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Ex3. • Prove β_N is biased.

$$E[\beta_N] - Var(z_1) = E\left[\frac{1}{N}\sum_{i=1}^{N}\left(z_i - \frac{1}{N}\sum_{i=1}^{N} z_i\right)^2\right] - Var(z_1)$$

Claim $E[z_i] = u$, $Var[z_1] = \sigma^2$

$$E[\beta_N] - Var(z_1) = E\left[\frac{1}{N}\sum_{i=1}^{N}\left((z_i - u) - \left(\frac{1}{N}\sum_{i=1}^{N} z_i - u\right)\right)^2\right] - Var(z_1)$$

$$= E\left[\frac{1}{N}\cdot\sum_{i=1}^{N}\left((z_i-u)^2 + \frac{1}{N^2}\left(\sum_{i=1}^{N} z_i - u\right)^2 - 2(z_i-u)\left(\frac{1}{N}\sum_{i=1}^{N} z_i - u\right)\right)\right] - Var(z_1)$$

$$= \frac{1}{N}\sum_{i=1}^{N}\left(E[(z_i-u)^2] + \frac{1}{N^2}E\left[\left(\sum_{i=1}^{N} z_i - u\right)^2\right] - 2E[z_i - u]\cdot E\left[\frac{1}{N}\sum_{i=1}^{N} z_i - u\right]\right)$$

$$- Var(z_1)$$

$\because E[(z_i-u)^2] = Var(z_i) = \sigma^2$

So, $E[\beta_N] - Var(z_1) = \sigma^2 - \frac{1}{n}\sigma^2 - \sigma^2 = -\frac{1}{n}\sigma^2$, which means that β_N is ~~not~~ biased

• Can you propose an unbiased estimator.

Solu:

$$\beta_N = \frac{1}{N-1}\sum_{i=1}^{N}(z_i - \bar{z}_N)^2 \quad \text{with } \bar{z}_N = \frac{1}{N}\sum_{i=1}^{N} z_i$$