Algorithms: COMP3121/3821/9101/9801

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INTRODUCTION

Introduction

What is this course about?

It is about **designing algorithms** for solving problems.

Why should you study algorithms design?

Can you find every algorithm you might need using Google?

Our goal:

To learn **techniques** which can be used to solve **new, unfamiliar** problems that arise in a rapidly changing field.

Course content:

- a survey of algorithms **design techniques**
- particular algorithms will be mostly used to illustrate design techniques
- emphasis on development of **algorithm design skills**

Textbooks

Textbook:

Kleinberg and Tardos: *Algorithm Design*
paperback edition available at the UNSW book store
good: very readable (and very pleasant to read!); an excellent textbook;
not so good: as a reference manual for later use.

An alternative textbook:

Cormen, Leiserson, Rivest and Stein: *Introduction to Algorithms*
preferably the third edition, should also be available at the bookstore
excellent: to be used later as a reference manual;
not so good: quite formalistic and written in a rather dry style.

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Only when this is not clear by common sense.

Example: MERGESORT

Merge-Sort(A,p,r) *sorting A[p..r]*

1. **if** $p < r$
2. **then** $q \leftarrow \lfloor \frac{p+r}{2} \rfloor$
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5. The above is essentially a proof by induction, but we will never bother formalizing proofs of (essentially) obvious facts.

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- **However, be very careful what you call trivial!!**

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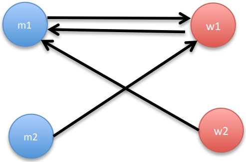
a set of n pairs $p = (m, w)$ of a man m and a woman w so that the following situation never happens:

for two pairs $p = (m, w)$ and $p' = (m', w')$:

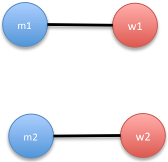
- man m prefers woman w' to woman w, **and**
- woman w' prefers man m to man m'.

Stable Matching Problem

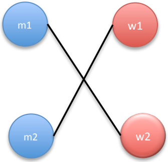
Preferences:



<u>Case1:</u>
stable matching



not stable



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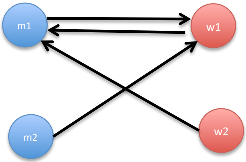

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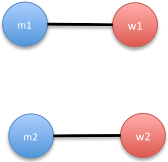

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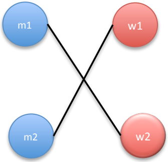

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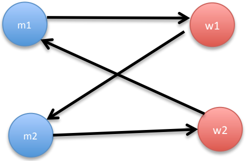

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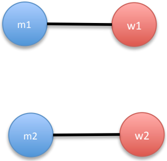

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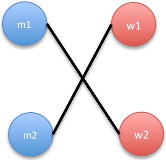

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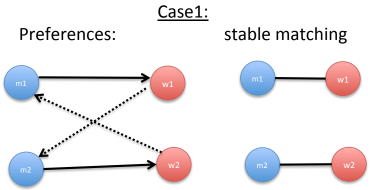

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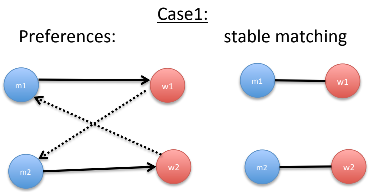

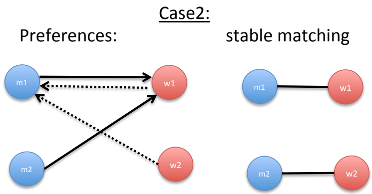

Stable Matching Problem: Gale - Shapley algorithm (invented to pair newly graduated physicians with US hospitals for residency training)

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        Else m is lower on her preference list than m';
            the proposal is rejected and m remains free.

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Note: this method is called "divide-and-conquer".

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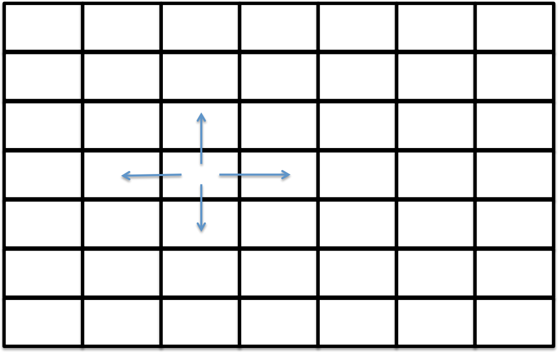
- More ways to hide than the number of all possible weighting outcomes! Thus, it is impossible to do it!

Puzzles!!!

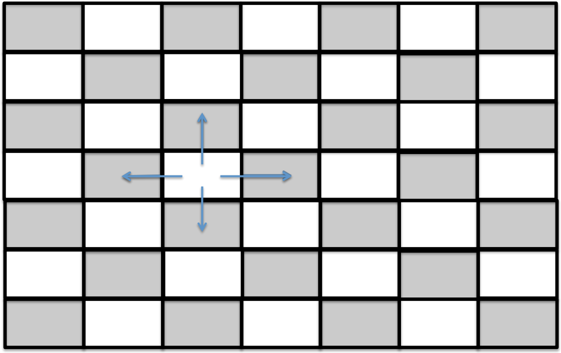
Problem: Consider a block of 7 X 7 houses:



The inhabitant of each house thinks that all four houses around him (to the left, right, top and bottom) are nicer than his house and would like to move to any of the four. Can you move the inhabitants around to make them all happier?

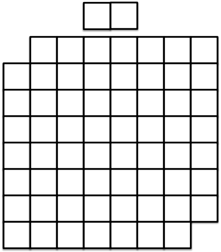
Puzzles!!!

Hint:



Puzzles!!!

Problem: Consider an 8×8 board with two diagonal squares missing, and an 1×2 domino:



Can you cover the entire board with 31 such dominoes?

Puzzles!!!

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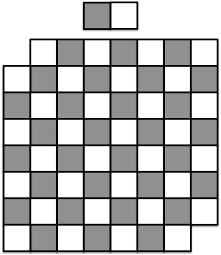

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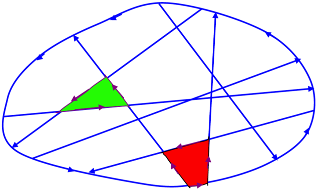

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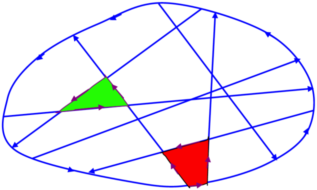

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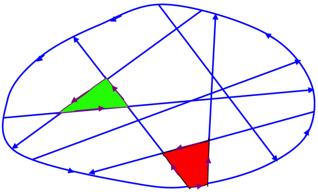

- A block is a part of the city that is not intersected by any street.
- Design an algorithm that, given a map of a city, finds a block (just one such block, not all such blocks) that can be circumnavigated while respecting all one-way signs.

(for example, the green block has such property, but not the red one)

Basics revisited: how do we add two numbers?

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- no asymptotically faster algorithm

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- "Simple" problems can actually turn out to be difficult!

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- AB can now be calculated as follows:

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What we mean is that the product AB can be calculated recursively by the following program:

function $\textsc{Mult}(A, B)$
  if $|A| = |B| = 1$ **then return** AB
  else
    $A_1 \leftarrow \text{MoreSignificantPart}(A)$;
    $A_0 \leftarrow \text{LessSignificantPart}(A)$;
    $B_1 \leftarrow \text{MoreSignificantPart}(B)$;
    $B_0 \leftarrow \text{LessSignificantPart}(B)$;
    $X \leftarrow \textsc{Mult}(\mathrm{A}_0, \mathrm{B}_0)$;
    $Y \leftarrow \textsc{Mult}(\mathrm{A}_0, \mathrm{B}_1)$;
    $Z \leftarrow \textsc{Mult}(\mathrm{A}_1, \mathrm{B}_0)$;
    $W \leftarrow \textsc{Mult}(\mathrm{A}_1, \mathrm{B}_1)$;
    return $W\,2^n + (Y + Z)\,2^{n/2} + X$
  end if
end function

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$$T(n) = 4T\left(\frac{n}{2}\right) + c\,n \tag{2}$$

Claim: if $T(n)$ satisfies

$$T(n) = 4T\left(\frac{n}{2}\right) + c\,n \tag{3}$$

then

$$T(n) = n^2(c+1) - c\,n$$

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Some history: In 1952, one of the most famous mathematicians of the 20^{th} century, Andrey Kolmogorov, conjectured that you cannot multiply in less than $\Omega(n^2)$ elementary operations. In 1960, Karatsuba, then a 23-year-old student, found an algorithm (later it was called "divide and conquer") that multiplies two n-digit numbers in $\Theta(n^{\log_2 3}) \approx \Theta(n^{1.58\ldots})$ elementary steps, thus disproving the conjecture!! Kolmogorov was shocked!

The Karatsuba trick

How did Karatsuba do it??

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Take again our two input numbers A and B, and split them into two halves:

$$A = A_1 2^{\frac{n}{2}} + A_0 \qquad \underbrace{XX\dots X}_{\frac{n}{2}}\underbrace{XX\dots X}_{\frac{n}{2}}$$

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Thus, the algorithm will look like this:

function $\textsc{Mult}(A, B)$
    if $|A| = |B| = 1$ **then return** AB
    else
        $A_1 \leftarrow \text{MoreSignificantPart}(A);$
        $A_0 \leftarrow \text{LessSignificantPart}(A);$
        $B_1 \leftarrow \text{MoreSignificantPart}(B);$
        $B_0 \leftarrow \text{LessSignificantPart}(B);$
        $U \leftarrow A_0 + A_1;$
        $V \leftarrow B_0 + B_1;$
        $X \leftarrow \textsc{Mult}(\mathrm{A}_0, \mathrm{B}_0);$
        $W \leftarrow \textsc{Mult}(\mathrm{A}_1, \mathrm{B}_1);$
        $Y \leftarrow \textsc{Mult}(\mathrm{U}, \mathrm{V});$
        return $W\,2^n + (Y - X - W)\,2^{n/2} + X$
    end if
end function

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$$\approx 3^{\log_2 n}T(1) + c\,n\frac{\left(\frac{3}{2}\right)^{\log_2 n} - 1}{\frac{3}{2} - 1} = 3^{\log_2 n}T(1) + 2c\,n\left(\left(\frac{3}{2}\right)^{\log_2 n} - 1\right)$$

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Please review the basic properties of logarithms and the asymptotic notation from the review material (the first item at the class webpage under "class resources".)

Next time:

1. Can we multiply large integers faster than $O\left(n^{\log_2 3}\right)$??

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2. Can we avoid messy computations like:

$$
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T(n) &= 3T\left(\frac{n}{2}\right) + cn = 3\left(3T\left(\frac{n}{2^2}\right) + c\frac{n}{2}\right) + cn = 3^2T\left(\frac{n}{2^2}\right) + c\frac{3n}{2} + cn \\
&= 3^2\left(3T\left(\frac{n}{2^3}\right) + c\frac{n}{2^2}\right) + c\frac{3n}{2} + cn = 3^3T\left(\frac{n}{2^3}\right) + c\frac{3^2n}{2^2} + c\frac{3n}{2} + cn \\
&= 3^3T\left(\frac{n}{2^3}\right) + cn\left(\frac{3^2}{2^2} + \frac{3}{2} + 1\right) = \\
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&\ldots \\
&= 3^{\lfloor \log_2 n \rfloor}T\left(\frac{n}{\lfloor 2^{\log_2 n} \rfloor}\right) + cn\left(\left(\frac{3}{2}\right)^{\lfloor \log_2 n \rfloor - 1} + \ldots + \frac{3^2}{2^2} + \frac{3}{2} + 1\right) \\
&\approx 3^{\log_2 n}T(1) + cn\frac{\left(\frac{3}{2}\right)^{\log_2 n} - 1}{\frac{3}{2} - 1} \\
&= 3^{\log_2 n}T(1) + 2cn\left(\left(\frac{3}{2}\right)^{\log_2 n} - 1\right)
\end{aligned}
$$

That's All, Folks!!