- Prove or disprove: Suppose f is convex and X is submartingale, prove that $g(t) = E[f(X_t)]$ is increasing.

Pf:

We could find a counter example to disprove it.

Let $f(X_t) = \frac{1}{X_t^2}$. Claim $t_1 < t_2$:

$g_{t_2} - g_{t_1} = E[f(X_{t_2})] - E[f(X_{t_1})] = E\left[\frac{X_{t_1}^2 - X_{t_2}^2}{X_{t_2}^2 X_{t_1}^2}\right] = E\left[\frac{(X_{t_1}-X_{t_2})(X_{t_1}+X_{t_2})}{X_{t_2}^2 X_{t_1}^2}\right]$

$\because$ X is a submartingale $\therefore E[X_{t_2}] > E[X_{t_1}]$, $t_2 > t_1$

So $E\left[\frac{(X_{t_1}-X_{t_2})(X_{t_1}+X_{t_2})}{X_{t_1}^2 X_{t_2}^2}\right] < 0$. That is $g_{t_2} - g_{t_1} < 0$

So $g(t)$ is decreasing.

- Let $t \mapsto e^{-rt} S_t$ be a martingale, then prove that $C(t) = E[e^{-rt}(S_t - K)^+]$ is increasing.

Pf.

When $S_t < K$: Claim $t_2 > t_1$, $E[e^{-rt_2}(S_{t_2}-K)^+ | \mathcal{F}_{t_1}]$

$= E[e^{-rt_2} \cdot 0] = 0$. Similarly, $E[e^{-rt_1}(S_{t_1}-K)^+] = 0$

When $S_t \geq K$: claim $t_2 > t_1$,

$E[e^{-rt_2}(S_{t_2}-K)^+ | \mathcal{F}_{t_1}] = E[e^{-rt_2}(S_{t_2}-K) | \mathcal{F}_{t_1}] = E[e^{-rt_2} \cdot S_{t_2} - K(e^{-rt_2} + e^{-rt_1} - e^{-rt_1}) | \mathcal{F}_{t_1}]$

$= e^{-rt_1} S_{t_1} - Ke^{-rt_1} + KE[e^{-rt_1} - e^{-rt_2}]$

$= e^{-rt_1} S_{t_1} - Ke^{-rt_1} - KE[e^{-rt_2} - e^{-rt_1}]$

$\because e^{-rt_2} < e^{-rt_1}$ $\therefore$ ~~$e^{-rt_1}(S_t - K)$~~

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$\therefore e^{-rt_1}(S_{t_1}-K)-K\cdot E[e^{-rt_2}-e^{-rt_1}] \geq e^{-rt_1}(S_{t_1}-K)$

And then $E[E[e^{-rt_2}(S_{t_2}-K)^+ | \mathcal{F}_{t_1}]] \geq E[e^{-rt_1}(S_{t_1}-K)]$

That is $E[e^{-rt_2}(S_{t_2}-K)^+] \geq E[e^{-rt_1}(S_{t_1}-K)] = E[e^{-rt_1}(S_{t_1}-K)^+]$

So. $c(t)$ is increasing.

- Suppose $r=0$ and S is martingale, prove that $p(t)=E[(S_t-K)^-]$ is increasing.

Pf:

~~When S_t~~ Claim $t_2>t_1$

when $S_t \geq K$, $p(t_2)-p(t_1)=0$

when $S_t<K$:

$E[K-S_{t_2}|\mathcal{F}_{t_1}] = K-S_{t_1} \geq (S_{t_1}-K)^-$

Then $E[E[K-S_{t_2}|\mathcal{F}_{t_1}]] \geq E[(S_{t_1}-K)^-]$

That is $E[K-S_{t_2}] = E[(S_{t_2}-K)^-] \geq E[(S_{t_1}-K)^-]$

So, $p(t)$ is increasing.

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