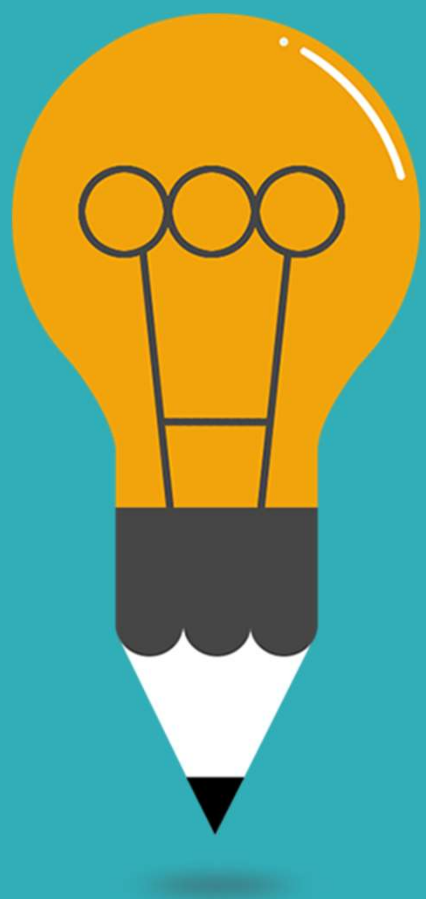
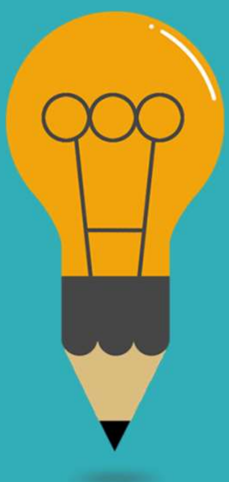


EIGEN VALUES

and

EIGEN VECTORS



Eigen values

If A is a matrix $n×n$, then a nonzero vector $\boldsymbol{x}$ on $\mathbb{R}n$ is called an eigenvect or (characteristic vector) of A if $A\boldsymbol{x}$ is a scalar multiple of $\boldsymbol{x}$; explained:

$$A\boldsymbol{x}=\lambda\boldsymbol{x}$$

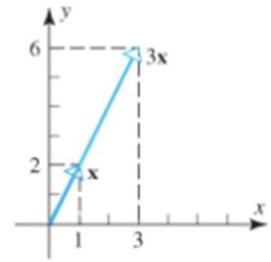
for any scalar λ.

This scalar λ is called the eigenvalue (characteristic value) of A, and

x is called the eigenvector (characteristic vector) of A associated with λ.

$$A\mathbf{x} = \begin{bmatrix} 3 & 0 \\ 8 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \end{bmatrix} = 3\mathbf{x}$$

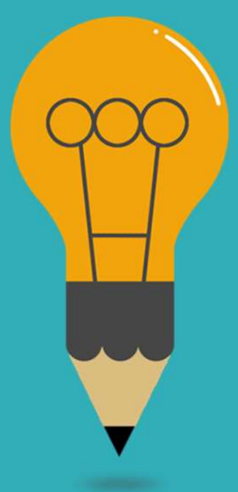
Eigenvalues - example



Diberikan vektor $\boldsymbol{x} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ dan matriks $A = \begin{bmatrix} 3 & 0 \\ 8 & -1 \end{bmatrix}$.

$$A\boldsymbol{x} = \begin{bmatrix} 3 & 0 \\ 8 & -1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ 2 \end{bmatrix} = 3\boldsymbol{x}$$

Maka, vektor $\boldsymbol{x} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ disebut vektor eigen dari matriks $A = \begin{bmatrix} 3 & 0 \\ 8 & -1 \end{bmatrix}$ yang terkait dengan nilai eigen $\lambda = 3$.



$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \times \begin{bmatrix} p \\ q \end{bmatrix} = \begin{bmatrix} a \cdot p + b \cdot q \\ c \cdot p + d \cdot q \end{bmatrix}$$

$$\mathbf{y} = A\mathbf{x}$$

$$\begin{aligned} \mathbf{y} &= \begin{bmatrix} 3 & 8 & 5 \\ 6 & 4 & 7 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} \\ &= \begin{bmatrix} 3.2 + 8.3 + 5.4 \\ 6.2 + 4.3 + 7.4 \end{bmatrix} \\ &= \begin{bmatrix} 50 \\ 52 \end{bmatrix} \end{aligned}$$

Eigen values

To obtain the eigenvalues of a matrix A of size n×n, eq

$A\boldsymbol{x}$=$\lambda\boldsymbol{x}$ can be rewritten as

$$A\boldsymbol{x}=\lambda I\boldsymbol{x}$$

$$A\boldsymbol{x}-\lambda I\boldsymbol{x}=\mathbf{0}$$

$$A-\lambda I\ \boldsymbol{x}=\mathbf{0}$$

For λ to be an eigenvalue, there must be one nonzero solution to this equation. This eq uation has a nonzero solution if and only if

$$\det\ A-\lambda I\ =0$$

The above equation is called the characteristic equation of the matrix A;

The scalars that satisfy this equation are the eigenvalues of the matrix A.

The characteristic equation above can also be written:

$$\det\ \lambda I-A\ =0$$

Matrix Determinant ordo 2x2

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \rightarrow det(A) = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

$$X = \begin{bmatrix} 4 & 2 \\ 3 & 5 \end{bmatrix} \rightarrow det(X) = \begin{vmatrix} 4 & 2 \\ 3 & 5 \end{vmatrix} = 4.5 - 2.3 = 20 - 6 = 14$$

$$P = \begin{bmatrix} 7 & 1 \\ 6 & 2 \end{bmatrix} \rightarrow det(P) = ...$$

matrix Determinant 3x3

$$\det(A) = \left|\begin{array}{ccc|cc} a_{11} & a_{12} & a_{13} & a_{11} & a_{12} \\ a_{21} & a_{22} & a_{23} & a_{21} & a_{22} \\ a_{31} & a_{32} & a_{33} & a_{31} & a_{32} \end{array}\right.$$

$$- \quad - \quad - \qquad + \quad + \quad +$$

Ilustrasi matriks (Dok. Arsip Zenius)

$det(A) = a_{11}.a_{22}.a_{33} + a_{12}.a_{23}.a_{31} + a_{13}.a_{21}.a_{32} - a_{13}.a_{22}.a_{31} - a_{11}.a_{23}.a_{32} - a_{12}.a_{21}.a_{33}$

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 4 \\ 3 & 1 & 2 \end{bmatrix} \rightarrow det(A) = \left|\begin{array}{ccc|cc} 1 & 2 & 3 & 1 & 2 \\ 2 & 1 & 4 & 2 & 1 \\ 3 & 1 & 2 & 3 & 1 \end{array}\right.$$

$det(A) = 1.1.2 + 2.4.3 + 3.2.1 - 3.1.3 - 1.4.1 - 2.2.2 = 2 + 24 + 6 - 9 - 4 - 8 = 11$

$$A = \begin{pmatrix} 3 & 2 & 1 \\ 5 & 4 & 2 \\ 2 & 1 & 3 \end{pmatrix}$$

Cara 1 : Aturan Sarrus

$$|A| = \left|\begin{array}{ccc|cc} 3 & 2 & 1 & 3 & 2 \\ 5 & 4 & 2 & 5 & 4 \\ 2 & 1 & 3 & 2 & 1 \end{array}\right.$$

$$= (36 + 8 + 5) - (8 + 6 + 30)$$

$$= 49 - 44$$

$$= 5$$

Cara 2 : Metode Minor-Kofaktor

$$A = \begin{pmatrix} \cancel{3} & \cancel{2} & 1 \\ 5 & 4 & \cancel{2} \\ 2 & 1 & \cancel{3} \end{pmatrix} \qquad \begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$$

$$|A| = 3\begin{vmatrix} 4 & 2 \\ 1 & 3 \end{vmatrix} - 2\begin{vmatrix} 5 & 2 \\ 2 & 3 \end{vmatrix} + 1\begin{vmatrix} 5 & 4 \\ 2 & 1 \end{vmatrix}$$

This uses the sarrus method, try calculating it using the cofactor method

THEOREM 1

If A is a triangular matrix (top/bottom) or diagonal matrix, then the eigenvalues of A are the entries that lie on the main diagonal of matrix A.

Example:
Determine the eigenvalues of the matrix

$$B = \begin{bmatrix} \frac{3}{4} & -2 & -\frac{7}{8} & 10 \\ 0 & \frac{2}{3} & 29 & 2 \\ 0 & 0 & -1 & 3 \\ 0 & 0 & 0 & -6 \end{bmatrix}$$

Based on Theorem 1, the eigenvalues of matrix B is $\lambda = 3/4$, $\lambda = 2/3$. $\lambda = -1$, $\lambda = -6$

TEOREMA 2

Jika A adalah suatu matriks $n \times n$ dan λ adalah suatu bilangan riil, maka pernyataan-pernyataan berikut ini adalah ekuivalen.

(1) λ adalah suatu nilai eigen dari A.

(2) Sistem persamaan $(A - \lambda I)\boldsymbol{x} = \mathbf{0}$ memiliki solusi nontrivial.

(3) Terdapat suatu vektor taknol x pada $\mathbb{R}^n$ sedemikian rupa sehingga $A\boldsymbol{x} = \lambda\boldsymbol{x}$.

(4) λ adalah suatu solusi dari persamaan karakteristik $\det(A - \lambda I) = 0$.

Eigenvalues - example

Tentukan nilai-nilai eigen dari

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 4 & -17 & 8 \end{bmatrix}$$

Pertama, cari dahulu matriks $A - \lambda I$.

$$A - \lambda I = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 4 & -17 & 8 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 4 & -17 & 8 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}$$

$$= \begin{bmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ 4 & -17 & 8 - \lambda \end{bmatrix}$$

Eigenvalues - Example

Selanjutnya, cari $\det(A-\lambda I)$.

$$\det(A-\lambda I)=(-\lambda)(-\lambda)(8-\lambda)+(1)(1)(4)+(0)(0)(-17)-(0)(-\lambda)(4)-(-\lambda)(1)(-17)-(1)(0)(8-\lambda)$$
$$=(8\lambda^2-\lambda^3)+4+0-0-17\lambda-0$$
$$=8\lambda^2-\lambda^3+4-17\lambda$$
$$=-\lambda^3+8\lambda^2-17\lambda+4$$

Dengan menggunakan persamaan karakteristik, diperoleh

$$\det(A-\lambda I)=0$$
$$-\lambda^3+8\lambda^2-17\lambda+4=0$$
$$\lambda^3-8\lambda^2+17\lambda-4=0$$
$$(\lambda-4)(\lambda^2-4\lambda+1)=0$$

Dengan menggunakan rumus kuadratik, maka solusi untuk $(\lambda^2-4\lambda+1)=0$ adalah $2+\sqrt{3}$ dan $2-\sqrt{3}$, sehingga didapatlah nilai-nilai eigen dari matriks A, yaitu:

$$\lambda=4, \qquad \lambda=2+\sqrt{3}, \qquad \lambda=2-\sqrt{3}$$

Eigenvectors - Example

After knowing how to find eigenvalues, next is to learn how to find eigenvectors. The eigenvectors of the matrix A associated with an eigenvalue λ are nonzero vectors $\boldsymbol{x}$ that satisfy the equation

$$A\boldsymbol{x} = \lambda\boldsymbol{x}$$

In other words, the eigenvectors associated with λ are vectors in the solution space $A - \lambda I\ \boldsymbol{x} = \mathbf{0}$. This solution space is called the eigenspace of the matrix A which is related to λ.

Eigenvectors

Determine the bases for the eigenspace of the matrix

$$A = \begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix}$$

Persamaan karakteristik dari matriks A adalah:

$$-\lambda^3 + 5\lambda^2 - 8\lambda + 4 = 0$$

Atau

$$\lambda^3 - 5\lambda^2 + 8\lambda - 4 = 0$$

Dengan menggunakan pemfaktoran, didapatlah:

$$(\lambda - 1)(\lambda - 2)^2 = 0$$

Vektor Eigen - Contoh

So, the eigenvalues of A are:

$$\lambda = 1 \ \& \ \lambda = 2$$

Based on the definition,

$$\boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

is an eigenvector of the matrix A associated with λ if and only if $A\boldsymbol{x} = \lambda\boldsymbol{x}$.

Vektor Eigen - Contoh

This means that $\boldsymbol{x}$ is said to be an eigenvector of the matrix A if and only if $\boldsymbol{x}$ is a non trivial solution of the equation $A - \lambda I\ \boldsymbol{x} = \mathbf{0}$, namely:

$$\begin{bmatrix} \lambda & 0 & -2 \\ -1 & \lambda - 2 & -1 \\ -1 & 0 & \lambda - 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Jika $\lambda = 2$, maka diperoleh

$$\begin{bmatrix} 2 & 0 & 2 \\ -1 & 0 & -1 \\ -1 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Dengan menggunakan operasi baris elementer, didapatlah

$$x_1 + x_3 = 0 \quad \rightarrow \quad x_1 = -x_3$$

Because from the results obtained, there is no information about $x2$, then $x2$ can be considered as a parameter; let's say $x2 = t$. And, let's also say that $x3 = s$, then: $x1 = -s$, $x2 = t$, $x3 = s$

so, the eigenvectors of A associated with $\lambda = 2$ are nonzero vectors of the form

Vektor Eigen - Contoh

$$\boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -s \\ t \\ s \end{bmatrix} = \begin{bmatrix} -s \\ 0 \\ s \end{bmatrix} + \begin{bmatrix} 0 \\ t \\ 0 \end{bmatrix} = s\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} + t\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

Karena

$$\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \quad \& \quad \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

linearly independent, these vectors form a basis for the eigenspace associated with $\lambda = 2$.

If $\lambda = 1$, then we get

$$\begin{bmatrix} 1 & 0 & 2 \\ -1 & -1 & -1 \\ -1 & 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Vektor Eigen - Contoh

By using elementary row operations, we get

$$x1 + 2x3 = 0 \rightarrow x1 = -2x3$$

$$x2 - x3 = 0 \rightarrow x2 = x3$$

Let $x3 = s$, then

$$x1 = -2s,\ x2 = s,\ x3 = s$$

so, the eigenvectors of A associated with $\lambda = 1$ are nonzero vectors shaped

$$\boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2s \\ s \\ s \end{bmatrix} = s\begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} \qquad \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$$

Because they are linearly independent, the above vectors form a basis related to $\lambda = 1$.

To determine the eigenvectors corresponding to the eigenvalues (λ), you must First determine the bases for the eigenspace.

Vektor Eigen

Look again at the example above. For the eigenvector of A associated with λ = 2 are nonzero vectors of the form

$$\boldsymbol{x} = s\begin{bmatrix}-1\\0\\1\end{bmatrix} + t\begin{bmatrix}0\\1\\0\end{bmatrix}$$

Misalkan $s = 1$ dan $t = 1$, maka didapatlah vektor eigen yang terkait dengan $\lambda = 2$ adalah:

$$\boldsymbol{x} = 1\cdot\begin{bmatrix}-1\\0\\1\end{bmatrix} + 1\cdot\begin{bmatrix}0\\1\\0\end{bmatrix} = \begin{bmatrix}-1\\0\\1\end{bmatrix} + \begin{bmatrix}0\\1\\0\end{bmatrix} = \begin{bmatrix}-1\\1\\1\end{bmatrix}$$

Sementara, untuk vektor eigen dari A yang terkait dengan $\lambda = 1$ adalah vektor-vektor taknol yang berbentuk

$$\boldsymbol{x} = s\begin{bmatrix}-2\\1\\1\end{bmatrix}$$

Misalkan $s = -2$, maka didapatlah vektor eigen yang terkait dengan $\lambda = 1$ adalah:

$$\boldsymbol{x} = -2\cdot\begin{bmatrix}-2\\1\\1\end{bmatrix} = \begin{bmatrix}4\\-2\\-2\end{bmatrix}$$

Eigenvectors

THEOREM 3
If k is a positive integer, λ is the eigenvalue of a matrix A, and $\boldsymbol{x}$ is the eigenvector associated with λ, then λk is the eigenvalue of Ak and $\boldsymbol{x}$ is the eigenvector associated with it.

Contoh:

Pada contoh sebelumnya telah ditunjukkan bahwa nilai-nilai eigen dari matriks

$$A = \begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix}$$

adalah λ = 2 dan λ = 1, sehingga berdasarkan Teorema 3, nilai-nilai eigen dari matriks A^7 adalah:

$\lambda = 2^7 = 128$ dan $\lambda = 1^7 = 1$

Vektor Eigen

In addition, it has also been shown that the eigenvector of A is related to λ = 2 are nonzero vectors of the form

$$\boldsymbol{x} = s\begin{bmatrix}-1\\0\\1\end{bmatrix} + t\begin{bmatrix}0\\1\\0\end{bmatrix}$$

Then, by Theorem 3, the eigenvectors of the matrix A are related to λ = 2 will be equal to the eigenvectors of the matrix A7 associated with λ = 27 = 128. Likewise for λ = 17 = 1.

Vektor Eigen

It has been shown that the eigenvectors of A associated with λ = 1 are nonzero vectors o f the form

$$\boldsymbol{x} = s\begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$$

Then, by Theorem 3, the eigenvectors of the matrix A are related to

λ = 1 will be equal to the eigenvectors of matrix A7 associated with λ=17=1.

Exercises

1. Find the eigenvalue of A or the val $A=\begin{bmatrix} 4 & 0 & 1 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$

2. Determine the eigenvalue with $A = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}$ x=$\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and draw it o n a 2-dimensional

3. Look for the use of eigen in decision support systems