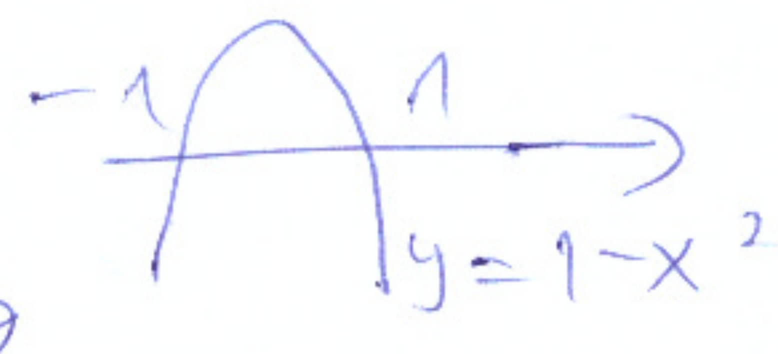
Analysis 2 | Week 4.

10./ Determine the global extreme values of f if:

a) $f(x) = \dfrac{x}{1+x^2} \quad (x \in \mathbb{R}) \implies$



$$f'(x) = \frac{1+x^2 - x \cdot 2x}{(1+x^2)^2} = \frac{1-x^2}{(1+x^2)^2} \quad (x \in \mathbb{R})$$

x	$-\infty$		-1		1		$+\infty$
$f'(x)$		$-$	0	$+$	0	$-$	
$f(x)$	0	$\searrow$	$-\frac{1}{2}$	$\nearrow$	$1/2$	$\searrow$	0

$f(-1) = \dfrac{-1}{2}$; $\displaystyle\lim_{x \to +\infty} \frac{x}{1+x^2} = 0$

$f(1) = \dfrac{1}{2}$ $\quad \displaystyle\lim_{x \to -\infty} \frac{x}{1+x^2} = 0$

So we have a $\boxed{\text{global min.}}$ at -1 with value $-\frac{1}{2}$ and a $\boxed{\text{global max}}$ at $x = 1$ with value $y = \frac{1}{2}$.

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b) $f(x) = \sin^4 x + \cos^4 x \quad \left(x \in \left[-\frac{2\pi}{3} ; \frac{\pi}{3}\right]\right)$.

Now the domain of f is a closed and bounded set (compact) so according to Weierstrass's theorem

we will surely have min / max.
Where can they be?
In inner points of the set where the derivative is 0 or the „endpoints"!
All we need to do is to find the possible points for min / max.

Here if $x \in \left(-\frac{2\pi}{3}, \frac{\pi}{3}\right) \Rightarrow$

$$f'(x) = 4\sin^3 x \cdot \cos x - 4\cos^3 x \cdot \sin x =$$

$$= 4\sin x \cos x \cdot \underbrace{(\sin^2 x - \cos^2 x)}_{} =$$

$$= +2 \cdot \sin 2x \cdot (-\cos 2x) =$$

$$= \underbrace{-\sin 4x}_{} = 0 \Leftrightarrow \sin 4x = 0 \Leftrightarrow$$

$$4x = k\pi \qquad x = \frac{k\pi}{4} \quad (k \in \mathbb{Z})$$

we need k values for which

$$\frac{k\pi}{4} \in \left(-\frac{2\pi}{3}, \frac{\pi}{3}\right)$$

so $\boxed{k = 0;\ 1;\ -1;\ -2}$

Compare the function values at the „inner" and „end" points:

$f(0) = 1$

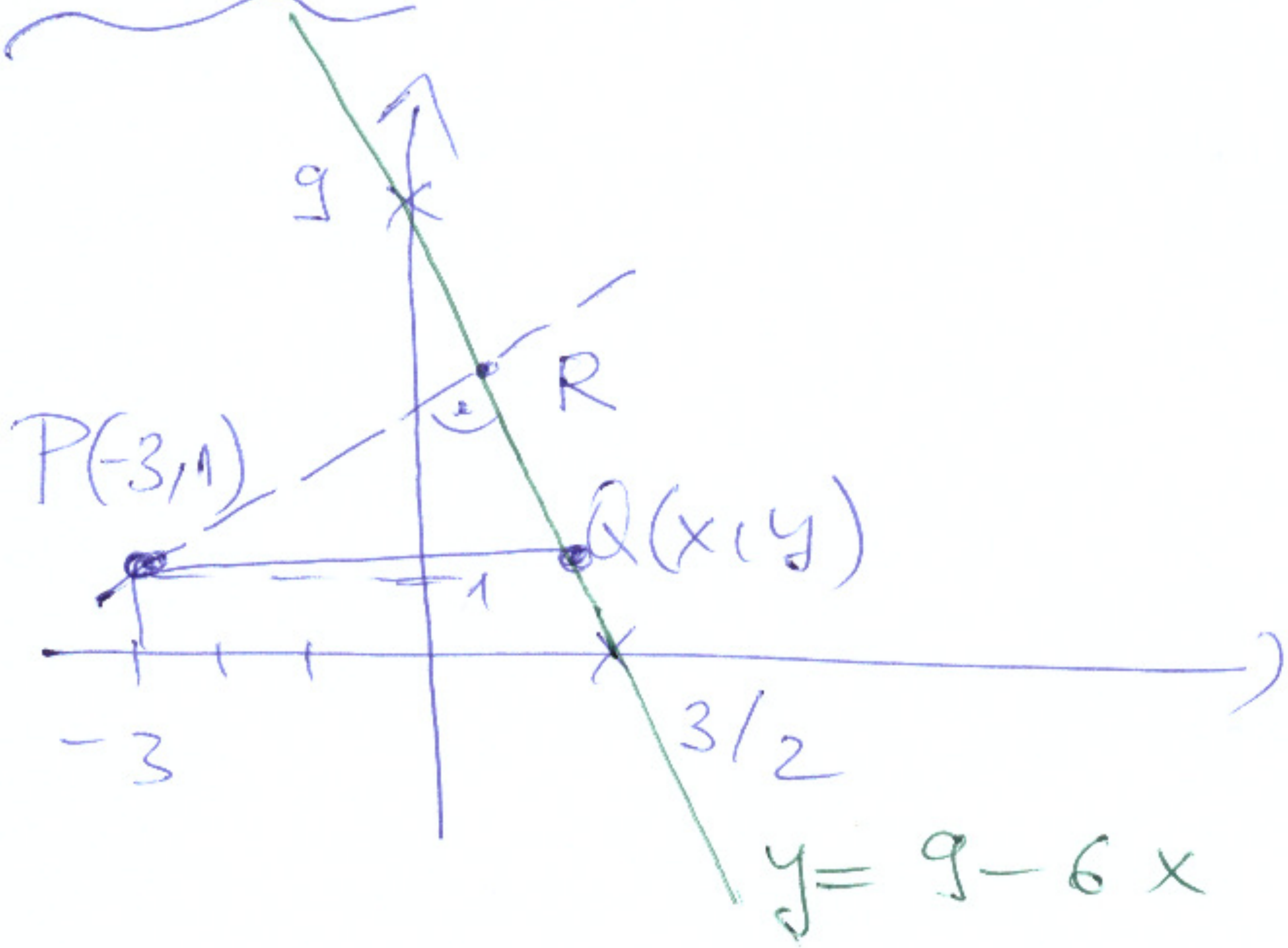
$$\left.\begin{array}{l} f\left(\frac{\pi}{4}\right) = \left(\frac{\sqrt{2}}{2}\right)^4 + \left(\frac{\sqrt{2}}{2}\right)^4 = 2 \cdot \frac{1}{4} = \frac{1}{2} \\ f\left(-\frac{\pi}{4}\right) = \left(\sin\left(-\frac{\pi}{4}\right)\right)^4 + \cos\left(-\frac{\pi}{4}\right)^4 = \frac{1}{2} \end{array}\right\} \underline{\min}$$

$f\left(-\frac{\pi}{2}\right) = 1$

$$\left.\begin{array}{l} f\left(-\frac{2\pi}{3}\right) = \left(\frac{\sqrt{3}}{2}\right)^4 + \left(-\frac{1}{2}\right)^4 = \frac{5}{4} \\ f\left(\frac{\pi}{3}\right) = \left(\frac{\sqrt{3}}{2}\right)^4 + \left(\frac{1}{2}\right)^4 = \frac{20}{16} = \frac{5}{4} \end{array}\right\} \underline{\max}.$$

$\boxed{11/4.}$ $\left.\begin{array}{l} 6x + y = 9 \\ P(-3, 1) \end{array}\right\}$ Which point of the given line is closest to the point P?

Solution:



Consider a point Q(x,y) of the line above and we minimalize the square of the distance PQ.

$$PQ^2=(x-(-3))^2+(y-1)^2=\underbrace{(x+3)^2+(y-1)^2}$$

we have the condition: $\underbrace{y=9-6x}$

By substituting into the square-distance function we get a one variable function

$$f(x):=(x+3)^2+(9-6x-1)^2$$

$$f(x)=(x+3)^2+(8-6x)^2 \quad (x\in\mathbb{R})$$

We want a global min value here

$D_f = \mathbb{R}$ is not compact (is not bounded set) so no Weierstrass can be applied here

$$f'(x) = 2(x+3) + 2\cdot(8-6x)\cdot(-6)$$

$$= 2x + 6 - 96 + 72x = 74x - 90 = 0$$

$$\Longleftrightarrow \quad x = \frac{90}{74} = \underline{\underline{\frac{45}{37}}}$$

x	$-\infty$		$\frac{45}{37}$		$+\infty$
$f'(x)$	$-$	$-$	0	$+$	$+$ $+$
$f(x)$	$\searrow$		$f\left(\frac{45}{37}\right)$		$\nearrow$

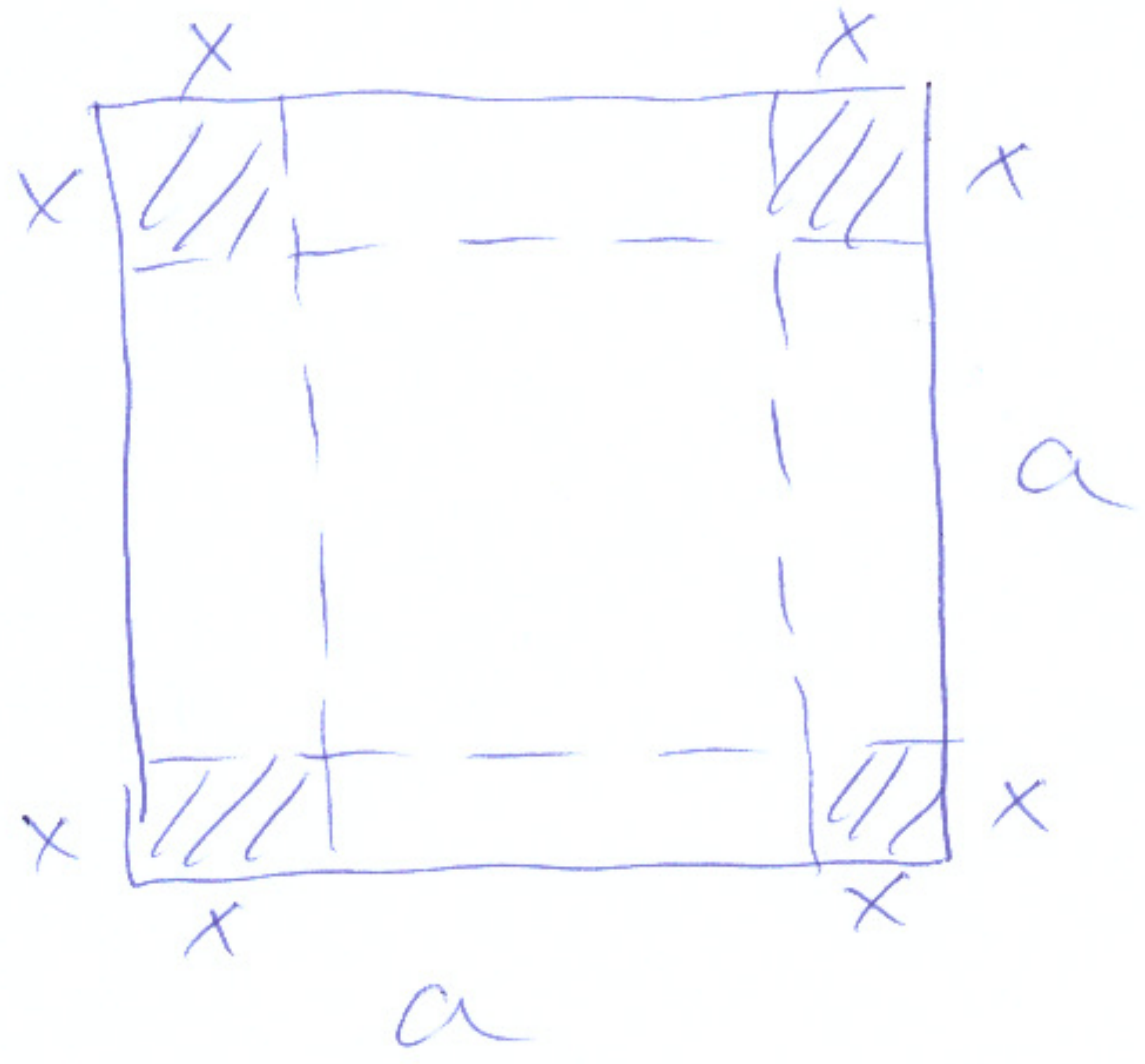
global min here

$$f\left(\frac{45}{37}\right) = \left(\frac{45}{37} + 3\right)^2 + \left(8 - 6\cdot\frac{45}{37}\right)^2$$ is the minimal value of f.

So point $\boxed{R\left(\frac{45}{37}; \frac{63}{37}\right)}$ is the closest to the given P.

$$y = 9 - 6\cdot\frac{45}{37} = \frac{63}{37}$$

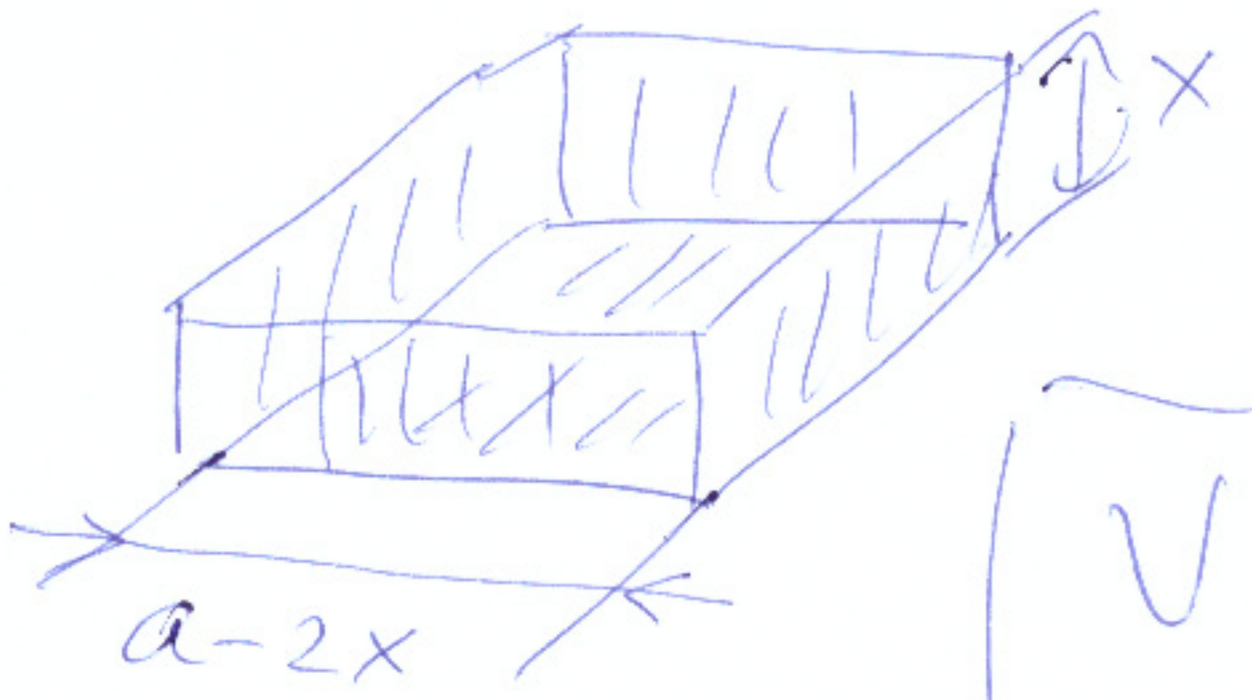
1 Find $x>0$ so that the box that can be made here to have a maximal volume.



$a>0$ is given

We cut off 4 squares of sides $0<x$ as in the picture is shown...

then we make a box, an open one from the top



The volume of this open box is:

$$\boxed{V(x) = (a-2x)^2 \cdot x \qquad x \in \left[0; \frac{a}{2}\right].}$$

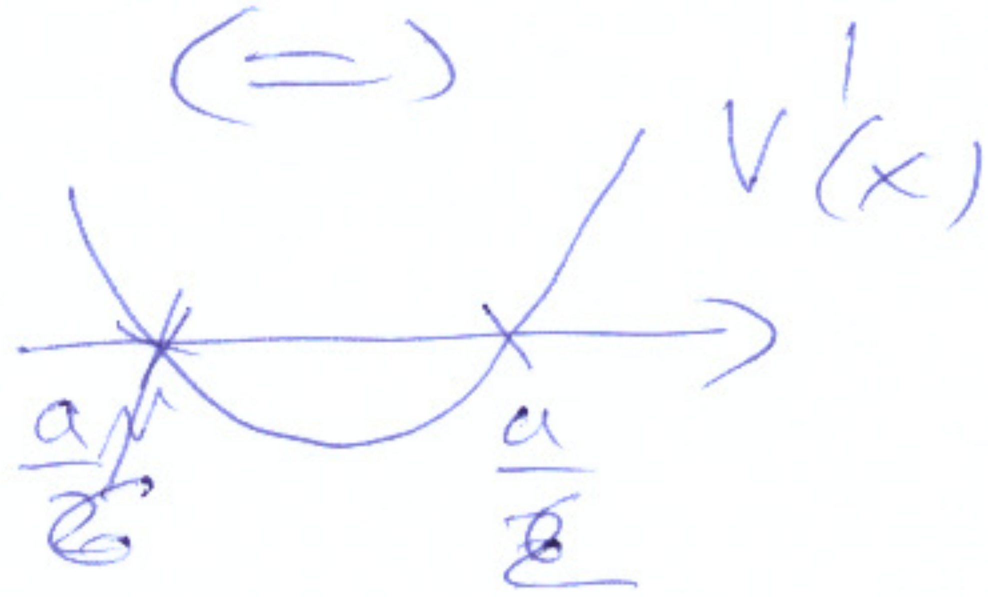
If $x \in \left(0, \frac{a}{2}\right)$ (inner points) $\Rightarrow$

$$V'(x) = 2 \cdot (a-2x) \cdot (-2) \cdot x + (a-2x)^2 \cdot 1 =$$

$$= (a-2x) \cdot (-4x + a - 2x) =$$

$$= (a-2x) \cdot (a-6x) = 0 \iff$$

$$x_1 = \frac{a}{2} \quad \text{or} \quad x_2 = \frac{a}{6}.$$



x	0		$\frac{a}{6}$		$\frac{a}{2}$
$V'(x)$	$+$	$+ +$	0	$- \; - \; -$	0
$V(x)$	0	$\nearrow$	$\frac{2a^3}{27}$	$\searrow$	0

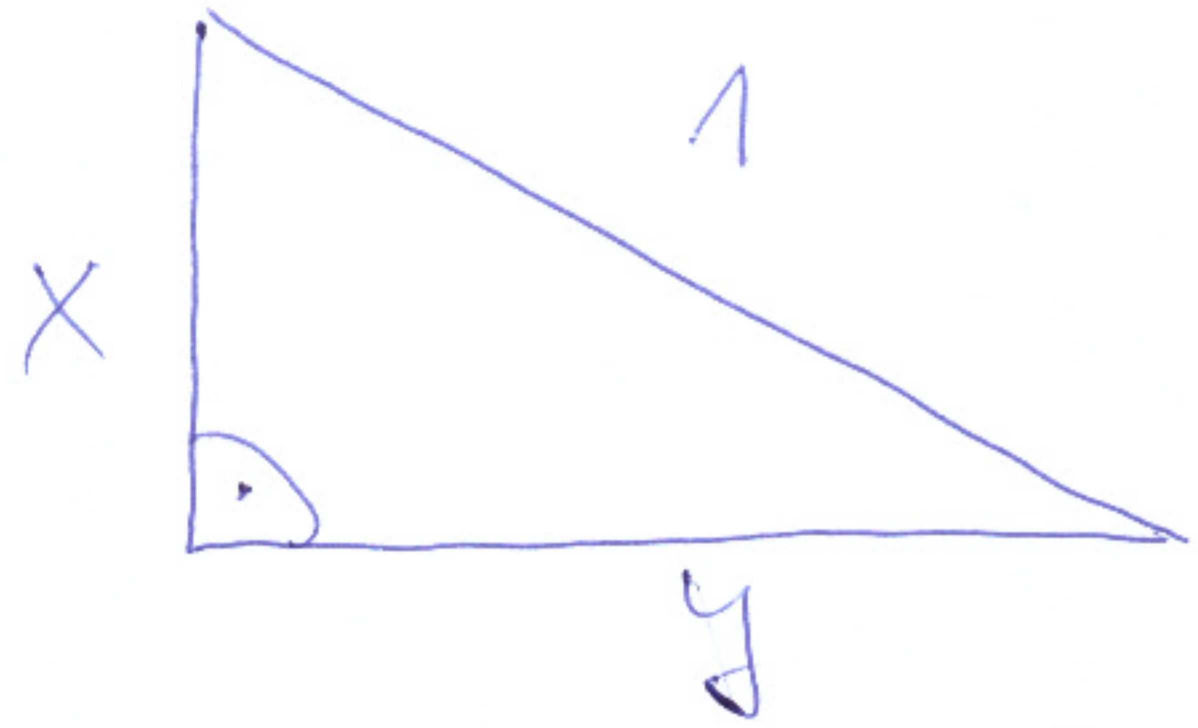
Abs. max. here

$$V\left(\frac{a}{6}\right) = \left(a - 2 \cdot \frac{a}{6}\right)^2 \cdot \frac{a}{6} = \left(a - \frac{a}{3}\right)^2 \cdot \frac{a}{6} =$$

$$= \left(\frac{2a}{3}\right)^2 \cdot \frac{a}{6} = \frac{4a^3}{9 \cdot 6} = \frac{2a^3}{9 \cdot 3} = \underline{\underline{\frac{2a^3}{27}}}$$

So $\boxed{x = \frac{a}{6}}$

$\boxed{11/e.}$



$$\boxed{\max(x+2y)=?}$$

By Pythagorean theorem we have:

$$x^2+y^2=1 \Rightarrow \text{since } y \geq 0$$

we have: $y=\sqrt{1-x^2}$

We make a function of <u>one variable</u>

$$f(x):=x+2\cdot\sqrt{1-x^2} \quad (x\in[0,1]).$$

$$\left.\begin{array}{l} D_f=[0,1] \text{ is compact;} \\ f\in C[0,1] \end{array}\right\} \Rightarrow$$

<u>Weierstrass</u> there is abs. max and abs. min. value.

If $x\in(0,1)$ (inner point) $\Rightarrow$

$$f'(x)=1+2\cdot\frac{1}{2\sqrt{1-x^2}}\cdot(-2x) =$$

$$= 1 - \frac{2x}{\sqrt{1-x^2}} = 0 \iff$$

$$\frac{2x}{\sqrt{1-x^2}} = 1 \qquad 0 \le 2x = \sqrt{1-x^2} \qquad (\;)^2$$

$$\Rightarrow \quad 4x^2 = 1 - x^2 \qquad 5x^2 = 1$$

$$0 < x = \frac{1}{\sqrt{5}};$$

So compare all possible places for extremal values:

$$f\left(\frac{1}{\sqrt{5}}\right) = \frac{1}{\sqrt{5}} + 2\cdot\sqrt{1-\frac{1}{5}} = \frac{1}{\sqrt{5}} + 2\sqrt{\frac{4}{5}} = \frac{5}{\sqrt{5}} = \underline{\underline{\sqrt{5}}}$$

$$f(0) = \underline{\underline{2}}$$

abs. max. value

$$f(1) = \underline{\underline{1}} \leftarrow \text{abs. min. value.}$$

So $\boxed{x = \frac{1}{\sqrt{5}};\ y = \frac{2}{\sqrt{5}}}$.