GJ

P1. 1. Pesudo code:

$\alpha = \frac{1}{2}$, $b_0 = 0$

for $k = 1, 2 \cdots n$ do

  Generate x_k

  $b \longleftarrow b + \alpha(x_k - b)$

Return b

2. Prove that $\lim_{n\to\infty} E b_n = b$.

Pf:

~~$b_{n+1} = (1-\alpha) b_n$~~ $\because b_{n+1} = b_n + \alpha(x_n - b_n)$

$\therefore b_{n+1} = (1-\alpha) b_n + \alpha x_n$

Then $E[b_{n+1}] = (1-\alpha) E[b_n] + \alpha E[x_n]$ (*)

$\because x_n \sim N(b, \sigma^2)$

$\therefore$ (*) is equal to that $E[b_{n+1}] = (1-\alpha) E[b_n] + \alpha b$ (**)

Claim $\lim_{n\to\infty} E[b_n] = x$

(**) is equal to that $x = (1-\alpha) x + \alpha b$

So $x = \lim_{n\to\infty} E[b_n] = b$

3. Can you prove or disprove that $b_n \to b$ in L^2?

Pf: I wanna prove that $b_n \to b$ in L^2.

$$\lim_{n\to\infty} E[|b_n - b|^2] = \lim_{n\to\infty} E[b_n^2 - 2bb_n + b^2] = \lim_{n\to\infty} \left(E[(\tfrac{1}{n}\sum_{i=1}^{n} x_i)^2] - 2b E[b_n]\right) + b^2$$

$$= \lim_{n\to\infty} \frac{1}{n^2}\left(\sum_{i=1}^{n} E[x_i^2] + \sum_{i\neq j} E[x_i]\cdot E[x_j]\right) - b^2$$

$$= \lim_{n\to\infty} \frac{1}{n^2} \cdot [n(b^2+\sigma^2) + (n^2-n)b^2] - b^2$$

$$= \lim_{n\to\infty} \frac{\sigma^2}{n} = 0$$ So $b_n \to b$ in L^2.

- Identify MRP with CFD:

$$V(x) = \gamma \left\{ l^h(x) + \sum_{i=1}^{d} p^h(x+h\vec{e}_i \mid x)\, V(x+h\vec{e}_i) + p^h(x-h\vec{e}_i \mid x) \cdot V(x-h\vec{e}_i) \right\}$$

Solu:

By CFD, we're able to get:

$$\gamma = \frac{2}{2+h^2}, \quad p^h(x \pm h e_i \mid x) = \frac{1}{4}, \quad l^h(x) = \frac{h^2}{2}\left(x_1^2 + x_2^2 - x_1 - x_2 - \frac{3}{2}\right)$$

By Bellman equation:

$$V(s) = R(s) + \gamma \cdot \sum_{s' \in S} P(s, s')\, V(s')$$

$$P(s,s') = \begin{cases} \frac{1}{4}, & \text{if } \|s'-s\| = h \\ 0, & \text{otherwise} \end{cases} ; \quad \gamma = \frac{2}{2+h^2}$$

$$R(s) = \gamma\, l^h(s) = \frac{h^2}{2+h^2}\left(x_1^2 + x_2^2 - x_1 - x_2 - \frac{3}{2}\right)$$

※ $\|\vec{x}\| \triangleq \sum_{i=1}^{d} |x_i|$