Determinant, Minor, Cofactor, Adjoint and Inverse

Determinant

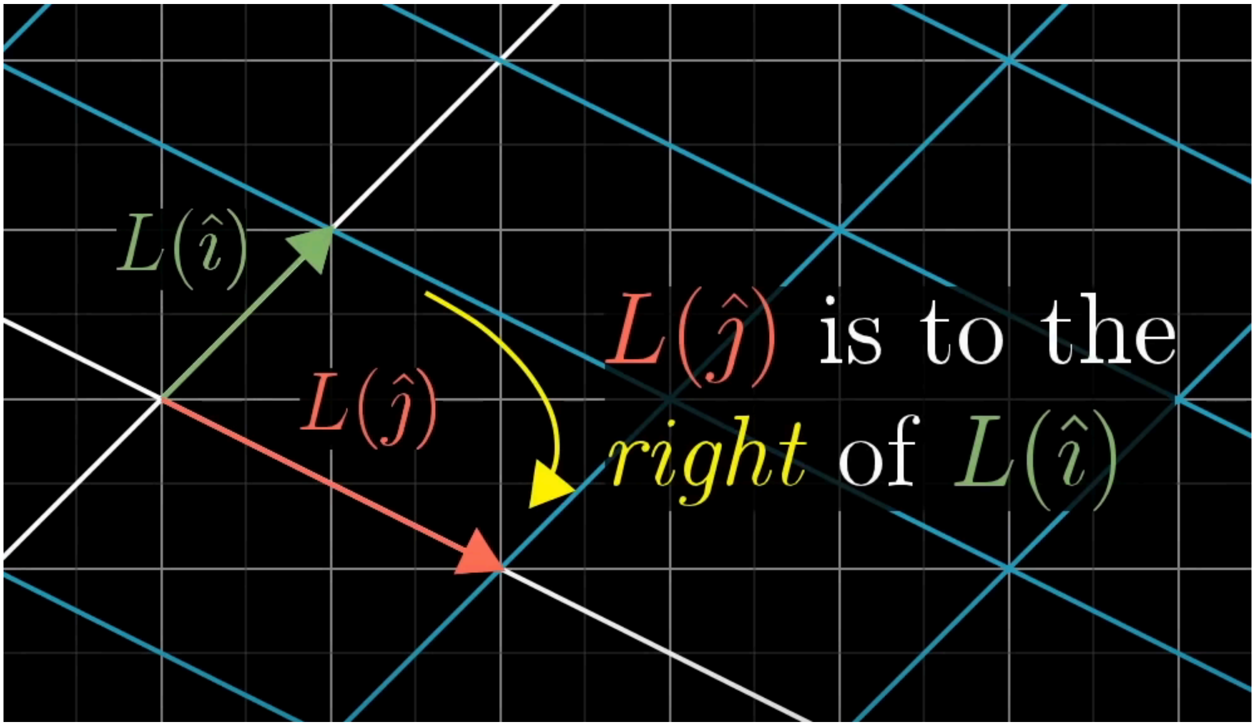
The **determinant** of a square matrix provides important information about the matrix. It's a scalar value that can be calculated for any square matrix. The determinant can tell us:

- Whether the matrix is **invertible** (non-zero determinant) or **singular** (zero determinant).
- The **area** or **volume** of the space defined by the matrix.

- **When you apply transformation, the area changes. Determinant tells us by what factor has the area changed.**
  - **The determinant of a transformation will be 3 if it increases the area by the factor of 3.**
  - **The determinant will be 0.5 if it squishes down the area by half.**
  - **The determinant will be 0 if it squishes all of space onto a line or a point.**

Determinants can have negative values

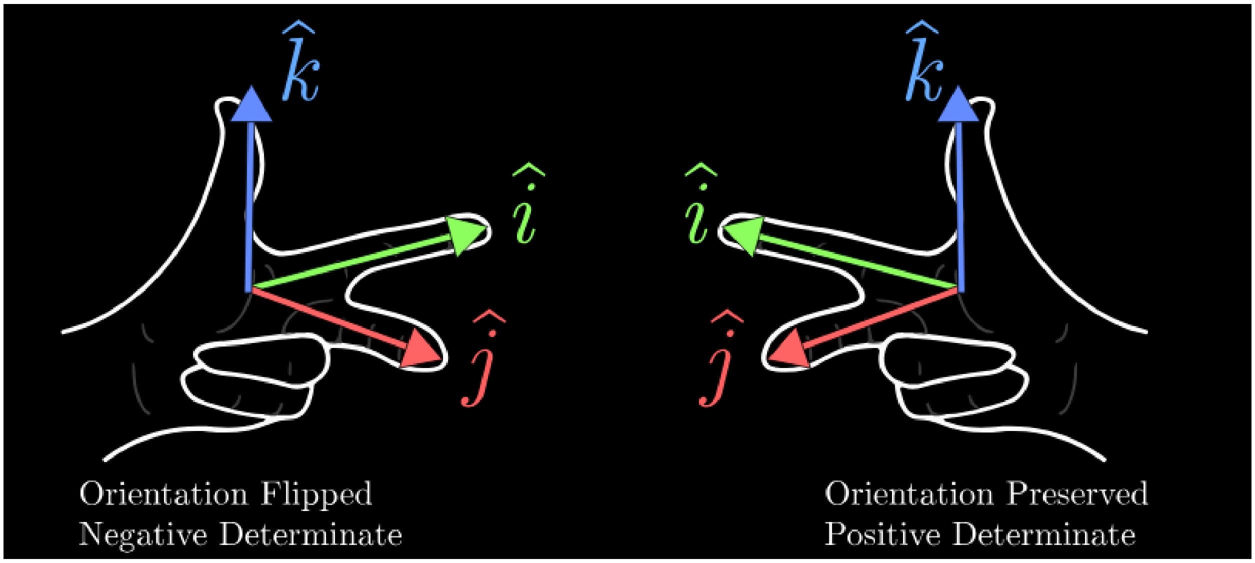
- It's about orientation
- It inverts the orientation of space
  - It flips the plane



For a 2x2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, the determinant is:

$$\det(A) = ad - bc$$

- In 3D, determinant tells information about the **Volume.**
- Right hand rule



$$A = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}$$

$$\det(A) = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

$$= a_1(b_2c_3 - b_3c_2) - a_2(b_1c_3 - b_3c_1) + a_3(b_1c_2 - b_2c_1)$$

$$\det(M_1 M_2) = \det(M_1)\det(M_2)$$

Eigenvalues and Eigenvectors

- An **eigenvector** is a vector that, when multiplied by a matrix, only gets **stretched or shrunk** (no change in direction). The amount of stretching or shrinking is called the **eigenvalue**.

Minor

What is a Minor?

- A **minor** of an element in a matrix is the **determinant of the smaller matrix** you get by removing the row and column that contain that element.
- Formed by deleting the row and column containing that element.
- It helps in calculating determinants, cofactors, and inverses.

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3. Example

Matrix A:

(2,3)

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

Find the Minor of a_{23} (Element in Row 2, Column 3):

1. **Delete Row 2 and Column 3:**

$$\text{Submatrix} = \begin{bmatrix} 1 & 2 \\ 7 & 8 \end{bmatrix}$$

2. **Compute Determinant:**

$$M_{23} = (1 \times 8) - (2 \times 7) = 8 - 14 = -6$$

Python code:

```
import numpy as np

def minor(matrix, i, j):
    # Delete the i-th row and j-th column
    submatrix = np.delete(np.delete(matrix, i, axis=0), j, axis=1)
    return np.linalg.det(submatrix)

# Example matrix
A = np.array([
    [1, 2, 3],
    [4, 5, 6],
    [7, 8, 9]
])

# Compute minor of element at row 1, column 2 (0-based indexing)
minor_23 = minor(A, 1, 2)
print(f"Minor of element at row 2, column 3: {minor_23}")

Output:
Minor of element at row 2, column 3: -6.0
```

Cofactor

- A **cofactor** is a number that is calculated based on the **minor** of a matrix element, with an additional sign factor depending on the position of the element.

How to Calculate a Cofactor?

1. Find a **Minor**

2. **Apply the Sign**: The **cofactor** is the minor, but you also apply a sign change based on the position of the element.
    - The sign follows a **checkerboard pattern** starting from the top-left corner:
    - The sign is **positive** for positions where **i+j is even**.
    - The sign is **negative** for positions where **i+j is odd**.

💡 **Multiply the minor by** $(-1)^{i+j}$ to adjust the sign.

This pattern looks like this:

$$\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$$

So, the cofactor of an element A_{ij} is calculated as:

$$C_{ij} = (-1)^{i+j} \cdot M_{ij}$$

Where:

- M_{ij} is the **minor** of the element A_{ij},
- $(-1)^{i+j}$ gives the sign based on the position of the element (using the checkerboard pattern).

Example

Given a **3×3 matrix:**

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

To find the **cofactor of 5** (which is at **row 2, column 2**, so $i = 2, j = 2$):

1. Remove **row 2** and **column 2**, leaving:

$$\begin{bmatrix} 1 & 3 \\ 7 & 9 \end{bmatrix}$$

2. Compute its **minor:**

$$(1 \times 9) - (3 \times 7) = 9 - 21 = -12$$

3. Compute the **cofactor:**

$$C_{22} = (-1)^{2+2} \times (-12) = 1 \times (-12) = -12$$

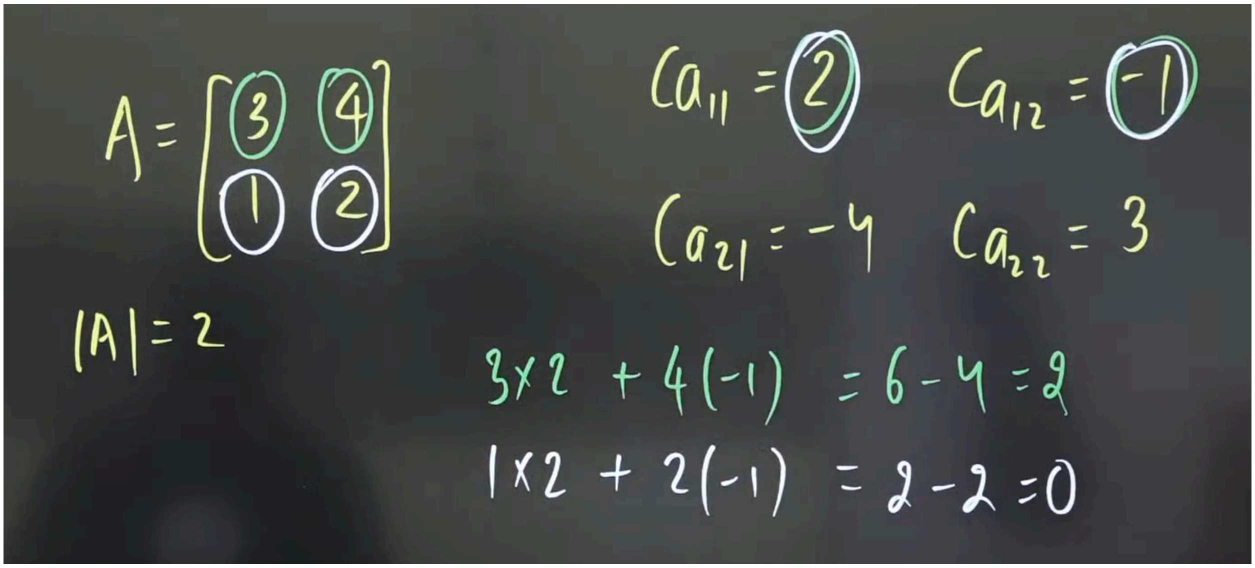
So, **the cofactor of 5 is -12**.

Why is Cofactor Important?

- **Used in Determinants** → Determinants are calculated using cofactors.
- **Used in Inverse Matrices** → The inverse of a matrix involves cofactors.
- **Used in Adjugates** → The adjugate (transpose of the cofactor matrix) is useful in computing the inverse.

💡 if we multiply a row by its own cofactor → Cofactor

If not → Zero



Adjoint/Adjugate

- Used particularly in finding the inverse of a matrix.

For a given square matrix A, the adjoint of A, denoted as $adj(A)$, is the transpose of the cofactor matrix of A.

How to calculate Adjoint?

- **Compute the Cofactor Matrix:** Determine the cofactor for each element of A.
- **Transpose the Cofactor Matrix:** Swap the rows and columns of the cofactor matrix to obtain the adjoint.

If A is an $n \times n$ matrix, the adjoint of A, denoted as $\text{adj}(A)$, is given by:

$$\text{adj}(A) = [C_{ij}]^{\text{T}}$$

Here, C_{ij} represents the cofactor of the element at the i-th row and j-th column of A, and $^{\text{T}}$ denotes the transpose operation.

Steps to Calculate the Adjoint:

1. **Find the Cofactor Matrix:**
   - For each element a_{ij} in A:
     - **Minor:** Eliminate the i-th row and j-th column to form a submatrix. Calculate its determinant.
     - **Cofactor:** Multiply the minor by $(-1)^{i+j}$ to account for the position's sign.
2. **Transpose the Cofactor Matrix:** Interchange rows and columns to obtain the adjoint.

Example:

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

1. **Minor Matrix**:
   - For element $a_{11} = 1$, remove the 1st row and 1st column to get:
   $$M_{11} = \det \begin{bmatrix} 4 \end{bmatrix} = 4$$
   - For element $a_{12} = 2$, remove the 1st row and 2nd column to get:
   $$M_{12} = \det \begin{bmatrix} 3 \end{bmatrix} = 3$$
   - For element $a_{21} = 3$, remove the 2nd row and 1st column to get:
   $$M_{21} = \det \begin{bmatrix} 2 \end{bmatrix} = 2$$
   - For element $a_{22} = 4$, remove the 2nd row and 2nd column to get:
   $$M_{22} = \det \begin{bmatrix} 1 \end{bmatrix} = 1$$
2. **Cofactor Matrix**: Apply the sign factor $(-1)^{i+j}$ to each minor:
   $$C = \begin{bmatrix} 4 & -3 \\ -2 & 1 \end{bmatrix}$$

3. **Adjoint Matrix** (Transpose of the Cofactor Matrix):
   $$\text{adj}(A) = C^T = \begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix}$$

Applications of Adjoint:

1. **Inverse of a Matrix**:
   - If the matrix A is **invertible**, the inverse of A can be computed using the adjoint:

$$A^{-1} = \frac{1}{\det(A)} \times \text{adj}(A)$$

   - This is particularly useful when you need to find the inverse of a matrix and you know its determinant is non-zero.
2. **Determinants**:
   - The adjoint matrix also has properties related to the determinant. For an $n \times n$ matrix A:

$$\det(\text{adj}(A)) = (\det(A))^{n-1}$$

3. **Solving Systems of Linear Equations**:
   - The adjoint can be used in methods for solving systems of linear equations, particularly when the system is represented in matrix form.

Inverse

💡 The **inverse** of a matrix is like the "opposite" of the matrix. Just like how multiplying 5 by

$\frac{1}{5}$ gives 1, multiplying a matrix by its inverse gives the **identity matrix I**.

$$A \times A^{-1} = A^{-1} \times A = I$$

Where I I is the **identity matrix** (like the number 1 in normal multiplication), which looks like this for a 2×2:

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

When Does a Matrix Have an Inverse?

A matrix **only** has an inverse if:

1. It is **square** (number of rows = number of columns).
2. Its **determinant** is **not zero** (if det(A)=0, then the matrix is **singular** and has no inverse).

How to Find the Inverse?

1. For a 2×2 Matrix

If:

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Then the inverse is given by:

$$\boxed{A^{-1} = \frac{1}{\det(A)} \times \operatorname{adj}(A)}$$

Where:

- **Determinant:**

$$\det(A) = ad - bc$$

- **Adjoint (Adjugate):**

$$\operatorname{adj}(A) = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

So:

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Example:

If

$$A = \begin{bmatrix} 4 & 7 \\ 2 & 6 \end{bmatrix}$$

1. Compute the determinant:

$$\det(A) = (4 \times 6) - (7 \times 2) = 24 - 14 = 10$$

Cofactor:

$$\begin{bmatrix} 6 & -2 \\ -7 & 4 \end{bmatrix}$$

2. Compute the adjugate:

Transpose of Cofactor

$$\operatorname{adj}(A) = \begin{bmatrix} 6 & -7 \\ -2 & 4 \end{bmatrix}$$

3. Multiply by $\frac{1}{\det(A)}$:

$$A^{-1} = \frac{1}{10} \begin{bmatrix} 6 & -7 \\ -2 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 0.6 & -0.7 \\ -0.2 & 0.4 \end{bmatrix}$$

2. For a 3×3 Matrix

1. **Find the determinant $\det(A)$.**
2. **Find the cofactor matrix** (each element's determinant, adjusted with $(-1)^{i+j}$).
3. **Transpose the cofactor matrix** (swap rows and columns).
4. **Divide by $\det(A)$.**

Inverse Using Python

```
import numpy as np

A = np.array([[4, 7], [2, 6]])
A_inv = np.linalg.inv(A)
```

```
print(A_inv)
```

Output:

[[0.6 -0.7]
[-0.2 0.4]]

Geometric Meaning

- The inverse of a **transformation matrix** undoes the transformation.
- For example, if a matrix rotates a point, its inverse rotates it back.
- If a matrix scales a vector, its inverse scales it back.

INVERSE=UNDO