



Algorithms: COMP3121/3821/9101/9801

Aleks Ignjatović

School of Computer Science and Engineering
University of New South Wales

TOPIC 2: FAST LARGE INTEGER MULTIPLICATION

Basics revisited: how do we multiply two numbers?

- The primary school algorithm:

```

      X X X X  <- first input integer
*   X X X X  <- second input integer
-----
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- Can we do it faster than in n^2 many steps??

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$$\begin{aligned} AB &= A_1 B_1 2^n + (A_1 B_0 + A_0 B_1) 2^{\frac{n}{2}} + A_0 B_0 \\ &= A_1 B_1 2^n + ((A_1 + A_0)(B_1 + B_0) - A_1 B_1 - A_0 B_0) 2^{\frac{n}{2}} + A_0 B_0 \end{aligned}$$

```

1: function MULT( $A, B$ )
2:   if  $|A| = |B| = 1$  then return  $AB$ 
3:   else
4:      $A_1 \leftarrow \text{MoreSignificantPart}(A)$ ;
5:      $A_0 \leftarrow \text{LessSignificantPart}(A)$ ;
6:      $B_1 \leftarrow \text{MoreSignificantPart}(B)$ ;
7:      $B_0 \leftarrow \text{LessSignificantPart}(B)$ ;
8:      $U \leftarrow A_0 + A_1$ ;
9:      $V \leftarrow B_0 + B_1$ ;
10:     $X \leftarrow \text{MULT}(A_0, B_0)$ ;
11:     $W \leftarrow \text{MULT}(A_1, B_1)$ ;
12:     $Y \leftarrow \text{MULT}(U, V)$ ;
13:    return  $W 2^n + (Y - X - W) 2^{n/2} + X$ 
14:  end if
15: end function

```

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- Thus, the first case of the Master Theorem applies.
- Consequently,

$$T(n) = \Theta(n^{\log_2 3}) < \Theta(n^{1.585})$$

without going through the messy calculations!

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- So,

$$AB = A_2 B_2 2^{4k} + (A_2 B_1 + A_1 B_2) 2^{3k} + (A_2 B_0 + A_1 B_1 + A_0 B_2) 2^{2k} + (A_1 B_0 + A_0 B_1) 2^k + A_0 B_0$$

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- Should we perhaps look at

$$(A_2 + A_1 + A_0)(B_2 + B_1 + B_0) = A_0B_0 + A_1B_0 + A_2B_0 + A_0B_1 + A_1B_1 + A_2B_1 + A_0B_2 + A_1B_2 + A_2B_2 \quad ???$$

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- Not clear at all how to get $C_0 - C_4$ with 5 multiplications only ...

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$$P_A(x) = A_2 x^2 + A_1 x + A_0;$$

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- Note that

$$A = A_2 (2^k)^2 + A_1 2^k + A_0 = P_A(2^k);$$

$$B = B_2 (2^k)^2 + B_1 2^k + B_0 = P_B(2^k).$$

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with only 5 multiplications, we can then obtain the product of numbers A and B simply as

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- We choose **the smallest possible 5 integer values** (smallest by their absolute value), i.e., $-2, -1, 0, 1, 2$.
- Thus, we compute $P_A(-2), P_A(-1), P_A(0), P_A(1), P_A(2)$
 $P_B(-2), P_B(-1), P_B(0), P_B(1), P_B(2)$

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- For $P_A(x) = A_2x^2 + A_1x + A_0$ we have

$$P_A(-2) = A_2(-2)^2 + A_1(-2) + A_0 = 4A_2 - 2A_1 + A_0$$

$$P_A(-1) = A_2(-1)^2 + A_1(-1) + A_0 = A_2 - A_1 + A_0$$

$$P_A(0) = A_20^2 + A_10 + A_0 = A_0$$

$$P_A(1) = A_21^2 + A_11 + A_0 = A_2 + A_1 + A_0$$

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- Similarly, for $P_B(x) = B_2x^2 + B_1x + B_0$ we have

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- These evaluations involve only additions because $2A = A + A$; $4A = 2A + 2A$.

The Karatsuba trick: slicing into 3 pieces

- Having obtained $P_A(-2), P_A(-1), P_A(0), P_A(1), P_A(2)$ and $P_B(-2), P_B(-1), P_B(0), P_B(1), P_B(2)$ we can now obtain $P_C(-2), P_C(-1), P_C(0), P_C(1), P_C(2)$ with only 5 multiplications of large numbers:

$$\begin{aligned}P_C(-2) &= P_A(-2)P_B(-2) \\ &= (A_0 - 2A_1 + 4A_2)(B_0 - 2B_1 + 4B_2)\end{aligned}$$

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$$\begin{aligned}P_C(0) &= P_A(0)P_B(0) \\ &= A_0B_0\end{aligned}$$

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$$\begin{aligned}P_C(2) &= P_A(2)P_B(2) \\ &= (A_0 + 2A_1 + 4A_2)(B_0 + 2B_1 + 4B_2)\end{aligned}$$

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- Thus, if we represent the product $C(x) = P_A(x)P_B(x)$ in the coefficient form as $C(x) = C_4x^4 + C_3x^3 + C_2x^2 + C_1x + C_0$ we get

$$C_4(-2)^4 + C_3(-2)^3 + C_2(-2)^2 + C_1(-2) + C_0 = P_C(-2) = P_A(-2)P_B(-2)$$

$$C_4(-1)^4 + C_3(-1)^3 + C_2(-1)^2 + C_1(-1) + C_0 = P_C(-1) = P_A(-1)P_B(-1)$$

$$C_40^4 + C_30^3 + C_20^2 + C_1 \cdot 0 + C_0 = P_C(0) = P_A(0)P_B(0)$$

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The Karatsuba trick: slicing into 3 pieces

- Thus, if we represent the product $C(x) = P_A(x)P_B(x)$ in the coefficient form as $C(x) = C_4x^4 + C_3x^3 + C_2x^2 + C_1x + C_0$ we get

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- Simplifying the left side we obtain

$$16C_4 - 8C_3 + 4C_2 - 2C_1 + C_0 = P_C(-2)$$

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- Here is the complete algorithm:

```

1: function MULT(A, B)
2:   obtain  $A_0, A_1, A_2$  and  $B_0, B_1, B_2$  such that  $A = A_2 2^{2^k} + A_1 2^k + A_0$ ;  $B = B_2 2^{2^k} + B_1 2^k + B_0$ ;
3:   form polynomials  $P_A(x) = A_2 x^2 + A_1 x + A_0$ ;  $P_B(x) = B_2 x^2 + B_1 x + B_0$ ;
4:
       $P_A(-2) \leftarrow 4A_2 - 2A_1 + A_0$ 
       $P_B(-2) \leftarrow 4B_2 - 2B_1 + B_0$ 
       $P_A(-1) \leftarrow A_2 - A_1 + A_0$ 
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5:
       $P_C(-2) \leftarrow \text{MULT}(P_A(-2), P_B(-2));$ 
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7:   form  $P_C(x) = C_4 x^4 + C_3 x^3 + C_2 x^2 + C_1 x + C_0$ ; compute
       $P_C(2^k) = C_4 2^{4k} + C_3 2^{3k} + C_2 2^{2k} + C_1 2^k + C_0$ 

8:   return  $P_C(2^k) = A \cdot B$ .

9: end function

```

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分成3份 5个多项式

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- Clearly, the first case of the MT applies and we get
 $T(n) = O(n^{\log_3 5}) < O(n^{1.47})$.

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- The answer is, in a sense, BOTH yes and no, so lets see what happens if we slice numbers into $n + 1$ many equal slices...

Generalizing Karatsuba's algorithm

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$$\begin{aligned} A &= A_n 2^{kn} + A_{n-1} 2^{k(n-1)} + \dots + A_0 \\ B &= B_n 2^{kn} + B_{n-1} 2^{k(n-1)} + \dots + B_0 \end{aligned}$$

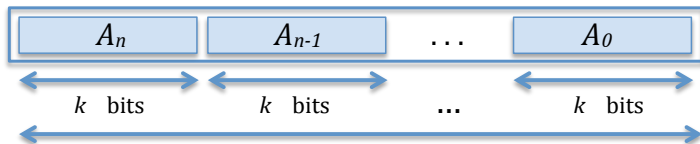
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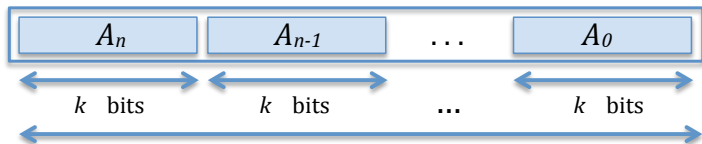
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- We form the naturally corresponding polynomials:

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$$A = P_A(2^k); \quad B = P_B(2^k); \quad AB = P_A(2^k)P_B(2^k) = (P_A(x) \cdot P_B(x))|_{x=2^k}$$

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Generalizing Karatsuba's algorithm

- Example:

$$\begin{aligned}(a_3x^3 + a_2x^2 + a_1x + a_0)(b_3x^3 + b_2x^2 + b_1x + b_0) = \\ a_3b_3x^6 + (a_2b_3 + a_3b_2)x^5 + (a_1b_3 + a_2b_2 + a_3b_1)x^4 \\ + (a_0b_3 + a_1b_2 + a_2b_1 + a_3b_0)x^3 + (a_0b_2 + a_1b_1 + a_2b_0)x^2 \\ + (a_0b_1 + a_1b_0)x + a_0b_0\end{aligned}$$

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$$\begin{aligned}(a_3x^3 + a_2x^2 + a_1x + a_0)(b_3x^3 + b_2x^2 + b_1x + b_0) = \\ a_3b_3x^6 + (a_2b_3 + a_3b_2)x^5 + (a_1b_3 + a_2b_2 + a_3b_1)x^4 \\ + (a_0b_3 + a_1b_2 + a_2b_1 + a_3b_0)x^3 + (a_0b_2 + a_1b_1 + a_2b_0)x^2 \\ + (a_0b_1 + a_1b_0)x + a_0b_0\end{aligned}$$

- In general: for

$$P_A(x) = A_nx^n + A_{n-1}x^{n-1} + \cdots + A_0$$

$$P_B(x) = B_nx^n + B_{n-1}x^{n-1} + \cdots + B_0$$

we have

$$P_A(x) \cdot P_B(x) = \sum_{j=0}^{2n} \left(\sum_{i+k=j} A_i B_k \right) x^j = \sum_{j=0}^{2n} C_j x^j$$

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- We need to find the coefficients $C_j = \sum_{i+k=j} A_i B_k$ without performing $(n+1)^2$ many multiplications necessary to get all products of the form $A_i B_k$.

A VERY IMPORTANT DIGRESSION:

If you have two sequences $\vec{A} = (A_0, A_1, \dots, A_{n-1}, A_n)$ and $\vec{B} = (B_0, B_1, \dots, B_{m-1}, B_m)$, and if you form the two corresponding polynomials

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then the sequence $\vec{C} = (C_0, C_1, \dots, C_{n+m})$ of the coefficients of the product polynomial, with these coefficients given by

$$C_j = \sum_{i+k=j} A_i B_k, \quad \text{for } 0 \leq j \leq n+m,$$

is **extremely important** and is called the **LINEAR CONVOLUTION** of sequences \vec{A} and \vec{B} and is denoted by $\vec{C} = \vec{A} \star \vec{B}$.

回旋

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- In signal processing these degrees can be greater than 1000.
- This is the main reason for us to study methods of fast computation of convolutions (aside of finding products of large integers, which is what we are doing at the moment).

Coefficient vs value representation of polynomials

- Every polynomial $P_A(x)$ of degree n is uniquely determined by its values at any $n + 1$ distinct input values x_0, x_1, \dots, x_n :

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- It can be shown that if x_i are all distinct then this matrix is invertible.
- Such a matrix is called *the Vandermonde matrix*.

Coefficient vs value representation of polynomials - ctd.

- Thus, if all x_i are all distinct, given any values $P_A(x_0), P_A(x_1), \dots, P_A(x_n)$ the coefficients A_0, A_1, \dots, A_n of the polynomial $P_A(x)$ are uniquely determined:

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Coefficient vs value representation of polynomials- ctd.

- If we fix the inputs x_0, x_1, \dots, x_n then commuting between a representation of a polynomial $P_A(x)$ via its coefficients and a representation via its values at these points is done via the following two matrix multiplications, with matrices made up from **constants**:

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- Thus, for fixed input values x_0, \dots, x_n this switch between the two kinds of representations is done in **linear time**!

Our strategy to multiply polynomials fast:

- 1 Given two polynomials of degree at most n ,

$$P_A(x) = A_n x^n + \dots + A_0; \quad P_B(x) = B_n x^n + \dots + B_0$$

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- 3 Convert such value representation of $P_C(x) = P_A(x)P_B(x)$ back to coefficient form

$$P_C(x) = C_{2n}x^{2n} + C_{2n-1}x^{2n-1} + \dots + C_1x + C_0;$$

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$$P_A(m) = A_n m^n + A_{n-1} m^{n-1} + \dots + A_0 : \quad -n \leq m \leq n,$$

$$P_B(m) = B_n m^n + B_{n-1} m^{n-1} + \dots + B_0 : \quad -n \leq m \leq n.$$

can be found in time linear in the number of bits of the input numbers!

Fast multiplication of polynomials - ctd.

- We now perform $2n + 1$ **multiplications of large numbers** to obtain

$$P_A(-n)P_B(-n), \dots, P_A(-1)P_B(-1), P_A(0)P_B(0), P_A(1)P_B(1), \dots, P_A(n)P_B(n)$$

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- For $P_C(x) = P_A(x)P_B(x)$ these products are $2n + 1$ many values of $P_C(x)$:

$$P_C(-n) = P_A(-n)P_B(-n), \dots, P_C(0) = P_A(0)P_B(0), \dots, P_C(n) = P_A(n)P_B(n)$$

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- Let C_0, C_1, \dots, C_{2n} be the coefficients of the product polynomial $C(x)$, i.e., let

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$$C_{2n}(-n)^{2n} + C_{2n-1}(-n)^{2n-1} + \dots + C_0 = P_C(-n)$$

$$C_{2n}(-(n-1))^{2n} + C_{2n-1}(-(n-1))^{2n-1} + \dots + C_0 = P_C(-(n-1))$$

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- This is just a system of linear equations, that can be solved for C_0, C_1, \dots, C_{2n} :

$$\begin{pmatrix} 1 & -n & (-n)^2 & \dots & (-n)^{2n} \\ 1 & -(n-1) & (-(n-1))^2 & \dots & (-(n-1))^{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & n & n^2 & \dots & n^{2n} \end{pmatrix} \begin{pmatrix} C_0 \\ C_1 \\ \vdots \\ C_{2n} \end{pmatrix} = \begin{pmatrix} P_C(-n) \\ P_C(-(n-1)) \\ \vdots \\ P_C(n) \end{pmatrix},$$

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- So here is the algorithm we have just described:

```

1: function MULT( $n, A, B$ )
2:   if  $|A| = |B| = 1$  then return  $AB$ 
3:   else
4:     obtain  $n + 1$  slices  $A_0, A_1, \dots, A_n$  and  $B_0, B_1, \dots, B_n$  such that

```

$$A = A_n 2^{n k} + A_{n-1} 2^{(n-1) k} + \dots + A_0$$

$$B = B_n 2^{n k} + B_{n-1} 2^{(n-1) k} + \dots + B_0$$

```

5:   form polynomials

```

$$P_A(x) = A_n x^n + A_{n-1} x^{(n-1)} + \dots + A_0$$

$$P_B(x) = B_n x^n + B_{n-1} x^{(n-1)} + \dots + B_0$$

```

6:   for  $m = -n$  to  $m = n$  do
7:     compute  $P_A(m)$  and  $P_B(m)$ ;
8:      $P_C(m) \leftarrow \text{MULT}(n, P_A(m)P_B(m))$ 
9:   end for
10:  compute  $C_0, C_1, \dots, C_{2n}$  via

```

$$\begin{pmatrix} C_0 \\ C_1 \\ \vdots \\ C_{2n} \end{pmatrix} = \begin{pmatrix} 1 & -n & (-n)^2 & \dots & (-n)^{2n} \\ 1 & -(n-1) & (-(n-1))^2 & \dots & (-(n-1))^{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & n & n^2 & \dots & n^{2n} \end{pmatrix}^{-1} \begin{pmatrix} P_C(-n) \\ P_C(-(n-1)) \\ \vdots \\ P_C(n) \end{pmatrix}.$$

```

11:   form  $P_C(x) = C_{2n} x^{2n} + \dots + C_0$  and compute  $P_C(2^k)$ 
12:   return  $P_C(2^k) = A \cdot B$ 
13: end if
14: end function

```

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- it is easy to see that the values of the two polynomials we are multiplying have at most $k + s$ bits where s is a constant which depends on n but does NOT depend on k :

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This is because each A_i is smaller than 2^k because each A_k has k bits; thus

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- Since s is constant, its impact can be neglected.

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- so we get:

$$T(N) = \Theta(N^{\log_b a}) = \Theta(N^{\log_{n+1}(2n+1)})$$

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$$\begin{aligned} N^{\log_{n+1}(2n+1)} &< N^{\log_{n+1} 2(n+1)} = N^{\log_{n+1} 2 + \log_{n+1}(n+1)} \\ &= N^{1 + \log_{n+1} 2} = N^{1 + \frac{1}{\log_2(n+1)}} \end{aligned}$$

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- Thus, we would have to slice the input numbers into $2^{10} = 1024$ pieces!!

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- Consequently, slicing the input numbers in more than just a few slices results in a hopelessly slow algorithm, despite the fact that the asymptotic bounds improve as we increase the number of slices!

- We would have to evaluate polynomials $P_A(x)$ and $P_B(x)$ both of degree n at values up to n .
- However, $n = 2^{10}$, so evaluating $P_A(n) = A_n n^n + \dots + A_0$ involves multiplication of A_n with $n^n = (2^{10})^{2^{10}} \approx 1.27 \times 10^{3079}$.
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- Consequently, slicing the input numbers in more than just a few slices results in a hopelessly slow algorithm, despite the fact that the asymptotic bounds improve as we increase the number of slices!
- ✓ The moral is: **In practice, asymptotic estimates are useless if the size of the constants hidden by the O -notation are not estimated and found to be reasonably small!!!**

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- After we study the FFT we will have a guest lecture by a Dolby engineer to demonstrate to you some cool applications of FFT.