

Week 10: Search Tree Algorithms 2

Tree Review

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Binary search trees ...

- data structures designed for $O(\log n)$ search
- consist of nodes containing item (incl. key) and two links
- can be viewed as recursive data structure (subtrees)
- have overall ordering ($\text{data}(\text{Left}) < \text{root} < \text{data}(\text{Right})$)
- insert new nodes as leaves (or as root), delete from anywhere
- have structure determined by insertion order (*worst*: $O(n)$)
- operations: insert, delete, search, rotate, rebalance, ...

Randomised BST Insertion

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Effects of order of insertion on BST shape:

- best case (for at-leaf insertion): keys inserted in pre-order (median key first, then median of lower half, median of upper half, etc.)
- worst case: keys inserted in ascending/descending order
- average case: keys inserted in *random* order $\Rightarrow O(\log_2 n)$

Tree ADT has no control over order that keys are supplied.

Can the algorithm itself introduce some *randomness*?

In the hope that this randomness helps to balance the tree ...

... Randomised BST Insertion

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How can a computer pick a number at random?

- it cannot

Software can only produce *pseudo random numbers*.

- a pseudo random number is one that is predictable
 - (although it may appear unpredictable)
- \Rightarrow implementation may deviate from expected theoretical behaviour
 - (more on this in week 12)

... Randomised BST Insertion

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- Pseudo random numbers in C:

```
rand() // generates random numbers in the range 0 .. RAND_MAX
```

where the constant `RAND_MAX` is defined in `stdlib.h`
(depends on the computer: on the CSE network, `RAND_MAX` = 2147483647)

To convert the return value of `rand()` to a number between 0 .. RANGE

- compute the remainder after division by `RANGE+1`

... Randomised BST Insertion

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Approach: normally do leaf insert, randomly do root insert.

```
insertRandom(tree,item)
|   Input   tree, item
|   Output tree with item randomly inserted
|
|   if tree is empty then
|       return new node containing item
|   end if
|   // p/q chance of doing root insert
|   if random number mod q < p then
|       return insertAtRoot(tree,item)
|   else
|       return insertAtLeaf(tree,item)
|   end if
```

E.g. 30% chance \Rightarrow choose $p=3, q=10$

... Randomised BST Insertion

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Cost analysis:

- similar to cost for inserting keys in random order: $O(\log_2 n)$
- does not rely on keys being supplied in random order

Approach can also be applied to deletion:

- standard method promotes inorder successor to root
- for the randomised method ...
 - promote inorder successor from right subtree, OR
 - promote inorder predecessor from left subtree

Splay Trees

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Splay Trees

A kind of "self-balancing" tree ...

Splay tree insertion modifies insertion-at-root method:

- by considering *parent-child-grandchild* (three level analysis)
- by performing double-rotations based on p-c-g orientation

The idea: appropriate double-rotations improve tree balance.

... Splay Trees

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Splay tree implementations also do *rotation-in-search*:

- by performing double-rotations also when searching

The idea: provides similar effect to periodic rebalance.

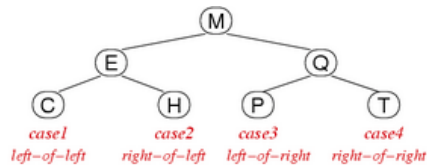
⇒ improves balance but makes search more expensive

... Splay Trees

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Cases for splay tree double-rotations:

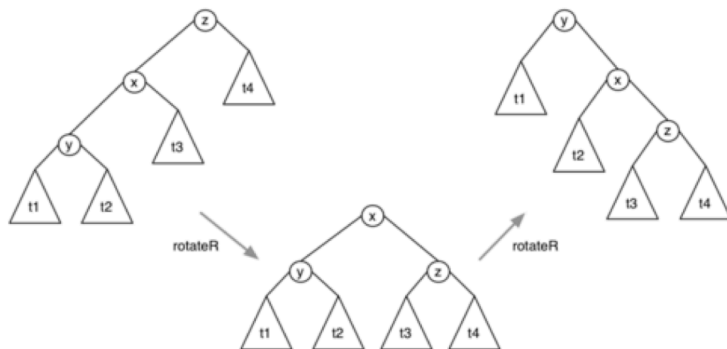
- case 1: grandchild is left-child of left-child ⇒ double right rotation from top
- case 2: grandchild is right-child of left-child
- case 3: grandchild is left-child of right-child
- case 4: grandchild is right-child of right-child ⇒ double left rotation from top



... Splay Trees

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Double-rotation case for left-child of left-child ("zig-zig"):

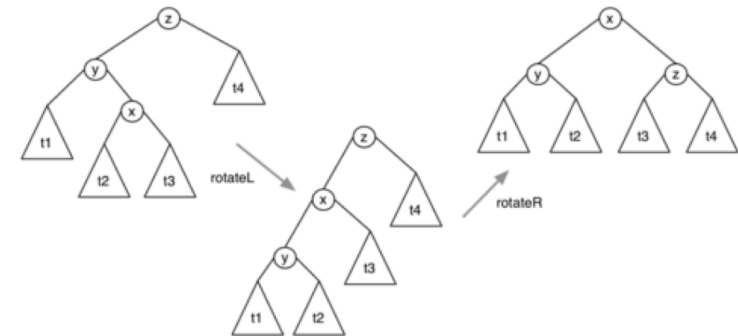


Note: both rotations at the root (unlike insertion-at-root)

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... Splay Trees

Double-rotation case for right-child of left-child ("zig-zag"):



Note: rotate subtree first (like insertion-at-root)

... Splay Trees

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Algorithm for splay tree insertion:

insertSplay(tree,item):

Input tree, item

Output tree with item splay-inserted

```

if tree is empty then return new node containing item
else if item=data(tree) then return tree
else if item<data(tree) then
    if left(tree) is empty then
        left(tree)=new node containing item
    else if item<data(left(tree)) then
        // Case 1: left-child of left-child "zig-zig"
        left(left(tree))=insertSplay(left(left(tree)),item)
        tree=rotateRight(tree)
    else if item>data(left(tree)) then
        // Case 2: right-child of left-child "zig-zag"
        right(left(tree))=insertSplay(right(left(tree)),item)
        left(tree)=rotateLeft(left(tree))
    end if
    return rotateRight(tree)
else // item>data(tree)
    if right(tree) is empty then
        right(tree)=new node containing item
    else if item<data(right(tree)) then
        // Case 3: left-child of right-child "zag-zig"
        left(right(tree))=insertSplay(left(right(tree)),item)
        right(tree)=rotateRight(right(tree))

```

```

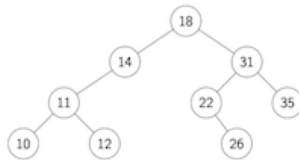
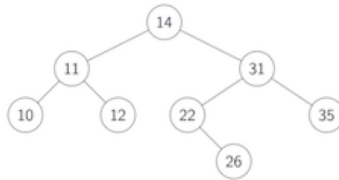
else if item > data(right(tree)) then
    // Case 4: right-child of right-child "zag-zag"
    right(right(tree)) = insertSplay(right(right(tree)), item)
    tree = rotateLeft(tree)
end if
return rotateLeft(tree)
end if

```

Exercise #1: Splay Trees

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Insert 18 into this splay tree:



... Splay Trees

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Searching in splay trees:

```

searchSplay(tree, item):
    Input  tree, item
    Output address of item if found in tree
           NULL otherwise

    if tree=NULL then
        return NULL
    else
        tree = splay(tree, item)
        if data(tree) = item then
            return tree
        else
            return NULL
        end if
    end if

```

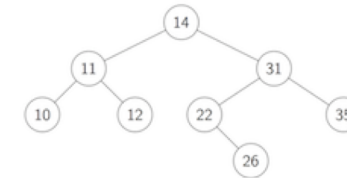
where `splay()` is similar to `insertSplay()`,

except that it doesn't add a node ... simply moves `item` to root if found, or nearest node if not found

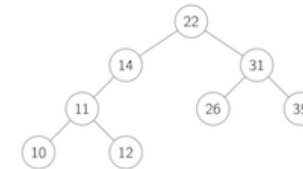
Exercise #2: Splay Trees

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If we search for 22 in the splay tree



... how does this affect the tree?



... Splay Trees

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Why take into account both child and grandchild?

- moves accessed node to the root
- *moves every ancestor of accessed node roughly halfway to the root*

⇒ better amortized cost than insert-at-root

... Splay Trees

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Analysis of splay tree performance:

- assume that we "splay" for both insert and search
- consider: m insert+search operations, n nodes
- *Theorem.* Total number of comparisons: average $O((n+m) \cdot \log(n+m))$

Gives good overall (amortized) cost.

- insert cost not significantly different to insert-at-root
- search cost increases, but ...
 - improves balance on each search

- moves frequently accessed nodes closer to root

But ... still has worst-case search cost $O(n)$

Real Balanced Trees

Better Balanced Binary Search Trees

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So far, we have seen ...

- randomised trees ... make poor performance unlikely
- occasional rebalance ... fix balance periodically
- splay trees ... reasonable amortized performance
- but both types still have $O(n)$ worst case

Ideally, we want both average/worst case to be $O(\log n)$

- AVL trees ... fix imbalances as soon as they occur
- 2-3-4 trees ... use varying-sized nodes to assist balance
- red-black trees ... isomorphic to 2-3-4, but binary nodes

AVL Trees

AVL Trees

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Invented by Georgy Adelson-Velsky and Evgenii Landis

Approach:

- insertion (at leaves) may cause imbalance
- repair balance as soon as we notice imbalance
- repairs done locally, not by overall tree restructure

A tree is unbalanced when: $\text{abs}(\text{height}(\text{left}) - \text{height}(\text{right})) > 1$

This can be repaired by at most two rotations:

- if left subtree too deep ...
 - if data inserted in left-right grandchild \Rightarrow left-rotate left subtree
 - rotate right
- if right subtree too deep ...
 - if data inserted in right-left grandchild \Rightarrow right-rotate right subtree
 - rotate left

Problem: determining height/depth of subtrees may be expensive.

... AVL Trees

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Implementation of AVL insertion

```
insertAVL(tree,item):
  Input  tree, item
  Output tree with item AVL-inserted

  if tree is empty then
    return new node containing item
  else if item=data(tree) then
    return tree
  else
    if item<data(tree) then
      left(tree)=insertAVL(left(tree),item)
    else if item>data(tree) then
      right(tree)=insertAVL(right(tree),item)
    end if
    if height(left(tree))-height(right(tree)) > 1 then
      if item>data(left(tree)) then
        left(tree)=rotateLeft(left(tree))
      end if
      tree=rotateRight(tree)
    else if height(right(tree))-height(left(tree)) > 1 then
      if item<data(right(tree)) then
        right(tree)=rotateRight(right(tree))
      end if
      tree=rotateLeft(tree)
    end if
    return tree
  end if
```

Exercise #3: AVL Trees

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Insert 27 into the AVL tree





What happens when you insert 24 now?

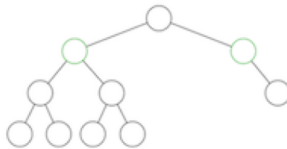
You may like the animation at www.cs.usfca.edu/~galles/visualization/AVLtree.html

... AVL Trees

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Analysis of AVL trees:

- trees are *height*-balanced; subtree depths differ by ± 1
- average/worst-case search performance of $O(\log n)$
- *require* extra data to be stored in each node ("height")
- may not be *weight*-balanced; subtree sizes may differ



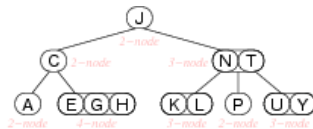
2-3-4 Trees

2-3-4 Trees

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2-3-4 trees have three kinds of nodes

- 2-nodes, with two children (same as normal BSTs)
- 3-nodes, two values and three children
- 4-nodes, three values and four children



... 2-3-4 Trees

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2-3-4 trees are ordered similarly to BSTs



In a *balanced* 2-3-4 tree:

- all leaves are at same distance from the root

2-3-4 trees grow "upwards" by splitting 4-nodes.

... 2-3-4 Trees

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Possible 2-3-4 tree data structure:

```
typedef struct node {
    int      order;      // 2, 3 or 4
    int      data[3];    // items in node
    struct node *child[4]; // links to subtrees
} node;
```

... 2-3-4 Trees

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Searching in 2-3-4 trees:

Search(tree,item):

```
Input  tree, item
Output address of item if found in 2-3-4 tree
        NULL otherwise

if tree is empty then
    return NULL
else
    i=0
    while i<tree.order-1 ^ item>tree.data[i] do
        i=i+1 // find relevant slot in data[]
    end while
    if item=tree.data[i] then // item found
        return address of tree.data[i]
    else // keep looking in relevant subtree
        return Search(tree.child[i],item)
    end if
end if
```

... 2-3-4 Trees

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2-3-4 tree searching cost analysis:

- as for other trees, worst case determined by height h
- 2-3-4 trees are always balanced \Rightarrow height is $O(\log n)$

- worst case for height: all nodes are 2-nodes
same case as for balanced BSTs, i.e. $h \approx \log_2 n$
- best case for height: all nodes are 4-nodes
balanced tree with branching factor 4, i.e. $h \approx \log_4 n$

Insertion into 2-3-4 Trees

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Starting with the root node:

repeat

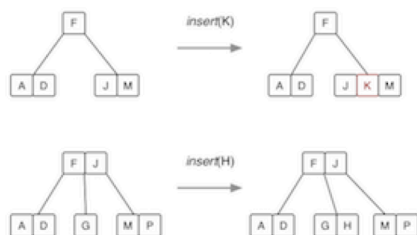
- if current node is full (i.e. contains 3 items)
 - split into two 2-nodes
 - promote middle element to parent
 - if no parent \Rightarrow middle element becomes the new root 2-node
 - go back to parent node
- if current node is a leaf
 - insert Item in this node, order++
- if current node is not a leaf
 - go to child where Item belongs

until Item inserted

... Insertion into 2-3-4 Trees

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Insertion into a 2-node or 3-node:



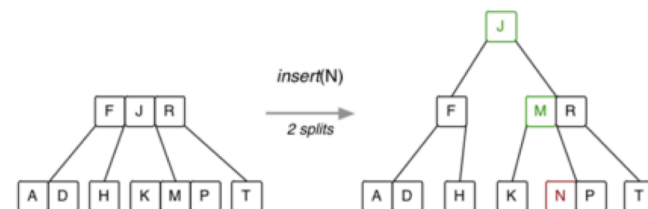
Insertion into a 4-node (requires a split):



... Insertion into 2-3-4 Trees

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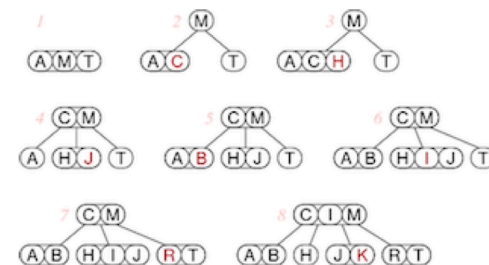
Splitting the root:



... Insertion into 2-3-4 Trees

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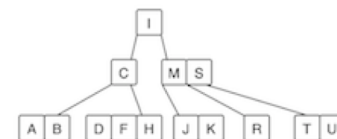
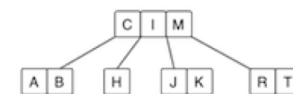
Building a 2-3-4 tree ... 7 insertions:



Exercise #4: Insertion into 2-3-4 Tree

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Show what happens when D, S, F, U are inserted into this tree:



... Insertion into 2-3-4 Trees

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Insertion algorithm:

```
insert(tree, item):
|   Input  2-3-4 tree, item
```

Output tree with item inserted

```
node=root(tree), parent=NULL
repeat
  if node.order=4 then
    promote = node.data[1]      // middle value
    nodeL   = new node containing node.data[0]
    nodeR   = new node containing node.data[2]
    if parent=NULL then
      make new 2-node root with promote,nodeL,nodeR
    else
      insert promote,nodeL,nodeR into parent
      increment parent.order
    end if
    node=parent
  end if
  if node is a leaf then
    insert item into node
    increment node.order
  else
    parent=node
    if item<node.data[0] then
      node=node.child[0]
    else if item<node.data[1] then
      node=node.child[1]
    else
      node=node.child[2]
    end if
  end if
until item inserted
```

... Insertion into 2-3-4 Trees

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Variations on 2-3-4 trees ...

Variation #1: why stop at 4? why not 2-3-4-5 trees? or M -way trees?

- allow nodes to hold up to $M-1$ items, and at least $M/2$
- if each node is a disk-page, then we have a *B-tree* (databases)
- for B-trees, depending on Item size, $M > 100/200/400$

Variation #2: don't have "variable-sized" nodes

- use standard BST nodes, augmented with one extra piece of data
- implement similar strategy as 2-3-4 trees \rightarrow red-black trees.

Red-Black Trees

Red-Black Trees

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Red-black trees are a representation of 2-3-4 trees using BST nodes.

- each node needs one extra value to encode link type
- but we no longer have to deal with different kinds of nodes

Link types:

- *red* links ... combine nodes to represent 3- and 4-nodes
- *black* links ... analogous to "ordinary" BST links (child links)

Advantages:

- standard BST search procedure works unmodified
- get benefits of 2-3-4 tree self-balancing (although deeper)

Red-Black Trees

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Definition of a *red-black tree*

- a BST in which each node is marked red or black
- no two red nodes appear consecutively on any path
- a red node corresponds to a 2-3-4 sibling of its parent
- a black node corresponds to a 2-3-4 child of its parent

Balanced red-black tree

- all paths from root to leaf have same number of black nodes

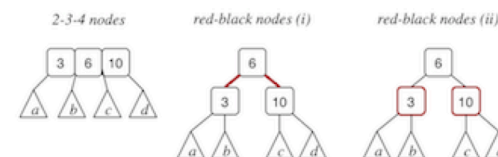
Insertion algorithm: avoids worst case $O(n)$ behaviour

Search algorithm: standard BST search

... Red-Black Trees

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Representing 4-nodes in red-black trees:

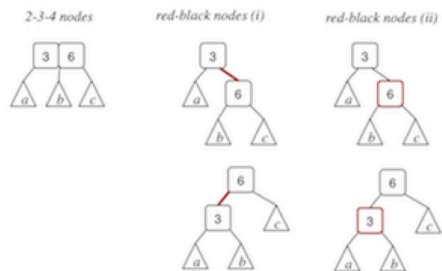


Some texts colour the links rather than the nodes.

... Red-Black Trees

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Representing 3-nodes in red-black trees (two possibilities):



... Red-Black Trees

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Equivalent trees (one 2-3-4, one red-black):



... Red-Black Trees

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Red-black tree implementation:

```
typedef enum {RED, BLACK} Colour;
typedef struct node *RBTree;
typedef struct node {
    int data; // actual data
    Colour colour; // relationship to parent
    RBTree left; // left subtree
    RBTree right; // right subtree
} node;

#define colour(tree) ((tree)->colour)
#define isRed(tree) ((tree) != NULL && (tree)->colour == RED)
```

RED = node is part of the same 2-3-4 node as its parent (sibling)

BLACK = node is a child of the 2-3-4 node containing the parent

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... Red-Black Trees

New nodes are always red:

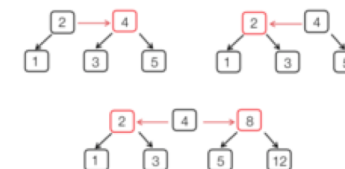
```
RBTree newNode(Item it) {
    RBTree new = malloc(sizeof(Node));
    assert(new != NULL);
    data(new) = it;
    colour(new) = RED;
    left(new) = right(new) = NULL;
    return new;
}
```

... Red-Black Trees

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Node.colour allows us to distinguish links

- black = parent node is a "real" parent
- red = parent node is a 2-3-4 neighbour



... Red-Black Trees

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Search method is standard BST search:

```
SearchRedBlack(tree, item):
    Input tree, item
    Output true if item found in red-black tree
           false otherwise

    if tree is empty then
        return false
    else if item < data(tree) then
        return SearchRedBlack(left(tree), item)
    else if item > data(tree) then
        return SearchRedBlack(right(tree), item)
    else // found
        return true
    end if
```

Red-Black Tree Insertion

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Insertion is more complex than for standard BSTs

- need to recall direction of last branch (L or R)
- need to recall whether parent link is red or black
- splitting/promoting implemented by rotateLeft/rotateRight
- several cases to consider depending on colour/direction combinations

... Red-Black Tree Insertion

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High-level description of insertion algorithm:

```
insertRB(tree,item,inRight):
  Input tree, item, inRight indicating direction of last branch
  Output tree with it inserted

  if tree is empty then
    return newNode(item)
  end if
  if left(tree) and right(tree) both are RED then
    split 4-node
  end if
  recursive insert cases (cf. regular BST)
  re-arrange links/colours after insert
  return modified tree
```

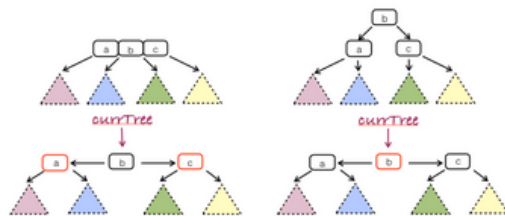
```
insertRedBlack(tree,item):
  Input red-black tree, item
  Output tree with item inserted

  tree=insertRB(tree,item,false)
  colour(tree)=BLACK
  return tree
```

... Red-Black Tree Insertion

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Splitting a 4-node, in a red-black tree:



Algorithm:

```
if isRed(left(currentTree)) and isRed(right(currentTree)) then
  colour(currentTree)=RED
  colour(left(currentTree))=BLACK
  colour(right(currentTree))=BLACK
end if
```

... Red-Black Tree Insertion

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Simple recursive insert (a la BST):



Algorithm:

```
if item < data(tree) then
  left(tree) = insertRB(left(tree), item, false)
  re-arrange links/colours after insert
else // item larger than data in root
  right(tree) = insertRB(right(tree), item, true)
  re-arrange links/colours after insert
end if
```

Not affected by colour of tree node.

... Red-Black Tree Insertion

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Re-arrange after insert (step 1): "normalise" direction of successive red links



Algorithm:

```
if inRight and currentTree is red and left(currentTree) is red then
  currentTree = rotateRight(currentTree)
end if
```

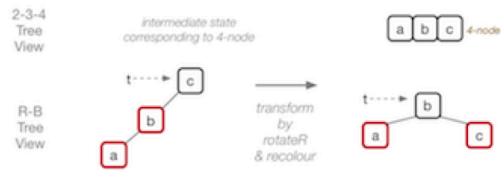
Symmetrically,

- if not inRight and both currentTree and right(currentTree) are red
⇒ left rotate currentTree

... Red-Black Tree Insertion

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Re-arrange after insert (step 2): two successive red links = newly-created 4-node



- Randomised insertion
- Self-adjusting trees
 - Splay trees
 - AVL trees
 - 2-3-4 trees
 - Red-black trees

Algorithm:

```

if isRed(left(currentTree)) and isRed(left(left(currentTree))) then
  currentTree=rotateRight(currentTree)
  colour(currentTree)=BLACK
  colour(right(currentTree))=RED
end if

```

Symmetrically,

- if both right(currentTree) and right(right(currentTree)) are red
 \Rightarrow left rotate currentTree, then re-colour currentTree and left(currentTree)

- Suggested reading:
 - Sedgewick, Ch.13.1-13.4

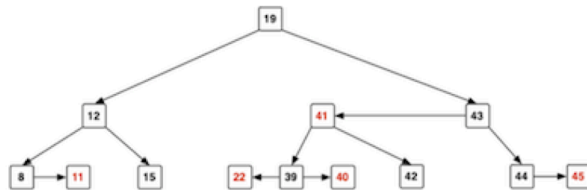
Produced: 30 Sep 2018

... Red-Black Tree Insertion

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Example of insertion, starting from empty tree:

22, 12, 8, 15, 11, 19, 43, 44, 45, 42, 41, 40, 39



Red-black Tree Performance

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Cost analysis for red-black trees:

- tree is well-balanced; worst case search is $O(\log_2 n)$
- insertion affects nodes down one path; max #rotations is $2 \cdot h$
 (where h is the height of the tree)

Only disadvantage is complexity of insertion/deletion code.

Note: red-black trees were popularised by Sedgewick.

Summary

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