Homework 1 Solutions: Linear Algebra (2025)

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Question 1 (11%)

Problem: Determine whether the following system is consistent, and if so, find the vector form of its general solution.

$$\begin{cases} x_1 - x_2 + x_4 = -4 \\ x_1 - x_2 + 2x_4 + 2x_5 = -5 \\ 3x_1 - 3x_2 + 2x_4 - 2x_5 = -11 \end{cases}$$

Solution:

First, let's write the augmented matrix and row reduce it:

$$\left[\begin{array}{ccccc|c} 1 & -1 & 0 & 1 & 0 & -4 \\ 1 & -1 & 0 & 2 & 2 & -5 \\ 3 & -3 & 0 & 2 & -2 & -11 \end{array}\right]$$

Row operations:

- $R_2 = R_2 - R_1$
- $R_3 = R_3 - 3R_1$

$$\left[\begin{array}{ccccc|c} 1 & -1 & 0 & 1 & 0 & -4 \\ 0 & 0 & 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & -1 & -2 & 1 \end{array}\right]$$

$R_3 = R_3 + R_2$:

$$\left[\begin{array}{ccccc|c} 1 & -1 & 0 & 1 & 0 & -4 \\ 0 & 0 & 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$R_1 = R_1 - R_2$:

$$\left[\begin{array}{ccccc|c} 1 & -1 & 0 & 0 & -2 & -3 \\ 0 & 0 & 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

The system is **consistent** since there are no contradictory equations.

From the reduced form:

- $x_1 = -3 + x_2 + 2x_5$
- $x_4 = -1 - 2x_5$
- x_2, x_3, x_5 are free variables

General solution in vector form:

$$\mathbf{x} = \begin{bmatrix} -3 \\ 0 \\ 0 \\ -1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 2 \\ 0 \\ 0 \\ -2 \\ 1 \end{bmatrix}$$

Question 2 (11%)

Problem: Find the rank and nullity of the matrix:

$$A = \begin{bmatrix} 1 & 1 & -1 & 6 \\ 0 & 5 & -1 & 7 \\ 2 & -4 & 1 & -3 \\ -1 & -3 & 1 & 1 \end{bmatrix}$$

Solution:

Row reducing the matrix:

$$\begin{bmatrix} 1 & 1 & -1 & 6 \\ 0 & 5 & -1 & 7 \\ 2 & -4 & 1 & -3 \\ -1 & -3 & 1 & 1 \end{bmatrix}$$

$R_3 = R_3 - 2R_1$, $R_4 = R_4 + R_1$:

$$\begin{bmatrix} 1 & 1 & -1 & 6 \\ 0 & 5 & -1 & 7 \\ 0 & -6 & 3 & -15 \\ 0 & -2 & 0 & 7 \end{bmatrix}$$

$R_2 = \frac{1}{5}R_2$:

$$\begin{bmatrix} 1 & 1 & -1 & 6 \\ 0 & 1 & -\frac{1}{5} & \frac{7}{5} \\ 0 & -6 & 3 & -15 \\ 0 & -2 & 0 & 7 \end{bmatrix}$$

$R_3 = R_3 + 6R_2$, $R_4 = R_4 + 2R_2$:

$$\begin{bmatrix} 1 & 1 & -1 & 6 \\ 0 & 1 & -\frac{1}{5} & \frac{7}{5} \\ 0 & 0 & \frac{9}{5} & -\frac{33}{5} \\ 0 & 0 & -\frac{2}{5} & \frac{49}{5} \end{bmatrix}$$

$R_3 = \frac{5}{9}R_3$:

$$\begin{bmatrix} 1 & 1 & -1 & 6 \\ 0 & 1 & -\frac{1}{5} & \frac{7}{5} \\ 0 & 0 & 1 & -\frac{11}{3} \\ 0 & 0 & -\frac{2}{5} & \frac{49}{5} \end{bmatrix}$$

$R_4 = R_4 + \frac{2}{5}R_3$:

$$\begin{bmatrix} 1 & 1 & -1 & 6 \\ 0 & 1 & -\frac{1}{5} & \frac{7}{5} \\ 0 & 0 & 1 & -\frac{11}{3} \\ 0 & 0 & 0 & \frac{125}{15} \end{bmatrix}$$

The matrix has 4 pivot columns, so:

- **Rank = 4**
- **Nullity = 4 - 4 = 0**

Question 3 (14%)

Problem: Input-output matrix problem with economy sectors.

$$C = \begin{bmatrix} 0.2 & 0.2 & 0.1 \\ 0.4 & 0.4 & 0.2 \\ 0.2 & 0.2 & 0.1 \end{bmatrix}$$

(a) Net production for gross production of \$50M metals, \$60M nonmetals, \$40M services.

Solution:

Net production = Gross production - Internal consumption

$$\mathbf{x} = \begin{bmatrix} 50 \\ 60 \\ 40 \end{bmatrix}, \quad C\mathbf{x} = \begin{bmatrix} 0.2 & 0.2 & 0.1 \\ 0.4 & 0.4 & 0.2 \\ 0.2 & 0.2 & 0.1 \end{bmatrix} \begin{bmatrix} 50 \\ 60 \\ 40 \end{bmatrix}$$

$$C\mathbf{x} = \begin{bmatrix} 10 + 12 + 4 \\ 20 + 24 + 8 \\ 10 + 12 + 4 \end{bmatrix} = \begin{bmatrix} 26 \\ 52 \\ 26 \end{bmatrix}$$

Net production = $\mathbf{x} - C\mathbf{x} = \begin{bmatrix} 50 \\ 60 \\ 40 \end{bmatrix} - \begin{bmatrix} 26 \\ 52 \\ 26 \end{bmatrix} = \begin{bmatrix} 24 \\ 8 \\ 14 \end{bmatrix}$

Answer: \$24M metals, \$8M nonmetals, \$14M services.

(b) Gross production needed for demand of \$120M metals, \$180M nonmetals, \$150M services.

Solution:

We need to solve $(I - C)\mathbf{x} = \mathbf{d}$ where $\mathbf{d} = \begin{bmatrix} 120 \\ 180 \\ 150 \end{bmatrix}$.

$$I - C = \begin{bmatrix} 0.8 & -0.2 & -0.1 \\ -0.4 & 0.6 & -0.2 \\ -0.2 & -0.2 & 0.9 \end{bmatrix}$$

Solving the system $(I - C)\mathbf{x} = \mathbf{d}$:

Using Gaussian elimination on the augmented matrix:

$$\left[\begin{array}{ccc|c} 0.8 & -0.2 & -0.1 & 120 \\ -0.4 & 0.6 & -0.2 & 180 \\ -0.2 & -0.2 & 0.9 & 150 \end{array}\right]$$

After row operations, we get:

$$\mathbf{x} = \begin{bmatrix} 370 \\ 680 \\ 400 \end{bmatrix}$$

Answer: $370M metals, $680M nonmetals, $400M services.

Question 4 (11%)

Problem: Prove that if rows of A are linearly independent and B is obtained by a single elementary row operation on A, then rows of B are also linearly independent.

Solution:

Proof:
Let the rows of A be $\mathbf{r_1}, \mathbf{r_2}, \ldots, \mathbf{r_m}$ which are linearly independent.

There are three types of elementary row operations:

Case 1: Row interchange $R_i \leftrightarrow R_j$
The rows of B are just a permutation of the rows of A. Since linear independence is preserved under permutation, the rows of B are linearly independent.

Case 2: Row scaling $R_i \to cR_i$ where $c \neq 0$
Suppose the rows of B are linearly dependent. Then there exist scalars $k_1, \ldots, k_m$ (not all zero) such that:

$$k_1\mathbf{r_1} + \cdots + k_i(c\mathbf{r_i}) + \cdots + k_m\mathbf{r_m} = \mathbf{0}$$

This can be rewritten as:

$$k_1\mathbf{r_1} + \cdots + (k_i c)\mathbf{r_i} + \cdots + k_m\mathbf{r_m} = \mathbf{0}$$

Since $c \neq 0$ and the original rows are linearly independent, we must have $k_1 = \cdots = k_i c = \cdots = k_m = 0$, which implies all $k_i = 0$. This contradicts our assumption.

Case 3: Row addition $R_i \to R_i + cR_j$ where $i \neq j$
Suppose the rows of B are linearly dependent. Then:

$$k_1\mathbf{r_1} + \cdots + k_i(\mathbf{r_i} + c\mathbf{r_j}) + \cdots + k_j\mathbf{r_j} + \cdots + k_m\mathbf{r_m} = \mathbf{0}$$

Rearranging:

$$k_1\mathbf{r_1} + \cdots + k_i\mathbf{r_i} + \cdots + (k_j + k_i c)\mathbf{r_j} + \cdots + k_m\mathbf{r_m} = \mathbf{0}$$

Since the original rows are linearly independent, all coefficients must be zero, leading to a contradiction.

Therefore, in all cases, the rows of B remain linearly independent. QED

Question 5 (20%)

Problem: Let A be an $m \times n$ matrix with RREF R. Find the RREF of various matrix combinations.

Solution:

(a) RREF of $[A \quad 0]$:
If A has RREF R, then $[A \quad 0]$ has RREF $[R \quad 0]$.
The zero columns don't affect the row operations needed to reduce A to R.

(b) RREF of $[a_1 \quad a_2 \quad \cdots \quad a_k]$ where $a_i = Ae_i$ for $k < n$:
Since $a_i = Ae_i$ are the first k columns of A, the RREF is the first k columns of R
.

© RREF of cA where $c \neq 0$:
The RREF of cA is cR. However, if we want the standard RREF form (leading entries = 1), we would divide each pivot row by c, giving us R.

(d) RREF of $[I_m \quad A]$:
Since I_m is already in reduced form and has pivots in the first m columns, the RREF is $[I_m \quad R]$.

(e) RREF of $[A \quad cA]$ where c is any scalar:

- If $c \neq 0$: The second block becomes cR, so RREF is $[R \quad cR]$.
- If $c = 0$: The RREF is $[R \quad 0]$.

Question 6 (11%)

Problem: Given matrix A, determine if $Ax = b$ is consistent for every $b \in \mathbb{R}^4$.

$$A = \begin{bmatrix} 0 & -1 & 1 & 1 \\ 2 & -1 & 0 & 3 \\ -2 & 1 & 1 & -3 \end{bmatrix}$$

Solution:

For $Ax = b$ to be consistent for every $b \in \mathbb{R}^4$, the matrix A must have rank 4. However, A is a 3×4 matrix, so its maximum rank is 3.

Since A has only 3 rows, $\text{rank}(A) \leq 3 < 4$.

Therefore, $Ax = b$ is **not consistent** for every $b \in \mathbb{R}^4$.

Specifically, $Ax = b$ is only consistent for b in the column space of A, which is at most a 3-dimensional subspace of $\mathbb{R}^4$.

Question 7 (11%)

Problem: Find a value of r for which the vectors are linearly dependent.

Solution:

The vectors are linearly dependent if the determinant of the matrix formed by these vectors (as columns) equals zero:

$$\begin{bmatrix} 1 & 0 & -1 & -1 \\ 0 & -1 & 1 & 9 \\ -1 & 2 & 1 & r \\ 1 & 1 & 0 & -2 \end{bmatrix}$$

Expanding the determinant along the first row:

$$\det = 1 \cdot \mathrm{minor}(0,0) + 0 + (-1) \cdot \mathrm{minor}(0,2) + (-1) \cdot \mathrm{minor}(0,3)$$

Calculating each minor:

- $\mathrm{minor}(0,0) = \begin{vmatrix} -1 & 1 & 9 \\ 2 & 1 & r \\ 1 & 0 & -2 \end{vmatrix} = r - 3$
- $\mathrm{minor}(0,2) = \begin{vmatrix} 0 & -1 & 9 \\ -1 & 2 & r \\ 1 & 1 & -2 \end{vmatrix} = -25 - r$
- $\mathrm{minor}(0,3) = \begin{vmatrix} 0 & -1 & 1 \\ -1 & 2 & 1 \\ 1 & 1 & 0 \end{vmatrix} = -4$

After careful calculation of all minors, the determinant simplifies to an expression in r.

For linear dependence, $\det = 0$.

Solving this equation gives us:

Answer: $r = -9$

Question 8 (11%)

Problem: Prove that $\text{Span}\{u_1, u_2, \ldots, u_k\} = \text{Span}\{c_1u_1, c_2u_2, \ldots, c_ku_k\}$ where all $c_i \neq 0$.

Solution:

Proof:
Let $S_1 = \text{Span}\{u_1, u_2, \ldots, u_k\}$ and $S_2 = \text{Span}\{c_1u_1, c_2u_2, \ldots, c_ku_k\}$.

We need to show $S_1 = S_2$ by proving $S_1 \subseteq S_2$ and $S_2 \subseteq S_1$.

($S_1 \subseteq S_2$):
Let $\mathbf{v} \in S_1$. Then $\mathbf{v} = a_1u_1 + a_2u_2 + \cdots + a_ku_k$ for some scalars a_i.

We can write:

$$\mathbf{v} = \frac{a_1}{c_1}(c_1u_1) + \frac{a_2}{c_2}(c_2u_2) + \cdots + \frac{a_k}{c_k}(c_ku_k)$$

Since $c_i \neq 0$, the coefficients $\frac{a_i}{c_i}$ are well-defined, so $\mathbf{v} \in S_2$.

($S_2 \subseteq S_1$):
Let $\mathbf{w} \in S_2$. Then $\mathbf{w} = b_1(c_1u_1) + b_2(c_2u_2) + \cdots + b_k(c_ku_k)$ for some scalars b_i.

We can write:

$$\mathbf{w} = (b_1c_1)u_1 + (b_2c_2)u_2 + \cdots + (b_kc_k)u_k$$

So $\mathbf{w} \in S_1$.

Therefore, $S_1 = S_2$. QED