1. Suppose that we wish to approximate the first derivative $u'(x)$ of a very smooth function with an error of only $O(h)^4$, where h is the step size. Which difference approximation could we use? (Hint: you may consider to use more than two points in the neighborhood)

   soln.

   $$f'(x) \simeq \frac{2}{3}\delta_{\pm h} f(x) - \frac{1}{12}\delta_{\pm 2h} f(x).$$

2. Let $f : \mathbb{R} \mapsto \mathbb{R}$ be a smooth even function satisfying $f(0) = 0$. Our objective is to approximate the second order derivative $f''(0)$.

   - Prove that $f'(0) = 0$.
   - Chenyu proposes the following estimator for $f''(0)$: for a step size h

     $$a_h = \frac{2f(h)}{h^2}.$$

     Please justify that Chenyu's estimation has its convergence $O(h^2)$.
   - Is there anyway to improve the above convergence to $O(h^4)$ in the form of

     $$b_h = \frac{c_1 f(h) + c_2 f(2h)}{h^2}$$

     for some constants c_1 and c_2?
   - If the above function f is odd and other properties remain the same, how do you want to find the $f''(0)$ efficiently?

   Solution

   - it can be directly shown from the definition of f'.
   - $f^n(0) = 0$ for all odd number n. Therefore, taylor expansion gives

     $$f(h) = \frac{1}{2}h^2 f''(0) + \frac{1}{24}h^4 f^{(4)}(0) + O(h^6),$$

     and the result follows.
   - we can combine the above taylor expansion with

     $$f(2h) = 2h^2 f''(0) + \frac{2}{3}h^4 f^{(4)}(0) + O(h^6).$$

     It yields that, with $c_1 = 8/3$ and $c_2 = -1/6$,

     $$c_1 f(h) + c_2 f(2h) = h^2 f''(0) + O(h^6).$$
   - If f is odd, then $f''(0) = 0$ and no estimate is needed any more.