Midterm test for Linear Algebra class@ csie.ntnu.edu.tw-2025

1. (a)(6%) Find the rank and nullity of matrix A, where $A = \begin{bmatrix} 1 & 1 & 0 & -1 & 6 \\ 5 & 2 & -1 & -1 & 7 \\ -4 & -1 & 1 & 1 & -3 \\ -3 & 0 & 1 & 1 & 1 \end{bmatrix}$.

   (b)(8%) Determine a vector form for the general solution of the system of linear equations $A\boldsymbol{x} = \boldsymbol{0}$, where $\boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_5 \end{bmatrix}$. Also find the pivot columns of A?

2. (10%) Let $\boldsymbol{u}$ and $\boldsymbol{v}$ be any vectors in $\mathcal{R}^n$. Prove that the spans of $\{\boldsymbol{u}, \boldsymbol{v}\}$ and $\{\boldsymbol{u}+\boldsymbol{v}, \boldsymbol{u}-\boldsymbol{v}\}$ are equal.

3. (9 %) Let A be an $m\times n$ matrix with reduced row echelon form R. Determine the reduced row echelon form of each of the following matrices:
   (a) $[\boldsymbol{a}_1 \quad \boldsymbol{a}_2 \quad \ldots \quad \boldsymbol{a}_k]$ for $k<n$, where $\boldsymbol{a}_i=A\boldsymbol{e}_{i};$
   (b) cA, where c is a nonzero scalar;
   (c) $[A \quad cA]$, where c is any scalar.

4. Let $A = \begin{bmatrix} 1 & 2 & 1 & -1 \\ 2 & 4 & 1 & 1 \\ 3 & 2 & -1 & -2 \\ 2 & 5 & 3 & 0 \end{bmatrix}$. Find
   (a) (6%) a permutation matrix P such that PA has an LU decomposition and
   (b) (8%) an LU decomposition of PA.

5. (12%) Suppose that T: $\mathcal{R}^3 \to \mathcal{R}^3$ is a linear transformation such that $T\left(\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, $T\left(\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$ and $T\left(\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}\right) = \begin{bmatrix} 5 \\ 4 \\ 3 \end{bmatrix}$. Determine $T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right)$ for any $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ in $\mathcal{R}^3$.

6. Linear transformations T: $\mathcal{R}^3 \to \mathcal{R}^2$ and U: $\mathcal{R}^2 \to \mathcal{R}^3$ are defined by $T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} 2x_1 + x_3 \\ x_1 + x_2 - x_3 \end{bmatrix}$ and $U\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} 3x_1 - x_2 \\ x_2 \\ x_1 + x_2 \end{bmatrix}$.
   (a) (9%) Determine the domain, codomain, and the rule for UT.
   (b) (6%) Find the standard matrix of UT.

7. (a) (6%) Evaluate the determinant of matrix A, where $A = \begin{bmatrix} 2 & 1 & 5 & 2 \\ 2 & 1 & 8 & 1 \\ 2 & -1 & 5 & 3 \\ 4 & -2 & 10 & 3 \end{bmatrix}$.

(b) (4 %) Evaluate det kA if A is an $n \times n$ matrix and k is a scalar.

(c) (6%) Suppose that an $n \times n$ matrix can be expressed in the form $A = PDP^{-1}$, where P is an invertible matrix and D is a diagonal matrix . Prove that the determinant of A equals the product of the diagonal entries of D.

8. (a) (4%) Let A and B be square matrices of order n satisfying $A\boldsymbol{x} = B\boldsymbol{x}$ for all $\boldsymbol{x} \in \mathfrak{R}^n$. Find the rank and nullity of $A - B$.

(b) (6%) Let A be an $m \times n$ matrix. Prove that Null $A \subset$ Null $A^T A$.