模拟试卷五答案

一、选择题

1. C　2. B　3. B　4. C　5. D

二、填空题

1. $2dy$

2. $(x-1)+2(y-1)+3(z-3)=0$

3. $[-3,3)$

4. $\frac{4}{3}\pi R^4$

5. π

三、计算题

1. 解：$z_x=-f_1'\sin x+f_2'\cdot 0+f_3'\frac{\partial u}{\partial x}=-f_1'\sin x+f_3'\frac{\partial u}{\partial x}$

$z_y=f_1'\cdot 0+f_2'\cdot 2y+f_3'\frac{\partial u}{\partial y}=f_2'\cdot 2y+f_3'\frac{\partial u}{\partial y}$

令 $F(x,y,z)=u^5-5xy+5u-1=0$

$F_x=-5y\qquad F_y=-5x\qquad F_u=5u^4+5=5(u^4+1)$

$\therefore \frac{\partial u}{\partial x}=-\frac{F_x}{F_u}=\frac{y}{u^4+1}\qquad \frac{\partial u}{\partial y}=\frac{x}{u^4+1}$

$\therefore \frac{\partial z}{\partial x}=-f_1'\sin x+f_3'\cdot\frac{y}{u^4+1}\qquad \frac{\partial z}{\partial y}=f_2'\cdot 2y+f_3'\cdot\frac{x}{u^4+1}$

2. 解：$grad\ u\big|_{P_0}=(y+z,x+z,y+x)\big|_{P_0}=(4,5,3)$

设 $P(x,y,z)$，$\overrightarrow{P_0P}=(x-2,y-1,z-3)$

$\vec{e}_x=(\pm1,0,0)\qquad \vec{e}_y=(0,\pm1,0)\qquad \vec{e}_z=(0,0,\pm1)$

$$\frac{\overrightarrow{P_0P}\cdot\vec{e_x}}{\left|\overrightarrow{P_0P}\right|}=\frac{\overrightarrow{P_0P}\cdot\vec{e_y}}{\left|\overrightarrow{P_0P}\right|}=\frac{\overrightarrow{P_0P}\cdot\vec{e_z}}{\left|\overrightarrow{P_0P}\right|}\Rightarrow \pm(x-2)=\pm(y-1)=\pm(z-3)$$

$$\cos\alpha=\frac{\pm(x-2)}{\sqrt{(x-2)^2+(y-1)^2+(z-3)^2}}=\pm\frac{\sqrt{3}}{3}$$

同理：$\cos\beta=\pm\dfrac{\sqrt{3}}{3}\qquad \cos\gamma=\pm\dfrac{\sqrt{3}}{3}$

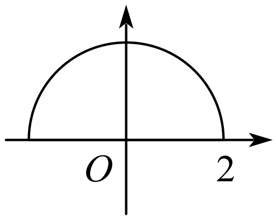
$$\therefore \vec{e}=\left(\pm\frac{\sqrt{3}}{3},\pm\frac{\sqrt{3}}{3},\pm\frac{\sqrt{3}}{3}\right)$$

$$\therefore \left.\frac{\partial u}{\partial l}\right|_{P_0}=(4,5,3)\cdot\left(\pm\frac{\sqrt{3}}{3},\pm\frac{\sqrt{3}}{3},\pm\frac{\sqrt{3}}{3}\right)$$

$$=4\cdot\left(\pm\frac{\sqrt{3}}{3}\right)+5\cdot\left(\pm\frac{\sqrt{3}}{3}\right)+3\cdot\left(\pm\frac{\sqrt{3}}{3}\right)=\pm4\sqrt{3}$$

3. 解：$D:\begin{cases}0\leqslant x\leqslant 2\\ 0\leqslant y\leqslant\sqrt{4-x^2}\end{cases}$　　$I=\int_0^{\frac{\pi}{2}}d\theta\int_0^2\ln(1+\rho^2)\rho d\rho$

$$=\frac{\pi}{4}\cdot\int_0^2\ln(1+\rho^2)d(1+\rho^2)$$



$$\xlongequal{1+\rho^2=t}\frac{\pi}{4}\int_1^5\ln t dt$$

$$=\frac{\pi}{4}\left[t\ln t\Big|_1^5-\int_1^5 t d\ln t\right]$$

$$=\frac{\pi}{4}[5\ln5-4]=\frac{5\pi\ln5}{4}-\pi$$

4. ① $\sum_{n=1}^{\infty}\left(1-\frac{1}{n}\right)^{n^2}$

$$\lim_{n\to\infty}\sqrt[n]{\left(1-\frac{1}{n}\right)^{n^2}}=\lim_{n\to\infty}\left(1-\frac{1}{n}\right)^{n^2\cdot\frac{1}{n}}=\lim_{n\to\infty}\left(1-\frac{1}{n}\right)^{n}$$

$$=\lim_{n\to\infty}\left(1-\frac{1}{n}\right)^{-n\cdot(-1)}=\frac{1}{e}<1\qquad \therefore\sum_{n=1}^{\infty}\left(1-\frac{1}{n}\right)^{n^2}\text{收敛}$$

② $\sum_{n=1}^{\infty}\dfrac{1}{\ln(n+1)}$

$$\lim_{n\to\infty}\frac{\frac{1}{\ln(n+1)}}{\frac{1}{n}}=\lim_{n\to\infty}\frac{n}{\ln(n+1)}=\lim_{x\to\infty}\frac{x}{\ln(x+1)}$$

$$=\lim_{x\to\infty}\frac{1}{\frac{1}{1+x}}=\lim_{x\to\infty}(1+x)=\infty \qquad \therefore \sum_{n=1}^{\infty}\frac{1}{\ln(n+1)}\text{发散}$$

5. 解：$f(x)=\dfrac{x-4+3}{4-x}=-1+\dfrac{3}{4-x}=-1+\dfrac{3}{3-(x-1)}$

$$=-1+\frac{1}{1-\frac{x-1}{3}}=-1+\sum_{n=0}^{\infty}\left(\frac{x-1}{3}\right)^n=-1+\sum_{n=0}^{\infty}\frac{1}{3^n}(x-1)^n$$

$$f(x)=f(1)+f'(1)(x-1)+\frac{f''(1)}{2}(x-1)^2+\dots+\frac{f^{(n)}(1)}{n!}(x-1)^n+\dots$$

由 $\dfrac{1}{3^n}(x-1)^n=\dfrac{f^{(n)}(1)}{n!}(x-1)^n$

$\therefore f^{(n)}(1)=\dfrac{n!}{3^n}$

四、解：设长 x、宽 y、高 z $(x>0,y>0,z>0)$

约束条件：$xyz=V_0$

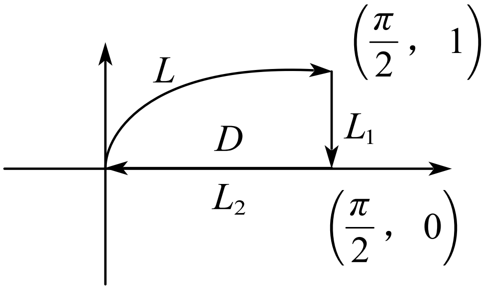
目标函数：$S=xy+2xz+2yz$

令 $F(x,y,z,\lambda)=xy+2xz+2yz+\lambda(xyz-V_0)$

$$\begin{cases} F_x=y+2z+yz\lambda=0 \text{ ①} \\ F_y=x+2z+xz\lambda=0 \text{ ②} \\ F_z=2x+2y+xy\lambda=0 \text{ ③} \\ F_\lambda=xyz-V_0=0 \text{ ④} \end{cases}$$

得长为 $2\sqrt[3]{\dfrac{V_0}{4}}$，宽为 $2\sqrt[3]{\dfrac{V_0}{4}}$，高为 $\sqrt[3]{\dfrac{V_0}{4}}$

五、解：



• • • $P(x,y)=2xy^3-y^2\sin x$

• • $\dfrac{\partial P}{\partial y}=6xy^2-2y\sin x$

• • $Q(x,y)=1+xy+2y\cos x+3x^2y^2$

• $\dfrac{\partial Q}{\partial x}=y-2y\sin x+6xy^2$

补充 L_1: $x=\dfrac{\pi}{2}$，从 $\left(\dfrac{\pi}{2},1\right)$ 到 $\left(\dfrac{\pi}{2},0\right)$，$L_2$: $y=0$，从 $\left(\dfrac{\pi}{2},0\right)$ 到 $(0,0)$

$$\therefore \oint_{L+L_1+L_2}=-\iint_D\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)dxdy=-\iint_D ydxdy$$

$$\bullet\ \bullet =-\int_0^1 dy\int_{\frac{\pi}{2}y^2}^{\frac{\pi}{2}} ydx$$

$$\bullet =-\int_0^1 y\cdot\frac{\pi}{2}(1-y^2)dy=-\frac{\pi}{2}\left(\frac{1}{2}-\frac{1}{4}\right)=-\frac{\pi}{8}$$

$$\int_{L_1}\overset{dx=0}{=\!=\!=}\int_1^0\left(1+\frac{\pi}{2}y+2y\cos\frac{\pi}{2}+3\cdot\frac{\pi^2}{4}y^2\right)dy$$

$$=-\left(1+\frac{\pi}{2}\cdot\frac{1}{2}+\frac{3\pi^2}{4}\cdot\frac{1}{3}\right)=-\left(1+\frac{\pi}{4}+\frac{\pi^2}{4}\right)$$

$$\int_{L_2}\overset{dy=0}{=\!=\!=}0$$

$$\therefore \int_L=\oint_{L+L_1+L_2}-\int_{L_1}-\int_{L_2}=-\frac{\pi}{8}+1+\frac{\pi}{4}+\frac{\pi^2}{4}$$

$$=1+\frac{\pi}{8}+\frac{\pi^2}{4}$$

六、$I=\iint\limits_{\Sigma}(x^2-y^2)dydz+(y^2-z^2)dzdx+(z^2-1)dxdy$，其中 Σ 是

$\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}+\dfrac{z^2}{c^2}=1(z\geqslant 0)$ 的上侧。

解：添加$\Sigma_1:\begin{cases} z=0 \\ \dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}\leqslant 1 \end{cases}$下侧

由高斯公式：

$$\oiint_{\Sigma+\Sigma_1} = \iiint_{\Omega}(2x+2y+2z)dV \xlongequal[\Omega\text{关于}xoz\text{平面对称}]{\Omega\text{关于}yoz\text{平面对称}} 2\iiint_{\Omega} zdV$$

$$\xlongequal{\text{先二后一}} 2\int_0^c zdz\iint_{D_z} dxdy = 2\int_0^c z\cdot\pi\cdot ab\left(1-\frac{z^2}{c^2}\right)dz$$

$$=2\pi ab\int_0^c\left(z-\frac{z^3}{c^2}\right)dz = 2\pi ab\left(\frac{c^2}{2}-\frac{1}{c^2}\cdot\frac{c^4}{4}\right)=2\pi ab\cdot\frac{c^2}{4}=\frac{\pi abc^2}{2}$$

而$\iint_{\Sigma_1}(x^2-y^2)dydz+(y^2-z^2)dzdx+(z^2-1^2)dxdy$

$$\xlongequal[dz=0]{z=0} -\iint_{\Sigma_1} dxdy = -\iint_{D_{xy}}(-dxdy) = \iint_{D_{xy}} dxdy = \pi ab$$

$$\therefore \iint_{\Sigma} = \oiint_{\Sigma_1+\Sigma_2} - \iint_{\Sigma_1} = \frac{\pi abc^2}{2}-\pi ab$$

七、解：$\sum\limits_{n=1}^{\infty}\left|\dfrac{(-1)^{n-1}x^{2n+1}}{2n+1}\right|$

$\lim\limits_{n\to\infty}\left|\dfrac{x^{2n+3}}{2n+3}\Big/\dfrac{x^{2n+1}}{2n+1}\right|=x^2<1$ $\quad\therefore R=1$，收敛区间$|x|<1$

当$x=\pm1$时，$\sum\limits_{n=1}^{\infty}\dfrac{(-1)^{n-1}(\pm1)^{2n+1}}{2n+1}=\pm\sum\limits_{n=1}^{\infty}\dfrac{(-1)^{n-1}}{2n+1}$收敛

$\therefore$收敛域$[-1,1]$

设$S(x)=\sum\limits_{n=1}^{\infty}\dfrac{(-1)^{n-1}x^{2n+1}}{2n+1}=\dfrac{x^3}{3}-\dfrac{x^5}{5}+\ldots+\dfrac{(-1)^{n-1}x^{2n+1}}{2n+1}+\ldots$,

$S(0)=0 \qquad S'(x)=x^2-x^4+\ldots+(-1)^{n-1}x^{2n}+\ldots=\dfrac{x^2}{1+x^2}$

$S(x)=\int_0^x S'(x)dx=\int_0^x\dfrac{x^2+1-1}{1+x^2}dx=x-\arctan x$

$$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{2n+1}=S(1)=1-\arctan 1=1-\frac{\pi}{4}$$

八、解：（1） $\sum_{n=1}^{\infty}|u_n|,\sum_{n=1}^{\infty}|v_n|$, 收敛， $|u_n+v_n|\leqslant|u_n|+|v_n|$

由 $\sum_{n=1}^{\infty}|u_n|+|v_n|$ 收敛

$\therefore\sum_{n=1}^{\infty}|u_n+v_n|$ 也收敛，则 $\sum_{n=1}^{\infty}(u_n+v_n)$ 绝对收敛

（2） $\sum_{n=1}^{\infty}u_n$ 收敛， $\sum_{n=1}^{\infty}v_n$ 收敛，则 $\sum_{n=1}^{\infty}u_n+v_n$ 收敛

反证：设 $\sum_{n=1}^{\infty}(u_n+v_n)$ 绝对收敛

$|v_n|=|u_n+v_n-u_n|\leqslant|u_n|+|u_n+v_n|$

由 $\sum_{n=1}^{\infty}|u_n+v_n|+|u_n|$ 收敛 $\Rightarrow\sum_{n=1}^{\infty}|v_n|$ 收敛， $\sum_{n=1}^{\infty}v_n$ 绝对收敛，

矛盾 $\therefore\sum_{n=1}^{\infty}v_n$ 条件收敛， $\sum_{n=1}^{\infty}(u_n+v_n)$ 条件收敛，得证