Gauss-Seidel Method

12TH WEEK

Gauss-Seidel

- Iterative or approximate methods provide an alternative to the elimination methods.
- The Gauss-Seidel method is the most commonly used iterative method.
- The iterative methods are more appropriate when the number of equations involved is large (typically of the order of 100 or more), or when the matrix is sparse.

Iterative Methods

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= b_1 \\
a_{21}x_1 + a_{22}x_2 + a_{23}x_3 &= b_2 \\
a_{31}x_1 + a_{32}x_2 + a_{33}x_3 &= b_3
\end{aligned}
\quad \Rightarrow \quad
\begin{aligned}
x_1 &= (b_1 - a_{12}x_2 - a_{13}x_3)/a_{11} \\
x_2 &= (b_2 - a_{21}x_1 - a_{23}x_3)/a_{22} \\
x_3 &= (b_3 - a_{31}x_1 - a_{32}x_2)/a_{33}
\end{aligned}
$$

Idea

- Starts with initial guesses for x_1, x_2, x_3
- A simple way to obtain initial guesses is to assume that they are zero.
- Iteratively substitute old values of x_1, x_2, x_3 in the right hand side of the equations to get updated values of x_1, x_2, x_3.

Gauss-Seidel

- Update the values of *x*'s one by one using the latest available set of *x*'s.

Starts with initial guesses $x_1^{(0)}, x_2^{(0)}, x_3^{(0)}$

- Compute $x_1^{(i+1)}$ as $x_1^{(i+1)} = (b_1 - a_{12}x_2^{(i)} - a_{13}x_3^{(i)}) / a_{11}$
- Compute $x_2^{(i+1)}$ as $x_2^{(i+1)} = (b_2 - a_{21}x_1^{(i+1)} - a_{23}x_3^{(i)}) / a_{22}$
- Compute $x_3^{(i+1)}$ as $x_3^{(i+1)} = (b_3 - a_{31}x_1^{(i+1)} - a_{32}x_2^{(i+1)}) / a_{33}$

Stopping Criteria

- When the iteration limit is reached, or
- When the estimated percentage relative error of every x's is less then the acceptable error.

$$\left|\varepsilon_{a,i}\right| = \left|\frac{x_i^{(j)} - x_i^{(j-1)}}{x_i^{(j)}}\right| 100\% < \varepsilon_s, \quad \text{for all } i$$

Example

$$6x_1 - x_2 - x_3 = 3$$
$$6x_1 + 9x_2 + x_3 = 40$$
$$-3x_1 + x_2 + 12x_3 = 50$$

$$x_1 = \frac{3 + x_2 + x_3}{6}$$

$$x_2 = \frac{40 - 6x_1 - x_3}{9}$$

$$x_3 = \frac{50 + 3x_1 - x_2}{12}$$

$$x_1 = \frac{3 + x_2 + x_3}{6}$$

$$x_2 = \frac{40 - 6x_1 - x_3}{9}$$

$$x_3 = \frac{50 + 3x_1 - x_2}{12}$$

x1	x2	x3	ea1	ea2	ea3	maximum ea%
0	0	0				
0.5	4.111111	3.949074				
1.843364	2.776749	4.396112	0.728757	0.480548	0.101689	72.88%
1.695477	2.82567	4.355063	0.087225	0.017313	0.009425	8.72%
1.696789	2.829356	4.355084	0.000773	0.001303	4.78E-06	0.13%