

Homework 5, for Linear Algebra class@ csie.ntnu.edu.tw-2025

1. (10%) Let A be any $m \times n$ matrix.

(a) Prove that $A^T A$ and A have the same null space.

(b) Use (a) to prove that $\text{rank } A^T A = \text{rank } A$.

2. (18%) Let $\mathcal{S} = \left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 1 \\ 5 \end{bmatrix} \right\}$.

(a) Apply the Gram-Schmidt process to replace the given linearly independent set \mathcal{S} by an orthogonal set of nonzero vectors with the same span, and obtain an orthonormal set with the same span as \mathcal{S} .

(b) Let A be the matrix whose columns are the vectors in \mathcal{S} . Determine the matrices Q and R in a QR factorization of A .

(c) Use the QR factorization to solve the system $Ax = \mathbf{b}$ where $\mathbf{b} = \begin{bmatrix} 8 \\ 0 \\ 1 \\ 11 \end{bmatrix}$.

3. (15%) Let $\{\mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_n\}$ be an orthonormal basis for \mathbb{R}^n . Prove that, for any vectors \mathbf{u} and \mathbf{v} in \mathbb{R}^n ,

(a) $\mathbf{u} + \mathbf{v} = (\mathbf{u} \cdot \mathbf{w}_1 + \mathbf{v} \cdot \mathbf{w}_1)\mathbf{w}_1 + (\mathbf{u} \cdot \mathbf{w}_2 + \mathbf{v} \cdot \mathbf{w}_2)\mathbf{w}_2 + \dots + (\mathbf{u} \cdot \mathbf{w}_n + \mathbf{v} \cdot \mathbf{w}_n)\mathbf{w}_n$.

(b) $\mathbf{u} \cdot \mathbf{v} = (\mathbf{u} \cdot \mathbf{w}_1)(\mathbf{v} \cdot \mathbf{w}_1) + (\mathbf{u} \cdot \mathbf{w}_2)(\mathbf{v} \cdot \mathbf{w}_2) + \dots + (\mathbf{u} \cdot \mathbf{w}_n)(\mathbf{v} \cdot \mathbf{w}_n)$. (This is known as *Parseval's identity*.)

(c) $\|\mathbf{u}\|^2 = (\mathbf{u} \cdot \mathbf{w}_1)^2 + (\mathbf{u} \cdot \mathbf{w}_2)^2 + \dots + (\mathbf{u} \cdot \mathbf{w}_n)^2$.

4. (15 %) Let $\mathbf{u} = \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix}$ and W be the solution set of the system of equations $\begin{aligned} x_1 + 2x_2 - 3x_3 &= 0 \\ x_1 + x_2 - 3x_3 &= 0 \end{aligned}$

(a) Find the orthogonal projection matrix P_W .

(b) Obtain the unique vectors \mathbf{w} in W and \mathbf{z} in W^\perp such that $\mathbf{u} = \mathbf{w} + \mathbf{z}$.

(c) Find the distance from \mathbf{u} to W .

5. (10 %) Let W be a subspace of \mathbb{R}^n . Prove that $P_W P_{W^\perp} = P_{W^\perp} P_W = O$, and hence $P_{W^\perp} = I_n - P_W$ (or $P_W + P_{W^\perp} = I_n$).

6. (12 %) An inconsistent system of linear equations $Ax = \mathbf{b}$ is given where $A = \begin{bmatrix} 1 & 2 & -1 \\ 1 & -1 & 2 \\ 2 & 1 & 1 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}$.

(a) Obtain the vector \mathbf{z} for which $\|A\mathbf{z} - \mathbf{b}\|$ is a minimum. (b) Find the vector \mathbf{z} of least norm for

which $\|A\mathbf{z} - \mathbf{b}\|$ is a minimum.

7. (10 %) Find an orthogonal operator T on \mathbb{R}^3 such that $T(\mathbf{v}) = \mathbf{w}$ where $\mathbf{v} = \frac{1}{\sqrt{10}} \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}$ and $\mathbf{w} = \frac{1}{\sqrt{5}} \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix}$.

8. (10 %) Given a symmetric matrix $A = \begin{bmatrix} 0 & 2 & 2 \\ 2 & 0 & 2 \\ 2 & 2 & 0 \end{bmatrix}$, find an orthonormal basis of eigenvectors and their

corresponding eigenvalues. Use this information to obtain a spectral decomposition of A .