1. **best straight best parabola type problems:**

4.

x_i	-3	-2	-1	1	2	3
y_i	1	0	3	0	-2	0

(a) The linear system for the coefficients of the polynomial $P_1(x) = a_1x + a_0$ is

$$\left[\begin{array}{cc|c} N & \sum x_i & \sum y_i \\ \sum x_i & \sum x_i^2 & \sum x_i y_i \end{array}\right] = \left[\begin{array}{cc|c} 6 & 0 & 2 \\ 0 & 28 & -10 \end{array}\right]$$

The solution is $a_0 = \frac{1}{3}$ and $a_1 = -\frac{5}{14}$, so the best straight line fitting to the points is

$$\boxed{P_1(x) = -\frac{5}{14}x + \frac{1}{3}}.$$

(3 points)

(b) The linear system for the coefficients of the polynomial $P_2(x) = a_2x^2 + a_1x + a_0$ is

$$\left[\begin{array}{ccc|c} N & \sum x_i & \sum x_i^2 & \sum y_i \\ \sum x_i & \sum x_i^2 & \sum x_i^3 & \sum x_i y_i \\ \sum x_i^2 & \sum x_i^3 & \sum x_i^4 & \sum x_i^2 y_i \end{array}\right] = \left[\begin{array}{ccc|c} 6 & 0 & 28 & 2 \\ 0 & 28 & 0 & -10 \\ 28 & 0 & 196 & 4 \end{array}\right]$$

From the second equation $a_1 = -\frac{5}{14}$.

For a_0 and a_2 we have we have the system

$$\left[\begin{array}{cc|c} 3 & 14 & 1 \\ 7 & 49 & 1 \end{array}\right] \sim \left[\begin{array}{cc|c} 3 & 14 & 1 \\ 1 & 21 & -1 \end{array}\right] \sim \left[\begin{array}{cc|c} 0 & -49 & 4 \\ 1 & 21 & -1 \end{array}\right]$$

It follows

$$a_2 = -\frac{4}{49}, \qquad a_0 = -1 + \frac{84}{49} = \frac{35}{49} = \frac{5}{7},$$

so the best parabola fitting to the points is

$$\boxed{P_2(x) = -\frac{4}{49}x^2 - \frac{5}{14}x + \frac{5}{7}x^2}.$$

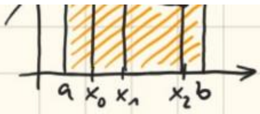
(6 points)

https://www.emathhelp.net/en/calculators/linear-algebra/gauss-jordan-elimination-calculator/

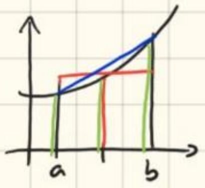
2. trapezoidal Simpson's formulae type question:

-Approximate integral

Recall Quadrature formulas $I(f) \approx Q(f)$



$$\int_a^b f = \int_a^b f(x)\,dx \approx \sum_{k=0}^{n} A_k \cdot f(x_k) = A_0 \cdot f(x_0) + A_1 \cdot f(x_1) + \dots + A_n \cdot f(x_n)$$



Midpoint formula $M(f) = (b-a)\cdot f\left(\frac{a+b}{2}\right)$ (1)

Trapezoid formula $T(f) = \frac{b-a}{2}\cdot\left(f(a)+f(b)\right)$ [1 1]

Simpson's formula $S(f) = \frac{b-a}{6}\cdot\left(f(a) + 4\cdot f\left(\frac{a+b}{2}\right) + f(b)\right)$ [1 4 1]

↳ all 3 of Newton–Cotes type : uniform division of [a,b] N-C

-estimate error bound:

Recall Error formulas $|I(f) - M_m(f)| \le \frac{M_2\cdot(b-a)^3}{24\cdot m^2}$ $M_2 = \|f''\|_\infty$

$|I(f) - T_m(f)| \le \frac{M_2\cdot(b-a)^3}{12\cdot m^2}$, $|I(f) - S_m(f)| \le \frac{M_4\cdot(b-a)^5}{180\cdot m^4}$

Remark: $m \to \infty \Rightarrow Q_m(f) \to I(f)$ $(f \in C^4)$, M_1, T_1, S_2 : the formula

$f(x) = \frac{1}{x} = x^{-1}$, $f'(x) = -x^{-2}$, $f''(x) = 2\cdot x^{-3} = \frac{2}{x^3}$, $f^{(3)}(x) = -6\cdot x^{-4}$, $f^{(4)}(x) = 24\cdot x^{-5} = \frac{24}{x^5}$

$M_2 = \max_{x\in[1,2]} |f''(x)| = \max_{x\in[1,2]} \left|\frac{2}{x^3}\right| \overset{x=1}{=} 2$, $M_4 = \max_{x\in[1,2]} |f^{(4)}(x)| = \max_{x\in[1,2]} \left|\frac{24}{x^5}\right| \overset{x=1}{=} 24$

$|I(f) - M_1(f)| \le \frac{M_2\cdot(b-a)^3}{24\cdot m^2} = \frac{2\cdot(2-1)^3}{24\cdot 1^2} = \frac{2\cdot 1^3}{24} = \frac{1}{12}$ $\Rightarrow \ln 2 \in \left[\frac{2}{3} - \frac{1}{12}, \frac{2}{3} + \frac{1}{12}\right]$

$|I(f) - T_1(f)| \le \frac{M_2\cdot(b-a)^3}{12\cdot m^2} = \frac{2\cdot(2-1)^3}{12\cdot 1^2} = \frac{2\cdot 1^3}{12} = \frac{1}{6}$

$|I(f) - S_2(f)| \le \frac{M_4\cdot(b-a)^5}{180\cdot m^4} = \frac{24\cdot 1^5}{180\cdot 2^4} = \frac{24}{180\cdot 16} = \frac{1}{120}$ $\Rightarrow \int_1^2 \frac{1}{x}\,dx = \ln 2 \approx \frac{25}{36} \pm \frac{1}{120}$

$$f(x) = e^{-x^2}, \quad f'(x) = -2xe^{-x^2}, \quad f''(x) = e^{-x^2}(4x^2 - 2)$$

For the error bound we need $M_2 = \max_{[0,1]} |f''(x)|$. Therefore we calculate the third derivative, as well.

$$\begin{aligned} f'''(x) &= e^{-x^2}(-2x)(4x^2 - 2) + 8xe^{-x^2} \\ &= 4xe^{-x^2}(-2x^2 + 3) = 0 \iff x = 0, \text{ or } x = \pm\sqrt{\frac{3}{2}} \end{aligned}$$

$x = 0$ is an endpoint of the interval, the other values are not in $[0, 1]$.

As $f'''(x) > 0$ for $x \in (0, 1]$, we have $f'' \searrow$ and therefore $M_2 = |f''(0)| = 2$.

The error bound is

$$|I - T(f)| \leq \frac{1}{12} M_2 = \frac{1}{6}.$$

-trapezoidal rule with n:

Recall Quadrature rules (composite formulas) composite rules

Midpoint rule $M_m(f) = \frac{b-a}{m}(f(x_1) + \dots + f(x_m))$

Trapezoid rule $T_m(f) = \frac{b-a}{2m}(1 \cdot f(a) + 2 \cdot f(x_1) + \dots + 2 \cdot f(x_{m-1}) + 1 \cdot f(b))$

Simpson's rule m even, $\frac{m}{2}$ formulas

(b) The composite trapezoidal rule with n=2

$$\frac{1}{4}\left[f(0) + 2f(\frac{1}{2}) + f(1)\right] = \frac{1}{4}\left[1 + 2e^{-\frac{1}{4}} + e^{-1}\right] = \frac{1}{2} \cdot 0,684 + \frac{1}{2}e^{-\frac{1}{4}} = 0,732.$$

(2 points)

https://www.symbolab.com/solver/trapezoidal-approximation-calculator/trapezoidal%20%5Cint_%7B0%7D%5E%7B1%7D%20e%5E%7B-x%5E2%7Ddx%2Cn%3D2?or=input

-integral within the error:

(c) Error bound for the composite trapezoidal rule:

$$\frac{1}{12} \cdot \frac{(b-a)^3}{n^2} \cdot M_2 = \frac{1}{6n^2} < 10^{-2} \iff \frac{100}{6} < n^2 \iff 4,08 = \frac{10}{\sqrt{6}} < n$$

So with $\boxed{n = 5}$ the error is within 10^{-2}. (3 points)

3. Horner's scheme

-write the polynomial

5.

$$P(x) = 2x^4 - 3x^2 + 3x - 4.$$ $(x+2)$ $a=-2$

The Horner's scheme:

	2	0	−3	3	−4
−2		−4	8	−10	14
	2	−4	5	−7	$\boxed{10}$
−2		−4	16	−42	
	2	−8	21	$\boxed{-49}$	
−2		−4	24		
	2	−12	$\boxed{45}$		
−2		−4			
	2	$\boxed{-16}$			
−2					
	$\boxed{2}$				

Végül

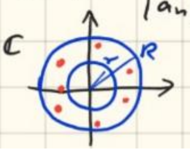
$$P(x) = 10 - 49(x+2) + 45(x+2)^2 - 16(x+2)^3 + 2(x+2)^4$$

-find all root:

Recall root estimates $P(x) = \underset{\neq 0}{a_n} \cdot x^n + a_{n-1} \cdot x^{n-1} + \dots + a_1 \cdot x + \underset{\neq 0}{a_0}$

$P(x_k) = 0 \Rightarrow r < |x_k| < R$ with $R = 1 + \dfrac{\max_{i=0}^{n-1} |a_i|}{|a_n|}$, $r = \left(1 + \dfrac{\max_{i=1}^{n} |a_i|}{|a_0|}\right)^{-1}$

complex ($\in \mathbb{C}$) roots may arise



Problem 3 No integer roots for $P(x) = \underset{\neq 0}{3}x^4 - x^3 - 5x^2 + 4x - \underset{\neq 0}{6}$.

$$R = 1 + \frac{\max\{|-1|, |-5|, |4|, |-6|\}}{|3|} = 1 + \frac{6}{3} = 2$$

$$r = \left(1 + \frac{\max\{|3|, |-1|, |-5|, |4|\}}{|-6|}\right)^{-1} = \left(1 + \frac{5}{6}\right)^{-1} = \frac{6}{11}$$

$$\frac{6}{11} < |x_k| < 2$$

Possible roots $\in \mathbb{Z}$: -1 and 1, but $P(-1) = -11 \neq 0$, $P(1) = -5 \neq 0$

4. quadrature formula

-weights has the highest possible degree:

3.

$$\int_0^1 f(x)\sqrt{x}\,dx \approx \underbrace{A_0 f(0)}_{1} + \underbrace{A_1 f\left(\frac{1}{2}\right)}_{2} + \underbrace{A_2 f(1)}_{3}$$

The system for the weights is

$$\begin{aligned}
1 \quad \int_0^1 1 \cdot \sqrt{x}\,dx &= \frac{2}{3} = A_0 + A_1 + A_2 \quad && 0^0=1 \quad \frac{1}{2}^0=1 \quad 1^0=1 \\
2 \quad \int_0^1 x^1 \cdot \sqrt{x}\,dx &= \frac{2}{5} = \frac{1}{2}A_1 + A_2 \quad && 0^1=0 \quad \frac{1}{2}^1=\frac{1}{2} \quad 1^1=1 \\
3 \quad \int_0^1 x^2 \cdot \sqrt{x}\,dx &= \frac{2}{7} = \frac{1}{4}A_1 + A_2 \quad && 0^2=0 \quad \frac{1}{2}^2=\frac{1}{4} \quad 1^2=1
\end{aligned}$$

$\int x^0$

(4 points)

From the second and third equations we get $A_1 = \frac{16}{35}$ and $A_2 = \frac{6}{35}$, substituting them into the first equation, we have $A_0 = \frac{4}{105}$. (2 points)

-Find x0, x1,c1:

(10 points)

6. Find x_0, x_1 and c_1 so that the quadrature formula

$$\int_0^1 f(x)\,dx \approx \frac{1}{2}f(x_0) + c_1 f(x_1)$$

has the highest possible degree of precision. (5 points)

$$\int_0^1 1\,dx = 1 = \frac{1}{2} + c_1 \qquad c_1 = \frac{1}{2}$$

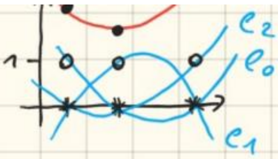
$$\int_0^1 x\,dx = \frac{1}{2} = x_0 \cdot \frac{1}{2} + x_1 \cdot \frac{1}{2}$$

$$\int_0^1 x^2\,dx = \frac{1}{3} = x_0^2\,\frac{1}{2} + x_1^2\,\frac{1}{2}$$

5. Lagrange and interpolation

-Lagrange form of the interpolation polynomial

Recall Lagrange's formula



var.

$$\ell_\ell(x) = \prod_{\substack{j=0 \\ j\neq \ell}}^{n} \frac{(x - x_j)}{(x_\ell - x_j)} \qquad \ell_\ell(x_i) = \delta_{i\ell} = \begin{cases} 1 & i=\ell \\ 0 & i\neq \ell \end{cases}$$

$$p(x) = L_n(x) = \sum_{\ell=0}^{n} y_\ell \cdot \ell_\ell(x)$$

ℓ_ℓ: Lagr. fundamental / base polynomials

Problem 1.(e) ($\sqrt{x}$; 1,4,9) Give the Lagrange form

$$\ell_0(x) = \frac{(x-4)(x-9)}{(1-4)(1-9)} = \frac{1}{24}(x-4)(x-9) = \frac{1}{24}(x^2-13x+36)$$

$$\ell_1(x) = \frac{(x-1)(x-9)}{(4-1)(4-9)} = -\frac{1}{15}(x-1)(x-9) = -\frac{1}{15}(x^2-10x+9)$$

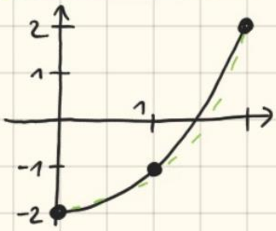
$$\ell_2(x) = \frac{(x-1)(x-4)}{(9-1)(9-4)} = \frac{1}{40}(x-1)(x-4) = \frac{1}{40}(x^2-5x+4)$$

$$L_2(x) = \overset{y_0}{1}\cdot \ell_0(x) + \overset{y_1}{2}\cdot \ell_1(x) + \overset{y_2}{3}\cdot \ell_3(x) = \overset{HW}{\dots} = -\frac{1}{60}(x^2-25x-36)$$

-inverse interpolation:

Problem 4

$f(x) = x^2 - 2 = 0$ ($\pm\sqrt{2}$), $x_0 = 0$, $x_1 = 1$, $x_2 = 2$, $x_3 = ?$



x	0	1	2
y	-2	-1	2

y_i	$f^{-1}(y_i)$		
-2	$\boxed{0}$	$\frac{1-0}{-1-(-2)} = \boxed{1}$	$\frac{1/3 - 1}{2-(-2)} = \frac{-2/3}{4} = \boxed{-1/6}$
-1	1	$\frac{2-1}{2-(-1)} = 1/3$	
2	2		

$y-(-2)$ $\quad$ $y-(-1)$

$$p_2(y) = 0 + 1\cdot(y+2) - \frac{1}{6}(y+2)(y+1)$$

$$\sqrt{2} \approx x_3 = p_2(0) = 0 + 1\cdot 2 - \frac{1}{6}\cdot 2\cdot 1 = 2 - \frac{1}{3} = \frac{5}{3} \approx 1{,}667$$

-estimate the error:

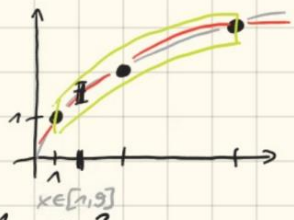
Recall Error formula $|f(x)-L_n(x)| \le \frac{M_{n+1}}{(n+1)!}\cdot|\omega_n(x)|$ $\left(\frac{M_n}{n!}|\omega_n|\right)$

with $\omega_n(x)=(x-x_0)\cdot(x-x_1)\cdots(x-x_n)=\prod_{i=0}^{n}(x-x_i)$ and $M_{n+1}=\max_{x\in[a,b]}|f^{(n+1)}(x)|$

natural approx.

Problem 1 Estimate error • $L_2(2)\approx f(2)=\sqrt{2}$ • $x\in[1,9]$

$|f(x)-L_2(x)| \le \frac{M_3}{3!}\cdot|\omega_2(x)|$, $3!=6$



$M_3=\max_{x\in[1,9]}|f'''(x)|$, $f(x)=\sqrt{x}=x^{1/2}$, $f'(x)=\frac{1}{2}x^{-\frac{1}{2}}$

$f''(x)=-\frac{1}{4}x^{-\frac{3}{2}}$, $f'''(x)=\frac{3}{8}x^{-\frac{5}{2}}$, $|f'''(x)|=\frac{3}{8}\cdot\frac{1}{\sqrt{x^5}} \underset{x=1}{\le} \frac{3}{8}=M_3$

$\omega_2(x)=(x-1)(x-4)(x-9)$, $\omega_2(2)=(2-1)(2-4)(2-9)=14$

(c) at $x=2$: $|f(2)-L_2(2)| \le \frac{M_3}{3!}\cdot|\omega_2(2)| = \frac{3}{8}\cdot\frac{1}{6}\cdot 14 = \frac{14}{16}=\frac{7}{8}$

6. Newton and convergence

-Newton's

Newton's method $x_{i+1}=x_i-\frac{f(x_i)}{f'(x_i)}$

find x, make $f(x)>0$ and <0

1. $f(x)=\sin x+x^2-1$, $f(0)=-1<0$, $f(\frac{\pi}{2})=(\frac{\pi}{2})^2>0$, therefore

$\therefore$ f has root in $[0,\frac{\pi}{2}]$, furthermore $f\in C^2[0,\frac{\pi}{2}]$ (2 points)

(a) $f'(x)=\cos x+2x>0$ for all $x\in[0,\frac{\pi}{2}]$

Newton's method

$$x_{n+1}=x_n-\frac{\sin x_n+x_n^2-1}{\cos x_n+2x_n}$$

don't change

(2 points)

$f\nearrow$ or $\searrow$ **(b)** $f''(x)=-\sin(x)+2>0$ for all $x\in[0,\frac{\pi}{2}]$, therefore f' and f'' do not change sign.

The conditions of the Monotone convergence theorem are fullfilled, so if $f(x_0)>0$ for $x_0\in[0,\frac{\pi}{2}]$, the iteration is convergent. As $f\nearrow$, the iteration is convergent for all $x_0\in[x^*,\frac{\pi}{2}]$, where x^* is the limit of the iteration. (3 points)

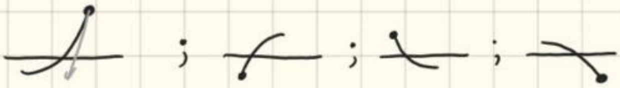
(c) For example $x_0=1$, $x_0=\frac{\pi}{2}$ are appropriate. (1 point)

https://www.desmos.com/calculator?lang=zh-CN

Recall convergence theorems for Newton's method

Monotone convergence theorem $f \in C^2[a,b]$, ① $\exists$ root in $[a,b]$,

② f' and f'' have constant signs, ③ $x_0 \in [a,b] : f(x_0) \cdot f''(x_0) > 0$ $\Rightarrow$ conv.

E.g., $f' > 0$, $f'' > 0$ 

Local convergence theorem $f \in C^2[a,b]$, ① $\exists$ root x^* in $[a,b]$,

② $f'(x) \neq 0$ ($x \in [a,b]$), ③ $x_0 \in [a,b] : |x_0 - x^*| < r := \min\left\{\frac{1}{M}, |x^* - a|, |x^* - b|\right\}$

with $M = \frac{M_2}{2m_1}$, $m_1 = \min_{x \in [a,b]} |f'(x)|$, $M_2 = \max_{x \in [a,b]} |f''(x)|$ $\Rightarrow$

quadratic convergence! to x^*, $|x_{\ell+1} - x^*| \leq M \cdot |x_\ell - x^*|^2$