Analysis II. practice 2

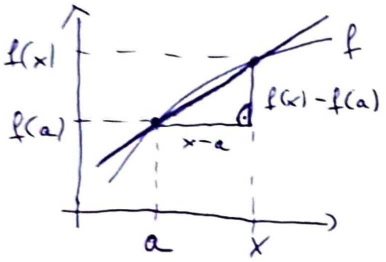
Differentiation of functions

<u>reminder</u>: $f \in \mathbb{R} \to \mathbb{R}$, $a \in \operatorname{int} \mathcal{D}_f$;

$$f \in D\{a\} \iff \exists \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \in \mathbb{R}$$

and then $f'(a) := \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$

the derivative of f at the point a



geometrical meaning:
the slope of the tangent line to the graph of f at the point a

$f \in D \Rightarrow f \in C$, but $f \in C \not\Rightarrow f \in D$, see:

$f(x) = |x|$: $f \notin D\{0\}$

derivatives of basic functions: see the table

<u>theorem</u> (differentiation rules):

i, $f, g \in D\{a\}$, $\lambda \in \mathbb{R} \Rightarrow (f + \lambda g)'(a) = f'(a) + \lambda g'(a)$

ii, $f, g \in D\{a\} \Rightarrow (f \cdot g)'(a) = f'(a) g(a) + f(a) g'(a)$

iii, $f, g \in D\{a\}$, $g(a) \neq 0 \Rightarrow \left(\frac{f}{g}\right)'(a) = \frac{f'(a) g(a) - f(a) g'(a)}{g^2(a)}$

iv, $g \in D\{a\}$, $f \in D\{g(a)\} \Rightarrow (f \circ g)'(a) = f'(g(a)) \cdot g'(a)$

1. determine $f'(a)$ by the def.

a) $f(x) = \sqrt{x}, \quad a = 3$

$$\lim_{x\to a} \frac{f(x)-f(a)}{x-a} = \lim_{x\to 3} \frac{\sqrt{x}-\sqrt{3}}{x-3} = \lim_{x\to 3} \frac{\sqrt{x}-\sqrt{3}}{(\sqrt{x}-\sqrt{3})(\sqrt{x}+\sqrt{3})} =$$

$$= \lim_{x\to 3} \frac{1}{\sqrt{x}+\sqrt{3}} = \frac{1}{2\sqrt{3}} \in \mathbb{R} \Rightarrow f \in D\{a\}, \; f'(a) = \frac{1}{2\sqrt{3}}$$

remark: by the basic derivative $(x^\alpha)' = \alpha x^{\alpha-1}$:

$$f'(x) = \frac{1}{2} x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}} \Rightarrow f'(3) = \frac{1}{2\sqrt{3}}$$

b) $f(x) = x^2 + 2x - 1, \quad a = 1$

$$\lim_{x\to a} \frac{f(x)-f(a)}{x-a} = \lim_{x\to 1} \frac{x^2+2x-1-2}{x-1} =$$

$$= \lim_{x\to 1} \frac{\cancel{(x-1)}(x+3)}{\cancel{x-1}} = 4 \in \mathbb{R} \Rightarrow f \in D\{a\}, \; f'(a) = 4$$

(check: $f'(x) = 2x+2 \Rightarrow f'(1) = 4$)

c) $f(x) = \dfrac{x+2}{x^2-9}, \quad a = -1$

$$\lim_{x\to a} \frac{f(x)-f(a)}{x-a} = \lim_{x\to -1} \frac{\frac{x+2}{x^2-9} + \frac{1}{8}}{x+1} =$$

$$= \lim_{x\to -1} \frac{8(x+2)+x^2-9}{8(x^2-9)(x+1)} = \lim_{x\to -1} \frac{x^2+8x+7}{8(x^2-9)(x+1)} =$$

$$= \lim_{x\to -1} \frac{\cancel{(x+1)}(x+7)}{8(x^2-9)\cancel{(x+1)}} = \frac{6}{8\cdot(-8)} = -\frac{3}{32} \Rightarrow f'(a) = -\frac{3}{32}$$

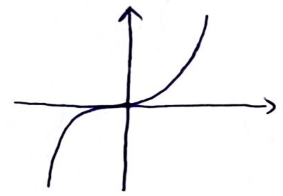
(check: $f'(x) = \dfrac{(x+2)'(x^2-9)-(x+2)(x^2-9)'}{(x^2-9)^2} =$

$$= \frac{x^2-9-2x(x+2)}{(x^2-9)^2} \Rightarrow f'(-1) = \frac{-8+2}{64} = -\frac{3}{32}$$)

2. discuss the differentiability

a, $f(x) := x \cdot |x| \quad (x \in \mathbb{R})$

$$\Rightarrow \quad f(x) = \begin{cases} x^2, & x \geq 0 \\ -x^2, & x < 0 \end{cases}$$



is it continuous in 0?

$$\lim_{0-0} f = \lim_{0+0} f = f(0) = 0 \quad \Rightarrow \quad f \in C\{0\}$$

is it „smooth" (differentiable)?

$x \in (0, +\infty)$: $\quad f \in D\{x\}, \; f'(x) = 2x$

$x \in (-\infty, 0)$: $\quad f \in D\{x\}, \; f'(x) = -2x$

$x = 0$: $\quad f'_+(0) = \lim_{x \to 0+0} \dfrac{f(x) - f(0)}{x - 0} =$

$$= \lim_{x \to 0+0} \frac{x^2 - (x^2)|_{x=0}}{x - 0} = (2x)|_{x=0} = 0$$

$$f'_-(0) = \lim_{x \to 0-0} \frac{f(x) - f(0)}{x - 0} =$$

$$= \lim_{x \to 0-0} \frac{(-x^2) - (x^2)|_{x=0}}{x - 0} = \lim_{x \to 0-0} \frac{(-x^2) - (-x^2)|_{x=0}}{x - 0} =$$

$$= (-2x)|_{x=0} = 0$$

$$f'_+(0) = f'_-(0) = 0 \quad \Rightarrow \quad f \in D\{0\}, \; f'(0) = 0$$

(right/left der.)

b, $f(x) := \begin{cases} 1 - x, & x < 0 \\ e^{-x}, & x \geq 0 \end{cases}$

$x \in (0, +\infty)$: $\quad f \in D\{x\}, \; f'(x) = -e^{-x}$

$x \in (-\infty, 0)$: $\quad f \in D\{x\}, \; f'(x) = -1$

$x = 0$: $\quad f \in D\{x\} \iff f \in C\{x\} \wedge f'_+(0) = f'_-(0)$

$$\lim_{x \to 0-0} (1 - x) = \lim_{x \to 0+0} (e^{-x}) = 1 = f(0) \quad \Rightarrow \quad f \in C\{0\}$$

$$f'_+(0) = (-e^{-x})_{|x=0} = -1$$

$$f'_-(0) = \lim_{x\to 0-0} \frac{(1-x)-(e^{-x})_{|x=0}}{x-0} = \lim_{x\to 0-0} \frac{(1-x)-(1-x)_{|x=0}}{x-0} =$$

$$= (1-x)'_{|x=0} = -1$$

$$\Rightarrow f \in D\{0\}, \quad f'(0) = -1$$

c, $f(x) := \begin{cases} \alpha x + x^2, & x<0 \\ x - x^2, & x \geq 0 \end{cases}$ $\quad (\alpha \in \mathbb{R})$

$x \in (0, +\infty)$: $\quad f \in D\{x\}, \; f'(x) = 1-2x$

$x \in (-\infty, 0)$: $\quad f \in D\{x\}, \; f'(x) = \alpha + 2x$

$$\left.\begin{array}{l} \lim\limits_{x\to 0-0} (\alpha x + x^2) = 0 \\ \lim\limits_{x\to 0+0} (x - x^2) = 0 = f(0) \end{array}\right\} \Rightarrow f \in C\{0\}$$

$$\left.\begin{array}{l} f'_+(0) = (1-2x)_{|x=0} = 1 \\ f'_-(0) = (\alpha+2x)_{|x=0} = \alpha \end{array}\right\} \Rightarrow f \in D\{0\} \Leftrightarrow \alpha = 1, \; f'(0) = 1$$

3. $f'(x)$?

a, $f(x) = 4x^5 - 3x^4 + 2x^3 - 7x^2 + 6x + 7$

$f'(x) = 20x^4 - 12x^3 + 6x^2 - 14x + 6$

b, $f(x) = x^2 \cdot \sqrt[3]{x} = x^{7/3}$

$f'(x) = \frac{7}{3} \cdot x^{4/3}$

c, $f(x) = \sqrt{x \cdot \sqrt[3]{x}} = x^{2/3} \Rightarrow f'(x) = \frac{2}{3} x^{-\frac{1}{3}}$

d, $f(x) = e^x \sin x$

$f'(x) = (e^x)' \sin x + e^x (\sin x)' = e^x \sin x + e^x \cos x$

e, $f(x) = (x^3 + \ln x) \cos x$

$f'(x) = (3x^2 + \frac{1}{x}) \cos x + (x^3 + \ln x)(-\sin x)$

f) $f(x) = \dfrac{2x^2+3x+1}{x^3+x^2+x+1}$

$$f'(x) = \frac{(2x^2+3x+1)'(x^3+x^2+x+1) - (2x^2+3x+1)(x^3+x^2+x+1)'}{(x^3+x^2+x+1)^2} =$$

$$= \frac{(4x+3)(x^3+x^2+x+1) - (2x^2+3x+1)(3x^2+2x+1)}{(x^3+x^2+x+1)^2}$$

g) $f(x) = \sin(x^3+\ln x)$

$$f'(x) = \cos(x^3+\ln x)\cdot(x^3+\ln x)' =$$

$$= \cos(x^3+\ln x)\cdot\left(3x^2+\frac{1}{x}\right)$$

h) $f(x) = e^{\sin^3 x} \Rightarrow f'(x) = e^{\sin^3 x}\cdot 3\sin^2 x\cdot\cos x$

i) $f(x) = \dfrac{1}{\sqrt[3]{x+\sqrt{x}}} = (x+\sqrt{x})^{-\frac{1}{3}}$

$$f'(x) = -\frac{1}{3}(x+\sqrt{x})^{-\frac{4}{3}}\cdot\left(1+\frac{1}{2\sqrt{x}}\right)$$

j) $f(x) = x^x = e^{\ln x^x} = e^{x\ln x}$

$$f'(x) = e^{x\ln x}\cdot(x\ln x)' = e^{x\ln x}\cdot\left(\ln x + x\cdot\frac{1}{x}\right) =$$

$$= x^x(\ln x+1)$$

k) $f(x) = (\sin x)^{\cos\sqrt{x}} = e^{\ln(\sin x)^{\cos\sqrt{x}}} = e^{\cos\sqrt{x}\cdot\ln(\sin x)}$

$$f'(x) = e^{\cos\sqrt{x}\cdot\ln(\sin x)}\cdot\left(-\sin\sqrt{x}\cdot\frac{1}{2\sqrt{x}}\cdot\ln(\sin x) + \right.$$

$$\left. + \cos\sqrt{x}\cdot\frac{1}{\sin x}\cdot\cos x\right)$$