Chapter 3 Determinants

- Information about a matrix.
- Cramer's rule
- The use of determinants as a computational tool has diminished!
  - It is used in Chapter 5 to determine the eigenvalues of a square matrix.

3.1 Cofactor Expansion

Consider $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $C = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.

Then, $AC =$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \begin{bmatrix} ad-bc & 0 \\ 0 & ad-bc \end{bmatrix} = (ad-bc)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

and similarly $CA =$

$$\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} ad-bc & 0 \\ 0 & ad-bc \end{bmatrix} = (ad-bc)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

- $\Longleftarrow$ If $ad-bc \neq 0$, $\frac{1}{ad-bc}C$ is the inverse of A and so A is invertible.
- $\Longrightarrow$ Conversely, suppose that $ad-bc = 0$. Then $AC=CA=0$.

  If A were invertible, then $C = CI_2 = C(AA^{-1}) = (CA)A^{-1} = OA^{-1} = O$ and so all entries of C equal to 0.
  It follows that $A=O$, contradicting that A is invertible.

The matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is invertible iff ad–bc≠0, in which case $A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.

- The scalar $ad-bc$ is thus called the **determinant** of A, denoted by det A or $|A|$.
- Here the principal use of determinant is to calculate the scalars c for which the matrix $A-cI_n$ is not invertible.
  (Or, via elementary row operations and Theorems 2.5∨2.6)
- Define the **determinant** of 1×1 matrix $[a]$ by det $[a]=a$.

The determinant of $n \times n$ matrix A=?(By mathematical induction)

- Define the $(n-1)\times(n-1)$ matrix A_{ij} to be the matrix obtained from A by deleting row i and column j.

$$A_{ij} = \begin{array}{cc} \begin{bmatrix} a_{11} & \cdots & a_{1j} & \cdots & a_{1n} \\ \vdots & & \vdots & & \vdots \\ a_{i1} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & & \vdots & & \vdots \\ a_{n1} & \cdots & a_{nj} & \cdots & a_{nn} \end{bmatrix} & \begin{array}{l} \\ \\ \leftarrow \text{row } i \\ \\ \\ \end{array} \\ \uparrow & \\ \text{Column j} & \end{array}$$

If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then A_{11}=[d] and A_{12}=[c]. Thus,

$$\det A = a \det A_{11} - b \det A_{12}.$$

- Define the determinant of an $n \times n$ matrix A for $n \geq 3$ by

$$\det A = a_{11} \det A_{11} - a_{12} \det A_{12} + \cdots + (-1)^{n+1} a_{1n} \det A_{1n}.$$

Let $c_{ij} = (-1)^{i+j} \det A_{ij}$. Then, the determinant of A can be written as

$$\det A = \sum_{j=1}^{n} a_{1j} c_{1j}.$$

The number c_{ij} is called the (i,j)-**cofactor** of A, and the above equation is called **cofactor expansion** of A along the first row.

Theorem 3.1 (**Cofactor expansion** of A along row i)

For any $i=1, 2, \ldots, n$, we have

$$\det A = \sum_{j=1}^{n} a_{ij} c_{ij}$$

where c_{ij} denotes the (i,j)-cofactor of A.

For any $m \times m$ matrix A and $m \times n$ matrix B,

$$\det \begin{bmatrix} A & B \\ O & I_n \end{bmatrix} = \det A.$$

Theorem 3.2

The determinant of an upper triangular $n \times n$ matrix or a lower triangular $n \times n$ matrix equals the product of its diagonal entries.

3.2 Properties of Determinants

Observations: The forward pass of Gaussian elimination algorithm transforms any matrix into an upper triangular matrix (in row echelon form) by a sequence of elementary row operations.

Theorem 3.3

Let A be an $n \times n$ matrix.

a) If B is a matrix obtained by interchanging two rows of A, then $\det B = -\det A$.
b) If B is a matrix obtained by multiplying each entry of some row of A by a scalar k, then $\det B = k \cdot \det A$.
c) If B is a matrix obtained by adding a multiple of some row of A to a different row, then $\det B = \det A$.
d) For any $n \times n$ elementary matrix E, we have $\det EA = (\det E)(\det A)$.

Proof: ???

- Let $A_n = I_n$ in Theorem 3.3. Then,

1) (a), (b) and (c) give the value of the determinant of each type of elementary matrix.
2) det $E=1$ if E performs a row addition operation
3) det $E=-1$ if E performs a row interchange operation.

- Suppose that an $n \times n$ matrix A is transformed into an upper triangular matrix U by a sequence of elementary row operations other than scaling operations. Thus there is a sequence of elementary matrices $E_1, E_2, \ldots, E_k$ such that $E_k \cdots E_2 E_1 A = U$. By Theorem 3.3 (d), we have

$$(\det E_k) \cdots (\det E_2)(\det E_1)(\det A) = \det U.$$

Thus $(-1)^r \det A = \det U$

where r is the number of row interchange operations that occur in the transformation of A into U.

If an $n \times n$ matrix A is transformed into an upper triangular matrix U by elementary row operations other than scaling operations, then

$$detA = (-1)^r u_{11} u_{22} \cdots u_{nn},$$

where r is the number of row interchanges performed and u_{ii} are the diagonal entries of U.

Theorem 3.4 (Four Properties of Determinants)

Let A and B be a square matrices of the same size. The following statements are true:

a) A is invertible iff $\det A \neq 0$.

b) $\det AB = (\det A)(\det B)$.

c) $\det A^T = \det A$.

d) If A is invertible, then $\det A^{-1} = \dfrac{1}{\det A}$.

Proof ???

Example:

Suppose that a matrix M can be partitioned as $\begin{bmatrix} A & B \\ 0 & C \end{bmatrix}$, where A is an $m \times m$ matrix, C is an $n \times n$ matrix, and O is the $n \times m$ zero matrix. Then, $\det M = (\det A)(\det C)$.

Proof: Note that $\begin{bmatrix} I_m & O' \\ O & C \end{bmatrix} \begin{bmatrix} A & B \\ O & I_n \end{bmatrix} = \begin{bmatrix} A & B \\ 0 & C \end{bmatrix}$, where O' is the $m \times n$ zero matrix.

- Theorem 3.4(c) implies that the determinant of A can be evaluated by cofactor expansion along any column, as well as any row.

Theorem 3.5 (Cramer's Rule, 1750)

Let A be an invertible $n\times n$ matrix, $\boldsymbol{b}$ be in R^n, and M_i be the matrix obtained from A by replacing column i of A by $\boldsymbol{b}$. Then $A\boldsymbol{x}=\boldsymbol{b}$ has a unique solution $\boldsymbol{u}$ in which the components are given by

$$u_i = \frac{\det M_i}{\det A} \text{ for } i=1, 2, \ldots, n.$$