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1. Pf:

$$u' = 1 - \frac{e^{\frac{x-1}{\sqrt{\varepsilon}}} \cdot \frac{1}{\sqrt{\varepsilon}} - e^{-\frac{x+1}{\sqrt{\varepsilon}}} \cdot \left(-\frac{1}{\sqrt{\varepsilon}}\right)}{1 - e^{-\frac{2}{\sqrt{\varepsilon}}}} = 1 - \frac{\frac{1}{\sqrt{\varepsilon}} \cdot e^{\frac{x-1}{\sqrt{\varepsilon}}} + \frac{1}{\sqrt{\varepsilon}} \cdot e^{-\frac{x+1}{\sqrt{\varepsilon}}}}{1 - e^{-\frac{2}{\sqrt{\varepsilon}}}}$$

$$u'' = - \frac{\frac{1}{\sqrt{\varepsilon}} \cdot e^{\frac{x-1}{\sqrt{\varepsilon}}} \cdot \frac{1}{\sqrt{\varepsilon}} + \frac{1}{\sqrt{\varepsilon}} \cdot e^{-\frac{x+1}{\sqrt{\varepsilon}}} \cdot \left(-\frac{1}{\sqrt{\varepsilon}}\right)}{1 - e^{-\frac{2}{\sqrt{\varepsilon}}}} = - \frac{\frac{1}{\varepsilon} \cdot e^{\frac{x-1}{\sqrt{\varepsilon}}} - \frac{1}{\varepsilon} \cdot e^{-\frac{x+1}{\sqrt{\varepsilon}}}}{1 - e^{-\frac{2}{\sqrt{\varepsilon}}}}$$

$$\text{So. } LHS = -\varepsilon u'' + u = -\varepsilon \times \left(- \frac{\frac{1}{\varepsilon} e^{\frac{x-1}{\sqrt{\varepsilon}}} - \frac{1}{\varepsilon} \cdot e^{-\frac{x+1}{\sqrt{\varepsilon}}}}{1 - e^{-\frac{2}{\sqrt{\varepsilon}}}} \right) + x - \frac{e^{\frac{x-1}{\sqrt{\varepsilon}}} - e^{-\frac{x+1}{\sqrt{\varepsilon}}}}{1 - e^{-\frac{2}{\sqrt{\varepsilon}}}}$$

$$= \frac{e^{\frac{x-1}{\sqrt{\varepsilon}}} - e^{-\frac{2}{\sqrt{\varepsilon}}}}{1 - e^{-\frac{2}{\sqrt{\varepsilon}}}} + x - \frac{e^{\frac{x-1}{\sqrt{\varepsilon}}} - e^{-\frac{x+1}{\sqrt{\varepsilon}}}}{1 - e^{-\frac{2}{\sqrt{\varepsilon}}}} = x = RHS$$

In conclusion, $u(x)$ is the solution of (1).

2. Claim $h = \frac{1}{5}$, $R_f^h = [0, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, 0]$

$$r_i = \frac{1}{h^2} + \frac{b_i}{2h} = \frac{1}{\frac{1}{25}} + \frac{0}{\frac{2}{5}} = 25$$

$$s_i = \frac{2}{h^2} + c_i = \frac{2}{\frac{1}{25}} + 1 = 51$$

$$t_i = \frac{1}{h^2} - \frac{b_i}{2h} = \frac{1}{\frac{1}{25}} - \frac{(-\varepsilon)}{\frac{2}{5}} = 25.000001$$

$$\text{So } L^h = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ -25 & 51 & -25.000001 & 0 & 0 & 0 \\ 0 & -25 & 51 & -25.000001 & 0 & 0 \\ 0 & 0 & -25 & 51 & -25.000001 & 0 \end{pmatrix}$$

$\to$ Next page

$$\begin{pmatrix} 0 & 0 & 0 & -25 & 51 & -25.000001 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

and $R^f\lambda = \begin{pmatrix} 0 \\ \frac{1}{5} \\ \frac{2}{5} \\ \frac{3}{5} \\ \frac{4}{5} \\ 0 \end{pmatrix}$

3. $(L^h R^h v)_0 = v_0 = 0$ and $(R^h L v)_0 = Lv(x_0) = 0$,

So $|(L^h R^h v)_0 - (R^h L v)_0| = 0$

Similarly, $|(L^h R^h v)_N - (R^h L v)_N| = 0$

When $1 \le i \le N-1$, we have

$$(L^h R^h v)_i = -\varepsilon \delta_h \delta_{-h} v_i + v_i$$

and $(R^h L v)_i = Lv(x_i) = -\varepsilon v''(x_i) + v(x_i)$

So $|(L^h R^h v)_i - (R^h L v)_i| = O(h^2)$

So L^h is consistent of $2=2$

Now, we WTS L^h is stable. For any $V \in \mathbb{R}^{N+1}$

$$\|L^h V\|_\infty \ge |(L^h V)_0| = |V_0| = \|V\|_\infty.$$

Similarly, $\|L^h V\|_\infty \ge |(L^h V)_N| = |V_N| = \|V\|_\infty$

When $1 \le i \le N-1$,

$$(L^h V)_i = -r(V_{i+1} - V_i) - r(V_{i-1} - V_i) + (s-2r) V_i.$$

When $-V_i = \|V\|_\infty$ for $1 \le i \le N-1$,

$$(L^h V)_i = -rV_{i+1} + sV_i - rV_{i-1} \le (s-2r)V_i = V_i < 0$$

So $\|L^h V\|_\infty \ge |(L^h V)_i| \ge |V_i| = \|V\|_\infty$

So $\|u^h - R^h u\|_\infty \le \|L^h (u^h - R^h u)\|_\infty$.