(4) Prove that the exact price is 0.02275.

$$v = E^{Q}[h(S_T)] = Q(W_1 < -\frac{b+u}{\sigma} = -2) = \phi(-2) \approx 0.02275.$$

$$W_1 \sim N(0,1)$$

(5) Prove or demonstrate IS is more efficient to OMC.

Pf: As we show in video:

$$\hat{V}_{omc} = \frac{1}{n}\sum_{i=1}^{n} I(X_i < -2), \quad \hat{V}_{IS} = \frac{1}{n}\sum_{i=1}^{n} 1(Y_i < -2) \cdot e^{\frac{1}{2}a^2 + aY_i}$$

$\because$ $\hat{V}_{omc}$ and $\hat{V}_{IS}$ are unbiased.

$$\therefore MSE(\hat{V}_{omc}) = Var(\hat{V}_{omc}) = \frac{1}{n}(\phi(-2) - \phi(-2)^2)$$

$$MSE(\hat{V}_{IS}) = Var(\hat{V}_{IS}) = \frac{1}{n}(\underbrace{E[1(Y_i<-2)e^{a^2+2aY_i}]}_{①} - \phi^2(-2)) \quad (*)$$

①: $$E[1(Y_i<-2)e^{a^2+2aY_i}] = \int_{-\infty}^{-2} e^{a^2+2ay} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{(y+a)^2}{2}} dy$$

$$= e^{a^2} \cdot \phi(-2-a)$$

So (*) can be transformed to:

$$MSE(\hat{V}_{IS}) = \frac{1}{n}(e^{a^2}\phi(-2-a) - \phi^2(-2)).$$

In conclusion: $$MSE(\hat{V}_{omc}) - MSE(\hat{V}_{IS}) = \frac{1}{n}(\phi(-2) - e^{a^2}\phi(-2-a)) \geq 0$$

IS is more efficient to OMC.