$$\min_{x \in \mathcal{M}} f(x)$$

s.t. $h(x) = 0$, $g(x) \leq 0$.

$\Downarrow$

how to implemente RIPM when $\mathcal{M}$ is product manifold. Say. $\mathcal{M} \triangleq \mathcal{M}_1 \times \mathcal{M}_2$.

A special problem:

$$\min \quad a_1^T x_1 + a_2^T x_2$$

s.t. $\|x_1\|_2 = 1$, $x_1 \geq 0$.

$x_2 \geq 0.5$ (entrywise)

P1

min $a_1^T x_1 + a_2^T x_2$

s.t. $\|x_1\|_2 = 1$,

$x_1 \geq 0$. $x_2 \geq 0.5$ (entrywise)

$x_1 \in \mathbb{R}^{n_1}$, $x_2 \in \mathbb{R}^{n_2}$

$\boxed{\begin{array}{l} x_1 \in \mathcal{M}_1 = \mathbb{R}^{n_1}, \\ x_2 \in \mathcal{M}_2 = \mathbb{R}^{n_2}, \end{array}}$ $\quad \mathcal{M}_1 \times \mathcal{M}_2 \subseteq \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \cong \mathbb{R}^{n_1+n_2}$.

$f(x_1, x_2) = a_1^T x_1 + a_2^T x_2$. $f: \mathcal{M}_1 \times \mathcal{M}_2 \to \mathbb{R}$.

$h(x_1, x_2) = x_1^T x_1 - 1 = 0$. $h: \mathcal{M}_1 \times \mathcal{M}_2 \to \mathbb{R}^1$.

$g(x_1, x_2) = \begin{pmatrix} -x_1 \\ -x_2 + 0.5 \end{pmatrix} \leq 0$. $g: \mathcal{M}_1 \times \mathcal{M}_2 \to \mathbb{R}^{n_1+n_2}$.

P2

min $a_1^T x_1 + a_2^T x_2$

s.t. $x_1 \geq 0$

$x_2 \geq 0.5$ (entrywise)

$x_1 \in S^{n_1}$ ($\Leftrightarrow \|x_1\|_2 = 1$, $x_1 \in \mathbb{R}^{n_1}$).

$x_2 \in \mathbb{R}^{n_2}$.

$\boxed{\begin{array}{l} x_1 \in \mathcal{M}_1 = S^{n_1}. \\ x_2 \in \mathcal{M}_2 = \mathbb{R}^{n_2}, \end{array}}$ $\quad \mathcal{M}_1 \times \mathcal{M}_2 \subseteq \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \cong \mathbb{R}^{n_1+n_2}$.

$f(x_1, x_2) = a_1^T x_1 + a_2^T x_2$. $f: \mathcal{M}_1 \times \mathcal{M}_2 \to \mathbb{R}$.

$\cancel{h(x_1, x_2) = x_1^T x_1 - 1 = 0. \; h: \mathcal{M}_1 \times \mathcal{M}_2 \to \mathbb{R}^1}$.

$g(x_1, x_2) = \begin{pmatrix} -x_1 \\ -x_2 + 0.5 \end{pmatrix} \leq 0$. $g: \mathcal{M}_1 \times \mathcal{M}_2 \to \mathbb{R}^{n_1+n_2}$

$(z_1, z_2) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} = E_{ineq}$.

$$\tilde{g}_z = \langle \underbrace{(z_1, z_2)}_{\text{Fix}}, g(x_1, x_2) \rangle = \left\langle \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}, \begin{pmatrix} -x_1 \\ -x_2 + 0.5 \cdot \mathbf{1} \end{pmatrix} \right\rangle$$

$\downarrow$

$\operatorname{grad}_{(x_1, x_2)}$.

$$= -z_1^T x_1 - z_2^T x_2 + (0.5) z_2^T \mathbf{1}.$$

$(x_1, x_2) \longmapsto \tilde{g}_z(x_1, x_2)$. $\tilde{g}_z : \mathcal{M}_1 \times \mathcal{M}_2 \to \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$.

$\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$.

$$\operatorname{grad} \tilde{g}_z(x_1, x_2) = \begin{bmatrix} -z_1 \\ -z_2 \end{bmatrix}$$

$$\left\langle \begin{pmatrix} -z_1 \\ -z_2 \end{pmatrix}, \begin{pmatrix} dx_1 \\ dx_2 \end{pmatrix} \right\rangle$$

$$\left\langle \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}, \begin{pmatrix} -dx_1 \\ -dx_2 \end{pmatrix} \right\rangle.$$