Solution guide for LA-Homework 2@ csie.ntnu.edu.tw-2025

1. (12%) A square matrix A is called **upper triangular** ... Prove that if A and B ...

   ANS:

   Suppose that A and B are both $n \times n$ upper triangular matrices. By the *row column rule*, the (i, j)-entry of AB is $\sum_{k=1}^{n} a_{i,k} b_{k,j}$.

   Consider that $i>j$.

   For $k=1,2, \ldots, n$ in the term $a_{i,k} b_{k,j}$ above

   (i) if $k > j$, then $b_{k,j}=0$ because B is upper triangular;

   (ii) if $k \leq j$, then $k < i$ and hence $a_{i,k}=0$ because A is upper triangular.

   $\rightarrow$Thus every term $a_{i,k} b_{k,j}$ for $k = 1,2,\cdots,n$ is 0 and therefore AB is upper triangular.

2. (12%) The trace of an … Prove that if A is an $m \times n$… $\text{trace}(AB) = \text{trace}(BA)$.

   ANS:

   The trace of AB is the sum of the diagonal entries of AB, that is,

$$trace(AB) = \begin{array}{l} [a_{11}b_{11} + a_{12}b_{21} + \cdots + a_{1n}b_{n1}] + \\ [a_{21}b_{12} + a_{22}b_{22} + \cdots + a_{2n}b_{n2}] + \\ \quad\vdots \\ [a_{m1}b_{1m} + a_{m2}b_{2m} + \cdots + a_{mn}b_{nm}] \end{array}$$

   By adding the first terms within each bracket, then the second terms, etc., we obtain the trace of BA..

   That is,

$$trace(BA) = \begin{array}{l} [b_{11}a_{11} + b_{12}a_{21} + \cdots + b_{1m}a_{m1}] + \\ [b_{21}a_{12} + b_{22}a_{22} + \cdots + b_{2m}a_{m2}] + \\ \quad\vdots \\ [b_{n1}a_{1n} + b_{n2}a_{2n} + \cdots + b_{nm}a_{mn}] \end{array}.$$

   Thus, $\text{trace}(AB) = \text{trace}(BA)$.

3. (12%) A and B are 3×3 invertible matrices given by … Find the value of $(A^T B^T)^{-1}$.

   ANS:

   Applying Theorem 2.2, we have $(A^T B^T)^{-1} = (B^T)^{-1} (A^T)^{-1} = (B^{-1})^T (A^{-1})^T = \begin{bmatrix} 2 & 0 & 3 \\ -1 & 0 & -2 \\ 3 & 4 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 \\ 2 & 0 & 1 \\ 3 & 1 & -1 \end{bmatrix} =$

   $\begin{bmatrix} 11 & 7 & -1 \\ -7 & -4 & 1 \\ 14 & 7 & 6 \end{bmatrix}$.

4. (12%) For a given matrix B, find columns $\boldsymbol{b_3}$ and $\boldsymbol{b_4}$ as a linear combination of …

   ANS:

   The reduced row echelon form of B is $R = \begin{bmatrix} 1 & 0 & 1 & -3 & 0 & 3 \\ 0 & 1 & -1 & 2 & 0 & -2 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$.

   By Column correspondence property,

   $\boldsymbol{b}_3 = 1\boldsymbol{b}_1 + (-1)\boldsymbol{b}_2 + 0\boldsymbol{b}_5$; $\quad \boldsymbol{b}_4 = (-3)\boldsymbol{b}_1 + 2\boldsymbol{b}_2 + 0\boldsymbol{b}_5$.

5. (14%) Determine (a) the reduced row echelon form R of A and (b) an invertible matrix P …

ANS:

Note that $P[A\ \vdots\ I_3] = [PA\ \vdots\ P] = [R\ \vdots\ P]$.

$$[A\ \vdots\ I_3] = \left[\begin{array}{ccccc|ccc} 1 & -1 & 0 & -1 & 2 & 1 & 0 & 0 \\ -1 & 1 & 1 & -2 & 1 & 0 & 1 & 0 \\ 5 & -5 & -3 & 4 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow[-5r_1+r_3\to r_3]{r_1+r_2\to r_2}$$

$$\left[\begin{array}{ccccc|ccc} 1 & -1 & 0 & -1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & -3 & 3 & 1 & 1 & 0 \\ 0 & 0 & -3 & 9 & -9 & -5 & 0 & 1 \end{array}\right] \xrightarrow{3r_2+r_3\to r_3}$$

$$\left[\begin{array}{ccccc|ccc} 1 & -1 & 0 & -1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & -3 & 3 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -2 & 3 & 1 \end{array}\right].$$

Thus, we have $R = \begin{bmatrix} 1 & -1 & 0 & -1 & 2 \\ 0 & 0 & 1 & -3 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ and $P = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ -2 & 3 & 1 \end{bmatrix}$.

6. (12%) Let A and B be $n\times n$ matrices. We say that A is **similar** to B if ... Prove the following …

ANS:

(a) Write $A = I_n^{-1} A I_n$ and let $P = I_n$.

(b) Since $B=P^{-1}AP$ for some invertible matrix P, it follows that $A= PBP^{-1}=(P^{-1})^{-1} BP^{-1}$. So, B is similar to A using P^{-1}as the invertible matrix.

(c) $B=P^{-1}AP$ and $C=Q^{-1}BQ$ for some invertible matrices P and Q.

$\to C=Q^{-1} P^{-1}AP\,Q=(PQ)^{-1}A(PQ)$. So, A is similar to C using PQ as the invertible matrix.

7. (12%) Assume that A, C and D are $n\times n$ matrices …

ANS:

We have $\begin{bmatrix} C & A \\ D & O \end{bmatrix}\begin{bmatrix} O & D^{-1} \\ A^{-1} & -A^{-1}CD^{-1} \end{bmatrix} = \begin{bmatrix} AA^{-1} & (CD^{-1} - AA^{-1}CD^{-1}) \\ O & DD^{-1} \end{bmatrix} = \begin{bmatrix} I_n & O \\ O & I_n \end{bmatrix} = I_{2n}$.

8. (14%) Let $A = \cdots$ Find (a) (6%) a permutation matrix … and (b) (8%) an LU decomposition …

ANS:

$$\begin{bmatrix} 2 & 4 & -6 & 0 \\ -2 & 1 & 3 & 2 \\ 2 & 9 & -9 & 1 \\ 4 & 3 & -3 & 0 \end{bmatrix} \xrightarrow[\substack{-r_1+r_3\to r_3 \\ -2r_1+r_4\to r_4}]{r_1+r_2\to r_2} \begin{bmatrix} 2 & 4 & -6 & 0 \\ (-1) & 5 & -3 & 2 \\ (1) & 5 & -3 & 1 \\ (2) & -5 & 9 & 0 \end{bmatrix}$$

$$\xrightarrow[r_2+r_4\to r_4]{-r_2+r_3\to r_3} \begin{bmatrix} 2 & 4 & -6 & 0 \\ (-1) & 5 & -3 & 2 \\ (1) & (1) & 0 & -1 \\ (2) & (-1) & 6 & 2 \end{bmatrix} \xrightarrow{r_3\leftrightarrow r_4} \begin{bmatrix} 2 & 4 & -6 & 0 \\ (-1) & 5 & -3 & 2 \\ (2) & (-1) & 6 & 2 \\ (1) & (1) & 0 & -1 \end{bmatrix}.$$

Therefore,

(a) $I_4 \xrightarrow{r_3\leftrightarrow r_4} P = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$ (b) $L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 2 & -1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix}$ and $U = \begin{bmatrix} 2 & 4 & -6 & 0 \\ 0 & 5 & -3 & 2 \\ 0 & 0 & 6 & 2 \\ 0 & 0 & 0 & -1 \end{bmatrix}$