Simulation and analysis for the temperature filed inside the bathtub

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Summary

Through the analysis based on the heat transfer in the bathtub, the authors set the function of the thermal field in this article. This article tries to determine how much water one should use and what strategy one should make to have a comfortable bath. Furthermore, this article analyzes the impact caused by the person's motion, bubble and temperature of hot water to determine the optimal strategy.
In order to evaluate the temperature distribution along time parameter, this model treats water as finite tiny cells firstly and calculates the temperature relation among these cells in a unit heat transfer process. Under such assumptions, the temperature of hot water from faucet, heat transfer coefficient between air and water (take bubble into consideration) and how much water added are variables that will be modified to obtain their influence on the thermal field. In particular, the model can not only simulate the thermal field in steady state but also at any time. Through the regression analysis method, the basic relationships between each variable are found.
Then the authors make a correction depending on the mobility of water, and further consider the human motion to improve this model. Finally, the interactional relationships among the yield of water, comfort degree (measured by mean and standard deviation of water temperature) and time can be known. Since different people have different preference between saving water and bathing in a uniform temprature, different strategies are given depending on the significance of these two expectations.
During model evaluation, strengths and weaknesses are analyzed. Error analysis and suggestions are also been discussed in this part.

Keywords: Heat expansion; Liquid cell; ODE; Thermal field;

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1 Introduction

The authors intended to model the water temperature inside a bathtub (a thermal field), analyze the heat transfer from faucet water to water in the bathtub, heat exchange between water and air, water and bathtub wall, as well as water and the person. Afterwards, the authors attempt to determine an optimal strategy that keeps the temperature of water in the bathtub even towards the following objectives:

- Minimize the total volume of water from the faucet.
- Make the distribution of water temperature in different parts of the bathtub as uniform as possible.
- Optimize the operation, i.e., lower the frequency of adjusting the faucet.

In this article, the authors decompose the problem into the following steps when modeling and simulating:

- Construct the model of temperature distribution of water inside the bathtub under the assumption that the person is stationary and the bathtub is a cuboid.
- Using the model to evaluate how the shape and volume of the bathtub and the bubble influence the optimal strategy to keep an even temperature of the water in the bathtub.
- Take the motion of the person into consideration and revise the model.

2 Basic Assumptions

1. The flow and temperature of the water from faucet are constants.
2. The sensibility of the person is precise, the interval of comfortable water temperature is strictly fixed.
3. Basic notations in the article are listed as follow in table1.

Symbol	Meaning
A_1	Area of contact surface between the water and air
A_2	Area of contact surface between the water and the bathtub wall
A_3	Area of contact surface between the water and the person
T	Temperature of water in the bathtub
T_1	Lowest temperature the person can bear
T_2	Highest temperature the person can bear
T_s	Temperature of the water from the faucet
α_w	Temperature of the bathtub wall
α_a	Temperature of the air
α_p	Temperature of the person
k_w	Heat transfer coefficient of the bathtub wall
k_a	Heat transfer coefficient of the air
k_p	Heat transfer coefficient of the person
t	time
x, y, z	Axis of the rectangular coordinates
d	Length of each water cell
Q	Heat
q	Heat flux
λ	Coefficient of thermal conductivity
c	Specific heat capacity
ρ	Density of water
h_w	Coefficient of convection heat transfer between water and the bathtub wall
h_a	Coefficient of convection heat transfer between water and air
h_p	Coefficient of convection heat transfer between water and the person
a, b, h	Length, width, and height of the bathtub.

Table 1: Basic Notation

2.1 Other Assumptions

- In the initial model, assume that the water velocity is small, so the convection of water in the bathtub is not taken into consideration. In other words, the heat transfer of water inside the bathtub is considered as a model of heat expansion among a solid medium. The authors adopt cellular automation in this model.
- Under the model of cellular automation, the authors assume that water inside the container can be treated as accumulation of infinitesimal cubic cells of water. More specifically, the pattern of the accumulation can be represented by Figure 1. During

the computation, the authors make the assumptions that the length of each cell is d and the time interval for every single simulation is Δt.

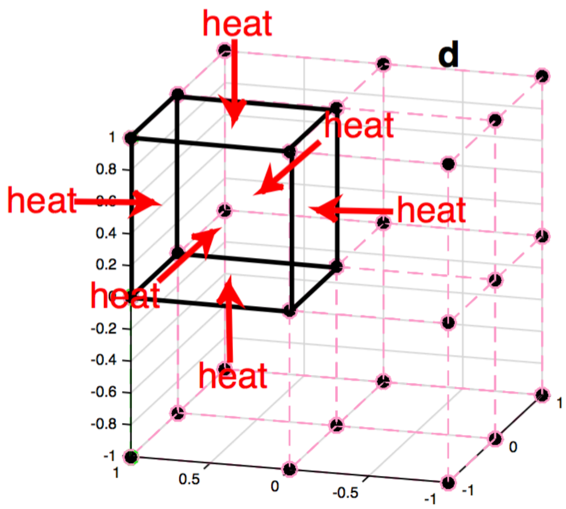


Figure 1: water cube

- In the initial model, the kinetic energy of water from faucet can be ignored, logically equivalently, thermal energy is the only energy type that the system can obtain from the faucet.
- It is assumed that temperature distribution in the bathtub is uniform in an extremely short period before the faucet is turned on for the first time.
- In the initial model, assume the shape of the bathtub is cuboid, the shape of the person is a cylinder, and the person is stationary.

3 Analysis of the Problem

In order to keep the water temperature even, the heat from the faucet should approximately equal to the heat leaving the bathtub. Firstly, the authors analyze the heat injection into the system and microscopically analyze the heat expansion between water cells in the bathtub. In order to simplify the problem, we assume the hot water faucet is placed on the left side of the bathtub, when the faucet is turned on, the water flow of the faucet can be represented by the number of water cells that get temperature T_s instantaneously. Moreover, among water cells, we only consider the heat expansion from left to right. By Fourier's Law, we can acquire the heat flux of each tiny water cube in unit time through unit cross section area.

Theorem 3.1. *Fourier's Law:* $q = -\lambda \cdot \frac{\partial T}{\partial d}$

Then the heat expansion across the cross section of a water cell during time Δt can be calculated by

$$Q = q \cdot d^2 \cdot \Delta t \tag{1}$$

Assume Δt is small enough, apply Fourier's Law and the definition of specific heat capacity to equation (1), let ΔT represent the temperature change of the water cell at right, for every two neighboring cells, let T_R represent the initial temperature of the water cell at right, let T_L represent the initial temperature of the water cell at left, we get the equation

$$c \cdot \rho \cdot d^3 \cdot \Delta T = -\lambda \cdot \frac{T_R - T_L}{d} \cdot d^2 \cdot \Delta t \tag{2}$$

transform the equation as:

$$\Delta T = \frac{\lambda \cdot \Delta t}{c \cdot d^2 \cdot \rho} \cdot (T_L - T_R) \tag{3}$$

Nextly, we use Newton's law of cooling to analyze the heat exchange process between water in the bathtub and the air/person/container wall.

Theorem 3.2. *Newton's law of cooling:* $\frac{dT}{dt}$ = $k{\cdot}(T\text{-}\alpha)$

Suppose the temperature of water is T, the initial temperature of water is T_0, the temperature of the air/person/bathtub wall is represented as α. Transform the equation of Newton's law for cooling in the following way:

$$\frac{dT}{T - \alpha} = k \cdot dt \tag{4}$$

$$ln(T - \alpha) = k \cdot t + C, \; (C \, is \, a \, constant) \tag{5}$$

$$T - \alpha = e^{k \cdot t + C} \tag{6}$$

By equation (4), when t = 0, C = ln(T_0 - α), apply this to equation (5), we get the equation:

$$T = \alpha + e^{k \cdot t + ln(T_0)} \tag{7}$$

By simplifying, we get the equation:

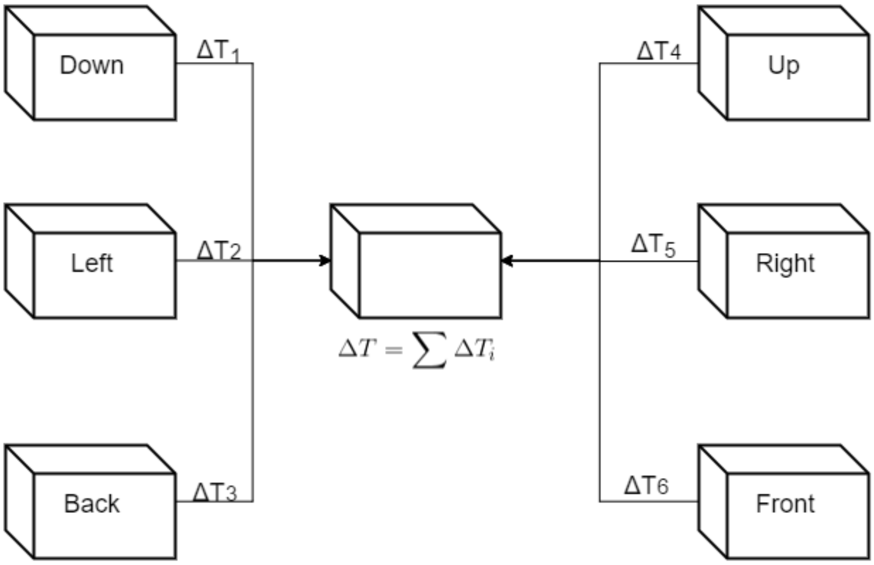
$$T - T_0 = (e^{k \cdot t} - 1) \cdot T_0 + (1 - e^{k \cdot t}) \cdot \alpha \tag{8}$$

In order to establish a model to evaluate the distribution of water temperature inside the container by equation (3) and (8), the author divide all the surfaces of water cubes into four categories:

- adjacent to another water cube
- adjacent to the container wall
- adjacent to the air
- adjacent to the person

The authors divide the water inside the bathtub into water cubes in the form of Figure 1, for each water cell inside the bathtub, calculate the heat exchange of its six surfaces to get its temperature after time period Δt, then the temperature of each water cube at the time Δt, 2·Δt, 3·Δt...will be obtained. Since the water cells and time interval Δt are small, then by calculating many times, the authors can mainly get the temperature distribution with respect to coordinate (x, y, z) and time.

4 Calculating and Simplifying the Model



Given that: $$\Delta T = \frac{\lambda \cdot \Delta t}{c \cdot d^2 \cdot \rho} \cdot (T_L - T_R) \quad \text{(Between water cells)}$$
$$= (e^{k \cdot t} - 1) \cdot T_0 + (1 - e^{k \cdot t}) \cdot \alpha \quad \text{(Between water and other medium)}$$

Figure 2: theory of algorithm

A dynamic array is the foundation of the implementation, allowing multiple simulations on various kinds of bathtubs. In order to simulate the structure of the model, firstly, an array of considerable size representing the whole bathtub will be created, within which each single cubic cell corresponds to infinitesimal of the mathematical model. Furthermore, the size of the array is flexible and able to conduct simulation on bathtubs of various sizes and shapes.

With the simplification and induction in part 3, the equation describing the temperature variation between consecutive simulation units has been deduced, and calculation is based on the equations. After every single operation, the temperature variation of a certain cell will be stored temporarily in memory for further operation. In the end, the temperature of each cell will be updated based on the data obtained from the previous step.

The implementation can simulate over 1000 times of heat transferring in one second. With massive calculation, simulation time in the unit of minute can be more precise and will lead to equilibrium of the system.

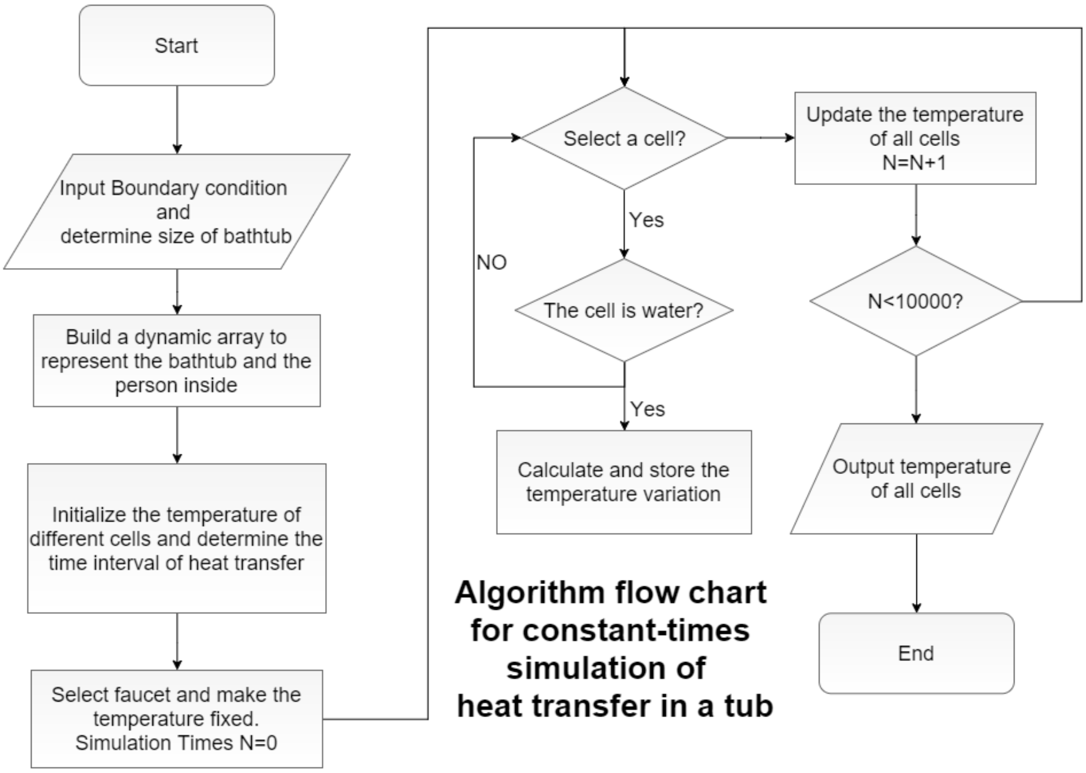


Figure 3: flow chart

5 Revise and improve the model

5.1 Influence of the shape and volume of the bathtub

In the model, the authors assume the heat transfer of water as heat exchange among solid cubes, however, water is fluid, so convection is an important factor that influences the heat transfer in water, so there will be a noticeable error when calculating the heat transfer of water inside the bathtub. In order to revise this error, when applying Fourier's Law, the authors add a term which represents the effect of fluid convection.

$$q = -\lambda \cdot \frac{\partial T}{\partial d} + q\prime \tag{9}$$

In equation (9), $q\prime$ is a variable that varies as the velocity of water or the shape/volume of the bathtub changes.
In order to determine the quantitive relation between the water velocity/shape and volume of the bathtub and $q\prime$, we apply the Flow rate continuation equation into this model.

Theorem 5.1. *Flow rate continuation equation:*

$$v_1 \cdot A_1 = v_2 \cdot A_2 = ... = v_i \cdot A_i = constant \tag{10}$$

where A_i denotes the area of cross section, v_i denotes the velocity of water crossing the cross section A_i.

We assume the bathtub bottom is flat with respect to y, denote the shape of the bathtub bottom by a function z = f(x), $h\prime$ denotes the maximum height of the bathtub, the length and height of the bathtub are the same as those in model one, g(x) denotes the area of the cross section of the person at the coordinate x, as shown in the figure. The velocity of water coming into and out of the bathtub are equal, denoted as v_0. Then for each x_0, the flow across the cross section whose x-coordinate is x_0 can be calculated by the equation:

$$v \cdot [(h\prime - f(x)) \cdot b - g(x)] = v_0 \cdot b \cdot h \tag{11}$$

We suppose that the volume of the bathtub equals to the cuboid bathtub in model one. Then we can build the equation:

$$b \cdot (\int_0^a (h\prime - f(x))dx) = a \cdot b \cdot \ h \tag{12}$$

In order to find a rough trend of the velocity fluctuation, the authors analyze the sine form of f(x). As shown in the Figure 4, we plot the graph of water velocity along x-coordinate.

By observing the figures, and known that q$\prime$ has a positive correlation between v, a

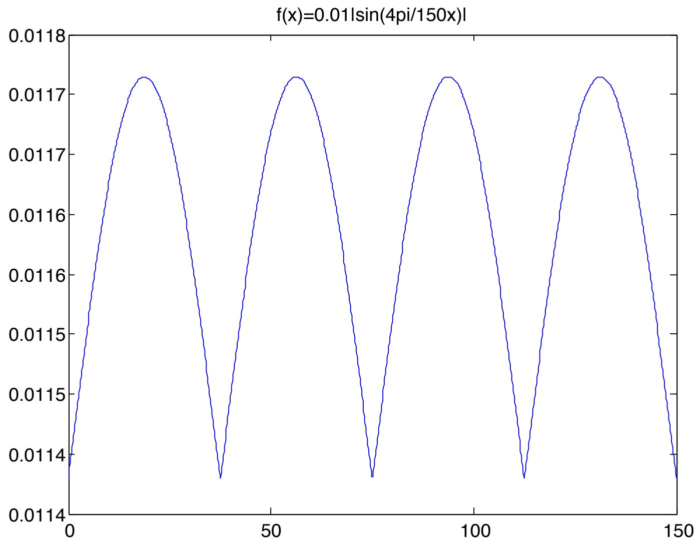


Figure 4: velocity of water when bottom is f(x)

qualitative conclusion can be drawn that when f(x) has several crests, the heat transfer of water in the bathtub is faster than that in a cuboid bathtub. More figures representing this trend will be shown in later chapters.

This fact implies that in order to keep the water temperature suitable and save water to a larger extent, the bottom of the bathtub should be a winding one. This is an important factor of the final optimal strategy.

5.2 Influence of the motion of the person

Under most realistic situations, the person is not stationary inside the bathtub, taken the person's motion into consideration, the velocity of heat transfer inside the water will increase since the water velocity is increased. The influence of the person's motion is similar to that of water convection.

5.3 Influence of the bubble layer

If the person uses bubble when taking a bath, then the bubble layer will obstruct the heat exchange between the water in the bathtub and the air, then in order to revise this error, the coefficient of convection heat transfer h_a between water and air should be adjusted to a smaller one. After fixing other factors, the heat injection will be unchanged, while the heat dissipation gets smaller.
We can acquire a qualitative result that adding bubbles into the bathtub is a way to slow the heat loss of water in the container, so the heat needed to be injected into the bathtub is smaller, thus saving water.

6 The Model Results

6.1 Velocity of water

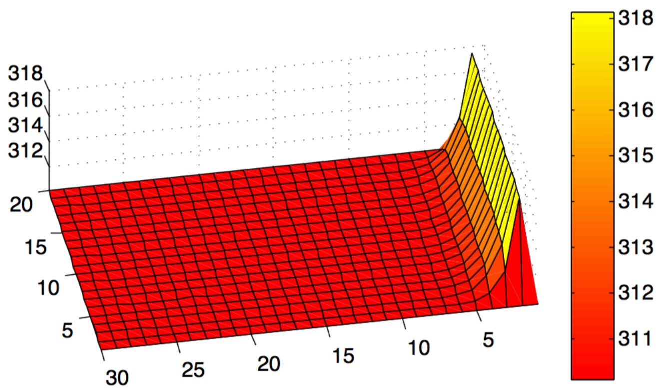


Figure 5: Steady mode plan view

Implement the process of heat transfer in 200 seconds, the statistics show that the difference of water temperature distribution between each cross section along the same axis is tiny, so the choices of layer to be plotted can be rather arbitrary. Figure 5 describes the distribution of temperature along a layer parallel to x-y plane when T_s = 318.15 K, t = 200s. Figure 6 describes the distribution of temperature along a layer parallel to y-z plane when T_s = 318.15 K, t = 200s. Figure 7 describes the distribution of temperature when the model reaches equilibrium when T_s = 318.15 K.
Taken water convection into consideration, as stated in section 5.1, the flux over unit area increases compared to the basic model, thus increasing the water velocity. Figure 8

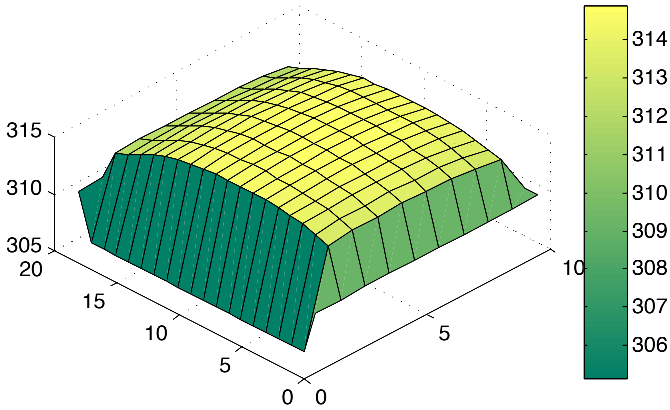


Figure 6: Steady mode lateral view

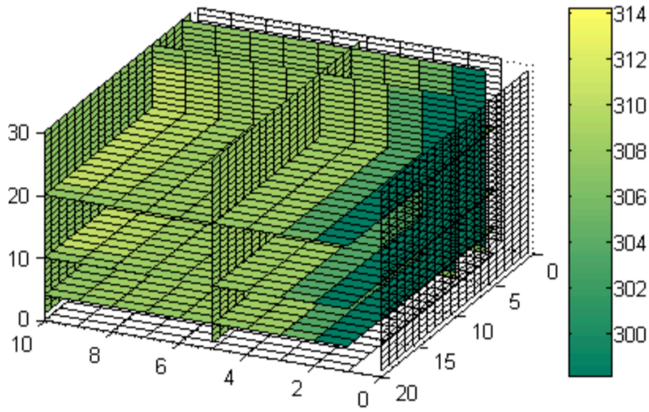


Figure 7: Terminal mode

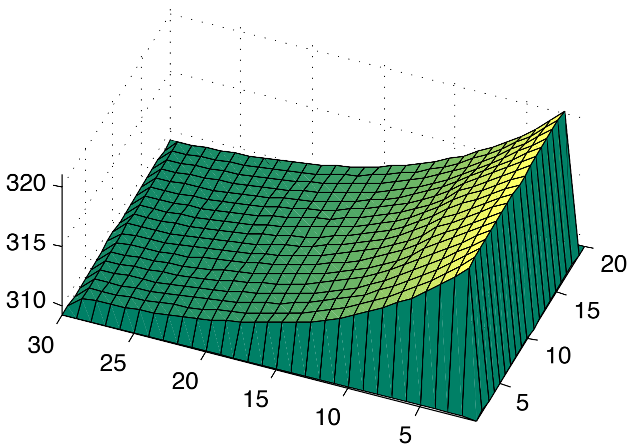


Figure 8: convection mode

describes the temperature distribution when T_s = 318.15 K, t = 200 s, convection taken into consideration. Observing the figure 5 and 8, it is obvious that as the velocity of water increases, the gradient of the temperature distribution graph becomes smaller, which implies that the temperature distribution becomes more uniform as velocity rises.

6.2 temperature of faucet water

In order to determine the relation between the temperature of faucet water and the temperature distribution inside the bathtub, factors other than temperature of faucet water should be kept unchanged. By fitting curves, a correlation can be found from Figure 9 that when temperature of the faucet water gets higher, it will take less time for the system to reach equilibrium. The figure also shows that in an equilibrium system, the standard deviation of water temperature increases as the water flow from the faucet increases. This result implies that increasing the temperature of faucet water makes the water temperature distribution inside the bathtub at equilibrium state less uniform, however, it takes less time for the model to reach its equilibrium.

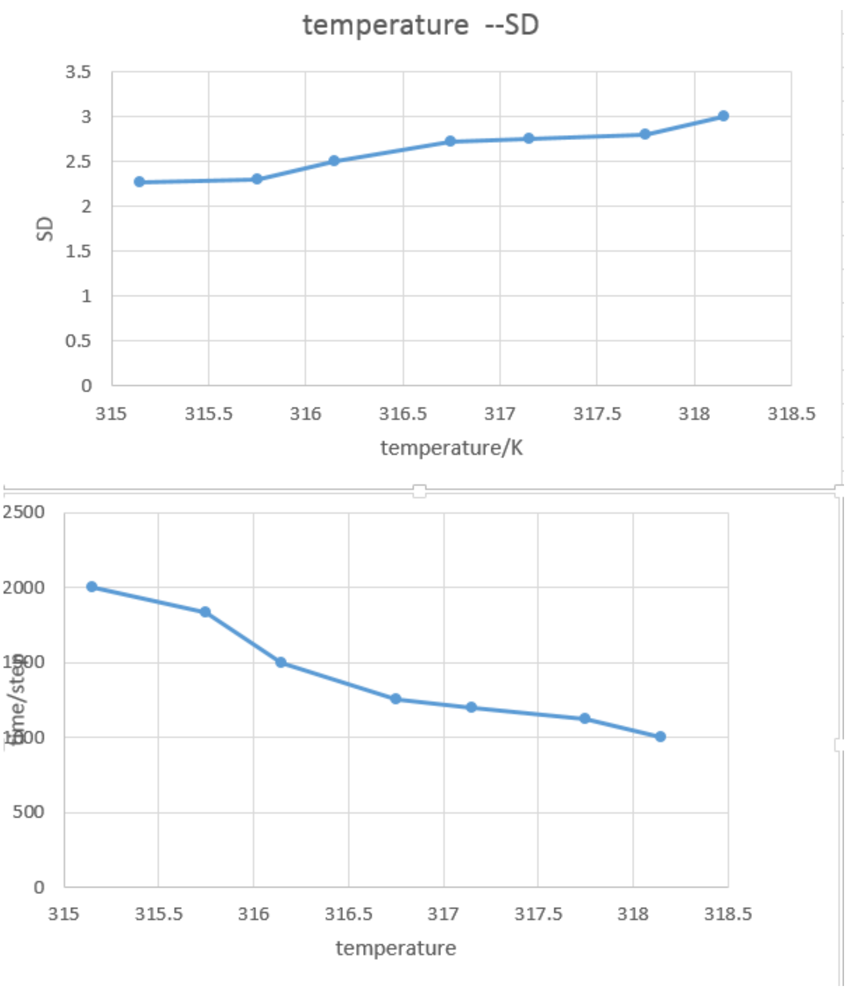


Figure 9: influence of temperature of faucet water

6.3 Water flow of the faucet

Keeping other factors unchanged, our algorithm control water flow of the faucet by adjusting the quantity of water cubes that get temperature T_s initially. Figure 10 describes the trend of standard deviation of the temperature distribution as the water flow increases. Figure 10 shows that as water flow rises, the standard deviation of water temperature distribution increases linearly. Figure 11 is a temperature distribution when T_s = 318.15 K, t = 200 s, and the water flow is the half of the flow of Figure 5. Comparing Figure 5 and Figure 11, it is obvious that by lowering water flow of the faucet, the temperature distribution will be more uniform.

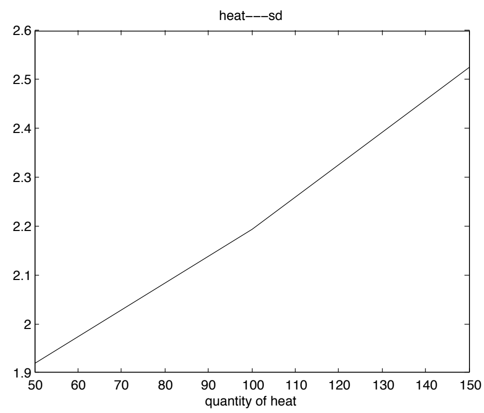


Figure 10: sd – water folw

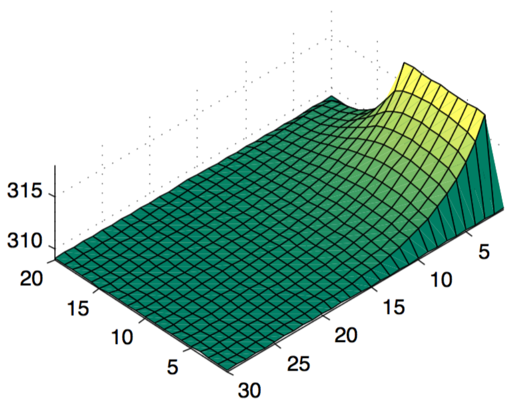


Figure 11: lower water flow

7 Conclusions

If the person wants to get a more uniform water temperature distribution inside the bathtub, the water flow should be relatively small, and the temperature of faucet water should be lower. On the other hand, if the person attempts to get an equilibrium sooner, the temperature of faucet water should be high, or the customer should choose a bathtub with a winding bottom. What's more, it will take more time for a large bathtub to reach

equilibrium and the heat needed will be more, thus wasting water.
For the convenience of making an optimal strategy, we introduce a new variable called preference coefficient, denoted as P. A group of equations of P :

$$\begin{cases} P = 4 \cdot \varphi \cdot sd + \psi \cdot V \\ \varphi + \psi = 10 \end{cases} \tag{13}$$

where sd represents the value of standard deviation of temperature of water cells, V represents the value of total volume of water consumed under the unit L, φ is a number between 0-10 that represents the person's tendentiousness of bathing in a uniform temperature system, ψ is a number between 0-10 that represents the person's tendentiousness of saving water. The standard deviation of temperature distribution and the consume of water are important factors that measures the quality of the model. By analyzing and comparing the statistics, it is found that four times the value of standard deviation of temperature is near to the value of volume of water consumed during bathing. Hence such a P is a representative variable to judge the optimal strategy for the customer. After φ and ψ are fixed by the customer, initialize a group of different T_s and water flow, then execute our algorithm by the process stated in section 4 and calculate P for each situation, compare the statistics, then choose the strategy that lead to a relative minimum P as the optimal strategy. Figure 12 describes the relation between water flow (unit: L/min) under the optimal strategy and the ratio φ/ψ

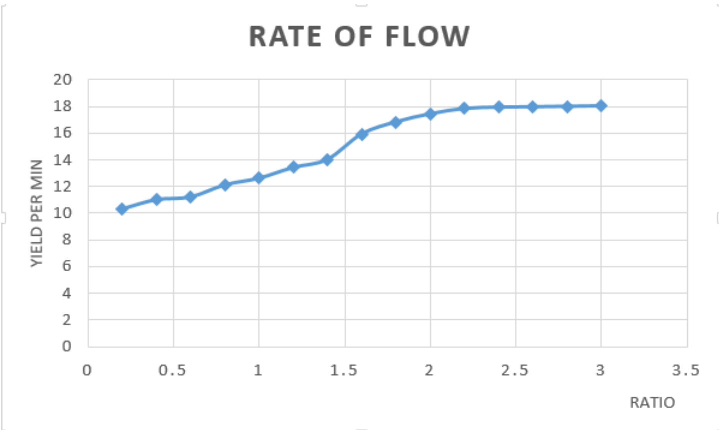


Figure 12: water flow-ratio

8 Evaluate of the Model

8.1 Strengths

- The authors analyze this problem with differential element method, since this method has a great generality, it's convenient to apply this model into other heat transfer cases.

- When implementing the model by the algorithm stated in section 4, the using of dynamic array guarantees the accuracy of the model as well as the convenience of implementation.

- This model can simulate the thermal field at any time. It is a practical model to solve the problem in which heat expansion velocity cannot be accurately described in a complex system. If the calculation in this model is done with a super computer, time error can be very small.

- The model can be used to deal with other problems on heat transfer. Application field of this model does not depend on many factors such as shape/volume/-mass/medium and so on. It can be applied widely, and keep accuracy in bulky system where other models will not work.

- The model shows the thermal field clearly. So it is easy to calculate the standard deviation and mean of the water temperature in the bathtub so that the whole bathtub can be analyzed in a visual mode.

- More potential improvement can be made to this model. With a more complicated algorithm, a group of neighboring unit cells can be used to simulate the person in motion, the shape and volume can be rather random, by implement the algorithm, the influence of the person's shape and volume will be shown intuitively.

8.2 Weaknesses

- The authors discrete the water in the bathtub into tiny cubes, and discrete a time period into small intervals. Hence if we want to get a more accurate distribution of temperature in space and time, we must narrow the volume of water cubes and the length of time interval. Unfortunately, when narrowing these parameters, the workload of the computing process will be very large, this algorithm is time consuming.
- When building this model, the kinetic energy of faucet water is ignored. Actually, as the water flow of the faucet increases, the kinetic energy of the faucet water also increases, the rate of flow inside the bathtub will be larger, thus accelerating the heat transfer inside the bathtub. In this case, the time-water flow figure(Figure 8) might not be accurate.
- The model assumes that the heat transfer only in the direction from left to right, this simulation weakens the actual influence of convection and can lead to significant deviation.
- The model takes little consideration on the person's shape and volume, so there will be errors caused by these factors.

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Appendices

Appendix A First appendix

Here are simulation programmes we used in our model as follow.

```
int mapp(int r, int c, int h){
        return (r*COL + c + ROW*COL*h);
}

void initializeTANK(Tub tank[]){
        //get surface and air respectively
        //get bottom(surface) and top(air)
        for (int i = 0; i < ROW; i++){
                for (int j = 0; j < COL; j++){
                        tank[mapp(i, j, 0)].updateType('s');
                        tank[mapp(i, j, 0)].updateTemper(ALPHAS);
                        tank[mapp(i, j,
0)].updatePreTemper(ALPHAS);
                        tank[mapp(i, j, HEI - 1)].updateType('a');
                        tank[mapp(i, j, HEI -
1)].updateTemper(ALPHAA);
                        tank[mapp(i, j, HEI -
1)].updatePreTemper(ALPHAA);
} }

        //get right and left surface
        for (int i = 0; i < ROW; i++){
                for (int k = 0; k < HEI; k++){
                        tank[mapp(i, 0, k)].updateType('s');
                        tank[mapp(i, 0, k)].updateTemper(ALPHAS);
                        tank[mapp(i, 0,
k)].updatePreTemper(ALPHAS);
                        tank[mapp(i, HEI - 1, k)].updateType('s');
                        tank[mapp(i, HEI - 1,
k)].updateTemper(ALPHAS);
                        tank[mapp(i, HEI - 1,
k)].updatePreTemper(ALPHAS);
} }

        //get front and back surface
        for (int j = 0; j < ROW; j++){
                for (int k = 0; k < HEI; k++){
                        tank[mapp(0, j, k)].updateType('s');
                        tank[mapp(0, j, k)].updateTemper(ALPHAS);
                        tank[mapp(0, j,
k)].updatePreTemper(ALPHAS);
                        tank[mapp(ROW - 1, j, k)].updateType('s');
                        tank[mapp(ROW - 1, j,
k)].updateTemper(ALPHAS);
                        tank[mapp(ROW - 1, j,
k)].updatePreTemper(ALPHAS);
} }

        //get body
        for (int i = 0; i < ROW; i++){
                for (int j = 0; j < COL; j++){
                        if (pow(i - ROW / 2.0, 2) + pow(j - COL /
2.0, 2) <=
2)){
k)].updateType('b');
k)].updateTemper(ALPHAB);
k)].updatePreTemper(ALPHAB);
pow(fmin(ROW / 4.5, COL / 4.5),
for (int k = 2; k < HEI - 2; k++){
        tank[mapp(i, j,
        tank[mapp(i, j,
        tank[mapp(i, j,
}

} }

} }
```

```
double calculateT(Tub tank[], int r_1, int c_1, int h_1, int r_2,
int c_2, int h_2){
h_2);

int index_1 = mapp(r_1, c_1, h_1), index_2 = mapp(r_2, c_2,
if (tank[index_1].getType() == 'w'){
        return tank[index_1].heatDiff(tank[index_2]);
} else

}

return 0;

void updateBlock(Tub tank[], int r, int c, int h){
        int aa[6] = { -1, 0, 0, 1, 0, 0 };
        int bb[6] = { 0, -1, 0, 0, 1, 0 };
        int cc[6] = { 0, 0, -1, 0, 0, 1 };
        double deltaT = 0;
        for (int i = 0; i < 6; i++){
                 deltaT += calculateT(tank, r, c, h, r + aa[i], c +
bb[i], h + cc[i]);
}

        tank[mapp(r, c, h)].updateTemper(tank[mapp(r, c,
h)].getTemper() + deltaT);
}

void updateAll(Tub tank[]){
        for (int i = 1; i < ROW - 1; i++){
                 for (int j = 1; j < COL - 1; j++){
                         for (int k = 1; k < HEI - 1; k++){
                                  updateBlock(tank, i, j, k);
} }

        }
        for (int i = 1; i < ROW - 1; i++){
                 for (int j = 1; j < COL - 1; j++){
                         for (int k = 1; k < HEI - 1; k++){
tank[mapp(i, j,
k)].updatePreTemper(tank[mapp(i, j, k)].getTemper());
} }

} }
```

Diffculty

To maintain equilibrium of the water temperature in the bathtub is not as simple as one can imagine. In other words, several factors make it difficult to keep the temperature throughout the bath water evenly distributed.

First of all, compared to the heat transfer rate between water and other material, that inside the water is relatively slow. During the process of heat transfer in the bathtub, the heat brought by the hot water from the faucet would take a long time to transfer through the tub. After a certain period, the temperature distribution would not be uniform but rather in the form of a slope.

In order to tackle the first impact factor, larger amount of hot water can be added into the tub, enhancing the heat transfer in a unit time. However, this "solution" can leads to another problem that excessive hot water can bring about over-heated situation and it cause unnecessary water waste.

Moreover, the motion of the person will raise considerable uncertain factors to the temperature distribution. The unpredictable motion not only influence the precision of theoretical simulation, but also break the equilibrium of real-life system.

Our inspired solutions

1. Keep the temperature of heat source inside the interval between 111.2 Fahrenheit and 114.8 Fahrenheit to maintain a reasonable heat transfer rate.

2. Try more bubble bath to minimize the heat dissipation between the air and water surface.

3. Enlarge the water flow of the faucet. It will be more comfortable as the water flow become large. However, it is recommend that the flow should be around 11 L/min.

4. Consider various shape of the bottom. Certain bottom type can be easier to reach and maintain uniform temperature distribution.