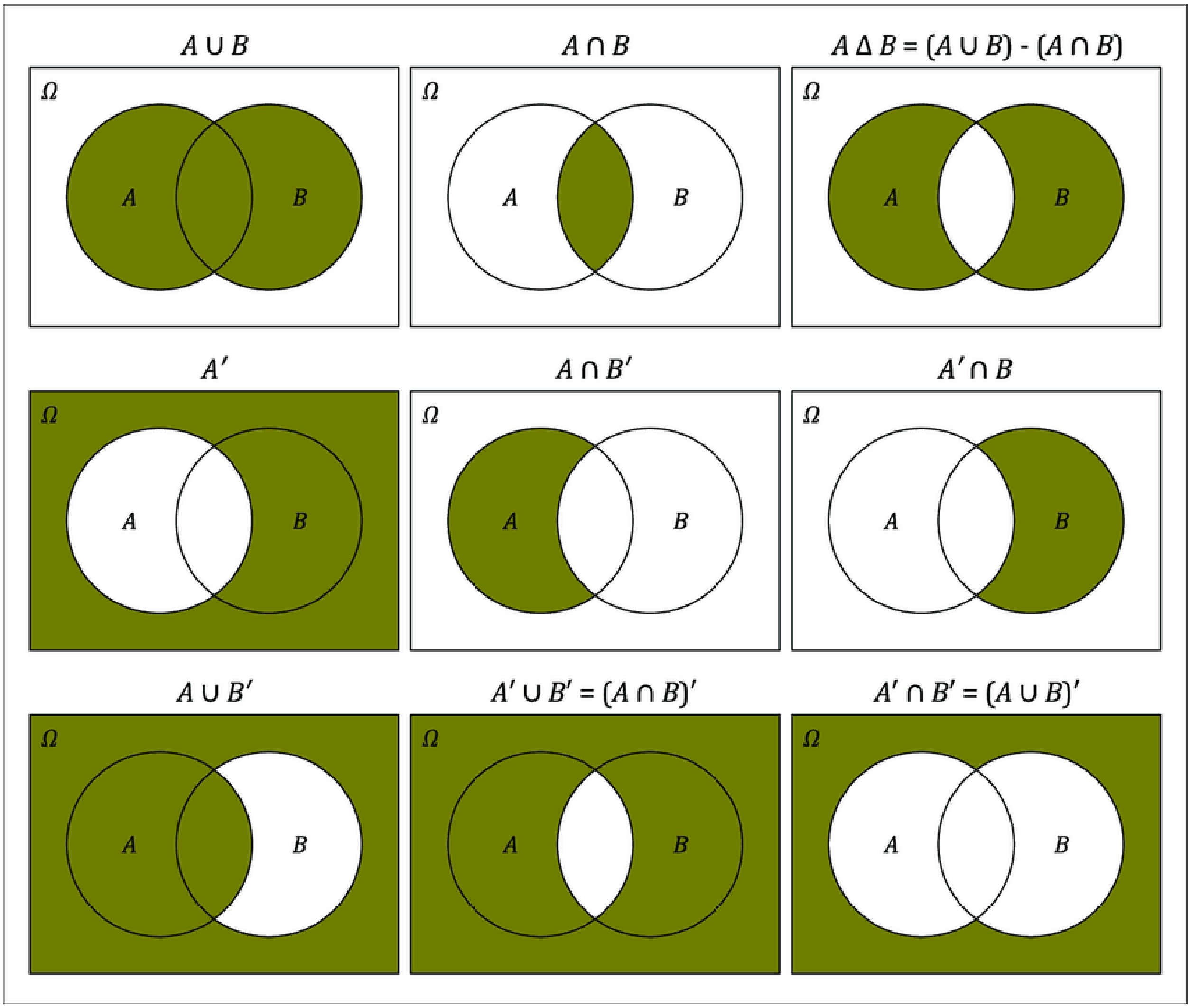
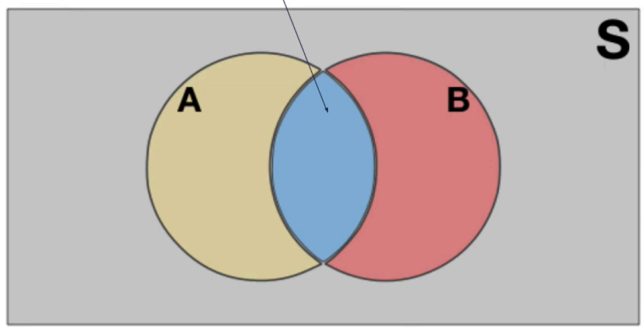
Joint | Marginal | Conditional Probability and Bayes' Theorem



$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

The diagram to the right shows the intersection as the the light blue shaded region.



Contingency Tables

	PC	Mac	Row Totals
Male	66 (0.296)	40 (0.179)	106
Female	30 (0.135)	87 (0.390)	117
Column Totals	96	127	223

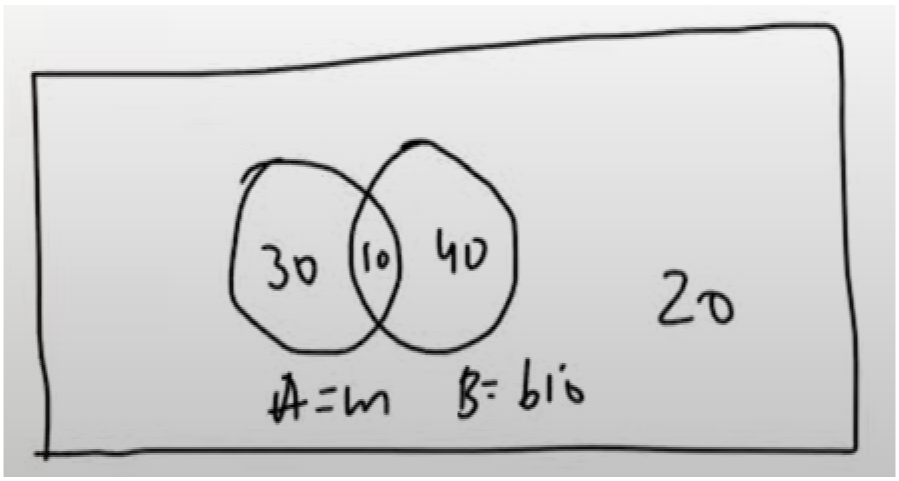
- Example- Rolling a die
  - Getting a number ≥ 4
  - Getting an even number

	Even	Not Even
≥ 4	2	1
<4	1	2

- Ex. 2 : 100 Students
  - 40 → Only Bio
  - 30 → Only Math

- 10 Both

	Math	Not Math
Bio	10	40
Not Bio	30	20



Types of Probability

1. **Joint Probability**
2. **Marginal Probability**
3. **Conditional Probability**

Joint Probability

- **Definition**: The probability of two or more events happening at the same time or in conjunction.
- It's used when we want to find the probability of the occurrence of multiple events.

$$P(A \cap B) = P(A) \cdot P(B|A)$$

- $P(A)$: Probability of event A.
- $P(B|A)$: Probability of event B given that A has occurred.

Example 1:

- Probability that a randomly chosen person is **male and a doctor**:
  → $P(Male \cap Doctor)$

Example 2:

- If you flip a coin and roll a die, the joint probability of getting **Heads** on the coin flip (event A) and a **3** on the die (event B) is:

$$P(A \cap B) = P(\text{Heads and 3}) = P(\text{Heads}) \times P(3) = \frac{1}{2} \times \frac{1}{6} = \frac{1}{12}$$

Here, the probability of getting heads on a coin and a 3 on a die simultaneously is $\frac{1}{12}$.

Python Code

```
df = pd.read_csv(r"https://raw.githubusercontent.com/pandas-dev/pandas/refs/heads/main/doc/data/titanic.csv")

pd.crosstab(df.Survived, df.Pclass)
```

Pclass	1	2	3
Survived			
0	80	97	372
1	136	87	119

- To find the probability → **Normalize**

```
pd.crosstab(df.Survived, df.Pclass, normalize='all')
```

Pclass	1	2	3
Survived			
0	0.089787	0.108866	0.417508
1	0.152637	0.097643	0.133558

- The sum of the above table is 1.
- **Individual number** → Joint Probability
- All numbers together → **Joint Probability Distribution**

Marginal/ Simple/ Unconditional Probability:

- The probability of an event occurring, regardless of the outcome of other variables.
- **Notation**: $P(A)$ or $P(B)$.

$$P(A) = \sum_B P(A, B)$$

Example:

- Probability that a person is **male**, regardless of profession:

$$P(\text{Male}) = P(\text{Male} \cap \text{Doctor}) + P(\text{Male} \cap \text{Engineer}) + \ldots$$

- In the same coin-flip and die-rolling example, the marginal probability of flipping heads (ignoring the outcome of the die) is simply:

$$P(A) = P(\text{Heads}) = \frac{1}{2}$$

And the marginal probability of rolling a 3 on the die (ignoring the coin flip) is:

$$P(B) = P(3) = \frac{1}{6}$$

Note: The marginal probability is obtained by summing (in the case of discrete events) or integrating (in the case of continuous events) over the other events.

Python Code

```
pd.crosstab(df.Survived, df.Pclass, margins=True)
```

Pclass	1	2	3	All
Survived				
0	80	97	372	549
1	136	87	119	342
All	216	184	491	891

Probability:

```
pd.crosstab(df.Survived, df.Pclass, normalize='all', margins=True)
```

Pclass	1	2	3	All
Survived				
0	0.089787	0.108866	0.417508	0.616162
1	0.152637	0.097643	0.133558	0.383838
All	0.242424	0.206510	0.551066	1.000000

- We divided each number with 891.

Conditional Probability

- **Definition**: The probability of an event occurring given that another event has already occurred.
- Notation: $P(A \mid B)$
    - Read as Probability of A given B.

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

- $P(A \cap B)$: Joint probability of A and B.
- $P(B)$: Marginal probability of B.

Example 1:

- Probability that a person is a **doctor**, given that they are **male:**

$$P(\text{Doctor}|\text{Male}) = \frac{P(\text{Male} \cap \text{Doctor})}{P(\text{Male})}$$

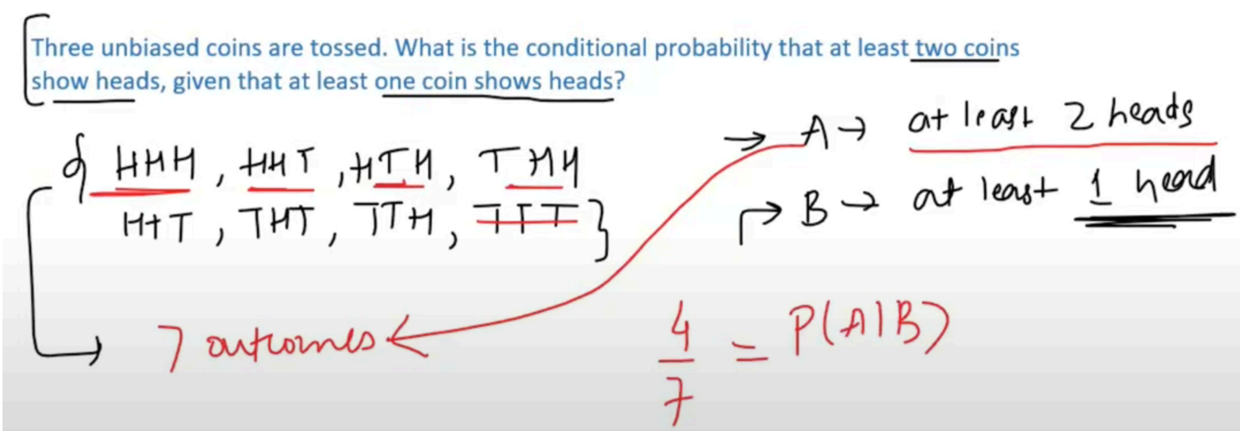
Example 2:

- **Example**: If you know that the coin flip was **Heads**, the conditional probability of rolling a 3 on the die (given that the coin flip was heads) is:

$$P(B \mid A) = \frac{P(A \cap B)}{P(A)} = \frac{P(\text{Heads and 3})}{P(\text{Heads})} = \frac{\frac{1}{12}}{\frac{1}{2}} = \frac{1}{6}$$

So, given that you flipped heads, the probability of rolling a 3 on the die is still $\frac{1}{6}$, because the two events are independent.

Example 3:



Python Code:

- **Probability of someone dying if they're travelling in PClass 3:**

`normalize='columns'`

```
pd.crosstab(df.Survived, df.Pclass, normalize='columns')
```

Pclass	1	2	3
Survived			
0	0.37037	0.527174	0.757637
1	0.62963	0.472826	0.242363

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

Pclass	1	2	3	All
Survived				
0	0.089787	0.108866	0.417508	0.616162
1	0.152637	0.097643	0.133558	0.383838
All	0.242424	0.206510	0.551066	1.000000

$$\frac{0.417508}{0.551066} = 0.75$$

- **Reverse the Above example: Probability of someone travelling in PClass 3 who has died :**

`normalize='index'`

```
pd.crosstab(df.Survived, df.Pclass, normalize='index')
```

Pclass	1	2	3
Survived			
0	0.145719	0.176685	0.677596
1	0.397661	0.254386	0.347953

- Here, we know that they are dead.
- We are finding the probability of they being in PClass 3

Independent vs Mutually Exclusive Events

Independent Events

🔷 **Definition**: Two events are independent if the occurrence of one does **not** affect the probability of the other.

$$P(A \cap B) = P(A) \cdot P(B)$$

🔷**Intuition**:

- If flipping one coin does **not** affect the outcome of another flip, the events are **independent**.

🔷**Example**:

- Rolling a die and flipping a coin. The probability of getting a 6 on the die (1/6) **does not change** whether the coin lands on heads or tails.

For independent events:

$$P(A|B) = P(A)$$
$$P(B|A) = P(B)$$

$$P(A|B) = P(A).$$
$$P(B|A) = P(B).$$

	If statistically independent	If mutually exclusive
$P(A \mid B) =$	$P(A)$	0
$P(B \mid A) =$	$P(B)$	0
$P(A \cap B) =$	$P(A)P(B)$	0

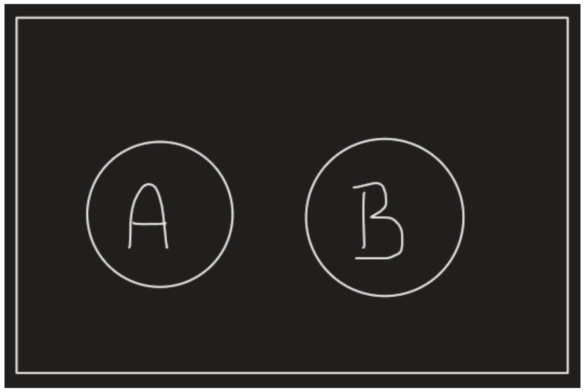
Dependent Events

- Occurrence of one event affects the occurrence of other.
- Ex. Drawing a card with replacement

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

Mutually Exclusive Events

- **Definition**: Two events are mutually exclusive if they cannot occur at the same time.
- **Notation**: A and B are mutually exclusive if $P(A \cap B) = 0$.



- **Example**:
  - Drawing a red card and drawing a black card from a deck in a single draw.
  - You cannot draw a card that is both red and black.

Feature	Independent Events	Mutually Exclusive Events
Definition	The occurrence of one event does not affect the occurrence of the other event.	Two events cannot occur at the same time.
Formula	$P(A \cap B) = P(A) \cdot P(B)$	$P(A \cap B) = 0$
Example	Flipping a coin and rolling a die.	Flipping a coin and getting Heads or Tails.
Can both events happen?	Yes, both events can occur together.	No, both events cannot happen together.
Relation	Events are **unrelated** to each other.	Events are **mutually exclusive**—one event excludes the other.

Bayes' Theorem

- **Definition**: A mathematical formula used to **update the probability** of an event based on **new information.**

- **Purpose**: Helps us revise our beliefs or predictions when we get additional **evidence**.
- It describes the probability of an event, based on prior knowledge of conditions that might be related to the event. Essentially, it allows us to update our beliefs about the probability of an event occurring, given new evidence.

$$P(A|B) = \frac{P(B|A) \cdot P(A)}{P(B)}$$

- $P(A \mid B)$ is the **posterior probability**: the probability of event A occurring given that B has occurred.
- $P(B \mid A)$ is the **likelihood**: the probability of event B occurring given that A has occurred.
- $P(A)$ is the **prior probability**: the initial probability of event A occurring before observing event B.
- $P(B)$ is the **marginal probability**: the total probability of event B occurring, regardless of event A

$$\underbrace{\boxed{P(A|B)}}_{\text{posterior}} = \underbrace{\boxed{P(A)}}_{\text{prior}} \times \frac{\boxed{P(B|A)}\ \text{likelihood}}{\boxed{P(B)}\ \text{marginal}}$$

📌 **Problem:** Suppose there's a rare disease affecting **1%** of the population.
A test correctly detects the disease **90%** of the time (True Positive Rate).
However, **5%** of healthy people **also** test positive (False Positive Rate).
If you test positive, what's the probability that you actually have the disease?

📌 **Solution using Bayes' Theorem**

- $P(D) = 0.01$ → Probability of having the disease (Prior probability)
- $P(\neg D) = 0.99$ → Probability of NOT having the disease
- $P(T|D) = 0.90$ → Probability of testing positive **if diseased** (True Positive)
- $P(T|\neg D) = 0.05$ → Probability of testing positive **if healthy** (False Positive)
- $P(T) = P(T|D)P(D) + P(T|\neg D)P(\neg D)$

Now apply Bayes' Theorem:

$$P(D|T) = \frac{(0.90 \times 0.01)}{(0.90 \times 0.01) + (0.05 \times 0.99)}$$

$$= \frac{0.009}{0.009 + 0.0495} = \frac{0.009}{0.0585} \approx 0.154$$

Final Answer:

Even after testing **positive**, the probability of actually having the disease is **only 15.4%**, not 90%! This is due to the **False Positives** dominating because the disease is rare.

Components:

- **Prior**: Initial belief.
- **Likelihood**: Probability of evidence given the event.
- **Posterior**: Updated belief after seeing evidence.

Video → https://youtu.be/cqTwHnNbc8g?si=m8jHimmMJ0bgJEsN

Ex You've been planning a picnic for your family. You're trying to decide whether to postpone due to rain. The chance of rain on any day is 15%. The morning of the picnic, it's cloudy. The prob. of it being cloudy is 25% and on days where it rains, it's cloudy in the morning 80% of the time.

Should you postpone the picnic?

$P(\text{rain}) = 0.15$

$P(\text{cloudy}) = 0.25$

$P(\text{cloudy}|\text{rain}) = 0.80$

$$P(A|B) = \frac{P(B|A) \cdot P(A)}{P(B)}$$

$$P(\text{rain}|\text{cloudy}) = \frac{P(\text{cloudy}|\text{rain}) \cdot P(\text{rain})}{P(\text{cloudy})} = \frac{0.8 \cdot 0.15}{0.25}$$

$$\boxed{P(\text{rain}|\text{cloudy}) = 0.48}$$