2022-2023 期末试题答案解析

一、填空题

1、7/8

2、

$$F_Z(z)=\begin{cases}0, & z<0\\ 2z-z^2, & 0\le z<1\\ 1 & z\ge 1\end{cases}$$

3、4.2

4、0 与 2.5

5、n 2

二、选择题

DADCC

三、

设事件 A_i $(i=1,\ 2)$ 表示第 i 次取到新球.

(1) 设事件 A 表示第二次才取到新球，则 $A=\bar{A}_1A_2$，由乘法公式可得

$$P(A)=P\left(\bar{A}_1A_2\right)=P\left(\bar{A}_1\right)P\left(A_2\middle|\bar{A}_1\right)=\frac{4}{10}\times\frac{6}{9}=\frac{4}{15}$$

(2) 设事件 A 表示其中之一是新球，B 表示2个都是新球，则

$$P(AB)=P\left(A_1A_2\right)=P\left(A_1\right)P\left(A_2\middle|A_1\right)=\frac{6}{10}\times\frac{5}{9}=\frac{1}{3}$$

$$\begin{aligned}P(A)&=P\left(A_1A_2\cup A_1\bar{A}_2\cup\bar{A}_1A_2\right)=P\left(A_1A_2\right)+P\left(A_1\bar{A}_2\right)+P\left(\bar{A}_1A_2\right)\\&=\frac{1}{3}+P\left(A_1\right)P\left(\bar{A}_2\middle|A_1\right)+P\left(\bar{A}_1\right)P\left(A_2\middle|\bar{A}_1\right)=\frac{1}{3}+\frac{6}{10}\times\frac{4}{9}+\frac{4}{10}\times\frac{6}{9}=\frac{13}{15}\end{aligned}$$

于是，所求概率为 $P\left(B\middle|A\right)=\dfrac{P(AB)}{P(A)}=\dfrac{5}{13}$

四、

(1) 因为 $1=\int_{-\infty}^{+\infty} f(x, y)\,\mathrm{d}x\,\mathrm{d}y=\iint\limits_{G} Axy\,\mathrm{d}x\,\mathrm{d}y=\int_{0}^{2} Ax\,\mathrm{d}x\int_{0}^{x^2} y\,\mathrm{d}y=\frac{A}{2}\int_{0}^{2} x^5\,\mathrm{d}x=\frac{16}{3}A$

所以 $A=\frac{3}{16}$

(2) $f_X(x)=\int_{-\infty}^{+\infty} f(x, y)\,\mathrm{d}y=\begin{cases}\int_{0}^{x^2}\frac{3}{16}xy\,\mathrm{d}y=\frac{3}{32}x^5, & 0<x<2\\ 0, & \text{其它}\end{cases}$

$f_Y(y)=\int_{-\infty}^{+\infty} f(x, y)\,\mathrm{d}x=\begin{cases}\int_{\sqrt{y}}^{2}\frac{3}{16}xy\,\mathrm{d}x=\frac{3}{32}y(4-y), & 0<y<4\\ 0, & \text{其它}\end{cases}$

(3) 当 $0<y<4$ 时，$f_{X|Y}(x|y)=\frac{f(x, y)}{f_Y(y)}=\begin{cases}\frac{2x}{4-y}, & \sqrt{y}<x<2\\ 0, & \text{其它}\end{cases}$

当 $0<x<2$ 时，$f_{Y|X}(y|x)=\frac{f(x, y)}{f_X(x)}=\begin{cases}\frac{2y}{x^4}, & 0<y<x^2\\ 0, & \text{其它}\end{cases}$

注意：将条件概率密度函数写成如下形式是错误的，

$f_{X|Y}(x|y)=\begin{cases}\frac{2x}{4-y}, & \sqrt{y}<x<2,\ 0<y<4\\ 0, & \text{其它}\end{cases}$

因为条件概率密度函数并不是二元函数.

五、

解:(1)当$0 < x < 1$时，

$$f_X(x) = \int_{-\infty}^{+\infty} f(x,y)dy = \int_0^x 24(1-x)ydy = 12(1-x)x^2,$$

$$f_X(x) = \begin{cases} 12(1-x)x^2, 0 < x < 1 \\ 0\text{其它} \end{cases}$$

当$0 < y < 1$时，$f_Y(y) = \int_y^1 24(1-x)ydx = -12y(1-x)^2 |_y^1 = 12y(1-y)^2$,

$$f_Y(y) = \begin{cases} 12y(1-y)^2, 0 < y < 1 \\ 0,\text{其他} \end{cases}$$

因为$f_X(x)f_Y(y) \neq f(x,y)$，所以X,Y不独立。

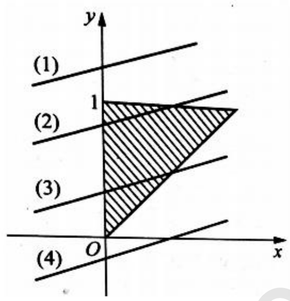
$$(2) f_{X|Y}(x|y) = \frac{f(x,y)}{f_Y(y)} = \begin{cases} \dfrac{2(1-x)}{(1-y)^2}, 0 < y < x < 1 \\ 0\text{，其他} \end{cases},$$

$$(3) f_{X|Y}(x|y=\frac{1}{2}) = \begin{cases} 8(1-x), \frac{1}{2} < x < 1 \\ 0\text{，其他} \end{cases}$$

$$P\{X \le \frac{3}{4} | Y = \frac{1}{2}\} = \int_{-\infty}^{\frac{3}{4}} f_{X|Y}(x|y=\frac{1}{2})dx = \int_{\frac{1}{2}}^{\frac{3}{4}} 8(1-x)dx = -4(1-x)^2 |_{\frac{1}{2}}^{\frac{3}{4}}$$

$$= 4[(\frac{1}{2})^2 - (\frac{1}{4})^2] = 4(\frac{1}{4} - \frac{1}{16}) = 1 - \frac{1}{4} = \frac{3}{4}.$$

六、



先求分布函数，如右图所示

$$F_Z(z)=P(Z\le z)=P(X-2Y\le z)=\iint\limits_{x-2y\le z} f(x,\ y)\,\mathrm{d}x\,\mathrm{d}y$$

$$=\begin{cases} 0, & z<-2 \\ \int_0^{z+2}\mathrm{d}x\int_{\frac{x-z}{2}}^{1}3y\,\mathrm{d}y=2+\dfrac{3}{2}z-\dfrac{1}{8}z^3, & -2\le z<-1 \\ \int_0^{z}\mathrm{d}x\int_{\frac{x-z}{2}}^{1}3y\,\mathrm{d}y+\int_z^1\mathrm{d}x\int_x^1 3y\,\mathrm{d}y=1+\dfrac{3}{8}z^3, & -1\le z\le 0 \\ 1, & z>0 \end{cases}$$

于是所求概率密度函数为 $f(z)=\dfrac{\mathrm{d}}{\mathrm{d}z}F_Z(z)=\begin{cases} \dfrac{3}{2}-\dfrac{3}{8}z^2, & -2\le z<-1 \\ \dfrac{9}{8}z^2, & -1\le z<0 \\ 0, & \text{其它} \end{cases}$

七、

(1) 由递推公式$Y_n = Y_{n-1} + X_n \ (n \geq 1)$可得

$$Y_n = Y_{n-1} + X_n = Y_{n-2} + X_{n-1} + X_n = Y_{n-3} + X_{n-2} + X_{n-1} + X_n = \cdots\cdots$$
$$= Y_0 + X_1 + X_2 + \cdots + X_{n-1} + X_n = Y_0 + \sum_{i=1}^{n} X_i$$

(2) 18天后该商品的价格为$Y_{18} = Y_0 + \sum_{i=1}^{18} X_i = 100 + \sum_{i=1}^{18} X_i$，其中$X_1, X_2, \ldots, X_{18}$相互独立，且$E(X_i) = 0$，$D(X_i) = 2 \ (n = 1, 2, \ldots, 18)$，故$E\left(\sum_{i=1}^{18} X_i\right) = \sum_{i=1}^{18} E(X_i) = 0$，

$D\left(\sum_{i=1}^{18} X_i\right) = \sum_{i=1}^{18} D(X_i) = 36$，由独立同分布的中心极限定理可得$\sum_{i=1}^{18} X_i \sim N(0, 36)$，

于是所求概率为

$$P(96 < Y_{18} < 104) = P\left(-4 < \sum_{i=1}^{18} X_i < 4\right) = \Phi\left(\frac{4}{6}\right) - \Phi\left(-\frac{4}{6}\right) = 2\Phi\left(\frac{2}{3}\right) - 1 \approx 0.4972$$

八、

(1) $f(x)$为偶函数，所以$xf(x)$为奇函数，于是

$E(X) = 0$

$$E(X^2) = \int_{-\infty}^{+\infty} x^2 f(x)\,\mathrm{d}x = \int_{-\infty}^{+\infty} x^2 \frac{1}{2\theta} \mathrm{e}^{-\frac{|x|}{\theta}}\,\mathrm{d}x = \int_{0}^{+\infty} \frac{x^2}{\theta} \mathrm{e}^{-\frac{x}{\theta}}\,\mathrm{d}x = -x^2 \mathrm{e}^{-\frac{x}{\theta}}\Big|_{-\infty}^{+\infty} + 2\int_{0}^{+\infty} x\mathrm{e}^{-\frac{x}{\theta}}\,\mathrm{d}x$$
$$= 2\int_{0}^{+\infty} -\theta x \mathrm{d}\left(\mathrm{e}^{-\frac{x}{\theta}}\right) = -2\theta x \mathrm{e}^{-\frac{x}{\theta}}\Big|_{0}^{+\infty} + 2\theta\int_{0}^{+\infty} \mathrm{e}^{-\frac{x}{\theta}}\,\mathrm{d}x = 2\theta^2$$

由矩估计的定义，令$E(X^2) = 2\theta^2 = \frac{1}{n}\sum_{i=1}^{n} X_i^2$，求得$\theta$的矩估计量$\hat{\theta} = \sqrt{\frac{1}{2n}\sum_{i=1}^{n} X_i^2}$.

(2) 似然函数 $L(\theta) = \prod_{i=1}^{n} f(x_i;\ \theta) = \prod_{i=1}^{n} \frac{1}{2\theta} \mathrm{e}^{-\frac{|x_i|}{\theta}} = (2\theta)^{-n} \mathrm{e}^{-\frac{1}{\theta}\sum_{i=1}^{n}|x_i|}$

取对数有 $\ln L(\theta) = -n\ln(2\theta) - \frac{1}{\theta}\sum_{i=1}^{n}|x_i|$，令$\frac{\mathrm{d}}{\mathrm{d}\theta}\ln L(\theta) = -\frac{n}{\theta} + \frac{1}{\theta^2}\sum_{i=1}^{n}|x_i| = 0$

得θ的最大似然估计量为 $\hat{\theta} = \frac{1}{n}\sum_{i=1}^{n}|X_i|$.

九、

$$
\begin{aligned}
D(XY) &= E\left[(XY)^2\right] - \left[E(XY)\right]^2 = E\left(X^2Y^2\right) - \left[E(X)E(Y)\right]^2 \\
&= E\left(X^2\right)E\left(Y^2\right) - \left[E(X)\right]^2\left[E(Y)\right]^2 \\
&= \left\{D(X) + \left[E(X)\right]^2\right\}\left\{D(Y) + \left[E(Y)\right]^2\right\} - \left[E(X)\right]^2\left[E(Y)\right]^2 \\
&= D(X)D(Y) + \left[E(X)\right]^2 D(Y) + \left[E(Y)\right]^2 D(X) \\
&\geq D(X)D(Y)
\end{aligned}
$$