P6: (1) $u(x) = x - \frac{e^{x-1} - e^{-x-1}}{1-e^{-2}}$ ①

$u'(x) = 1 - \frac{1}{1-e^{-2}} \cdot (e^{x-1} - e^{-x-1} \cdot (-1)) = 1 - \frac{e^{x-1}+e^{-x-1}}{1-e^{-2}}$ ②

$u''(x) = -\frac{1}{1-e^{-2}} \cdot (e^{x-1} - e^{-x-1})$ ③

plug ①.②.③ into $-u'' + u = x$

$$\frac{1}{1-e^{-2}}(e^{x-1} - e^{-x-1}) + x - \frac{e^{x-1}-e^{-x-1}}{1-e^{-2}} = x$$

So. $u(x)$ is the unique solution.

(2) $Lu(x) = \begin{cases} -u''(x) + u(x) & , 0<x<1 \\ u(x) & , x=0 \end{cases}$ $\forall u \in C^2$ and $\hat{f}(x) = \begin{cases} f(x) & , 0<x<1 \\ 0 & , x=0.1 \end{cases}$

Claim $h = \frac{1}{n}$, since UFD is similar with CFD, ~~so~~

So,

$$L^h u = \begin{pmatrix} u_0 \\ -ru_0 + su_1 - tu_2 \\ \vdots \\ -ru_{i-1} + su_i - tu_{i+1} \\ \vdots \\ -ru_{n-2} + su_{n-1} - tu_n \\ u_n \end{pmatrix}, \quad u \in R^{n+1} \qquad r = \frac{1}{h^2}, \quad s = \frac{2}{h^2} + 1, \quad t = \frac{1}{h^2}$$

Also, $R^h f = \begin{pmatrix} f(x_0) \\ f(x_1) \\ \vdots \\ f(x_n) \end{pmatrix}$, $\forall f \in C([0.1])$

(3) ~~App~~ Apply UFD method for $-u'' + u = x$ , we're able to get that

$$-u_{i-1}^{h} \cdot \frac{1}{h^2} + u_i^h \left(\frac{2}{h^2} + 1\right) - u_{i+1}^{h} \cdot \frac{1}{h^2} = \hat{f}_i = x_i \quad (*)$$

~~By the UFD numerical method discussed in the class, we got:~~

$$\cancel{\begin{cases} u_0^h = 0 \\ -r u_{i-1}^h + s u_i^h - t u_{i+1}^h = \hat{f}_i , \quad i = 1, 2, \cdots, N-1 \end{cases}}$$

Transform $(*)$ into: $\left(\frac{2}{h^2} + 1\right) u_i^h = \frac{1}{h^2}\left(u_{i-1}^h + u_{i+1}^h\right) + x_i$,

That is $u_i^h = \frac{2}{2+h^2}\left(\frac{h^2}{2} x_i + \frac{1}{2} u_{i-1}^h + \frac{1}{2} u_{i+1}^h\right)$ ,

where we claim that $\gamma = \frac{2}{2+h^2}$ , $\ell^h(x) = \frac{h^2}{2} x$ , $p^h(x+h|x) = \frac{1}{2}$,

$p^h(x-h|x) = \frac{1}{2}$

So, the Markovian Reward Process is that:

$$u(x) = \gamma \cdot \left[\ell^h(x) + p^h(x+h|x)\, u(x+h) + p^h(x-h|x) \cdot u(x-h)\right]$$

(4) Pseudocode:

Initialize $\tilde{\varepsilon}, h$ ; Input $\{ v(x) : x \in O^h \}$ ; $v(x)$ is the initial value.

```
A ← 1 , n ← 0
while flag do:
    ε ← 0 ; n ← n+1
    for x ∈ O^h do:
        u(x) ← v(x)
        v(x) ← F^h u(x)
        ε ← max{|u(x) − v(x)|}
        if ε < ε̃ then flag ← 0
print {v(x) : x ∈ O^h}
```

(5) Input $\{V(x) : x \in O^h\}$

Total $\leftarrow 0$

for $x \in O^h$ do :

for $i = 1, 2, 3 \cdots, n$ do :

$W_i \leftarrow \{S_0 = x, R_1, S_1 \cdots R_T, S_T, R_{T+1}\}$

$G \leftarrow \sum_{i=1}^{T+1} r^{i-1} R_i$

Total $\leftarrow$ Total $+ G$

~~return~~ $V(x) \leftarrow \frac{\text{Total}}{n}$

print ( $\{V(x) : x \in O^h\}$ )

(6) WTS Consistency, so we have to show that: $\exists \alpha > 0$ s.t.

$\|L^h R^h v - R^h L v\|_\infty \le K h^\alpha, \forall v \in C^2(0,1) \cap C[0,1]$

So ① when $i = 0$, we have: $(L^h R^h v)_i = (R^h v)_0 = v(x_0) = 0$,

Also, $(R^h L v)_i = \cancel{R^h v}\, Lv(x_0) = v(x_0) = 0$.

② when $i = n$ : $(L^h R^h v)_i = (R^h v)_n = v(x_n) = 0$

$(R^h L v)_i = Lv(x_n) = v(x_n) = 0$

③ when $1 \le i \le n-1$ : $(L^h R^h v)_i = -\delta_h \delta_{-h} v(x_i) + v(x_i)$

$(R^h L v)_i = -v''(x_i) + v(x_i)$

By taking account into ①.②.③ and Taylor theorem:

$\|L^h R^h v - R^h L v\|_\infty \le K h^2, \forall v \in C^2(0,1) \cap C[0,1]$

So, we proved the consistency.

(7) WTS stability, so we have to show that $\exists K$ s.t. $\|v\|_\infty < K \|L^h v\|_\infty$

① when $|v_0| = \|v\|_\infty$, then $\|L^h v\|_\infty \ge |(L^h v)_0| = |v_0| = \|v\|_\infty$

② when $|v_n| = \|v\|_\infty$, $\|L^h v\|_\infty \ge |(L^h v)_n| = |v_n| = \|v\|_\infty$

③ When $|v_i| = \|v\|_\infty$ , $1 \le i \le n-1$ ,

$$\|L^h v\|_\infty \ge |(L^h v)_i| = |-r v_{i-1} + s v_i - t v_{i+1}| \ge s|v_i| - r|v_{i-1}| - t|v_{i+1}|$$

$$\ge (s-r-t)|v_i| \ge |v_i| = \|v\|_\infty$$

By taking into account ①②③, we proved stability.

(8) Let $u^h$ be the numerical solution.

$$\|u^h - R^h u\|_\infty \le \|L^h (u^h - R^h u)\|_\infty = \|L^h u^h - L^h R^h u\|_\infty = \|\hat{R}^h f - L^h R^h u\|_\infty$$

$$= \|R^h L u - L^h R^h u\|_\infty = O(h^2).$$