Solution guide for LA Homework 3@ csie.ntnu.edu.tw-2025

1. (16 %) Find generating sets for the range and null space …

   ANS: The standard matrix of L.T. T is $A = [T(\boldsymbol{e}_1) \quad T(\boldsymbol{e}_2) \quad T(\boldsymbol{e}_3)] = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & -1 \\ 1 & 2 & 1 \end{bmatrix}$. The reduced row echelon form of A is $R = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

   (i) $\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \\ 1 \end{bmatrix} \right\}$ (or simply $\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \\ 2 \end{bmatrix} \right\}$) is a generating set for the range of T.

   (ii) $\left\{ \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \right\}$ is a generating set for the null space of T .

2. (12%) Given a linear transformation T …, (a) Find a basis …; (b) If the null space …, find a basis for …

   ANS:

   (a) The standard matrix of T is $A = \begin{bmatrix} -2 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 5 \end{bmatrix}$. The reduced row echelon form of A is $R = \begin{bmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

   The range of T equals the column space of A. Hence the set of pivots of A, $\{\boldsymbol{a}_1, \boldsymbol{a}_2\}= \left\{ \begin{bmatrix} -2 \\ 0 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 2 \\ 3 \end{bmatrix} \right\}$ is a basis for the range of T.

   (b) The null space of T is the same as the null space of A, $\{\boldsymbol{x} \in R^4; A\boldsymbol{x} = 0\}$. The vector form of the general solution of $A\boldsymbol{x} = 0$ is

   $$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 2 \\ -3 \\ 0 \\ 1 \end{bmatrix}.$$

   Hence $\left\{ \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ 0 \\ 1 \end{bmatrix} \right\}$ is a basis for the null space of T.

3. (12%) Let $\mathcal{L} =$ … Find a basis for the subspace V …

   ANS:

$$\begin{bmatrix} 0 & 1 & -1 & -3 & 1 \\ 0 & -1 & 1 & 3 & 2 \\ 1 & -3 & 1 & -1 & -1 \\ 0 & 2 & -2 & -6 & 1 \end{bmatrix} \to \text{a row echelon form: } \begin{bmatrix} 1 & -3 & 1 & -1 & 1 \\ 0 & -1 & 1 & 3 & 2 \\ 0 & 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Then, $\left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ -3 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ -1 \\ 1 \end{bmatrix} \right\}$ is a basis for the subspace V that contains $\mathcal{L}$.

4. (12%) Given a linearly independent subset ..., find a basis for the subspace V …

ANS: From the pivot columns of the augmented matrix $\begin{bmatrix} -1 & 5 & 1 & 1 & 0 \\ -1 & -9 & -2 & -1 & 1 \\ 6 & -2 & 0 & -2 & -2 \\ -7 & -1 & 1 & 3 & 10 \end{bmatrix}$ which has a row echelon form $R = \begin{bmatrix} -1 & 5 & 1 & 1 & 0 \\ 0 & -14 & -3 & -2 & 1 \\ 0 & 0 & 12/7 & 8/7 & 52/7 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$, we therefore get the basis $\left\{ \begin{bmatrix} -1 \\ -1 \\ 6 \\ -7 \end{bmatrix}, \begin{bmatrix} 5 \\ -9 \\ -2 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ 0 \\ 1 \end{bmatrix} \right\}$ for the subspace V that contains the subset $\mathcal{L}$.

5. (12%) For the linear … determine (a)the dimension … ; (b)the dimension of the null … ; (c)whether …

ANS: (a) 3; (b) 0; (c) both one-to-one and onto (bijective or bijection)

(The standard matrix of T is $A = \begin{bmatrix} -1 & -1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 0 \end{bmatrix}$ which has the row echelon form $\begin{bmatrix} -1 & -1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$.

Thus, rank A=3.)

6. (12%) Find the unique representation of $\boldsymbol{u} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ as a linear combination ….

ANS:

By Theorem 4.11, $[\boldsymbol{u}]_{\mathcal{B}} = B^{-1}\boldsymbol{u}$, where $B = [\boldsymbol{b_1} \quad \boldsymbol{b_2} \quad \boldsymbol{b_3}]$.

Therefore, $[\boldsymbol{u}]_{\mathcal{B}} = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 1 \\ 2 & 2 & 1 \end{bmatrix}^{-1} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \cdots = \begin{bmatrix} -2a - b + c \\ 3a + b - c \\ -2a \quad + c \end{bmatrix}$.

That is, $\boldsymbol{u} = (-2a - b + c)\boldsymbol{b_1} + (3a + b - c)\boldsymbol{b_2} + (-2a + c)\boldsymbol{b_3}$

7. (12 %) Let $\mathcal{B}$={$\boldsymbol{b}_1$, $\boldsymbol{b}_2$, $\boldsymbol{b}_3$} … (a) Show that…. (b)Determine the matrix… (c)What is the relationship

ANS:

(a) Let B=[$\boldsymbol{b}_1$ $\boldsymbol{b}_2$ $\boldsymbol{b}_3$]. The reduced row echelon form of B is $R_B=I_3$. Since rank R_B =3, $\mathcal{B}$ is a linearly independent subset of $\mathcal{R}^3$ containing 3 vectors.

(b) $A=[B^{-1}\boldsymbol{e}_1 \quad B^{-1}\boldsymbol{e}_2 \quad B^{-1}\boldsymbol{e}_3]= B^{-1}I_3= B^{-1}=\begin{bmatrix} -3 & -1 & 2 \\ -4 & -1 & 2 \\ 3 & 1 & -1 \end{bmatrix}$

(c) $A= B^{-1}$ as shown in (b).

8. (12%) Let $\mathcal{A}$={$\boldsymbol{u}_1$, $\boldsymbol{u}_2$, …, $\boldsymbol{u}_n$} be a basis for $\mathcal{R}^n$…, compute $[\boldsymbol{v}]_{\mathcal{B}}$.

ANS:

Let A=[$\boldsymbol{u}_1$ $\boldsymbol{u}_2$ ... $\boldsymbol{u}_n$] and B=[$\boldsymbol{u}_1$ $\boldsymbol{u}_1$+$\boldsymbol{u}_2$ ... $\boldsymbol{u}_1$+$\boldsymbol{u}_n$].

$$\text{Then, } B = A\begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}.$$

$$\text{Since } [\boldsymbol{v}]_{\mathcal{B}} = B^{-1}\boldsymbol{v} = \left(A\begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}\right)^{\mathbf{-1}} \boldsymbol{v} =$$

$$\left(\begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}\right)^{\mathbf{-1}} A^{-1}\boldsymbol{v} = \begin{bmatrix} 1 & -1 & -1 & \cdots & -1 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix} [\boldsymbol{v}]_{\mathcal{A}} =$$

$$\begin{bmatrix} 1 & -1 & -1 & \cdots & -1 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} a_1 - a_2 - \cdots - a_n \\ a_2 \\ \vdots \\ a_n \end{bmatrix}.$$