Homework 3 Solutions: Linear Algebra (2025)

Question 1 (16%)

Problem: Find generating sets for the range and null space of linear transformation T defined as

$$T\left(\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}\right)=\begin{bmatrix}x_1+x_2\\x_2+x_3\\x_1-x_3\\x_1+2x_2+x_3\end{bmatrix}$$

Solution:

First, we find the standard matrix A for the linear transformation T.

$$A=\begin{bmatrix}1&1&0\\0&1&1\\1&0&-1\\1&2&1\end{bmatrix}$$

Range of T: The range of T is the column space of A. A generating set for the range is the set of column vectors of A.

$$\text{Generating set for Range(T)}=\left\{\begin{bmatrix}1\\0\\1\\1\end{bmatrix},\begin{bmatrix}1\\1\\0\\2\end{bmatrix},\begin{bmatrix}0\\1\\-1\\1\end{bmatrix}\right\}$$

To find a basis, we row reduce A:

$$\begin{bmatrix}1&1&0\\0&1&1\\1&0&-1\\1&2&1\end{bmatrix}\to\begin{bmatrix}1&0&-1\\0&1&1\\0&0&0\\0&0&0\end{bmatrix}$$

The first two columns are pivot columns, so a basis for the range is:

$$\text{Basis for Range(T)}=\left\{\begin{bmatrix}1\\0\\1\\1\end{bmatrix},\begin{bmatrix}1\\1\\0\\2\end{bmatrix}\right\}$$

Null Space of T: We need to solve $Ax=0$. From the reduced row echelon form: $x_1-x_3=0\Rightarrow x_1=x_3$ $x_2+x_3=0\Rightarrow x_2=-x_3$ x_3 is a free variable.

The general solution is:

$$\mathbf{x}=\begin{bmatrix}x_3\\-x_3\\x_3\end{bmatrix}=x_3\begin{bmatrix}1\\-1\\1\end{bmatrix}$$

A generating set (and a basis) for the null space is:

$$\text{Generating set for Null(T)}=\left\{\begin{bmatrix}1\\-1\\1\end{bmatrix}\right\}$$

Question 2 (12%)

Problem: Given a linear transformation T defined as

$$T\left(\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix}\right)=\begin{bmatrix}-2x_1-x_2+x_4\\0\\x_1+2x_2+3x_3+4x_4\\2x_1+3x_2+4x_3+5x_4\end{bmatrix}$$

(a) Find a basis for the range of T; (b) If the null space of T is nonzero, find a basis for the null space of T.

Solution: The standard matrix A is:

$$A = \begin{bmatrix} -2 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 5 \end{bmatrix}$$

Row reducing A:

$$\begin{bmatrix} -2 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 5 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

(a) Basis for the range of T: The pivot columns are 1 and 2. So a basis for the range is the first two columns of the original matrix A.

$$\text{Basis for Range(T)} = \left\{ \begin{bmatrix} -2 \\ 0 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 2 \\ 3 \end{bmatrix} \right\}$$

(b) Basis for the null space of T: From the RREF of A, we have: $x_1 - x_3 - 2x_4 = 0 \Rightarrow x_1 = x_3 + 2x_4$ $x_2 + 2x_3 + 3x_4 = 0 \Rightarrow x_2 = -2x_3 - 3x_4$ x_3 and x_4 are free variables.

The solution in vector form is:

$$\mathbf{x} = \begin{bmatrix} x_3 + 2x_4 \\ -2x_3 - 3x_4 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 2 \\ -3 \\ 0 \\ 1 \end{bmatrix}$$

The null space is nonzero. A basis for the null space is:

$$\text{Basis for Null(T)} = \left\{ \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ 0 \\ 1 \end{bmatrix} \right\}$$

Question 3 (12%)

Problem: Let $\mathcal{L} = \left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}$ and $V = \text{Col} \begin{bmatrix} 1 & -1 & 1 & -3 & 3 & 2 \\ 1 & -1 & -3 & 1 & -1 & -1 \\ 2 & -2 & 1 & -6 & -1 & -1 \\ -1 & 1 & 6 & -7 & 6 & -7 \end{bmatrix}$. Find a basis for the subspace V that contains the given linearly independent subset L of V.

Solution: Let the vector in $\mathcal{L}$ be l_1 and the columns of the matrix be $v_1, ..., v_6$. We want to find a basis for $V = \text{Span}\{v_1, ..., v_6\}$ that includes l_1. We form a matrix with l_1 and the columns of the matrix for V, and then find the pivot columns. The corresponding vectors will form the basis.

$$[l_1 \quad v_1 \quad v_2 \quad v_3 \quad v_4 \quad v_5 \quad v_6] = \begin{bmatrix} 0 & 1 & -1 & 1 & -3 & 3 & 2 \\ 0 & 1 & -1 & -3 & 1 & -1 & -1 \\ 1 & 2 & -2 & 1 & -6 & -1 & -1 \\ 0 & -1 & 1 & 6 & -7 & 6 & -7 \end{bmatrix}$$

Row reducing this matrix:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

The pivot columns are 1, 2, 4, and 5. These correspond to the vectors l_1, v_1, v_3, v_4. A basis for V containing L is:

$$\left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ -3 \\ 1 \\ 6 \end{bmatrix}, \begin{bmatrix} -3 \\ 1 \\ -6 \\ -7 \end{bmatrix} \right\}$$

Question 4 (12%)

Problem: Given a linearly independent subset $\mathcal{L} = \left\{ \begin{bmatrix} 5 \\ -9 \\ -2 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ 0 \\ 1 \end{bmatrix} \right\}$ of the subspace $V =$ Span $\left\{ \begin{bmatrix} 1 \\ -1 \\ -2 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -2 \\ 10 \end{bmatrix} \right\}$, find a basis for the subspace V that contains the given linearly independent subset L of V.

Solution: Let the vectors in $\mathcal{L}$ be l_1, l_2 and the spanning vectors of V be v_1, v_2. We form a matrix $[l_1\ l_2\ v_1\ v_2]$ and row reduce to find the pivot columns.

$$\begin{bmatrix} 5 & 1 & 1 & 0 \\ -9 & -2 & -1 & 1 \\ -2 & 0 & -2 & -2 \\ -1 & 1 & 3 & 10 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The pivot columns are 1, 2, and 3. This means the vectors l_1, l_2, v_1 form a basis for the space spanned by all four vectors. Since V is a subspace of this, and L is in V, this set is a basis for V. A basis for V containing L is:

$$\left\{ \begin{bmatrix} 5 \\ -9 \\ -2 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ -2 \\ 3 \end{bmatrix} \right\}$$

Question 5 (12%)

Problem: For the linear transformation defined by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \right) = \begin{bmatrix} -x_1 - x_2 + x_3 \\ x_1 + 2x_2 + x_3 \\ x_1 + x_2 \end{bmatrix}$$

determine (a) the dimension of the range of T (b) the dimension of the null space of T (c) whether T is one-to-one or on-to.

Solution: The standard matrix A is:

$$A = \begin{bmatrix} -1 & -1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

Row reducing A:

$$\begin{bmatrix} -1 & -1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The matrix has 3 pivots. (a) The dimension of the range of T is the rank of A, which is 3. (b) The dimension of the null space of T is the nullity of A, which is the number of columns minus the rank. Nullity = 3 - 3 = 0. (c) T is one-to-one because the nullity is 0. T is onto because the rank (3) is equal to the dimension of the codomain $\mathbb{R}^3$.

Question 6 (12%)

Problem: Find the unique representation of $u = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ as a linear combination of $b_1 = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$, $b_2 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$, and $b_3 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$.

Solution: We want to find scalars x_1, x_2, x_3 such that $x_1b_1 + x_2b_2 + x_3b_3 = u$. We can write this as an augmented matrix and row reduce.

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & a \\ -1 & 0 & 1 & b \\ 2 & 2 & 1 & c \end{array}\right]$$

Row operations: $R_2 = R_2 + R_1$ $R_3 = R_3 - 2R_1$

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & a \\ 0 & 1 & 1 & a+b \\ 0 & 0 & 1 & c-2a \end{array}\right]$$

Now we back substitute: $x_3 = c - 2a$ $x_2 = a + b - x_3 = a + b - (c - 2a) = 3a + b - c$ $x_1 = a - x_2 = a - (3a + b - c) = -2a - b + c$

The unique representation is:

$$u = (-2a - b + c)b_1 + (3a + b - c)b_2 + (c - 2a)b_3$$

Question 7 (12%)

Problem: Let $\mathcal{B} = \{b_1, b_2, b_3\}$, where $b_1 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$, $b_2 = \begin{bmatrix} -1 \\ 3 \\ 0 \end{bmatrix}$, and $b_3 = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$. (a) Show that $\mathcal{B}$ is a basis for $\mathbb{R}^3$. (b) Determine the matrix $A = [[e_1]_{\mathcal{B}} \quad [e_2]_{\mathcal{B}} \quad [e_3]_{\mathcal{B}}]$. (c) What is the relationship between A and $B = [b_1 \quad b_2 \quad b_3]$?

Solution: (a) To show that $\mathcal{B}$ is a basis for $\mathbb{R}^3$, we can form a matrix B with these vectors as columns and show it is invertible.

$$B = \begin{bmatrix} 1 & -1 & 0 \\ -2 & 3 & 2 \\ 1 & 0 & 1 \end{bmatrix}$$

The determinant of B is $1(3) - (-1)(-2-2) + 0 = 3 - 4 = -1$. Since the determinant is non-zero, the matrix is invertible, and the vectors are linearly independent and span $\mathbb{R}^3$. Thus, $\mathcal{B}$ is a basis for $\mathbb{R}^3$.

(b) The columns of A are the coordinate vectors of the standard basis vectors with respect to $\mathcal{B}$. This means we need to solve $Bx = e_i$ for $i = 1, 2, 3$. This is equivalent to finding B^{-1}. We find the inverse of B by augmenting it with the identity matrix and row reducing.

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ -2 & 3 & 2 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -3 & -1 & 2 \\ 0 & 1 & 0 & -4 & -1 & 2 \\ 0 & 0 & 1 & 3 & 1 & -1 \end{array}\right]$$

So, $A = B^{-1} = \begin{bmatrix} -3 & -1 & 2 \\ -4 & -1 & 2 \\ 3 & 1 & -1 \end{bmatrix}$.

(c) The relationship between A and B is that A is the inverse of B. $A = B^{-1}$.

Question 8 (12%)

Problem: Let $\mathcal{A} = \{u_1, u_2, \dots, u_n\}$ be a basis for $\mathbb{R}^n$. Then, $\mathcal{B} = \{u_1, u_1 + u_2, \dots, u_1 + u_n\}$ is also a basis for $\mathbb{R}^n$. If v is a vector in $\mathbb{R}^n$ and $[v]_{\mathcal{A}} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$, compute $[v]_{\mathcal{B}}$.

Solution: Let $[v]_{\mathcal{B}} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$. From the definition of coordinate vectors, we have: $v = a_1 u_1 + a_2 u_2 + \cdots + a_n u_n$
$v = b_1(u_1) + b_2(u_1 + u_2) + \cdots + b_n(u_1 + \cdots + u_n)$

Expanding the second expression for v: $v = (b_1 + b_2 + \cdots + b_n)u_1 + (b_2 + \cdots + b_n)u_2 + \cdots + b_n u_n$

Since $\mathcal{A}$ is a basis, the representation is unique. We can equate the coefficients of u_i: $a_1 = b_1 + b_2 + \cdots + b_n$ $a_2 = b_2 + \cdots + b_n$ ... $a_n = b_n$

Now we can solve for the b_i terms starting from the bottom: $b_n = a_n$ $b_{n-1} = a_{n-1} - (b_n) = a_{n-1} - a_n$ $b_{n-2} = a_{n-2} - (b_{n-1} + b_n) = a_{n-2} - (a_{n-1} - a_n + a_n) = a_{n-2} - a_{n-1}$ ... $b_1 = a_1 - (b_2 + \cdots + b_n) = a_1 - a_2$

So, for $i = 1, \dots, n-1$, $b_i = a_i - a_{i+1}$, and $b_n = a_n$. Therefore,

$$[v]_{\mathcal{B}} = \begin{bmatrix} a_1 - a_2 \\ a_2 - a_3 \\ \vdots \\ a_{n-1} - a_n \\ a_n \end{bmatrix}$$