Chapter 4: Subspaces and their properties

4.1 Subspaces

Definition A set W of vectors in R^n is called a **subspace** of R^n if it has the following three properties:

1. $\boldsymbol{0} \in W$.
2. $\boldsymbol{u} + \boldsymbol{v} \in W$ for every $\boldsymbol{u}, \boldsymbol{v} \in W$. ($W$ is **closed under (vector) addition**)
3. $c\boldsymbol{u} \in W$ for every $\boldsymbol{u} \in W$ and scalar c. (W is **closed under scalar multiplication**.)

Properties 2 and 3 define the *closure* of a set.

Examples:

- The set of R^n is a subspace of itself.
- $W=\{\boldsymbol{0}\}$ in R^n is a subspace of R^n, called the **zero subspace**.
- A subspace of R^n other than $\{\boldsymbol{0}\}$ is called a **nonzero subspace**.

Theorem 4.1

The span of a finite nonempty subset of R^n is a subspace of R^n.

- *The only sets of vectors in R^n that have generating sets are subspaces of R^n* . (See example 6)
- A generating set for a subspace V must consist of vectors from V.

Subspaces associated with a matrix

Definition The null space of a matrix A is the solution set of $A\boldsymbol{x}=\boldsymbol{0}$. It is denoted by Null A.

- For an $m \times n$ matrix A, Null $A=\{\boldsymbol{v} \in R^n : A\boldsymbol{v}=\boldsymbol{0}\}$.
- The solution set of any homogeneous system of linear equations equals the null space of the coefficient matrix of that system.

Theorem 4.2

If A is an $m \times n$ matrix, then Null A is a subspace of R^n.

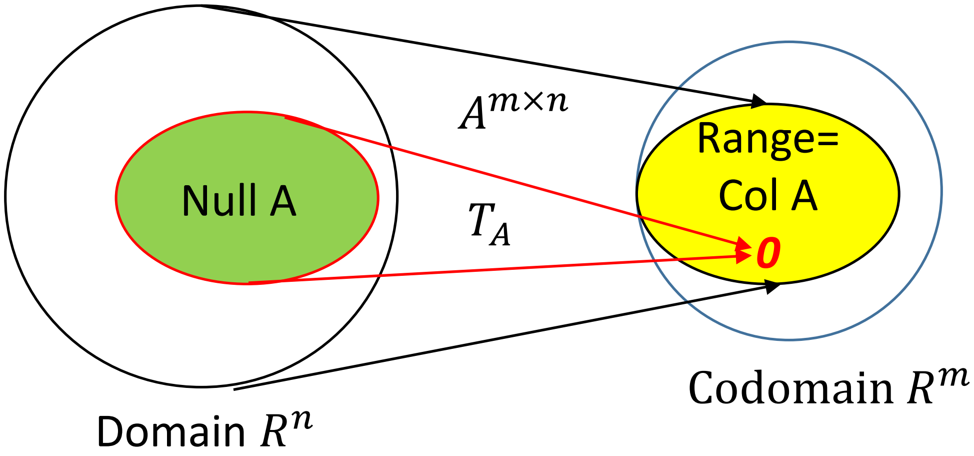
Definition The column space of a matrix A is the span of its columns. It is denoted by Col A.

For an $m \times n$ matrix A, Col $A = \{A\mathbf{v} : \mathbf{v} \in R^n\}$.

- The column space of A is a subspace of R^m (by Theorem 4.1).
- The null space of A is a subspace of R^n.

The **row space** of a matrix A is defined to be the span of its rows, denoted by Row A.

- Row A = Col A^T. So, Row A is a subspace of R^n.



Subspace associated with a L. T.

The range of a linear transformation is the same as the column space of its standard matrix.

Therefore, the range of a linear transformation $T: \mathcal{R}^n \to \mathcal{R}^m$ is a subspace of $\mathcal{R}^m$.

The null space of a linear transformation is the same as the null space of its standard matrix.

This implies that the null space of a linear transformation $T: \mathcal{R}^n \to \mathcal{R}^m$ is a subspace of $\mathcal{R}^n$.

4.2 Basis and Dimension

Definition Let V be a nonzero subspace of R^n. A **basis** for V is a linearly independent generating set for V.

The set of standard vectors $\mathcal{E}=\{\boldsymbol{e}_1, \boldsymbol{e}_2, \ldots, \boldsymbol{e}_n\}$ in R^n is both a linearly independent set and a generating set for R^n, and is thus a basis for R^n. This basis is called the **standard basis** for R^n.

The pivot columns of a matrix form a basis for its column space. (by Theorem 2.4)

- The column space of a matrix is usually different from that of its reduced row echelon form. (See example 1)

Theorem 4.3 (Reduction Theorem)

Let $\mathcal{S}$ be a finite generating set for a nonzero subspace V of R^n. Then $\mathcal{S}$ can be reduced to a basis for V by removing vectors from $\mathcal{S}$.

Let $\mathcal{S}$ be a finite subset of R^n. Then the following are true.

1. If $\mathcal{S}$ is a generating set for R^n, then $\mathcal{S}$ contains at least n vectors. (by Theorem 1.6)
2. If $\mathcal{S}$ is linearly independent, then $\mathcal{S}$ contains at most n vectors. (by Property 4 following Theorem 1.9)
3. If $\mathcal{S}$ is a basis for R^n, then $\mathcal{S}$ contains exactly n vectors.

Theorem 4.4 (**Extension Theorem**)

Let $\mathcal{S}$ be a linearly independent subset of a nonzero subspace V of R^n. Then $\mathcal{S}$ can be extended to a basis for V by including additional vectors. In particular, every nonzero subspace has a basis.

Theorem 4.5

Let V be a nonzero subspace of R^n. Then any two bases for V contain the same number of vectors.

A basis is a generating set for a subspace containing the fewest possible vectors. (By Theorem 1.7 and Property 5 following Theorem 1.9)

A basis is a linearly independent subset of a subspace that is as large as possible. (By Theorems 4.4 and 4.5)

Definition The number of vectors in a basis for a nonzero subspace V of R^n is called the **dimension** of V and is denoted by dim V. It is convenient to define the dimension of the zero subspace of R^n to be 0.

Theorem 4.6

Let V be a subspace of R^n with dimension k. Then every linearly independent subset of V contains at most k vectors; or, equivalently, any finite subset of V contains more than k vectors is linearly dependent.

Theorem 4.7

Let V be a k-dimensional subspace of R^n. Suppose that $\mathcal{S}$ is a subset of V with exactly k vectors. Then $\mathcal{S}$ is a basis for V if either $\mathcal{S}$ is linearly independent or $\mathcal{S}$ is a generating set for V.

Steps to show that a set $\mathcal{B}$ is a basis for a subspace V of R^n .

1. Show that $\mathcal{B}$ is contained in V.
2. Show that $\mathcal{B}$ is linearly independent (or that $\mathcal{B}$ is a generating set for V).
3. Compute the dimension of V, and confirm that the number of vectors in $\mathcal{B}$ equals the dimension of V.

4.3 The Dimension of Subspaces Associated with a Matrix

The dimension of the column space of a matrix equals the rank of the matrix.

The dimension of the null space of a matrix equals the nullity of the matrix.

Because the row space of a matrix A equals the column space of A^T, a basis for Row A can be obtained as follows:

a) Form the transpose of A, whose columns are the rows of A.

b) Find the pivot columns of A^T, which forms a basis for the column space of A^T.

- The dimension of Row A is the rank of A^T.
- Unlike the column space of a matrix, the row space is unaffected by elementary row operations.

Theorem 4.8

The nonzero rows of the reduced row echelon form of a matrix constitute a basis for the row space of the matrix.

The dimension of the row space of a matrix equals its rank.

The row and column spaces of a matrix are rarely equal. Nevertheless, their dimensions (numbers) are always the same:

$$\dim(\text{Row } A) = \dim(\text{Col } A) = \dim(\text{Row } A^T).$$

The rank of any matrix equals the rank of its transpose.

Recall that

- The null space of a linear transformation T: $\mathcal{R}^n \rightarrow \mathcal{R}^m$ is equal to that of its standard matrix A, Null A.
- The range of T is equal to Col A.

Hence,

- The dimension of the null space of T is the nullity of A.
- The dimension of the range of T is the rank of A.

$\Rightarrow$ The sum of the dimensions of the null space and range of T equals the dimension of the domain of T.

Theorem 4.9

If V and W are the subspaces of R^n with V contained in W, then dim $V \leq$ dim W. Moreover, if V and W also have the same dimension, then $V = W$.

The dimensions of the subspaces associated with an $m \times n$ matrix A.(This table also applies to an L. T. $T: R^n \rightarrow R^m$ by taking A to be the standard matrix of T.)

Subspace	**Containing space**	**Dimension**
Col A	R^m	rank A
Null A	R^n	nullity $A = n -$ rank A
Row A	R^n	rank A

Bases for the Subspaces Associated with a Matrix *A*	
Col A	The pivot columns of A form a basis for Col A.
Null A	The vectors in the vector form of the solution of $A\mathbf{x}=\mathbf{0}$ constitute a basis for Null A.
Row A	The nonzero rows of the reduced row echelon form of A constitute a basis for Row A.

4.4 Coordinate Systems

Theorem 4.10

Let $\mathcal{B}=\{\boldsymbol{b}_1, \boldsymbol{b}_2, \ldots, \boldsymbol{b}_{\mathrm{k}}\}$ be a basis for a subspace V of R^n. Any vector $\boldsymbol{v}$ in V can be uniquely represented as a linear combination of the vectors in $\mathcal{B}$; that is, there are unique scalars $a_1, a_2, \ldots, a_{\mathrm{k}}$ such that $\boldsymbol{v} = a_1\boldsymbol{b}_1 + a_2\boldsymbol{b}_2 + \ldots + a_k\boldsymbol{b}_k$.

Definition Let $\mathcal{B}=\{\boldsymbol{b}_1, \boldsymbol{b}_2, \ldots, \boldsymbol{b}_n\}$ be a basis for R^n . For each $\boldsymbol{v}$ in R^n, there are unique scalars $c_1, c_2, \ldots, c_n$ such that $\boldsymbol{v} = c_1\boldsymbol{b}_1 + c_2\boldsymbol{b}_2 + \ldots + c_n\boldsymbol{b}_n$. The vector

$$\begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$$

in R^n is called the **coordinate vector** of $\boldsymbol{v}$ relative to $\mathcal{B}$, or the $\mathcal{B}$-**coordinate vector** of $\boldsymbol{v}$. We denote the $\mathcal{B}$-coordinate vector of $\boldsymbol{v}$ by $[\boldsymbol{v}]_{\mathcal{B}}$.

Theorem 4.11

Let $\mathcal{B}$ be a basis for R^n and B be the matrix whose columns are the vectors in $\mathcal{B}$. Then B is invertible, and for every vector $\boldsymbol{v}$ in R^n, $B[\boldsymbol{v}]_{\mathcal{B}}=\boldsymbol{v}$, or equivalently $[\boldsymbol{v}]_{\mathcal{B}}=B^{-1}\boldsymbol{v}$.

4.5 Matrix representations of Linear Operators

A L.T. where the domain and codomain equal R^n is called a **linear operator** on R^n.

Definition Let T be a linear operator on R^n and $\mathcal{B}=\{\boldsymbol{b}_1, \boldsymbol{b}_2, \ldots, \boldsymbol{b}_n\}$ be a basis for R^n . The matrix

$$[[T(\boldsymbol{b}_1)]_{\mathcal{B}} \quad [T(\boldsymbol{b}_2)]_{\mathcal{B}} \quad \cdots \quad [T(\boldsymbol{b}_n)]_{\mathcal{B}}]$$

is called the **matrix representation of *T* with respect to** $\mathcal{B}$, or the $\mathcal{B}$**-matrix of *T***. It is denoted by $[T]_{\mathcal{B}}$.

- The j-th column of the $\mathcal{B}$-matrix of T is the $\mathcal{B}$-**coordinate vector** of $T(\boldsymbol{b}_j)$, the image of the j-th vector in $\mathcal{B}$.
- When $\mathcal{B}=\mathcal{E}$, the $\mathcal{B}$-matrix of T is

$$[T]_{\mathcal{E}}= [[T(\boldsymbol{b}_1)]_{\mathcal{E}} \quad [T(\boldsymbol{b}_2)]_{\mathcal{E}} \quad \cdots \quad [T(\boldsymbol{b}_n)]_{\mathcal{E}}] = [T(\boldsymbol{e}_1) \quad T(\boldsymbol{e}_2) \quad \cdots \quad T(\boldsymbol{e}_n)]$$

which is the standard matrix of T.
- $[T]_{\mathcal{E}}[\boldsymbol{v}]_{\mathcal{E}}=A\boldsymbol{v}=T(\boldsymbol{v})=[T(\boldsymbol{v})]_{\mathcal{E}}$, where A is the standard matrix of linear operator T on R^n .
- If T is a linear operator on R^n and $\mathcal{B}$ is a basis for R^n , then the $\mathcal{B}$-matrix of T is the unique $n\times n$ matrix such that $[T(\boldsymbol{v})]_{\mathcal{B}}=[T]_{\mathcal{B}}[\boldsymbol{v}]_{\mathcal{B}}$.

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\begin{aligned}
[T(\boldsymbol{v})]_{\mathcal{B}} = [A\boldsymbol{v}]_{\mathcal{B}} = B^{-1}(A\boldsymbol{v}) = B^{-1}AI_n\boldsymbol{v} &= B^{-1}ABB^{-1}\boldsymbol{v} \\
&= B^{-1}AB[\mathbf{v}]_{\mathcal{B}} \\
&= B^{-1}A[\boldsymbol{b}_1\ \boldsymbol{b}_2\ \cdots\ \boldsymbol{b}_{\mathrm{n}}][\mathbf{v}]_{\mathcal{B}} \\
&= [B^{-1}A\boldsymbol{b}_1\ \ B^{-1}A\,\boldsymbol{b}_2\ \ \cdots\ \ B^{-1}A\boldsymbol{b}_{\mathrm{n}}][\mathbf{v}]_{\mathcal{B}} \\
&= [B^{-1}T(\boldsymbol{b}_1)\ \ B^{-1}T(\boldsymbol{b}_2)\ \ \cdots\ \ B^{-1}T(\boldsymbol{b}_{\mathrm{n}})][\mathbf{v}]_{\mathcal{B}} \\
&= [[T(\boldsymbol{b}_1)]_{\mathcal{B}}\ \ [T(\boldsymbol{b}_2)]_{\mathcal{B}}\ \ \cdots\ \ [T(\boldsymbol{b}_{\mathrm{n}})]_{\mathcal{B}}\ ][\mathbf{v}]_{\mathcal{B}} \\
&= [T]_{\mathcal{B}}\,[\mathbf{v}]_{\mathcal{B}}
\end{aligned}
$$

$$\downarrow$$

Theorem 4.12

Let T be a linear operator on R^n, $\mathcal{B}$ a basis for R^n , B the matrix whose columns are the vector in $\mathcal{B}$, and A the standard matrix of T. Then $[T]_{\mathcal{B}} = B^{-1} AB$ or equivalently, $A = B[T]_{\mathcal{B}}B^{-1}$.

If two square matrices A and B are such that $B = P^{-1}AP$ for some invertible matrix P, then A is said to be **similar** to B.

- A is similar to B iff B is similar to A.
- Theorem 4.12 shows that the $\mathcal{B}$-matrix representation of a linear operator on R^n is similar to its standard matrix.