Homework 4 Solutions: Linear Algebra (2025)

information

- 41173058h
- 鍾詠傑

Question 1 (20%)

Problem: Let T_w be the reflection of $\mathbb{R}^3$ about the plane W in $\mathbb{R}^3$ with equation $x + 2y - 3z = 0$ and let

$$\mathcal{B} = \left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix} \right\}.$$

Note that the first two vectors in $\mathcal{B}$ lie in W and the third vector is perpendicular (normal) to W.

(a) Find $T_w(\mathbf{v})$ for each vector $\mathbf{v}$ in $\mathcal{B}$.

Solution:
Let $\mathbf{b}_1 = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$, $\mathbf{b}_2 = \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}$, and $\mathbf{b}_3 = \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}$.
Since $\mathbf{b}_1$ and $\mathbf{b}_2$ lie in the plane W, the reflection leaves them unchanged:

$$T_w(\mathbf{b}_1) = \mathbf{b}_1$$

$$T_w(\mathbf{b}_2) = \mathbf{b}_2$$

Since $\mathbf{b}_3$ is normal to the plane W, the reflection maps it to its negative:

$$T_w(\mathbf{b}_3) = -\mathbf{b}_3$$

(b) Show that $\mathcal{B}$ is a basis for $\mathbb{R}^3$.

Solution:
The set $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$ consists of three vectors in $\mathbb{R}^3$. To show they form a basis, we need to show they are linearly independent.
Since $\mathbf{b}_1$ and $\mathbf{b}_2$ are non-parallel vectors in W (they are linearly independent), and $\mathbf{b}_3$ is normal to W (and thus not in W), $\mathbf{b}_3$ cannot be written as a linear combination of $\mathbf{b}_1$ and $\mathbf{b}_2$.
Alternatively, we can compute the determinant of the matrix formed by these vectors:

$$\det \begin{bmatrix} -2 & 3 & 1 \\ 1 & 0 & 2 \\ 0 & 1 & -3 \end{bmatrix} = -2(0-2) - 3(-3-0) + 1(1-0) = 4 + 9 + 1 = 14 \neq 0$$

Since the determinant is non-zero, the vectors are linearly independent and form a basis for $\mathbb{R}^3$.

© Find $[T_w]_{\mathcal{B}}$.

Solution:
Using the results from (a):

$$[T_w(\mathbf{b}_1)]_{\mathcal{B}} = [\mathbf{b}_1]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$[T_w(\mathbf{b}_2)]_{\mathcal{B}} = [\mathbf{b}_2]_{\mathcal{B}} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$[T_w(\mathbf{b}_3)]_{\mathcal{B}} = [-\mathbf{b}_3]_{\mathcal{B}} = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}$$

Thus, the matrix representation is:

$$[T_w]_{\mathcal{B}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

(d) Find the standard matrix of T_w.

Solution:
Let A be the standard matrix of T_w. We know that $A = P[T_w]_{\mathcal{B}}P^{-1}$, where P is the transition matrix from $\mathcal{B}$ to the standard basis, i.e., $P = [\mathbf{b}_1 \quad \mathbf{b}_2 \quad \mathbf{b}_3]$.

$$P = \begin{bmatrix} -2 & 3 & 1 \\ 1 & 0 & 2 \\ 0 & 1 & -3 \end{bmatrix}$$

Computing A:

$$A = P \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} P^{-1}$$

Using computational tools, we find:

$$A = \begin{bmatrix} 6/7 & -2/7 & 3/7 \\ -2/7 & 3/7 & 6/7 \\ 3/7 & 6/7 & -2/7 \end{bmatrix} = \frac{1}{7} \begin{bmatrix} 6 & -2 & 3 \\ -2 & 3 & 6 \\ 3 & 6 & -2 \end{bmatrix}$$

(e) Determine an explicit formula for $T_w \left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \right)$.

Solution:
Using the standard matrix A:

$$T_w \left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \right) = A \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \frac{1}{7} \begin{bmatrix} 6x_1 - 2x_2 + 3x_3 \\ -2x_1 + 3x_2 + 6x_3 \\ 3x_1 + 6x_2 - 2x_3 \end{bmatrix}$$

Question 2 (12%)

Problem: Prove the following properties about matrix representations of linear transformations.

(a) If A is the standard matrix of T, then $[T]_{\mathcal{B}}^{\mathcal{C}} = C^{-1}AB$.

Proof:
By definition, the columns of $[T]_{\mathcal{B}}^{\mathcal{C}}$ are coordinate vectors of $T(\mathbf{b}_j)$ with respect to $\mathcal{C}$.

$$[T]_{\mathcal{B}}^{\mathcal{C}} = [[T(\mathbf{b}_1)]_{\mathcal{C}} \dots [T(\mathbf{b}_n)]_{\mathcal{C}}]$$

We know that $T(\mathbf{x}) = A\mathbf{x}$. So $T(\mathbf{b}_j) = A\mathbf{b}_j$.
The coordinate vector $[\mathbf{v}]_{\mathcal{C}}$ is obtained by $C^{-1}\mathbf{v}$.
Thus, $[T(\mathbf{b}_j)]_{\mathcal{C}} = C^{-1}A\mathbf{b}_j$.
Therefore,

$$[T]_{\mathcal{B}}^{\mathcal{C}} = \left[C^{-1}A\mathbf{b}_1 \dots C^{-1}A\mathbf{b}_n\right] = C^{-1}A\left[\mathbf{b}_1 \dots \mathbf{b}_n\right] = C^{-1}AB$$

(b) \([T(\mathbf{v})]_{\mathcal{C}} = [T]_{\mathcal{B}}^{\mathcal{C}} [\mathbf{v}]_{\mathcal{B}}\) for any vector \(\mathbf{v}\) in \(\mathbb{R}^n\).

Proof:
Let $[\mathbf{v}]_{\mathcal{B}} = \mathbf{x}$. Then $\mathbf{v} = B\mathbf{x}$.

$$T(\mathbf{v}) = T(B\mathbf{x}) = A(B\mathbf{x}) = (AB)\mathbf{x}$$

We want to find $[T(\mathbf{v})]_{\mathcal{C}}$, which is $C^{-1}T(\mathbf{v})$.

$$[T(\mathbf{v})]_{\mathcal{C}} = C^{-1}(AB)\mathbf{x} = (C^{-1}AB)\mathbf{x}$$

From part (a), $[T]_{\mathcal{B}}^{\mathcal{C}} = C^{-1}AB$.

So,

$$[T(\mathbf{v})]_{\mathcal{C}} = [T]_{\mathcal{B}}^{\mathcal{C}}[\mathbf{v}]_{\mathcal{B}}$$

© $[UT]_{\mathcal{B}}^{\mathcal{D}} = [U]_{\mathcal{C}}^{\mathcal{D}}[T]_{\mathcal{B}}^{\mathcal{C}}$.

Proof:

Let A be the standard matrix of T and M be the standard matrix of U.
From (a), $[T]_{\mathcal{B}}^{\mathcal{C}} = C^{-1}AB$ and $[U]_{\mathcal{C}}^{\mathcal{D}} = D^{-1}MC$.
The standard matrix of the composition UT is MA.
Using (a) for UT:

$$[UT]_{\mathcal{B}}^{\mathcal{D}} = D^{-1}(MA)B$$

Now compute the product of the individual representations:

$$[U]_{\mathcal{C}}^{\mathcal{D}}[T]_{\mathcal{B}}^{\mathcal{C}} = (D^{-1}MC)(C^{-1}AB) = D^{-1}M(CC^{-1})AB = D^{-1}MIAB = D^{-1}(MA)B$$

Thus,

$$[UT]_{\mathcal{B}}^{\mathcal{D}} = [U]_{\mathcal{C}}^{\mathcal{D}}[T]_{\mathcal{B}}^{\mathcal{C}}$$

Question 3 (12%)

Problem: Find eigenvalues and basis for each eigenspace of T.

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} -4x_1 + 6x_2 \\ 2x_2 \\ -5x_1 + 5x_2 + x_3 \end{bmatrix}$$

Solution:

The standard matrix of T is:

$$A = \begin{bmatrix} -4 & 6 & 0 \\ 0 & 2 & 0 \\ -5 & 5 & 1 \end{bmatrix}$$

The characteristic polynomial is $\det(A - \lambda I)$:

$$\det \begin{bmatrix} -4-\lambda & 6 & 0 \\ 0 & 2-\lambda & 0 \\ -5 & 5 & 1-\lambda \end{bmatrix} = (2-\lambda)\det \begin{bmatrix} -4-\lambda & 0 \\ -5 & 1-\lambda \end{bmatrix} = (2-\lambda)(-4-\lambda)(1-\lambda)$$

The eigenvalues are $\lambda = 2, -4, 1$.

For $\lambda = 2$:

$$A - 2I = \begin{bmatrix} -6 & 6 & 0 \\ 0 & 0 & 0 \\ -5 & 5 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$x_1 = x_2$, $x_3 = 0$. Basis: $\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right\}$.

For $\lambda = -4$:

$$A + 4I = \begin{bmatrix} 0 & 6 & 0 \\ 0 & 6 & 0 \\ -5 & 5 & 5 \end{bmatrix} \sim \begin{bmatrix} 0 & 1 & 0 \\ -1 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$x_2 = 0$, $x_1 = x_3$. Basis: $\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$.

For $\lambda = 1$:

$$A - I = \begin{bmatrix} -5 & 6 & 0 \\ 0 & 1 & 0 \\ -5 & 5 & 0 \end{bmatrix} \sim \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$x_1 = 0$, $x_2 = 0$, x_3 is free. Basis: $\left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$.

Question 4 (12%)

Problem: Given $A = \begin{bmatrix} -2 & 3 & 6 \\ -2 & -8 & -2 \\ 6 & -1 & 4 \end{bmatrix}$ and characteristic polynomial $-(t+5)(t+4)(t+2)$, find P and D.

Solution:
There is a discrepancy in the problem statement. The characteristic polynomial of the matrix A as given is $f(\lambda) = -(\lambda^3 + 6\lambda^2 - 56\lambda - 356)$, which has non-integer roots (approximately $5.36, -3.11, -8.24$). The provided polynomial $-(t+5)(t+4)(t+2)$ has roots $-5, -4, -2$, which do not match the matrix.
As it is impossible to find a matrix P of eigenvectors for the provided polynomial without the correct corresponding matrix, and the matrix provided does not have simple integer eigenvalues suitable for manual diagonalization (and likely has a typo), we conclude that the problem cannot be solved as stated due to the contradiction.
If the matrix were such that its eigenvalues were $-2, -4, -5$, we would find the eigenvectors for each to form P and set $D = \text{diag}(-2, -4, -5)$.

Question 5 (10%)

Problem: Prove Cayley-Hamilton theorem for diagonalizable matrix A: $f(A) = O$.

Proof:
Since A is diagonalizable, there exists an invertible matrix P and a diagonal matrix D such that $A = PDP^{-1}$.
The characteristic polynomial is $f(t)$. Let $\lambda_1, \ldots, \lambda_n$ be the eigenvalues of A. These are the diagonal entries of D.
We compute $f(A)$:

$$f(A) = a_n A^n + \cdots + a_0 I = a_n (PDP^{-1})^n + \cdots + a_0 I$$

Since $(PDP^{-1})^k = PD^kP^{-1}$, we have:

$$f(A) = P(a_n D^n + \cdots + a_0 I)P^{-1} = Pf(D)P^{-1}$$

The matrix $f(D)$ is a diagonal matrix with entries $f(\lambda_i)$ on the diagonal.

$$f(D) = \mathrm{diag}(f(\lambda_1), \ldots, f(\lambda_n))$$

Since each λ_i is a root of the characteristic polynomial $f(t)$, we have $f(\lambda_i) = 0$ for all i.
Thus, $f(D)$ is the zero matrix.

$$f(A) = P \cdot O \cdot P^{-1} = O$$

Question 6 (10%)

Problem: Determine if T is diagonalizable, where $T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} 4x_1 - 5x_2 \\ -x_2 \\ -x_3 \end{bmatrix}$.

Solution:
The standard matrix is $A = \begin{bmatrix} 4 & -5 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$.
Since A is upper triangular, the eigenvalues are the diagonal entries:
$\lambda_1 = 4, \lambda_2 = -1, \lambda_3 = -1$.
Algebraic multiplicities: $\lambda = 4$ is 1, $\lambda = -1$ is 2.
We check the geometric multiplicity of $\lambda = -1$.

$$A - (-1)I = \begin{bmatrix} 5 & -5 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

The system $x_1 - x_2 = 0$ has two free variables (x_2 and x_3). Thus the dimension of the eigenspace is 2.
Since the geometric multiplicity (2) equals the algebraic multiplicity (2) for $\lambda = -1$, and the multiplicities match for $\lambda = 4$ (both 1), the operator T is **diagonalizable**.

A basis $\mathcal{B}$ can be formed by the union of basis vectors for the eigenspaces:

For $\lambda = -1$: Basis $\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$.

For $\lambda = 4$: Basis $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}$.

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}$$

Question 7 (12%)

Problem: Find c for which T is not diagonalizable.

$A = \begin{bmatrix} 1 & 2 & -1 \\ 0 & c & 0 \\ 6 & -1 & 6 \end{bmatrix}$, $f(t) = -(t-c)(t-3)(t-4)$.

Solution:

The eigenvalues are $c, 3, 4$.

T is diagonalizable if the geometric multiplicity equals the algebraic multiplicity for every eigenvalue.

If $c, 3, 4$ are distinct ($c \neq 3$ and $c \neq 4$), T is diagonalizable.

We check the cases where eigenvalues are repeated:

Case 1: $c = 3$.

Eigenvalues are $3, 3, 4$. Algebraic multiplicity of $\lambda = 3$ is 2.

We check the rank of $A - 3I$:

$$A - 3I = \begin{bmatrix} -2 & 2 & -1 \\ 0 & 0 & 0 \\ 6 & -1 & 3 \end{bmatrix}$$

Row 3 is not a multiple of Row 1 (independent). The rank is 2.

Geometric multiplicity = $3 - \text{Rank} = 3 - 2 = 1$.

Since $1 < 2$, T is **not diagonalizable** for $c = 3$.

Case 2: $c = 4$.

Eigenvalues are $4, 4, 3$. Algebraic multiplicity of $\lambda = 4$ is 2.

We check the rank of $A - 4I$:

$$A - 4I = \begin{bmatrix} -3 & 2 & -1 \\ 0 & 0 & 0 \\ 6 & -1 & 2 \end{bmatrix}$$

Row 3 is not a multiple of Row 1. Rank is 2.

Geometric multiplicity = $3 - 2 = 1$.

Since $1 < 2$, T is **not diagonalizable** for $c = 4$.

Answer: $c = 3$ and $c = 4$.

Question 8 (12%)

Problem: $T(a\mathbf{u} + b\mathbf{v} + c\mathbf{w}) = a\mathbf{u} + b\mathbf{v} - c\mathbf{w}$ for basis $\mathcal{B} = \{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$.

(a) Find eigenvalues and basis for each eigenspace.

Solution:
From the definition:
$T(\mathbf{u}) = 1\mathbf{u}$
$T(\mathbf{v}) = 1\mathbf{v}$
$T(\mathbf{w}) = -1\mathbf{w}$
The eigenvalues are $\lambda = 1$ and $\lambda = -1$.
Eigenspace for $\lambda = 1$: Spanned by $\{\mathbf{u}, \mathbf{v}\}$.
Eigenspace for $\lambda = -1$: Spanned by $\{\mathbf{w}\}$.

(b) Is T diagonalizable?

Solution:
Yes. The sum of dimensions of the eigenspaces is $2 + 1 = 3$, which equals the dimension of the space $\mathbb{R}^3$.
Also, the matrix representation relative to $\mathcal{B}$ is diagonal: $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$.