

Homework 5, for Linear Algebra class@ csie.ntnu.edu.tw-2025

1. (10%) Let A be any $m \times n$ matrix.
 - (a) Prove that $A^T A$ and A have the same null space.
 - (b) Use (a) to prove that $\text{rank } A^T A = \text{rank } A$.
2. (18%) Let $\mathcal{S} = \left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 1 \\ 5 \end{bmatrix} \right\}$.
 - (a) Apply the Gram-Schmidt process to replace the given linearly independent set \mathcal{S} by an orthogonal set of nonzero vectors with the same span, and obtain an orthonormal set with the same span as \mathcal{S} .
 - (b) Let A be the matrix whose columns are the vectors in \mathcal{S} . Determine the matrices Q and R in a QR factorization of A .
 - (c) Use the QR factorization to solve the system $Ax=b$ where $b = \begin{bmatrix} 8 \\ 0 \\ 1 \\ 11 \end{bmatrix}$.
3. (15%) Let $\{w_1, w_2, \dots, w_n\}$ be an orthonormal basis for \mathcal{R}^n . Prove that, for any vectors u and v in \mathcal{R}^n ,
 - (a) $u+v = (u \cdot w_1 + v \cdot w_1)w_1 + (u \cdot w_2 + v \cdot w_2)w_2 + \dots + (u \cdot w_n + v \cdot w_n)w_n$.
 - (b) $u \cdot v = (u \cdot w_1)(v \cdot w_1) + (u \cdot w_2)(v \cdot w_2) + \dots + (u \cdot w_n)(v \cdot w_n)$. (This is known as *Parseval's identity*.)
 - (c) $\|u\|^2 = (u \cdot w_1)^2 + (u \cdot w_2)^2 + \dots + (u \cdot w_n)^2$.
4. (15 %) Let $u = \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix}$ and W be the solution set of the system of equations

$$\begin{aligned} x_1 + 2x_2 - 3x_3 &= 0 \\ x_1 + x_2 - 3x_3 &= 0 \end{aligned}$$
 - (a) Find the orthogonal projection matrix P_W .
 - (b) Obtain the unique vectors w in W and z in W^\perp such that $u = w + z$.
 - (c) Find the distance from u to W .
5. (10 %) Let W be a subspace of \mathcal{R}^n . Prove that $P_W P_{W^\perp} = P_{W^\perp} P_W = O$, and hence $P_{W^\perp} = I_n - P_W$ (or $P_W + P_{W^\perp} = I_n$).
6. (12 %) An inconsistent system of linear equations $Ax=b$ is given where $A = \begin{bmatrix} 1 & 2 & -1 \\ 1 & -1 & 2 \\ 2 & 1 & 1 \end{bmatrix}$ and $b = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}$.
 - (a) Obtain the vector z for which $\|Az-b\|$ is a minimum.
 - (b) Find the vector z of least norm for which $\|Az-b\|$ is a minimum.
7. (10 %) Find an orthogonal operator T on \mathcal{R}^3 such that $T(v)=w$ where $v = \frac{1}{\sqrt{10}} \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}$ and $w = \frac{1}{\sqrt{5}} \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix}$.
8. (10 %) Given a symmetric matrix $A = \begin{bmatrix} 0 & 2 & 2 \\ 2 & 0 & 2 \\ 2 & 2 & 0 \end{bmatrix}$, find an orthonormal basis of eigenvectors and their corresponding eigenvalues. Use this information to obtain a spectral decomposition of A .