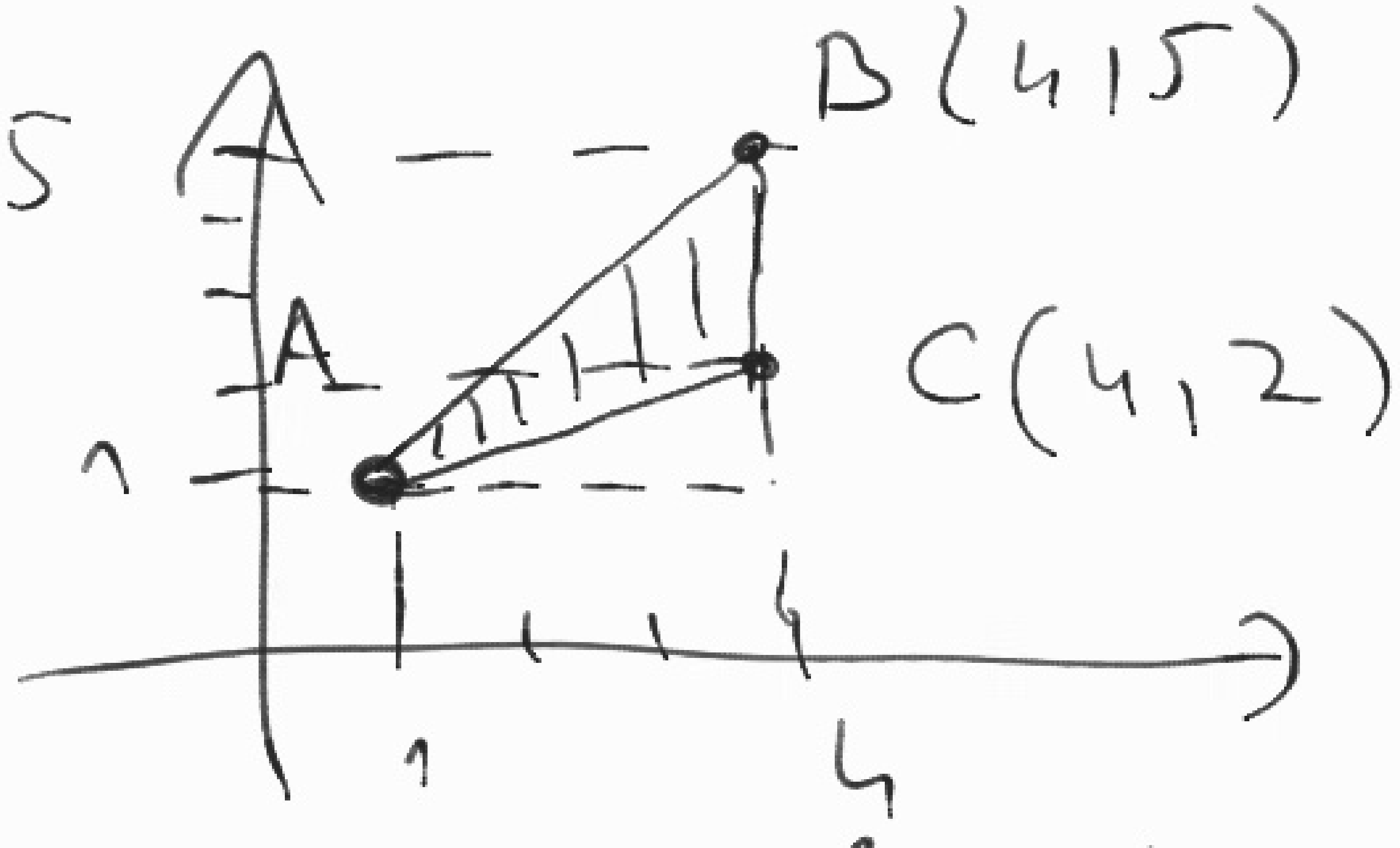
① $\iint\limits_D xy\,dx\,dy$

D: the triangle with vertices

$A(1,1)$, $B(4,5)$, $C(4,2)$




Equations of lines:

$AC: \quad \ell(t) =$

$$= (1,1) + t \cdot \big( (4,2) - (1,1) \big) =$$

$$= (1,1) + t(3,1) =$$

$$= (3t+1, t+1) \quad 10$$

$$\begin{cases} x = 3t+1 \Rightarrow \\ y = t+1 \end{cases} \Rightarrow$$

$$t = y - 1 \Rightarrow$$

$$x = 3(y-1) + 1 \Rightarrow$$

$$y = \frac{1}{3}(x-1) + 1$$

$$y = \frac{1}{3}x + 1 - \frac{1}{3}$$

$$\boxed{y = \frac{1}{3}x + \frac{2}{3}}$$

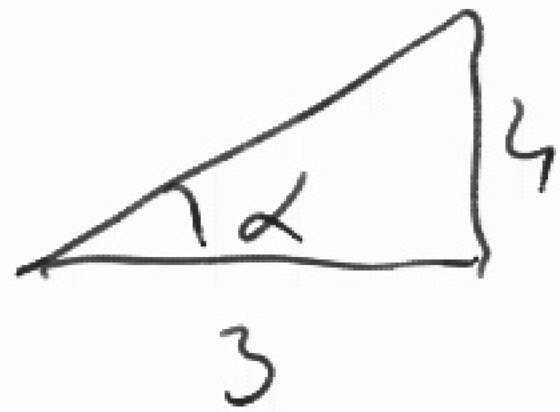
and for line AB

$y = ax + b$ with

$(1|1) \in AB$
$(4|5) \in AB$

$$\begin{cases} 1 = a + b \\ 5 = 4a + b \end{cases} \Rightarrow$$

$3a=4 \Rightarrow a=\frac{4}{3}$




$= \operatorname{tg}\alpha$

$b = 1-a = 1-\frac{4}{3} = -\frac{1}{3} < 0$

$$\boxed{y=\frac{4}{3}x-\frac{1}{3}}$$

$f\in C(ABC) \Rightarrow f\in R(ABC)$

$$\iint_{ABC} xy\,dx\,dy = \int_{1}^{4}\int_{\frac{1}{3}x+\frac{2}{3}}^{\frac{4}{3}x-\frac{1}{3}} xy\,dy\,dx =$$

$$= \int_1^4 x \cdot \left[\frac{y^2}{2}\right]_{\frac{x}{3}+\frac{2}{3}}^{\frac{4}{3}x-\frac{1}{3}} dx =$$

$$= \frac{1}{2} \cdot \int_1^4 x \left[\left(\frac{4}{3}x - \frac{1}{3}\right)^2 - \left(\frac{x}{3} + \frac{2}{3}\right)^2\right] dx$$

$$= \frac{1}{2} \cdot \int_1^4 x \cdot \left(\frac{4}{3}x - \frac{1}{3} - \frac{x}{3} - \frac{2}{3}\right) \cdot$$

$$\cdot \left(\frac{4}{3}x - \frac{1}{3} + \frac{x}{3} + \frac{2}{3}\right) dx =$$

$$= \frac{1}{2} \int_1^4 x \cdot (x-1) \cdot \left(\frac{5}{3}x + \frac{1}{3}\right) dx =$$

$$= \frac{1}{6} \cdot \int_1^4 x(x-1)(5x+1)\, dx =$$

$$= \frac{1}{6} \int_1^4 (x^2 - x)(5x+1)\, dx =$$

$$= \frac{1}{6} \int_1^4 (5x^3 + x^2 - 5x^2 - x)\, dx$$

$$= \frac{1}{6} \cdot \int_1^4 (5x^3 - 4x^2 - x)\, dx =$$

$$= \frac{1}{6}\cdot\left[5\frac{x^4}{4} - 4\frac{x^3}{3} - \frac{x^2}{2}\right]_1^4 =$$

$$= \frac{1}{6}\cdot\left[5\frac{4^4}{4} - 4\frac{4^3}{3} - \frac{4^2}{2} \underline{- \frac{5}{4} + 4} \underline{+ \frac{1}{2}}\right]$$

$$= \frac{1}{6}\left[5\cdot 4^3 - 4^3\cdot\frac{4}{3} \underbrace{- 8 + 4}_{-4} - \frac{3}{4}\right]$$

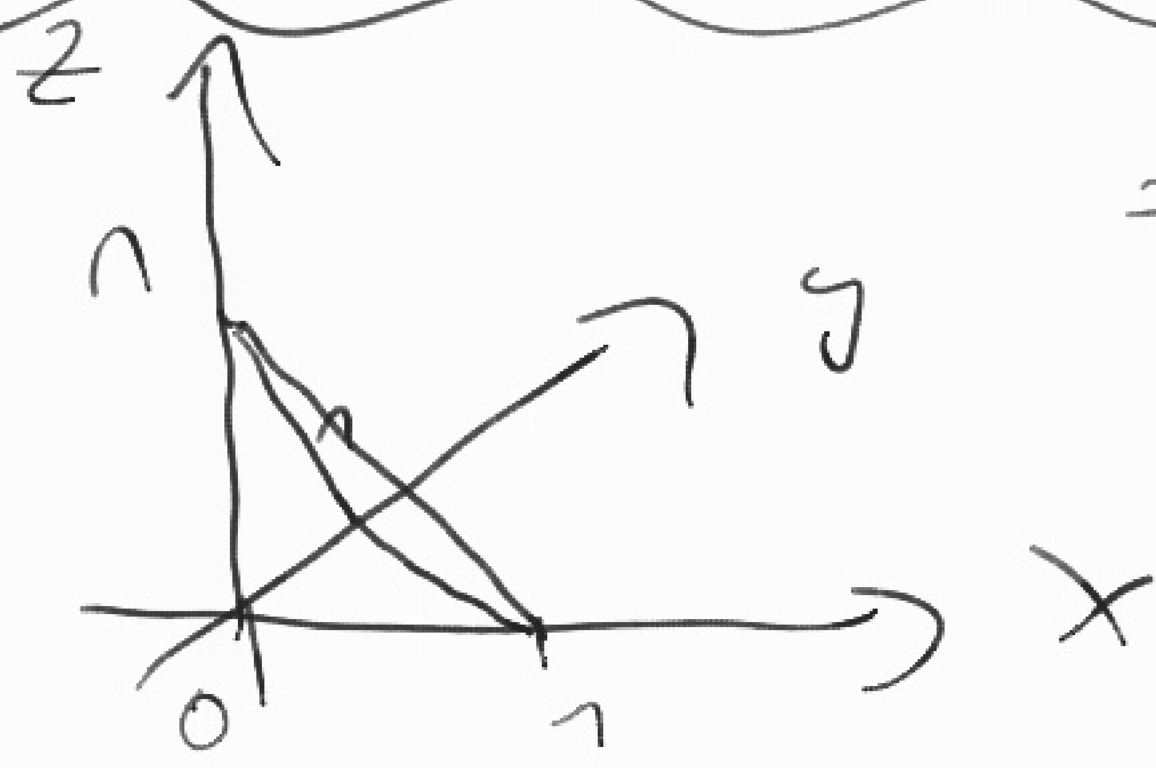
$$= \frac{1}{6}\cdot\left[4^3\left(5 - \frac{4}{3}\right) - \frac{19}{4}\right]$$

$$= \frac{1}{6}\left[\frac{11\cdot 64}{3} - \frac{19}{4}\right] =$$

$$= \frac{1}{6} \cdot \frac{2816 - 57}{3 \cdot 4} = \frac{2759}{72};$$

(2) $f(x,y,z) = 2xy$

$$\begin{cases} (x,y,z) \in \mathbb{R}^3 \\ x \geq 0, \quad y \geq 0, \quad z \geq 0, \\ x+y+z \leq 1. \end{cases}$$




$\Longrightarrow$

$$f \in C(D) \Rightarrow f \in R(D) \text{ \underline{and}}$$

$$\iiint_D f = \int_0^1 \int_0^{1-x} \int_0^{1-x-y} 2xy \, dz \, dy \, dx$$

$$= 2 \cdot \int_0^1 x \cdot \int_0^{1-x} y \left( \int_0^{1-x-y} 1 \, dz \right) dy \, dx$$

$$= 2 \int_0^1 x \cdot \int_0^{1-x} \left[ z \right]_0^{1-x-y} \cdot y \, dy \, dx$$

$$= 2 \int_0^1 x \cdot \int_0^{1-x} (1-x-y) y \, dy \, dx$$

$$= 2 \int_0^1 x \cdot \int_0^{1-x} (y - xy - y^2)\, dy\, dx$$

$$= 2 \int_0^1 x \cdot \left[ \frac{(1-x) \cdot y^2}{2} - \frac{y^3}{3} \right]_0^{1-x} dx =$$

$$= 2 \cdot \int_0^1 x \cdot \left( \frac{(1-x)^3}{2} - \frac{(1-x)^3}{3} \right) dx$$

$$= \frac{2}{6} \cdot \int_0^1 x\, (1-x)^3\, dx$$

$$= \frac{1}{3} \cdot \int_0^1 x (1 - 3x + 3x^2 - x^3)\, dx$$

$$= \frac{1}{3} \cdot \int_0^1 \left(x - 3x^2 + 3x^3 - x^4\right) dx$$

$$= \frac{1}{3} \left[ \frac{x^2}{2} - x^3 + 3\frac{x^4}{4} - \frac{x^5}{5} \right]_0^1 =$$

$$= \frac{1}{3} \left( \frac{1}{2} - 1 + \frac{3}{4} - \frac{1}{5} \right) =$$

$$= \frac{1}{3} \, \frac{10 - 20 + 15 - 4}{20} =$$

$$= \frac{1}{3} \, \frac{1}{20} = \boxed{\frac{1}{60}}$$