NUMERICAL METHODS

2020/2021 Spring Semester

HOMEWORK #2

Deadline of submission: midnight before the second midterm

1. Find an iteration method to evaluate the value

$$\sqrt{2+\sqrt{2+\sqrt{2+\dots}}}$$

   Prove its convergence and give the order of convergence.

2. Consider the iteration $x_{n+1} = \sqrt{\cos(x_n)}$. Study its convergence. Find an interval for x_0 such that the fixed point iteration converges.

3. Apply Newton's method to approximate the solution of the equation $x - e^{-x} = 0$. Show that the method converges for all $0 < x_0 < 1$.

4. Show that the function $y(x)$ which fulfils the equation

$$\begin{vmatrix} y(x) & 1 & x & x^2 & \dots & x^n \\ f(x_0) & 1 & x_0 & x_0^2 & \dots & x_0^n \\ f(x_1) & 1 & x_1 & x_1^2 & \dots & x_1^n \\ & & \vdots & & & \\ f(x_n) & 1 & x_n & x_n^2 & \dots & x_n^n \end{vmatrix} = 0$$

   is $y(x) = L_n(x; f)$, the interpolating polynomial with respect to the nodes $x_0, x_1, \dots, x_n$.

5. Show that on the nodes $x_0, x_1, \dots, x_n$ $(n \geq 2)$

$$\sum_{j=0}^{n} (x - x_j)^k \ell_j(x) = \begin{cases} 0, & \text{ha } 1 \leq k \leq n \\ 1, & \text{ha } k = 0 \end{cases},$$

   where $\ell_j(x)$ denotes the fundamental polynomial of Lagrange interpolation with respect to x_j.

6. Show that on the nodes $x_0, x_1, \dots, x_n$

$$\sum_{j=0}^{n} \frac{x_j^k}{\omega_n'(x_j)} = \begin{cases} 0, & \text{ha } 0 \leq k \leq n-1 \\ 1, & \text{ha } k = n \end{cases},$$

   where $\omega_n(x) = (x - x_0)(x - x_1) \cdots (x - x_n)$.

7. Show that on the nodes $x_0, x_1, \ldots, x_n$

$$x^{n+1} - \sum_{j=0}^{n} x_j^{n+1} \ell_j(x) = \omega_n(x).$$

8. It is known that

$$\sum_{j=0}^{n} j^3 = P_4(n)$$

where $P_4(x)$ is a polynomial of degree 4. Find the polynomial $P_4(x)$ using interpolation.

9. Consider the

$$\int_0^1 f(x)\sqrt{x}\, dx \approx \sum_{k=1}^{n} A_k^{(n)} \cdot f\left(\frac{k}{n}\right)$$

quadrature formula, which is exact for the polynomials of degree at most n. Find the coefficients of the formula for $n = 3$ and apply it to approximate the integral

$$\int_0^1 \sqrt{x} \cos x\, dx.$$

10. Consider the

$$\int_0^1 f(x)\, dx \approx A_0 f(x_0) + A_1 f(x_1)$$

quadrature formula Find A_0, A_1 and x_0, x_1 for highest degree of precision.