String amplitudes and automorphic representations (Lecture 1)

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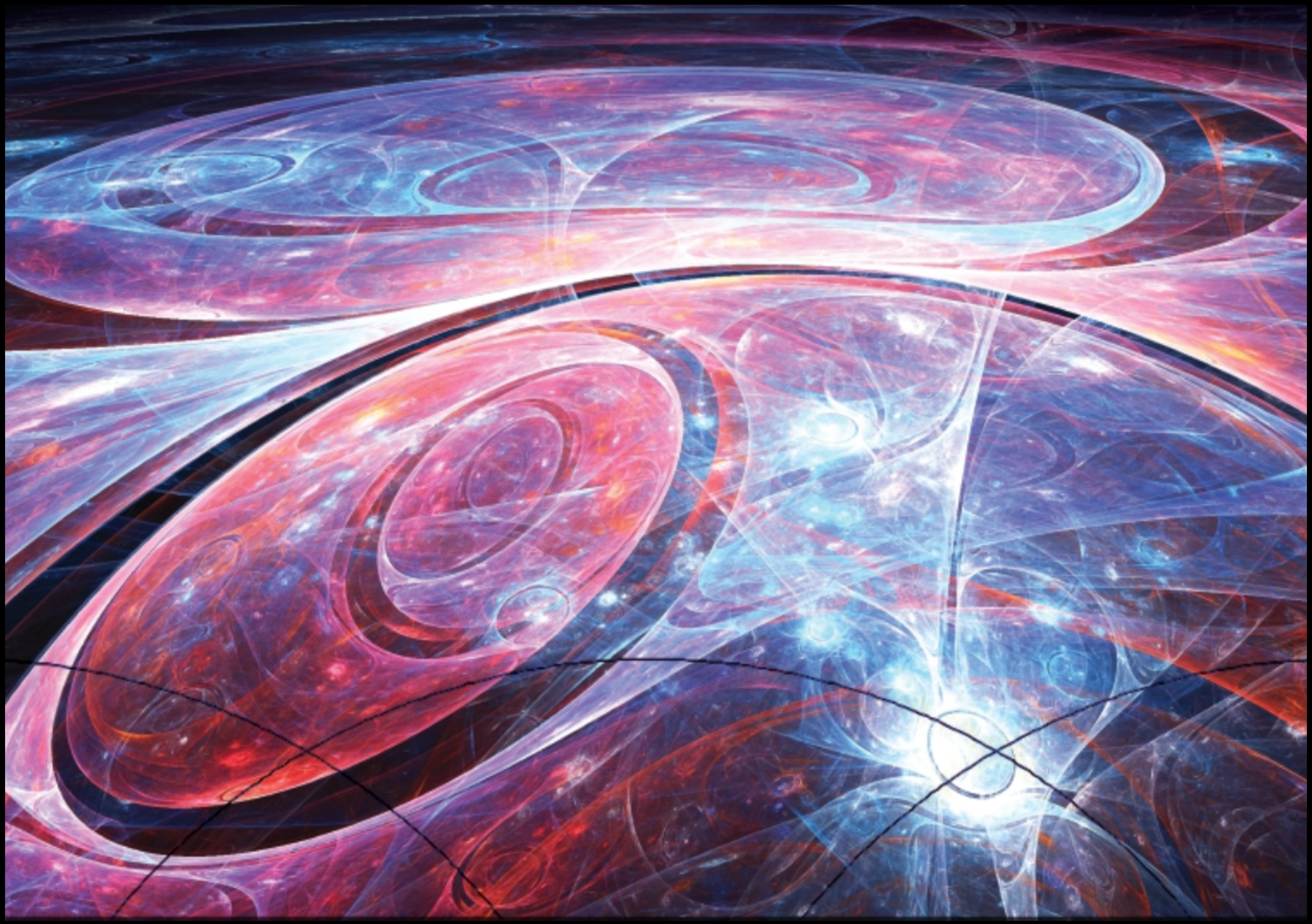
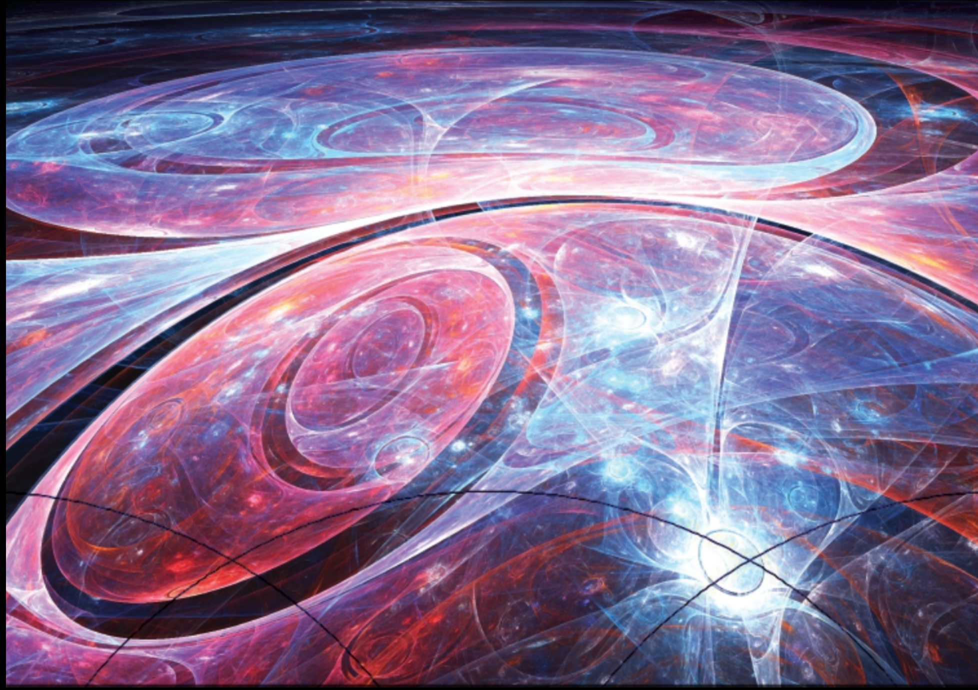


Bhaskara Math Seminars — Harish-Chandra Lecture Series
August 26, 2024

Outline



1. Motivation

2. Automorphic forms and representation theory

3. Small representations

4. Outlook

Suggested literature

Survey: "String scattering amplitudes and small automorphic representations"

w/ Guillaume Bossard and Axel Kleinschmidt, 2024

(Waseda number theory workshop - proceedings)

Book: "Eisenstein series and automorphic representations - with applications in string theory"

w/ Philipp Fleig, Axel Kleinschmidt and Henrik Gustafsson

Cambridge University Press, 2018

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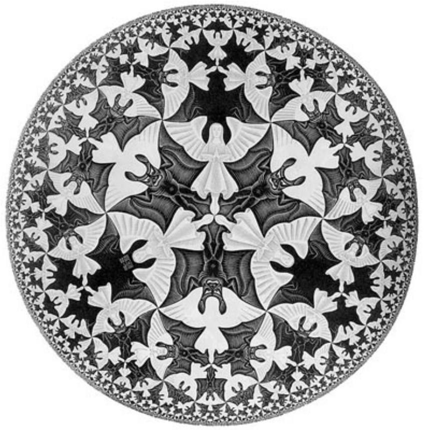
Ahlén, Marcus Berg, Guillaume Bossard, Philipp Fleig, Solomon Friedberg, David Ginzburg, Dmitry Gourevitch, Michael Green, Henrik Gustafsson, Axel Kleinschmidt, Liu,Steve Miller, Manish Patnaik, Boris Pioline, Siddhartha Sahi, Pierre Vanhove

1. Motivation

Fourier coefficients of automorphic forms



- Fourier coefficients of **classical modular forms** encode deep number-theoretic information (counting points on elliptic curves etc..)

- **Moonshine:** relations with finite sporadic groups and CFT/string theory

- **Enumerative geometry:** rational curves on K3, GW-theory…

- Higher rank groups: **Langlands program** (automorphic L-functions, functoriality…)

See **Sol Friedberg**'s talk in this series!

- The Fourier coefficients of Eisenstein series also encode **string theory scattering amplitudes**

QFT amplitudes

Compute the probability of a scattering process between elementary particles

$$\mathcal{A}(p_1, p_2, p_3, p_4) \in \mathbb{C}$$

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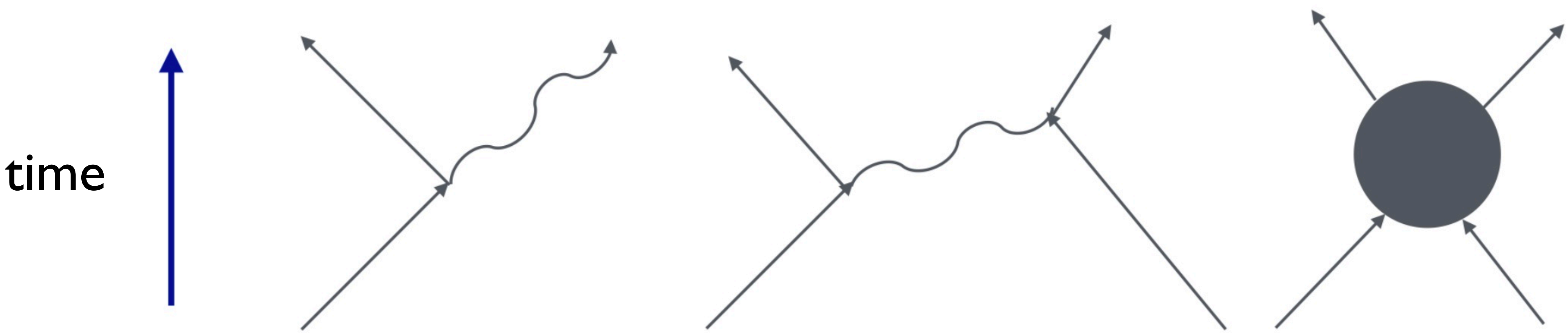

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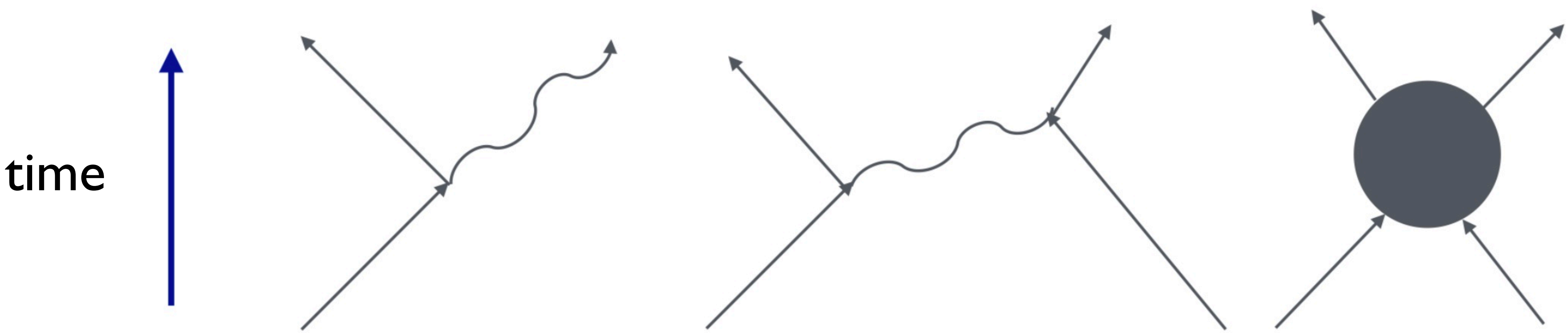


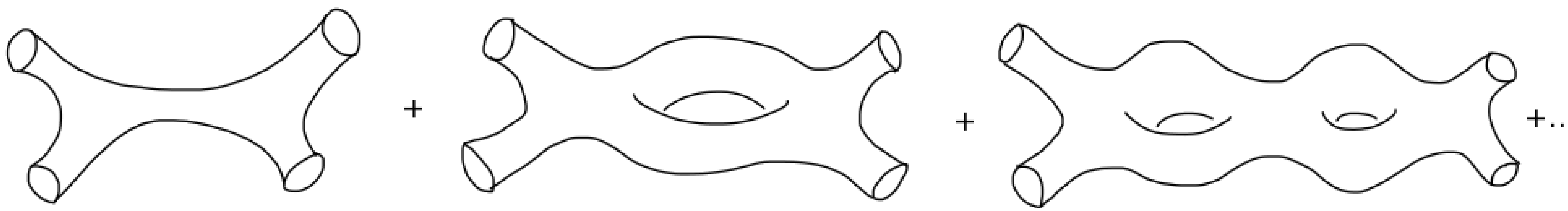
Perturbation theory: expand for small coupling g

$$\mathcal{A}(p_1, p_2, p_3, p_4) = \sum_{n \geq 0} g^n \mathcal{A}_n(p_1, p_2, p_3, p_4) + \mathcal{A}^{\text{non-pert.}}(p_1, p_2, p_3 . p_4)$$

$$\sim e^{-1/g}$$

String amplitudes

Understand the structure of **string interactions**



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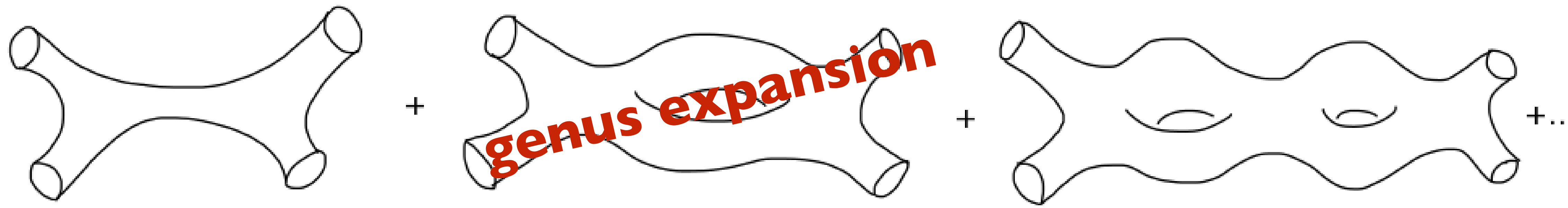

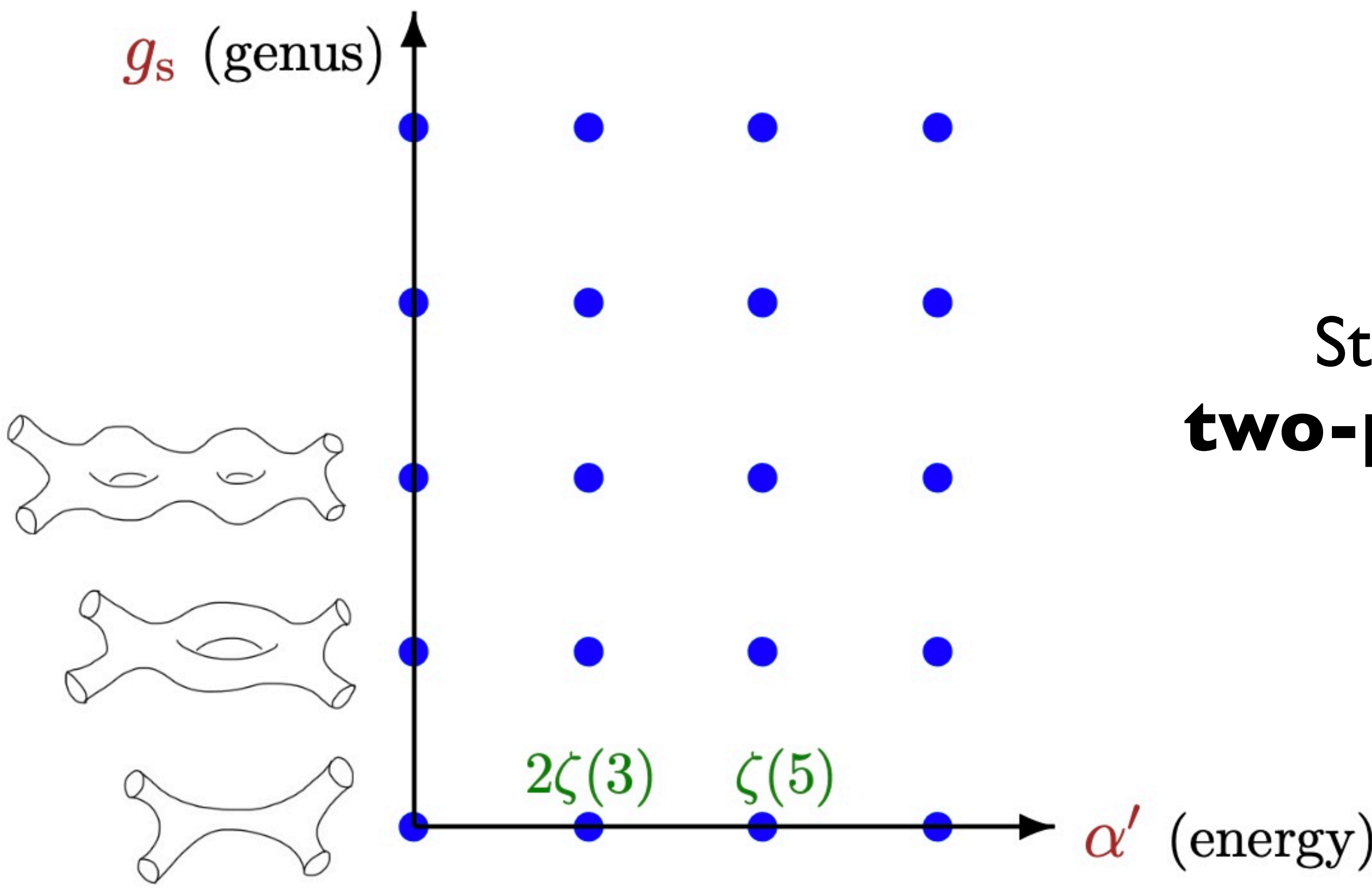

String amplitudes have a
two-parameter expansion

String amplitudes

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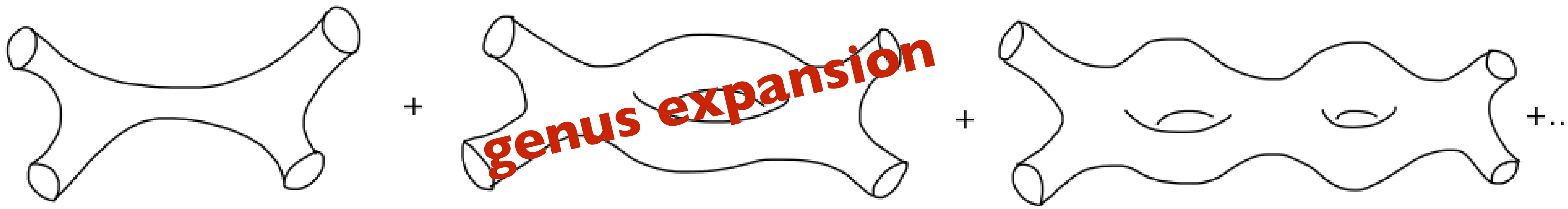


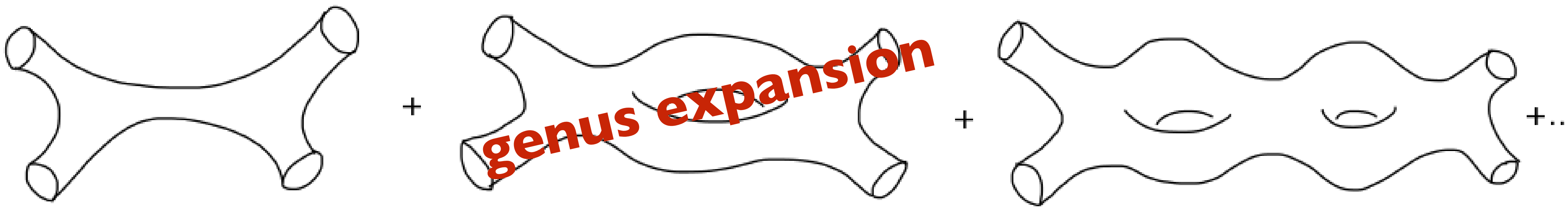
Strongly constrained by **symmetries**!

- supersymmetry
- U-duality

amplitudes have intricate
arithmetic structure $G(\mathbb{Z})$

String amplitudes

Understand the structure of **string interactions**



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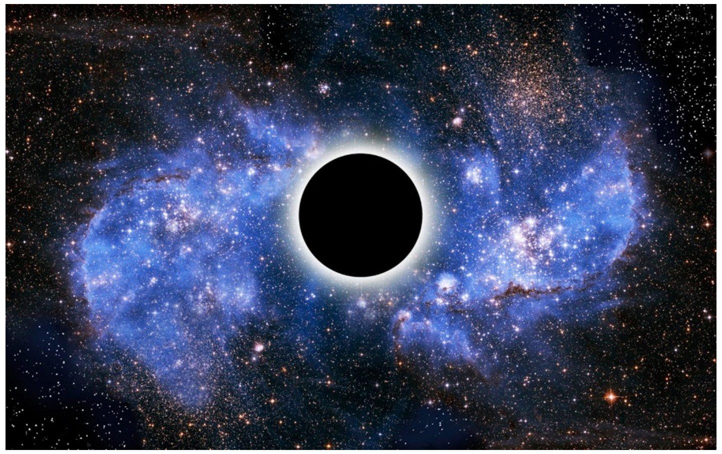
amplitudes have intricate **arithmetic structure** $G(\mathbb{Z})$

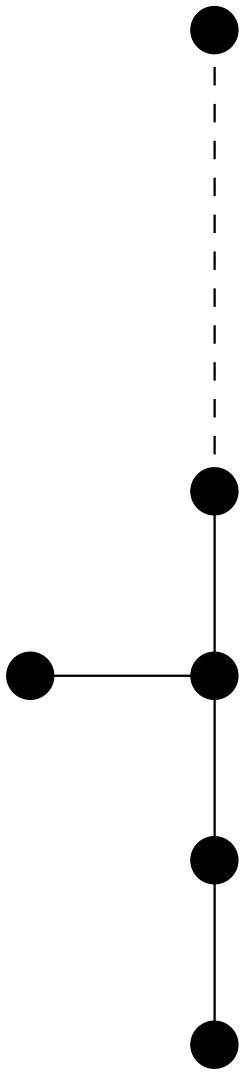
Symmetry constrains interactions, leads to insights about:

- ultraviolet properties of gravity
- non-perturbative effects (black holes, instantons)
- novel mathematical predictions from physics



Toroidal compactifications yield the chain of **U-duality groups**

[Cremmer, Julia][Hull, Townsend]



D	G	K	$G(\mathbb{Z})$
10	$\mathrm{SL}(2,\mathbb{R})$	$\mathrm{SO}(2)$	$\mathrm{SL}(2,\mathbb{Z})$
9	$\mathrm{SL}(2,\mathbb{R}) \times \mathbb{R}^+$	$\mathrm{SO}(2)$	$\mathrm{SL}(2,\mathbb{Z})$
8	$\mathrm{SL}(3,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$	$\mathrm{SO}(3) \times \mathrm{SO}(2)$	$\mathrm{SL}(3,\mathbb{Z}) \times \mathrm{SL}(2,\mathbb{Z})$
7	$\mathrm{SL}(5,\mathbb{R})$	$\mathrm{SO}(5)$	$\mathrm{SL}(5,\mathbb{Z})$
6	$\mathrm{Spin}(5,5,\mathbb{R})$	$\big(\mathrm{Spin}(5) \times \mathrm{Spin}(5)\big)/\mathbb{Z}_2$	$\mathrm{Spin}(5,5,\mathbb{Z})$
5	$\mathrm{E}_6(\mathbb{R})$	$\mathrm{USp}(8)/\mathbb{Z}_2$	$\mathrm{E}_6(\mathbb{Z})$
4	$\mathrm{E}_7(\mathbb{R})$	$\mathrm{SU}(8)/\mathbb{Z}_2$	$\mathrm{E}_7(\mathbb{Z})$
3	$\mathrm{E}_8(\mathbb{R})$	$\mathrm{Spin}(16)/\mathbb{Z}_2$	$\mathrm{E}_8(\mathbb{Z})$

Amplitudes are given by **functions** on

$$G(\mathbb{Z})\backslash G(\mathbb{R})/K$$

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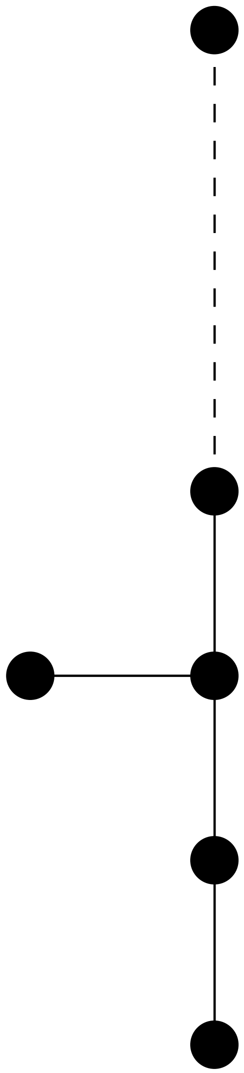

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Amplitudes $\longrightarrow$ **Action functional**

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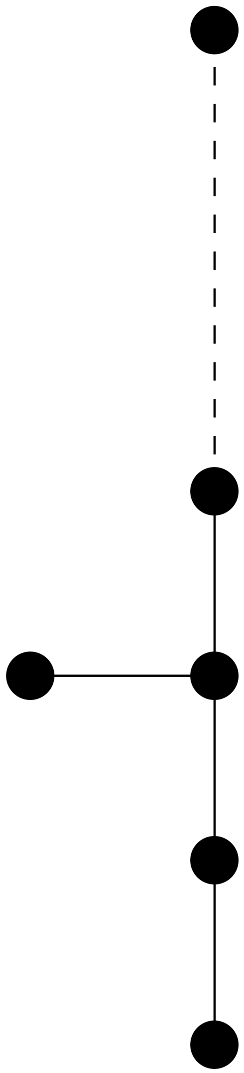

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Amplitudes $\longrightarrow$ **Action functional**

Einstein-Hilbert action $\quad \displaystyle\int d^4x\sqrt{G}R$ $\longrightarrow$ Einstein's equations

Higher-derivative action in type II string theory

Action functional:

$$\int d^{10-n}x\sqrt{G}\left[(\alpha')^3 f_0(g)\mathcal{R}^4 + (\alpha')^5 f_4(g)\partial^4\mathcal{R}^4 + \cdots\right]$$

Higher-derivative action in type II string theory

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contraction of four Riemann tensors

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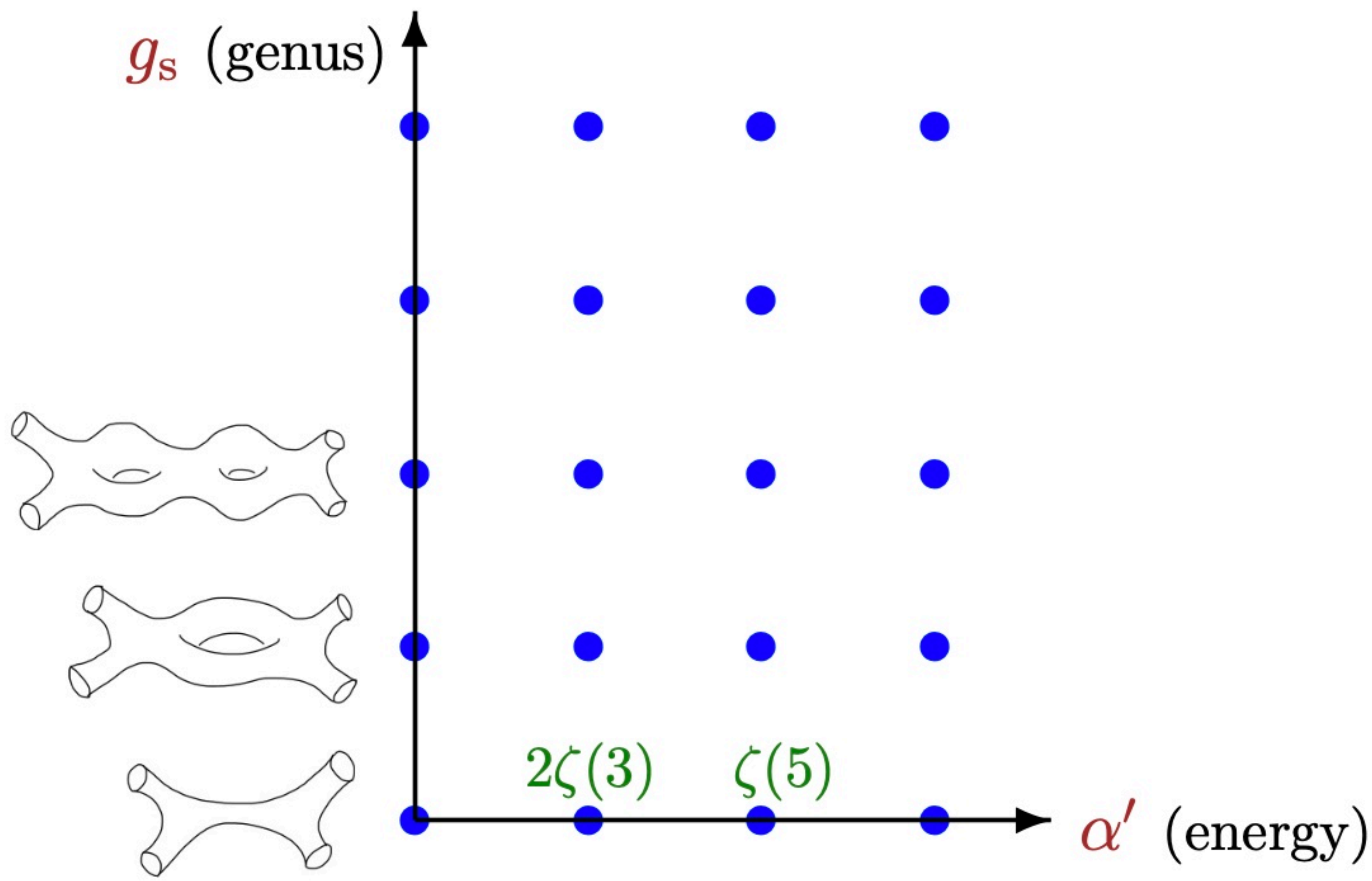

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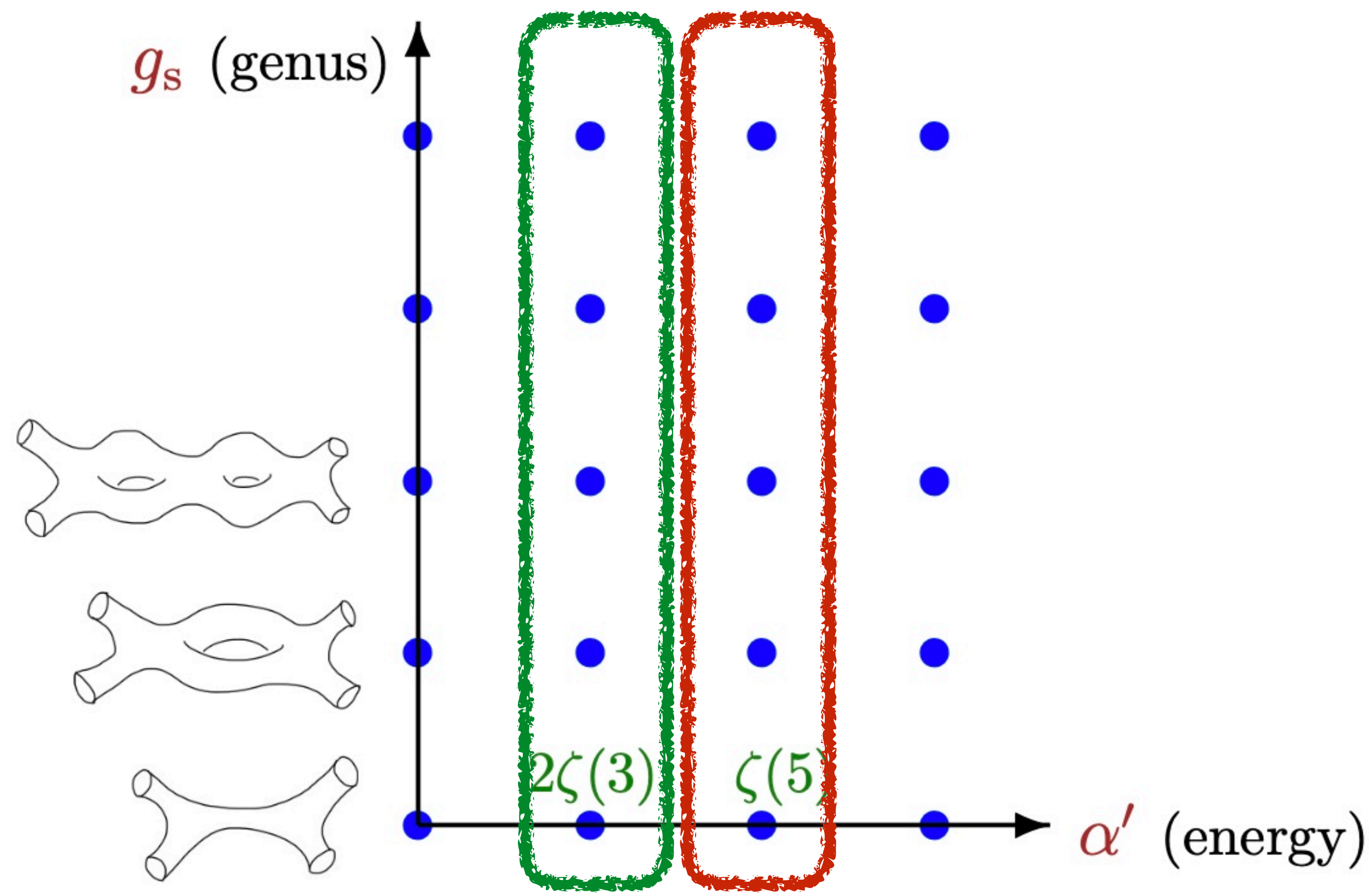

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- $f_0(g), f_4(g)$ are functions of $g \in E_{n+1}(\mathbb{R})/K$
- must be **invariant** under U-duality $E_{n+1}(\mathbb{Z})$
- supersymmetry requires that they are **Laplacian eigenfunctions**
- well-defined **weak-coupling expansions** as $g_s \to 0$

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defining properties of an **automorphic form!**

Example: type IIB in D=10

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unique solution!

$$\boxed{f_0(\tau) = \sum_{(m,n)\neq(0,0)} \frac{y^{3/2}}{|m+n\tau|^3}}$$

[Green, Gutperle]
[Green, Sethi]
[Pioline]

Non-holomorphic Eisenstein series

Consider the sum:

$$E_s(\tau) = \sum_{(c,d)=1} \frac{y^s}{|c\tau + d|^{2s}}$$

non-holomorphic Eisenstein series

$s \in \mathbb{C}$

→ a function on $\mathbb{H} = \{\tau = x + iy \in \mathbb{C} \,|\, y > 0\}$

→ invariant under $\tau \mapsto \gamma \cdot \tau = \dfrac{a\tau + b}{c\tau + d}$

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z})$$

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→ invariant under $\tau \mapsto \gamma \cdot \tau = \dfrac{a\tau + b}{c\tau + d}$

→ converges absolutely for $\Re s > 1$

→ $\Delta_{\mathbb{H}} E_s = s(s-1) E_s$

Invariance under $\tau \mapsto \tau + 1$ yields the **Fourier expansion**

$$E_s(\tau) = \underbrace{C(y;s)}_{\substack{\text{constant term} \\ \text{zero mode}}} + \underbrace{\sum_{n\neq 0} F_n(y;s) e^{2\pi i n x}}_{\text{non-zero mode}}$$

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$$\rightarrow \quad C(y;s) = y^s + \frac{\xi(2s-1)}{\xi(2s)} y^{1-s}$$

completed zeta-function:

$$\xi(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s)$$

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divisor sum: $\mu_s(n) = \sum_{d|n} d^s$

modified Bessel function

Example: $G = SL(2,\mathbb{R})$

$$\int d^{10}x\sqrt{G}f_0(\tau)\mathcal{R}^4$$

$$f_0(\tau) = \overbrace{\underbrace{2\zeta(3)y^{3/2}}_{\text{tree-level}} + \underbrace{4\zeta(2)y^{-1/2}}_{\text{one-loop}}}^{\text{perturbative terms}} + \overbrace{2\pi\sum_{m\neq 0}\sqrt{|m|}\sigma_{-2}(m)e^{-S_{\text{inst}}(z)}\left[1+\mathcal{O}(y^{-1})\right]}^{\text{non-perturbative terms}}$$

amplitudes in the presence of instantons

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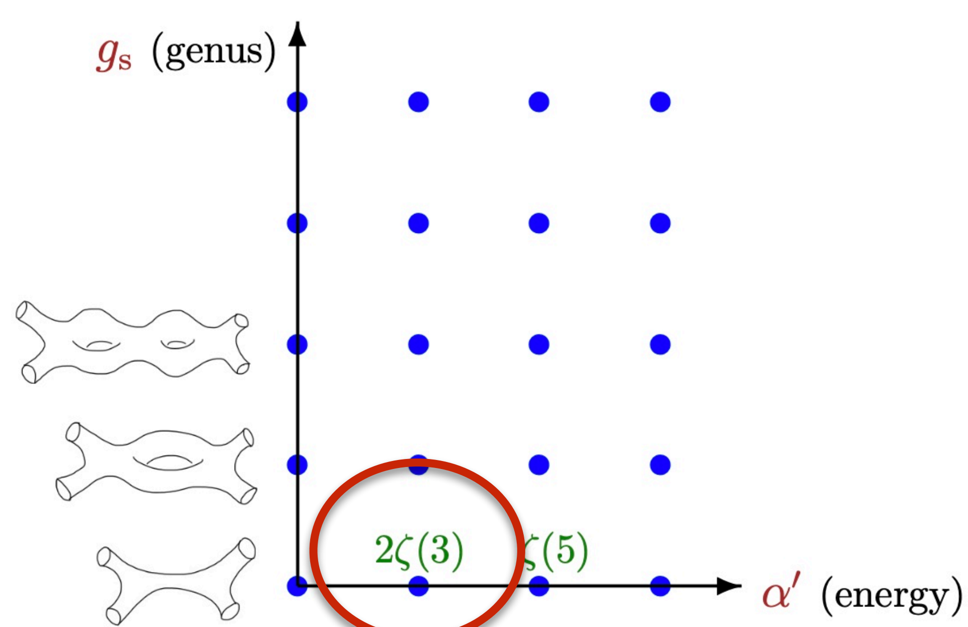

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amplitudes in the presence of instantons

instanton action

$$S_{\text{inst}}(z) := 2\pi\,|m|\,y - 2\pi i m x$$

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$$S_{\text{inst}}(z) := 2\pi\,|m|\,y - 2\pi i m x$$

instanton measure

$$\sigma_{-2}(m) = \sum_{d|m} d^{-2}$$

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amplitudes in the presence of instantons

Fourier coefficients of automorphic forms encode information about scattering amplitudes!

2. Automorphic forms and representation theory

Data:

- $G(\mathbb{R})$ real simple Lie group (e.g. $SL(n,\mathbb{R})$)
- $G(\mathbb{Z}) \subset G$ arithmetic subgroup (e.g. $SL(n,\mathbb{Z})$)

Definition:

An **automorphic form** is a smooth function $\varphi : G \longrightarrow \mathbb{C}$ satisfying

1. Automorphy: $\forall \gamma \in G(\mathbb{Z}),\ \ \varphi(\gamma g) = \varphi(g)$
2. φ is an eigenfunction of the ring of inv. diff. operators on G
3. φ has well-behaved growth conditions

Sometimes we also add a condition involving the maximal compact subgroup (K -finiteness)

Eisenstein series on semi-simple Lie groups

The **Langlands Eisenstein series** on a semi-simple Lie group is defined by:

$$\boxed{E(\lambda, g) = \sum_{\gamma \in B(\mathbb{Z})\backslash G(\mathbb{Z})} e^{\langle \lambda + \rho | H(\gamma g) \rangle}}$$

Eisenstein series on semi-simple Lie groups

The **Langlands Eisenstein series** on a semi-simple Lie group is defined by:

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Iwasawa decomposition: $G = BK = NAK$

$$A \sim \begin{pmatrix} * & & \\ & * & \\ & & * \end{pmatrix} \qquad N \sim \begin{pmatrix} 1 & * & * \\ & 1 & * \\ & & 1 \end{pmatrix}$$

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Weight: $\lambda \in \mathfrak{h}^\star \otimes \mathbb{C}$

Weyl vector: $\rho = \frac{1}{2} \sum_{\alpha > 0} \alpha$

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→ Converges absolutely on a subspace of $\mathfrak{h}^\star \otimes \mathbb{C}$

Godement's domain
$\{\lambda | \langle \lambda, \alpha \rangle > 1, \forall \alpha \in \Pi\}$

→ Can be continued to a meromorphic function on all of $\mathfrak{h}^\star \otimes \mathbb{C}$ [Langlands]

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- Can be continued to a meromorphic function on all of $\mathfrak{h}^\star \otimes \mathbb{C}$ [Langlands]
- Invariant: $E(\lambda, \gamma g k) = E(\lambda, g) \qquad \gamma \in G(\mathbb{Z}) \qquad k \in K$
- Eigenfunction of the Laplacian: $\Delta_{G/K} E(\lambda, g) = \frac{1}{2}(\langle \lambda | \lambda \rangle - \langle \rho | \rho \rangle) E(\lambda, g)$
- Functional relation: $E(\lambda, g) = M(w, \lambda) E(w\lambda, g), \quad \forall w \in W(\mathfrak{g})$

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Example: $G = SL(2,\mathbb{R})$

$$E(s,g) = \sum_{(c,d)=1} \frac{y^s}{|c\tau + d|^{2s}} = \sum_{\gamma \in B(\mathbb{Z})\backslash SL(2,\mathbb{Z})} e^{\langle \lambda+\rho | H(\gamma g)\rangle}$$

$\lambda + \rho = 2s\Lambda$ (fundamental weight: $\Lambda = \alpha/2$)

$H(a) = H(e^{yH_\alpha}) = yH_\alpha$ $\langle \Lambda | H_\alpha \rangle = 1$

Automorphic representations

Eisenstein series are attached to the (non-unitary) **principal series:**

$$I(\lambda) = \mathrm{Ind}_B^G \chi = \{ f : G \to \mathbb{C} \,|\, f(bg) = \chi(b) f(g), b \in B \}$$

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The theory of Eisenstein series then defines a map

$$E \, : \, I(\lambda) \to \mathcal{A}(G(\mathbb{Z}) \backslash G(\mathbb{R}))$$

from the principal series to **the space of automorphic forms on** $G(\mathbb{R})$

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G acts on $\mathcal{A}(G(\mathbb{Z})\backslash G(\mathbb{R}))$ by **right-translation:**

$$[\rho(h) f](g) = f(gh)$$

The irreducible constituents in the decomposition of $\mathcal{A}(G(\mathbb{Z})\backslash G(\mathbb{R}))$ under this action are called **automorphic representations**

[Gelfand, Graev, Piatetski-Shapiro][Langlands]...

Toy model: Fourier analysis on $\mathbb{Z}\backslash\mathbb{R} \cong S^1$

Any function $f \in C^\infty(\mathbb{Z}\backslash\mathbb{R})$ can be decomposed into a Fourier series:

$$f(x) = \sum_{k\in\mathbb{Z}} c_k \psi_k(x)$$

$\psi_k \,:\, \mathbb{Z}\backslash\mathbb{R} \to U(1)$ $\qquad$ $\psi_k(x) = e^{2\pi i k x} \quad k \in \mathbb{Z},\, x \in \mathbb{R}$

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Moderate growth: restrict to **square integrable functions**

$$L^2(\mathbb{Z}\backslash\mathbb{R}) = \{f \in C^\infty(\mathbb{Z}\backslash\mathbb{R}) \mid \sum_{k\in\infty} |c_k|^2 < \infty\}$$

$G = \mathbb{R}$ acts on $L^2(\mathbb{Z}\backslash\mathbb{R})$ via the regular representation

$$(\rho(y)f)(x) = f(x+y)$$

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"automorphic representation"

Automorphic representations

There is an important notion of "size" of an automorphic representation, called the **Gelfand-Kirillov dimension**.

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This is important for physics, since we have the correspondence:

number of independent **physical charges** (e.g. electric, magnetic) $\longleftrightarrow$ **Gelfand-Kirillilov dimension** of the associated automorphic representation

3. Fourier coefficients

Fourier coefficients

The periodicity $f(\tau+1)=f(\tau)$ generalises to

$$E(\lambda, ng) = E(\lambda, g) \qquad n \in N(\mathbb{Z})$$

Much more complicated since $N(\mathbb{Z})$ is **non-abelian**.

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General structure:

$$E(\lambda, g) = E^{\text{const}}(\lambda, g) + \sum_{\psi} W_{\psi}(\lambda, g) + \cdots$$

constant term
(zero-mode)
perturbative effects

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Whittaker coefficient

$$\psi \, : \, N(\mathbb{Z})\backslash N(\mathbb{R}) \to U(1)$$

unitary character on $N(\mathbb{R})$
trivial on $N(\mathbb{Z})$

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unitary character on $N(\mathbb{R})$

$$\longrightarrow \quad \psi(n) = e^{2\pi i \sum_j m_j x_j}$$

(simple roots)

$m_j \in \mathbb{Z}$

$x_j \in \mathbb{R}$

if all $m_j \neq 0$ then ψ is **generic**

if some $m_j = 0$ then ψ is **degenerate**

Holomorphic modular form $f(\tau)$ $\tau \in \mathbb{H} \cong SL(2,\mathbb{R})/U(1)$

$$\psi\left(\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix}\right) = \psi(e^{xE_\alpha}) = e^{2\pi i m x}$$

$x \in \mathbb{R}$ $m \in \mathbb{Z}$

ψ generic $\longleftrightarrow$ $m \neq 0$

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$$W_m(\tau) = \int_0^1 f(\tau+1)e^{-2\pi i m u}\,du$$

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$$= c(m)e^{2\pi i m \tau}$$

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Non-holomorphic Eisenstein series

$$E(s,\tau) = \sum_{\substack{(m,n)\in\mathbb{Z}^2 \\ (m,n)=1}} \frac{y^s}{|m+n\tau|^{2s}}$$

$s \in \mathbb{C}$

$\tau = x + iy \in \mathbb{H}$

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$\tau = x + iy \in \mathbb{H}$

$$W_m(\tau) = \int_0^1 E(s,\tau+u) e^{-2\pi i m u} du \;=\; \frac{\sqrt{y}}{\xi(2s)} \sigma_{1-2s}(m) K_{s-1/2}(2\pi|m|y) e^{2\pi i m x}$$

$$\sigma_{1-2s}(m) = \sum_{d|m} d^{1-2s}$$

(modified) Bessel function

Euler products

Before we proceed with the higher rank case we mention some further properties of the Fourier expansion, namely that it decomposes into Euler products

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archimedean Whittaker coefficient

$$W_\infty(g) = \frac{2\pi^s}{\Gamma(s)} y|m|^{s-1/2} K_{s-1/2}(2\pi|m|y) e^{2\pi i m x}$$

$$g = nak = \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} \begin{pmatrix} y^{1/2} & \\ & y^{-1/2} \end{pmatrix} k \in SL(2,\mathbb{R})$$

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p -adic Whittaker coefficient

$$W_p(1) = (1-p^{-2s}) \frac{1-p^{-2s+1}|m|_p^{2s-1}}{1-p^{-2s+1}}$$

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p -adic Whittaker coefficient

$1 \in SL(2, \mathbb{Q}_p)$

$$W_p(1) = (1 - p^{-2s}) \frac{1 - p^{-2s+1} |m|_p^{2s-1}}{1 - p^{-2s+1}}$$

This is the basis for the **adelic** formulation of automorphic forms

Adelic framework

> *An efficient, but abstract, way to approach the subject of automorphic forms is by the introduction of adeles, rather ungainly objects that nevertheless, once familiar, spare much unnecessary thought and many useless calculations.*

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$$
\begin{tikzcd}
\text{Adelic Eisenstein series} \arrow[r, "\text{Compute}"] & \text{Adelic Fourier coefficient} \arrow[d, "\text{Restrict}"] \\
\text{Eisenstein series} \arrow[u, "\text{Lift}"'] & \text{Fourier coefficient}
\end{tikzcd}
$$

Adelic framework

arithmetic groups	$G(\mathbb{Z}) \subset G(\mathbb{R})$	$\longrightarrow$	$G(\mathbb{Q}) \subset G(\mathbb{A})$
space of automorphic forms	$\mathcal{A}\big(G(\mathbb{Z})\backslash G(\mathbb{R})\big)$	$\longrightarrow$	$\mathcal{A}\big(G(\mathbb{Q})\backslash G(\mathbb{A})\big)$

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	$\bigcup$		$\bigcup$
Eisenstein series	$\sum\limits_{\gamma\in B(\mathbb{Z})\backslash G(\mathbb{Z})} e^{\langle\lambda+\rho\vert H(\gamma g)\rangle}$	$\longrightarrow$	$\sum\limits_{\gamma\in B(\mathbb{Q})\backslash G(\mathbb{Q})} e^{\langle\lambda+\rho\vert H(\gamma g)\rangle}$

$\lambda \in \mathfrak{h}^\star \otimes \mathbb{C}$ $\qquad$ $H \ : G \to \mathfrak{h}$

Theorem [Jacquet, Langlands]**:** The generic Whittaker coefficient is Eulerian

$$W_\psi(\lambda, g) = \int_{N(\mathbb{A})} \chi(w_0 n g)\, \overline{\psi(n)}\, dn = \prod_p W_{\psi_p}$$

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$$w_0 = \text{longest element of } W(\mathfrak{g})$$

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$$W_{\psi_\infty} = \int_{N(\mathbb{R})} \chi_\infty(w_0 n a_\infty)\overline{\psi_\infty(n)}\, dn$$

$$W_{\psi_p} = \int_{N(\mathbb{Q}_p)} \chi_p(w_0 n a_p)\overline{\psi_p(n)}\, dn$$

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Theorem [Shintani, Casselman-Shalika]**:** When $p < \infty$ we have

$$W_{\psi_p} = \text{Weyl character formula for } {}^L G$$

Perturbative limit - choices of unipotent subgroups

$$P = LU$$

Levi subgroups:

→ Decompactification limit

$$L = E_7$$

- perturbative parameter: radius of decompactified circle
- non-perturbative effects: KK-instantons, BPS-instantons

→ String perturbation limit

$$L = D_7$$

- perturbative parameter: string coupling
- non-perturbative effects: D-instantons, NS5-instantons

→ M-theory limit

$$L = A_7$$

- perturbative parameter: volume of M-theory torus
- non-perturbative effects: M2- & M5-instantons

Example $SL(3,\mathbb{R})$

$$N = \left\{ \begin{pmatrix} 1 & \star & \star \\ & 1 & \star \\ & & 1 \end{pmatrix} \right\}$$

$$U_{\alpha_1} = \left\{ \begin{pmatrix} 1 & \star & \star \\ & 1 & \\ & & 1 \end{pmatrix} \right\}, \qquad U_{\alpha_2} = \left\{ \begin{pmatrix} 1 & & \star \\ & 1 & \star \\ & & 1 \end{pmatrix} \right\}$$

In tomorrow's lecture we will talk about:

Automorphic forms on Kac-Moody groups

Small representations of exceptional groups

Open questions/future research

Thank you!