Gaussian Elimination

12TH WEEK

Gaussian Elimination

- The basic idea is to reduce the linear equations until only a single unknown is left, because such equations are trivial to solve
- Such reduction is achieved by manipulating the equations
- These manipulations are called **elementary row operations**

Elementary Row Operations

01

Reordering the equations by interchanging both sides of the i_{th} and j_{th} equation in the system

02

Multiplying both sides of an equation by a scalar

03

Adding an equation to a second equation, and the result substituted for the original equation

Gaussian Elimination

- Gaussian Elimination consists of 2 steps:
  - Forward elimination
  - Backward substitution

$$
\left.
\begin{array}{c}
\left[\begin{array}{ccc|c}
a_{11} & a_{12} & a_{13} & b_1 \\
a_{21} & a_{22} & a_{23} & b_2 \\
a_{31} & a_{32} & a_{33} & b_3
\end{array}\right] \\
\Downarrow \\
\left[\begin{array}{ccc|c}
a_{11} & a_{12} & a_{13} & b_1 \\
 & a'_{22} & a'_{23} & b'_2 \\
 & & a''_{33} & b''_3
\end{array}\right]
\end{array}
\right\} \text{Forward elimination}
$$

$$\Downarrow$$

$$
\left.
\begin{array}{r}
x_3 = b''_3 / a''_{33} \\
x_2 = (b'_2 - a'_{23}x_3)/a'_{22} \\
x_1 = (b_1 - a_{12}x_2 - a_{13}x_3)/a_{11}
\end{array}
\right\} \text{Back substitution}
$$

Example

$$
\begin{aligned}
2x &+ 4y - 2z = 2 \\
4x &+ 9y - 3z = 8 \\
-2x &- 3y + 7z = 10
\end{aligned}
$$

Example

$$\begin{array}{rcrcrcr} 2x & + & 4y & - & 2z & = & 2 \\ 4x & + & 9y & - & 3z & = & 8 \\ -2x & - & 3y & + & 7z & = & 10 \end{array}$$

- Using the first equation to eliminate x from the next two equations

Example

$$
\begin{aligned}
2x + 4y - 2z &= 2 \\
y + z &= 4 \\
y + 5z &= 12
\end{aligned}
$$

Example

$$
\begin{aligned}
2x + 4y - 2z &= 2 \\
y + z &= 4 \\
\boxed{y} + 5z &= 12
\end{aligned}
$$

- Using the second equation to eliminate *y* from the third equation

Example

$$
\begin{aligned}
2x \;+\; 4y \;-\; 2z &= 2 \\
y \;+\; z &= 4 \\
4z &= 8
\end{aligned}
$$

Example

- We now have a triangular system which is easily solved using a technique called ***Backward-Substitution***.

$$\begin{array}{rcrcrcl} 2x & + & 4y & - & 2z & = & 2 \\ & & y & + & z & = & 4 \\ & & & & 4z & = & 8 \end{array}$$

Example

- If A is upper triangular, we can solve $Ax = b$ by:

$$x_n = b_n / A_{nn}$$

$$x_i = \left(b_i - \sum_{j=i+1}^{n} A_{ij} x_j \right) / A_{ii}, \quad i = n-1, \ldots, 1$$

Example

- From the previous work, we have

$$\begin{array}{rcrcrcr} 2x & + & 4y & - & 2z & = & 2 \\ & & y & + & z & = & 4 \\ & & & & z & = & 2 \end{array}$$

- And substitute z in the first two equations

Example

$$
\begin{aligned}
2x + 4y - 4 &= 2 \\
y + 2 &= 4 \\
z &= 2
\end{aligned}
$$

- We can solve y

Example

$$\begin{aligned} 2x + 4y - 4 &= 2 \\ y &= 2 \\ z &= 2 \end{aligned}$$

- Substitute y to the first equation

Example

$$\begin{aligned} 2x + 8 - 4 &= 2 \\ y &= 2 \\ z &= 2 \end{aligned}$$

- We can solve the first equation

Example

$$
\begin{aligned}
x \qquad\qquad &= -1 \\
y \qquad &= 2 \\
z &= 2
\end{aligned}
$$

Gaussian Elimination with Augmented Matrix

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 6 \\ 2x_1 + 5x_2 + 10x_3 &= 17 \\ x_1 + 3x_2 + 10x_3 &= 18 \end{aligned}$$

$$\begin{bmatrix} 1 & 2 & 3 & 6 \\ 2 & 5 & 10 & 17 \\ 1 & 3 & 10 & 17 \end{bmatrix}$$

Gaussian Elimination with Augmented Matrix

$$\begin{bmatrix} 1 & 2 & 3 & 6 \\ 2 & 5 & 10 & 17 \\ 1 & 3 & 10 & 17 \end{bmatrix}$$

$$B_2 = B_2 - 2B_1$$
$$B_3 = B_3 - B_1$$

$$\begin{bmatrix} 1 & 2 & 3 & 6 \\ 0 & 1 & 4 & 5 \\ 0 & 1 & 7 & 11 \end{bmatrix}$$

Gaussian Elimination with Augmented Matrix

$$\begin{bmatrix} 1 & 2 & 3 & 6 \\ 0 & 1 & 4 & 5 \\ 0 & 1 & 7 & 11 \end{bmatrix} \qquad B_3 = B_3 - B_2 \qquad \begin{bmatrix} 1 & 2 & 3 & 6 \\ 0 & 1 & 4 & 5 \\ 0 & 0 & 3 & 6 \end{bmatrix}$$

Gaussian Elimination
with Augmented Matrix

$$\begin{bmatrix} 1 & 2 & 3 & 6 \\ 0 & 1 & 4 & 5 \\ 0 & 0 & 3 & 6 \end{bmatrix}$$

$$3x_3 = 6,$$
$$x_3 = 2$$

$$x_2 + 4x_3 = 5$$
$$x_2 + 4(2) = 5$$
$$x_2 = -3$$

$$x_1 + 2x_2 + 3x_3 = 6$$
$$x_1 + 2(-3) + 3(2) = 6$$
$$x_1 = 6$$