned}$$

Since we try to maximize the loading weight, it has to be set on a constraint to prevent the weights from bloating into excessive values. We try to limit the weights by normalizing its distance into 1:

$$\|w_1\| = \sum_n w_{n1}^2 = 1$$

This singly constrained linear programming can be solved by differentiating the Lagrangian equation of the model with respect to the w_1

$$\mathcal{L}(w_1) = \text{Var}(z_1) - \alpha(\|w_1\| - 1)$$

$$= \text{Var}(w_1^T x) - \alpha(\|w_1\| - 1)$$

$$= w_1^T \Sigma w_1 - \alpha(w_1^T w_1 - 1)$$

$$\frac{\partial \mathcal{L}(w_1)}{\partial w_1} = \Sigma w_1 - \alpha w_1 = 0$$

$$(\Sigma - \alpha I)w_1 = 0$$

This implies α , as a scalar variable, should be the eigenvalue and w_1 should be a eigenvector for Σ , a nxn matrix where n is the dimension of x , to be reduced into the formula.

1 b)

PCA can lower the dimension of the variable vector, hence saving the calculation times and resources for the model prediction.

2 c)

$$X = \begin{bmatrix} x & y & z \end{bmatrix} = \begin{bmatrix} -1 & -1 & 2 \\ -2 & 1 & 1 \end{bmatrix}$$

$$\mu_x = \frac{\sum_n x_n}{n} = \frac{-1 - 2}{2} = -1.5$$

$$\mu_y = \frac{\sum_n x_n}{n} = \frac{-1 + 1}{2} = 0$$

$$\mu_z = \frac{\sum_n x_n}{n} = \frac{2 + 1}{2} = 1.5$$

Calculate the Covariance

$$\Sigma = \frac{1}{N} \sum_n (x_n - \mu)(x_n - \mu)^T$$

$$= \begin{bmatrix} var(x) & cov(x, y) & cov(x, z) \\ cov(y, x) & var(y) & cov(y, z) \\ cov(z, x) & cov(z, y) & var(z) \end{bmatrix}$$

$$= \begin{bmatrix} \frac{(-1+1.5)^2 + (-2+1.5)^2}{2} & \frac{(-1+1.5)(-1-0) + (-2+1.5)(1-0)}{2} & \frac{(-1+1.5)(2-1.5) + (-2+1.5)(1-1.5)}{2} \\ \frac{(-1-0)(-1-1.5) + (1-0)(-2-1.5)}{2} & \frac{(-1-0)^2 + (1-0)^2}{2} & \frac{(-1-0)(2-1.5) + (1-0)(1-1.5)}{2} \\ \frac{(2-1.5)(-1+1.5) + (1-1.5)(-2+1.5)}{2} & \frac{(2-1.5)(-1-0) + (1-1.5)(1-0)}{2} & \frac{(2-1.5)^2 + (1-1.5)^2}{2} \end{bmatrix}$$

$$= \begin{bmatrix} 0.25 & -0.5 & 0.25 \\ -0.25 & 1 & -0.25 \\ 0.25 & -0.5 & 0.25 \end{bmatrix}$$

The derivative of the Lagrangian will yield the following equation with the eigenvalue to be solved:

$$\Sigma w = \alpha w$$

Rewrite the eigen equation:

$$(\Sigma - \alpha I)w = 0$$

$$\begin{vmatrix} 0.25 - \alpha & -0.5 & 0.25 \\ -0.25 & 1 - \alpha & -0.25 \\ 0.25 & -0.5 & 0.25 - \alpha \end{vmatrix} = 0$$

$$= (0.25 - \alpha) \begin{vmatrix} 1 - \alpha & -0.25 \\ -0.5 & 0.25 - \alpha \end{vmatrix} - 0.5 \begin{vmatrix} -0.25 & -0.25 \\ 0.25 - \alpha & 0.25 \end{vmatrix} + 0.25 \begin{vmatrix} -0.25 & 1 - \alpha \\ 0.25 & -0.5 \end{vmatrix}$$

$$(0.25 - \alpha)(\alpha^2 - 1.25\alpha + 0.25 - 0.125) - 0.5(-0.0625 + 0.625 - 0.25\alpha) + 0.25(0.125 - 0.25 + 0.25\alpha)$$

$$= -\alpha^3 + 1.5\alpha^2 - 0.75\alpha$$

$$\alpha = 0$$

is the only real solution

Substitute back to the eigen equation

$$(\Sigma - \alpha I)w = \begin{pmatrix} 0.25 & -0.5 & 0.25 \\ -0.25 & 1 & -0.25 \\ 0.25 & -0.5 & 0.25 \end{pmatrix} w = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} w = \begin{bmatrix} 0.25w_1 - 0.5w_2 + 0.25w_3 \\ -0.25w_1 + w_2 - 0.25w_3 \\ 0.25w_1 - 0.5w_2 + 0.25w_3 \end{bmatrix} = 0$$

Assume $w_1 = k$, from the eigenequation

$$w_2 = 0, \quad w_3 = -k$$

$$w = k \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

Normalize the vector s.t. $\|w\| = 1$

$$\|w\| = k\sqrt{1^2 + 0^2 + (-1)^2} = 1$$

$$k = \frac{1}{\sqrt{2}}$$

$$w = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ -\frac{1}{\sqrt{2}} \end{bmatrix}$$