Computer Science BSc; Analysis-1
Questions for the Theoretical Exam (100 questions)
2019/2020 Spring

1. State Newton's identity and the the Binomial Theorem

Newton's identity:

$$\sum_{i=1}^{n} x_i^k = x_1^k + x_2^k + \dots + x_n^k$$

Binomial Theorem:

For any $a, b \in \mathbb{R}$ and $n \in \mathbb{N}$ holds

$$(a+b)^n = \binom{n}{0}a^n + \binom{n}{1}a^{n-1}b + \dots \binom{n}{n-1}ab^{n-1} + \binom{n}{n}b^n =$$
$$= \sum_{k=0}^{n} \binom{n}{k}a^{n-k}b^k = \sum_{k=0}^{n} \binom{n}{k}a^k b^{n-k}$$

2. State the Triangle Inequalities in R

For any real numbers $x,\ y \in \mathbb{R}$ hold

a) $|x+y| \leq |x| + |y|$ *(first triangle inequality)*

b) $|x-y| \geq \Big| |x| - |y| \Big|$ *(second triangle inequality)*

3. State the theorem about the Inequality between the Arithmetic and the Geometric means

Let $n \in \mathbb{N},\ n \geq 2$ and $x_1, \dots, x_n \in \mathbb{R}^+$. Then $G_n \leq A_n$, that is

$$\sqrt[n]{x_1 \cdot \ldots \cdot x_n} \leq \frac{x_1 + \ldots + x_n}{n},$$

or equivalently $G_n^n \leq A_n^n$, that is:

$$x_1 \cdot \ldots \cdot x_n \leq \left(\frac{x_1 + \ldots + x_n}{n}\right)^n.$$

The equality holds if and only if $x_1 = \ldots = x_n$.

4. Define the upper bound and the lower bound of a set

1.9. Definition Let $\emptyset \neq H \subseteq \mathbb{R}$ and $K,\ L \in \mathbb{R}$. We say that

a) K is an upper bound of H if $\forall x \in H: \quad x \leq K$,

b) L is a lower bound of H if $\forall x \in H: \quad x \geq L$.

5. Define the following concepts: "a set is bounded above", "a set is bounded below"

1.10. Definition Let $\emptyset \neq H \subseteq \mathbb{R}$. We say that

a) H is bounded above if it has an upper bound, that is $\exists K \in \mathbb{R} \, \forall x \in H: \quad x \leq K$,

b) H is bounded below if it has a lower bound, that is $\exists L \in \mathbb{R} \, \forall x \in H: \quad x \geq L$,

c) H is bounded if it is bounded above and it is bounded below.

6. Define the minimal element of a set and the maximal element of a set

1.12. Definition Let $\emptyset \neq H \subseteq \mathbb{R}$ and $a \in \mathbb{R}$. We say that

- a is the minimal element (or: least element) of H if $a \in H$ and $\forall x \in H: \quad x \geq a$. Notation: $a = \min H$.
- a is the maximal element (or: greatest element) of H if $a \in H$ and $\forall x \in H$: $x \leq a$. Notation: $a = \max H$.

7. Define the least upper bound (sup) of a set. What is the least upper bound of a set which is not bounded above?

1.13. Theorem *[the Existence of the Least Upper Bound]*
Let $\emptyset \neq H \subseteq \mathbb{R}$ and suppose that H is bounded above. Then the set of its upper bounds

$$B := \{K \in \mathbb{R} \mid K \text{ is upper bound of } H\}$$

has minimal element. This minimal element is called the least upper bound of H and is denoted by $\sup H$ or $\operatorname{lub} H$. So

$$\sup H = \operatorname{lub} H := \min B.$$

1.17. Definition Let $\emptyset \neq H \subseteq \mathbb{R}$ and suppose that H is unbounded above. Then $\sup H := +\infty$.

8. Define the greatest lower bound (inf) of a set. What is the greatest lower bound of a set which is not bounded below?

1.15. Theorem *[the Existence of the Greatest Lower Bound]*
Let $\emptyset \neq H \subseteq \mathbb{R}$ and suppose that H is bounded below. Then the set of its lower bounds

$$A := \{K \in \mathbb{R} \mid K \text{ is lower bound of } H\}$$

has maximal element. This maximal element is called the greatest lower bound of H and is denoted by $\inf H$ or $\operatorname{glb} H$. So

$$\inf H = \operatorname{glb} H := \max A.$$

Let $\emptyset \neq H \subseteq \mathbb{R}$ and suppose that H is unbounded below. Then $\inf H := -\infty$

9. Define the index-sequence and the subsequence. Give an example for them.

3.3. Definition The sequence $n_k \in \mathbb{N} \quad (k \in \mathbb{N})$ is called index sequence if it is strictly monotone increasing, that is

$$\forall n \in N : \quad n_k < n_{k+1}.$$

3.4. Definition Let $a : \mathbb{N} \to H$ be a sequence and let (n_k) be an index sequence. Then the sequence

$$a_{n_k} \in H \quad (k \in \mathbb{N})$$

is called the subsequence of (a_n) (composed with the index sequence (n_k)).

3.5. Example

If $a_n = \dfrac{1}{n} \quad (n \in \mathbb{N})$ and $n_k = 2^k \quad (k \in \mathbb{N})$, then

$$a_{n_k} = a_{2^k} = \frac{1}{2^k} = \left(\frac{1}{2}\right)^k \qquad (k \in \mathbb{N}).$$

10. Define the neighbourhood (environment) of a number in K. What does it mean geometrically in R and in C?

3.7. Definition Let $a \in \mathbb{K}$ and $r > 0$. The neighbourhood (or ball or environment) of a with radius r is the set

$$B(a, r) := \{x \in \mathbb{K} \mid \ |x - a| < r\} \subset \mathbb{K}.$$

3.8. Remarks.

1. If $\mathbb{K} = \mathbb{R}$, then the neighbourhood $B(a, r) = \{x \in \mathbb{R} \mid \ |x - a| < r\}$ is equal to the open interval $(a - r,\ a + r)$.
2. If $\mathbb{K} = \mathbb{C}$, then the neighbourhood $B(a, r) = \{z \in \mathbb{C} \mid \ |z - a| < r\}$ is equal to the open circular disk with centre a and with radius r on the complex number plane. Really, let

   $$a = u + vi \quad \text{and} \quad z = x + yi.$$

   Then

   $$|z - a| = |(x - u) + (y - v)i| = \sqrt{(x - u)^2 + (y - v)^2},$$

   thus the inequality $|z - a| < r$ is equivalent to

   $$(x - u)^2 + (y - v)^2 < r^2,$$

11. Define the concept of convergence of a sequence with neighbourhoods, and define the limit of a convergent sequence.

3.10. Definition The number sequence $a_n \in \mathbb{K} \quad (n \in \mathbb{N})$ is named convergent if

$$\exists A \in \mathbb{K}\ \forall \varepsilon > 0\ \exists N \in \mathbb{N}\ \forall n \geq N : \quad a_n \in B(A, \varepsilon)\,.$$

3.12. Definition Let $a_n \in \mathbb{K} \quad (n \in \mathbb{N})$ be a convergent number sequence. The unique number A in the definition 3.10 is called the limit of the sequence (a_n), and is denoted in one of the following ways:

$$\lim a = A\,, \quad \lim a_n = A\,, \quad \lim_{n\to\infty} a_n = A\,, \quad a_n \to A \quad (n \to \infty)\,,$$

$$\lim(a_n) = A\,, \quad (a_n) \to A \quad (n \to \infty)\,.$$

12. Define the concept of convergence of a sequence with inequalities, and define the limit of a convergent sequence.

3.10. Definition The number sequence $a_n \in \mathbb{K} \quad (n \in \mathbb{N})$ is named convergent if

$$\exists A \in \mathbb{K}\ \forall \varepsilon > 0\ \exists N \in \mathbb{N}\ \forall n \geq N : \quad |a_n - A| < \varepsilon\,.$$

3.12. Definition Let $a_n \in \mathbb{K} \quad (n \in \mathbb{N})$ be a convergent number sequence. The unique number A in the definition 3.10 is called the limit of the sequence (a_n), and is denoted in one of the following ways:

$$\lim a = A\,, \quad \lim a_n = A\,, \quad \lim_{n\to\infty} a_n = A\,, \quad a_n \to A \quad (n \to \infty)\,,$$

$$\lim(a_n) = A\,, \quad (a_n) \to A \quad (n \to \infty)\,.$$

13. Define the concept: a sequence is bounded

3.18. Definition The sequence $a_n \in \mathbb{K}$ $(n \in \mathbb{N})$ is called bounded if

$$\exists\, M > 0\ \forall\, n \in \mathbb{N} : \quad |a_n| \leq M\,.$$

14. State and the theorem about the connection between the convergent and the bounded sequences

3.21. Theorem *Every convergent number sequence is bounded.*

Extra info (just in case):

Proof. Let $a_n \in \mathbb{K}$ $(n \in \mathbb{N})$ be a convergent sequence and $A = \lim\limits_{n\to\infty} a_n \in \mathbb{K}$. Apply the definition of convergency with $\varepsilon = 1$:

$$\exists\, N \in \mathbb{N}\ \forall\, n \geq N : \quad |a_n - A| < 1\,.$$

Use the second triangle inequality:

$$|a_n| - |A| \leq \big|\,|a_n| - |A|\,\big| \leq |a_n - A| < 1\,,$$

from where we have after rearranging

$$|a_n| < 1 + |A| \qquad (n \geq N)\,.$$

Thus obviously

$$|a_n| \leq M \quad (n \in \mathbb{N}) \quad \text{where} \quad M := \max\{|a_1|,\, |a_2|,\, \ldots,\, |a_{N-1}|,\, 1 + |A|\}\,.$$

15. Define the zero-sequence

3.23. Definition The number sequence $a_n \in \mathbb{K}$ $(n \in \mathbb{N})$ is called zero sequence if it is convergent and $\lim_{n\to\infty} a_n = 0$.

16. State the five theorems (Th1-Th5) in connection with zero sequences

3.24. Theorem *[T1] Let $a_n \in \mathbb{K}$ $(n \in \mathbb{N})$ and $A \in \mathbb{K}$. Then*

$$\lim_{n\to\infty} a_n = A \Leftrightarrow \lim_{n\to\infty} (a_n - A) = 0\,.$$

3.25. Theorem *[T2] Let $a_n \in \mathbb{K}$ $(n \in \mathbb{N})$. Then*

$$\lim_{n\to\infty} a_n = 0 \Leftrightarrow \lim_{n\to\infty} |a_n| = 0\,.$$

3.26. Theorem *[T3, Majorant Principle] Let $a_n \in \mathbb{K}$ $(n \in \mathbb{N})$ and $b_n \in \mathbb{R}$ $(n \in \mathbb{N})$. Suppose that (b_n) is a zero sequence and that*

$$\exists\, N_0 \in \mathbb{N}\ \forall n \geq N_0 : \quad |a_n| \leq b_n\,,$$

3.27. Theorem *[T4, Sum] Let $a_n, b_n \in \mathbb{K}$ $(n \in \mathbb{N})$ be zero sequences. Then their sum $(a_n + b_n)$ is also a zero sequence.*

3.28. Theorem *[T5, Product] Let $a_n \in \mathbb{K}$ $(n \in \mathbb{N})$ be a zero sequence and $b_n \in \mathbb{K}$ $(n \in \mathbb{N})$ be a bounded sequence. Then their product $(a_n b_n)$ is a zero sequence.*

17. State the theorem about the operations with convergent sequences

4.1. Theorem *[Absolute Value]*

Let $a_n \in \mathbb{K}$ $(n \in \mathbb{N})$ be a convergent sequence. Then its absolute value sequence $(|a_n|)$ is also convergent and

$$\lim_{n\to\infty} |a_n| = |\lim_{n\to\infty} a_n|\,.$$

4.3. Theorem *[Addition]*

Let $a_n, b_n \in \mathbb{K}$ $(n \in \mathbb{N})$ be convergent sequences. Then their sum $(a_n + b_n)$ is also convergent and

$$\lim_{n\to\infty} (a_n + b_n) = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n\,.$$

4.4. Theorem *[Multiplication]*

Let $a_n, b_n \in \mathbb{K}$ $(n \in \mathbb{N})$ be convergent sequences. Then their product $(a_n b_n)$ is also convergent and

$$\lim_{n\to\infty} (a_n b_n) = (\lim_{n\to\infty} a_n) \cdot (\lim_{n\to\infty} b_n)\,.$$

4.8. Theorem *[Reciprocal]*

Let $b_n \in \mathbb{K} \setminus \{0\}$ $(n \in \mathbb{N})$ be a convergent sequence. Suppose that $B := \lim_{n\to\infty} b_n \neq 0$. Then

a) The sequence $\left(\frac{1}{b_n}\right)$ is bounded

b) The sequence $\left(\frac{1}{b_n}\right)$ is convergent and

$$\lim_{n\to\infty} \frac{1}{b_n} = \frac{1}{B}\,.$$

4.10. Theorem *[Division]*

Let $a_n \in \mathbb{K}$ $(n \in \mathbb{N})$ $b_n \in \mathbb{K} \setminus \{0\}$ $(n \in \mathbb{N})$ *be convergent sequences. Suppose that* $\lim\limits_{n\to\infty} b_n \neq 0$. *Then their quotient* $\left(\dfrac{a_n}{b_n}\right)$ *is also convergent and*

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{\lim\limits_{n\to\infty} a_n}{\lim\limits_{n\to\infty} b_n}\,.$$

4.12. Theorem *[q-th root]*

Let $q \in \mathbb{N}$, $q \geq 2$ *and* $a_n \in \mathbb{R}$ $(n \in \mathbb{N})$ *be a convergent sequence. Suppose that* $a_n \geq 0$ $(n \in \mathbb{N})$. *Then its q-th root sequence* $(\sqrt[q]{a_n})$ *is also convergent, and*

$$\lim_{n\to\infty} \sqrt[q]{a_n} = \sqrt[q]{\lim_{n\to\infty} a_n}\,.$$

18. State the Sandwich Theorem

4.14. Theorem *[Sandwich Theorem]*

Let $a_n,\ b_n,\ c_n \in \mathbb{R}$ $(n \in \mathbb{N})$ *be real number sequences and suppose that*

a) $\exists\, N_0 \in \mathbb{N}\ \forall\, n \geq N_0 : \quad a_n \leq b_n \leq c_n$ *and that*

b) (a_n) *and* (c_n) *are convergent and* $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} c_n =: A$.

Then (b_n) *is also convergent and* $\lim\limits_{n\to\infty} b_n = A$.

19. State the theorem about the connection between the convergence and the ordering relations

3.15. Theorem *Let* $a_n,\ b_n \in \mathbb{R}$ $(n \in \mathbb{N})$ *be convergent sequences and suppose that*

$$\lim_{n\to\infty} a_n < \lim_{n\to\infty} b_n\,.$$

Then

$$\exists\, N \in \mathbb{N}\ \forall\, n \geq N : \quad a_n < b_n\,.$$

20. Define the geometric sequence. State the theorem about the convergence of geometric sequences

4.15. Definition Let $q \in \mathbb{K}$ be a fixed number. Then the sequence

$$a_n := q^n \quad (n \in \mathbb{N})$$

is called a geometric sequence (with base q or with quotient q).

4.16. Theorem *The geometric sequence is convergent if and only if* $|q| < 1$ *or* $q = 1$. *In this case*

$$\lim_{n\to\infty} q^n = \begin{cases} 0 & \text{if } |q| < 1 \\ 1 & \text{if } q = 1 \end{cases}$$

21. State the theorem about the convergence of $\sqrt[n]{a}$ and of $\sqrt[n]{n}$

4.17. Theorem *Let $a \in \mathbb{R}$, $a > 0$ be fixed. Then*

$$\lim_{n\to\infty} \sqrt[n]{a} = 1\,.$$

4.18. Theorem

$$\lim_{n\to\infty} \sqrt[n]{n} = 1\,.$$

22. State the theorem about the convergence of $n^k \cdot q^n$ and of $\dfrac{n^k}{a^n}$

4.20. Theorem *Let $q \in \mathbb{K}$ and $k \in \mathbb{N}$ be fixed. Then*

$$\lim_{n\to\infty} n^k \cdot q^n = 0\,.$$

$$\lim_{n\to\infty} \frac{n^k}{a^n} = 0$$

23. State and prove the theorem about the convergence of $\dfrac{x^n}{n!}$

4.22. Theorem *Let $x \in \mathbb{K}$ be fixed. Then*

$$\lim_{n\to\infty} \frac{x^n}{n!} = 0\,.$$

Proof. Let $n \in \mathbb{N}$, $n > |x|$. (There exists such n by the Archimedean property of ordering.) Then we have for any $n \geq N+2$:

$$\left|\frac{x^n}{n!}\right| = \frac{|x|^n}{n!} = \frac{\overbrace{|x|\cdot\;\ldots\;\cdot|x|}^{N\text{ factors}}\cdot|x|\cdot\;\ldots\;\cdot|x|}{\underbrace{1\cdot\;\ldots\;\cdot N}_{N\text{ factors}}\cdot(N+1)\cdot\;\ldots\;\cdot n} = \frac{|x|^N}{N!}\cdot\frac{|x|}{N+1}\cdot\;\cdots\;\frac{|x|}{n-1}\cdot\frac{|x|}{n} \leq$$

$$\leq \frac{|x|^N}{N!}\cdot\frac{|x|}{n} \longrightarrow 0 \qquad (n\to\infty)\,.$$

In the above estimation we have used that

$$\frac{|x|}{N+1} < 1, \quad \ldots, \quad \frac{|x|}{n-1} < 1\,,$$

and that the other factors of the product are nonnegative.

Finally, applying the majorant principle for zero sequences (see: Theorem 3.26) we obtain that $(\dfrac{x^n}{n!})$ is a zero sequence. $\square$

24. Define the monotonic sequences (i.e. the different types of monotonicity)

5.1. Definition Let $a_n \in \mathbb{R}$ $(n \in \mathbb{N})$ be a real number sequence. We say that this sequence is

- monotonically increasing if $\forall n \in \mathbb{N}: \quad a_n \leq a_{n+1}$
- strictly monotonically increasing if $\forall n \in \mathbb{N}: \quad a_n < a_{n+1}$
- monotonically decreasing if $\forall n \in \mathbb{N}: \quad a_n \geq a_{n+1}$
- strictly monotonically decreasing if $\forall n \in \mathbb{N}: \quad a_n > a_{n+1}$
- monotone if it is either monotonically increasing or monotonically decreasing
- strictly monotone if it is either strictly monotonically increasing or strictly monotonically decreasing

25. State the theorem about the convergence of a monotonically increasing sequence

5.4. Theorem *Let $a_n \in \mathbb{R}$ $(n \in \mathbb{N})$ be a monotone real number sequence. Then it is convergent if and only if it is bounded. Moreover*

$$\lim_{n\to\infty} a_n = \begin{cases} \sup\{a_n \mid n \in \mathbb{N}\} & \text{if } (a_n) \text{ is monotonically increasing} \\ \inf\{a_n \mid n \in \mathbb{N}\} & \text{if } (a_n) \text{ is monotonically decreasing} \end{cases}$$

26. State the theorem about the convergence of a monotonically decreasing sequence was not, but may be for 4,5

5.4. Theorem *Let $a_n \in \mathbb{R}$ $(n \in \mathbb{N})$ be a monotone real number sequence. Then it is convergent if and only if it is bounded. Moreover*

$$\lim_{n\to\infty} a_n = \begin{cases} \sup\{a_n \mid n \in \mathbb{N}\} & \text{if } (a_n) \text{ is monotonically increasing} \\ \inf\{a_n \mid n \in \mathbb{N}\} & \text{if } (a_n) \text{ is monotonically decreasing} \end{cases}$$

27. State that the sequence $\left(\left(1+\frac{1}{n}\right)^n\right)$ is convergent and define the Euler's number e

$$e := \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$$

28. State the theorem about the existence of monotonic subsequence

5.3. Theorem *Every real number sequence has a monotone subsequence.*

29. State the Bolzano-Weierstrass Theorem, and prove it in R

5.6. Theorem *[Bolzano-Weierstrass]*
Every bounded number sequence contains a convergent subsequence.

Proof. In the first step we will prove the statement for real number sequences. Let $a_n \in \mathbb{R}$ $(n \in \mathbb{N})$ be a bounded real number sequence. By Theorem 5.3 it contains a monotone subsequence (a_{n_k}). The subsequence (a_{n_k}) is obviously bounded (with the same bound as (a_n)), therefore it is a monotone and bounded sequence. Consequently – by the previous theorem – (a_{n_k}) is convergent.

In the second step we will prove the statement for complex number sequences. Let $z_n = a_n + b_n i \in \mathbb{C}$ $(n \in \mathbb{N})$ be a bounded complex number sequence. Then by Theorem 4.26 the real sequences (a_n) and (b_n) are bounded. Applying the proved part of the theorem for the real part sequence (a_n), it has a convergent subsequence $(a_{n_k},\ k \in \mathbb{N})$. However, the subsequence $(b_{n_k},\ k \in \mathbb{N})$ of the imaginary part sequence (b_n) is also bounded, so – applying once more the proved part of the theorem – (b_{n_k}) has a convergent subsequence $(b_{n_{k_s}},\ s \in \mathbb{N})$. Hence – using Theorem 4.27 – the complex number sequence

$$z_{n_{k_s}} = a_{n_{k_s}} + b_{n_{k_s}} i \qquad (s \in \mathbb{N})$$

is convergent, and obviously it is a subsequence of (z_n). □

30. Define the Cauchy-sequence

5.10. Definition The number sequence $a_n \in \mathbb{K}$ $(n \in \mathbb{N})$ is called a Cauchy sequence if

$$\forall \varepsilon > 0\ \exists N \in \mathbb{N}\ \forall m, n \geq N: \quad |a_n - a_m| < \varepsilon .$$

31. State the Cauchy's Convergence Test

5.12. Theorem *[Cauchy's Convergence Test]*
The number sequence $a_n \in \mathbb{K}$ $(n \in \mathbb{N})$ is convergent if and only if it is a Cauchy sequence.

32. Define $\lim\limits_{n\to\infty} a_n = +\infty$ and $\lim\limits_{n\to\infty} a_n = -\infty$

5.14. Definition Let $a_n \in \mathbb{R}$ $(n \in \mathbb{N})$ be a real number sequence. We say that (a_n) tends to $+\infty$ if

$$\forall P > 0\ \exists N \in \mathbb{N}\ \forall n \geq N: \quad a_n > P .$$

5.15. Definition Let $a_n \in \mathbb{R}$ $(n \in \mathbb{N})$ be a real number sequence. We say that (a_n) tends to $-\infty$ if

$$\forall P < 0\ \exists N \in \mathbb{N}\ \forall n \geq N: \quad a_n < P .$$

33. State the theorem about the limit of a monotonically increasing unbounded sequence

5.19. Theorem *a) Let $a_n \in \mathbb{R}$ $(n \in \mathbb{N})$ be a monotonically increasing sequence. Suppose that it is not bounded above. Then*

$$\lim_{n\to\infty} a_n = +\infty .$$

34. Give an example for sequence which is not monotonically increasing, but its limit is $+\infty$ for adv.level

$a_n = n$ *when n is odd*

$a_n = n^2$ *when n is even*

Here a_n tends to infinity but it is not monotonically increasing:

$a_1 = 1$, $a_2 = 4$, $a_3 = 3$, $a_4 = 16$, $a_5 = 5$ ...

35. State the theorem about the connection between the addition and the infinite limit. Give the table of addition in $\overline{\mathbb{R}}$

5.21. Theorem *[Addition]*

Let $a_n, b_n \in \mathbb{R}$ $(n \in \mathbb{N})$. *Suppose that* $\lim\limits_{n\to\infty} a_n = A$ *where* $-\infty < A \leq +\infty$ *and* $\lim\limits_{n\to\infty} b_n = +\infty$. *Then*

$$\lim_{n\to\infty}(a_n + b_n) = +\infty\,.$$

$x+y$	$y=-\infty$	$y\in\mathbb{R}$	$y=+\infty$
$x=-\infty$	$-\infty$	$-\infty$	not defined
$x\in\mathbb{R}$	$-\infty$	$x+y$	$+\infty$
$x=+\infty$	not defined	$+\infty$	$+\infty$

36. Why $(+\infty) + (-\infty)$ is not defined? Make it clear via some examples

For example, if a_n = n and b_n = -n, then a_n tends to infinity, b_n tends to -infinity, and a_n + b_n tends to zero. But let's say that now a_n = n+1 and b_n = -n, then they still tend to + and - infinity, but now a_n + b_n tends to 1.

37. State the theorem about the connection between the multiplication and the infinite limit.

5.24. Theorem *[Multiplication]*

Let $a_n, b_n \in \mathbb{R}$ $(n \in \mathbb{N})$. *Suppose that* $\lim\limits_{n\to\infty} a_n = A$ *where* $A \in \overline{\mathbb{R}} \setminus \{0\}$ *and* $\lim\limits_{n\to\infty} b_n = +\infty$. *Then*

$$\lim_{n\to\infty}(a_n b_n) = \begin{cases} +\infty & \textit{if} \quad A > 0 \\ \\ -\infty & \textit{if} \quad A < 0 \end{cases}$$

38. Define the concept of series and its partial sum

6.1. Definition Let $a_n \in \mathbb{K}$ $(n \in \mathbb{N})$ be a number sequence. The expression

$$a_1 + a_2 + a_3 + \ldots = \sum_{n=1}^{\infty} a_n$$

is called an infinite numerical sum or an infinite numerical series. The numbers a_n are the terms of the series. The sequence

$$S_n := a_1 + a_2 + \ldots + a_n = \sum_{k=1}^{n} a_k \qquad (n \in \mathbb{N})$$

is called the partial sum sequence of the series. S_n is the n-th partial sum.

39. Define the convergence and the sum of a numerical series, and define the divergence of a numerical series

6.3. Definition The series $\sum\limits_{n=1}^{\infty} a_n$ is called convergent if its partial sum sequence (S_n) is convergent. In this case the limit of the partial sum sequence is called the sum of the series. For the sum of the series we will use the same symbol as for the series itself:

$$\sum_{n=1}^{\infty} a_n := \lim_{n\to\infty} S_n = \lim_{n\to\infty} \sum_{k=1}^{n} a_k\,.$$

A series is called divergent if it is not convergent. In this case the sum of the infinitely many terms a_n is undefined.

40. State the theorem about the addition and scalar multiplication of convergent series

6.4. Theorem *Let $a_n, b_n \in \mathbb{K}$ $(n \in \mathbb{N})$ and suppose that the series $\sum\limits_{n=1}^{\infty} a_n$ and $\sum\limits_{n=1}^{\infty} b_n$ are convergent. Then*

a) The series $\sum\limits_{n=1}^{\infty} (a_n + b_n)$ is also convergent and

$$\sum_{n=1}^{\infty} (a_n + b_n) = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n\,.$$

b) For any $c \in \mathbb{K}$ the series $\sum\limits_{n=1}^{\infty} (c \cdot a_n)$ is also convergent and

$$\sum_{n=1}^{\infty} (c \cdot a_n) = c \cdot \sum_{n=1}^{\infty} a_n\,.$$

41. State the theorem about the convergence and the sum of the geometric series

6.8. Theorem *The geometric series*

$$1+q+q^2+q^3+\dots=\sum_{n=0}^{\infty} q^n$$

is convergent if and only if $|q|<1$. *In this case*

$$\sum_{n=0}^{\infty} q^n=\frac{1}{1-q}\,.$$

42. State the Zero-sequence Test

6.9. Theorem *[Zero Sequence Test]*

Let $a_n\in\mathbb{K}$ $(n\in\mathbb{N})$ *and suppose that the series* $\sum\limits_{n=1}^{\infty} a_n$ *is convergent. Then*

$$\lim_{n\to\infty} a_n=0\,.$$

43. Define the positive term series. What is an important property of their partial sums?

6.14. Definition Let $a_n\in\mathbb{R}$ $(n\in\mathbb{N})$. The series $\sum\limits_{n=1}^{\infty} a_n$ is called a positive term series if $a_n\geq 0$ $(n\in\mathbb{N})$.

6.15. Theorem *A positive term series is convergent if and only if its partial sum sequence is bounded above.*

44. State the theorem about the Comparison Tests (Major Test, Minor Test)

6.17. Theorem *[Direct Comparison Tests]*

Let $a_n, b_n\in\mathbb{R}$ $(n\in\mathbb{N})$ *and suppose that* $0\leq a_n\leq b_n$ $(n\in\mathbb{N})$. *Then*

a) If $\sum\limits_{n=1}^{\infty} b_n<\infty$, *then* $\sum\limits_{n=1}^{\infty} a_n<\infty$ *(Majorant Criterion)*

b) If $\sum\limits_{n=1}^{\infty} a_n=\infty$, *then* $\sum\limits_{n=1}^{\infty} b_n=\infty$ *(Minorant Criterion)*

45. Define the hyperharmonic series. State the theorem about its convergence.

6.19. Definition Let $p>0$ be a fixed real number. The positive term series

$$\sum_{n=1}^{\infty}\frac{1}{n^p}$$

6.21. Theorem *Let* $p>0$. *The hyperharmonic series*

$$\sum_{n=1}^{\infty}\frac{1}{n^p}$$

is convergent if and only if $p>1$.

46. State the theorem about the alternating series (Leibniz's Test)

6.24. Theorem *[Leibniz Criterion]*

The series of Leibniz type

$$\sum_{n=1}^{\infty}(-1)^{n-1}\cdot a_n$$

is convergent if and only if $\lim_{n\to\infty} a_n = 0$.

47. Define the absolute and the conditional convergence of a series. Give examples for them

7.1. Definition Let $a_n \in \mathbb{K}$ $(n \in \mathbb{N})$. The series $\sum_{n=1}^{\infty} a_n$ is called absolutely convergent, if the series of absolute values

$$\sum_{n=1}^{\infty} |a_n|$$

is convergent.

7.4. Definition A numerical series is called conditionally convergent if it is convergent, but not absolutely convergent.

7.5. Example

The alternating hyperharmonic series $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^p}$ is absolutely convergent if $p > 1$, and it is conditionally convergent if $0 < p \leq 1$.

48. State the theorem about the connection between the convergence and the absolute convergence

7.2. Theorem *Let* $a_n \in \mathbb{K}$ $(n \in \mathbb{N})$. *If the series* $\sum_{n=1}^{\infty} a_n$ *is absolutely convergent, then it is convergent.*

49. Define the rearrangement of a series. State (without proof) the theorem about it

7.7. Theorem *Let* $a_n \in \mathbb{K}$ $(n \in \mathbb{N})$, *and* $p : \mathbb{N} \to \mathbb{N}$ *be a bijection. If the series* $\sum_{n=1}^{\infty} a_n$ *is absolutely convergent, then the series*

$$\sum_{i=1}^{\infty} a_{p(i)}$$

is absolutely convergent.

The series $\sum_{i=1}^{\infty} a_{p(i)}$ *is called a rearrangement of* $\sum_{n=1}^{\infty} a_n$.

50. State the Root Test. Give some examples for the indeterminate case

7.9. Theorem *[Root Test]*
Let $a_n \in \mathbb{K}$ $(n \in \mathbb{N})$ and suppose that the limit

$$L := \lim_{n\to\infty} \sqrt[n]{|a_n|} \in [0\,,\,+\infty]$$

exists. Then

a) If $L < 1$, then the series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent

b) If $L > 1$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent. Moreover, $\lim_{n\to\infty} |a_n| = +\infty$.

The theorem does not say anything about the case $L = 1$. This is the indeterminate case. In this case anything can happen. For example, at each of the following sequences $L = 1$, but

a) $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is absolutely convergent

b) $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$ is conditionally convergent

c) $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent, the terms tend to 0.

d) $\sum_{n=1}^{\infty} 1$ is divergent, the terms do not tend to 0, the terms form a bounded sequence

e) $\sum_{n=1}^{\infty} n$ is divergent, the terms do not tend to 0, the terms form an unbounded sequence

51. State the Ratio Test. Give some examples for the indeterminate case

7.11. Theorem *[Ratio Test]*
Let $a_n \in \mathbb{K} \setminus \{0\}$ $(n \in \mathbb{N})$ and suppose that the limit

$$L := \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| \in [0\,,\,+\infty]$$

exists. Then

a) If $L < 1$, then the series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent

b) If $L > 1$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent. Moreover, $\lim_{n\to\infty} |a_n| = +\infty$.

The theorem does not say anything about the case $L = 1$. This is the indeterminate case. In this case anything can happen. For example, at each of the following sequences $L = 1$, but

a) $\sum\limits_{n=1}^{\infty} \frac{1}{n^2}$ is absolutely convergent

b) $\sum\limits_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$ is conditionally convergent

c) $\sum\limits_{n=1}^{\infty} \frac{1}{n}$ is divergent, the terms tend to 0.

d) $\sum\limits_{n=1}^{\infty} 1$ is divergent, the terms do not tend to 0, the terms form a bounded sequence

e) $\sum\limits_{n=1}^{\infty} n$ is divergent, the terms do not tend to 0, the terms form an unbounded sequence

52. Define the Cauchy's Product of two series

7.14. Definition Let $a_n, b_n \in \mathbb{K}$ $(n \in \mathbb{N} \cup \{0\})$. Then the series

$$\sum_{n=0}^{\infty} \sum_{\substack{i,j \in \mathbb{N} \cup \{0\} \\ i+j=n}} a_i b_j$$

is called the Cauchy product of the series $\sum\limits_{i=0}^{\infty} a_i$ and $\sum\limits_{j=0}^{\infty} b_j$.

The above formula often is written shortly as

$$\sum_{n=0}^{\infty} \sum_{i+j=n} a_i b_j \,.$$

53. State the theorems about the Cauchy's Product of two absolutely convergent series

7.16. Theorem *Let $a_n, b_n \in \mathbb{K}$ $(n \in \mathbb{N} \cup \{0\})$. If the series $\sum\limits_{n=0}^{\infty} a_n$ and $\sum\limits_{n=0}^{\infty} b_n$ are absolutely convergent, then their Cauchy product is absolutely convergent.*

54. Define the power series and give an example for it

8.7. Definition Let $a_n \in \mathbb{K}$ $(n \in \mathbb{N} \cup \{0\})$ be a number sequence and let $x_0 \in \mathbb{K}$. The series

$$\sum_{n=0}^{\infty} a_n \cdot (x - x_0)^n = a_0 + a_1 \cdot (x - x_0) + a_2 \cdot (x - x_0)^2 + \dots$$

is called a power series. The numbers a_n are the coefficients, the number x_0 is the centre of the power series. The symbol x is the variable of the power series.

8.8. Examples

1. $\sum\limits_{n=0}^{\infty} x^n$. Here $x_0 = 0$, $a_n = 1$.
2. $\sum\limits_{n=0}^{\infty} \dfrac{x^n}{n!}$. Here $x_0 = 0$, $a_n = \dfrac{1}{n!}$.
3. $\sum\limits_{n=0}^{\infty} \dfrac{(x-5)^n}{n \cdot 2^n}$. Here $x_0 = 5$, $a_n = \dfrac{1}{n \cdot 2^n}$.

55. Define the convergence set of a power series

8.9. Theorem *Let $\sum\limits_{n=0}^{\infty} a_n \cdot (x - x_0)^n$ be a power series and denote by S its convergence set. Suppose that the following limit exists:*

$$L := \lim_{n \to \infty} \sqrt[n]{|a_n|} \in [0, +\infty] .$$

Then

a) If $L = 0$, then the power series is absolutely convergent for any $x \in \mathbb{K}$. Thus $S = \mathbb{K}$.

b) If $L = +\infty$, then the power series is

- *absolutely convergent at $x = x_0$,*
- *divergent at any $x \neq x_0$.*

Thus $S = \{x_0\}$.

c) If $0 < L < +\infty$, then the power series is

- *absolutely convergent at any $x \in \mathbb{K}$ for which holds $|x - x_0| < \dfrac{1}{L}$,*
- *divergent at any $x \in \mathbb{K}$ for which holds $|x - x_0| > \dfrac{1}{L}$.*

Thus $B(x_0, \frac{1}{L}) \subseteq S \subseteq \overline{B}(x_0, \frac{1}{L})$.

56. State the theorem about the convergence set of a power series (using Root Test)

8.10. Theorem *Let* $\sum\limits_{n=0}^{\infty} a_n \cdot (x - x_0)^n$ *be a power series with coefficients* $a_n \neq 0$ $(n \in \mathbb{N} \cup \{0\})$, *and denote by* S *its convergence set. Suppose that the following limit exists:*

$$L := \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| \in [0, +\infty] \,.$$

Then

a) *If* $L = 0$, *then the power series is absolutely convergent for any* $x \in \mathbb{K}$. *Thus* $S = \mathbb{K}$.

b) *If* $L = +\infty$, *then the power series is*

- *absolutely convergent at* $x = x_0$,
- *divergent at any* $x \neq x_0$.

Thus $S = \{x_0\}$.

c) *If* $0 < L < +\infty$, *then the power series is*

- *absolutely convergent at any* $x \in \mathbb{K}$ *for which holds* $|x - x_0| < \dfrac{1}{L}$,
- *divergent at any* $x \in \mathbb{K}$ *for which holds* $|x - x_0| > \dfrac{1}{L}$.

Thus $B(x_0, \frac{1}{L}) \subseteq S \subseteq \overline{B}(x_0, \frac{1}{L})$.

57. Define the Radius of Convergence of a power series

8.13. Definition Using the foregoing notations, suppose that the limit L exists. Then the radius of convergence is defined as follows:

$$R := \begin{cases} +\infty & \text{if} \quad L = 0, \\ 0 & \text{if} \quad L = +\infty, \\ \dfrac{1}{L} & \text{if} \quad 0 < L < +\infty \,. \end{cases}$$

Shortly, $R = \dfrac{1}{L}$ if we agree that in this formula $\dfrac{1}{0} = +\infty$ and $\dfrac{1}{+\infty} = 0$.

58. Define the exponential function (exp) (with power series)

a) The function $\exp : \mathbb{K} \to \mathbb{K}$,

$$\exp(x) := 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots = \sum_{n=0}^{\infty} \frac{x^n}{n!} \qquad (x \in \mathbb{K})$$

59. Define the cosine (cos) function (with power series)

c) The function $\cos : \mathbb{K} \to \mathbb{K}$,

$$\cos(x) := 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots = \sum_{n=0}^{\infty} (-1)^n \cdot \frac{x^{2n}}{(2n)!} \qquad (x \in \mathbb{K})$$

is called the cosine (or: cosinus) function.

60. Define the sine (sin) function (with power series)

b) The function $\sin : \mathbb{K} \to \mathbb{K}$,

$$\sin(x) := x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots = \sum_{n=0}^{\infty} (-1)^n \cdot \frac{x^{2n+1}}{(2n+1)!} \qquad (x \in \mathbb{K})$$

61. Define the hyperbolic cosine (cosh) function (with power series)

e) The function $\cosh : \mathbb{K} \to \mathbb{K}$,

$$\cosh(x) := 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \ldots = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!} \qquad (x \in \mathbb{K})$$

62. Define the hyperbolic sine (sinh) function (with power series)

d) The function $\sinh : \mathbb{K} \to \mathbb{K}$,

$$\sinh(x) := x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \ldots = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!} \qquad (x \in \mathbb{K})$$

63. Write the simple properties of exp, cos, sin, cosh, sinh

9.4. Theorem *[The Simplest Properties of the Above defined Five Analytical Functions]*

a) $\exp 0 = 1,\ \sin 0 = 0,\ \cos 0 = 1,\ \sinh 0 = 0,\ \cosh 0 = 1$

b) For any $x \in \mathbb{K}$ hold:

$$\sin(-x) = -\sin(x) \qquad \cos(-x) = \cos(x)$$

$$\sinh(-x) = -\sinh(x) \qquad \cosh(-x) = \cosh(x)$$

Part b) means that sin *and* sinh *are odd functions,* cos *and* cosh *are even functions.*

64. State the Addition Theorem of the exponential function

9.7. Theorem *[Addition Formula of the Exponential Function]*
For any $x, y \in \mathbb{K}$ holds

$$\exp(x+y) = (\exp x) \cdot (\exp y)\,.$$

65. State the Addition Theorems of cos, sin, cosh, sinh

9.9. Theorem *[Addition Formulas of* cos sin cosh sinh*]*
For any $x, y \in \mathbb{K}$ holds

a) $\cos(x+y) = (\cos x)(\cos y) - (\sin x)(\sin y)$

b) $\sin(x+y) = (\sin x)(\cos y) + (\cos x)(\sin y)$

c) $\cosh(x+y) = (\cosh x)(\cosh y) + (\sinh x)(\sinh y)$

d) $\sinh(x+y) = (\sinh x)(\cosh y) + (\cosh x)(\sinh y)$

66. State some important consequences of the Addition Theorems of cos, sin

9.10. Corollary. 1. If we apply the Addition Formulas in the previous theorem for $y = x$, then we obtain:

$$\cos(2x) = \cos^2 x - \sin^2 x$$

$$\sin(2x) = 2\sin x \cos x$$

2. If we apply the Addition Formulas of cos in the previous theorem for $y = -x$, then we obtain:

$$1 = \cos^2 x + \sin^2 x$$

67. State some important consequences of the Addition Theorems of cosh, sinh

9.10. Corollary. 1. If we apply the Addition Formulas in the previous theorem for $y = x$, then we obtain:

$$\cosh(2x) = \cosh^2 x + \sinh^2 x$$

$$\sinh(2x) = 2\sinh x \cosh x$$

2. If we apply the Addition Formulas of cosh in the previous theorem for $y = -x$, then we obtain:

$$1 = \cosh^2 x - \sinh^2 x$$

68. Define the neighborhoods (environments) in R ($B(a, r)$, $B(+\infty, r)$, $B(-\infty, r)$)

a) The neighbourhoods of a finite number:

$$B(a,r) := \{x \in \mathbb{R} \mid |x-a| < r\} = (a-r, a+r) \subset \mathbb{R}.$$

b) The neighbourhoods of $+\infty$:

$$B(+\infty,r) := \{x \in \mathbb{R} \mid x > \frac{1}{r}\} = (\frac{1}{r}, +\infty) \subset \mathbb{R}.$$

c) The neighbourhoods of $-\infty$:

$$B(-\infty,r) := \{x \in \mathbb{R} \mid x < -\frac{1}{r}\} = (-\infty, -\frac{1}{r}) \subset \mathbb{R}.$$

69. Define the concept of accumulation point and of isolated point

10.1. Definition (Accumulation Point and Isolated Point) Let $\emptyset \neq H \subseteq \mathbb{R}$ and $a \in \overline{\mathbb{R}}$. The point a is called an accumulation point of H if

$$\forall r > 0 : \quad (B(a,r) \setminus \{a\}) \cap H \neq \emptyset .$$

70. Define the limit of a function using environments

10.3. Definition Let $f \in \mathbb{R} \to \mathbb{R}$, $a \in D'_f$. We say that f has a limit at the point a if

$$\exists\, A \in \overline{\mathbb{R}}\ \forall \varepsilon > 0\ \exists \delta > 0\ \forall x \in (B(a,\delta) \setminus \{a\}) \cap D_f : \quad f(x) \in B(A,\varepsilon) .$$

As in the case of Theorem 3.11 it can be proved that A in this definition is unique. This unique A is called the limit of the function f at the point a. The following notations are used to express this fact:

$$A = \lim_a f, \quad A = \lim_{x \to a} f(x), \quad f(x) \to A\ (x \to a) .$$

71. Define the finite limit at a finite place using inequalities

$$\lim_{x \to a} f(x) = A \quad \Longleftrightarrow \quad \forall \varepsilon > 0\ \exists \delta > 0\ \forall x \in D_f,\ 0 < |x-a| < \delta : \quad |f(x)-A| < \varepsilon$$

72. Define the +∞ limit at a finite place using inequalities

$$\lim_{x \to a} f(x) = +\infty \quad \Longleftrightarrow \quad \forall P > 0\ \exists \delta > 0\ \forall x \in D_f,\ 0 < |x-a| < \delta : \quad f(x) > P$$

73. State the theorem about the Transference Principle for limit

10.6. Theorem *[Transference Principle]*
Let $f \in \mathbb{R} \to \mathbb{R}$, $a \in D'_f$ and $A \in \overline{\mathbb{R}}$. Then

$$\lim_{x \to a} f(x) = A \quad \Leftrightarrow \quad \forall x_n \in D_f \setminus \{a\} \quad (n \in \mathbb{N}),\ \lim x_n = a : \quad \lim f(x_n) = A .$$

A sequence (x_n) with the properties

$$x_n \in D_f \setminus \{a\} \quad (n \in \mathbb{N}), \qquad \lim_{n \to \infty} x_n = a$$

is called: allowed sequence (more precisely: allowed sequence of f with respect to a).

74. Define the left-hand side limit

11.4. Definition Let $f \in \mathbb{R} \to \mathbb{R}$, Suppose that

Let $f \in \mathbb{R} \to \mathbb{R}$, $a \in \overline{\mathbb{R}}$, $-\infty < a \leq +\infty$. Suppose that a is an accumulation point of the set $(-\infty,\, a) \cap D_f$ (we say that a is a left-hand accumulation point of D_f). Then the left-hand limit of f at a is denoted by $\lim\limits_{a-} f$ and it is defined as follows:

$$\lim_{a-} f := \lim_a f_{|(-\infty,\, a) \cap D_f} .$$

75. Define the right-hand side limit

11.1. Definition Let $f \in \mathbb{R} \to \mathbb{R}$, $a \in \overline{\mathbb{R}}$, $-\infty \leq a < +\infty$. Suppose that a is an accumulation point of the set $(a, +\infty) \cap D_f$ (we say that a is a right-hand accumulation point of D_f). Then the right-hand limit of f at a is denoted by $\lim\limits_{a+} f$ and it is defined as follows:

$$\lim_{a+} f := \lim_{a} f_{|(a,+\infty)\cap D_f}\,.$$

76. Define the monotonically increasing and the monotonically decreasing function

11.9. Definition Let $f \in \mathbb{R} \to \mathbb{R}$. We say that f is

- monotonically increasing if $\quad \forall\, x_1, x_2 \in D_f,\ x_1 < x_2 : \quad f(x_1) \leq f(x_2)$
- monotonically decreasing if $\quad \forall\, x_1, x_2 \in D_f,\ x_1 < x_2 : \quad f(x_1) \geq f(x_2)$

77. Define the strictly monotonically increasing and the strictly monotonically decreasing function

11.9. Definition Let $f \in \mathbb{R} \to \mathbb{R}$. We say that f is

- strictly monotonically increasing if $\quad \forall\, x_1, x_2 \in D_f,\ x_1 < x_2 : \quad f(x_1) < f(x_2)$
- strictly monotonically decreasing if $\quad \forall\, x_1, x_2 \in D_f,\ x_1 < x_2 : \quad f(x_1) > f(x_2)$

78. State the theorem about the one-sided limits of a monotonically increasing function

11.11. Theorem *Let $f \in \mathbb{R} \to \mathbb{R}$ a monotonically increasing function and let $a \in \overline{\mathbb{R}}$.*

a) If $a < +\infty$ and $a \in ((a, +\infty) \cap D_f)'$, then

$$\lim_{a+} f = \inf f[(a, +\infty) \cap D_f] = \inf\{f(x) \in \mathbb{R} \mid x \in D_f,\ x > a\}\,.$$

b) If $a > -\infty$ and $a \in ((-\infty, a) \cap D_f)'$, then

$$\lim_{a-} f = \sup f[(-\infty, a) \cap D_f] = \sup\{f(x) \in \mathbb{R} \mid x \in D_f,\ x < a\}\,.$$

79. Define the continuity of a function at a point using environments

1.1. Definition Let $f \in \mathbb{R} \to \mathbb{R}$, $a \in D_f$. f is continuous at "a" $\overset{\text{df}}{\Leftrightarrow}$
$\forall\, \varepsilon > 0 \quad \exists\, \delta > 0 \quad \forall\, x \in B(a, \delta) \cap D_f : \quad f(x) \in B(f(a), \varepsilon)$.
Let us denote the set of functions that are continuous at "a" by $C(a)$.

From the definition it follows immediately that

- if "a" is an isolated point of D_f then f is continuous at "a".
- if "a" is an accumulation point of D_f then

$$f \quad \text{is continuous at "}a\text{"} \quad \Leftrightarrow \quad \lim_{x \to a} f(x) = f(a).$$

80. Define the continuity of a function at a point using inequalities

1.1. Definition Let $f \in \mathbb{R} \to \mathbb{R}$, $a \in D_f$. f is continuous at "a" $\overset{\text{df}}{\Leftrightarrow}$
$\forall \varepsilon > 0 \quad \exists \delta > 0 \quad \forall x \in D_f \quad |x - a| < \delta : \ |f(x) - f(a)| < \varepsilon$
Let us denote the set of functions that are continuous at "a" by $C(a)$.

From the definition it follows immediately that

- if "a" is an isolated point of D_f then f is continuous at "a".
- if "a" is an accumulation point of D_f then

$$f \quad \text{is continuous at "}a\text{"} \quad \Leftrightarrow \quad \lim_{x \to a} f(x) = f(a).$$

81. State the theorem about Transference Principle for continuity

1.3. Theorem *[the Transference Theorem for continuity] Using our notations:*

$$f \in C(a) \quad \Leftrightarrow \quad \forall x_n \in D_f \quad (n \in \mathbb{N}),\ \lim x_n = a : \quad \lim f(x_n) = f(a).$$

The proof of the Transference Theorem is similar to that of the case of limit.

Using the Transference Theorem it is easy to see that

$$f,\ g \in C(a),\ c \in \mathbb{R} \quad \Rightarrow \quad f + g,\ f - g,\ f \cdot g,\ f/g,\ c \cdot f \in C(a),$$

moreover

$$g \in C(a),\ f \in C(g(a)) \quad \Rightarrow \quad f \circ g \in C(a).$$

82. Define the types of discontinuities

1.5. Definition Let $f \in \mathbb{R} \to \mathbb{R}$, $a \in D_f$, $f \notin C(a)$. We say that "a" is a point of

- removable discontinuity $\Leftrightarrow \exists \lim\limits_a f$, but $\lim\limits_a f \neq f(a)$.
- jump $\Leftrightarrow \exists \lim\limits_{a-} f$ and $\exists \lim\limits_{a+} f$, but $\lim\limits_{a-} f \neq \lim\limits_{a+} f$.
- discontinuity of second kind $\Leftrightarrow \nexists \lim\limits_{a-} f$ or $\nexists \lim\limits_{a+} f$.

83. Define the following concepts: interior point, exterior point, boundary point

1.7. Definition Let $\emptyset \neq H \subset \mathbb{R}$, $x_0 \in \mathbb{R}$. Then

1. x_0 is an interior point of H, if $\exists r > 0 : \quad B(x_0, r) \subseteq H$
2. x_0 is an exterior point of H, if $\exists r > 0 : \quad B(x_0, r) \cap H = \emptyset$ that is $B(x_0, r) \subseteq \overline{H}$. Here $\overline{H}$ denotes the complement of H that is $\overline{H} = \mathbb{R} \setminus H$.
3. x_0 is a boundary point of H, if
$\forall r > 0 : \quad B(x_0, r) \cap H \neq \emptyset$ and $B(x_0, r) \cap \overline{H} \neq \emptyset$.

84. Define *intH*, *extH*, ∂H

1.9. Definition 1. The set of the interior points of H is called the interior of H and is denoted by $intH$. So

$$intH := \{x_0 \in \mathbb{R} \mid \exists\, r > 0 : \quad B(x_0, r) \subseteq H\} \subseteq H.$$

2. The set of the exterior points of H is called the exterior of H and is denoted by $extH$. So

$$extH := \{x_0 \in \mathbb{R} \mid \exists\, r > 0 : \quad B(x_0, r) \subseteq \overline{H}\} \subseteq \overline{H}.$$

3. The set of the boundary points of H is called the bound of H and is denoted by ∂H. So

$$\partial H := \{x_0 \in \mathbb{R} \mid \forall\, r > 0 : \quad B(x_0, r) \cap H \neq \emptyset \quad \text{and} \quad B(x_0, r) \cap \overline{H} \neq \emptyset\} \subset \mathbb{R}.$$

85. Define the concept of open set and of closed set

1.11. Definition Let $H \subseteq \mathbb{R}$. Then

1. H is called an open set $\overset{\text{df}}{\Leftrightarrow} \partial H \subseteq \overline{H}$.

2. H is called a closed set $\overset{\text{df}}{\Leftrightarrow} \partial H \subseteq H$.

86. State the theorem about the characterization of the closeness of a set with sequences

1.13. Theorem *Let $\emptyset \neq H \subseteq \mathbb{R}$. Then H is closed if and only if*

$$\forall\, x_n \in H \;\; (n \in \mathbb{N}) \;\; \textit{convergent sequence} : \; \lim_{n\to\infty} x_n \in H\,.$$

87. Define the compact set

1.14. Definition Let $\emptyset \neq H \subseteq \mathbb{R}$. H is called a compact set if

$$\forall\, x_n \in H \ (n \in \mathbb{N}) \text{ sequence } \quad \exists\, (x_{\nu_n}) \text{ subsequence}: \quad (x_{\nu_n}) \text{ is convergent and } \lim_{n\to\infty} x_{\nu_n} \in H\,.$$

The $\emptyset$ is called to be compact by definition.

Remark that from the definition it follows immediately that a compact set is closed.

88. State the theorem about the connection between the compact sets and the closed and bounded sets

1.15. Theorem *Let $\emptyset \neq H \subseteq \mathbb{R}$. Then H is compact if and only if it is closed and bounded.*

89. State the theorem about the minimal and maximal elements of a compact set

1.17. Theorem *Let $\emptyset \neq H \subseteq \mathbb{R}$ be a compact set in $\mathbb{R}$. Then H has minimal element $\min H$ and maximal element $\max H$.*

90. State the theorem about the compactness of the image

2.1. Theorem *[the continuous image of a compact set is compact]*
Let $f \in \mathbb{R} \to \mathbb{R}$ be continuous function ($f \in C$) and suppose that D_f is compact. Then R_f is compact.

91. State the minimax theorem of Weierstrass

2.3. Theorem *[Theorem of Weierstrass] Let $f \in \mathbb{R} \to \mathbb{R}$, $f \in C$, D_f compact. Then $\exists \min f$ and $\exists \max f$.*

92. State the Intermediate Value Theorem (Bolzano’s Theorem)

2.8. Theorem *[Intermediate Value Theorem, Theorem of Bolzano] Let $f : [a,b] \to \mathbb{R}$, $f \in C$. Suppose that $f(a) \neq f(b)$, for example $f(a) < f(b)$ (the discussion of the case $f(a) > f(b)$ is similar). Then*

$$\forall\, c \in (f(a), f(b)) \quad \exists\, \xi \in (a,b): \quad f(\xi) = c\,.$$

93. Define the natural logarithm function and define the functions a^x and $\log_a x$

2.15. Definition The inverse function of the real exponential function is called natural logarithm function and is denoted by ln. So

$$\ln := \exp^{-1}.$$

From this definition one can simply deduce the following basic properties of the natural logarithmic function:

$$D_{\ln} = R_{\exp} = (0, +\infty), \quad R_{\ln} = D_{\exp} = \mathbb{R}, \quad \ln 1 = 0, \quad \ln(xy) = \ln x + \ln y,$$

ln is strictly monotonically increasing, $\lim_{x\to+\infty} \ln x = +\infty$, $\lim_{x\to 0-0} \ln x = -\infty$.

2.20. Definition Let $a > 0$, $a \neq 1$. The inverse function of the exponential function with the base a is called the logarithm function with base a and is denoted by $\log_a$. So $\log_a := \exp_a^{-1}$.

94. Define the functions arcsin and arctan .

4.11. Definition $\arcsin := \sin^{-1}_{\big|[-\frac{\pi}{2},\frac{\pi}{2}]} : [-1,1] \to \left[-\frac{\pi}{2}, \frac{\pi}{2}\right].$

$$\operatorname{arc\,tg} := \operatorname{tg}^{-1}_{\big|(-\frac{\pi}{2},\frac{\pi}{2})} : \mathbb{R} \to \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$

95. Define the differentiability and the derivative of a function at a point

3.1. Definition Let $f \in \mathbb{R} \to \mathbb{R}$, $a \in intD_f$. f is differentiable at "a" $\overset{\text{df}}{\Leftrightarrow}$

$$\exists \lim_{x\to a} \frac{f(x) - f(a)}{x - a} \in \mathbb{R}.$$

In this case $f'(a) := \lim_{a} \frac{f(x) - f(a)}{x - a}$. This number is called the derivative of f at the point "a".

Let us denote the set of functions that are differentiable at "a" by $D(a)$.

96. What is the of the derivative?

2. The geometrical meaning of the derivative is: the slope of the tangent line to the graph of f at the point $(a, f(a))$.

97. State the theorem about the connection between the differentiability and the continuity

3.4. Theorem $f \in D(a) \Rightarrow f \in C(a)$.

98. State and the derivative of the sum, of the product and of the quotient

3.5. Theorem *Let $f, g \in D(x)$. Then $f + g \in D(x)$ and*

$$(f+g)'(x) = f'(x) + g'(x)\,.$$

$$(fg)'(x) = f'(x) \cdot g(x) + f(x) \cdot g'(x)\,.$$

$$\left(\frac{f}{g}\right)'(x) = \frac{f'(x) \cdot g(x) - f(x) \cdot g'(x)}{(g(x))^2}\,.$$

99. State the derivative of the composition (Chain Rule)

3.8. Theorem *Let $g \in \mathbb{R} \to \mathbb{R}$, $g \in D(x)$, $f \in \mathbb{R} \to \mathbb{R}$, $f \in D(g(x))$. Then $f \circ g \in D(x)$ and*

$$(f \circ g)'(x) = f'(g(x)) \cdot g'(x)\,.$$

100. State the derivative of the inverse (Inverse Rule)

3.9. Theorem *Let $I \subseteq R$ be an open interval, $f : I \to \mathbb{R}$, $f \in D$, be an (strictly) increasing function. Furthermore suppose that $f'(x) \neq 0$ $(x \in I)$. Then $f^{-1} \in D(J)$ where $J = R_f$ (we know that R_f is an open interval) and*

$$(f^{-1})'(y) = \frac{1}{f'(f^{-1}(y))} \quad (y \in J)\,.$$