Algorithms: COMP3121/3821/9101/9801

Aleks Ignjatović

School of Computer Science and Engineering
University of New South Wales

TOPIC 5: DYNAMIC PROGRAMMING

Dynamic Programming

- **The main idea of Dynamic Programming:** build an optimal solution to the problem from optimal solutions for (carefully chosen) subproblems;

1.how to generate subproblem
2.how to order them

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- Subproblems are chosen in a way which allows recursive construction of optimal solutions to problems from optimal solutions to smaller size problems.
- Efficiency of DP comes from the fact that that the sets of subproblems needed to solve larger problems heavily overlap; each subproblem is solved only once and its solution is stored in a table for multiple use for solving many larger problems.

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- Note: the role of condition 2 is to simplify recursion.

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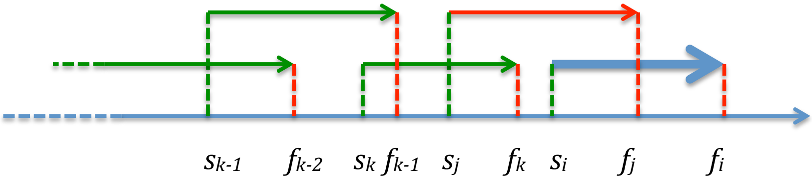

- In the table, besides $T(i)$, we also store j for which the above max is achieved.

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- Why? We apply the same "cut and paste" argument which we used to prove the optimality of the greedy solutions!
- If there were a sequence S^* of a larger total duration than the duration of sequence S' and also ending with activity $a_{k_{m-1}}$, we could obtain a sequence $\hat{S}$ by extending the sequence S^* with activity a_{k_m} and obtain a solution for subproblem $P(i)$ with a longer total duration than the total duration of sequence S, contradicting the optimality of S.

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- Time complexity: having sorted the activities by their finishing times in time $O(n \log n)$, we need to examine n intervals in the role of the last activity in an optimal sub-sequence and for each such interval we have to find all preceding compatible intervals and their optimal solutions (to be looked up in a table). Thus, the time complexity is $O(n^2)$.

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- Again, the condition for the sequence to end with $A[i]$ is not restrictive because if the optimal solution ends with some $A[m]$, it would have been constructed as the solution for $P(m)$.
- Time complexity: $O(n^2)$.
- Exercise: (somewhat tough, but very useful) Design an algorithm for solving this problem which runs in time $n \log n$.

- **Making Change.** You are given n types of coin denominations of values $v(1) < v(2) < ... < v(n)$ (all integers). Assume $v(1) = 1$, so that you can always make change for any integer amount. Give an algorithm which makes change for any given integer amount C with as few coins as possible, assuming that you have an unlimited supply of coins of each denomination.

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- Assume we have found optimal solutions for every amount $j < i$ and now want to find an optimal solution for amount i.

More Dynamic Programming Problems

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- Why does this produce an optimal solution for amount $i \leq C$?
- Consider an optimal solution for amount $i \leq C$; and say such solution includes at least one coin of denomination $v(m)$ for some $1 \leq m \leq n$.

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- Why does this produce an optimal solution for amount $i \leq C$?

- Consider an optimal solution for amount $i \leq C$; and say such solution includes at least one coin of denomination $v(m)$ for some $1 \leq m \leq n$. But then removing such a coin must produce an optimal solution for the amount $i - v(m)$ again by our cut-and-paste argument.

More Dynamic Programming Problems

- We consider optimal solutions $opt(i - v(k))$ for every amount of the form $i - v(k)$, where k ranges from 1 to n. (Recall $v(1), \dots, v(n)$ are all of the available denominations.)

- Among all of these optimal solutions (which we find in the table we are constructing recursively!) we pick one which uses the fewest number of coins, say this is $opt(i - v(m))$ for some m, $1 \leq m \leq n$.

- We obtain an optimal solution $opt(i)$ for amount i by adding to $opt(i - v(m))$ one coin of denomination $v(m)$.

- Why does this produce an optimal solution for amount $i \leq C$?

- Consider an optimal solution for amount $i \leq C$; and say such solution includes at least one coin of denomination $v(m)$ for some $1 \leq m \leq n$. But then removing such a coin must produce an optimal solution for the amount $i - v(m)$ again by our cut-and-paste argument.

- However, we do not know which coins the optimal solution includes, so we try all the available coins and then pick m for which the optimal solution for amount $i - v(m)$ uses the fewest number of coins.

- It is enough to store in the i^{th} slot of the table such m and $opt(i)$ because this allows us to reconstruct the optimal solution by looking at m_1 stored in the i^{th} slot, then look at m_2 stored in the slot $i - v(m_1)$, then look at m_2 stored in the slot $i - v(m_1) - v(m_2)$, etc.

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More Dynamic Programming Problems

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- But this is the best what we can do...

More Dynamic Programming Problems

Integer Knapsack Problem (Duplicate Items Allowed) You have n types of items; all items of kind i are identical and of weight w_i and value v_i. You also have a knapsack of capacity C. Choose a combination of available items which all fit in the knapsack and whose value is as large as possible. You can take any number of items of each kind.

More Dynamic Programming Problems

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- Add to such optimal solution for the knapsack of size $i - w_m$ item m to obtain a packing of a knapsack of size i of the highest possible value.

More Dynamic Programming Problems

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- **Integer Knapsack Problem (Duplicate Items NOT Allowed)** You have n items (some of which can be identical); item I_i is of weight w_i and value v_i. You also have a knapsack of capacity C. Choose a combination of available items which all fit in the knapsack and whose value is as large as possible.

More Dynamic Programming Problems

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- This is an example of a "2D" recursion; we will be filling a table of size $n \times C$, row by row; subproblems $P(i, c)$ for all $i \leq n$ and $c \leq C$ will be of the form:

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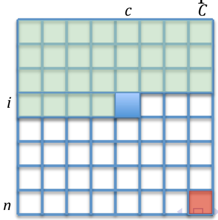

More Dynamic Programming Problems

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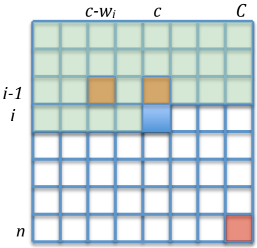

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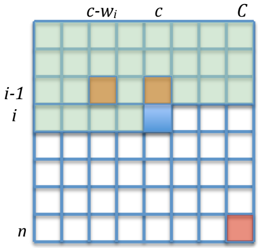

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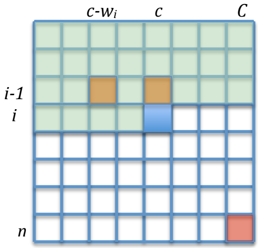

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  else $opt(i, c) = opt(i-1, c)$.
- Final solution will be given by $opt(n, C)$.

More Dynamic Programming Problems

- **Balanced Partition** You have a set of n integers. Partition these integers into two subsets such that you minimise $|S_1 - S_2|$, where S_1 and S_2 denote the sums of the elements in each of the two subsets.

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- Why? Since $S = S_1 + S_2$ we obtain

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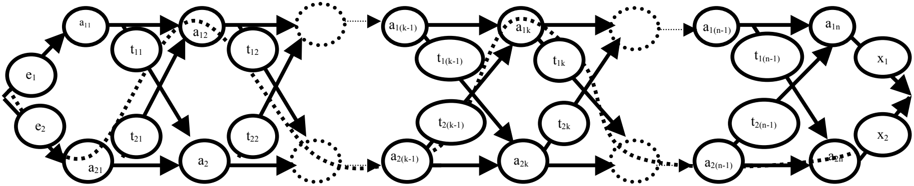
- Thus, minimising $S/2 - S_1$ will minimise $S_2 - S_1$.

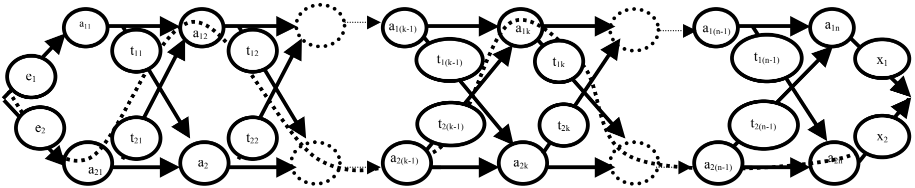
- So, all we have to do is find a subset of these numbers with the largest possible total sum which fits inside a knapsack of size $S/2$.

Dynamic Programming: Assembly line scheduling



Instance: Two assembly lines with workstations for n jobs.

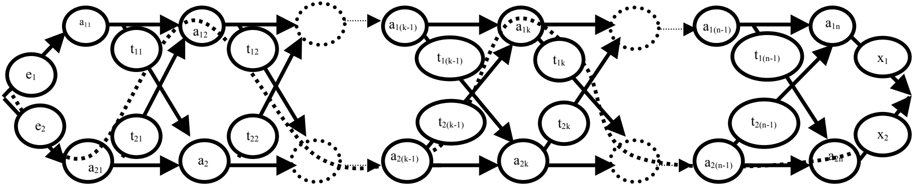
Dynamic Programming: Assembly line scheduling



Instance: Two assembly lines with workstations for n jobs.

- On the first assembly line the k^{th} job takes $a_{1,k}$ $(1 \leq k \leq n)$ units of time to complete; on the second assembly line the same job takes $a_{2,k}$ units of time.

Dynamic Programming: Assembly line scheduling



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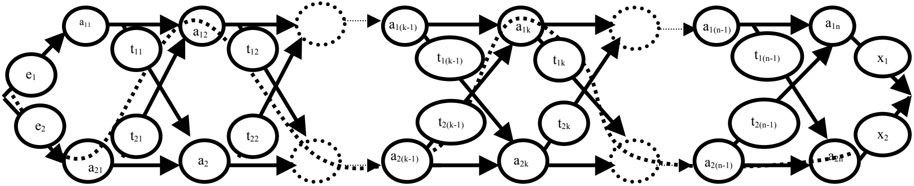
- On the first assembly line the k^{th} job takes $a_{1,k}$ $(1 \leq k \leq n)$ units of time to complete; on the second assembly line the same job takes $a_{2,k}$ units of time.
- To move the product from station $k-1$ on the first assembly line to station k on the second line it takes $t_{1,k-1}$ units of time.

Dynamic Programming: Assembly line scheduling



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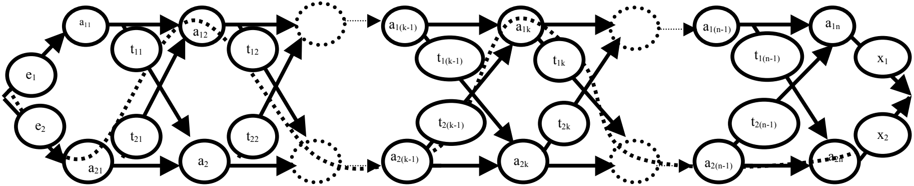
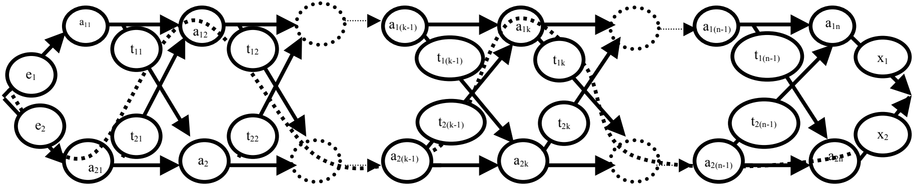
- To move the product from station $k-1$ on the first assembly line to station k on the second line it takes $t_{1,k-1}$ units of time.
- Likewise, to move the product from station $k-1$ on the second assembly line to station k on the first assembly line it takes $t_{2,k-1}$ units of time.



- To bring an unfinished product to the first assembly line it takes e_1 units of time.

Dynamic Programming: Assembly line scheduling



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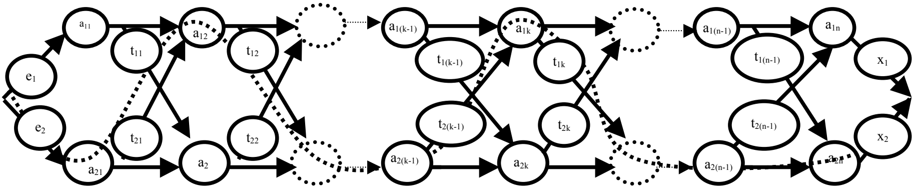
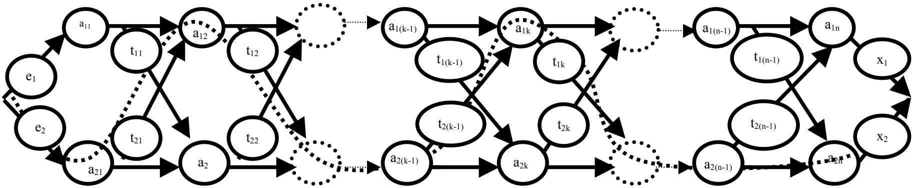
Dynamic Programming: Assembly line scheduling



- To bring an unfinished product to the first assembly line it takes e_1 units of time.
- To bring an unfinished product to the second assembly line it takes e_2 units of time.

- To get a finished product from the first assembly line to the warehouse it takes x_1 units of time;

Dynamic Programming: Assembly line scheduling



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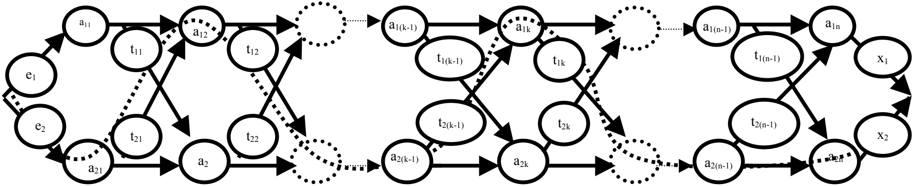

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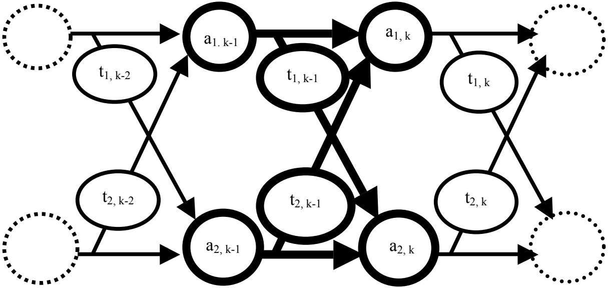
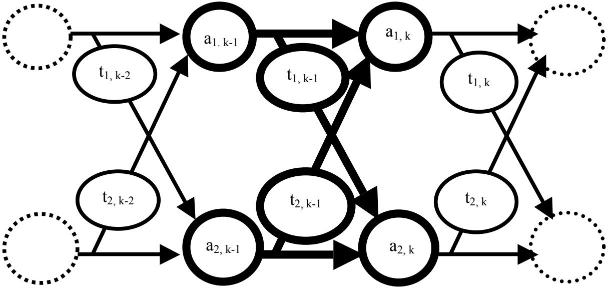
- **Task:** Find a *fastest way* to assemble a product using both lines as necessary.

Dynamic Programming: Assembly line scheduling



- For each $k \leq n$, we solve subproblems $P(1, k)$ and $P(2, k)$ by a **simultaneous recursion** on k:

Dynamic Programming: Assembly line scheduling



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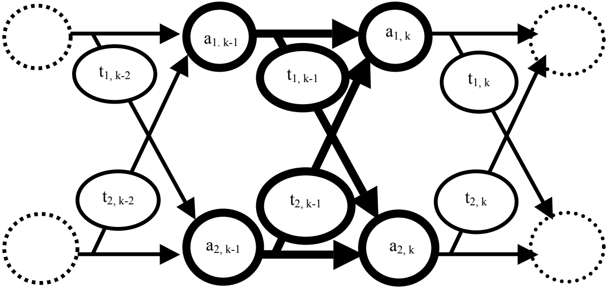

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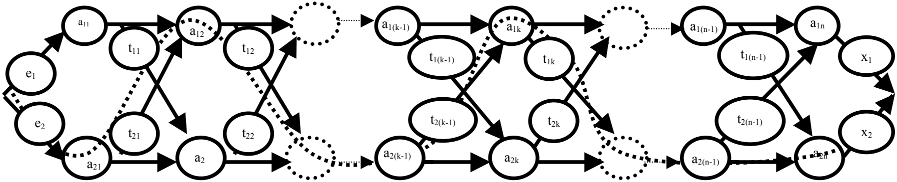

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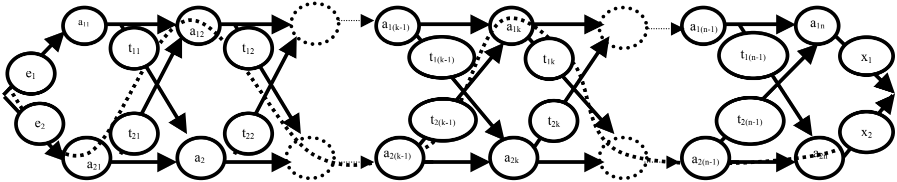

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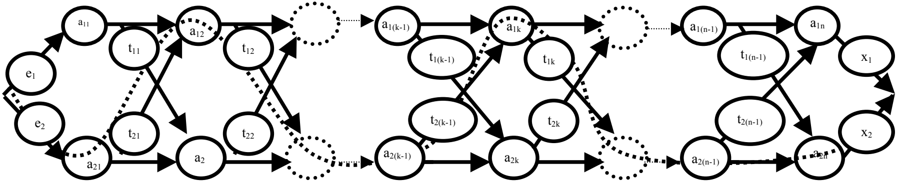
Dynamic Programming



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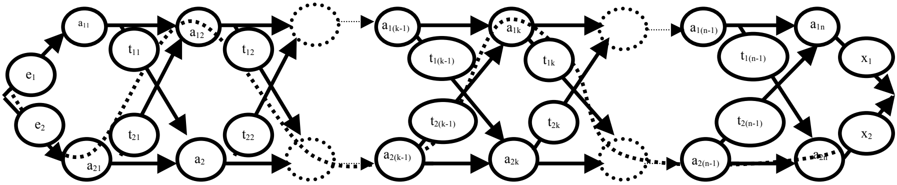

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- Finally, after obtaining $m(1,n)$ and $m(2,n)$ we choose

$$opt = \min\{m(1,n) + x_1, \quad m(2,n) + x_2\}.$$

Dynamic Programming: Matrix chain multiplication

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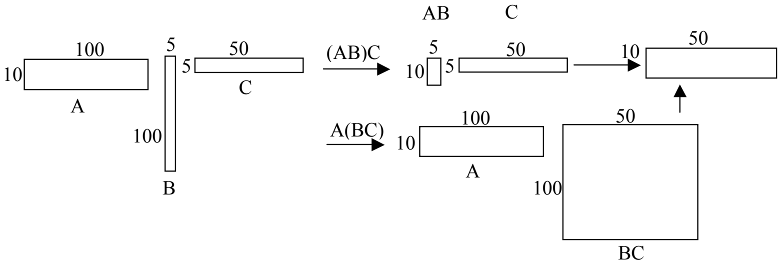

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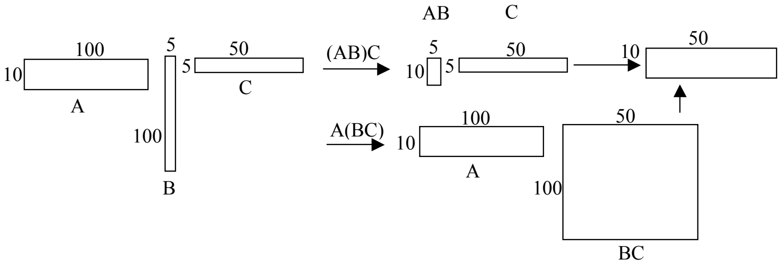

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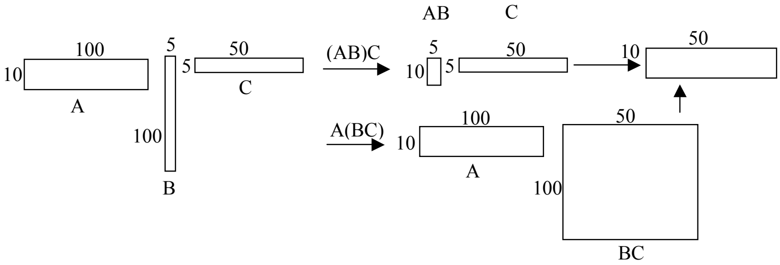

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- At each recursive step m we solve all subproblems $P(i, j)$ for which $j - i = m$.

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- Let $m(i,j)$ denote the minimal number of multiplications needed to compute the product $A_i A_{i+1} \dots A_{j-1} A_j$; let also the size of matrix A_i be $s_{i-1} \times s_i$.

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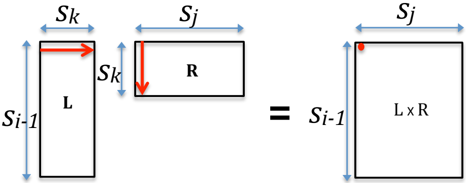
- To multiply an $s_{i-1} \times s_k$ matrix L and an $s_k \times s_j$ matrix R it takes $s_{i-1} s_k s_j$ many multiplications:



Total number of multiplications: $S_{i-1}\ S_j\ \ S_k$

- The recursion:

$$m(i,j) = \min\{m(i,k) + m(k+1,j) + s_{i-1}s_js_k \ : \ i \leq k \leq j-1\}$$

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- Thus, in the m^{th} slot of the table we are constructing we store all pairs $(m(i,j),k)$ for which $j-i=m$.

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- Given two sequences S and S^* a sequence s is a **Longest Common Subsequence** of S, S^* if s is a common subsequence of both S and S^* and is of maximal possible length.

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$$c[i,j] = \begin{cases} 0, & \text{if } \; i = 0 \text{ or } j = 0; \\ c[i-1, j-1] + 1 & \text{if } i, j > 0 \text{ and } a_i = b_j; \\ \max\{c[i-1, j], c[i, j-1]\} & \text{if } i, j > 0 \text{ and } a_i \neq b_j. \end{cases}$$

Retrieving a longest common subsequence:

```
LCS-LENGTH(X, Y)
  m ← length[X]
  n ← length[Y]
  for i ← 1 to m
       do c[i, 0] ← 0
  for j ← 0 to n
       do c[0, j] ← 0
  for i ← 1 to m
       do for j ← 1 to n
              do if x_i = y_j
                    then c[i, j] ← c[i − 1, j − 1] + 1
                         b[i, j] ← "↖"
                    else if c[i − 1, j] ≥ c[i, j − 1]
                            then c[i, j] ← c[i − 1, j]
                                 b[i, j] ← "↑"
                            else c[i, j] ← c[i, j − 1]
                                 b[i, j] ← "←"
  return c and b
```

	j	0	1	2	3	4	5	6
i		y_j	B	D	C	A	B	A
0	x_i	0	0	0	0	0	0	0
1	A	0	↑ 0	↑ 0	↑ 0	↖ 1	← 1	↖ 1
2	B	0	↖ 1	← 1	← 1	↑ 1	↖ 2	← 2
3	C	0	↑ 1	↑ 1	↖ 2	← 2	↑ 2	↑ 2
4	B	0	↖ 1	↑ 1	↑ 2	↑ 2	↖ 3	← 3
5	D	0	↑ 1	↖ 2	↑ 2	↑ 2	↑ 3	↑ 3
6	A	0	↑ 1	↑ 2	↑ 2	↖ 3	↑ 3	↖ 4
7	B	0	↖ 1	↑ 2	↑ 2	↑ 3	↖ 4	↑ 4

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- Can we do $\mathrm{LCS}(\mathrm{LCS}(S_1, S_2), S_3)$?
- Not necessarily! Consider

$$
\begin{aligned}
S_1 &= ABCDEGG & \quad \mathrm{LCS}(S_1, S_2) &= ABEG \\
S_2 &= ACBEEFG & \quad \mathrm{LCS}S_2, S_3) &= ACEF \\
S_3 &= ACCEDGF & \quad \mathrm{LCS}(S_1, S_3) &= ACDG
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{LCS}(\mathrm{LCS}(S_1, S_2), S_3) &= \mathrm{LCS}(ABEG, ACCEDGF) = AEG \\
\mathrm{LCS}(\mathrm{LCS}(S_2, S_3), S_1) &= \mathrm{LCS}(ACEF, ABCDEGG) = ACE \\
\mathrm{LCS}(\mathrm{LCS}(S_1, S_3), S_2) &= \mathrm{LCS}(ACDG, ACBEEFG) = ACG
\end{aligned}
$$

But

$$\mathrm{LCS}(S_1, S_2, S_3) = ACEG$$

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S_1 &= ABCDEGG & \qquad \text{LCS}(S_1, S_2) &= ABEG \\
S_2 &= ACBEEFG & \qquad \text{LCS}S_2, S_3) &= ACEF \\
S_3 &= ACCEDGF & \qquad \text{LCS}(S_1, S_3) &= ACDG
\end{aligned}
$$

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\text{LCS}(\text{LCS}(S_1, S_2), S_3) &= \text{LCS}(ABEG, ACCEDGF) = AEG \\
\text{LCS}(\text{LCS}(S_2, S_3), S_1) &= \text{LCS}(ACEF, ABCDEGG) = ACE \\
\text{LCS}(\text{LCS}(S_1, S_3), S_2) &= \text{LCS}(ACDG, ACBEEFG) = ACG
\end{aligned}
$$

But

$$\text{LCS}(S_1, S_2, S_3) = ACEG$$

- So how would you design an algorithm which computes correctly $\text{LCS}(S_1, S_2, S_3)$?

Dynamic Programming: Longest Common Subsequence

- **Instance:** Three sequences $S = \langle a_1, a_2, \ldots a_n \rangle$, $S^* = \langle b_1, b_2, \ldots, b_m \rangle$ and $S^{**} = \langle c_1, c_2, \ldots, c_k \rangle$.

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- for all $1 \leq i \leq n$, all $1 \leq j \leq m$ and all $l \leq k$, let $d[i, j, l]$ be the length of the longest common subsequence of the truncated sequences $S_i = \langle a_1, a_2, \dots a_i \rangle$, $S_j^* = \langle b_1, b_2, \dots, b_j \rangle$ and $S_l^* = \langle c_1, c_2, \dots, c_l \rangle$.

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- Recursion:

$$d[i, j, l] = \begin{cases} 0, & \text{if } \; i = 0 \text{ or } j = 0 \text{ or } l = 0; \\ d[i-1, j-1, l-1] + 1 & \text{if } i, j, l > 0 \text{ and } a_i = b_j = c_l; \\ \max\{d[i-1, j, l], d[i, j-1, l], d[i, j, l-1]\} & \text{otherwise.} \end{cases}$$

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- **Solution:** Find the longest common subsequence $LCS(s, s^*)$ of s and s^* and then add differing elements of the two sequences at the right places, in any order; for example:

$$\begin{aligned} s &= a\mathbf{b}a\mathbf{cad}a \\ s^* &= x\mathbf{b}y\mathbf{ca}z\mathbf{d} \\ LCS(s, s^*) &= \mathbf{bcad} \\ \text{shortest super-sequence } S &= ax\mathbf{b}ya\mathbf{ca}z\mathbf{d}a \end{aligned}$$

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- Our goal is to find for every vertex $t \in G$ the value of $\mathrm{opt}(n - 1, t)$ and the path which achieves such a length.
- Note that if the shortest path from a vertex v to t is $(v, p_1, p_2, \ldots, p_k, t)$ then $(p_1, p_2, \ldots, p_k, t)$ must be the shortest path from p_1 to t, and $(v, p_1, p_2, \ldots, p_k)$ must also be the shortest path from v to p_k.

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- **Recursion:** $\mathrm{opt}(i, v) = \min(\mathrm{opt}(i-1, v), \min_{p \in V}\{\mathrm{opt}(i-1, p) + \mathrm{w}(e(p, v))\}$.
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- As an exercise, given any directed weighted graph G, explain how we can use the above algorithm to decide if G has any cycles of negative weight.

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- Subproblems: Let $C(i, j)$ be the minimum cost of transforming the sequence $A[1..i]$ into the sequence $B[1..j]$ for all $i \leq n$ and all $j \leq m$.

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  1. if $A[i]$ is already equal to $B[j]$ do nothing, thus incurring a cost of only $C(i-1,j-1)$;

Dynamic Programming: Edit Distance

- **Subproblems $P(i,j)$:** Find the minimum cost $C(i,j)$ of transforming the sequence $A[1..i]$ into the sequence $B[1..j]$ for all $i \leq n$ and all $j \leq m$.

- **Recursion:** we again fill the table of solutions $C(i,j)$ for subproblems $P(i,j)$ row by row (why is this OK?):

$$C(i,j) = \min \begin{cases} c_d + C(i-1,j) \\ C(i,j-1) + c_i \\ \begin{cases} C(i-1,j-1) & \text{if } \ A[i] = B[j] \\ C(i-1,j-1) + c_r & \text{if } \ A[i] \neq B[j] \end{cases} \end{cases}$$

- cost $c_d + C(i-1,j)$ corresponds to the option if you first delete $A[i]$ and then recursively transform $A[1..i-1]$ into $B[1..j]$;
- cost $C(i,j-1) + c_i$ corresponds to the option if you first transform $A[1..i]$ to $B[1..j-1]$ and then append $B[j]$ at the end;
- the third option corresponds to first transforming $A[1..i-1]$ to $B[1..j-1]$ and
  1. if $A[i]$ is already equal to $B[j]$ do nothing, thus incurring a cost of only $C(i-1,j-1)$;
  2. if $A[i]$ is not equal to $B[j]$ replace $A[i]$ by $B[j]$ with a total cost of $C(i-1,j-1) + c_r$.

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- Exercise: write the exact recursion for this problem.

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- **Hint:** Order turtles in an increasing order of the sum of their weight and their strength, and proceed by recursion. You might want to first solve the longest increasing subsequence of numbers problem by a solution which runs in time $n \log n$ because both problems use similar tricks...