

FFT Examples

Let's calculate the Discrete Fourier Transform (DFT) of the sequence $\vec{s} = \langle 1, 2, 3, 4, 5, 6, 7, 8 \rangle$ using the Fast Fourier Transform (FFT).

(Remember, the FFT is just a fast algorithm for computing the DFT, it is not a transform, just a method for computing the DFT efficiently!)

The associated polynomial is

$$P(x) = 1 + 2x + 3x^2 + 4x^3 + 5x^4 + 6x^5 + 7x^6 + 8x^7;$$

the $DFT(\vec{s})$ is just the sequence of values

$$DFT(\vec{s}) = \langle P(\omega_8^0), P(\omega_8^1), P(\omega_8^2), P(\omega_8^3), P(\omega_8^4), P(\omega_8^5), P(\omega_8^6), P(\omega_8^7) \rangle$$

Note that we have

$$\begin{aligned} \omega_8^0 &= 1; & \omega_8^1 &= \frac{\sqrt{2}}{2} + i \frac{\sqrt{2}}{2}; & \omega_8^2 &= i; & \omega_8^3 &= -\frac{\sqrt{2}}{2} + i \frac{\sqrt{2}}{2}; \\ \omega_8^4 &= -1; & \omega_8^5 &= -\frac{\sqrt{2}}{2} - i \frac{\sqrt{2}}{2}; & \omega_8^6 &= -i; & \omega_8^7 &= \frac{\sqrt{2}}{2} - i \frac{\sqrt{2}}{2}; \end{aligned}$$

see Figure 1:

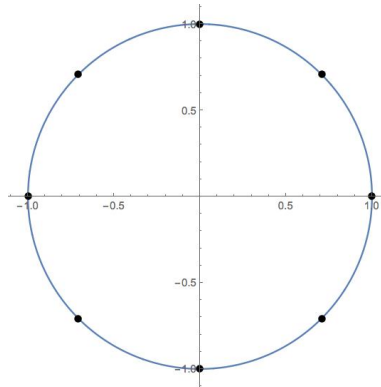


FIGURE 1.

Also,

$$\omega_4^0 = 1; \quad \omega_4^1 = i; \quad \omega_4^2 = -1; \quad \omega_4^3 = -i$$

$$\omega_2^0 = 1; \quad \omega_2^1 = -1.$$

We now proceed with the “Divide-And-Conquer” procedure: letting $y = x^2$ we get

$$\begin{aligned} P(x) &= 1 + 2x + 3x^2 + 4x^3 + 5x^4 + 6x^5 + 7x^6 + 8x^7 \\ &= (1 + 3x^2 + 5x^4 + 7x^6) + x(2 + 4x^2 + 6x^4 + 8x^6) \\ &= (1 + 3y + 5y^2 + 7y^3) + x(2 + 4y + 6y^2 + 8y^3). \end{aligned}$$

We now let

$$P_e(y) = 1 + 3y + 5y^2 + 7y^3; \quad P_o(y) = 2 + 4y + 6y^2 + 8y^3;$$

Continuing with “Divide-And-Conquer” strategy we get, letting $z = y^2$

$$\begin{aligned} P_e(y) &= (1 + 5y^2) + y(3 + 7y^2); & P_o(y) &= 2 + 6y^2 + y(4 + 8y^2); \\ &= (1 + 5z) + y(3 + 7z); & &= 2 + 6z + y(4 + 8z). \end{aligned}$$

We now let:

$$\begin{aligned} P_{ee}(z) &= 1 + 5z; & P_{eo}(z) &= 3 + 7z; \\ P_{oe}(z) &= 2 + 6z; & P_{oo}(z) &= 4 + 8z. \end{aligned}$$

Note that, as powers of ω_n rotate once around the circle, the powers of the squares $\omega_n^2 = \omega_{\frac{n}{2}}$ rotate twice through only $\frac{n}{2}$ many distinct values; for example, for $n=8$, using the cancelation Lemma, i.e., $(\omega_n^k)^2 = \omega_{\frac{n}{2}}^k$ we get:

$$\begin{array}{cccccccc}
x = & \omega_8^0 & \omega_8^1 & \omega_8^2 & \omega_8^3 & \omega_8^4 & \omega_8^5 & \omega_8^6 & \omega_8^7 \\
x^2 = & (\omega_8^0)^2 & (\omega_8^1)^2 & (\omega_8^2)^2 & (\omega_8^3)^2 & (\omega_8^4)^2 & (\omega_8^5)^2 & (\omega_8^6)^2 & (\omega_8^7)^2 \\
& \parallel & \parallel & \parallel & \parallel & \parallel & \parallel & \parallel & \parallel \\
& \omega_4^0 & \omega_4^1 & \omega_4^2 & \omega_4^3 & \omega_4^4 & \omega_4^5 & \omega_4^6 & \omega_4^7 \\
& \parallel & \parallel & \parallel & \parallel & \parallel & \parallel & \parallel & \parallel \\
& \omega_4^0 & \omega_4^1 & \omega_4^2 & \omega_4^3 & \omega_4^0 & \omega_4^1 & \omega_4^2 & \omega_4^3
\end{array}$$

Thus, if

$$(1) \quad P(x) = P^e(x^2) + x P^o(x^2)$$

and $P(x)$ is of degree $n - 1$, then for $k = 0$ to $\frac{n}{2} - 1$

$$(2) \quad P^e((\omega_n^k)^2) = P^e(\omega_{\frac{n}{2}}^k) \quad \text{and also} \quad P^e((\omega_{\frac{n}{2}}^{\frac{n}{2}+k})^2) = P^e(\omega_{\frac{n}{2}}^k);$$

$$(3) \quad P^o((\omega_n^k)^2) = P^o(\omega_{\frac{n}{2}}^k) \quad \text{and also} \quad P^o((\omega_{\frac{n}{2}}^{\frac{n}{2}+k})^2) = P^o(\omega_{\frac{n}{2}}^k);$$

$$(4) \quad \omega_{\frac{n}{2}}^{\frac{n}{2}+k} = \omega_{\frac{n}{2}}^{\frac{n}{2}} \omega_n^k = -\omega_n^k \quad .$$

Consequently, from (1)–(3) we have the following recursion for every n of the form $n = 2^m$:

if

- $\vec{s} = \langle s_0, s_1, s_2, s_3, \dots, s_n \rangle$;
 - $\vec{s}^e = \langle s_0, s_2, s_4, \dots, s_n \rangle$ and $\vec{s}^o = \langle s_1, s_3, s_5, \dots, s_{n-1} \rangle$
- (note: both of these sequences are of length $n/2$)

and if

- $DFT(\vec{s}) = \langle f_0, f_1, f_2, f_3, \dots, f_n \rangle$
- $DFT(\vec{s}^e) = \langle f_0^e, f_1^e, \dots, f_{\frac{n}{2}}^e \rangle$ and $DFT(\vec{s}^o) = \langle f_0^o, f_1^o, \dots, f_{\frac{n}{2}}^o \rangle$;

then for all $0 \leq k \leq \frac{n}{2} - 1$ we have

$$(5) \quad f_k = f_k^e + \omega_n^k f_k^o; \quad f_{\frac{n}{2}+k} = f_k^e + \omega_{\frac{n}{2}+k}^k f_k^o = f_k^e - \omega_n^k f_k^o;$$

We now compute

$$\begin{aligned} DFT(\langle 1, 5 \rangle) &= \langle P_{ee}(\omega_2^0), P_{ee}(\omega_2^1) \rangle = \langle P_{ee}(1), P_{ee}(-1) \rangle \\ &= \langle 1 + 5 \cdot 1, 1 + 5 \cdot (-1) \rangle = \langle 6, -4 \rangle \\ DFT(\langle 3, 7 \rangle) &= \langle P_{eo}(\omega_2^0), P_{eo}(\omega_2^1) \rangle = \langle P_{eo}(1), P_{eo}(-1) \rangle \\ &= \langle 3 + 7 \cdot 1, 3 + 7 \cdot (-1) \rangle = \langle 10, -4 \rangle \\ DFT(\langle 2, 6 \rangle) &= \langle P_{oe}(\omega_2^0), P_{oe}(\omega_2^1) \rangle = \langle P_{oe}(1), P_{oe}(-1) \rangle \\ &= \langle 2 + 6 \cdot 1, 2 + 6 \cdot (-1) \rangle = \langle 8, -4 \rangle \\ DFT(\langle 4, 8 \rangle) &= \langle P_{oo}(\omega_2^0), P_{oo}(\omega_2^1) \rangle = \langle P_{oo}(1), P_{oo}(-1) \rangle \\ &= \langle 4 + 8 \cdot 1, 4 + 8 \cdot (-1) \rangle = \langle 12, -4 \rangle \end{aligned}$$

We now use equations (5) with $n = 4$ to start putting things together:

$$\begin{aligned} DFT(\langle 1, 3, 5, 7 \rangle) &= \langle 6 + \omega_4^0 \cdot 10, -4 + \omega_4^1(-4), 6 - \omega_4^0 \cdot 10, -4 - \omega_4^1(-4) \rangle \\ &= \langle 6 + 10, -4 + i(-4), 6 - 10, -4 - i(-4) \rangle \\ &= \langle 16, -4 - 4i, -4, -4 + 4i \rangle \\ DFT(\langle 2, 4, 6, 8 \rangle) &= \langle 8 + \omega_4^0 \cdot 12, -4 + \omega_4^1(-4), 8 - \omega_4^0 \cdot 12, -4 - \omega_4^1(-4) \rangle \\ &= \langle 8 + 12, -4 + i(-4), 8 - 12, -4 - i(-4) \rangle \\ &= \langle 20, -4 - 4i, -4, -4 + 4i \rangle \end{aligned}$$

Finally we obtain from (5) with $n = 8$,

$$\begin{aligned} DFT(\langle 1, 2, 3, 4, 5, 6, 7, 8 \rangle) &= \langle 16 + \omega_8^0 \cdot 20, -4 - 4i + \omega_8^1(-4 - 4i), -4 + \omega_8^2 \cdot (-4), -4 + 4i + \omega_8^3(-4 + 4i), \\ &\quad 16 - \omega_8^0 \cdot 20, -4 - 4i - \omega_8^1(-4 - 4i), -4 - \omega_8^2 \cdot (-4), -4 + 4i - \omega_8^3(-4 + 4i) \rangle \end{aligned}$$

$$= \left\langle \begin{array}{cccccc} 16 + 1 \cdot 20, & -4 - 4i + \left(\frac{\sqrt{2}}{2} + i \frac{\sqrt{2}}{2} \right) (-4 - 4i), & -4 + i \cdot (-4), & -4 + 4i + \left(-\frac{\sqrt{2}}{2} + i \frac{\sqrt{2}}{2} \right) (-4 + 4i), \\ 16 - 1 \cdot 20, & -4 - 4i - \left(\frac{\sqrt{2}}{2} + i \frac{\sqrt{2}}{2} \right) (-4 - 4i), & -4 - i \cdot (-4), & -4 + 4i - \left(-\frac{\sqrt{2}}{2} + i \frac{\sqrt{2}}{2} \right) (-4 + 4i) \end{array} \right\rangle$$

$$= \langle 36, \quad -4 - 4i(1 + \sqrt{2}), \quad -4 - 4i, \quad -4 + 4i(1 - \sqrt{2}), \quad -4, \quad -4 - 4i(1 - \sqrt{2}), \quad -4 + 4i, \quad -4 + 4i(1 + \sqrt{2}) \rangle$$

Another example

Previous example might have resulted in confusingly symmetric calculations, so let us compute the DFT of the sequence $\langle 1, 8, 4, 3, 2, 5, 6, 7 \rangle$. The corresponding polynomial is now

$$\begin{aligned} Q(x) &= 1 + 8x + 4x^2 + 3x^3 + 2x^4 + 5x^5 + 6x^6 + 7x^7 \\ &= 1 + 4x^2 + 2x^4 + 6x^6 + x(8 + 3x^2 + 5x^4 + 7x^6). \end{aligned}$$

Substituting $y = x^2$ we get

$$Q(x) = 1 + 4y + 2y^2 + 6y^3 + x(8 + 3y + 5y^2 + 7y^3).$$

We let

$$Q_e(y) = 1 + 4y + 2y^2 + 6y^3; \quad Q_o(y) = 8 + 3y + 5y^2 + 7y^3;$$

We now have with substitution $z = y^2$

$$\begin{aligned} Q_e(y) &= 1 + 2y^2 + y(4 + 6y^2); & Q_o(y) &= 8 + 5y^2 + y(3 + 7y^2); \\ &= 1 + 2z + y(4 + 6z); & &= 8 + 5z + y(3 + 7z); \end{aligned}$$

We let

$$\begin{aligned} Q_{ee}(y) &= 1 + 2z; & Q_{eo}(y) &= 4 + 6z; \\ Q_{oe}(y) &= 8 + 5z; & Q_{oo}(y) &= 3 + 7z; \end{aligned}$$

We now have:

$$DFT(\langle 1, 2 \rangle) = \langle Q_{ee}(\omega_2^0), Q_{ee}(\omega_2) \rangle = \langle Q_{ee}(1), Q_{ee}(-1) \rangle = \langle 3, -1 \rangle;$$

$$DFT(\langle 4, 6 \rangle) = \langle Q_{eo}(\omega_2^0), Q_{eo}(\omega_2) \rangle = \langle Q_{eo}(1), Q_{eo}(-1) \rangle = \langle 10, -2 \rangle;$$

$$DFT(\langle 8, 5 \rangle) = \langle Q_{oe}(\omega_2^0), Q_{oe}(\omega_2) \rangle = \langle Q_{ee}(1), Q_{ee}(-1) \rangle = \langle 13, 3 \rangle;$$

$$DFT(\langle 3, 7 \rangle) = \langle Q_{oo}(\omega_2^0), Q_{oo}(\omega_2) \rangle = \langle Q_{ee}(1), Q_{ee}(-1) \rangle = \langle 10, -4 \rangle.$$

We now use equations (5) with $n = 4$ to start putting things together:

$$DFT(\langle 1, 4, 2, 6 \rangle) = \langle 3 + \omega_4^0 \cdot 10, \quad -1 + \omega_4^1(-2), \quad 3 - \omega_4^0 \cdot 10, \quad -1 - \omega_4^1(-2) \rangle$$

$$= \langle 3 + 10, \quad -1 + i(-2), \quad 3 - 10, \quad -1 - i(-2) \rangle$$

$$= \langle 13, \quad -1 - 2i, \quad -7, \quad -1 + 2i \rangle$$

$$DFT(\langle 8, 3, 5, 7 \rangle) = \langle 13 + \omega_4^0 \cdot 10, \quad 3 + \omega_4^1(-4), \quad 13 - \omega_4^0 \cdot 10, \quad 3 - \omega_4^1(-4) \rangle$$

$$= \langle 13 + 10, \quad 3 + i(-4), \quad 13 - 10, \quad 3 - i(-4) \rangle$$

$$= \langle 23, \quad 3 - 4i, \quad 3, \quad 3 + 4i \rangle$$

Finally, from (5) with $n = 8$ we obtain

$$DFT(\langle 1, 8, 4, 3, 2, 5, 6, 7 \rangle) = \langle 13 + \omega_8^0 \cdot 23, \quad -1 - 2i + \omega_8^1(3 - 4i), \quad -7 + \omega_8^2 \cdot 3, \quad -1 + 2i + \omega_8^3(3 + 4i),$$

$$13 - \omega_8^0 \cdot 23, \quad -1 - 2i - \omega_8^1(3 - 4i), \quad -7 - \omega_8^2 \cdot 3, \quad -1 + 2i - \omega_8^3(3 + 4i) \rangle$$

$$= \left\langle 13 + 1 \cdot 23, \quad -1 - 2i + \left(\frac{\sqrt{2}}{2} + i \frac{\sqrt{2}}{2} \right) (3 - 4i), \quad -7 + i \cdot 3, \quad -1 + 2i + \left(-\frac{\sqrt{2}}{2} + i \frac{\sqrt{2}}{2} \right) (3 + 4i), \right. \\ \left. 13 - 1 \cdot 23, \quad -1 - 2i - \left(\frac{\sqrt{2}}{2} + i \frac{\sqrt{2}}{2} \right) (3 - 4i), \quad -7 - i \cdot 3, \quad -1 + 2i - \left(-\frac{\sqrt{2}}{2} + i \frac{\sqrt{2}}{2} \right) (3 + 4i) \right\rangle$$

$$= \left\langle 36, \quad \frac{-2 + 7\sqrt{2}}{2} - i \frac{4 + \sqrt{2}}{2}, \quad -7 + 3i, \quad \frac{-2 - 7\sqrt{2}}{2} + i \frac{4 - \sqrt{2}}{2}, \right. \\ \left. -10, \quad \frac{-2 - 7\sqrt{2}}{2} - i \frac{4 - \sqrt{2}}{2}, \quad -7 - 3i, \quad \frac{-2 + 7\sqrt{2}}{2} + i \frac{4 + \sqrt{2}}{2} \right\rangle$$