Homework 1, for Linear Algebra class@ csie.ntnu.edu.tw-2025

1. (11%) Determine whether the following system is consistent, and if so, find the vector form of its general solution.
$$\begin{aligned} x_1 - x_2 + x_4 &= -4 \\ x_1 - x_2 + 2x_4 + 2x_5 &= -5 \\ 3x_1 - 3x_2 + 2x_4 - 2x_5 &= -11 \end{aligned}$$

2. (11%) Find the rank and nullity of matrix $\begin{bmatrix} 1 & 0 & 1 & -1 & 6 \\ 2 & -1 & 5 & -1 & 7 \\ -1 & 1 & -4 & 1 & -3 \\ 0 & 1 & -3 & 1 & 1 \end{bmatrix}$

3. (14%) The input-output matrix for an economy with sectors of metals, nonmetals, and services follows:
$$\begin{array}{cl} \begin{array}{ccc} \text{Met.} & \text{Nonm} & \text{Svcs} \end{array} & \\ \begin{bmatrix} .2 & .2 & .1 \\ .4 & .4 & .2 \\ .2 & .2 & .1 \end{bmatrix} & \begin{array}{l} \text{Metal} \\ \text{Nonmetals} \\ \text{Services} \end{array} \end{array}$$
   (a) What is the net production corresponding to a gross production of \$50 million of metals, \$60 million of nonmetals, and \$40 million of services?
   (b) What gross production is required to satisfy exactly a demand for \$120 million of metals, \$180 million of nonmetals, and \$150 million of services?

4. (11%) Let A and B be $m\times n$ matrices such that B can be obtained by performing a single elementary row operation on A. Prove that if the rows of A are linearly independent, then the rows of B are also linearly independent.

5. (20%) Let A be an m×n matrix with reduced row echelon form R. Determine the reduced row echelon form of each of the following matrices:
   (a) $[A \quad \mathbf{0}]$
   (b) $[\boldsymbol{a}_1 \quad \boldsymbol{a}_2 \quad \ldots \quad \boldsymbol{a}_k]$ for $k<n$, where $\boldsymbol{a}_i=A\boldsymbol{e}_i$.
   (c) cA, where c is a nonzero scalar
   (d) $[I_m \quad A]$
   (e) $[A \quad cA]$, where c is any scalar

6. (11%) Given $A = \begin{bmatrix} 1 & 0 & -1 \\ 2 & -1 & 1 \\ 0 & 3 & -2 \\ 1 & 1 & -3 \end{bmatrix}$, determine whether the equation $A\boldsymbol{x}=\boldsymbol{b}$ is consistent for every $\boldsymbol{b}$ in R^4.

7. (11%) Determine, if possible, a value of r for which the following set of vectors is linearly dependent:
$$\left\{\begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 9 \\ r \\ -2 \end{bmatrix}\right\}$$

8. (11%) Let $\boldsymbol{u}_1, \boldsymbol{u}_2, \cdots, \boldsymbol{u}_k$ be vectors in $\mathcal{R}^n$ and $c_1, c_2, \cdots, c_k$ be nonzero scalars. Prove that $\text{Span}\{\boldsymbol{u}_1, \boldsymbol{u}_2, \cdots, \boldsymbol{u}_k\} = \text{Span}\ \{c_1\boldsymbol{u}_1, c_2\boldsymbol{u}_2, \cdots, c_k\boldsymbol{u}_k\}$.