Analysis II. practice 12

$f \in \mathbb{R}^2 \to \mathbb{R}$, $a \in \operatorname{int} \mathcal{D}_f$ partial derivatives:

$$\partial_1 f(a) := \lim_{h \to 0} \frac{f(a_1+h, a_2) - f(a_1, a_2)}{h}$$

$$\partial_2 f(a) := \lim_{h \to 0} \frac{f(a_1, a_2+h) - f(a_1, a_2)}{h}$$

„we fix every variable except one and differentiate by that variable"

1. a) $f(x,y) = \ln(xy^2) - x^3 y^2 \cos(x^2+y^2) \qquad (x>0,\ y \neq 0)$

$$\partial_1 f(x,y) = \frac{y^2}{xy^2} - y^2 \cdot \left(3x^2 \cos(x^2+y^2) - x^3 \sin(x^2+y^2) \cdot 2x\right) =$$

$$= \frac{1}{x} - 3x^2y^2 \cos(x^2+y^2) + 2x^4 y^2 \sin(x^2+y^2)$$

$$\partial_2 f(x,y) = \frac{2xy}{xy^2} - x^3\left(2y \cos(x^2+y^2) - y^2 \sin(x^2+y^2) 2y\right) =$$

$$= \frac{2}{y} - 2x^3 y \cos(x^2+y^2) + 2x^3y^3 \sin(x^2+y^2)$$

b) $f(x,y) = \operatorname{arctg}\left(\frac{y}{x}\right) \qquad (y \in \mathbb{R},\ x \neq 0)$

$$\partial_1 f(x,y) = \frac{1}{1+\left(\frac{y}{x}\right)^2} \cdot \left(-\frac{y}{x^2}\right) = -\frac{y}{x^2+y^2}$$

$$\partial_2 f(x,y) = \frac{1}{1+\left(\frac{y}{x}\right)^2} \cdot \frac{1}{x} = \frac{x}{x^2+y^2}$$

Local extreme values of $\mathbb{R}^2 \to \mathbb{R}$ functions:

- theorem: $f \in D\{a\}$, f has local extremum in a

$$\Rightarrow \partial_1 f(a) = 0, \quad \partial_2 f(a) = 0$$

$$(f'(a) = (\partial_1 f(a), \partial_2 f(a)) = (0,0))$$

- theorem: $f \in D^2\{a\}$, $f'(a) = 0$;

$$f''(a) := \begin{pmatrix} \partial_{11} f(a) & \partial_{12} f(a) \\ \partial_{21} f(a) & \partial_{22} f(a) \end{pmatrix}$$

1. $f''(a)$ is positive definite ($\partial_{11} f(a) > 0$ and $\det f''(a) > 0$) $\Rightarrow$ loc. min. in a
2. $f''(a)$ is negative definite ($\partial_{11} f(a) < 0$ and $\det f''(a) > 0$) $\Rightarrow$ loc. max. in a
3. $f''(a)$ is indefinite ($\det f''(a) < 0$) $\Rightarrow$ no local extr. in a

2. local extreme values?

a, $f(x,y) = x^2 - 4xy + y^3 + 4y$

$$\partial_1 f(x,y) = 2x - 4y = 0 \quad \Rightarrow x = 2y$$
$$\partial_2 f(x,y) = -4x + 3y^2 + 4 = 0$$
$$3y^2 - 8y + 4 = 0$$
$$y_{1,2} = \frac{8 \pm \sqrt{64-48}}{6} \begin{cases} 2 & \Rightarrow x = 4 \\ 2/3 & \Rightarrow x = 4/3 \end{cases}$$

$\Rightarrow$ possible local extrema in $(4,2)$ and $\left(\frac{4}{3}, \frac{2}{3}\right)$

$$f''(x,y) = \begin{pmatrix} 2 & -4 \\ -4 & 6y \end{pmatrix}$$

$$f''(4,2) = \begin{pmatrix} 2 & -4 \\ -4 & 12 \end{pmatrix}$$

$2 > 0$, $\det = 8 > 0$
$\Rightarrow$ loc. min. in $(4,2)$,
$f(4,2) = 0$

$$f''\left(\frac{4}{3}, \frac{2}{3}\right) = \begin{pmatrix} 2 & -4 \\ -4 & 4 \end{pmatrix}$$

$\det = -8 < 0$
$\Rightarrow$ no local extr.

b) $f(x,y) = x^4 - 4xy + y^4$

$\partial_1 f(x,y) = 4x^3 - 4y = 0 \quad \Rightarrow y = x^3$

$\partial_2 f(x,y) = -4x + 4y^3 = 0$

$-4x + 4x^9 = 0$

$4x(x^8 - 1) = 0$ — $x = 0 \Rightarrow y = 0$; $x = 1 \Rightarrow y = 1$; $x = -1 \Rightarrow y = -1$

$$f''(x,y) = \begin{pmatrix} 12x^2 & -4 \\ -4 & 12y^2 \end{pmatrix}$$

$f''(0,0) = \begin{pmatrix} 0 & -4 \\ -4 & 0 \end{pmatrix}$ $\quad$ det $= -16 < 0 \Rightarrow$ no local extr.

$f''(1,1) = f''(-1,-1) = \begin{pmatrix} 12 & -4 \\ -4 & 12 \end{pmatrix}$ $\quad$ $12 > 0$, det $= 128 > 0$

$\Rightarrow$ loc. min. in $(1,1)$ and $(-1,-1)$,

$f(1,1) = f(-1,-1) = -2$

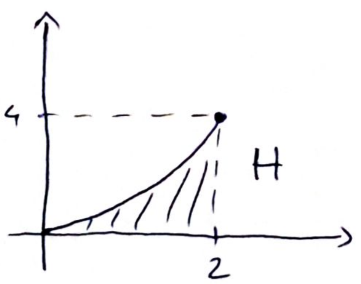
<u>global extreme values</u> on compact sets:

Weierstrass: $f \in C$, $\mathcal{D}_f$ compact $\Rightarrow \exists \min \mathcal{R}_f, \max \mathcal{R}_f$

possible places for global extrema: — inside $\mathcal{D}_f$ where $f' = 0$; — on the bounds of $\mathcal{D}_f$

(3.) $f(x,y) = 2xy - 3y$ global extrema on the set

$H = \{(x,y) \in \mathbb{R}^2 \mid 0 \le x \le 2,\ 0 \le y \le x^2\}$



1. $(x,y) \in \operatorname{int} H$:

$$\partial_1 f(x,y) = 2y = 0 \Rightarrow y = 0$$
$$\partial_2 f(x,y) = 2x - 3 = 0 \Rightarrow x = \frac{3}{2}$$
$$\Rightarrow \left(\frac{3}{2}, 0\right) \notin \operatorname{int} H$$

2. $0 \leq x \leq 2,\ y = 0$:

$$f(x,y) = 0 =: g(x) \quad (x \in [0,2])$$
$\Rightarrow$ possible extrema in points $(x,0)$ $(x \in [0,2])$

3. $x = 2,\ 0 \leq y \leq 4$:

$$f(x,y) = 4y - 3y = y =: h(y) \quad (y \in [0,4])$$
$y \in (0,4)$: $h'(y) = 1 \neq 0$

$\Rightarrow$ possible extrema in $(2,0), (2,4)$

4. $0 \leq x \leq 2,\ y = x^2$:

$$f(x,y) = 2x^3 - 3x^2 =: l(x) \quad (x \in [0,2])$$
$x \in (0,2)$: $l'(x) = 6x^2 - 6x = 0 \Leftrightarrow x = 1$

$\Rightarrow$ possible extrema: $(1,1), (0,0), (2,4)$

$f \in C$, H is compact $\to$ $\exists$ global min./max.

$f(x,0) = 0 \quad (x \in [0,2])$

$f(2,4) = 4 \quad \Rightarrow$ global max.

$f(1,1) = -1 \quad \Rightarrow$ global min.