Algorithms: COMP3121/3821/9101/9801

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LECTURE 9: LINEAR PROGRAMMING

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  - the price of all food per day is as low as possible.

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- Note that here all the equalities and inequalities, as well as the objective function, are **linear**. This problem is a typical example of a **Linear Programming problem**.

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- You wish to win the election by cleverly making a promise that **appears** that it will blow as small hole in the budget as possible.

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0.05x_b + 0.12x_p &\geq 0.51 \quad \text{(securing majority of city votes)}\\
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- In fact, Integer Linear Programming problems are NP hard!

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- Let the boldface $\mathbf{x}$ represent a (column) vector, $\mathbf{x} = \langle x_1 \dots x_n \rangle^{\intercal}$.
- To get a more compact representation of linear programs we introduce a partial ordering on vectors $\mathbf{x} \in \mathbf{R}^n$ by $\mathbf{x} \leq \mathbf{y}$ if and only if the corresponding inequalities hold coordinate-wise, i.e., if and only if $x_j \leq y_j$ for all $1 \leq j \leq n$.

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  - maximize $\mathbf{c}^\intercal \mathbf{x}$
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- Thus, to specify a Linear Programming optimisation problem we just have to provide a triplet $(A, \mathbf{b}, \mathbf{c})$;
- This is the usual form which is accepted by most standard LP solvers.

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- Equality constraints of the form $\sum_{i=1}^{n} a_{ij}x_i = b_j$ can be replaced by two inequalities: $\sum_{i=1}^{n} a_{ij}x_i \geq b_j$ and $\sum_{i=1}^{n} a_{ij}x_i \leq b_j$; thus, we can assume that all constraints are inequalities.

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- However, in the standard form such constraints are required for all of the variables.
- This poses no problem, because each occurrence of an unconstrained variable x_j can be replaced by the expression $x'_j - x^*_j$ where x'_j, x^*_j are new variables satisfying the constraints $x'_j \geq 0,\ x^*_j \geq 0$.

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- Also, some problems are naturally translated into constraints of the form $|A\mathbf{x}| \leq \mathbf{b}$. This also poses no problem because we can replace such constraints with two linear constraints: $A\mathbf{x} \leq \mathbf{b}$ and $-A\mathbf{x} \leq \mathbf{b}$ because $|x| \leq y$ if and only if $x \leq y$ and $-x \leq y$.

Linear Programming - Standard Form

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- Any vector $\mathbf{x}$ which satisfies the two constraints is called a *feasible solution*, regardless of what the corresponding objective value $\mathbf{c}^\intercal \mathbf{x}$ might be.

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$$\text{maximize} \quad z(x_1, x_2, x_3) = 3x_1 + x_2 + 2x_3 \tag{3}$$

subject to the constraints

$$x_1 + x_2 + 3x_3 \leq 30 \tag{4}$$

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y_1 + 2y_2 + 4y_3 &\ge 3\\
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Combining this with (9) we get:

$$30y_1 + 24y_2 + 36y_3 \geq 3x_1 + x_2 + 2x_3 = z(x_1, x_2, x_3)$$

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- Consequently, in order to find as tight upper bound for our objective $z(x_1, x_2, x_3)$ of the problem P:

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we have to look for y_1, y_2, y_3 which

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- The new problem is called the *dual problem* P^* for the original problem P.

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- It is now useful to remember how we proved that the Ford - Fulkerson Max Flow algorithm in fact produces a maximal flow, by showing that it terminates only when we reach the capacity of a (minimal) cut...

Linear Programming - primal/dual problem forms

- The original, *primal* Linear Program P and its *dual* Linear Program can be easily described in the most general case:

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\begin{aligned}
P: \quad & \text{maximize} & z(\mathbf{x}) = \sum_{j=1}^{n} c_j x_j, & \\
& \text{subject to the constraints} & \sum_{j=1}^{n} a_{ij} x_j \leq b_i; & \quad 1 \leq i \leq m \\
& & x_1, \dots, x_n \geq 0; &
\end{aligned}
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or, in matrix form,

$$
\begin{aligned}
P: \quad & \text{maximize} \quad z(\mathbf{x}) = \mathbf{c}^\intercal \mathbf{x}, \ \text{subject to the constraints} \quad A\mathbf{x} \le \mathbf{b} \ \text{and} \ \mathbf{x} \ge 0; \\
P^*: \quad & \text{minimize} \quad z^*(\mathbf{y}) = \mathbf{b}^\intercal \mathbf{y}, \ \text{subject to the constraints} \quad A^\intercal \mathbf{y} \ge \mathbf{c} \ \text{and} \ \mathbf{y} \ge 0.
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Weak Duality Theorem

- Recall that any vector $\mathbf{x}$ which satisfies the two constraints, $A\mathbf{x} \leq \mathbf{b}$ and $\mathbf{x} \geq 0$ is called a *feasible solution*, regardless of what the corresponding objective value $\mathbf{c}^\intercal \mathbf{x}$ might be.

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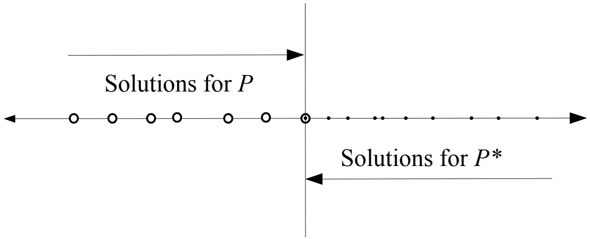

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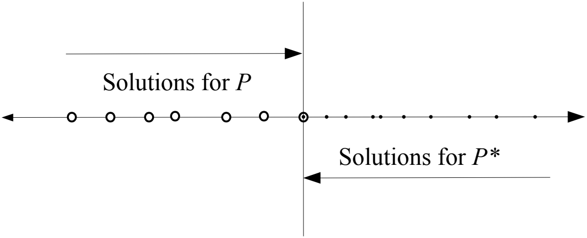

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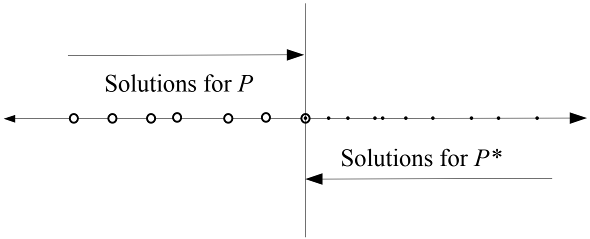

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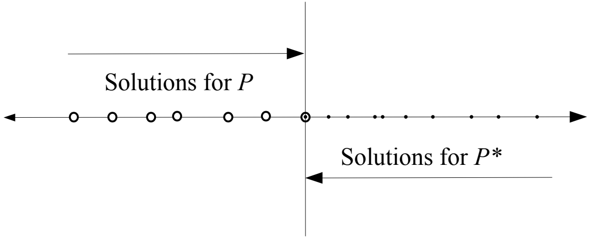

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- See the Lecture Notes for the details and an example of how the SIMPLEX algorithm runs.

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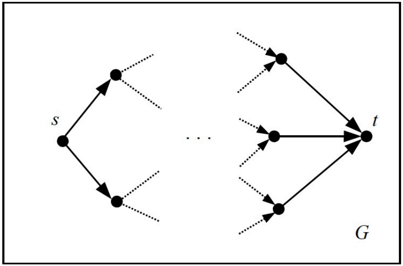

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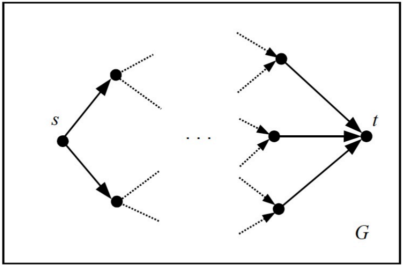

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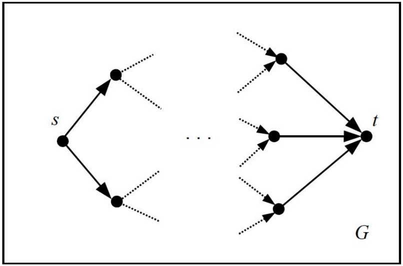

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$$C^* \begin{cases} f_{ij} & \leq \kappa_{ij}; \ (i, j) \in G; \quad \text{(flow smaller than pipe's capacity)} \\ \sum_{i\,:\,(i,j)\in G} f_{ij} & = \sum_{k\,:\,(j,k)\in G} f_{jk}; \ j \in G - \{s, t\}; \quad \text{(incoming flow equals outgoing)} \\ f_{ij} & \geq 0; \ (i, j) \in G \text{ (no negative flows).} \end{cases}$$

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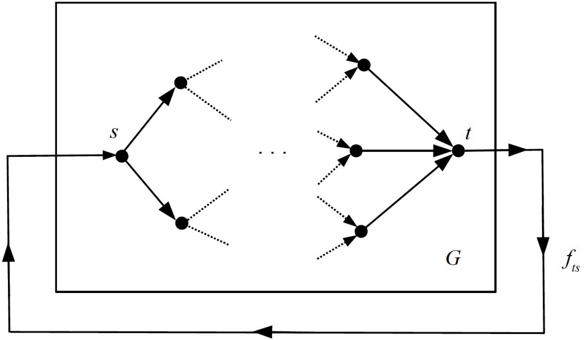
- We now have a new graph G', $G \subset G'$, with an additional edge $(t, s) \in G'$ with capacity ∞.



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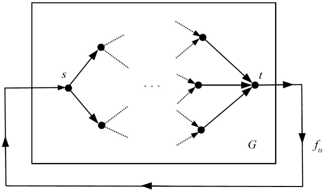

- We can now formulate the Max Flow problem as a Linear Program by replacing the equality in the second constraint with a single but equivalent inequality:

P: *maximize:* f_{ts}
subject to the constraints:

$$\begin{aligned} f_{ij} &\leq \kappa_{ij}; & (i,j) &\in G; \\ \sum_{i\,:\,(i,j)\in G'} f_{ij} - \sum_{k\,:\,(j,k)\in G'} f_{jk} &\leq 0; & j &\in G; \\ f_{ij} &\geq 0; & (i,j) &\in G'. \end{aligned}$$

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- The coefficients c_{ij} of the objective of the primal P are zero for all variables f_{ij} except for f_{ts} which is equal to 1, i.e.,

$$z(\mathbf{f}) = \sum_{ij} 0 \cdot f_{ij} + 1 \cdot f_{ts} \tag{19}$$

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- To obtain the dual of P we look for coefficients d_{ij}, $(i,j) \in G$, corresponding to the first set of constraints, and coefficients p_j, $j \in G$ corresponding to the second set of constraints to use as multipliers:

$$f_{ij} d_{ij} \leq \kappa_{ij}\, d_{ij}; \qquad (i,j) \in G; \tag{20}$$

$$\sum_{i\,:\,(i,j)\in G'} f_{ij} p_j - \sum_{k\,:\,(j,k)\in G'} f_{jk} p_j \leq 0; \qquad j \in G. \tag{21}$$

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- Summing up all inequalities in (20) and (21) and factoring out, we get

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- Thus, the minimum value of the dual objective $\sum_{(i,j)\in G} \kappa_{ij} d_{ij}$ precisely corresponds to the capacity of the cut defined by A, B.
- Since such value of the dual problem is equal to the maximal value of the flow defined by the primal problem, we have obtained a maximal flow and a minimal cut in G!

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- In fact, one can show that in the general case the ILP problems are "NP hard", and what this means will be explained next week.

Examples of dual programs: Max Flow

- In this particular case of the program P^*, due to the particular form of the constraints, the extremum of the objective was achieved with all the variables d_{ij} and $p_{i,j}$ taking integer values.
- This is usually NOT the case, even if all the coefficients in the constraints are integers.
- If we impose an additional constraint that all the variables must take integer values (i.e., if the problem is an ILP problem), the problem is most likely intractable, i.e., it cannot be solved in polynomial time.
- In fact, one can show that in the general case the ILP problems are "NP hard", and what this means will be explained next week.
- Even for such problems there are quite a few solvers (mostly based on heuristic search), but of course they are not guaranteed to return true optimum value in polynomial time...