Homework 4, for Linear Algebra class@csie.ntnu.edu.tw-2025

1. (20%) Let T_W be the reflection of $\mathcal{R}^3$ about the plane W in $\mathcal{R}^3$ with equation $x+2y-3z=0$ and let $\mathcal{B} = \left\{\begin{bmatrix}-2\\1\\0\end{bmatrix}, \begin{bmatrix}3\\0\\1\end{bmatrix}, \begin{bmatrix}1\\2\\-3\end{bmatrix}\right\}$.

   Note that the first two vectors in $\mathcal{B}$ lies W and the third vector is perpendicular (normal) to W (because $\begin{bmatrix}a\\b\\c\end{bmatrix}^T\begin{bmatrix}x\\y\\z\end{bmatrix} = ax+by+cz=0 \quad where \quad \begin{bmatrix}a\\b\\c\end{bmatrix} = \begin{bmatrix}1\\2\\-3\end{bmatrix}$.)

   (a) Find $T_W(\boldsymbol{v})$ for each vector $\boldsymbol{v}$ in $\mathcal{B}$.
   (b) Show that $\mathcal{B}$ is a basis for $\mathcal{R}^3$.
   (c) Find $[T_W]_{\mathcal{B}}$.
   (d) Find the standard matrix of T_W.
   (e) Determine an explicit formula for $T_W\left(\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}\right)$.

2. (12%) (Refer to the definition after this problem.) Let T: $\mathcal{R}^n \to \mathcal{R}^m$ be a linear transformation, and $\mathcal{B}=\{\boldsymbol{b}_1, \boldsymbol{b}_2, \ldots, \boldsymbol{b}_n\}$ and $\mathcal{C}=\{\boldsymbol{c}_1, \boldsymbol{c}_2, \ldots, \boldsymbol{c}_m\}$ be bases for $\mathcal{R}^n$ and $\mathcal{R}^m$, respectively. Let B and C be the matrices whose columns are the vectors in $\mathcal{B}$ and $\mathcal{C}$, respectively. Prove the following:
   (a) If A is the standard matrix of T, then $[T]_{\mathcal{B}}^{\mathcal{C}} = C^{-1}AB$.
   (b) $[T(\boldsymbol{v})]_{\mathcal{C}} = [T]_{\mathcal{B}}^{\mathcal{C}}[\boldsymbol{v}]_{\mathcal{B}}$ for any vector $\boldsymbol{v}$ in $\mathcal{R}^n$.
   (c) Let U: $\mathcal{R}^m \to \mathcal{R}^p$ be linear, and let $\mathcal{D}$ be a basis for $\mathcal{R}^p$. Then $[UT]_{\mathcal{B}}^{\mathcal{D}} = [U]_{\mathcal{C}}^{\mathcal{D}}[T]_{\mathcal{B}}^{\mathcal{C}}$.

Definition

Let T: $\mathcal{R}^n \to \mathcal{R}^m$ be a linear transformation, and let $\mathcal{B}=\{\boldsymbol{b}_1, \boldsymbol{b}_2, \ldots, \boldsymbol{b}_n\}$ and $\mathcal{C}=\{\boldsymbol{c}_1, \boldsymbol{c}_2, \ldots, \boldsymbol{c}_m\}$ be bases for $\mathcal{R}^n$ and $\mathcal{R}^m$, respectively. The Matrix $[[T(\boldsymbol{b}_1)]_{\mathcal{C}} \quad [T(\boldsymbol{b}_2)]_{\mathcal{C}} \quad \ldots \quad [T(\boldsymbol{b}_n)]_{\mathcal{C}}]$ is called the matrix representation of T with respective to $\mathcal{B}$ and $\mathcal{C}$. It is denoted by $[T]_{\mathcal{B}}^{\mathcal{C}}$.

3. (12%) Find the eigenvalues of linear operator T and determine a basis for each eigenspace, where $T\left(\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}\right) = \begin{bmatrix}-4x_1+6x_2\\2x_2\\-5x_1+5x_2+x_3\end{bmatrix}$.

4. (12%) Given a matrix $A = \begin{bmatrix}-2 & 6 & 3\\-2 & -8 & -2\\4 & 6 & -1\end{bmatrix}$ and its characteristic polynomial $-(t+5)(t+4)(t+2)$, find, if possible, an invertible matrix P and its diagonal matrix D such that $A = PDP^{-1}$.

5. (10%) Let A be a diagonalizable $n \times n$ matrix. Prove that if the characteristic polynomial of A is $f(t) = a_n t^n + a_{n-1}t^{n-1} + \cdots + a_1 t + a_0$, then $f(A) = O$, where $f(A) = a_n A^n + a_{n-1}A^{n-1} + \cdots + a_1 A + a_0 I_n$ (This is called the *Cayley-Hamilton theorem*.)

6. (10%) A linear operator T on $\mathcal{R}^n$ is given in the following. Find, if possible, a basis $\mathcal{B}$ for $\mathcal{R}^n$ such that $[T]_{\mathcal{B}}$ is a diagonal matrix. If no such basis exists, explain why. $T\left(\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}\right) = \begin{bmatrix}4x_1-5x_2\\-x_2\\-x_3\end{bmatrix}$.

7. (12%) Given a linear operator T and its characteristic polynomial $f(t)$, determine all the values of the scalar c for which T on $\mathcal{R}^3$ is not diagonalizable, where $T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_1 + 2x_2 - x_3 \\ cx_2 \\ 6x_1 - x_2 + 6x_3 \end{bmatrix}$ and $f(t) = -(t-c)(t-3)(t-4)$.

8. (12%) Let $\{\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}\}$ be a basis for $\mathcal{R}^3$, and let T be the linear operator on $\mathcal{R}^3$ defined by $T(a\boldsymbol{u} + b\boldsymbol{v} + c\boldsymbol{w}) = a\boldsymbol{u} + b\boldsymbol{v} - c\boldsymbol{w}$ for all scalars a, b and c.

(a) Find the eigenvalues of T and determine a basis for each eigenspace.

(b) Is T diagonalizable? Justify your answer.