On the work of Martin Raum

Wallenberg prize in mathematics 2022

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modular forms

moonshine

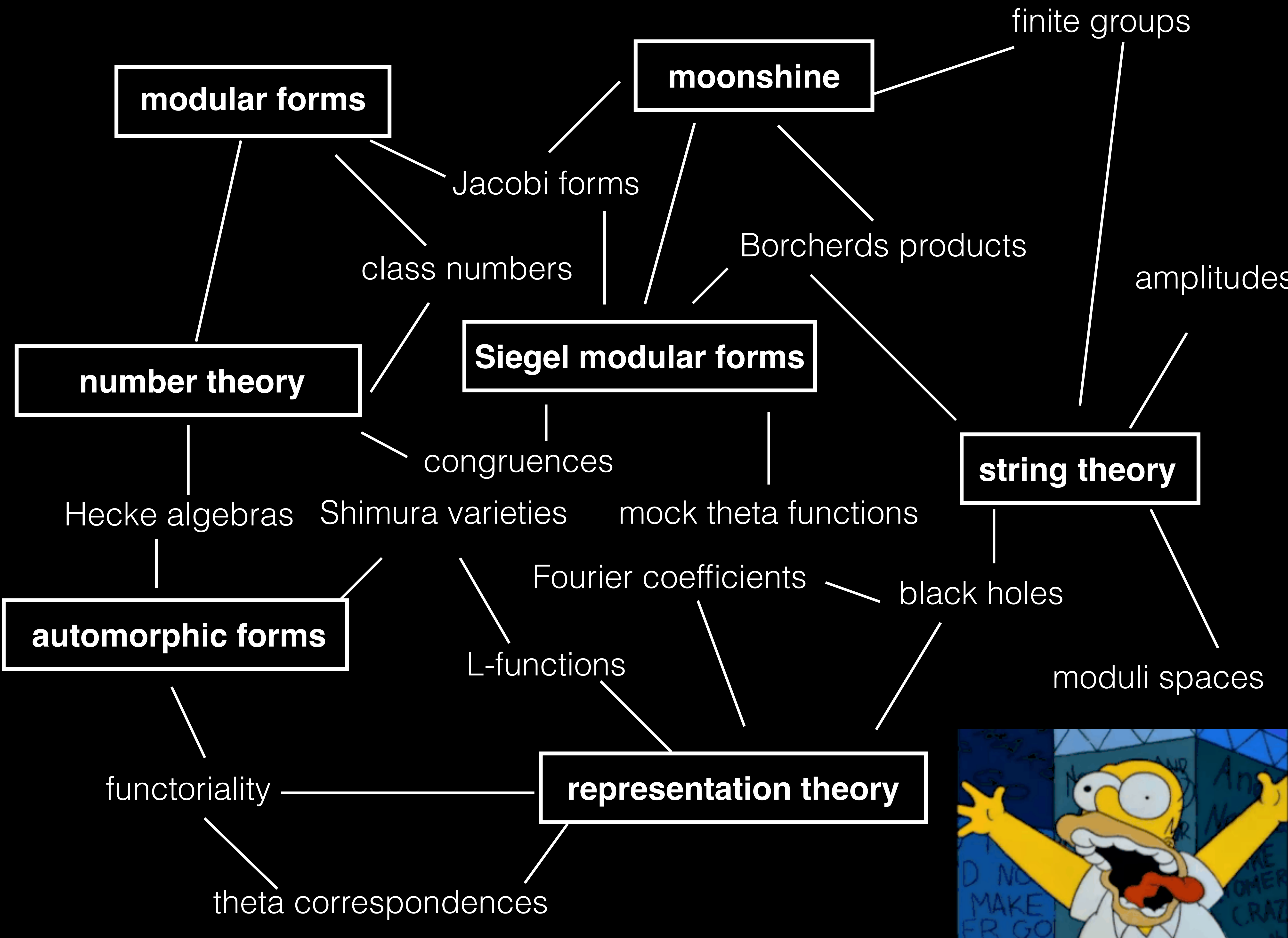
number theory

Siegel modular forms

string theory

automorphic forms

representation theory


modular forms
moonshine
finite groups
Jacobi forms
class numbers
Borcherds products
amplitudes
number theory
Siegel modular forms
string theory
congruences
Hecke algebras
Shimura varieties
mock theta functions
Fourier coefficients
black holes
automorphic forms
L-functions
moduli spaces
functoriality
representation theory
theta correspondences

Committee's Motivation

"Martin Raum tilldelas priset för sina många viktiga och inflytelserika bidrag till teorin för modulära och automorfa former, bland annat beviset av Kudlas modularitetsförmodan."

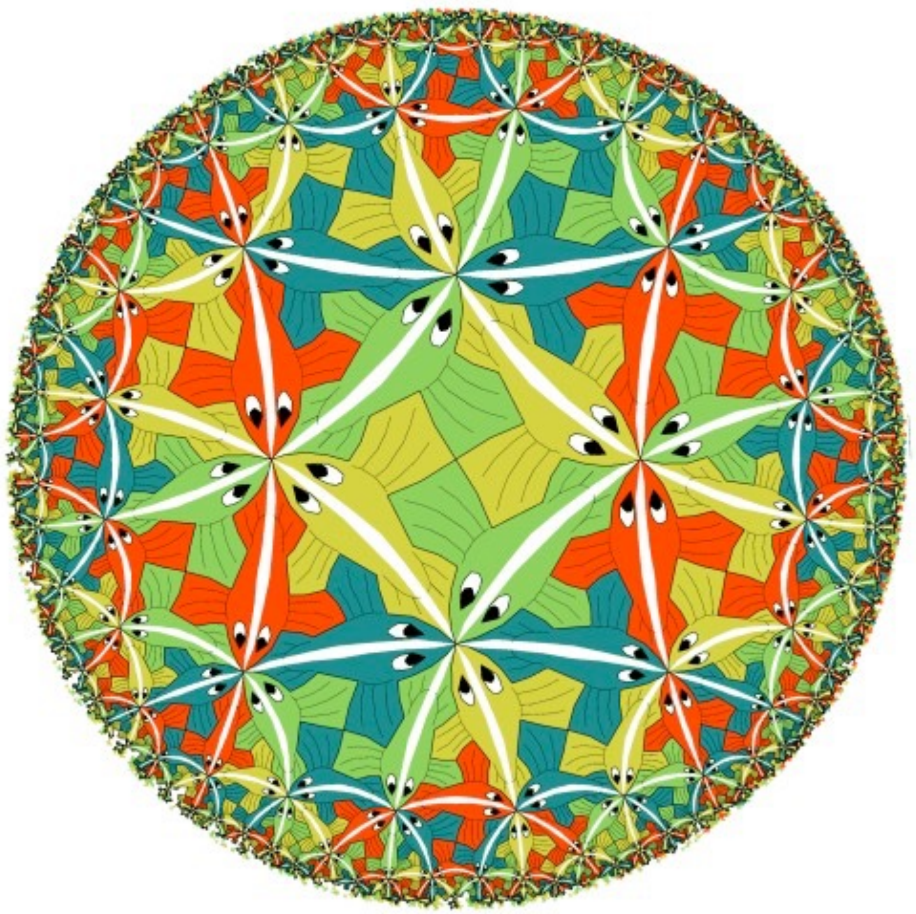
$$f : \mathbb{H} \to \mathbb{C}$$

modular forms

$$f\left(\frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^w f(\tau)$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z})$$

$$(\text{or } \Gamma \subset SL(2, \mathbb{Z}))$$



$\tau \in \mathbb{H}$

modular forms

Periodicity:

$$f(\tau+1) = f(\tau)$$

q - expansion

$$= \sum_{n\in\mathbb{Z}} c(n) q^n \qquad q = e^{2\pi i\tau}$$

Fourier coefficients of modular forms often encode interesting **arithmetic** information

Example: Eisenstein series

$$G_{2w}(\tau) = \sum_{(m,n)\in\mathbb{Z}^2\setminus(0,0)} \frac{1}{(m+n\tau)^{2w}}$$

Fourier expansion:

$$\frac{G_{2w}(\tau)}{2\zeta(2w)} = 1 + \frac{2}{\zeta(1-2w)} \sum_{k=1}^{\infty} \sigma_{2w-1}(k) e^{2\pi i k\tau}$$

divisor sum:

$$\sigma_{2w-1}(k) = \sum_{d|k} d^{2w-1}$$

Siegel modular forms

Modular forms with respect to $Sp(4,\mathbb{Z})$

(genus 2 case)

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(genus 2 case)

$$\Omega = \begin{pmatrix} \tau & z \\ z & \sigma \end{pmatrix} \qquad \tau, \sigma \in \mathbb{H} \qquad z \in \mathbb{C}$$

$$\boxed{f(\Omega) = \sum_{n\in\mathbb{Z}} \phi_m(\sigma, z) q^n}$$

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Fourier coefficients are ***also modular forms*** ("Jacobi forms")

Siegel modular forms

$\uparrow$

modular forms

Siegel modular forms

Saito-Kurokawa lift

modular forms

Siegel modular forms

Also used to study quantum properties of black holes

Saito-Kurokawa lift

modular forms

Kudla's conjecture

Shimura variety X

Higher-dimensional generalisations of the modular curve
$\Gamma\backslash SL(2,\mathbb{R})/SO(2)$

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Shimura variety X

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Example: $\Gamma \setminus SO(2,n)/(SO(2)\times SO(n))$

Kudla's conjecture

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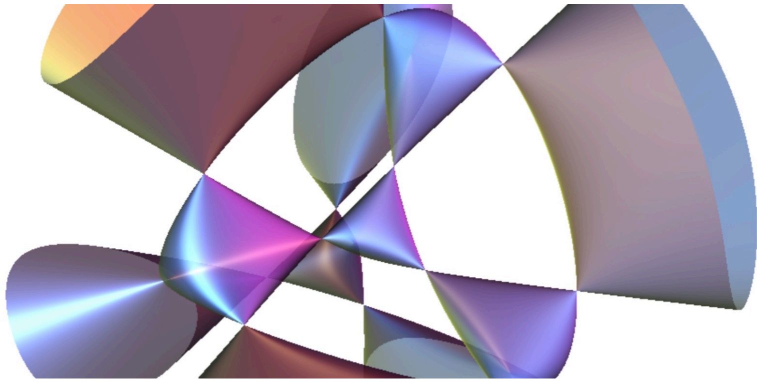
Shimura variety X

Example: $\Gamma \setminus SO(2,n)/(SO(2)\times SO(n))$

Cycles are subvarieties of algebraic varieties

Kudla constructed ***special cycles*** on X
(generalising Heegner points on the modular curve)

Example: $\Gamma \setminus D$ for D a divisor



Kudla's conjecture

Higher-dimensional generalisations of the modular curve $\Gamma\backslash SL(2,\mathbb{R})/SO(2)$

Shimura variety X

Kudla constructed ***special cycles*** on X
(generalising Heegner points on the modular curve)

Conjecture (Kudla):
The generating function of special cycles of codimension r is a ***Siegel modular form*** of degree r

The **generating function** is of the form

$$\phi_r = \sum_n [C_{n,r}] q^n$$

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- Borcherds proved (1999) that these are elliptic modular forms, thereby establishing the $r = 1$ case of Kudla's conjecture

- The $r = 2$ was proven independently by Raum and Bruinier in 2013.

Then Bruinier and Raum joined forces and in 2014 they proved:

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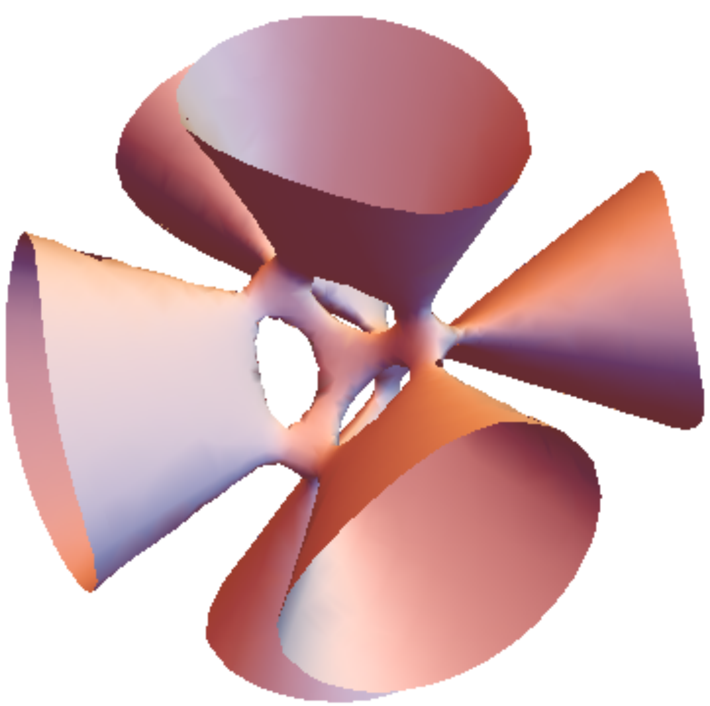
- The strategy of proof was to prove a very general modularity result for certain formal Fourier-Jacobi expansions, from which the theorem could be deduced.

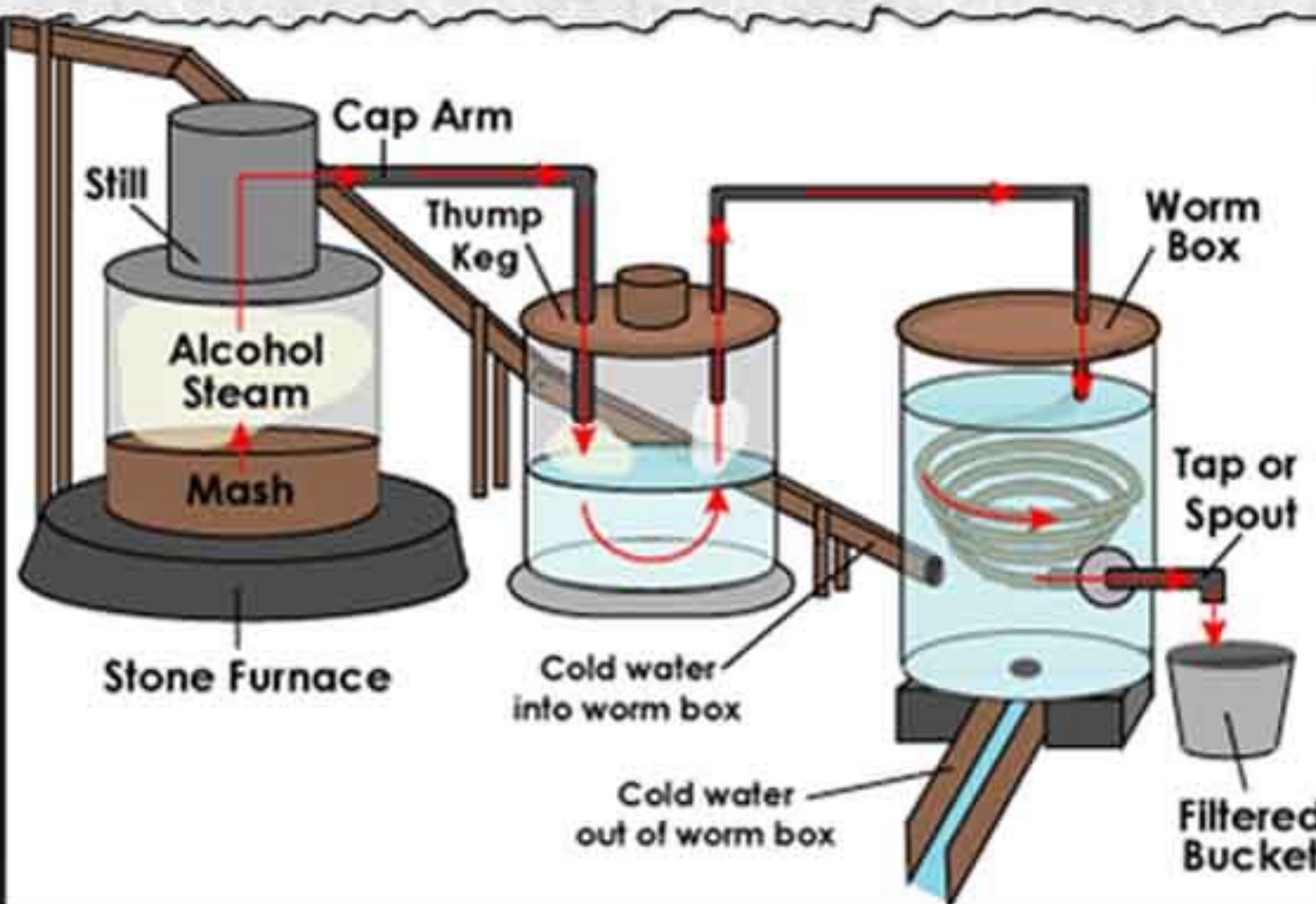
- Their result also gives a powerful algorithm for calculating Fourier coefficients of Siegel modular forms.

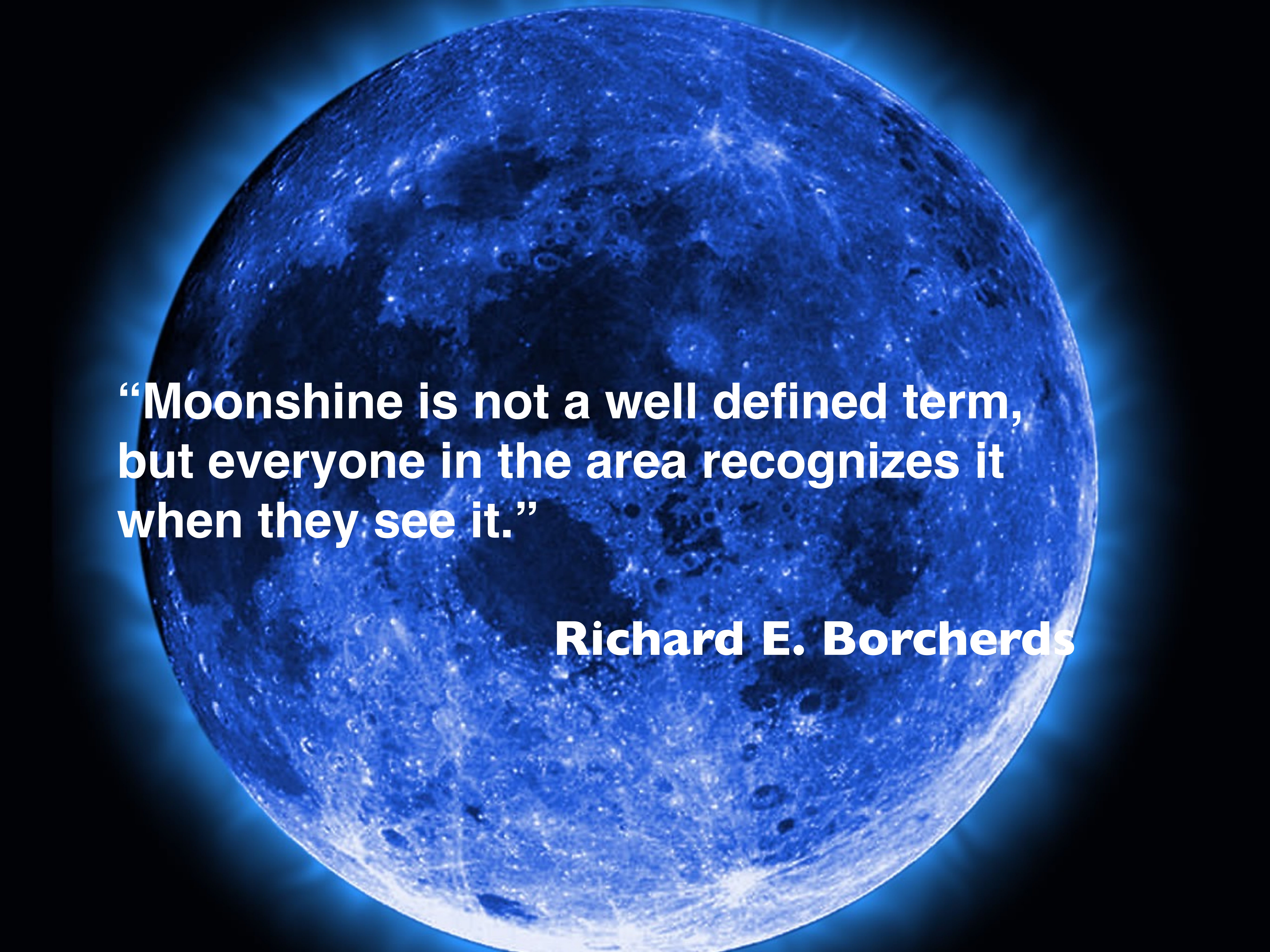
These results were later used by Raum and a team of mathematical physicists to calculate a class of scattering amplitudes in string theory



Oberdieck used the results of Bruinier & Raum in the context of enumerative geometry (Gromov-Witten invariants of K3-surfaces)

Step By Step Guide To Making Moonshine




“Moonshine is not a well defined term, but everyone in the area recognizes it when they see it.”
Richard E. Borcherds

What is **Moonshine**?

The term "**moonshine**" generally refers to surprising connections between a priori unrelated parts of mathematics and physics, involving:

representation theory of finite groups — **generalised Kac-Moody algebras**

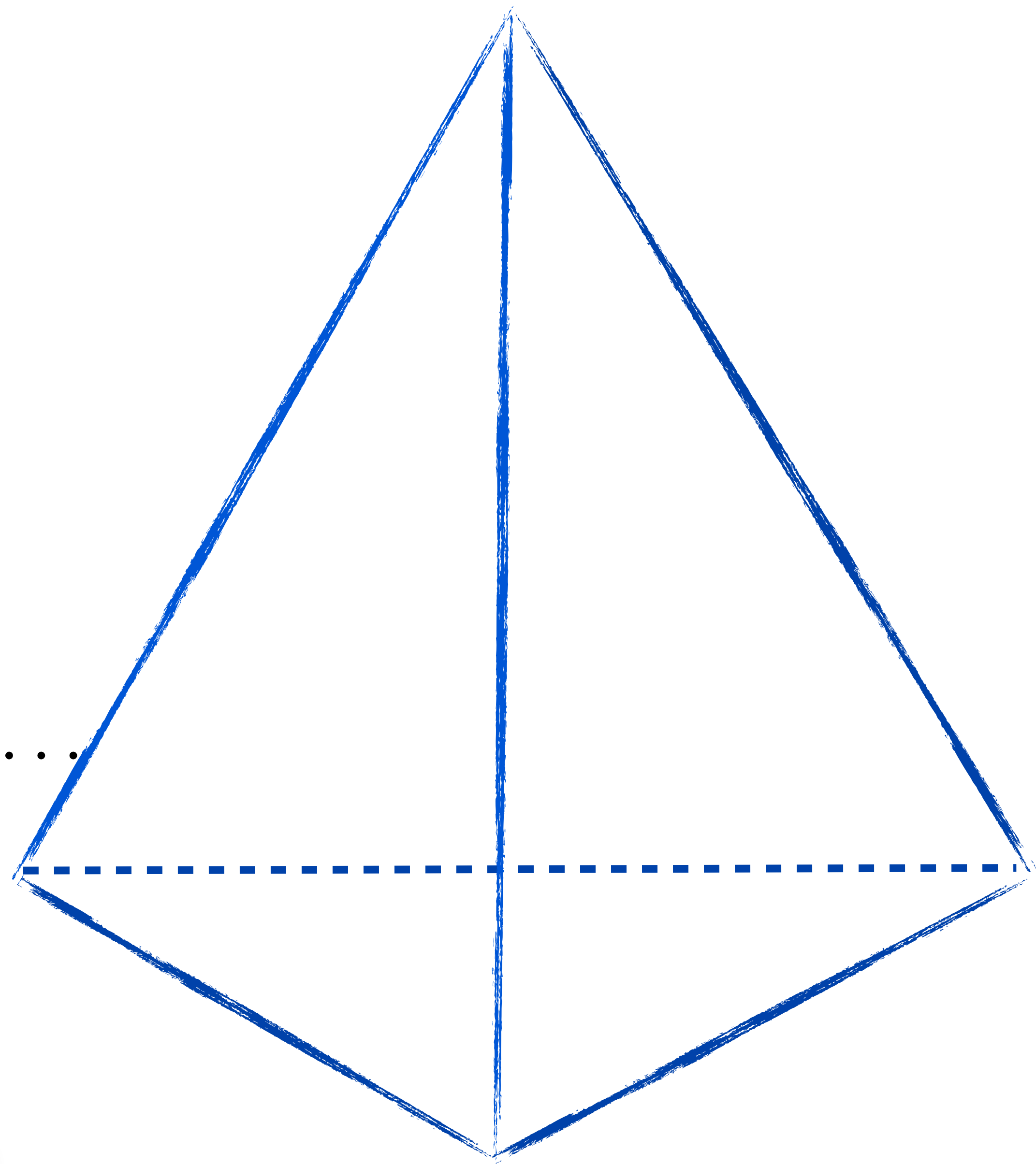
modular forms — **vertex algebras (conformal field theory)**

The most famous example is **Monstrous Moonshine**.

Monstrous Moonshine

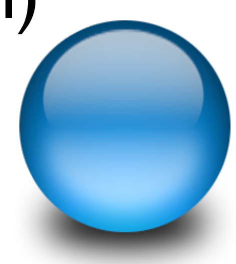
monster group

$\mathbb{M}$



$J(\tau) = q^{-1} + 196884q + \cdots$

modular function
(Hauptmodul)

$SL(2,\mathbb{Z})\backslash\mathbb{H} \sim$ 

bosonic string theory on
$(\mathbb{R}^{24}/\Lambda_{\text{Leech}})/\mathbb{Z}_2$
(holomorphic VOA $V^\natural$)

monster Lie algebra $\mathfrak{m}$

Enter Mathieu moonshine

In 2010, Eguchi, Ooguri, Tachikawa conjectured that there is Moonshine in the elliptic genus of K3 connected to the finite sporadic group $M_{24} \subset S_{24}$

A completely new moonshine phenomenon to explore!

Monstrous moonshine was all about ***modular*** forms.

In the new Mathieu moonshine we see the appearance of **mock modular forms.**

This story actually begins with Ramanujan…

The Last Letter...

3 months before his death, Ramanujan wrote a final letter to Hardy, including a list of 17 mysterious functions:

"Mock Theta Functions"

$$f(q) = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(1+q)^2 \cdots (1+q^n)^2},$$

$$\phi(q) = \sum_{n=0}^{\infty} \frac{(-q)^{n^2}}{(1+q^2)(1+q^4)\cdots(1+q^{2n})},$$

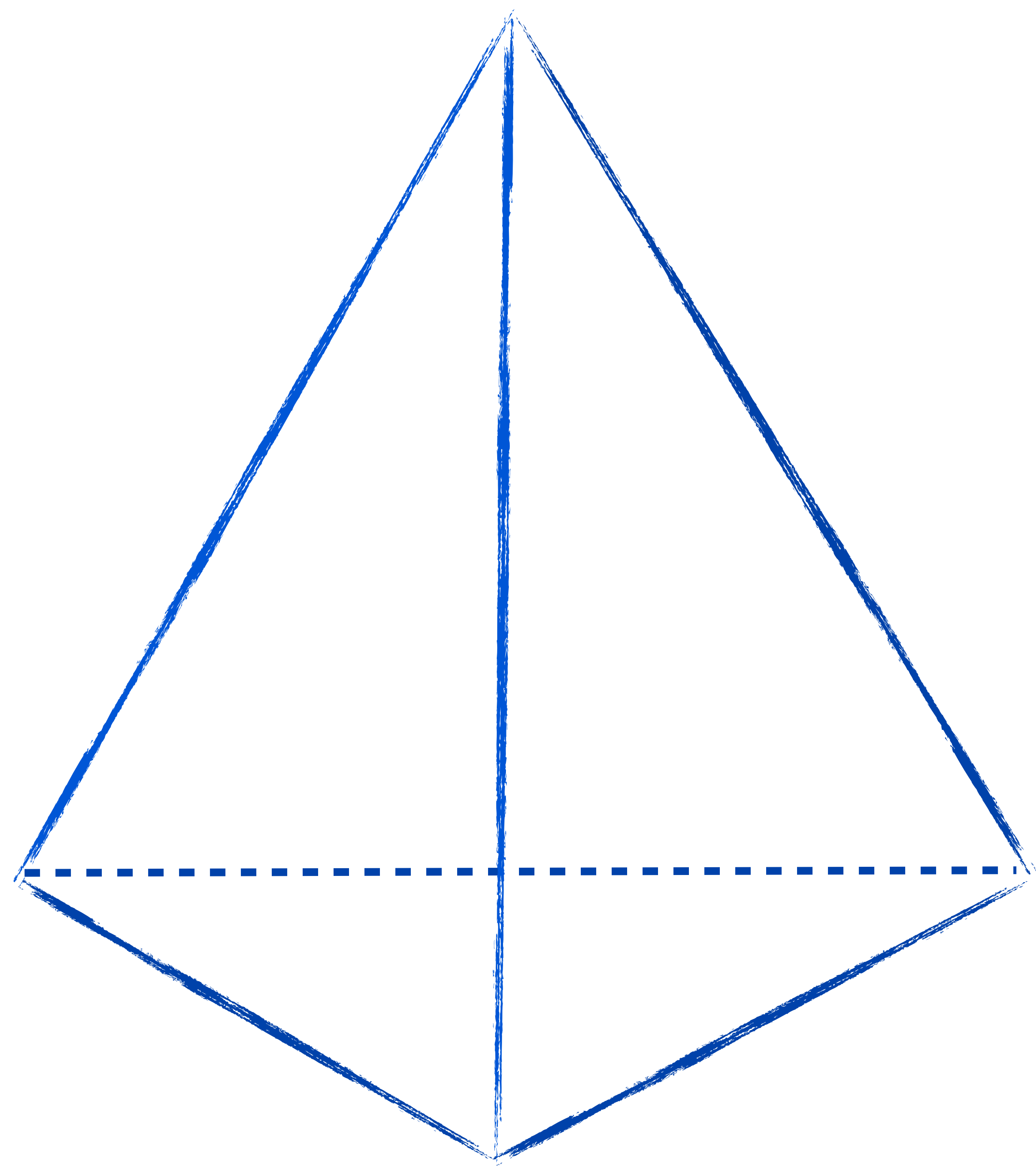
$$\psi(q) = \sum_{n=1}^{\infty} \frac{(-q)^{n^2}}{(1+q)(1+q^3)\cdots(1+q^{2n-1})}.$$

It took nearly 100 years, and the efforts of generations of mathematicians before Ramanujan's theory was finally understood...

Mathieu Moonshine

Mathieu group

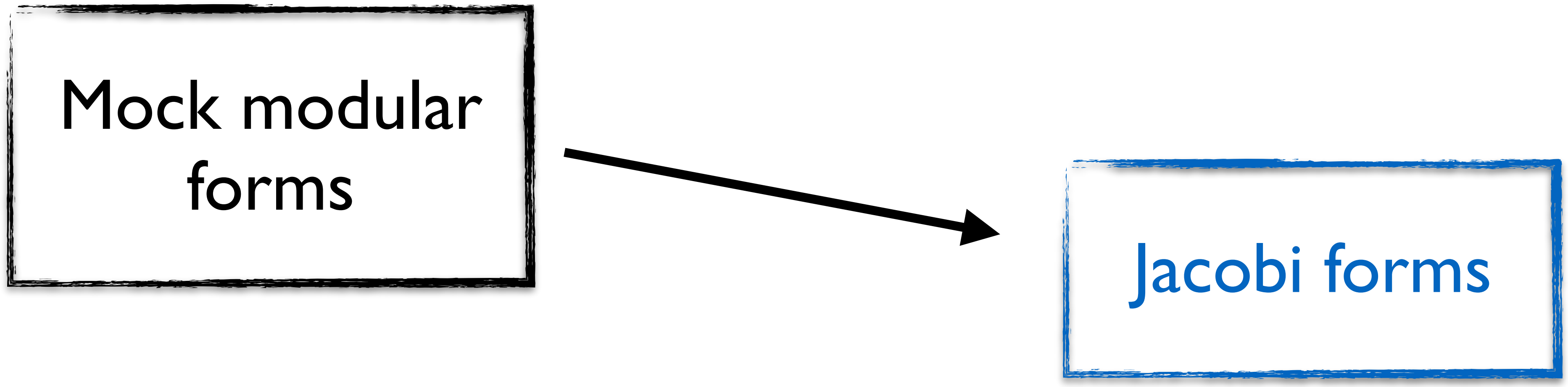
M_{24}

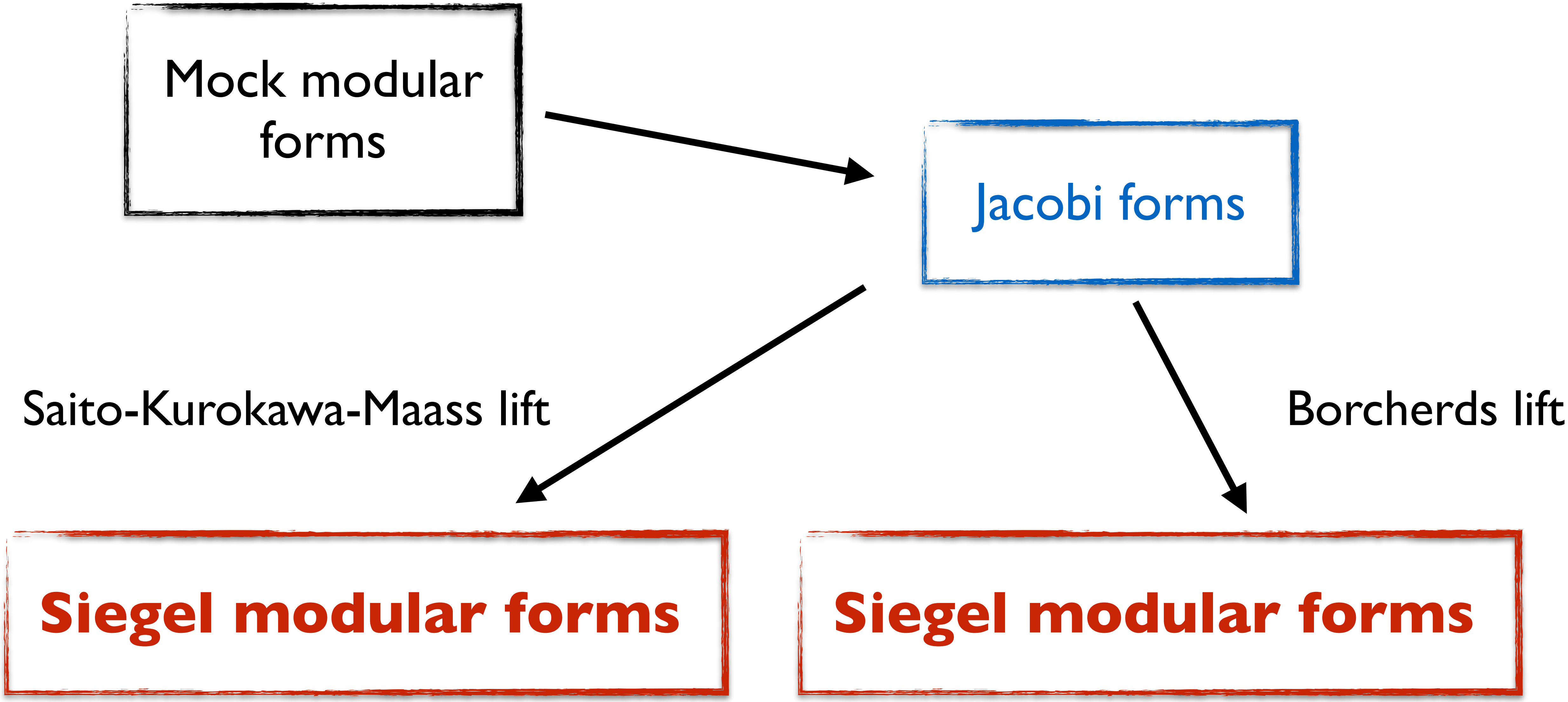


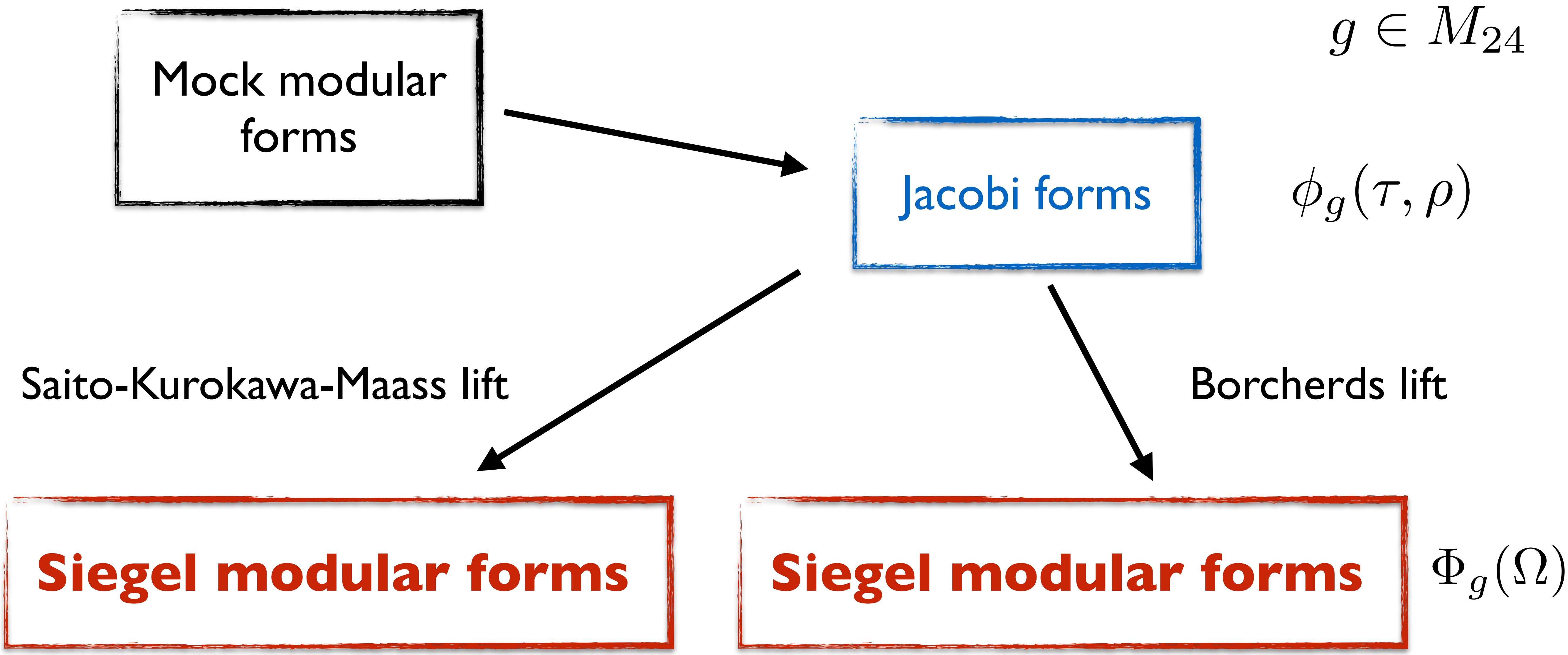
mock modular forms
(weak Jacobi forms)

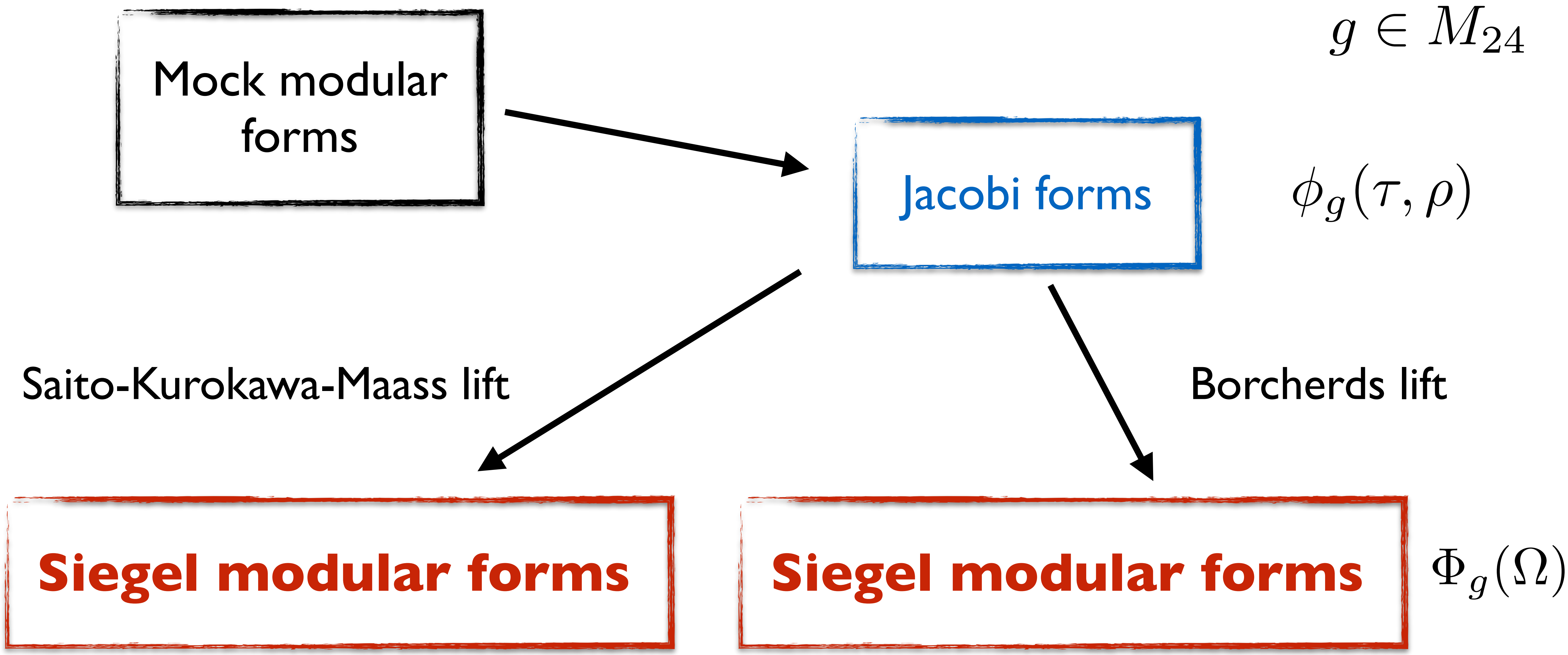
vertex algebra associated
with K3-surfaces
??

algebraic structure ???


Mock modular forms
Jacobi forms


Mock modular forms
Jacobi forms
Saito-Kurokawa-Maass lift
Borcherds lift
Siegel modular forms
Siegel modular forms


$g \in M_{24}$
Mock modular forms
Jacobi forms
$\phi_g(\tau, \rho)$
Saito-Kurokawa-Maass lift
Borcherds lift
Siegel modular forms
Siegel modular forms
$\Phi_g(\Omega)$



Theorem (Raum): For each $g \in M_{24}$ the infinite products

$$\Phi_g(\Omega) = qp\zeta \prod_{d|n_g} \prod_{(n,r,m)>0} \left(1 - q^{nd} p^{rd} \zeta^{md}\right)^{c_{g,d}(4mn-r^2)}$$

are Siegel modular forms (Borcherds lift).

(The $g = 1$ case corresponds to the Igusa cusp form of weight 10.)

Mock modular forms

$g \in M_{24}$

Jacobi forms

$\phi_g(\tau, \rho)$

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Fourier coefficients

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The reciprocal

$$\frac{1}{\Phi_g(\Omega)}$$

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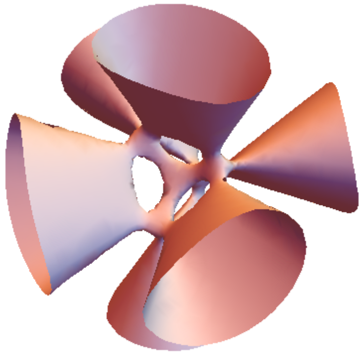
- denominator formula for a class of generalized Kac-Moody algebras

  [Cheng et al.]

- partition function of black holes in string theory

  [Sen et al.]



- generating function of equivariant Donaldson-Thomas invariants of K3-surfaces

  [Oberdieck et al.]

Raum's work cover numerous other areas:

- classification of harmonic weak Maass forms
- Ramanujan congruences
- vector-valued modular forms
- high-performance computing
- ...


Congratulations
Martin!!!