Can you prove or disprove that $b_n \to b$ in L^2?

Pf:

$b_n \not\to b$ in L^2.

Assume $b_n \to b$ in L^2, which means that $\lim\limits_{n\to\infty} E[(b_n - b)^2] = 0$.

Then, $\lim\limits_{n\to\infty} E[(b_n - E(b_n))^2] + \lim\limits_{n\to\infty} (E(b_n) - b)^2 = 0$

$\because \lim\limits_{n\to\infty} E(b_n) = b$, so $\lim\limits_{n\to\infty} E[(b_n - E(b_n))^2] = \lim\limits_{n\to\infty} Var(b_n) = 0$ (*)

$\because b_{n+1} = (1-\alpha) b_n + \alpha X_n$

$\therefore b_n = \sum\limits_{k=0}^{n-1} (1-\alpha)^{n-k-1} \cdot \alpha X_k$ with $\{X_n\}$ is iid of $N(b, \sigma^2)$

So $Var(b_n) = \sigma^2 \alpha^2 \sum\limits_{k=0}^{n-1} (1-\alpha)^{2k} = \sigma^2 \alpha^2 \dfrac{1-(1-\alpha)^{2n}}{1-(1-\alpha)^2}$

That is $\lim\limits_{n\to\infty} Var(b_n) = \dfrac{\sigma^2 \alpha}{2-\alpha}$, which contradicts with ~~the fact~~ (*).

So we know that b_n doesn't converge to b in L^2.