Analysis II. practice 3

1. determine f'

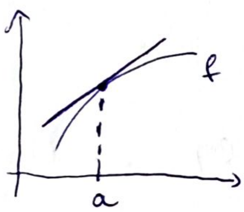
a, $f(x) = \arcsin(2x^2 - \sqrt{x})$

$$f'(x) = \frac{1}{\sqrt{1-(2x^2-\sqrt{x})^2}} \cdot \left(4x - \frac{1}{2\sqrt{x}}\right)$$

b, $f(x) = \dfrac{\arcsin x}{\operatorname{arctg} x}$

$$f'(x) = \frac{\frac{1}{\sqrt{1-x^2}} \cdot \operatorname{arctg} x - \arcsin x \cdot \frac{1}{1+x^2}}{\operatorname{arctg}^2 x}$$

reminder:



tangent line:
$y = f(a) + f'(a)(x-a)$

2. determine the equation of the tangent line

a, $y = \dfrac{x}{x^2-2}$, $a = 2$

$$f(x) := \frac{x}{x^2-2} \Rightarrow f'(x) = \frac{x^2-2-2x^2}{(x^2-2)^2} = \frac{-x^2-2}{(x^2-2)^2}$$

$f(2) = 1$, $f'(2) = -\frac{3}{2}$

equation: $y = f(2) + f'(2)(x-2) =$

$$= 1 - \frac{3}{2}(x-2) = -\frac{3}{2}x + 4$$

b, $y = e^x + e^{2x}$, $a = 0$

$f(x) := e^x + e^{2x} \Rightarrow f'(x) = e^x + 2e^{2x}$

$\Rightarrow \quad y = f(0) + f'(0)(x-0) =$

$= 2 + 3x$

c, $x^2 y = 2y + x^{x+1}$, $P(1,-1)$

$y(x^2-2) = x^{x+1}$

$y = \dfrac{x^{x+1}}{x^2-2}$ ($x^2-2 \neq 0$ when x is „around 1")

$f(x) := \dfrac{x^{x+1}}{x^2-2}$; $a := 1 \Rightarrow f(a) = -1$

$x^{x+1} = e^{(x+1)\ln x} \Rightarrow (x^{x+1})' = x^{x+1}\left(\ln x + \frac{x+1}{x}\right)$

$$f'(x) = \frac{x^{x+1}\left(\ln x + \frac{x+1}{x}\right)(x^2-2) - x^{x+1}\,2x}{(x^2-2)^2}$$

equation: $y = f(1) + f'(1)(x-1) =$

$= -1 - 4(x-1) = -4x + 3$

<u>reminder</u>:

(1.) $f: I \to \mathbb{R}$ ($I \subset \mathbb{R}$ open interval), $f \in D$:

$f' \geq 0 \Leftrightarrow f \nearrow$; $f' \leq 0 \Leftrightarrow f \searrow$

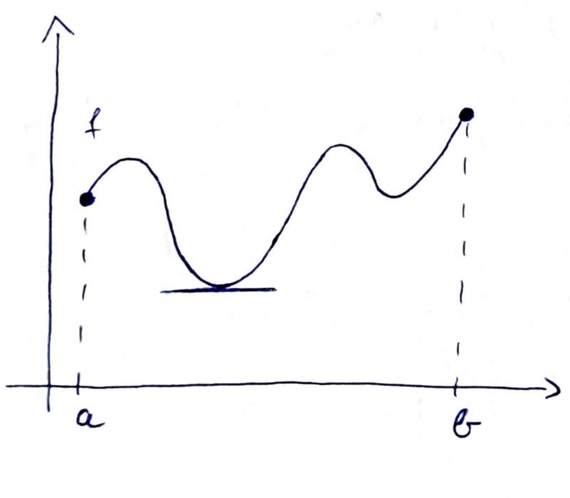
$f' > 0 \Rightarrow f \uparrow$; $f' < 0 \Rightarrow f \downarrow$

(2.) $f \in D\{a\}$, f has local extremum in a $\Rightarrow f'(a) = 0$

(3.) $f \in D\{a\}$, $f'(a) = 0$, f' changes sign in a $\Rightarrow$

f has local extremum in a ($+ \to -$: max, $- \to +$: min)

(4.) Weierstrass theorem: $f \in C[a,b] \Rightarrow \exists$ global min/max



3. monotonicity, extreme values (local/global)

a, $f(x) = 1 - 4x - x^2 \quad (x \in \mathbb{R})$

$f'(x) = -2x - 4 \quad = 0 \iff x = -2$

$\qquad\qquad\qquad\quad > 0 \iff x < -2$

$\qquad\qquad\qquad\quad < 0 \iff x > -2$

x		-2	
f'	$+$	0	$-$
f	$\nearrow$		$\searrow$

$\Rightarrow f \uparrow$ on $(-\infty, -2)$, $f \downarrow$ on $(-2, +\infty)$

$\Rightarrow$ loc. max. in -2, $f(-2) = 5$ (also global max.)

$\lim\limits_{\pm\infty} f = \lim\limits_{x \to \pm\infty} \left(x^2\left(-1 - \frac{4}{x} + \frac{1}{x^2}\right)\right) = -\infty \quad \Rightarrow$ no global min.

b, $f(x) = \dfrac{x}{x^2 - 6x - 16} \quad (x \in \mathbb{R} \setminus \{-2, 8\})$

$f'(x) = \dfrac{x^2 - 6x - 16 - x(2x-6)}{(x^2-6x-16)^2} = \dfrac{-x^2 - 16}{(x^2 - 6x - 16)^2} < 0 \quad (x \in \mathbb{R} \setminus \{-2, 8\})$

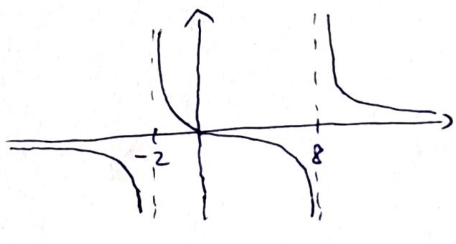
$\Rightarrow f \downarrow$ on the intervals $(-\infty, -2)$, $(-2, 8)$, $(8, +\infty)$;

there are no local extrema

$\Rightarrow$ no global either (since D_f is open)

$\left(\lim\limits_{x \to -2 \pm 0} f(x) = \pm\infty, \quad \lim\limits_{8 \pm 0} f = \pm\infty \right.$

$\left. \lim\limits_{\pm\infty} f = 0 \right)$



c, $f(x) = e^{x^2 - 4x} \quad (x \in \mathbb{R})$

$f'(x) = e^{x^2 - 4x}(2x - 4) \quad = 0 \iff x = 2$

$\qquad\qquad\qquad\qquad\quad > 0 \iff x > 2$

$\qquad\qquad\qquad\qquad\quad < 0 \iff x < 2$

x		2	
f'	$- -$	0	$+ +$
f	$\searrow$		$\nearrow$

$f \downarrow$ on $(-\infty, 2)$, $\uparrow$ on $(2, +\infty)$

$f(2) = e^{-4}$ local and global min.

no global max.

d, $f(x) = x \cdot \ln x \quad (x>0)$

$$f'(x) = \ln x + 1 \quad \begin{aligned} &= 0 \iff x = \tfrac{1}{e} \\ &> 0 \iff x > \tfrac{1}{e} \\ &< 0 \iff 0 < x < \tfrac{1}{e} \end{aligned}$$

x		$\frac{1}{e}$	
f'	$- -$	0	$+ +$
f	$\searrow$		$\nearrow$

$f(\frac{1}{e}) = -\frac{1}{e}$ local and global min.

no global max

e, $f(x) = x^3 - 3x^2 + 3x + 2 \quad (x \in \mathbb{R})$

$$f'(x) = 3x^2 - 6x + 3 = 3(x-1)^2 \quad \begin{aligned} &= 0 \iff x = 1 \\ &> 0 \iff x \in \mathbb{R} \setminus \{1\} \end{aligned}$$

x		1	
f'	$+ +$	0	$+ +$
f	$\nearrow$		$\nearrow$

no local or global extrema

f, $f(x) = x^2 e^{-x} \quad (x \in \mathbb{R})$

$$f'(x) = 2xe^{-x} - x^2 e^{-x} = e^{-x} x(2-x) \quad \begin{aligned} &= 0 \iff x = 0, x = 2 \\ &> 0 \iff 0 < x < 2 \\ &< 0 \iff x < 0, x > 2 \end{aligned}$$

x		0		2	
f'	$-$	0	$+$	0	$-$
f	$\searrow$		$\nearrow$		$\searrow$

$$\left.\begin{aligned} f(0) &= 0 \quad \text{loc. min.} \\ f(2) &= \tfrac{4}{e^2} \quad \text{loc. max.} \end{aligned}\right\} \text{global?}$$

$$\lim_{x \to -\infty} (x^2 e^{-x}) = +\infty \implies \text{no global max.}$$

$$f(x) = x^2 e^{-x} \geq f(0) = 0 \quad (x \in \mathbb{R})$$

$$\implies f(0) = 0 \text{ global min.}$$

(remark: $\displaystyle\lim_{x \to +\infty} \frac{x^2}{e^x} = 0$, see later L'Hospital's rule)

g, $f(x) = x - \ln(1+x) \quad (x > -1)$

homework :)