Extended Algorithms Courses COMP3821/9801

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Linear Time Deterministic Order Statistics

Deterministic Linear Time Algorithm for Order Statistics

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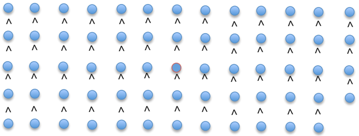

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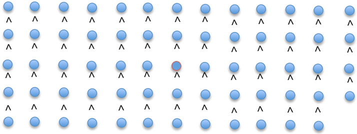
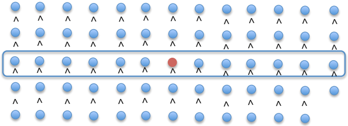


  - Take the collection of all $\lfloor \frac{n}{5} \rfloor$ middle elements of each group (i.e., the medians of each group of five)



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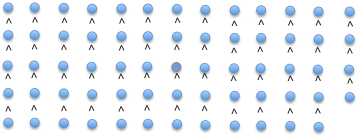

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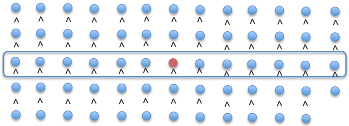

  - Apply recursively SELECT algorithm to find the median p of this collection;

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- **Note:** This algorithm is the same as RAND-SELECT except for the way how we chose the pivot.
  - instead of choosing pivot randomly we called recursively the very same algorithm to pick the pivot as the median of the middle elements of the groups of five elements.

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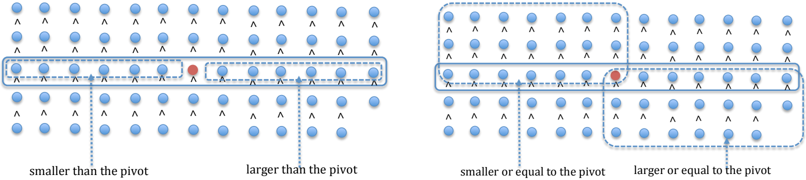

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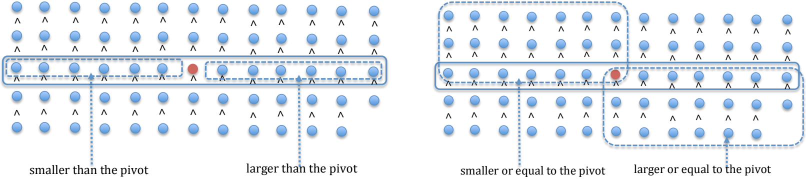

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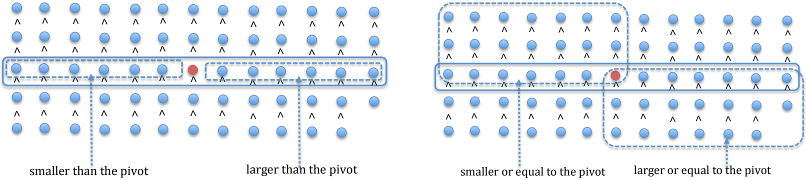

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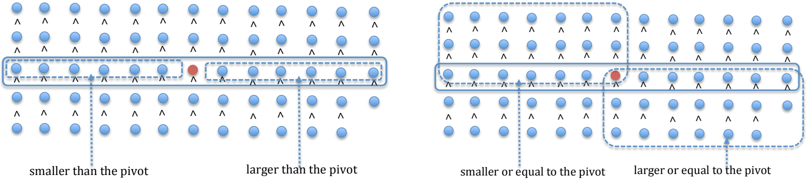

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- But this implies that at least $\lfloor 3n/10 \rfloor$ of the total number of elements are smaller than the pivot, and that many elements larger than the pivot.
- (the same caveat: we are assuming all elements are distinct; otherwise we have to slightly tweak the algorithm to split all elements equal to the pivot evenly between the two groups.)

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$$\begin{aligned} T(n) &\leq T(n/5) + T(7n/10) + Cn < 11C \cdot \frac{n}{5} + 11C \cdot \frac{7n}{10} + Cn \\ &= 109\frac{Cn}{10} < 11C \end{aligned}$$

which proves out statement that $T(n) < 11C \cdot n$.

- Note that this algorithm is a genuine recursion (rather than just an iteration) so its execution involves lots of traffic on the machine stack, which makes this algorithm slow in practice; the randomised version of it, RAND-SELECT, significantly outperforms it.

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- Similarly RAND-QUICKSORT in practice outperforms MERGESORT, which, unlike RAND-QUICKSORT, is guaranteed to run in time $O(n \log n)$.

Partitioning robust for having many repetitions in the array:

PARTITION(A, m, n, p) *(partitioning $A[m..n]$ around a pivot p)*

$i \leftarrow m - 1;\quad fl \leftarrow True;$

for $j = m$ **to** n

    if $A[j] < p$

        then $i + +;$

                exchange $A[i] \leftrightarrow A[j];$

    else if $A[j] = p$

            then if $fl = True$

                        $i + +;$

                        exchange $A[i] \leftrightarrow A[j];$

                        $fl = False;$

                    else $fl = True;$

return i