

1. best straight best parabola type problems:

4.

x_i	-3	-2	-1	1	2	3
y_i	1	0	3	0	-2	0

(a) The linear system for the coefficients of the polynomial $P_1(x) = a_1x + a_0$ is

$$\left[\begin{array}{cc|c} N & \sum x_i & \sum y_i \\ \sum x_i & \sum x_i^2 & \sum x_i y_i \end{array} \right] = \left[\begin{array}{cc|c} 6 & 0 & 2 \\ 0 & 28 & -10 \end{array} \right]$$

The solution is $a_0 = \frac{1}{3}$ and $a_1 = -\frac{5}{14}$, so the best straight line fitting to the points is

$$P_1(x) = -\frac{5}{14}x + \frac{1}{3}.$$

(3 points)

(b) The linear system for the coefficients of the polynomial $P_2(x) = a_2x^2 + a_1x + a_0$ is

$$\left[\begin{array}{ccc|c} N & \sum x_i & \sum x_i^2 & \sum y_i \\ \sum x_i & \sum x_i^2 & \sum x_i^3 & \sum x_i y_i \\ \sum x_i^2 & \sum x_i^3 & \sum x_i^4 & \sum x_i^2 y_i \end{array} \right] = \left[\begin{array}{ccc|c} 6 & 0 & 28 & 2 \\ 0 & 28 & 0 & -10 \\ 28 & 0 & 196 & 4 \end{array} \right]$$

From the second equation $a_1 = -\frac{5}{14}$.

For a_0 and a_2 we have we have the system

$$\left[\begin{array}{cc|c} 3 & 14 & 1 \\ 7 & 49 & 1 \end{array} \right] \sim \left[\begin{array}{cc|c} 3 & 14 & 1 \\ 1 & 21 & -1 \end{array} \right] \sim \left[\begin{array}{cc|c} 0 & -49 & 4 \\ 1 & 21 & -1 \end{array} \right]$$

It follows

$$a_2 = -\frac{4}{49}, \quad a_0 = -1 + \frac{84}{49} = \frac{35}{49} = \frac{5}{7},$$

so the best parabola fitting to the points is


$$P_2(x) = -\frac{4}{49}x^2 - \frac{5}{14}x + \frac{5}{7}.$$

(6 points)

2. trapezoidal Simpson's formulae type question:

-Approximate integral

Recall Quadrature formulas $I(f) \approx Q(f)$




$$\int_a^b f(x) dx \approx \sum_{\ell=0}^n A_{\ell} \cdot f(x_{\ell}) = A_0 \cdot f(x_0) + A_1 \cdot f(x_1) + \dots + A_n \cdot f(x_n)$$

Midpoint formula $M(f) = (b-a) \cdot f\left(\frac{a+b}{2}\right)$

Trapezoid formula $T(f) = \frac{b-a}{2} \cdot (f(a) + f(b))$

Simpson's formula $S(f) = \frac{b-a}{6} \cdot (f(a) + 4 \cdot f\left(\frac{a+b}{2}\right) + f(b))$



↑ all 3 of Newton-Cotes type: uniform division of $[a, b]$ N-C

-estimate error bound:

Recall Error formulas $|I(f) - M_m(f)| \leq \frac{M_2 \cdot (b-a)^3}{24 \cdot m^2}$ $M_2 = \|f''\|_{\infty}$

$$|I(f) - T_m(f)| \leq \frac{M_2 \cdot (b-a)^3}{12 \cdot m^2}, \quad |I(f) - S_m(f)| \leq \frac{M_4 \cdot (b-a)^5}{180 \cdot m^4}$$

Remark: $m \rightarrow \infty \Rightarrow Q_m(f) \rightarrow I(f)$ ($f \in C^4$), M_1, T_1, S_2 : the formula

$f(x) = \frac{1}{x} = x^{-1}$, $f'(x) = -x^{-2}$, $f''(x) = 2 \cdot x^{-3} = \frac{2}{x^3}$, $f^{(3)}(x) = -6 \cdot x^{-4}$, $f^{(4)}(x) = 24 \cdot x^{-5} = \frac{24}{x^5}$

$M_2 = \max_{x \in [1, 2]} |f''(x)| = \max_{x \in [1, 2]} \left| \frac{2}{x^3} \right| \stackrel{x=1}{=} 2$, $M_4 = \max_{x \in [1, 2]} |f^{(4)}(x)| = \max_{x \in [1, 2]} \left| \frac{24}{x^5} \right| \stackrel{x=1}{=} 24$

① $|I(f) - M_1(f)| \leq \frac{M_2 \cdot (b-a)^3}{24 \cdot m^2} = \frac{2 \cdot (2-1)^3}{24 \cdot 1^2} = \frac{2 \cdot 1^3}{24} = \frac{1}{12} \Rightarrow \rho_{1,2} \in \left[\frac{2}{3} - \frac{1}{12}, \frac{2}{3} + \frac{1}{12} \right]$

② $|I(f) - T_1(f)| \leq \frac{M_2 \cdot (b-a)^3}{12 \cdot m^2} = \frac{2 \cdot (2-1)^3}{12 \cdot 1^2} = \frac{2 \cdot 1^3}{12} = \frac{1}{6}$

③ $|I(f) - S_2(f)| \leq \frac{M_4 \cdot (b-a)^5}{180 \cdot m^4} = \frac{24 \cdot 1^5}{180 \cdot 2^4} = \frac{24}{180 \cdot 16} = \frac{1}{120} \Rightarrow \int_1^2 \frac{1}{x} dx = \rho_{1,2} \approx \frac{25}{26} \pm \frac{1}{120}$

$$f(x) = e^{-x^2}, \quad f'(x) = -2xe^{-x^2}, \quad f''(x) = e^{-x^2}(4x^2 - 2)$$

For the error bound we need $M_2 = \max_{[0,1]} |f''(x)|$. Therefore we calculate the third derivative, as well.

$$\begin{aligned} f'''(x) &= e^{-x^2}(-2x)(4x^2 - 2) + 8xe^{-x^2} \\ &= 4xe^{-x^2}(-2x^2 + 3) = 0 \iff x = 0, \text{ or } x = \pm\sqrt{\frac{3}{2}} \end{aligned}$$

$x = 0$ is an endpoint of the interval, the other values are not in $[0, 1]$.

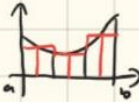

As $f'''(x) > 0$ for $x \in (0, 1]$, we have $f'' \searrow$ and therefore $M_2 = |f''(0)| = 2$.

The error bound is

$$|I - T(f)| \leq \frac{1}{12} M_2 = \frac{1}{6}.$$

-trapezoidal rule with n:

Recall Quadrature rules (composite formulas) composite rules

Midpoint rule	$M_m(f) = \frac{b-a}{m} (f(x_1) + \dots + f(x_m))$	
Trapezoid rule	$T_m(f) = \frac{b-a}{2m} (1 \cdot f(a) + 2 \cdot f(x_1) + \dots + 2 \cdot f(x_{m-1}) + 1 \cdot f(b))$	
Simpson's rule	\checkmark m even, $\frac{m}{2}$ formulas	

(b) The composite trapezoidal rule with $n=2$

$$\frac{1}{4} \left[f(0) + 2f\left(\frac{1}{2}\right) + f(1) \right] = \frac{1}{4} \left[1 + 2e^{-\frac{1}{4}} + e^{-1} \right] = \frac{1}{2} \cdot 0,684 + \frac{1}{2}e^{-\frac{1}{4}} = 0,732.$$

(2 points)

https://www.symbolab.com/solver/trapezoidal-approximation-calculator/trapezoidal%20%5Cint_%7B0%7D%5E%7B1%7D%20e%5E%7B-x%5E2%7Ddx%2Cn%3D2?or=input

-integral within the error:

(c) Error bound for the composite trapezoidal rule:

$$\frac{1}{12} \cdot \frac{(b-a)^3}{n^2} \cdot M_2 = \frac{1}{6n^2} < 10^{-2} \iff \frac{100}{6} < n^2 \iff 4,08 = \frac{10}{\sqrt{6}} < n$$

So with $\boxed{n=5}$ the error is within 10^{-2} .

(3 points)

3. Horner's scheme

-write the polynomial

5.

$$P(x) = 2x^4 - 3x^2 + 3x - 4. \quad (x+2) \quad a = -2$$

The Horner's scheme:

	2	0	-3	3	-4
-2		-4	8	-10	14
	2	-4	5	-7	10
-2		-4	16	-42	
	2	-8	21	-49	
-2		-4	24		
	2	-12	45		
-2		-4			
	2	-16			
-2					
	2				

Végül

$$P(x) = 10 - 49(x+2) + 45(x+2)^2 - 16(x+2)^3 + 2(x+2)^4$$

-find all root:

Recall root estimates $P(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$

$P(x_2) = 0 \Rightarrow r < |x_2| < R$ with $R = 1 + \frac{\max_{i=0}^{n-1} |a_i|}{|a_n|}$, $r = \left(1 + \frac{\max_{i=1}^n |a_i|}{|a_0|}\right)^{-1}$

complex ($\in \mathbb{C}$) roots may arise

Problem 3 No integer roots for $P(x) = 3x^4 - x^3 - 5x^2 + 4x - 6$.

$R = 1 + \frac{\max\{|-1|, |-5|, |4|, |-6|\}}{|3|} = 1 + \frac{6}{3} = 2$

$r = \left(1 + \frac{\max\{|3|, |-1|, |-5|, |4|\}}{|-6|}\right)^{-1} = \left(1 + \frac{5}{6}\right)^{-1} = \frac{6}{11}$

$\frac{6}{11} < |x_2| < 2$

Possible roots $\in \mathbb{Z}$: -1 and 1, but $P(-1) = -11 \neq 0$, $P(1) = -5 \neq 0$

4. quadrature formula

-weights has the highest possible degree:

3.

$$\int_0^1 f(x) \sqrt{x} dx \approx \underbrace{A_0 f(0)}_1 + \underbrace{A_1 f\left(\frac{1}{2}\right)}_2 + \underbrace{A_2 f(1)}_3$$

The system for the weights is

$$\begin{aligned} 1 \quad \int_0^1 1 \cdot \sqrt{x} dx &= \frac{2}{3} = A_0 + A_1 + A_2 & 0^0=1 & \frac{1}{2}^0=1 & 1^0=1 \\ 2 \quad \int_0^1 x \cdot \sqrt{x} dx &= \frac{2}{5} = \frac{1}{2} A_1 + A_2 & 0^1=0 & \frac{1}{2}^1=\frac{1}{2} & 1^1=1 \\ 3 \quad \int_0^1 x^2 \cdot \sqrt{x} dx &= \frac{2}{7} = \frac{1}{4} A_1 + A_2 & 0^2=0 & \frac{1}{2}^2=\frac{1}{4} & 1^2=1 \end{aligned}$$

(4 points)

From the second and third equations we get $A_1 = \frac{16}{35}$ and $A_2 = \frac{6}{35}$, substituting them into the first equation, we have $A_0 = \frac{4}{105}$. (2 points)

-Find x_0, x_1, c_1 :

(10 points)

6. Find x_0, x_1 and c_1 so that the quadrature formula

$$\int_0^1 f(x) dx \approx \frac{1}{2} f(x_0) + \underline{\quad} f(x_1)$$

has the highest possible degree of precision.

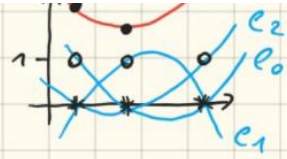
(5 points)

$$\begin{aligned} \int_0^1 1 dx &= 1 = \frac{1}{2} + c_1 & c_1 &= \frac{1}{2} \\ \int_0^1 x dx &= \frac{1}{2} = x_0 \cdot \frac{1}{2} + x_1 \cdot \frac{1}{2} \\ \int_0^1 x^2 dx &= \frac{1}{3} = x_0^2 \cdot \frac{1}{2} + x_1^2 \cdot \frac{1}{2} \end{aligned}$$

5. Lagrange and interpolation

-Lagrange form of the interpolation polynomial

Recall *Lagrange's formula*



$$l_i(x) = \prod_{j=0, j \neq i}^n \frac{(x - x_j)}{(x_i - x_j)}$$

$$l_i(x_i) = \delta_{i,i} = \begin{cases} 1 & i=i \\ 0 & i \neq i \end{cases}$$

$$p(x) = L_n(x) = \sum_{i=0}^n y_i \cdot l_i(x) \quad l_i: \text{Lagrange fundamental / base polynomials}$$

Problem 1.(c) $(x_i; 1, 4, 9)$ Give the Lagrange form

$$l_0(x) = \frac{(x-4)(x-9)}{(1-4)(1-9)} = \frac{1}{24} (x-4)(x-9) = \frac{1}{24} (x^2 - 13x + 36)$$

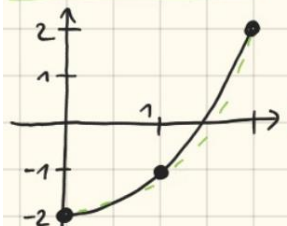
$$l_1(x) = \frac{(x-1)(x-9)}{(4-1)(4-9)} = -\frac{1}{15} (x-1)(x-9) = -\frac{1}{15} (x^2 - 10x + 9)$$

$$l_2(x) = \frac{(x-1)(x-4)}{(9-1)(9-4)} = \frac{1}{40} (x-1)(x-4) = \frac{1}{40} (x^2 - 5x + 4)$$

$$L_2(x) = \overset{y_0}{1} \cdot l_0(x) + \overset{y_1}{2} \cdot l_1(x) + \overset{y_2}{3} \cdot l_2(x) = \dots = \underline{\underline{-\frac{1}{60} (x^2 - 25x - 36)}}$$

-inverse interpolation:

Problem 4



$f(x) = x^2 - 2 = 0 \quad (\sqrt{2})$, $x_0 = 0, x_1 = 1, x_2 = 2$, $x_3 = ?$

$y_i: f^{-1}(y_i)$

-2	0	$\frac{1-0}{-1-(-2)} = 1$	$\frac{2-1}{2-(-2)} = \frac{1}{4} = -\frac{1}{6}$
-1	1	$\frac{2-1}{2-(-1)} = \frac{1}{3}$	
2	2		

$p_2(y) = 0 + 1 \cdot (y+2) - \frac{1}{6} (y+2)(y+1)$

$\sqrt{2} \approx x_3 = p_2(0) = 0 + 1 \cdot 2 - \frac{1}{6} \cdot 2 \cdot 1 = 2 - \frac{1}{3} = \underline{\underline{\frac{5}{3} \approx 1.667}}$

x	0	1	2
y	-2	-1	2

-estimate the error:

Recall Error formula $|f(x) - L_n(x)| \leq \frac{M_{n+1}}{(n+1)!} \cdot |\omega_n(x)|$ ($\frac{M_{n+1}}{(n+1)!} |\omega_n(x)|$)

with $\omega_n(x) = (x-x_0) \cdot (x-x_1) \cdots (x-x_n) = \prod_{i=0}^n (x-x_i)$ and $M_{n+1} = \max_{x \in [a,b]} |f^{(n+1)}(x)|$
natural approx.

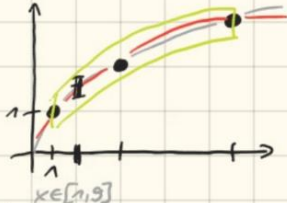
Problem 1 Estimate error • $L_2(2) \approx f(2) = \sqrt{2}$ • $x \in [1, 9]$

$|f(x) - L_2(x)| \leq \frac{M_3}{3!} \cdot |\omega_2(x)|$, $3! = 6$

$M_3 = \max_{x \in [1,9]} |f'''(x)|$, $f(x) = \sqrt{x} = x^{\frac{1}{2}}$, $f'(x) = \frac{1}{2} x^{-\frac{1}{2}}$
 $f''(x) = -\frac{1}{4} x^{-\frac{3}{2}}$, $f'''(x) = \frac{3}{8} x^{-\frac{5}{2}}$, $|f'''(x)| = \frac{3}{8} \cdot \frac{1}{x^{\frac{5}{2}}} \leq \frac{3}{8} = M_3$ at $x=1$

$\omega_2(x) = (x-1)(x-4)(x-9)$, $\omega_2(2) = (2-1)(2-4)(2-9) = 14$

(c) at $x=2$: $|f(2) - L_2(2)| \leq \frac{M_3}{3!} \cdot |\omega_2(2)| = \frac{3}{8} \cdot \frac{1}{6} \cdot 14 = \frac{14}{16} = \frac{7}{8}$



6. Newton and convergence

-Newton's

Newton's method $x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$

1. $f(x) = \sin x + x^2 - 1$, $f(0) = -1 < 0$, $f(\frac{\pi}{2}) = (\frac{\pi}{2})^2 > 0$, therefore

f has root in $[0, \frac{\pi}{2}]$, furthermore $f \in C^2[0, \frac{\pi}{2}]$ (2 points)

(a) $f'(x) = \cos x + 2x > 0$ for all $x \in [0, \frac{\pi}{2}]$

Newton's method

$$x_{n+1} = x_n - \frac{\sin x_n + x_n^2 - 1}{\cos x_n + 2x_n}$$

(2 points)

-convergence:

<https://www.desmos.com/calculator?lang=zh-CN>