Homework 3, for Linear Algebra class@ csie.ntnu.edu.tw-2025

1. (16 %) Find generating sets for the range and null space of linear transformation T defined as $T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_1 + x_2 \\ x_2 + x_3 \\ x_1 - x_3 \\ x_1 + 2x_2 + x_3 \end{bmatrix}$.

2. (12%) Given A linear transformation T defined as $T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}\right) = \begin{bmatrix} -2x_1 - x_2 + x_4 \\ 0 \\ x_1 + 2x_2 + 3x_3 + 4x_4 \\ 2x_1 + 3x_2 + 4x_3 + 5x_4 \end{bmatrix}$,
   (a) Find a basis for the range of T;
   (b) If the null space of T is nonzero, find a basis for the null space of T.

3. (12%) Let $\mathcal{L} = \left\{\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}\right\}$ and $V = \text{Col}\begin{bmatrix} 1 & -1 & -3 & 1 \\ -1 & 1 & 3 & 2 \\ -3 & 1 & -1 & -1 \\ 2 & -2 & -6 & 1 \end{bmatrix}$. Find a basis for the subspace V that contains the given linearly independent subset $\mathcal{L}$ of V.

4. (12%) Given a linearly independent subset $\mathcal{L} = \left\{\begin{bmatrix} -1 \\ -1 \\ 6 \\ -7 \end{bmatrix}, \begin{bmatrix} 5 \\ -9 \\ -2 \\ -1 \end{bmatrix}\right\}$ of the subspace $V = \text{Span}\left\{\begin{bmatrix} 1 \\ -2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ -2 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -2 \\ 10 \end{bmatrix}\right\}$, find a basis for the subspace V that contains the given linearly independent subset $\mathcal{L}$ of V.

5. (12%) For the linear transformation defined by $T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} -x_1 - x_2 + x_3 \\ x_1 + 2x_2 + x_3 \\ x_1 + x_2 \end{bmatrix}$, determine
   (a) the dimension of the range of T
   (b) the dimension of the null space of T
   (c) whether T is one-to-one or on-to.

6. (12%) Find the unique representation of $\boldsymbol{u} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ as a linear combination of $\boldsymbol{b_1} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$, $\boldsymbol{b_2} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$, and $\boldsymbol{b_3} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$.

7. (12 %) Let $\mathcal{B}=\{\boldsymbol{b}_1, \boldsymbol{b}_2, \boldsymbol{b}_3\}$, where $\boldsymbol{b_1} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$, $\boldsymbol{b_2} = \begin{bmatrix} -1 \\ 3 \\ 0 \end{bmatrix}$, and $\boldsymbol{b_3} = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$.
   (a) Show that $\mathcal{B}$ is a basis for $\mathcal{R}^3$.
   (b) Determine the matrix $A=[[\boldsymbol{e}_1]_{\mathcal{B}} \quad [\boldsymbol{e}_2]_{\mathcal{B}} \quad [\boldsymbol{e}_3]_{\mathcal{B}}]$.
   (c) What is the relationship between A and $B=[\boldsymbol{b}_1\ \boldsymbol{b}_2\ \boldsymbol{b}_3]$?

8. (12%) Let $\mathcal{A}=\{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_n\}$ be a basis for $\mathcal{R}^n$. Then, $\mathcal{B}=\{\boldsymbol{u}_1, \boldsymbol{u}_1+\boldsymbol{u}_2, \ldots, \boldsymbol{u}_1+\boldsymbol{u}_n\}$ is also a basis for $\mathcal{R}^n$. If $\boldsymbol{v}$ is a vector in $\mathcal{R}^n$ and $[\boldsymbol{v}]_{\mathcal{A}} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$, compute $[\boldsymbol{v}]_{\mathcal{B}}$.