Automorphic forms and string amplitudes (Lecture 2)

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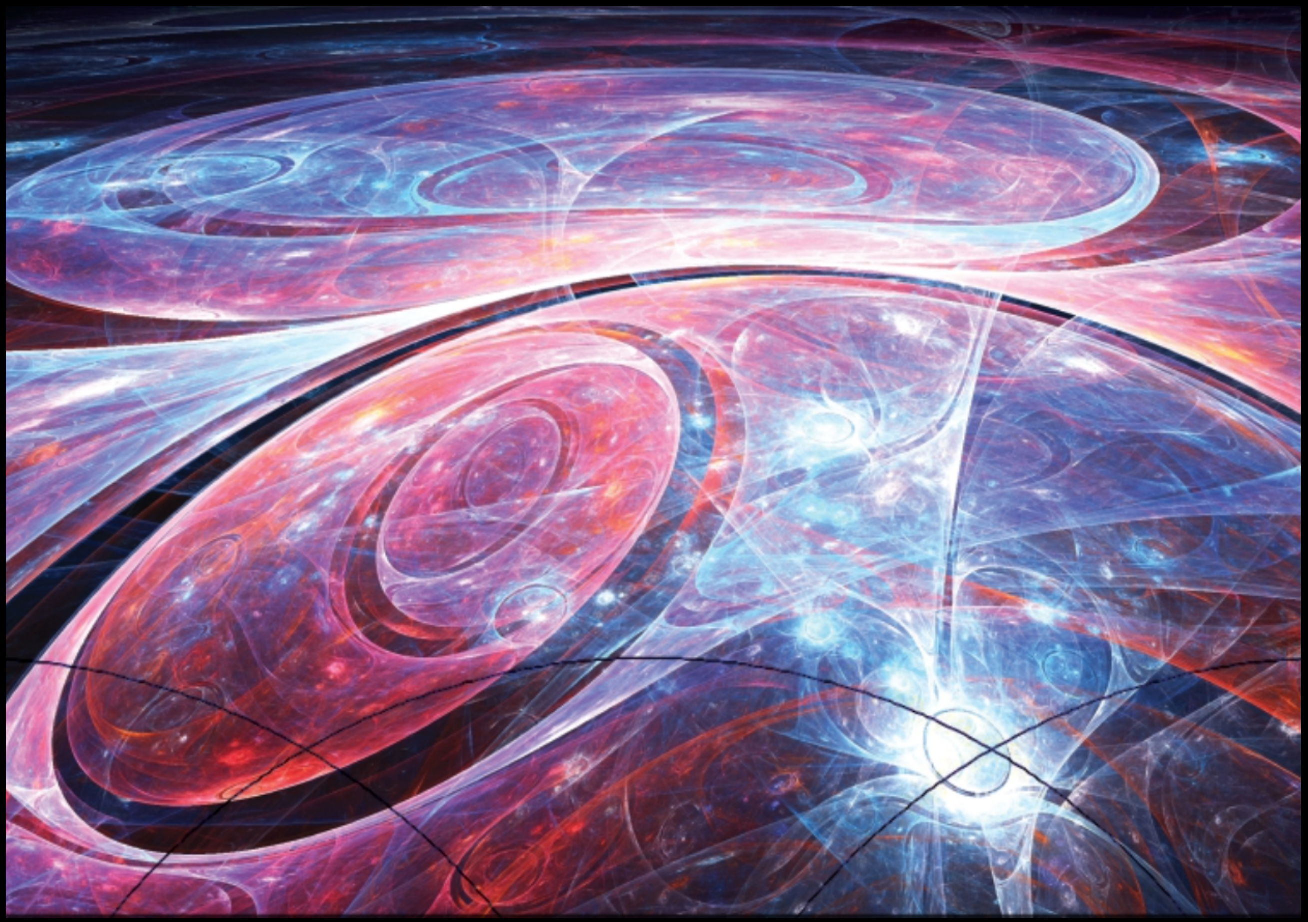

Bhaskara Math Seminars — Harish-Chandra Lecture Series
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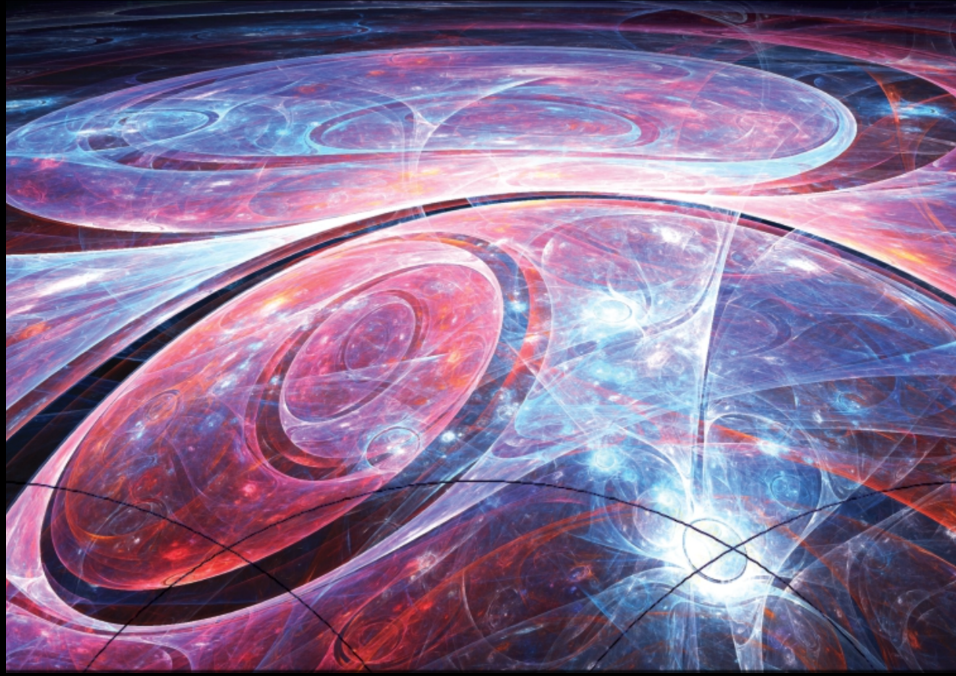

Outline

1. General Fourier coefficients

2. Small representations

3. Eisenstein series on Kac-Moody groups

4. Outlook

1. General Fourier coefficients

Recall: Whittaker coefficients

$$E(\lambda, g) = E^{\text{const}}(\lambda, g) + \sum_{\psi} W_{\psi}(\lambda, g) + \cdots$$

$$W_{\psi}(g) = \int_{N(\mathbb{Z})\backslash N(\mathbb{R})} E(\lambda, ng)\overline{\psi(n)}dn$$

Whittaker coefficient

$$\psi \, : \, N(\mathbb{Z})\backslash N(\mathbb{R}) \to U(1)$$

unitary character on $N(\mathbb{R})$

$$\longrightarrow \quad \psi(n) = e^{2\pi i \sum_j m_j x_j}$$

(simple roots)

$m_j \in \mathbb{Z}$

$x_j \in \mathbb{R}$

if all $m_j \neq 0$ then ψ is **generic**

if some $m_j = 0$ then ψ is **degenerate**

General Fourier coefficients

- $P = LU$ **standard parabolic** of G
- **unitary character** $\psi_U : U(\mathbb{Q})\backslash U(\mathbb{A}) \to U(1)$

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- $P = LU$ **standard parabolic** of G
- **unitary character** $\psi_U : U(\mathbb{Q})\backslash U(\mathbb{A}) \to U(1)$
- For any automorphic form φ we have the U ***-coefficient***

$$F_{\psi_U}(g) = \int_{U(\mathbb{Q})\backslash U(\mathbb{A})} \varphi(ug)\overline{\psi_U(u)}du$$

Also known as "unipotent period integrals".

$$F_{\psi_U}(g) = \int_{U(\mathbb{Q})\backslash U(\mathbb{A})} \varphi(ug)\overline{\psi_U(u)}du$$

- These are **not Eulerian** in general (no CS-formula)
- $F_{\psi_U}(ug) = \psi_U(u)F_{\psi_U}(g) \qquad \forall u \in U$
- Very difficult to compute
- Idea: consider **special types** of automorphic representations
- Also motivated by their role in string theory

2. Small representations

Minimal automorphic representations

Definition: *An automorphic representation*

$$\pi = \bigotimes_{p \leq \infty} \pi_p$$

is minimal if each factor π_p *has smallest non-trivial Gelfand-Kirillov dimension.*

[Joseph][Brylinski, Kostant][Ginzburg, Rallis, Soudry][Kazhdan, Savin]....

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Automorphic forms $\varphi \in \pi_{min}$ are characterised by having **very few non-vanishing Fourier coefficients**.

[Ginzburg, Rallis, Soudry]

Physical couplings

$g \in E_n(\mathbb{R})$

$$\int d^{11-n}x\sqrt{G}f_0(g)\mathcal{R}^4 \qquad f_0(g) = E(3/2, g) \qquad s = 3/2$$

$$\int d^{11-n}\sqrt{G}f_4(g)\partial^4\mathcal{R}^4 \qquad f_4(g) = E(5/2, g) \qquad s = 5/2$$

These partition functions are Eisenstein series attached to **small automorphic representations** of G.

[Green, Miller, Vanhove][Pioline]

minimal automorphic representation π_{min}	**next-to-minimal automorphic representation** π_{ntm}
1/2 - BPS	1/4 - BPS

Minimal automorphic representations of $SL(n, \mathbb{A})$

[w/ Ahlén, Gustafsson, Kleinschmidt, Liu]

$\mathrm{GKdim}(\pi_{min}) = n - 1$

Borel subgroup $B = NA$

F number field

$\mathbb{A} = \mathbb{A}_F$ adeles

$(F = \mathbb{Q}, \mathbb{A} = \mathbb{A}_{\mathbb{Q}})$

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Theorem: For any $\varphi \in \pi_{min}$ we have the Fourier expansion:

$$\varphi(g) = \int_{N(F)\backslash N(\mathbb{A})} \varphi(ng)dn + \sum_{i=1}^{n-1} \sum_{\gamma \in \Gamma_i} \int_{N(F)\backslash N(\mathbb{A})} \varphi(n\gamma g)\overline{\psi_{\alpha_i}(n)}dn$$

$$\Gamma_i = \mathrm{Stab}_{\hat{e}_i} \backslash SL(n-i, F)$$

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This is the **complete** expansion, including all **non-abelian coefficients.**

Analogue to the **Piatetski-Shapiro-Shalika expansion** of cusp forms.

$P = LU \subset SL(n, \mathbb{A})$ maximal parabolic

What can we say about the U **-coefficient?**

$$F_{\psi_U}(g) = \int_{U(\mathbb{Q})\backslash U(\mathbb{A})} \varphi(ug)\overline{\psi_U(u)}du$$

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Theorem [AGKLP]: For $\varphi \in \pi_{min}$ we have

$\mathrm{rank}(\psi_U) > 1$ $\quad F_{\psi_U} = 0$

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$$F_{\psi_U}(g) = \int_{N(F)\backslash N(\mathbb{A})} \varphi(ng)\overline{\psi_\alpha(n)}dn$$

$$= \prod_{p\leq\infty} F_p$$

Eulerian

Example for $SL(5, \mathbb{A})$

$$P = LU \qquad L = SL(4) \times GL(1) \qquad U = \left\{ \begin{pmatrix} 1 & & & & * \\ & 1 & & & * \\ & & 1 & & * \\ & & & 1 & * \\ & & & & 1 \end{pmatrix} \right\}$$

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$$\psi_U\left(\begin{pmatrix} 1 & & & & * \\ & 1 & & & * \\ & & 1 & & * \\ & & & 1 & x \\ & & & & 1 \end{pmatrix}\right) = e^{2\pi i k x} \qquad x\in\mathbb{A},\ k\in\mathbb{Q} \qquad \operatorname{rank}(\psi_U) = 1$$

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$$F_{\psi_U}(1) = \frac{2}{\xi(2s)} \sigma_{2s-4}(k)|k|^{2-s} K_{s-2}(2\pi|k|)$$

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For $s = 3/2$ this captures the contributions from **M2-brane instantons** in M-theory compactified on T^4 [Green, Miller, Vanhove]

Instanton measure: $\sigma_{2s-4}(k) = \sum_{d|k} d^{2s-4}$

Theorem: w/ [Gustafsson, Gourevitch, Kleinschmidt, Sahi]

Let G be a ***semisimple, simply laced Lie group****.*

Then all Fourier coefficients of $\varphi \in \pi_{ntm}$ are completely determined by degenerate Whittaker vectors of the form

$$W_{\psi_\alpha}(\varphi, g) = \int_{N(\mathbb{Q})\backslash N(\mathbb{A})} \varphi(ng)\overline{\psi_\alpha(n)}dn$$

$$W_{\psi_{\alpha,\beta}}(\varphi, g) = \int_{N(\mathbb{Q})\backslash N(\mathbb{A})} \varphi(ng)\overline{\psi_{\alpha,\beta}(n)}dn$$

where (α, β) are commuting simple roots.

This generalises earlier results of [Ginzburg, Rallis, Soudry][Miller, Sahi]

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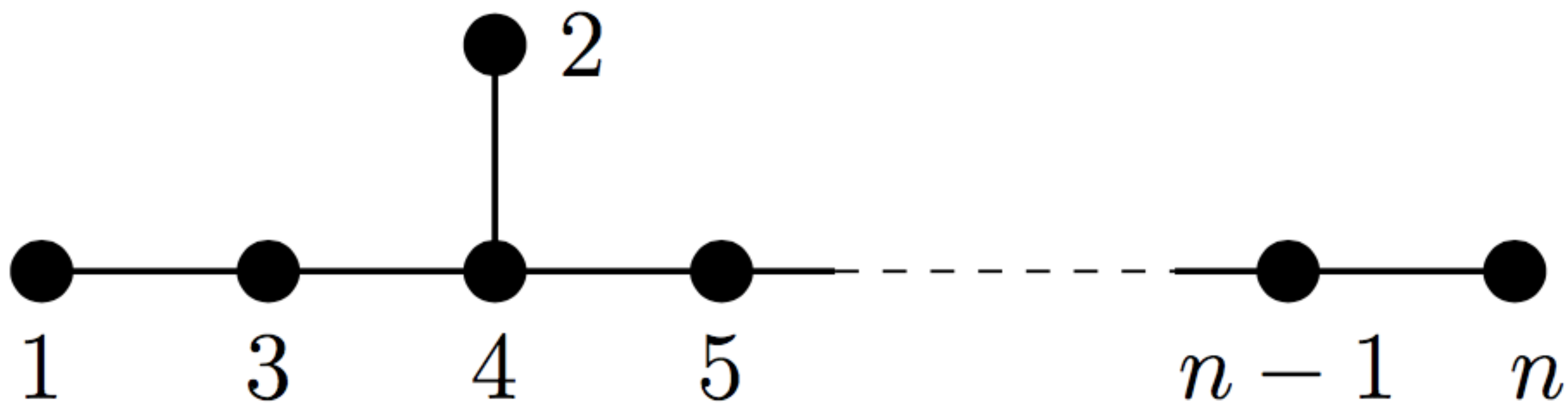
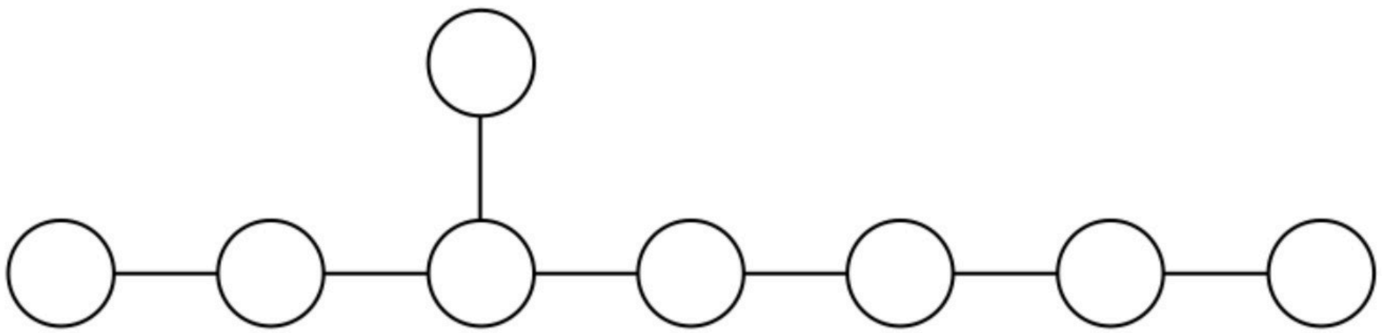
This allows to extract instanton effects to 1/4-BPS couplings

Focus on exceptional groups



Functional dimension of minimal representations:

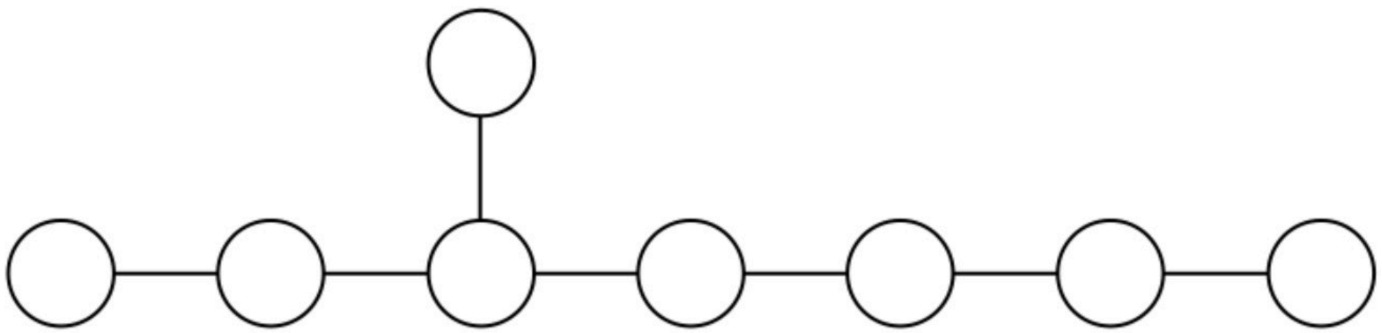
$$\mathrm{GKdim}\,\pi_{min} = \begin{cases} 11, & E_6 \\ 17, & E_7 \\ 29, & E_8 \end{cases}$$



$$\mathrm{GKdim}(\pi_{min}) = 29$$

(dim of generic rep: 120)

$$E_8 = \mathbf{1} \oplus \mathbf{56} \oplus (E_7 \oplus \mathbf{1}) \oplus \mathbf{56} \oplus \mathbf{1}$$



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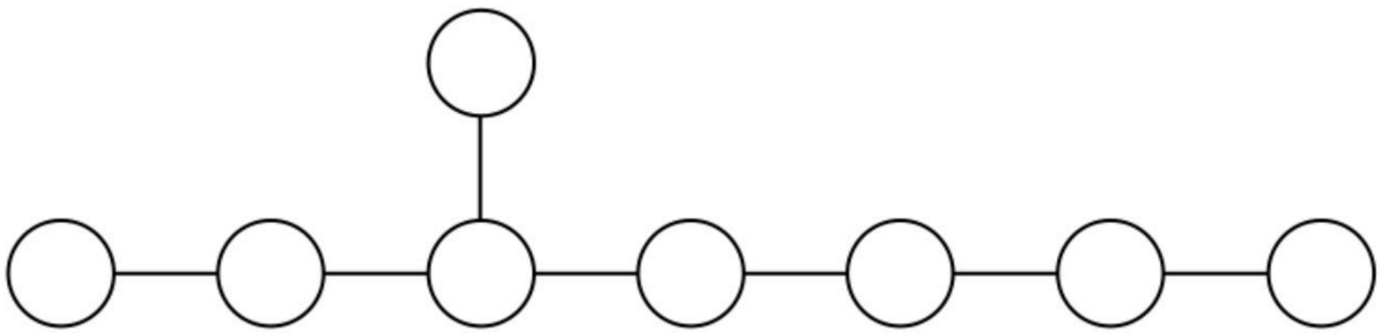
$$\begin{array}{c} \mathbf{56} \oplus \mathbf{1} \\ \bigcup \\ \mathbf{28} \oplus \mathbf{1} \end{array}$$



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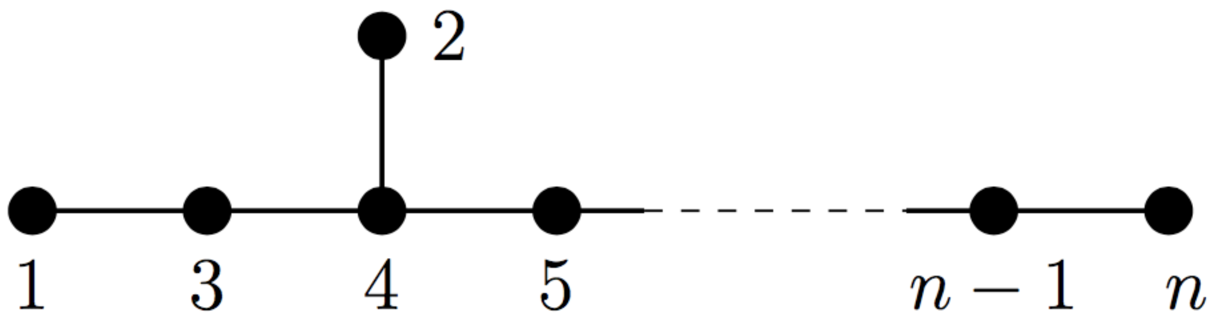
Minimal representation realized by the group action on functions of these 29 variables

Minimal representation of exceptional groups



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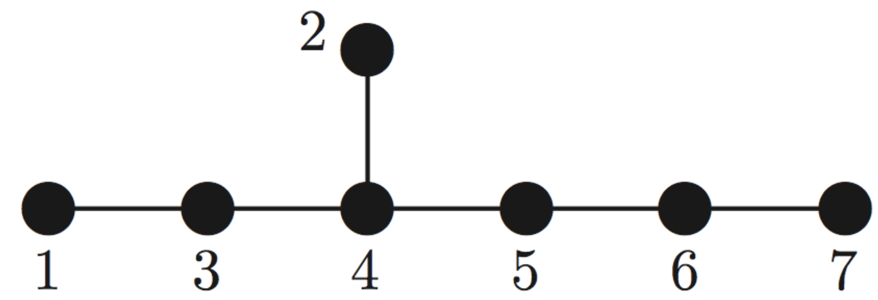
Minimal automorphic forms can be obtained as a **special values of Eisenstein series** [Ginzburg, Rallis, Soudry] (also [Green, Miller, Vanhove])

Theorem: $E(2s-\Lambda_1, g) \in \pi_{min} \quad \text{when} \quad s = 3/2$

Example: $G = E_7$



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Theorem: $$\pi_{min} \ni \varphi(g) = \varphi_U + \sum_{\gamma \in \mathrm{Stab}_L(\psi_{\alpha_7})\backslash L(\mathbb{Q})} F_{\psi_{\alpha_7}}(\gamma g)$$

[Ginzburg, Rallis, Soudry]

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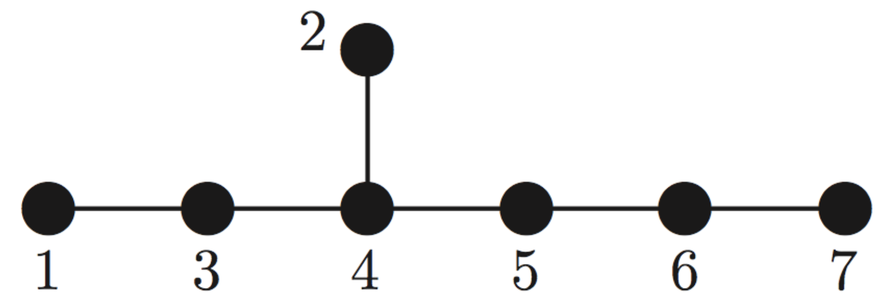

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[Ginzburg, Rallis, Soudry]

The complete expansion is given in terms of the **smallest non-trivial** character variety orbit

$$\mathrm{Stab}_L(\psi_{\alpha_7}) \backslash L \cong (SO(5,5) \times \mathbb{Q}^{16}) \backslash E_6(\mathbb{Q})$$

$$\dim = 17$$

[Miller, Sahi]

dimension of the minimal representation!

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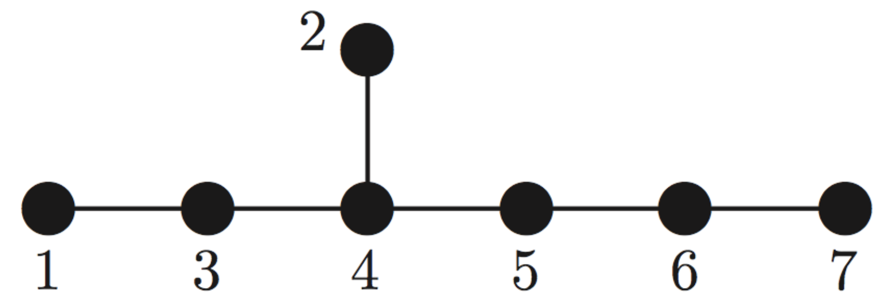

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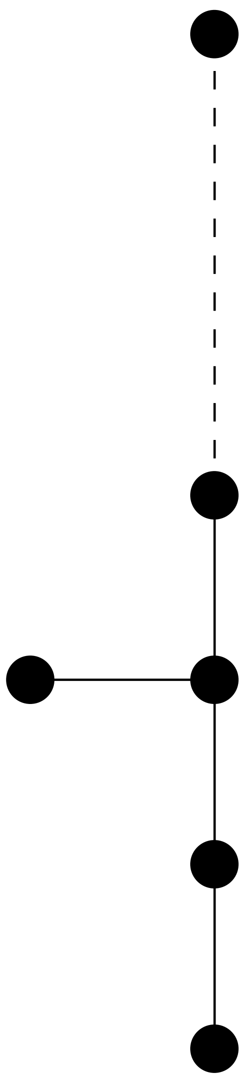
dimension of the minimal representation!

This is the complete **Fourier expansion**!

3. Eisenstein series on Kac-Moody groups

Toroidal compactifications yield the chain of **U-duality groups**

[Cremmer, Julia][Hull, Townsend]

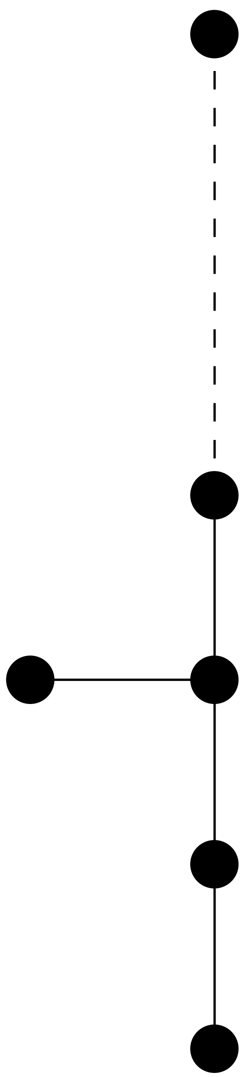


D	G	K	$G(\mathbb{Z})$
10	$\mathrm{SL}(2,\mathbb{R})$	$\mathrm{SO}(2)$	$\mathrm{SL}(2,\mathbb{Z})$
9	$\mathrm{SL}(2,\mathbb{R}) \times \mathbb{R}^+$	$\mathrm{SO}(2)$	$\mathrm{SL}(2,\mathbb{Z})$
8	$\mathrm{SL}(3,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$	$\mathrm{SO}(3) \times \mathrm{SO}(2)$	$\mathrm{SL}(3,\mathbb{Z}) \times \mathrm{SL}(2,\mathbb{Z})$
7	$\mathrm{SL}(5,\mathbb{R})$	$\mathrm{SO}(5)$	$\mathrm{SL}(5,\mathbb{Z})$
6	$\mathrm{Spin}(5,5,\mathbb{R})$	$\big(\mathrm{Spin}(5) \times \mathrm{Spin}(5)\big)/\mathbb{Z}_2$	$\mathrm{Spin}(5,5,\mathbb{Z})$
5	$\mathrm{E}_6(\mathbb{R})$	$\mathrm{USp}(8)/\mathbb{Z}_2$	$\mathrm{E}_6(\mathbb{Z})$
4	$\mathrm{E}_7(\mathbb{R})$	$\mathrm{SU}(8)/\mathbb{Z}_2$	$\mathrm{E}_7(\mathbb{Z})$
3	$\mathrm{E}_8(\mathbb{R})$	$\mathrm{Spin}(16)/\mathbb{Z}_2$	$\mathrm{E}_8(\mathbb{Z})$

Physical couplings are given by **functions** on

$$G(\mathbb{Z})\backslash G(\mathbb{R})/K$$

Below D=3 U-duality conjecturally becomes **infinite-dimensional**



D	$G(\mathbb{R})$
2	$E_9(\mathbb{R})$
1	$E_{10}(\mathbb{R})$
0	$E_{11}(\mathbb{R})$

[Julia][Nicolai]
[Damour, Henneaux, Nicolai][West]

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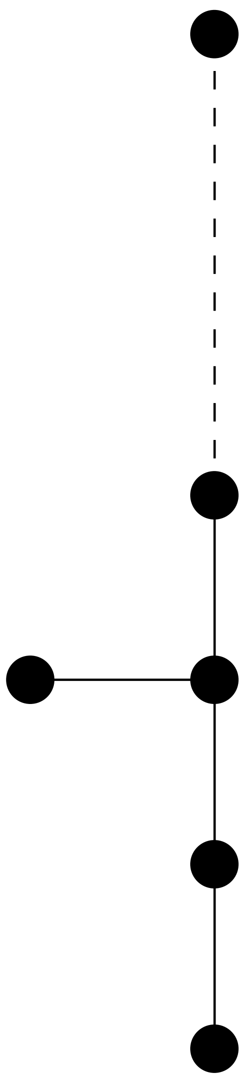


D	$G(\mathbb{R})$	$G(\mathbb{Z})$
2	$E_9(\mathbb{R})$	$E_9(\mathbb{Z})$
1	$E_{10}(\mathbb{R})$	$E_{10}(\mathbb{Z})$
0	$E_{11}(\mathbb{R})$	$E_{11}(\mathbb{Z})$

Can we understand physical couplings in terms of automorphic forms on **Kac-Moody groups**?

Yes, but many issues to be overcome:

- mathematical theory much less developed

- new exotic instanton effects

- unclear how to define U-duality invariant effective actions

Automorphic forms on Kac-Moody groups?

Kac-Moody groups are infinite-dimensional generalisations of Lie groups

- (Mid 90's): **Kapranov** constructed certain "geometric Eisenstein series on loop groups", motivated by Vafa-Witten theory
- (Late 90's): **Garland** constructed Langlands-type Eisenstein series on loop groups over number fields
- (2005): **Shahidi** suggested that Eisenstein series on Kac-Moody groups might provide a source of new L-functions
- (2008): **Patnaik** showed in his thesis that Kapranov's and Garland's constructions agree in the function field case
- (2013): **Fleig, DP, Kleinschmidt** calculate explicit Fourier expansion for Eisenstein serien on certain "exceptional" Kac-Moody groups (motivated by string theory)

Kac-Moody algebras

- Simple and affine Lie algebras are a subclass

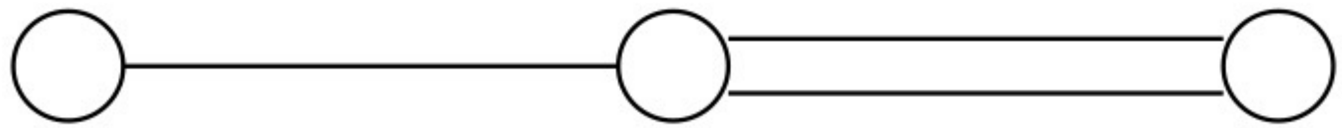
- Hyperbolic KM-algebras have been classified
  (For rank > 2 there are 238 of them) [Carbone et al]

- No classification of indefinite KM-algebras beyond hyperbolic.

- Root lattice is of indefinite signature so we have "imaginary roots", i.e. roots such that

$$(\alpha|\alpha) \leq 0$$

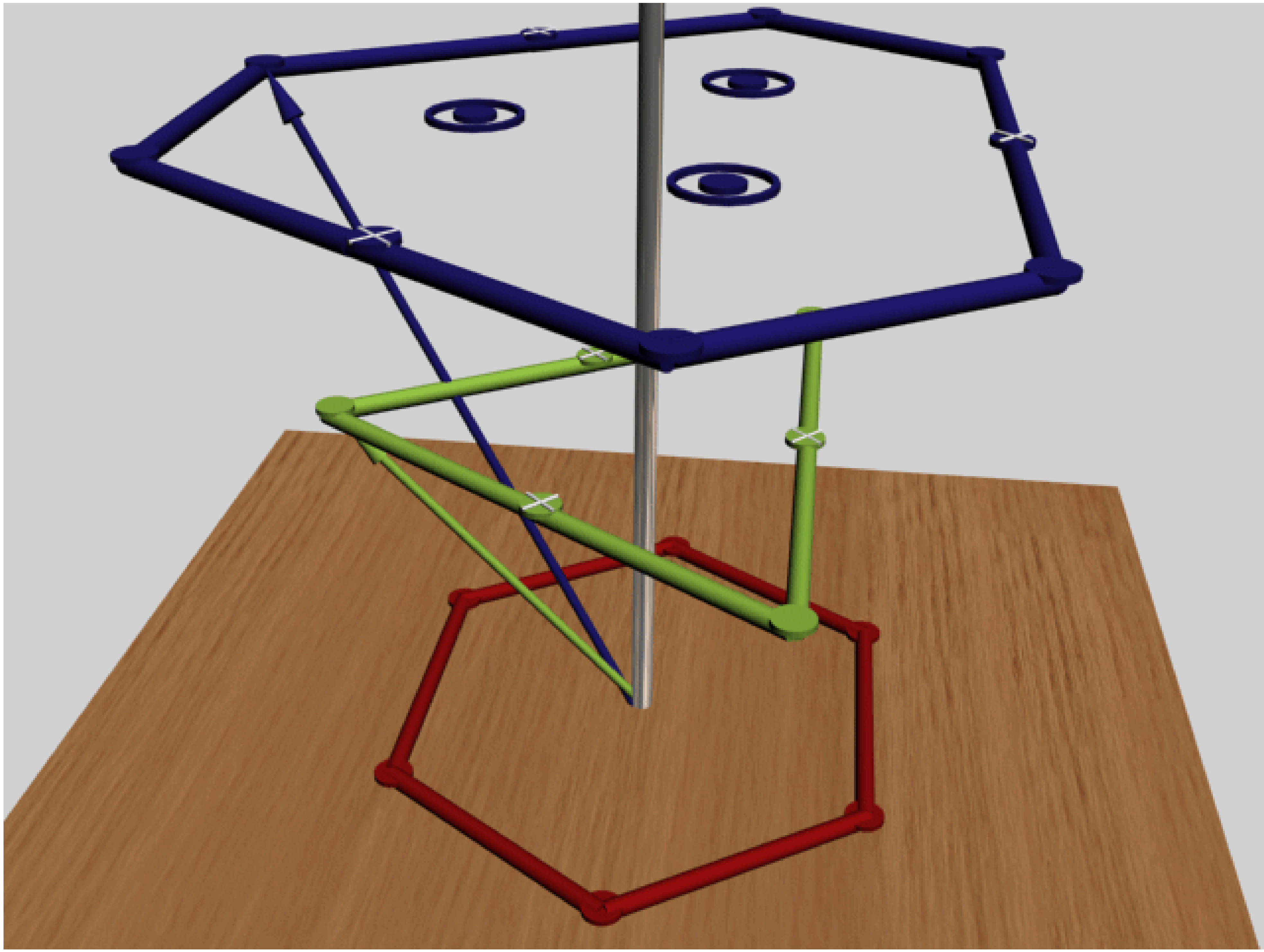
Example: Hyperbolic KM-algebra A_1^{++}



$$A(A_1^{++}) = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -2 \\ 0 & -2 & 2 \end{pmatrix}$$

Root lattice has **Lorentzian** signature

Roots with with crosses are **on the lightcone**



Eisenstein series can formally be defined for any **Kac-Moody group** G

$$E(\lambda, g) = \sum_{\gamma \in B(\mathbb{Z})\backslash G(\mathbb{Z})} e^{\langle \lambda+\rho | H(\gamma g)\rangle}$$

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$$E(\lambda, g) = \sum_{\gamma \in B(\mathbb{Z}) \backslash G(\mathbb{Z})} e^{\langle \lambda + \rho | H(\gamma g) \rangle}$$

→ $G(\mathbb{Z}) \subset G(\mathbb{R})$ defined as a **Chevalley group** [Garland, Carbone…]
(through exponentiation of Lie algebra generators)

$$G(F) := \langle u_{\alpha_i}(s), u_{-\alpha_i}(t) \,|\, s, t \in F, i \in \{1, \dots, r\} \rangle$$

→ **convergence** established by Garland in the affine case and by Carbone, Lee, Liu for rank 2 hyperbolic. General case wide open!

→ generalization of Langlands **constant term formula** established by Garland in the (untwisted) affine case

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→ **Function field analogue** studied by Braverman, Kazhdan, Patnaik.

→ **Cuspidal Eisenstein series** studied by Garland, Miller, Patnaik

→ **Fourier coefficients** studied by Lee, Liu, Patnaik, Fleig, Kleinschmidt, D.P. ...

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A **crucial difference** from the finite case:

$$B = B^{+} \quad \text{and} \quad B^{-} \quad \text{not conjugate!}$$

This is due to the **absence** of a longest Weyl word: w_0

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$$\boxed{\begin{array}{c}\textbf{\textit{Positive}}\\ \text{Eisenstein series}\end{array}} \quad \overset{?}{\longleftrightarrow} \quad \boxed{\begin{array}{c}\textbf{\textit{Negative}}\\ \text{Eisenstein series}\end{array}}$$

Over function fields these are related by a **functional equation** [Braverman, Kazhdan]

Whittaker coefficients

$$E(\lambda, g) = E^{\text{const}}(\lambda, g) + \sum_{\psi} W_{\psi}(\lambda, g) + \cdots$$

$$W_{\psi}(g) = \int_{N(\mathbb{Z})\backslash N(\mathbb{R})} E(\lambda, ng)\overline{\psi(n)}dn$$

Whittaker coefficient

For **Kac-Moody groups**

$$W_{\psi} = 0$$

for ψ **generic**

due to the **lack of a longest Weyl word** in the Weyl group $W(\mathfrak{g})$

Recall **finite-dim** case: $\chi(g) = e^{\langle \lambda+\rho | H(g) \rangle}$

$$W_\psi(\lambda, g) = \int_{N(\mathbb{A})} \chi(w_0 n g)\, \overline{\psi(n)}\, dn = \prod_p W_{\psi_p}$$

$$W_{\psi_p} = \int_{N(\mathbb{Q}_p)} \chi_p(w_0 n a_p) \overline{\psi_p(n)}\, dn$$

Recall **finite-dim** case:

$$\chi(g) = e^{\langle \lambda+\rho | H(g) \rangle}$$

$$W_\psi(\lambda, g) = \int_{N(\mathbb{A})} \chi(w_0 n g)\, \overline{\psi(n)}\, dn = \prod_p W_{\psi_p}$$

$$W_{\psi_p} = \int_{N(\mathbb{Q}_p)} \chi_p(w_0 n a_p) \overline{\psi_p(n)}\, dn$$

$$= \int_{N^-(\mathbb{Q}_p)} \chi_p(n a_p) \overline{\psi_p(n)} dn$$

opposite unipotent

Braverman-Kazhdan: Add this “by hand” in the Kac-Moody case

$$\int_{N^-(\mathbb{Q}_p)} \chi_p(na_p)\overline{\psi_p(n)}dn$$

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- Affine generalization of **Casselman-Shalika formula** $(p < \infty)$ [Patnaik]

- Not known how to do this for $\mathbb{Q}_\infty = \mathbb{R}$

- Beyond the affine case not understood

Instead, we focus on **degenerate Whittaker coefficients**

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Theorem [Fleig, Kleinschmidt, D.P.] $(m_j = 0 \text{ for some } j)$

For **degenerate** characters (and assuming convergence) we have

$$\boxed{W_\psi(\lambda, a) = \sum_{ww_0' \in \mathcal{C}_\psi} e^{\langle (ww_0')^{-1}\lambda + \rho | a \rangle} M(w^{-1}, \lambda) W_\psi^{G'}(w^{-1}, \mathbb{I})}$$

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- Subgroup $G' \subset G$ selected by $m_j \neq 0$
- w_0' longest Weyl word of $W(\mathfrak{g}')$
- $W_\psi^{G'}$ generic Whittaker vector of G'
- $\mathcal{C}_\psi \subset W(\mathfrak{g})$

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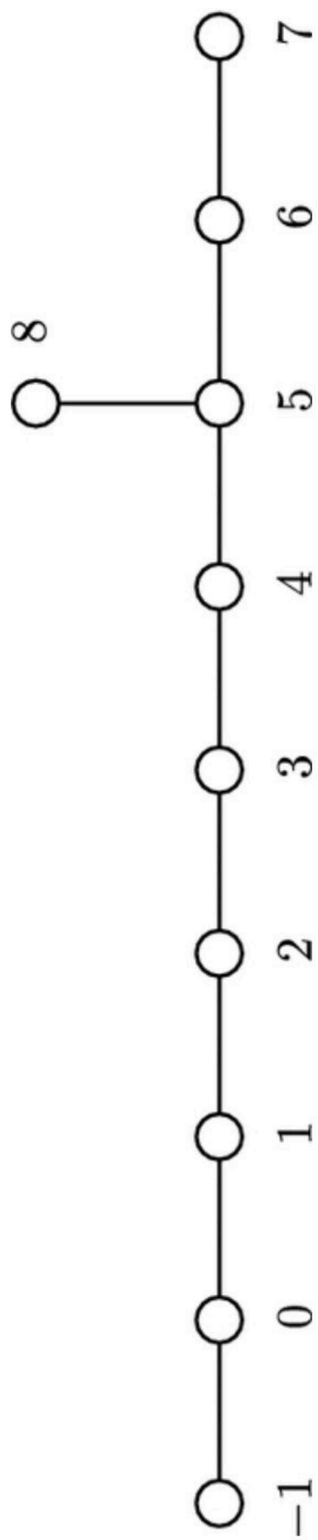
$$M(w, \lambda) = \prod_{\alpha > 0 \,|\, w\alpha < 0} \frac{\xi(\langle \lambda | \alpha \rangle)}{\xi(1 + \langle \lambda | \alpha \rangle)}$$

intertwiner

$$G = E_{10}(\mathbb{R})$$



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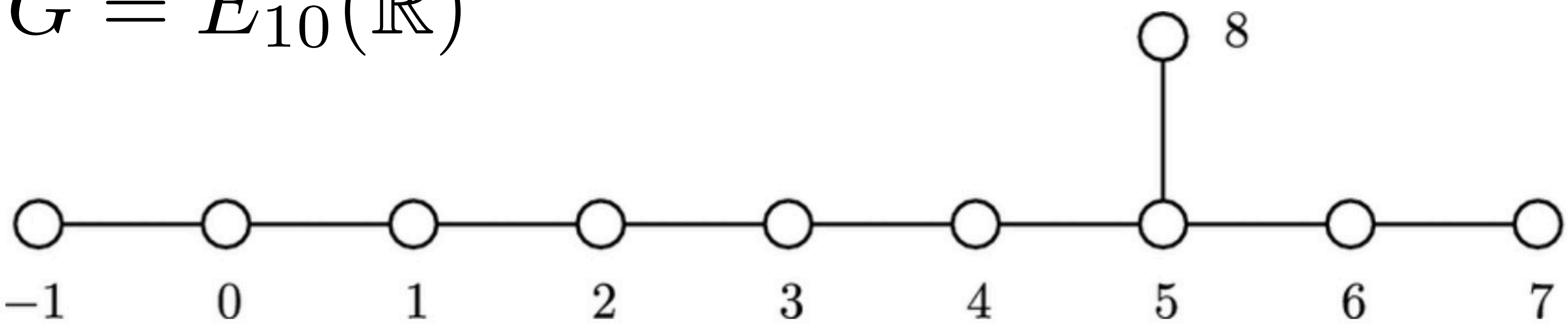

Eisenstein series: $$E(\lambda, g) = \sum_{\gamma \in B(\mathbb{Z}) \backslash G(\mathbb{Z})} e^{\langle \lambda + \rho | H(\gamma g) \rangle}$$

Generically, this has an **infinite number** of Fourier coefficients for each character ψ

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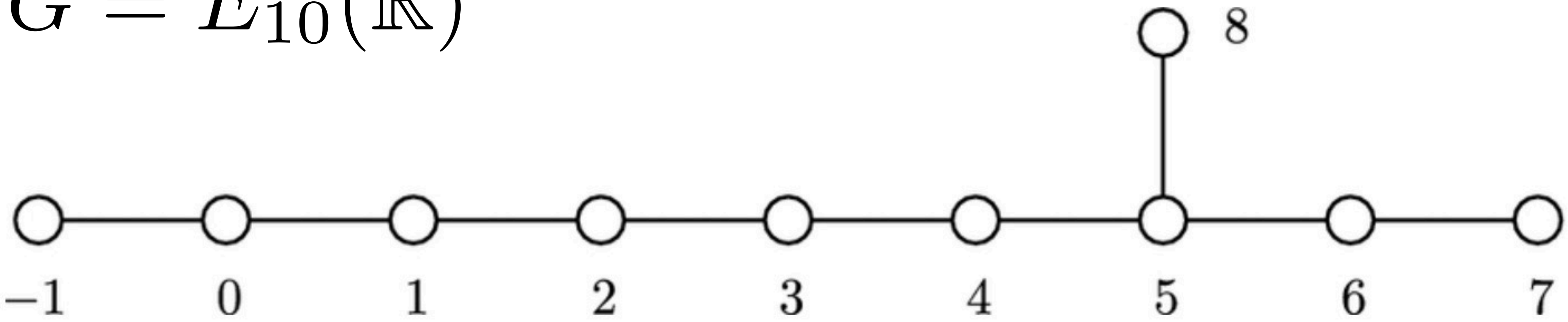

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Originally motivated by string amplitudes we **study the special value**:

$$s = 3/2$$

$$\lambda = 3\Lambda_1 - \rho$$

For the special value $\lambda = 3\Lambda_1 - \rho$ the sum **collapses**!

ψ	$W_\psi(\chi_{3/2}, a)$
$(m,0,0,0,0,0,0,0,0,0)$	$v_3^2 v_1^{-1} B_{3/2,m}\left(v_1^2 v_3^{-1}\right)$
$(0,m,0,0,0,0,0,0,0,0)$	$\frac{v_2^2 \tilde{B}_{0,m}(v_2^2 v_4^{-1})}{\xi(3)}$
$(0,0,m,0,0,0,0,0,0,0)$	$\frac{\xi(2) v_4 B_{1,m}\left(v_3^2 v_1^{-1} v_4^{-1}\right)}{\xi(3)}$
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$(0,0,0,0,m,0,0,0,0,0)$	$\frac{v_5^2 \tilde{B}_{0,m}(v_5^2 v_4^{-1} v_6^{-1})}{\xi(3) v_6}$
$(0,0,0,0,0,m,0,0,0,0)$	$\frac{\xi(2) v_6^3 B_{-1/2,m}\left(v_6^2 v_5^{-1} v_7^{-1}\right)}{\xi(3) v_7^2}$
$(0,0,0,0,0,0,m,0,0,0)$	$v_7^4 v_8^{-3} B_{-1,m}\left(v_7^2 v_6^{-1} v_8^{-1}\right)$
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$$\boxed{B_{s,m}(a) \sim \Big(\sum_{d|m} d^s\Big) K_s(ma)}$$

This is due to the **vanishing** properties of the intertwiner

$$M(w, \lambda)$$

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$(0,0,0,0,0,m,0,0,0,0)$	$\xi(2) v_6^3 B_{-1/2}$ [illegible]
$(0,0,0,0,0,0,m,0,0$ [illegible]	[illegible]
$(0,0,0,0,0,0$ [illegible]	[illegible] $\left(v_8^2 v_7^{-1} v_9^{-1}\right)$ / $\xi(3)$
$(0,0,0,0,0,0,$ [illegible]	$\frac{\xi(5) v_9^6 v_{10}^{-5} B_{-2,m}\left(v_9^2 v_8^{-1} v_{10}^{-1}\right)}{\xi(3)}$
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These are the only non-vanishing coefficients!

[illegible] to the [illegible]**ing** properties of the intertwiner

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What does our result mean mathematically?

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Conjecture: *For* $G = E_9, E_{10}, E_{11}$ *the Eisenstein series* $E(3\Lambda_1 - \rho, g)$ *is attached to the* ***minimal representation*** π_{min}

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These representations have not yet been defined for Kac-Moody groups

(no notion of minimal nilpotent orbit etc…)

→ Work in progress with Bossard, Kleinschmidt, Patnaik, Sahi

Minimal automorphic representations

There exists special representations with smallest non-trivial **Gelfand-Kirillov** dimension

The GK-dimension gives a notion of the **size** of a representation on a function space

Example:

$$\mathrm{GKdim}\big(L^2(\mathbb{R}^n)\big) = n$$

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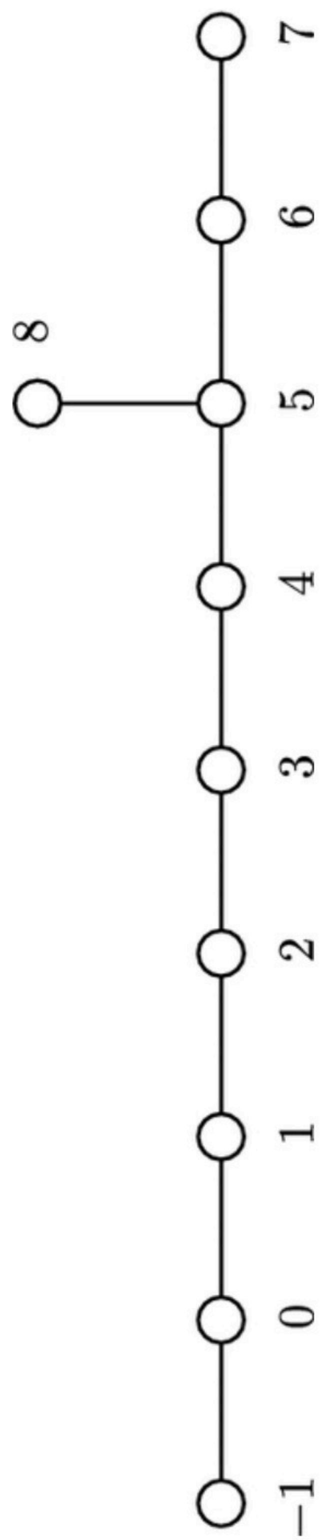
Minimal representations are generalizations of the **Weil representation**

→ **Theta series** $\theta(\tau) = \sum_{n\in\mathbb{Z}} e^{i\pi\tau n^2}$

In general, minimal automorphic forms are characterised by having **very few non-zero Fourier coefficients**

[Ginzburg, Rallis, Soudry]

Back to E_{10}



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Implications for the Kac-Moody case

- The min rep is Eulerian $\pi_{min} = \bigotimes_p \pi_p$

- The only non-vanishing Whittaker coefficients are **maximally degenerate**:

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- The only non-vanishing Whittaker coefficients are **maximally degenerate**:

$$\int_{N(F)\backslash N(\mathbb{A})} E(s, ng)\psi_\alpha^{-1}(n)dn = F_\infty \times \prod_{p<\infty} F_p$$

and **Eulerian**

Open questions

- $p-$ adic representation π_p : Related to reps of the **DAHA**? (Double Affine Hecke Algebra)
- archimedean representation π_∞ : analogue of the **Joseph ideal**?
- What is the **Gelfand-Kirillov** dimension of π_{min} ?
- Minimal nilpotent co-adjoint orbit $\mathcal{O}_{min}$? $\boxed{G \cdot E_\alpha}$
- Theta correspondences for Kac-Moody groups? [Garland, Liu]
- Uniqueness?

Open questions

For higher derivative couplings $\partial^n \mathcal{R}^4$ string theory predicts a **new type of automorphic object**

For example, $\partial^6 \mathcal{R}^4$ in D=10 predicts:

$$(\Delta_\tau - 12)\mathcal{F}(\tau) = -\big(E_{3/2}(\tau)\big)^2$$

[Green, Vanhove][Green, Miller, Vanhove][Fedosova, Klinger-Logan, Radchenko]

These are not eigenfunctions of the Laplace operators, but satisfy Poisson-type equations.

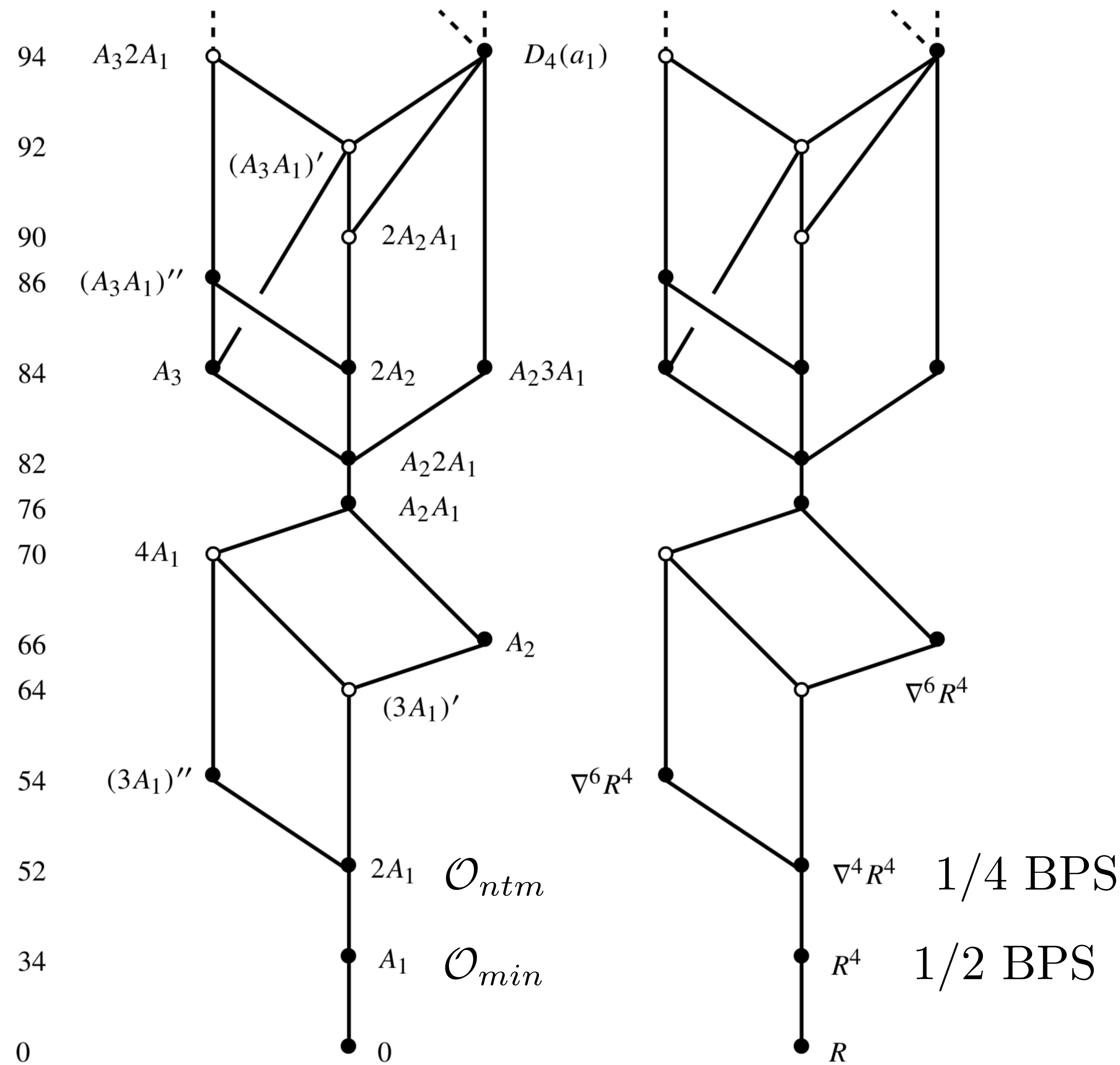
(Violates $\mathcal{Z}$ -finiteness)

[In progress w/ Bossard, Gourevitch, Friedberg, Kleinschmidt]

Is there a representation-theoretic origin?

Thank you!

Hasse diagram for E_7



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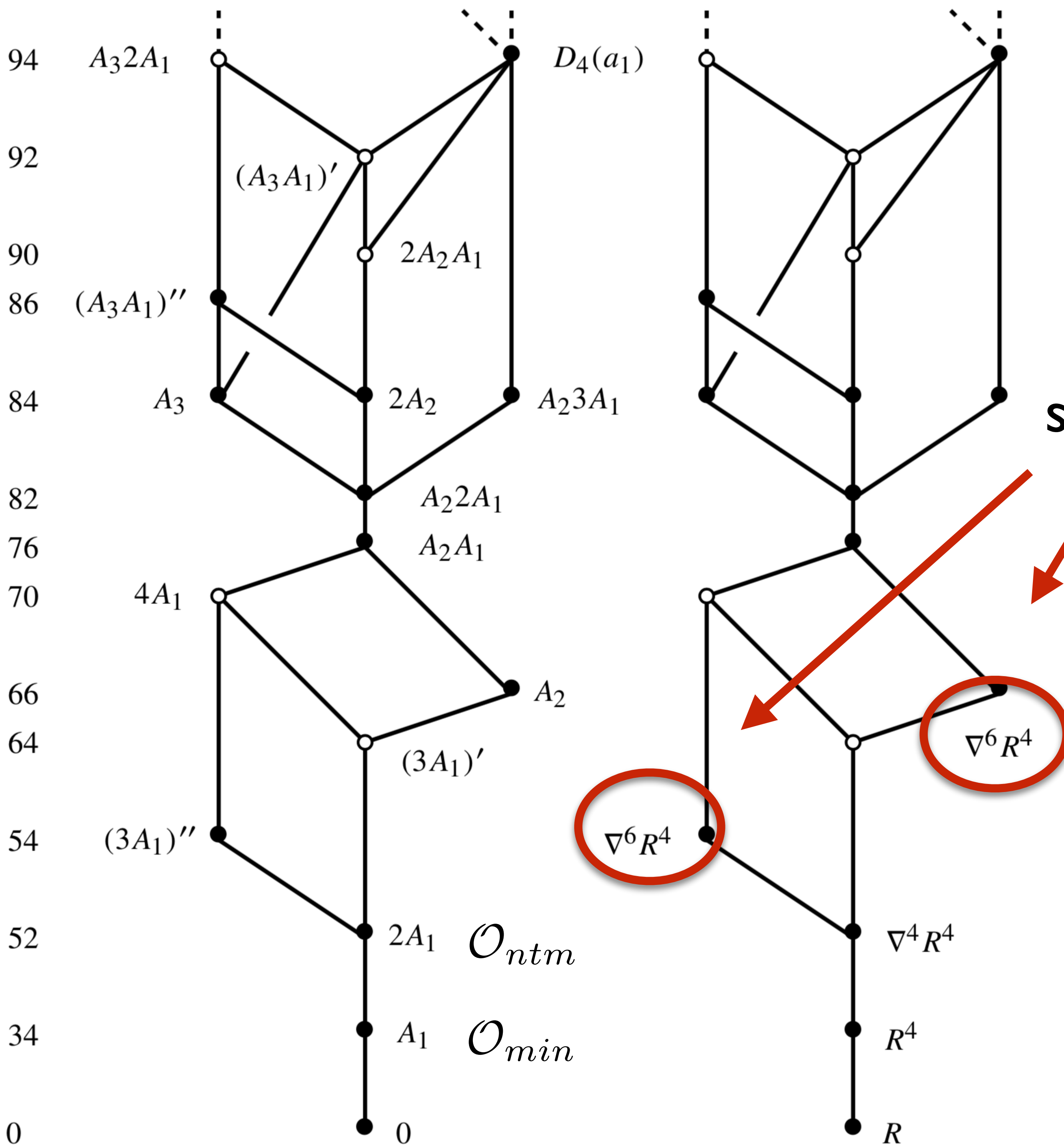

1/8 BPS-contributions:

New automorphic objects, satifying **Poisson-type** equations

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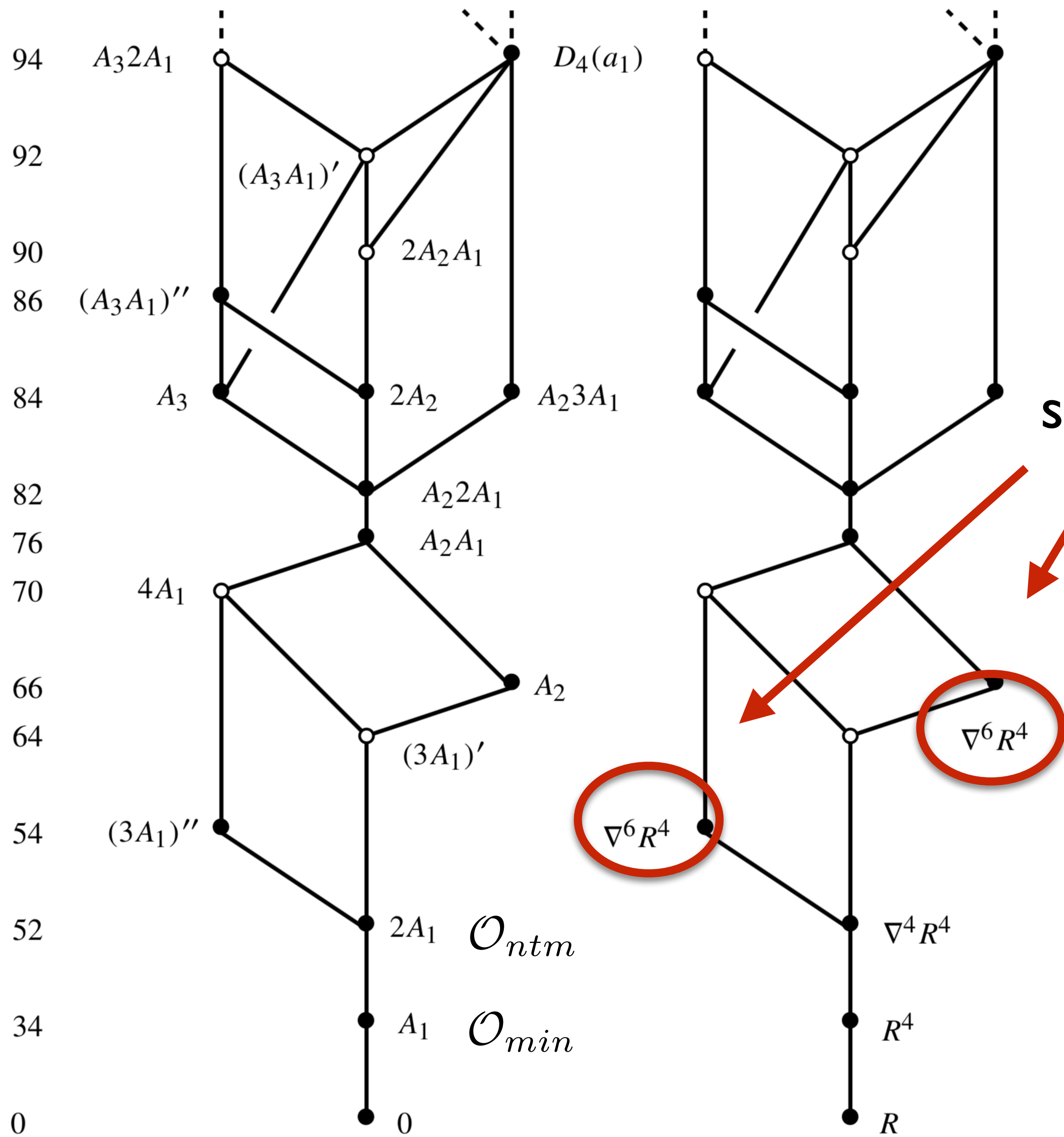

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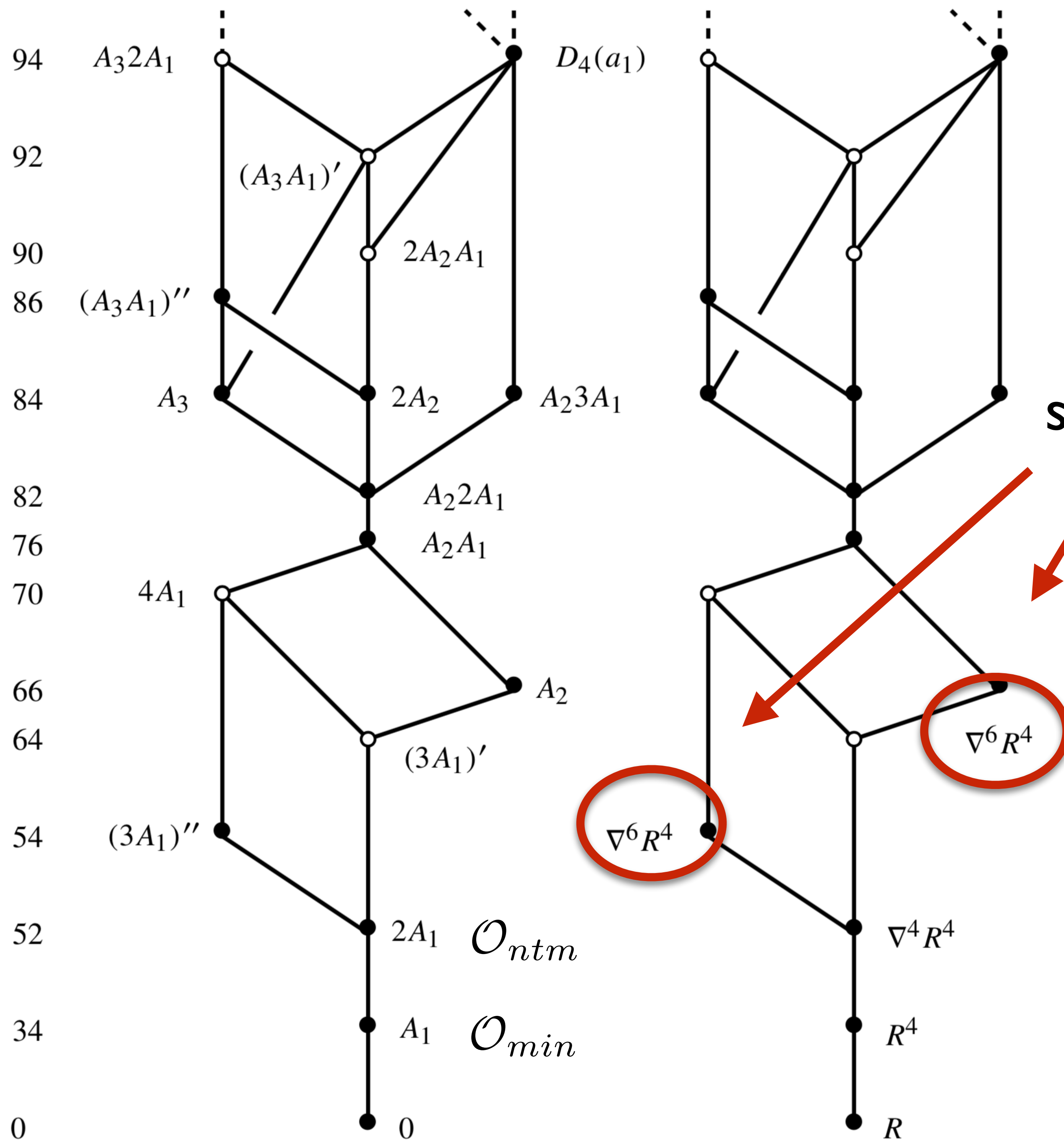

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