Solution guide for LA-Homework 1@ csie.ntnu.edu.tw-2025

1. (11%) Determine whether the following system is consistent ….

   ANS:

   Apply Gaussian elimination method to the augmented matrix $[A \;\; \boldsymbol{b}] =$

   $$\begin{bmatrix} 1 & -1 & 0 & 1 & 0 & -4 \\ 1 & -1 & 0 & 2 & 2 & -5 \\ 3 & -3 & 0 & 2 & -2 & -11 \end{bmatrix} \dots \to [R \;\; \boldsymbol{c}] = \begin{bmatrix} 1 & -1 & 0 & 0 & -2 & -3 \\ 0 & 0 & 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$ Thus, it is consistent and the solution is

   $$\begin{cases} x_1 = -3 + x_2 + 2x_5 \\ x_2 \quad free \\ x_3 \quad free \\ x_4 = -1 - 2x_5 \\ x_5 \quad free \end{cases} \quad \text{or} \quad \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -3 \\ 0 \\ 0 \\ -1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 2 \\ 0 \\ 0 \\ -2 \\ 1 \end{bmatrix}$$

2. (11%) Find the rank and nullity of matrix ….

   ANS:

   The rank is 3, and the nullity is 2.

   The row echelon form of the given matrix is $\begin{bmatrix} 1 & 0 & 1 & -1 & 6 \\ 0 & -1 & 3 & 1 & -5 \\ 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$, which has 3 pivot columns or rows. Thus the rank is 3.

3. (14%) The input-output matrix for an economy with sectors of metals, nonmetals, and services …

   (a) What is the net production …. ?

   (b) What gross production is required to … ?

   ANS:

   (a) The gross production vector is $\boldsymbol{x} = \begin{bmatrix} 50 \\ 60 \\ 40 \end{bmatrix}$ and the input-output matrix is $C = \begin{bmatrix} .2 & .2 & .1 \\ .4 & .4 & .2 \\ .2 & .2 & .1 \end{bmatrix}$. The net production vector is $\boldsymbol{x} - \boldsymbol{C}\boldsymbol{x} = \begin{bmatrix} 50 \\ 60 \\ 40 \end{bmatrix} - \begin{bmatrix} .2 & .2 & .1 \\ .4 & .4 & .2 \\ .2 & .2 & .1 \end{bmatrix} \begin{bmatrix} 50 \\ 60 \\ 40 \end{bmatrix} = \begin{bmatrix} 24 \\ 8 \\ 14 \end{bmatrix}$, ie. \$24 million of metals, \$8 million of nonmetals, and \$14 million of services.

   (b) Solve the system of linear equations $\boldsymbol{x}(\boldsymbol{I} - \boldsymbol{C}) = \begin{bmatrix} 120 \\ 180 \\ 150 \end{bmatrix}$ to get the solution $\boldsymbol{x} = \begin{bmatrix} 370 \\ 680 \\ 400 \end{bmatrix}$, ie the gross productions required are \$370 million of metals, \$680 millions of nonmetals, and \$400 million of services.

4. (11%) Let A and B be $m \times n$ matrices such that B can be obtained by performing a single elementary …

   ANS:

   Fact1: If B is obtained from A by a row interchange, then the sets of rows of A and B are the same, so the result is immediate.

   Fact 2: If B is obtained from A by a scaling operation, the result follows by the fact that if vectors $\{\boldsymbol{u}_1, \boldsymbol{u}_2, \dots, \boldsymbol{u}_m\}$ are linearly independent, then $\{c_1\boldsymbol{u}_1, c_2\boldsymbol{u}_2, \dots, c_m\boldsymbol{u}_m\}$, $c_i \neq 0$ for all $i = 1, \dots, m$, are also linearly independent.

With facts 1 and 2, we only need to prove the result for row addition operations.

Let the rows of A be denoted by $\boldsymbol{u_1}, \boldsymbol{u_2}, \ldots, \boldsymbol{u_m}$. Suppose that B is obtained from A by adding d times of the j-th row to k-th row of A, where $j<k \leq m$. Then the rows of B are given by $\boldsymbol{u_1}, \boldsymbol{u_2}, \ldots, \boldsymbol{u_{k-1}}, d\boldsymbol{u_j} + \boldsymbol{u_k}$, $\boldsymbol{u_{k+1}}, \ldots, \boldsymbol{u_m}$. Suppose for scalars $c_1, c_2, \ldots, c_m$, we have

$$c_1\boldsymbol{u}_1 + c_2\boldsymbol{u}_2 + \cdots + c_{k-1}\boldsymbol{u}_{k-1} + c_k\left(d\boldsymbol{u}_j + \boldsymbol{u}_k\right) + c_{k+1}\boldsymbol{u}_{k+1} + \cdots + c_m\boldsymbol{u}_m = 0.$$

Then,

$$c_1\boldsymbol{u}_1 + c_2\boldsymbol{u}_2 + \cdots + c_{j-1}\boldsymbol{u}_{j-1} + \left(c_j + c_k d\right)\boldsymbol{u}_j + c_{j+1}\boldsymbol{u}_{j+1} + \cdots + c_{k-1}\boldsymbol{u}_{k-1} + c_k\boldsymbol{u}_k + c_{k+1}\boldsymbol{u}_{k+1} + \cdots + c_m\boldsymbol{u}_m = 0.$$

Because the rows of A are linearly independent, we have $c_1 = c_2 = \cdots = c_{j-1} = \left(c_j + c_k d\right) = c_{j+1} = \cdots = c_{k-1} = c_k = c_{k+1} = \cdots = c_m = 0$. It follows that $c_1 = c_2 = \cdots = c_m = 0$ and so the rows of B are linearly independent.

5. (20 %) Let A be an m×n matrix with reduced row echelon form R. Determine …..

   ANS:

   (a) $[R \quad \boldsymbol{0}]$

   (b) $[\boldsymbol{r}_1 \quad \boldsymbol{r}_2 \quad \ldots \quad \boldsymbol{r}_k]$, where $r_i = R\boldsymbol{e_i}$

   (c) R

   (d) $[I_m \quad A]$

   (e) $[R \quad cR]$

6. (11%) Given $A = \cdots$, determine whether the equation ….

   ANS: No. ← By Theorem 1.6 (d) and (b), the reduced row echelon form of A has one zero row.

7. (11%) Determine, if possible, a value of r for which ……

   ANS:

   Apply elementary row operations to the matrix whose columns are the given vectors. We have $\begin{bmatrix} 1 & 0 & -1 & -1 \\ 0 & -1 & 1 & 9 \\ 0 & 0 & 2 & r+17 \\ 0 & 0 & 0 & -r-9 \end{bmatrix}$. The rank of the matrix is less than 4 iff $r = -9$. By Theorem 1.8, the vectors are linearly dependent iff $r = -9$.

8. (11%) Let $\boldsymbol{u}_1, \boldsymbol{u}_2, \cdots, \boldsymbol{u}_k$ be vectors in $\mathcal{R}^n$ and…..

   ANS:

   Let $\boldsymbol{S} = Span\{\boldsymbol{u}_1, \boldsymbol{u}_2, \cdots, \boldsymbol{u}_k\}$ and $\boldsymbol{S}' = \text{Span } \{c_1\boldsymbol{u}_1, c_2\boldsymbol{u}_2, \cdots, c_k\boldsymbol{u}_k\}$. Then S consists of vectors $a_1\boldsymbol{u}_1 + a\boldsymbol{u}_2 + \cdots + a_k\boldsymbol{u}_k$ for all scalars $a_1, a_2, \ldots, a_k$.

   By the relationship

   $$a_1\boldsymbol{u}_1 + a\boldsymbol{u}_2 + \cdots + a_k\boldsymbol{u}_k = \frac{a_1}{c_1}(c_1\boldsymbol{u}_1) + \frac{a_2}{c_2}(c_2\boldsymbol{u}_2) + \cdots + \frac{a_k}{c_k}(c_k\boldsymbol{u}_k), \quad \text{hence } \boldsymbol{S} \subset \boldsymbol{S}'. \qquad (1)$$

   Similarly, $\boldsymbol{S}'$ consists of vectors $d_1(c_1\boldsymbol{u}_1) + d_2(c_2\boldsymbol{u}_2) + \cdots + c_k(c_k\boldsymbol{u}_k)$ for all scalars $d_1, d_2, \ldots, d_k$.
   But, $d_1(c_1\boldsymbol{u}_1) + d_2(c_2\boldsymbol{u}_2) + \cdots + c_k(c_k\boldsymbol{u}_k) = (d_1c_1)\boldsymbol{u}_1 + (d_2c_2)\boldsymbol{u}_2 + \cdots + (d_kc_k)\boldsymbol{u}_k$, which implies $\boldsymbol{S}' \subset \boldsymbol{S}$ **(2)**

   From relationships (1) and (2), we have $\boldsymbol{S}' = \boldsymbol{S}$.