Solution guide for LA Homework 4@csie.ntnu.edu.tw-2025

1. (20 %) Let T_w be the reflection of $\mathcal{R}^3$ about the plane W in $\mathcal{R}^3$ …. (a) Find $T_w(\boldsymbol{v})$ … (b) Show that $\mathcal{B}$ … (c) Find $[T_w]_{\mathcal{B}}$ … (d) Find …. (e) Determine an explicit …

   ANS:

   (a) $T_W\left(\begin{bmatrix}-2\\1\\0\end{bmatrix}\right)=\begin{bmatrix}-2\\1\\0\end{bmatrix}$, $T_W\left(\begin{bmatrix}3\\0\\1\end{bmatrix}\right)=\begin{bmatrix}3\\0\\1\end{bmatrix}$ and $T_W\left(\begin{bmatrix}1\\2\\-3\end{bmatrix}\right)=-\begin{bmatrix}1\\2\\-3\end{bmatrix}$ (or $T_w(\boldsymbol{b}_1)=\boldsymbol{b}_1$, $T_w(\boldsymbol{b}_2)=\boldsymbol{b}_2$ and $T_w(\boldsymbol{b}_3)=-\boldsymbol{b}_3$ where $\boldsymbol{b}_1=\begin{bmatrix}-2\\1\\0\end{bmatrix}$, $\boldsymbol{b}_2=\begin{bmatrix}3\\0\\1\end{bmatrix}$ and $\boldsymbol{b}_3=\begin{bmatrix}1\\2\\-3\end{bmatrix}$).

   (b) $\mathcal{B}$ is a linearly independent set of 3 vectors in $\mathcal{R}^3$.

   (c) $[T_w]_{\mathcal{B}}=[[T_w(\boldsymbol{b}_1)]_{\mathcal{B}}\quad [T_w(\boldsymbol{b}_2)]_{\mathcal{B}}\quad [T_w(\boldsymbol{b}_3)]_{\mathcal{B}}]=\begin{bmatrix}1&0&0\\0&1&0\\0&0&-1\end{bmatrix}$

   (d) Let A be the standard matrix of T_w. Then,

   $$A=B[T_w]_{\mathcal{B}}B^{-1}=\begin{bmatrix}-2&3&1\\1&0&2\\0&1&-3\end{bmatrix}\begin{bmatrix}1&0&0\\0&1&0\\0&0&-1\end{bmatrix}\begin{bmatrix}-2&3&1\\1&0&2\\0&1&-3\end{bmatrix}^{-1}=\frac{1}{7}\begin{bmatrix}6&-2&3\\-2&3&6\\3&6&-2\end{bmatrix}.$$

   (e) $$T_w\left(\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}\right)=A\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\frac{1}{7}\begin{bmatrix}6&-2&3\\-2&3&6\\3&6&-2\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\frac{1}{7}\begin{bmatrix}6x_1-2x_2+3x_3\\-2x_1+3x_2+6x_3\\3x_1+6x_2-2x_3\end{bmatrix}.$$

2. (12 %) … Let T: $\mathcal{R}^n\to\mathcal{R}^m$ be a linear transformation … Prove the following: (a) If A is the standard matrix … (b) $[T(\boldsymbol{v})]_{\mathcal{C}}=[T]_{\mathcal{B}}^{\mathcal{C}}[\boldsymbol{v}]_{\mathcal{B}}$ … (c) Let U: $\mathcal{R}^m\to\mathcal{R}^p$ be linear, and …

   ANS:

   (a) Let A be the standard matrix of T.

   Then $[T]_{\mathcal{B}}^{\mathcal{C}}=[[T(\boldsymbol{b}_1)]_{\mathcal{C}}\quad [T(\boldsymbol{b}_2)]_{\mathcal{C}}\quad \dots\quad [T(\boldsymbol{b}_n)]_{\mathcal{C}}]$
   $=[C^{-1}T(\boldsymbol{b}_1)\quad C^{-1}T(\boldsymbol{b}_2)\ \dots C^{-1}T(\boldsymbol{b}_n)]=C^{-1}[T(\boldsymbol{b}_1)\quad T(\boldsymbol{b}_2)\ \dots T(\boldsymbol{b}_n)]$
   $=C^{-1}[A\boldsymbol{b}_1\quad A\boldsymbol{b}_2\ \dots A\boldsymbol{b}_n]=C^{-1}A[\boldsymbol{b}_1\quad \boldsymbol{b}_2\ \dots \boldsymbol{b}_n]=C^{-1}AB.$

   (b) For any $\boldsymbol{v}$ in $\mathcal{R}^n$,

   $[T(\boldsymbol{v})]_{\mathcal{C}}=C^{-1}T(\boldsymbol{v})=C^{-1}A\boldsymbol{v}=C^{-1}ABB^{-1}\boldsymbol{v}=C^{-1}AB(B^{-1}\boldsymbol{v})=C^{-1}AB[\boldsymbol{v}]_{\mathcal{B}}=[T]_{\mathcal{B}}^{\mathcal{C}}[\boldsymbol{v}]_{\mathcal{B}}.$

   (c) Let D be the matrix whose columns are the vectors in $\mathcal{D}$ and let P be the standard matrix of U.
   The standard matrix of the composition UT is PA (by theorem 2.12).
   By (a), we have $[UT]_{\mathcal{B}}^{\mathcal{D}}=D^{-1}PAB=D^{-1}P(CC^{-1})AB=(D^{-1}PC)(C^{-1}AB)=[U]_{\mathcal{C}}^{\mathcal{D}}[T]_{\mathcal{B}}^{\mathcal{C}}.$

3. (12 %) Find the eigenvalues of linear operator T and determine a basis …

   ANS:

   The standard matrix of T is $A=\begin{bmatrix}-4&6&0\\0&2&0\\-5&5&1\end{bmatrix}$. The characteristic polynomial of A is

$$det(A-\lambda I_3) = det\begin{bmatrix} -4-\lambda & 6 & 0 \\ 0 & 2-\lambda & 0 \\ -5 & 5 & 1-\lambda \end{bmatrix} = (-4-\lambda)(2-\lambda)(1-\lambda).$$

The reduced row echelon form of $A - 1 \times I_3$ is $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

Then a basis for the eigenspace corresponding to eigenvalue $\lambda = 1$ is $\left\{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right\}$. ....

Therefore, three eigenvalues and a basis consisting of one eigenvector for each eigenspace are :

$-4, \left\{\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}\right\};$ $1, \left\{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right\};$ $2, \left\{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}\right\}$ (Some multiple of each eigenvector is also OK)

4. (12%) Given a matrix $A = \cdots$ and its characteristic polynomial …, find …
ANS:

$$P = \begin{bmatrix} -1 & 0 & 1 \\ 0 & -1 & -1 \\ 1 & 2 & 2 \end{bmatrix}, D = \begin{bmatrix} -5 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$ (The solution for *P* is not unique.)

5. (10 %) Let *A* be a diagonalizable $n \times n$ matrix. Prove that …

ANS:

- Let λ be an eigenvalue of *A*, then $f(\lambda)=0$.
- Let $A= PDP^{-1}$ where $D = \begin{bmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ \vdots & 0 & \ddots & \vdots \\ 0 & 0 & 0 & \lambda_n \end{bmatrix}$. Then $f(D) = a_n D^n + a_{n-1}D^{n-1} + \cdots +$

$$a_1 D + a_0 I = \begin{bmatrix} f(\lambda_1) & 0 & 0 & 0 \\ 0 & f(\lambda_2) & 0 & 0 \\ \vdots & 0 & \ddots & \vdots \\ 0 & 0 & 0 & f(\lambda_n) \end{bmatrix} = O.$$

Hence $f(A)=f(PDP^{-1}) = a_n(PDP^{-1})^n + a_{n-1}(PDP^{-1})^{n-1} + \cdots + a_1 PDP^{-1} + a_0 P I_n P^{-1} = \cdots = P f(D) P^{-1} = POP^{-1} = O$.

6. (10%) A linear operator *T* on $\mathcal{R}^n$ is given in the following. Find …
ANS:

The standard matrix of *T* is $A = \begin{bmatrix} 4 & -5 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$ and its characteristic polynomial is

$-(\lambda+1)^2(\lambda-4)$.

Bases for the eigenspaces of T corresponding to eigenvalues -1 and 4 are $\mathcal{B}_1 = \left\{\begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}\right\}$

and $\mathcal{B}_2 = \left\{\begin{bmatrix}1\\0\\0\end{bmatrix}\right\}$, respectively.

So, combining these two sets produces a basis $\mathcal{B}$ for $\mathcal{R}^3$, where $\mathcal{B} = \left\{\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}\right\}$

7. (12%) Given a linear operator T and its characteristic polynomial $f(t)$, determine ….
ANS: c =3 or 4

The standard matrix of T is $A = \begin{bmatrix}1 & 2 & -1\\0 & c & 0\\6 & -1 & 6\end{bmatrix}$.

- For eigenvalue value $t = 3$, $A - 3I_3 = \begin{bmatrix}-2 & 2 & -1\\0 & c-3 & 0\\6 & -1 & 3\end{bmatrix} \xrightarrow{R} \begin{bmatrix}-2 & 2 & -1\\0 & 5 & 0\\0 & c-3 & 0\end{bmatrix}$.

  If $c = 3$, the nullity of $A - 3I_3$ is "one", not equal to the multiplicity of eigenvalue $t = 3$.

- For eigenvalue value $t = 4$, $A - 4I_3 = \begin{bmatrix}-3 & 2 & -1\\0 & c-4 & 0\\6 & -1 & 2\end{bmatrix} \xrightarrow{R} \begin{bmatrix}-3 & 2 & -1\\0 & 3 & 0\\0 & c-4 & 0\end{bmatrix}$.

  If $c = 4$, the nullity of $A - 3I_3$ is "one", not equal to the multiplicity of eigenvalue $t = 4$.

8. (12 %) Let $\{\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}\}$ be a basis … (a) Find the eigenvalues of T and …. (b) Is T diagonalizable?
ANS:

(a) Let $\mathcal{B}=\{\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}\}$. Obviously, $T(\boldsymbol{u}) = \boldsymbol{u}$, $T(\boldsymbol{v}) = \boldsymbol{v}$ and $T(\boldsymbol{w}) = -\boldsymbol{w}$. Hence $\boldsymbol{u}$ and $\boldsymbol{v}$ are eigenvectors of T with corresponding eigenvalue 1, and $\boldsymbol{w}$ is an eigenvector of T with corresponding eigenvalue -1.

* Thus, $\{\boldsymbol{u}, \boldsymbol{v}\}$ is a basis for the eigenspace of T corresponding to eigenvalue 1, and

** $\{\boldsymbol{w}\}$ is a basis for the eigenspace of T corresponding to eigenvalue -1.

(b) By (a), there is a basis $\mathcal{B}$ for $\mathcal{R}^3$ consisting of eigenvectors of T. Thus, T is diagonalizable.