Analysis II ; Practice 7-8.

INDEFINITE INTEGRALS

ANTIDERIVATES

① Review: Let $I \neq \emptyset$, $I \subseteq \mathbb{R}$ be an open interval and $f: I \to \mathbb{R}$. We say that f has an antiderivate or a primitive function if there exists a function $F: I \to \mathbb{R}$, $F \in D(I)$ and $F'(x) = f(x)$ $(\forall x \in I)$.

ii) If F is an antiderivate of f

Then $\forall c \in \mathbb{R}$ $(F+c)' = F' + 0 =$
$= f$ so $F+c$ is also an ant.
der.

iii) if F_1, F_2 are two antideri-
vates of f (in the interval
I) $\Rightarrow$ $\exists c \in \mathbb{R}$ so that

$$F_2 = F_1 + c$$

iv) $\int f(x)\,dx := \{ F : I \to \mathbb{R} \mid F \in D$
and $F' = f \} = F(x) + c$
$(x \in I,\ c \in \mathbb{R})$
the set of all antiderivates
of f, or the so called

indefinite integral of f.

<u>Basic methods</u> to find integrals:

$\boxed{\text{I}}$ $\boxed{\text{Basic integrals}}$

① $\int (6x^2 - 8x + 3)\, dx =$

(the integral is linear:

$$\int (f(x) + g(x))\, dx = \int f(x)\, dx + \int g(x)\, dx \text{ ; } \underline{\text{and}}$$

If $\lambda \in \mathbb{R} \Rightarrow \int \lambda \cdot f(x)\, dx = \lambda \cdot \int f(x)\, dx$)

$$= 6 \cdot \int x^2\, dx - 8 \cdot \int x\, dx + 3 \cdot \int 1\, dx$$

$$= 6 \cdot \frac{x^3}{3} - 8 \cdot \frac{x^2}{2} + 3 \cdot x + c =$$

$$= \underbrace{2x^3 - 4x^2 + 3x + c}\quad (x \in \mathbb{R},\ c \in \mathbb{R})$$

Here we used the power rule, so:

$$\int x^\alpha dx = \frac{x^{\alpha+1}}{\alpha+1} + c$$

if $\alpha \in \mathbb{R} \setminus \{-1\}$; $x > 0$

(for special exponents $x \in \mathbb{R}$ can be)

② $$\int \left(2x + \frac{5}{\sqrt{1-x^2}}\right) dx =$$

$x \in (-1, 1) =: \underline{I}$

$$= 2\int x\,dx + 5\int \frac{1}{\sqrt{1-x^2}}\,dx =$$

$$= 2\cdot\frac{x^2}{2} + 5\cdot \arcsin x + c$$

$$= x^2 + 5\cdot \arcsin x + c$$

$(x \in (-1,1) =: I \;,\; c \in \mathbb{R})$

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③ $$\int \frac{x^2}{x^2+1}\,dx = \int \frac{x^2+1-1}{x^2+1}\,dx$$

$(x \in I := \mathbb{R})$

$$= \int \left(1 - \frac{1}{x^2+1}\right)dx =$$

decomposition into sums" =

"

$$= \int 1\,dx - \int \frac{1}{1+x^2}\,dx =$$

$$= \underbrace{x - \arctan x + C}_{(C \in \mathbb{R}\,;\ x \in \mathbb{R})}$$

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d) $$\int \frac{\cos^2 x - 5}{1+\cos 2x}\,dx =$$

$$\left(x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) =: I\right)$$

$$= \int \frac{\cos^2 x - 5}{\underbrace{\sin^2 x + \cos^2 x}_{1} + \underbrace{\cos^2 x - \sin^2 x}_{\cos 2x}}\,dx$$

$$= \int \frac{\cos^2 x - 5}{2\cos^2 x}\,dx =$$

$$= \frac{1}{2}\int\left(1 - \frac{5}{\cos^2 x}\right)dx =$$

$$= \frac{1}{2}x - \frac{5}{2}\int \frac{1}{\cos^2 x}\,dx =$$

$$= \frac{1}{2}x - \frac{5}{2}\tan x + c \qquad c \in \mathbb{R}$$

e) $$\int \frac{1}{\sqrt{x}}\,dx = \int x^{-\frac{1}{2}}\,dx =$$

$(x > 0)$

= power rule =

$$= \frac{x^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} + c = \frac{\sqrt{x}}{1/2} + c$$

$$= \underline{2\sqrt{x} + C}, \quad (C \in \mathbb{R}, x > 0)$$

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f) $\int x \cdot \sqrt{x}\, dx = \int x \cdot x^{\frac{1}{2}}\, dx$

$(x > 0)$ $= \int x^{3/2}\, dx =$ p.r.

$$= \frac{x^{3/2+1}}{3/2+1} + C = \frac{x^{5/2}}{5/2} + C =$$

$$= \frac{2}{5} \cdot \sqrt{x^5} + C^{\in \mathbb{R}} \quad (x > 0)$$

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g) $\int \frac{(x+1)^2}{\sqrt{x}}\, dx =$ decompose it + power rule $=$

$(x > 0)$

$$= \int \frac{x^2+2x+1}{\sqrt{x}}\,dx =$$

$$= \int \frac{x^2}{x^{1/2}}\,dx + 2\int \frac{x}{x^{1/2}}\,dx +$$

$$+ \int \frac{1}{x^{1/2}}\,dx =$$

$$= \int x^{2-\frac{1}{2}}\,dx + 2\cdot\int x^{1-\frac{1}{2}}\,dx +$$

$$+ \int x^{-1/2}\,dx =$$

$$= \int x^{3/2}\,dx + 2\int x^{1/2}\,dx$$

$$+ 2\sqrt{x} + C$$

$$= \frac{x^{5/2}}{5/2} + 2\cdot\frac{x^{3/2}}{3/2} + 2\sqrt{x}$$

$$+ C =$$

$$= \underbrace{\frac{2}{5}\cdot\sqrt{x^5} + \frac{4}{3}\cdot\sqrt{x^3} + 2\sqrt{x}}$$

$$\underbrace{+ C \in \mathbb{R}} \quad \left(x \in I := (0,\infty)\right)$$

h) $\int \frac{1}{x}\,dx = \underbrace{\ln(x)} + \underset{\mathbb{R}}{\underset{\cap}{C}}$

$(x > 0)$

i) $\int \frac{1}{x}\,dx = \underbrace{\ln(-x)} + \underset{\mathbb{R}}{\underset{\cap}{C}}$

$(x < 0)$

II Linear substitution

__Review__: Suppose that $\int f \neq \emptyset$, $a, b \in \mathbb{R}$, $a \neq 0 \Rightarrow$

$$\int f(ax+b)\,dx = \frac{F(ax+b)}{a} + c$$

with any $F \in \int f$.

1° Find the integrals using the rule above:

a) $$\int (3x+2)^4\,dx = \frac{(3x+2)^5}{5 \cdot 3} + c$$

$$= \frac{1}{15} \cdot \underbrace{(3x+2)^5 + c}\ , c \in \mathbb{R}\ (x \in \mathbb{R})$$

b) $\int \frac{1}{\sqrt{1-2x^2}}\,dx =$

$x \in \left(-1/\sqrt{2}, 1/\sqrt{2}\right)$

$= \int \frac{1}{\sqrt{1-(\sqrt{2}x)^2}}\,dx =$

$a = \sqrt{2}$
$b = 0$

$= \frac{\arcsin(\sqrt{2}x)}{\sqrt{2}} + c^{\in\mathbb{R}}$

c) $\int \frac{2}{3+2x^2}\,dx =$

$\underline{(x \in \mathbb{R})}$

$\left(\int f(x)\,dx = \int \frac{1}{1+x^2}\,dx = \arctan x + c\right)$

$$= \frac{2}{3} \cdot \int \frac{1}{1+\frac{2}{3}x^2}\, dx =$$

$$= \frac{2}{3} \cdot \int \frac{1}{1+\left(\sqrt{\frac{2}{3}}x\right)^2}\, dx =$$

$$= \frac{2}{3} \cdot \frac{\arctan\left(\sqrt{\frac{2}{3}}x\right)}{\sqrt{\frac{2}{3}}} + \underset{\mathbb{R}}{\overset{\in}{C}}$$

d) $\displaystyle\int \frac{1}{4x^2-4x+2}\, dx =$

$(x \in \mathbb{R})$ $\displaystyle = \int \frac{1}{1+(2x-1)^2}\, dx$

$$= \frac{\arctan(2x-1)}{2} + C \in \mathbb{R}$$

e) $\int \sin^2 x \, dx =$ linearizáción

$$= \int \frac{1-\cos 2x}{2} dx =$$

$$= \frac{x}{2} - \frac{1}{2} \int \cos 2x \, dx =$$

$$= \frac{x}{2} - \frac{1}{2} \cdot \frac{\sin 2x}{2} + c$$

$$= \frac{x}{2} - \frac{1}{4} \sin 2x + c \in \mathbb{R}$$

f) $\int 5^{3-2x} dx = \frac{5^{3-2x}}{\ln 5 \cdot (-2)}$ $+ c \in \mathbb{R}$

$(x \in \mathbb{R})$

$\boxed{\text{III}}$ Integrals of form:

$\alpha \neq -1$: $\int f'(x) \cdot f^{\alpha}(x)\, dx$ or

$\alpha = -1$: $\int \frac{f'(x)}{f(x)}\, dx$

It's easy to check that:

If $\alpha \in \mathbb{R} \setminus \{-1\}$:

$$\int f'(x) \cdot f^{\alpha}(x)\, dx = \frac{f^{\alpha+1}(x)}{\alpha+1} + c$$

$(c \in \mathbb{R}\,;\, x \in I)$

If $\alpha = -1 \Rightarrow$

$$\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)| + c$$

$(x \in I, c \in \mathbb{R})$

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① Evaluate the integrals of this type:

a) $\int x^2 (2x^3+4)^{2020}\,dx =$

$(x \in \mathbb{R})$, $\alpha = 2020$

$f(x) = 2x^3+4$

$\Rightarrow f'(x) = 6x^2$

$= \frac{1}{6} \cdot \int (2x^3+4)' \cdot (2x^3+4)^{2020}\,dx$

$$= \frac{1}{6} \cdot \frac{(2x^3+4)^{2021}}{2021} + C^{\in\mathbb{R}}$$

b) $$\int \frac{x^2}{\sqrt{6x^3+4}}\,dx =$$

$(x>1)$

$$= \int x^2 \cdot \Big(\underbrace{6x^3+4}_{= f(x)}\Big)^{\overbrace{-\frac{1}{2}}^{= \alpha}} dx =$$

$$\Rightarrow f'(x) = (6x^3+4)' = 1\ x^2$$

$$= \frac{1}{18} \int 18x^2 \cdot (6x^3+4)^{-1/2}\, dx$$

$$= \frac{1}{18} \int (6x^3+4)^{-1/2} \cdot (6x^3+4)' \, dx$$

$$= \frac{1}{18} \cdot \frac{(6x^3+4)^{-1/2+1}}{-1/2+1} + C =$$

$$= \frac{1}{18} \cdot 2 \cdot \sqrt{6x^3+4} + \overset{\in \mathbb{R}}{C}$$

c) $\int e^x (1-e^x)^{20} \, dx = (*)$

$x \in \mathbb{R}$ $\Rightarrow \alpha = 20$;

$f(x) := 1 - e^x \Rightarrow$

$f'(x) = (1-e^x)' = -e^x$

$$(*) = -\int (1-e^x)' (1-e^x)^{20} \, dx =$$

$$= -\frac{(1-e^x)^{21}}{21} + c \quad c \in \mathbb{R}$$

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d) $\int x \cdot \sqrt{3x-1} \, dx =$

$(x > 1/3)$ let's use substitution:

$$\sqrt{3x-1} = t > 0$$

$$\Rightarrow 3x-1 = t^2 \overset{(1)'}{\Rightarrow}$$

$$x = \frac{t^2+1}{3} \Rightarrow$$

$$(x)'\,dx = \left(\frac{t^2+1}{3}\right)'dt$$

So: $\boxed{dx = \frac{2}{3}t\,dt}$

$$\Rightarrow \int x\sqrt{3x-1}\,dx = .$$

$$= \int \frac{t^2+1}{3}\cdot t\cdot \frac{2}{3}t\,dt$$

$t = \sqrt{3x-1}$

So the new integral:

$$= \frac{2}{9} \int (t^4 + t^2)\, dt =$$

$$= \frac{2}{9} \cdot \frac{t^5}{5} + \frac{2}{9} \cdot \frac{t^3}{3} + c =$$

$$= \frac{2}{45} t^5 + \frac{2}{27} \cdot t^3 + c$$

$\boxed{\text{So}}$ "back to x"

$$\int x\sqrt{3x-1}\, dx = \frac{2}{15}\left(\sqrt{3x-1}\right)^5 + \frac{2}{27} \cdot \left(\sqrt{3x-1}\right)^3 + c \in \mathbb{R}.$$

e) Integrals of type

$$\int \sin^n x \cdot \cos^m x \, dx$$

$(n, m \in \mathbb{Z})$ (can be extended to other powers as well)

Case 1: If one of n or m is even (or both of them) $\Rightarrow$ then we choose the smallest one) then we make the substitution

$t = \cos x$ if n is even

and

$t = \sin x$ if m is even.

Case 2: If n, m both are odd $\Rightarrow$ Linearization in many steps if needed.

Formulas for Linearization:

$$\boxed{\sin^2 x = \frac{1-\cos 2x}{2} \qquad (x\in\mathbb{R}) \qquad \cos^2 x = \frac{1+\cos 2x}{2}}$$

(i) $\int \sin^3 x \, dx =$

$(x\in\mathbb{R})$; $n=3$, $m=0$

$= \int \sin^2 x \cdot \underbrace{\sin x}_{} \, dx$

<u>idea</u> $= -(\cos x)'$

$$\cos x = t \Rightarrow ()'$$

$$-\sin x \, dx = 1 \, dt$$

So $\sin x \, dx = -dt$

$$\int \sin^2 x \cdot \sin x \, dx =$$

$$= \int (1 - \cos^2 x) \sin x \, dx =$$

$$= \int (1 - t^2)(-dt) \Big|^{t=\cos x}$$

$$= \frac{t^3}{3} - t + C \Big|_{t=\cos x} =$$

$$= \frac{\cos^3 x}{3} - \cos x + C \in \mathbb{R}$$

ii) $\int \sin^5 x \cdot \cos^3 x \, dx =$

$n = 5 > m = 3$

$$= \int \sin^5 x \cdot \underbrace{\cos^2 x}_{(1-\sin^2 t)} \cdot \underbrace{\cos x \, dx}_{= (\sin x)'}$$

So $t = \sin x \Rightarrow$

$dt = \cos x \, dx \Rightarrow$

$$\int t^5 \cdot (1-t^2)\,dt \Bigg|\, t = \sin x$$

The new integral:

$$\int (t^5 - t^7)\,dt = \frac{t^6}{6} - \frac{t^8}{8} + c$$

$\Rightarrow$ „Back to x"

$$\int \sin^5 x \cos^3 x\,dx = \frac{\sin^6 x}{6} - \frac{\sin^8 x}{8} + c, \quad (x, c \in \mathbb{R})$$

iii) $$\int \sin^2 x \cdot \cos^4 x \, dx =$$

$$= \int \underbrace{\sin^2 x \, \cos^2 x}_{} \cdot \cos^2 x \, dx$$

$$= \frac{1}{4} \int (\sin 2x)^2 \cdot \frac{1 + \cos 2x}{2} \, dx$$

$$= \frac{1}{8} \int \sin^2 2x \, dx +$$

$$+ \frac{1}{8} \int (\sin 2x)^2 \cos 2x \, dx$$

= Linearisation =

$$= \frac{1}{8} \int \frac{1-\cos 4x}{2} dx +$$

$$+ \frac{1}{16} \int \underbrace{(\sin 2x)'}_{f'} \cdot \underbrace{(\sin 2x)^2}_{f^\alpha} dt$$

$$= \frac{1}{16} x - \frac{\sin 4x}{64} +$$

$$+ \frac{1}{16} \frac{\sin^3 2x}{3} + C =$$

$$= \frac{1}{16} x - \frac{\sin 4x}{64} + \frac{1}{48} \sin^3 2x$$

$$+ C, \quad (x, C \in \mathbb{R})$$

f) $\int \frac{x}{x^2+3}\,dx = \frac{1}{2} \cdot \int \frac{2x}{x^2+3}\,dx =$

$(x \in \mathbb{R})$ $\quad (x^2+3)' = 2x$

$= \frac{1}{2} \int \frac{(x^2+3)'}{x^2+3}\,dx =$

$= \frac{1}{2} \ln |\underbrace{x^2+3}_{+}| + c =$

$= \frac{1}{2} \ln(x^2+3) + c \in \mathbb{R}$

g) $\int \frac{1}{x \ln x}\,dx =$

$0 < x < 1$ or $x > 1$

$$= \int \frac{1}{x} \cdot (\ln x)^{-1} dx =$$

$$= \int \frac{(\ln x)'}{\ln x} dx =$$

$$= \ln |\ln x| + c =$$

$$= \begin{cases} \ln(\ln x) + c_1 \text{ if } x > 1 \\ \ln(-\ln x) + c_1 \text{ if } 0 < x < 1 \end{cases}$$

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h) $\int \sin^2 x \cos^3 x \, dx =$

$$= \int \sin^2 x \cdot \cos^2 x \cdot \cos x \, dx$$

$$\overset{\text{I}}{=} \int \underbrace{t^2(1-t^2)}_{t^2-t^4}\,dt \Bigg|_{t=\sin x} =$$

$$\sin x =: t \quad \Big)\,()'$$
$$\cos x\,dx = dt$$

$$= \frac{t^3}{3} - \frac{t^5}{5} + c \Bigg|_{t=\sin x} =$$

$$= \frac{\sin^3 x}{3} - \frac{\sin^5 x}{5} + c \quad c \in \mathbb{R}$$

v) $\displaystyle\int \frac{\arctan^2 x}{1+x^2}\,dx =$

$(x \in \mathbb{R})$

$$= \int \underbrace{(\arctan x)'}_{f'} \underbrace{(\arctan x)^2}_{f^\alpha} dx$$

$$= \frac{(\arctan x)^3}{3} + C \in \mathbb{R}$$

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g) $$\int \frac{1}{(x^2+1)\cdot \arctan x}\, dx =$$

$(x \in (-\infty, 0))$

$$= \int \frac{(\arctan x)'}{\arctan x}\, dx =$$

$$= \ln|\overbrace{\arctan x}^{-}| + C =$$

$(x<0)$ $\qquad =$

$= \ln(-\arctan x) + C \in \mathbb{R}$ i

IV Integrals of rational functions

Just cases when we can decompose them in basic fractions like

$$\int \frac{1}{(ax+b)^n}\,dx \quad (1 \le n \in \mathbb{N})$$

(1) Find the integrals:

a) $\boxed{\displaystyle\int \frac{1}{x^2+2x-8}\,dx \qquad x \in (-4, 2)}$

$$\frac{1}{x^2+2x-8} = \frac{1}{(x-2)(x+4)} =$$

= partial decomposition =

$$= \frac{A}{x-2} + \frac{B}{x+4}, \quad A, B = ? \in \mathbb{R}$$

$$\Rightarrow \quad 1 = A(x+4) + B(x-2) \quad (*)$$

$$1 = (A+B)x + 4A - 2B$$

So equal coefficients:

$$\begin{cases} A+B=0 \quad |\cdot 2 : & x^1 \\ 4A-2B=1 & x^0 \end{cases}$$

$\Rightarrow 6A = 1 \quad \boxed{A=\frac{1}{6}}$ and $\boxed{B=-\frac{1}{6}}$

Remark: the same if in (*) we substitute special x values:

$$1 = A(x+4) + B(x-2)$$

$x=2 \Rightarrow \quad 1 = 6A \Rightarrow A=\frac{1}{6}$

$x=-4 \Rightarrow \quad 1 = -6B \Rightarrow B=-\frac{1}{6}$

So: $\displaystyle\int \frac{\frac{1}{6}}{x-2}\,dx + \int \frac{-\frac{1}{6}}{x+4}\,dx$

$$= \frac{1}{6} \int \frac{1}{x-2}\,dx - \frac{1}{6}\int \frac{1}{x+4}\,dx$$

$$= \frac{1}{6} \ln|x-2| - \frac{1}{6} \ln|x+4|$$

$$+ c = (-4 < x < 2) =$$

$$= \frac{1}{6}\left[\ln(2-x) - \ln(x+4)\right] + c$$

$$= \frac{1}{6} \ln \frac{2-x}{x+4} + c \in \mathbb{R}$$

b) $$\int \frac{4x^2+13x-9}{x^3+2x^2-3x}\,dx$$
$(x \in (0,1))$

$$\frac{4x^2+13x-9}{x(x^2+2x-3)} = \frac{4x^2+13x-9}{x(x+3)(x-1)}$$

$$= \frac{A}{x} + \frac{B}{x+3} + \frac{C}{x-1} \quad \Big| \cdot (x+3) \cdot (x-1) \cdot x$$

$$4x^2+13x-9 = A(x+3)(x-1) + B\,x(x-1) + C\cdot x(x+3)$$

If $x=0 \Rightarrow -9=-9A \Rightarrow$

$$\boxed{A=1}$$

If $x=-3 \Rightarrow$

$36-39-9=12B$

$-12=12B \Rightarrow \boxed{B=-1}$

If $x=1 \Rightarrow$

$8=4C \Rightarrow \boxed{C=2}$

So :

$$\int\left(\frac{1}{x}-\frac{1}{x+3}+\frac{2}{x-1}\right)dx$$

$= \ln|x| - \ln|x+3| +$

$+ 2\ln|x-1| + C =$

$= \ln x - \ln(x+3) +$

$+ 2\ln(1-x) + C$

$0 < x < 1$

$= \ln \dfrac{x}{x+3} + 2\ln(1-x) +$

$+ C \in \mathbb{R}.$

$\boxed{c)} \displaystyle\int \frac{1}{x^3+2x^2+x}\,dx$

$(x < -1)$

$$= \int \frac{1}{x\,(x^2+2x+1)}\,dx =$$

$$= \int \frac{1}{x\cdot(x+1)^2}\,dx =$$

$$= \int \left(\frac{A}{x} + \frac{B}{x+1} + \frac{C}{(x+1)^2} \right) dx$$

So here:

$$1 = A(x+1)^2 + B\cdot x(x+1) +$$

$$Cx$$

$$1 = (A+B)\cdot x^2 + (2A+B+C)\,x$$

$$+ A$$

So we make the coefficients equal:

- for x^2: $A+B=0$
- for x^1: $2A+B+C=0$
- for x^0: $\boxed{A=1}$

$\Rightarrow \boxed{B=-1} \Rightarrow C=-2+B$

$\boxed{C=-1}$

So the integral =

$$= \int \left(\frac{1}{x} - \frac{1}{x+1} - \frac{1}{(x+1)^2} \right) dx =$$

$(x+1)^{-2}$

$$= \ln|x| - \ln|x+1| + \frac{1}{x+1} + c$$

$$= (x<-1) = \ln(-x) -$$

$$- \ln(-x-1) + \frac{1}{x+1} + c =$$

$$= \ln\frac{-x}{-x-1} + \frac{1}{x+1} + c =$$

$$= \ln\frac{x}{x+1} + \frac{1}{x+1} + \overset{\mathbb{R}}{\overset{\in}{C}}.$$

$$\boxed{d)\ \int \frac{x^5+2x^3-x+1}{x^4-2x^2+1}\,dx \quad (x>1)}$$

$\int \frac{P(x)}{Q(x)}\,dx$, if $\deg P \geq \deg Q$

we devide first.

So:

$(x^5 + 2x^3 - x + 1) : (x^4 - 2x^2 + 1) = x$

$x^5 - 2x^3 + x$ $\ominus$

$/ \; 4x^3 - 2x + 1$ CANNOT GO ON

$$\Rightarrow \int \left(x + \frac{4x^3 - 2x + 1}{x^4 - 2x^2 + 1} \right) dx =$$

$= \frac{x^2}{2} + I(x)$, where

$$I(x) = \int \frac{4x^3 - 2x + 1}{(x^2-1)^2}\, dx =$$

$$= \int \frac{4x^3 - 2x + 1}{(x-1)^2 \cdot (x+1)^2}\, dx =$$

$$= \int \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x+1} + \frac{D}{(x+1)^2}\, dx$$

$$\Rightarrow 4x^3 - 2x + 1 = A(x-1)(x+1)^2 +$$

$$+ B(x+1)^2 + C(x+1)(x-1)^2 +$$

$$+ D(x-1)^2$$

$$\Rightarrow$$

If $x=1 \Rightarrow 3=4B \Rightarrow \boxed{B=\frac{3}{4}}$

If $x=-1 \Rightarrow -1=4D \Rightarrow \boxed{D=-\frac{1}{4}}$

If $x=0 \Rightarrow 1=-A+B+C+D$

$\Rightarrow 1=-A+\frac{3}{4}+C-\frac{1}{4}$

$1=-A+C+\frac{1}{2} \Rightarrow$

$\boxed{-A+C=\frac{1}{2}} \quad \ldots (1)$

If $x=2 \Rightarrow 29=9A+9B+$
$3C+D$

$$\Rightarrow 9A + \frac{27}{4} + 3C - \frac{1}{4} = 29$$

$$9A + 3C + \frac{26}{4} = 29$$

$$9A + 3C = 29 - \frac{26}{4} =$$

$$= \frac{116 - 26}{4} = \frac{90}{4}$$

$$\begin{cases} 3A + C = \dfrac{30}{4} \\ -A + C = \dfrac{1}{2} \end{cases} \oplus \quad \Big| \cdot 3$$

$$4C = \frac{42}{4}$$

$$\boxed{C = \frac{21}{8}}$$

$$A = C - \frac{1}{2} = \frac{21}{8} - \frac{4}{8} = \frac{17}{8}.$$

So:

$$\frac{17}{8} \cdot \int \frac{1}{x-1}\,dx + \frac{3}{4} \int \frac{1}{(x-1)^2}\,dx$$

$$+ \frac{21}{8} \int \frac{1}{x+1}\,dx - \frac{1}{4} \int \frac{1}{(x+1)^2}\,dx$$

$$= \frac{17}{8} \ln|\overbrace{x-1}^{+}| - \frac{3}{4} \cdot \frac{1}{x-1}$$

$$+ \frac{21}{8} \ln|\overbrace{x+1}^{+}| + \frac{1}{4} \frac{1}{x+1} + C$$

$$= \frac{17}{8}\ln(x-1) + \frac{21}{8}\ln(x+1) -$$

$$-\frac{3}{4}\cdot\frac{1}{x-1} + \frac{1}{4}\cdot\frac{1}{x+1} + c$$

$(c \in \mathbb{R},\ x > 1)$

THE END.