Sol guide for 2025 LA-Midterm test problems@ csie.ntnu.edu.tw

1. (a)(6%) Find the rank and nullity of matrix A, where …

   ANS: (Rank A=3, Nullity=2) because $R = \begin{bmatrix} 1 & 0 & -\frac{1}{3} & 0 & -1 \\ 0 & 1 & \frac{1}{3} & 0 & 5 \\ 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

   (b)(8%) Determine a vector form …. Also find the pivot columns of ..

   ANS: $\boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = x_3 \begin{bmatrix} 1/3 \\ -1/3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 1 \\ -5 \\ 0 \\ 2 \\ 1 \end{bmatrix}$. There are three pivot columns of A, which are the three vectors in the set $S = \left\{ \begin{bmatrix} 1 \\ 5 \\ -4 \\ -3 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ 1 \\ 1 \end{bmatrix} \right\}$.

2. (10%) Let $\boldsymbol{u}$ and $\boldsymbol{v}$ be any vectors in $\mathcal{R}^n$. Prove that the spans ….

   ANS:

   $\Leftarrow$ Suppose that $\boldsymbol{w} \in \text{Span}\{\boldsymbol{u}+\boldsymbol{v}, \boldsymbol{u}-\boldsymbol{v}\}$. Then, there exist some scalars a and b such that $\boldsymbol{w}=a(\boldsymbol{u}+\boldsymbol{v})+b(\boldsymbol{u}-\boldsymbol{v})=(a+b)\boldsymbol{u}+(a-b)\boldsymbol{v}$. So, $\boldsymbol{w} \in \text{Span}\{\boldsymbol{u},\boldsymbol{v}\}$.

   Therefore, $\text{Span}\{\boldsymbol{u}+\boldsymbol{v}, \boldsymbol{u}-\boldsymbol{v}\} \subset \text{Span}\{\boldsymbol{u},\boldsymbol{v}\}$ ……(1)

   $\Rightarrow$ Suppose $z \in \text{Span}\{\boldsymbol{u}, \boldsymbol{v}\}$. Then, for some scalars c and d, we have $z=c\boldsymbol{u}+d\boldsymbol{v}=\frac{c+d}{2}(\boldsymbol{u}+\boldsymbol{v})+\frac{c-d}{2}(\boldsymbol{u}-\boldsymbol{v})$. Thus $z \in \text{Span}\{\boldsymbol{u}+\boldsymbol{v}, \boldsymbol{u}-\boldsymbol{v}\}$.

   Therefore, $\text{Span}\{\boldsymbol{u},\boldsymbol{v}\} \subset \text{Span}\{\boldsymbol{u}+\boldsymbol{v}, \boldsymbol{u}-\boldsymbol{v}\}$……(2)

   By (1) and (2), we have $\text{Span}\{\boldsymbol{u},\boldsymbol{v}\} = \text{Span}\{\boldsymbol{u}+\boldsymbol{v}, \boldsymbol{u}-\boldsymbol{v}\}$.

3. (9 %) Let A be an m×n .. Determine the reduced row echelon …(a) ...; (b) …; (c) ...

   ANS:

   (a) $[\boldsymbol{r}_1 \quad \boldsymbol{r}_2 \quad \ldots \quad \boldsymbol{r}_k]$, where $r_i=R\boldsymbol{e}_i$

   (b) R

   (c) $[R \quad cR]$

4. Let $A = \cdots$ Find (a) (6%) a permutation matrix P … and (b) (8%) an LU…

   ANS:

$$\begin{bmatrix} 1 & 2 & 1 & -1 \\ 2 & 4 & 1 & 1 \\ 3 & 2 & -1 & -2 \\ 2 & 5 & 3 & 0 \end{bmatrix} \xrightarrow[\substack{(-3)r_1+r_3 \to r_3 \\ (-2)r_1+r_4 \to r_4}]{(-2)r_1+r_2 \to r_2} \begin{bmatrix} 1 & 2 & 1 & -1 \\ (2) & 0 & -1 & 3 \\ (3) & -4 & -4 & 1 \\ (2) & 1 & 1 & 2 \end{bmatrix} \xrightarrow{r_2 \leftrightarrow r_4} \begin{bmatrix} 1 & 2 & 1 & -1 \\ (2) & 1 & 1 & 2 \\ (3) & -4 & -4 & 1 \\ (2) & 0 & -1 & 3 \end{bmatrix}$$

$$\xrightarrow{4r_2+r_3 \to r_3} \begin{bmatrix} 1 & 2 & 1 & -1 \\ (2) & 1 & 1 & 2 \\ (3) & (-4) & 0 & 9 \\ (2) & 0 & -1 & 3 \end{bmatrix} \xrightarrow{r_3 \leftrightarrow r_4} \begin{bmatrix} 1 & 2 & 1 & -1 \\ (2) & 1 & 1 & 2 \\ (2) & 0 & -1 & 3 \\ (3) & (-4) & 0 & 9 \end{bmatrix}$$

(a) $P = [\boldsymbol{e}_1 \quad \boldsymbol{e}_2 \quad \boldsymbol{e}_4 \quad \boldsymbol{e}_3][\boldsymbol{e}_1 \quad \boldsymbol{e}_4 \quad \boldsymbol{e}_3 \quad \boldsymbol{e}_2] = [\boldsymbol{e}_1 \quad \boldsymbol{e}_3 \quad \boldsymbol{e}_4 \quad \boldsymbol{e}_2] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$,

(b) $L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 2 & 0 & 1 & 0 \\ 3 & -4 & 0 & 1 \end{bmatrix}$, $U = \begin{bmatrix} 1 & 2 & 1 & -1 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & -1 & 3 \\ 0 & 0 & 0 & 9 \end{bmatrix}$.

5. (12%) Suppose that T: $\mathcal{R}^3 \to \mathcal{R}^3$ is a linear transformation … Determine $T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right)$ ….

ANS: $T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_1 + 3x_2 - x_3 \\ 2x_1 + 3x_2 + x_3 \\ 2x_1 + 3x_2 + 2x_3 \end{bmatrix}$ $\Leftarrow$

Let A be the standard matrix of T, $B = \begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}$, and $C = \begin{bmatrix} 1 & -3 & 5 \\ 2 & 0 & 4 \\ 3 & 1 & 3 \end{bmatrix}$. Then $AB=C$ and thus

$A=CB^{-1}$. Therefore $T(\boldsymbol{x})=A\boldsymbol{x}$, where $A = \begin{bmatrix} 1 & 3 & -1 \\ 2 & 3 & 1 \\ 2 & 3 & 2 \end{bmatrix}$.

(Or, notice that

(i) $T(\boldsymbol{e}_1) = \frac{1}{2}\left[T\left(\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}\right) + T\left(\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}\right)\right] = \frac{1}{2}\left(\begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 5 \\ 4 \\ 3 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$;

(ii) $T(\boldsymbol{e}_2) = \frac{1}{2}\left[T\left(\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}\right) + T\left(\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}\right)\right] = \frac{1}{2}\left(\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + \begin{bmatrix} 5 \\ 4 \\ 3 \end{bmatrix}\right) = \begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix}$; and

(iii) $T(\boldsymbol{e}_3) = \frac{1}{2}\left[T\left(\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}\right) + T\left(\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}\right)\right] = \frac{1}{2}\left(\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}$.

Thus $A = [T(\boldsymbol{e}_1) \quad T(\boldsymbol{e}_2) \quad T(\boldsymbol{e}_3)] = \begin{bmatrix} 1 & 3 & -1 \\ 2 & 3 & 1 \\ 2 & 3 & 2 \end{bmatrix}$.)

6. Linear transformations …. (a) (9%) Determine … and the rule for.. (b) (6%) Find the standard …

ANS

(a) The domain and codomain are both $\mathcal{R}^3$. $UT\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} 5x_1 - x_2 + 4x_3 \\ x_1 + x_2 - x_3 \\ 3x_1 + x_2 \end{bmatrix}$.

(b) $\begin{bmatrix} 5 & -1 & 4 \\ 1 & 1 & -1 \\ 3 & 1 & 0 \end{bmatrix}$.

7. (a) (6%) Evaluate the … (b) (4 %) Evaluate det kA …(c) (6%) … Prove that the determinant …

ANS:

(a) -36 ;

(b) $k^n \det A$ (or $-36 \times k^4$, as an extension of Problem 7(a));

(c) We have

$$\det A = \det(PDP^{-1}) = (\det P)(\det D)\ (\det P^{-1}) = (\det P)(\det D)\ (\frac{1}{\det P}) = \det D.$$ Since D is a diagonal matrix, $\det D$ is product of its diagonal entries.

8. (a) (4%) Let A and B be square matrices. Find …(b) (6%)… Prove that Null $A \subset$ Null $A^T A$.

ANS:

(a) Its rank is 0. Its nullity is n.

(b) Note that Null $A = \{\boldsymbol{x} \in \mathcal{R}^n : A\boldsymbol{x} = \boldsymbol{0}\}$ and Null $A^T A = \{\boldsymbol{x} \in \mathcal{R}^n : A^T A\boldsymbol{x} = \boldsymbol{0}\}$.

For each vector $\boldsymbol{v} \in$ Null A, $A\boldsymbol{v} = \boldsymbol{0}$ and hence $A^T A\boldsymbol{v} = A^T\boldsymbol{0} = \boldsymbol{0}.$

Thus, it is also true that $\boldsymbol{v} \in$ Null $A^T A$.

Therefore Null $A \subset$ Null $A^T A$.