Jing Guo

1. Pf:

$$u' = 1 - \frac{e^{\frac{x-1}{\sqrt{\varepsilon}}} \cdot \frac{1}{\sqrt{\varepsilon}} - e^{-\frac{x+1}{\sqrt{\varepsilon}}} \cdot \left(-\frac{1}{\sqrt{\varepsilon}}\right)}{1 - e^{-\frac{2}{\sqrt{\varepsilon}}}} = 1 - \frac{\frac{1}{\sqrt{\varepsilon}} \cdot e^{\frac{x-1}{\sqrt{\varepsilon}}} + \frac{1}{\sqrt{\varepsilon}} \cdot e^{-\frac{x+1}{\sqrt{\varepsilon}}}}{1 - e^{-\frac{2}{\sqrt{\varepsilon}}}}$$

$$u'' = -\frac{\frac{1}{\sqrt{\varepsilon}} \cdot e^{\frac{x-1}{\sqrt{\varepsilon}}} \cdot \frac{1}{\sqrt{\varepsilon}} + \frac{1}{\sqrt{\varepsilon}} \cdot e^{-\frac{x+1}{\sqrt{\varepsilon}}} \cdot \left(-\frac{1}{\sqrt{\varepsilon}}\right)}{1 - e^{-\frac{2}{\sqrt{\varepsilon}}}} = -\frac{\frac{1}{\varepsilon} \cdot e^{\frac{x-1}{\sqrt{\varepsilon}}} - \frac{1}{\varepsilon} \cdot e^{-\frac{x+1}{\sqrt{\varepsilon}}}}{1 - e^{-\frac{2}{\sqrt{\varepsilon}}}}$$

So, $$\text{LHS} = -\varepsilon u'' + u = -\varepsilon \times \left(- \frac{\frac{1}{\varepsilon} e^{\frac{x-1}{\sqrt{\varepsilon}}} - \frac{1}{\varepsilon} \cdot e^{-\frac{x+1}{\sqrt{\varepsilon}}}}{1 - e^{-\frac{2}{\sqrt{\varepsilon}}}} \right) + x - \frac{e^{\frac{x-1}{\sqrt{\varepsilon}}} - e^{-\frac{x+1}{\sqrt{\varepsilon}}}}{1 - e^{-\frac{2}{\sqrt{\varepsilon}}}}$$

$$= \frac{e^{\frac{x-1}{\sqrt{\varepsilon}}} - e^{-\frac{2}{\sqrt{\varepsilon}}}}{1 - e^{-\frac{2}{\sqrt{\varepsilon}}}} + x - \frac{e^{\frac{x-1}{\sqrt{\varepsilon}}} - e^{-\frac{x+1}{\sqrt{\varepsilon}}}}{1 - e^{-\frac{2}{\sqrt{\varepsilon}}}} = x = \text{RHS}$$

In conclusion, $u(x)$ is the solution of (1).

Now we want to show the solution is unique.

$$-u'' = \frac{x-u}{\varepsilon} = -\frac{1}{\varepsilon} u(x) + \frac{1}{\varepsilon} \cdot x, \quad u(0) = u(1) = 0$$

Claim $f(u,x) = -\frac{1}{\varepsilon}u + \frac{1}{\varepsilon}x$

WTS $f(u,x)$ satisfy the Libschiz condition w.r.t. u, which means that $\exists K$ s.t. $|f(a,x) - f(b,x)| < K|a-b|$, $0<b<a<1$

So $\exists G \in \mathbb{R}$ s.t. $G < \varepsilon$

$$|f(a,x) - f(b,x)| = \left|-\frac{1}{\varepsilon}a + \frac{1}{\varepsilon}x + \frac{1}{\varepsilon}b - \frac{1}{\varepsilon}x\right| = \left|\frac{1}{\varepsilon}(b-a)\right|$$

$$= \left|\frac{1}{\varepsilon}\right| \cdot |b-a| \leq \left|\frac{1}{G}\right| \cdot |b-a|$$

which means that $\exists K = \left|\frac{1}{G}\right|$, s.t. $|f(a,x) - f(b,x)| < K|a-b|$

In conclusion, $u(x)$ is the unique solution.