Lectures on NS5-branes & hypermultiplets

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<u>Goal</u>: To highlight interesting mathematical structures that emerge from the study of hypermultiplet moduli spaces in string theory.

- Quantum mirror symmetry
- Quaternion-Kähler geometry
- Wall-crossing & DT-invariants
- NS5-branes
  (no math description known)
- Automorphic forms
- (Quantization of) integrable systems

Disclaimer: Most of this won't be rigorous mathematics, but it can hopefully be turned into mathematics with your help!

Main references

Alexandrov, D.P., Pioline:

1009.3026 } NS5
1010.5792

1110.0466 } Wall-crossing & QK/HK-correspondence

⇒ Recent review:

Alexandrov, Manschot, D.P., Pioline (to appear)

Background refs

Gaiotto, Moore, Neitzke 0807.4723
Alexandrov, Pioline, Saueressig, Vandoren 0812.4219
Kontsevich, Soibelman 0910.4315
(Joyce-Song 0810.5645)

Outline

1. General set-up & overview (quantum mirror symmetry)

2. Hypermultiplet moduli spaces

3. D-instantons, DT-invariants and wall-crossing

4. NS5-instantons I: the partition function

5. NS5-instantons II: S-duality and the topological string

6. NS5-instantons III: quantization of cluster varieties

I. Overview of quantum mirror symmetry

Let (X, Y) be a pair of CY_3-folds.

String theory associates to this pair two moduli spaces:

Type IIA sector:

$$\mathcal{M}_A(X) = \mathcal{M}_K(X) \times \mathcal{M}_H(X)$$

Type IIB sector:

$$\mathcal{M}_B(Y) = \mathcal{M}_C(Y) \times \mathcal{M}_H(Y)$$

The various factors are:

$\mathcal{M}_K(X)$ = complexified Kähler moduli space of X

$\subset H^2(X, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{C}^*$

(vector multiplet moduli space in physics)

$\mathcal{M}_H(X)$ = hypermultiplet moduli space in type IIA.

[No mathematical definition, but knows about complex struct of X.]

$\mathcal{M}_c(\cancel{X})$ = complex structure moduli space of Y

(vector multiplet moduli space in IIB)

$\mathcal{M}_H(Y)$ = hypermultiplet moduli space in type IIB

[Knows about Kähler structure of Y]

when $(X, Y) = (X, \hat{X})$ is a mirror pair then mirror symmetry asserts that there is an isomorphism

$$\mathcal{M}_K(X) \cong \mathcal{M}_C(\hat{X})$$

$\Rightarrow$ Symplectic geometry of X related to complex geometry of $\hat{X}$.

<u>Def</u>: The isomorphism above is called <u>classical mirror symmetry</u>.

<u>Remark</u>: Since there is still no satisfactory mathematical def. of the "stringy Kähler moduli space" $\mathcal{M}_K(X)$ the above relation is often taken as a definition of this space.

Remark: Physically, the metric on $\mathcal{M}_C(\hat{X})$ is classically exact, while the metric on $\mathcal{M}_K(X)$ receives non-perturbative correction from worldsheet instantons.

Mathematically these are related to ~~moduli~~ GW-invariants.

Our main interest in these lectures will be the hypermultiplet moduli spaces $\mathcal{M}_H(X)$ and $\hat{\mathcal{M}}_H(\hat{X})$.

In particular, we wish to emphasize what appears to be interesting mathematical structures that deserve further study.

Key features of $\mathcal{M}_H$

- $\mathcal{M}_H$ carries a quaternion-Kähler metric
- $\mathcal{M}_C(X) \subset \mathcal{M}_H(X)$
- $\mathcal{M}_K(\hat{X}) \subset \hat{\mathcal{M}}_H(\hat{X})$
- Receives D-instanton corrections. Mathematically, these are related to (generalized) DT-invariants
- Receives NS5-instanton corrections. These have no known mathematical interpretation.

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	IIA / X		IIB / $\hat{X}$	
moduli space	$\mathcal{M}_K(X)$	$\times$ $\mathcal{M}_H(X)$	$\mathcal{M}_C(\hat{X})$	$\times$ $\hat{\mathcal{M}}_H(\hat{X})$
metric	special Kähler	QK	special Kähler	QK
worldsheet instantons (GW-invariants)	Yes	No	No	Yes
D-instantons (DT-invariants)	No	Yes (sLags $\subset X$)	No	Yes (coherent sheaves on $\hat{X}$)
NS5-instant.	No	Yes	No	Yes

Quantum Mirror Symmetry

QMS asserts that there should be an equivalence of theories

$$\mathrm{IIA}/X \cong \mathrm{IIB}/\hat{X}$$

including <u>all quantum corrections</u>

In terms of moduli spaces this implies:

$$\mathcal{M}_H(X) \cong \hat{\mathcal{M}}_H(\hat{X})$$

[Becker, Becker, Strominger]

Mathematically, D-instanton correction to $\mathcal{M}_H(X)$ correspond to sLags, i.e. to (semi-)stable objects in the (bounded derived) Fukaya category $D^b Fuk(X)$.

Similarly, D-instantons in $\mathcal{M}_{H}(\hat{X})$ correspond to coherent sheaves on $\hat{X}$, or, more precisely, to Bridgeland (semi)-stable objects in the (bounded derived) category of coherent sheaves $D^b Coh(\hat{X})$.

Hence, Quantum Mirror Symmetry includes Kontsevich's Homological Mirror Symmetry conjecture:

$$D^b Fuk(X) \cong D^b Coh(\hat{X})$$

(quasi-isomorphism of triangulated A_∞-categories)

But QMS contains more

$$NS5/_X \cong NS5/_{\hat{X}}$$

What is the mathematical description of these effects?

Is there an extension of homological mirror symmetry that would be equivalent to quantum mirror symmetry?

Heuristically, we expect something like the following to hold true:

$$\text{HMS} + \text{S-duality} = \text{QMS}$$

But more about this later...

2. Hypermultiplet moduli spaces

For definiteness we shall restrict to the type IIA picture, and only comment on the IIB sector when needed.

In what follows, X will be a compact CY_3-fold, i.e. a compact Kähler manifold with a nowhere vanishing holomorphic 3-form $\Omega^{3,0}$ that trivializes the canonical bundle, $K_X = 0$.

Type IIA string theory associates to X a real $4b_{3,1}$-dimensional manifold

$$\mathcal{M}_H = \mathcal{M}_H(X)$$

equipped with a quaternion-Kähler metric.

2.1 Topology of $\mathcal{M}_H$

The hypermultiplet moduli space $\mathcal{M}_H(X)$ is a $\mathbb{C}^*$-bundle

$$\begin{array}{ccc} \mathbb{C}^* & \longrightarrow & \mathcal{M}_H(X) \\ & & \downarrow \\ & & J_W(X) \end{array}$$

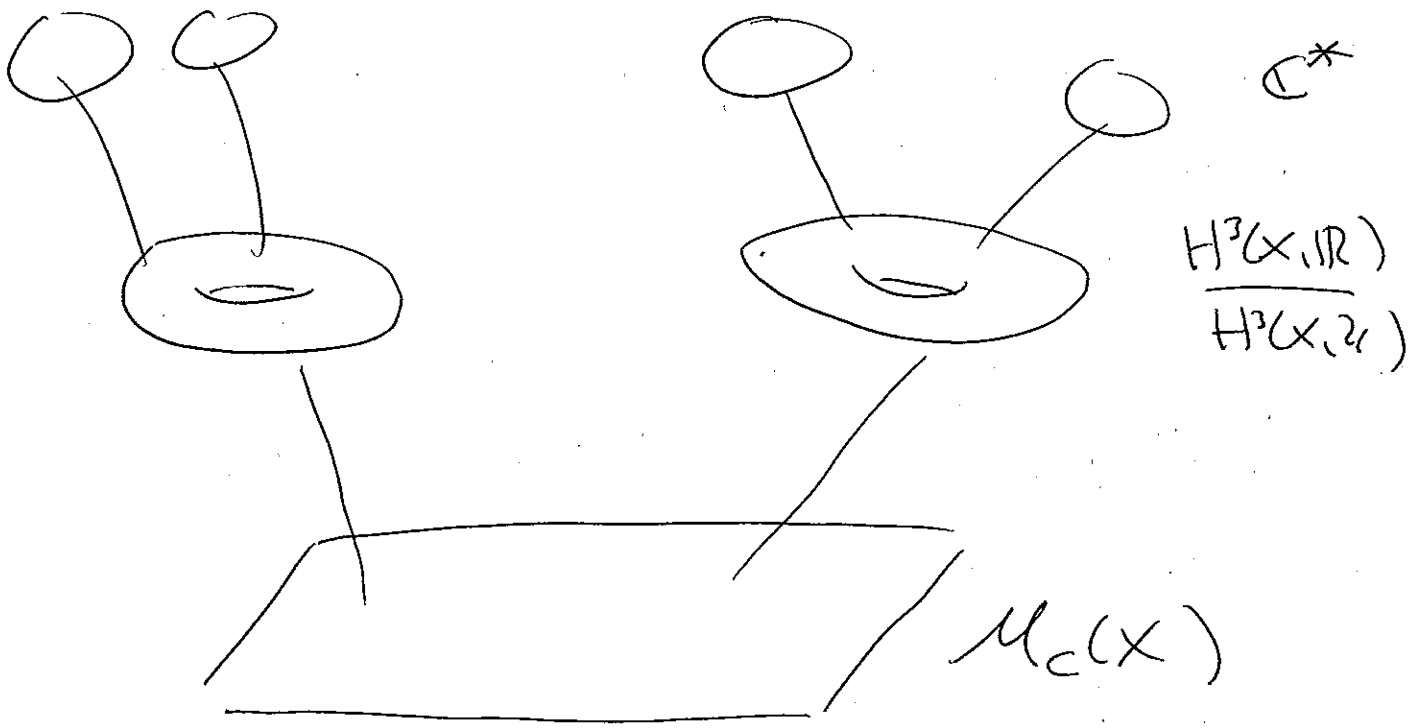
The base of the fibration is a bundle of tori

$$\begin{array}{ccc} T & \longrightarrow & J_W(X) \\ & & \downarrow \\ & & \mathcal{M}_c(X) \end{array}$$

where T is the intermediate Jacobian of X, equipped with the Weil complex structure:

$$T = \frac{H^3(X,\mathbb{R})}{H^3(X,\mathbb{Z})} = \frac{H^{3,0}(X) \oplus H^{1,2}(X)}{H^3(X,\mathbb{Z})}$$

we call $J_W(X) \to \mathcal{M}_c(X)$ the family of Weil intermediate Jacobians.



Let us now see how this structure arises from the physics.

Consider the moduli space of pairs

$$\mathcal{L}_X = \left(\text{choice of complex struct}, \Omega^{3,0}\right).$$

This is a $\mathbb{C}^\times$- bundle

$$\mathcal{L}_X \to \mathcal{M}_c(X)$$

($\Omega^{3,0}$ is defined up to a non-vanishing complex rescaling)

Fix a symplectic basis ("marking") (A^Λ, B_Λ) of $\Gamma \equiv H_3(X, \mathbb{Z})$ $(\Lambda = 0, 1, \ldots, h_{2,1})$

The period map

$$\Omega^{3,0} \longmapsto \left(\int_{A^\Lambda} \Omega^{3,0}, \int_{B_\Lambda} \Omega^{3,0}\right) = \left(X^\Lambda, F_\Lambda\right)$$

realize $\mathcal{L}_X$ as a complex Lagrangian cone in $H^3(X, \mathbb{C})$

Locally one may express

$$F_\Lambda = \partial_\Lambda F(X^\Lambda)$$

where $F(X^\Lambda)$ is a holomorphic function, homogeneous of degree 2, known as the prepotential.

Away from $X^0 = 0$ we have coordinates on $\mathcal{M}_C(X)$ given by the ratios

$$Z^a \equiv X^a / X^0 \qquad a = 1, \ldots, h_{2,1}.$$

[So $\mathcal{M}_X$ is really an orbifold $\mathcal{L}_X / \mathbb{C}^*$, or "moduli stack"]

The ~~[illegible]~~ coordinates Z^a are scalar fields in the 4d, N=2 effective action. These originate from the spacetime metric in 10d.

In IIA string theory we also have a (RR-form), that is a 3-form potential C, whose periods

$$C \mapsto \left(\int_{A^\Lambda} C, \int_{B_\Lambda} C \right) \equiv (\xi^\Lambda, \tilde{\xi}_\Lambda)$$

yield additional scalar fields in the action.
We shall abuse notation and write

$$C = (\xi^\Lambda, \tilde{\xi}_\Lambda) \in H^3(X, \mathbb{R})$$

Invariance under large gauge transformations

$$C \to C + H$$

with $H \in H^3(X, \mathbb{Z})$ imply that $(\xi^\Lambda, \tilde{\xi}_\Lambda)$ ~~para~~ are periodic and hence parametrize point on

$$\mathcal{T} = \frac{H^3(X, \mathbb{R})}{H^3(X, \mathbb{Z})}$$

Under monodromies in $\mathcal{M}_C(X)$ the vector C transform by a symplectic rotation. Hence τ is non-trivially fibered over $\mathcal{M}_C(X)$ and the total space is the family $J_W(X) \to \mathcal{M}_C(X)$ of Weil intermediate Jacobians.

Finally, the 4d dilaton field e^{ϕ} and the Poincaré dual σ to the B-field in 4d yield a complex scalar field

$$\sigma + i e^{-\phi} \in \mathbb{C}^*$$

that parametrize the fiber of $\mathcal{M}_H(X) \to J_C(X)$.

2.2 Heisenberg symmetry

Large gauge transformations of the B-field translate into the periodicity

$$\sigma \to \sigma + 2K, \quad K \in \mathbb{Z}.$$

But a gauge transformation $C \to C + H$ also act on σ and in total we have the transformation rule:

$$(C, \sigma) \mapsto \left(C+H, \sigma + 2K + \langle C, H \rangle + 2c(H)\right)$$

where $\langle\, , \rangle : \Gamma \times \Gamma \to \mathbb{Z}$ is the symplectic pairing.

Here $c(H)$ parametrize a choice of quadratic refinement

$$\lambda : \Gamma \to \{\pm 1\}$$

via $\lambda(H) \equiv (-1)^{2c(H)}$.

A quadratic refinement is defined by the condition

$$\lambda(H+H') = (-1)^{\langle H, H' \rangle} \lambda(H)\lambda(H')$$

Given a choice of basis (A^Λ, B_Λ) of $H^3(X,\mathbb{Z})$ we can solve this condition by

$$\lambda(H) = e^{-i\pi m_\Lambda n^\Lambda + 2\pi i(m_\Lambda \theta^\Lambda - n^\Lambda \phi_\Lambda)}$$

where $H = m_\Lambda A^\Lambda - n^\Lambda B_\Lambda \in H^3(X,\mathbb{Z})$ and $\Theta \equiv (\theta^\Lambda, \phi_\Lambda)$ are called "characteristics".

The extra shift of $2c(H)$ is required for the closure of th group action. We thus have a discrete Heisenberg group $Heis_{\mathbb{Z}}$ acting on $\mathcal{M}_H$.

$$\mathcal{M}_H(X) \circlearrowleft Heis_{\mathbb{Z}}$$

2.3 Perturbative metric

The perturbative expansion of string theory is govern by the dilaton

$$g_s = e^{\phi} = r \quad (\text{really } \langle\phi\rangle)$$

The metric on $\mathcal{M}_H$ is perturbatively exact at one-loop, with explicit metric given by:

$$g_{pert} = \frac{r+2c}{r^2(r+c)}\, dr^2 + \frac{4(r+c)}{r}\, g_{\mathcal{M}_c} + \frac{1}{r}\, g_{\mathcal{T}}(c)$$

$$+ \frac{r+c}{16r^2(r+2c)} \left(d\sigma + A(c) \right)^2$$

* $c = -\chi(X)/192\pi$ (one-loop correction)

* $g_{\mathcal{T}}(c)$ = Weil metric on $\mathcal{T}$, perturbed by c.

* $g_{\mathcal{M}_c}$ = ~~…~~ Special Kähler metric on $\mathcal{M}_c$.

Some details.

* $$g_\tau = -\frac{1}{2} d\omega_\Lambda \, \mathrm{Im}\, \mathcal{N}^{\Lambda\Sigma} d\bar{\omega}_\Sigma$$

where $\omega_\Lambda = \xi_\Lambda - \bar{\mathcal{N}}_{\Lambda\Sigma} \xi^\Lambda$

and the Weil period matrix is

$$\mathcal{N}_{\Lambda\Lambda'} = \bar{\tau}_{\Lambda\Lambda'} + 2i \frac{[\mathrm{Im}\,\tau \cdot X]_\Lambda [\mathrm{Im}\,\tau \cdot X]_{\Lambda'}}{X^\Sigma \, \mathrm{Im}\,\tau_{\Sigma\Sigma'} \, X^{\Sigma'}}$$

with $\tau_{\Lambda\Lambda}$ the Griffiths period matrix

$$\tau_{\Lambda\Lambda'} = \partial_\Lambda \partial_{\Lambda'} F(X^\Sigma)$$

* The metric $g_{\mathcal{M}_c}$ follows from the Kähler potential

$$K = -\log\left[i \int_X \Omega^{3,0} \wedge \bar{\Omega}^{3,0}\right]$$

$$= \log\left[i(\bar{X}^\Lambda F_\Lambda - X^\Lambda \bar{F}_\Lambda)\right]$$

Most importantly, $A(c)$ is a connection on the circle bundle $\mathcal{C}_\sigma \to \mathcal{J}_W(X)$ whose fiber is parametrized by σ.

$$A(c) = \tilde{\zeta}_\Lambda d\zeta^\Lambda - \zeta^\Lambda d\tilde{\zeta}_\Lambda + 8c\, A_K$$

with A_K the Kähler connection on $\mathcal{L}_X \to \mathcal{M}_c$:

$$A_K = \frac{i}{2}\left(\partial_{z^a} K\, dz^a - \partial_{\bar{z}^{\bar{a}}} K\, d\bar{z}^{\bar{a}} \right)$$

The curvature of the connection yields

$$c_1(\mathcal{C}_\sigma) = \omega_\mathcal{T} + \frac{\chi(X)}{24}\omega_c$$

where

$$\begin{cases} \omega_\mathcal{T} = d\tilde{\zeta}_\Lambda \wedge d\zeta^\Lambda \\ \omega_c = -\frac{1}{2\pi} dA_K = c_1(\mathcal{L}_X) \end{cases}$$

So the circle bundle $\mathcal{C}_\sigma$ is non-trivially fibered both over $\mathcal{T}$ as well as $\mathcal{M}_c$.

We will return to the topology of $\mathcal{C}_\sigma$ later when we discuss NS5-branes.

Remarks:

* The tree-level metric is recovered for $c=0$; this is the local c-map metric first discovered by Cecotti et al.

* The one-loop metric g_{pert} has a curvature singularity at $r=-2c$ ($r=0$ & $r=-c$ are coord. singularities)
It is expected that this is resolved when all instanton corrections are present.

* The tree-level metric obtained for $c=0$ is the supergravity analogue of the semi-flat hyperkähler metric on the Coulomb branch of Seiberg-Witten theory on $\mathbb{R}^3 \times S^1$, studied by GMN.

Example: rigid CY_3

For a rigid CY_3-fold there are no cplx structure deformation ~~and~~ $h_{2,1}=0$, and $\mathcal{M}_C(X)$ is trivial. The period matrix is a complex number

$$\tau = \frac{\int_B \Omega^{3,0}}{\int_A \Omega^{3,0}}$$

The intermediate Jacobian is an elliptic curve

$$J(X) = \mathbb{C} / \mathbb{Z} + \tau \mathbb{Z}$$

In this case the QK metric on $\mathcal{M}_H$ becomes

$$g = d\phi^2 + e^{2\phi} \frac{|d\tilde{\xi} + \tau d\xi|^2}{\operatorname{Im}\tau}$$
$$+ e^{4\phi} (d\sigma + \xi d\tilde{\xi} - \tilde{\xi} d\xi)^2$$

This is the invariant metric on the symmetric space

$$\mathcal{M}_H^{cl}(X) = SU(2,1) / SU(2) \times U(1)$$

Quantum corrections will deform the metric away from a symmetric space.

<u>Conjecture</u>:

$$\mathcal{M}_H(X) \circlearrowleft SU(2,1;\mathbb{Z}[i])$$

(Bao, Kleinschidt, Nilsson, D.P., Pioline)

2.4 Twistor space description

Let $\mathcal{M}$ be a QK-manifold
of real dimension $4n$.

$$\Rightarrow \quad \mathrm{Hol}(\mathcal{M}) \subset Usp(n) \times SU(2) \subset SO(4n)$$

$\exists$ a triplet of almost complex
structures $\vec{J} = (J_1, J_2, J_3)$
(defined locally up to $SU(2)$ rotations)
and an associated triplet of
2-forms $\vec{\omega} = (\omega_1, \omega_2, \omega_3)$.

$\Rightarrow$ Globally defined closed
4-form

$$\vec{\omega} \wedge \vec{\omega}$$

The J_i's are not integrable
unless the scalar curvature
of $\mathcal{M}$ vanishes $\Rightarrow$ hyperkähler.

Although M is not a complex manifold, one can describe its metric complex analytically by passing to its twistor space Z.

<u>Theorem</u> [Salamon, LeBrun]

For M a QK-manifold there exists a non-trivial fibration

$$\mathbb{P}^1 \to Z \xrightarrow{p} M$$

such that the total space Z carries a canonical $\mathbb{C}$-contact structure.

This implies that there exists on (an open patch of) Z a holomorphic 1-form $\mathcal{X}$ such that

$$\mathcal{X} \wedge (d\mathcal{X})^n \neq 0 \quad \underline{\text{everywhere}}$$

This one-form is analogous to the ~~...~~ symplectic 2-form in symplectic geometry.

The contact structure can be determined as the kernel of the $(1,0)$-form

$$Dt = dt + p_+ - i p_3 t + p_- t^2$$

where $t \in \mathbb{P}^1$ and $\vec{p} = (p_+, p_-, p_3)$ are the components of the $SU(2)$ Levi-Civita connection.

Locally on Z there exists a ~~holomorphic~~ function

$$\Phi : Z \to \mathbb{C}$$

which is holomorphic along the fiber $p^{-1}(x)$, $x \in M$, and such that

$$\mathcal{X} = -4i e^{\Phi} \frac{Dt}{t}$$

Darboux coordinates

Locally on Z there exists complex Darboux coordinates

$$(\xi^\Lambda, \tilde{\xi}_\Lambda, \alpha) \qquad (\Lambda = 0, \ldots, n-1)$$

such that the contact one-form takes the canonical form:

$$\mathcal{X} = d\alpha + \xi^\Lambda d\tilde{\xi}_\Lambda .$$

It will be convenient to instead work with

$$\Sigma \equiv (\xi^\Lambda, \tilde{\xi}_\Lambda) ; \quad \tilde{\alpha} \equiv -2\alpha - \tilde{\xi}_\Lambda \xi^\Lambda$$

for which

$$\mathcal{X} = -\frac{1}{2}\left(d\tilde{\alpha} + \tilde{\xi}_\Lambda d\xi^\Lambda - \xi^\Lambda d\tilde{\xi}_\Lambda \right)$$

$$= -\frac{1}{2}\left(d\tilde{\alpha} + \langle \Sigma, d\Sigma \rangle \right)$$

Transition functions

On the overlap $\mathcal{U}_i \cap \mathcal{U}_j$ of two patches the Darboux coordinates are related by complex contact transformations preserving the contact one-form up to a holomorphic rescaling:

$$\mathcal{X}^{[i]} = f_{ij}^{2}\, \mathcal{X}^{[j]}$$

The global structure of $\mathcal{Z}$ can be specified by providing generating functions

$$H^{[ij]}\left(\xi^{\Lambda}_{[i]},\ \tilde{\xi}^{[j]}_{\Lambda},\ \alpha^{[j]}\right)$$

for contact transformations.

The metric on $\mathcal{M}$ can be recovered from the knowledge of the Darboux coordinates as function of $x^\mu \in \mathcal{M}$ and $t \in \mathbb{P}^1$

$$\xi^\Lambda(t, x^\mu) \,, \quad \tilde{\xi}_\Lambda(t, x^\mu) \,, \quad \alpha(t, x^\mu)$$

These are called "twistor lines".

Plugging these into the contact one-form $\mathcal{X}$ and expand in a ~~Taylor~~ Laurent series in t allows to extract $\vec{p} = (p_+, p_-, p_3)$:

$$\mathcal{X} \sim \frac{dt}{t} + p_+ t^{-1} - i p_3 + p_- t$$

From $d\vec{p} + \frac{1}{2} \vec{p} \wedge \vec{p} = \frac{\nu}{2} \vec{\omega}$ one gets the metric by contracting

$$g(u, v) = \omega_3(u, J_3 v)$$

Darboux coordinates for g_{pert}

Consider the patch $\mathcal{U}_0 \subset \mathcal{Z}$ around the equator on $\mathbb{P}^1$

$$\mathcal{U}_0 = \mathbb{P}^1 \setminus \{0, \infty\}$$

The perturbative hypermultiplet metric on $\mathcal{M}_H$ can then be recovered from

$$\begin{cases} \xi^\Lambda = \zeta^\Lambda + 2\sqrt{r}\, e^{K/2}\left(t^{-1} X^\Lambda - t \bar{X}^\Lambda\right) \\ \tilde{\xi}_\Lambda = \tilde{\zeta}_\Lambda + 2\sqrt{r}\, e^{K/2}\left(t^{-1} F_\Lambda - t \bar{F}_\Lambda\right) \\ \alpha = \sigma + 2\sqrt{r}\, e^{K/2}\left(t^{-1} W - t \bar{W}\right) \\ \qquad - 8ic \log t \end{cases}$$

where we defined

$$W = F_\Lambda \zeta^\Lambda - X^\Lambda \tilde{\zeta}_\Lambda$$

Key properties of Z

(i) Isometries of $\mathcal{M}$ lift to a holomorphic action on Z. [Galicki]

This implies that on $Z \to \mathcal{M}_H$ we have an action of Heis_{2l} given by

$$(\Sigma, \tilde{\alpha}) \mapsto \left(\Sigma + H, \tilde{\alpha} + 2K_{\xi} + \langle \Sigma, H \rangle + 2c(H)\right)$$

where $H \in \Gamma$ as before.

(ii) Linear deformations of a QK-manifold $\mathcal{M}$ are classified by the non-abelian group $H^1(Z, \mathcal{O}(2))$. [LeBrun].

Concretely this means that generating function of contact transformations may be deformed by sections

$$H^{[ij]}(\xi, \tilde{\xi}, \tilde{\alpha}) \in H^1(Z, \mathcal{O}(2)).$$

3. D-instantons, DT-invariants and wall-crossing

Away from the strict zero-coupling limit $g_s = e^{\phi} = r \to 0$, the metric g_{pert} receives non-perturbative corrections of the order e^{-1/g_s}.

These are due to D-instantons and take the schematic form.

$$g_D \sim \sum_{\gamma \in \Gamma} \lambda_D(\gamma)\, \bar{\Omega}(\gamma, z)\, e^{-2\pi \frac{|Z_\gamma|}{g_s} - 2\pi i \langle \gamma, c \rangle}$$

* $\lambda_D : \Gamma \to \{\pm 1\}$ (quadratic refinement)
* $\bar{\Omega} : \Gamma \to \mathbb{Q}$ (instanton measure)
* $Z_\gamma : \Gamma \to \mathbb{C}$ (central charge function)

By T-duality we expect that

$$\Omega(\gamma) \equiv \sum_{d \mid \gamma} \frac{1}{d^2}\, \mu(d)\, \bar{\Omega}(\gamma/d) \in \mathbb{Z}$$

is the BPS-index (second helicity supertrace) counting supersymmetric D-branes of charge $\gamma \in \Gamma$.

sector	Physics	Math 1.0	Math 2.0
IIA/X	D2-branes wrapping 3-cycles in X	sLags $\subset X$	semi-stable objects in $D^b\mathrm{Fuk}(X)$
$\mathrm{IIB}/\hat{X}$	D(-1)-D1-D3-D5 wrapping even cycles in $\hat{X}$	coherent sheaves on $\hat{X}$	semi-stable objects in $D^b\mathrm{Coh}(\hat{X})$

Conjecture: [GMN][KS][Joyce][...

BPS-invariant $\Omega(\gamma)$ = generalized DT-invariant counting semi-stable objects in $D^b\mathrm{Fuk}(X)$ or $D^b\mathrm{Coh}(\hat{X})$

3.1 Stability

The BPS-index $\Omega(\gamma; z)$ is locally constant as a function of $z \in B$ $(= \mathcal{M}_c(X)$ or $\mathcal{M}_K(\hat{X}))$ but jumps at walls of marginal stability where BPS states may decay.

Stability is measured by the central charge function $Z_\gamma(z)$.

$$\text{IIA}: \quad Z_\gamma(z) = e^{K/2} \int_\gamma \Omega^{3,0}$$

$$\text{IIB}: \quad Z_\gamma(z) = \int_{\hat{X}} e^{B+iJ}\, ch(\mathcal{E}) \sqrt{Td\hat{X}}$$

$(\mathcal{E} \in D^b Coh(\hat{X}))$

Decay can only happen when

$$|Z_\gamma| = |Z_{\gamma_1}| + |Z_{\gamma_2}|$$

i.e for $\arg Z_{\gamma_1} = \arg Z_{\gamma_2}$.

This defines real codimension 1 walls in B:

$$W(\gamma_1, \gamma_2) \equiv \left\{ z \in B \mid \arg Z_{\gamma_1}(z) = \arg Z_{\gamma_2}(z) \right\}$$

<u>Mathematical description of stability</u>

Let A be an abelian category with central charge function

$$Z : K(A) \to \mathbb{C}$$

where $K(A)$ is the Grothendieck group.

Let $\varphi(\gamma)$ be the argument (phase) of Z_γ.

<u>Def</u>: An object $F \in A$ with charge $\gamma \in K(A)$ is called semi-stable if for every subobject $F' \subset F$, with charge $\gamma' \in K(A)$, one has

$$\varphi(\gamma') \leq \varphi(\gamma)$$

The object is called stable if the inequality is strict.

One can extend the notion of stability in terms of subobjects to triangulated categories D (like $D^b \mathrm{Fuk}(X)$ or $D^b \mathrm{Coh}(\check{X})$) by considering abelian subcategories $A \subset D$ ("heart of a t-structure"). [Bridgeland]

The moduli space M_{ss} has $\dim < \infty$ (although singular).

$\Omega(\gamma)$ = (weighted) Euler characteristic of $M_{ss}(\gamma)$. [Behrend].

3.2 Wall-crossing formula

Kontsevich-Soibelman (KS) and Joyce-Song have written formulas for how $\Omega(\gamma; z)$ jumps, for arbitrary decay $\gamma \to M\gamma_1 + N\gamma_2$.

Here we consider the KS-approach since it has a clear geometric meaning in the hypermultiplet story.

Introduce the Lie algebra of infinitesimal symplectomorphisms of the complex torus $\Gamma \otimes_{\mathbb{Z}} \mathbb{C}^\times$ generated by $(e_\gamma)_{\gamma \in \Gamma}$ such that:

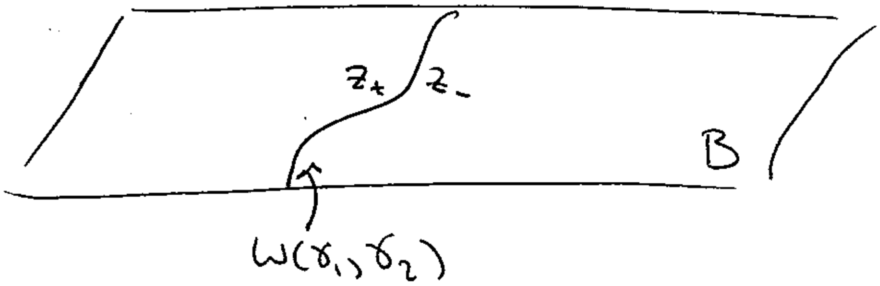
$$[e_\gamma, e_{\gamma'}] = (-1)^{\langle \gamma, \gamma' \rangle} \langle \gamma, \gamma' \rangle \, e_{\gamma+\gamma'}$$

Group element

$$U_\gamma = \exp \sum_{n=1}^{\infty} \frac{e_{n\gamma}}{n^2} = \exp \mathrm{Li}_2(e_\gamma)$$

KS-formula:

$$\prod_{\substack{\gamma = M\gamma_1 + N\gamma_2 \\ M \geq 0,\, N \geq 0}}^{\curvearrowleft} U_\gamma^{\Omega(\gamma;\, z_+)} = \prod_{\substack{\gamma = M\gamma_1 + N\gamma_2 \\ M \geq 0,\, N \geq 0}}^{\curvearrowright} U_\gamma^{\Omega(\gamma;\, z_-)}$$



This allows to determine

$$\Delta\Omega = \Omega(\gamma;\, z_+) - \Omega(\gamma;\, z_-)$$

for any $\gamma = M\gamma_1 + N\gamma_2$.

3.3. D-instantons in twistor space.

GMN interpreted the KS-formula as ensuring the smoothness of the hyperkähler metric on the Coulomb branch of SW-theory on $\mathbb{R}^3 \times S^1$, across walls of marginal stability.

Our aim is to generalize this construction to the QK-case.

D-instanton corrections modify the contact structure on Z by replacing the patch $\mathcal{U}_0 \subset Z$ by an infinite set of angular sectors, separated by "BPS-rays"

$$\ell_\gamma = \left\{ t \in \mathbb{P}^1 \,\middle|\, t^{-1} Z_\gamma(z) \in i\mathbb{R}_- \right\}$$

Across such a ray the Darboux coordinates $(\xi^\Lambda, \tilde{\xi}_\Lambda, \alpha)$ jump by a contact transf.

To write formula for the ~~xxx~~ discontinuity we introduce holomorphic Fourier modes

$$\mathcal{X}_\gamma \equiv \mathbb{E}\left(-\langle \gamma, \Sigma \rangle\right) = e^{-2\pi i (q_\Lambda \xi^\Lambda - p^\Lambda \tilde{\xi}_\Lambda)}$$

$$\left(\mathbb{E}(x) = e^{2\pi i x}\right)$$

we postulate that across ℓ_γ the jump is given by the KS-symplectomorphism:

$$U_\gamma^{\Omega(\gamma)} : \mathcal{X}_{\gamma'} \longmapsto \mathcal{X}_{\gamma'} \left(1 - \sigma_D(\gamma) \mathcal{X}_\gamma\right)^{\Omega(\gamma)\langle \gamma, \gamma' \rangle}$$

Requiring that the contact 1-form is invariant determines the discontinuity in $\hat{\alpha}$.

Define:

$$\tilde{\Upsilon} = e^{i\pi\hat{\alpha}}$$

Then the complete contact transformation across ℓ_γ is

$$V_\gamma : (\mathcal{X}_{\gamma'}, \tilde{\Upsilon}) \mapsto \left(U_\gamma^{\Omega(\gamma)} \mathcal{X}_{\gamma'}, \tilde{\Upsilon} e^{-\frac{\Omega(\gamma)}{2\pi i} L_{\sigma(\gamma)}(\sigma(\gamma)\mathcal{X}_\gamma)}\right)$$

where $L_\epsilon(x)$ is

$$L_\epsilon(x) \equiv \mathrm{Li}_2(x) + \frac{1}{2}\log(\epsilon^{-1}x)\log(1-x)$$

$$\epsilon = 1 \Rightarrow L_1(x) = L(x)$$ Rogers dilog

Riemann-Hilbert problem

Require that $\mathcal{X}_\gamma$ & $\tilde{\Upsilon}$ reduces to the semi-flat Darboux coordinate in the limit $t \to 0, \infty$:

$$\mathcal{X}^{sf}_\gamma(t) = \mathbb{E}\left(-\langle\gamma, c\rangle - 2\sqrt{r}\, e^{-K/2}\left(t^{-1} Z_\gamma - t \bar{Z}_\gamma\right)\right)$$

$$\hat{\Upsilon}^{sf}(t) = \mathbb{E}\left(\frac{\sigma}{2} + \sqrt{3}\, e^{K/2}\left(t^{-1} W - t \bar{W}\right)\right)$$

The gluing condition across patch then becomes a system of integral equations

$$\mathcal{X}_\gamma(t) = \mathcal{X}^{sf}_\gamma(t) \exp\left[\frac{1}{4\pi i} \sum_{\gamma'} \Omega(\gamma') \langle\gamma, \gamma'\rangle \right.$$

$$\left. \times \int_{\ell_{\gamma'}} \frac{dt'}{t'} \frac{t+t'}{t-t'} \log\left[1 - \lambda_D(\gamma') \mathcal{X}_{\gamma'}(t')\right]\right]$$

One can solve this iteratively by inserting $\mathcal{X}_\gamma = \mathcal{X}_\gamma^{sf}$ on the rhs. This generates a multi-instanton expansion

$$\sum_{\gamma'} \Omega(\gamma') + \sum_{\gamma,\gamma'} \Omega(\gamma)\Omega(\gamma') + \dots$$

which corrects the semi-flat coordinates and thereby the metric on $\mathcal{M}_H$.

The D-instanton corrections generate corrections to $\tilde{\alpha}$, represented by:

$$\tilde{\Upsilon}(t) = \exp\Big[i\pi\Big(\sigma + t^{-1}\mathcal{W} - t\bar{\mathcal{W}} + \frac{i\chi(X)}{24\pi}\log t\Big) + \frac{i}{8\pi^3}\sum_\gamma \Omega(\gamma) \int_{\ell_\gamma} \frac{dt'}{t'}\frac{t+t'}{t-t'} L_{\lambda_D(\gamma)}\big(\lambda_D(\gamma)\mathcal{X}_\gamma\big)\Big)\Big]$$

Remarks

* The gluing conditions hold only in an open set, away from walls of marginal stability.

  The consistency of the construction globally require some analogue of the KS formulas for contact transformations.

* In contrast to the hyperkähler situation, the twistor space is a non-trivial fibration

$$\pi : Z_H \to \mathbb{P}^1$$

  and the fibers $\pi^{-1}(t)$ are not complex manifold.

  To circumvent these problem we make use of a duality between QK and HK geometry.

3.4 Wall-crossing of D-instanton and the QK/HK correspondence

For QK-manifolds $\mathcal{M}$ with a quaternionic circle action (preserving $\vec{\omega} \wedge \vec{\omega}$) there exists a dual description in terms of a HK manifold $\mathcal{M}'$, of the same real dimension, ~~with~~ with a hyperholomorphic line bundle

$$\mathcal{L} \to \mathcal{M}'!$$

[Hyperholomorphic: $c_1(\mathcal{L})$ of type (1,1) with respect to ~~all~~ the whole $\mathbb{P}^1$ worth of complex structures.]

[Haydys] [Alexandrov, D.P., Pioline] [Hitchin]

Details on QK/HK

Let M be a $4n$-dimensional QK-manifold with a quaternionic circle action, i.e that there exists a Killing vector Y such that

$$\mathcal{L}_Y (\vec{\omega} \wedge \vec{\omega}) = 0$$

Theorem:

Given M one can construct a dual HK-manifold M', of $\dim_{\mathbb{R}} = 4n$, with a Killing vector Y' that fixes J_3' and rotates (J_1', J_2').

Conversely, given a HK-manifold M' with a circle action Y' as above, and equipped with a hyperholomorphic connection λ with curvature

$$F_\lambda = 2i\partial\bar{\partial}\mu - \omega_3'$$

where μ is the moment map of Y' : $i_{Y'}(\omega_3') = d\mu$, ∂ is the Dolbeault derivation in J_3;

λ is a connection on a hyperholomorphic line bundle $\mathcal{L}$ with

$$c_1(\mathcal{L}) = \left[\frac{F_2}{2\pi}\right] \in H^2(M', \mathbb{Z}).$$

At the level of twistor spaces this means that we can trade the construction of $Z \to M$ for the construction of the pair $(Z', \mathcal{L}_{Z'})$ where

$$\mathcal{L}_{Z'} \to Z'$$

is a holomorphic $\mathbb{C}^\times$-bundle over $Z' \to M'$.

Wall-crossing of D-instantons

Since D-instantons preserve the axion σ, the moduli space $\mathcal{M}_H$ retains one quaternionic isometry ∂_σ. This lifts to a holomorphic action on $\mathcal{Z}$ generated by ∂_α.

$\Rightarrow$ QK/HK valid?

Effectively, the dual description trivializes the bundle $\pi: \mathcal{Z} \to \mathbb{P}^1$ such that $(\xi^\Lambda, \tilde{\xi}_\Lambda)$ now parametrize a complex torus

$$\pi^{-1}(t) = \Gamma \otimes_{\mathbb{Z}} \mathbb{C}^*$$

The remaining coordinate $\tilde{\alpha}$ parametrizes the fiber of $\mathcal{L}_{\mathcal{Z}'} \to \mathcal{Z}'$.

The contact one-form

$$\chi = d\alpha + \xi^{\wedge} d\hat{\xi}_{\wedge}$$

becomes a holomorphic connection ~~on~~ on $\mathcal{L}_{Z'}$.

We get a holomorphic section on $\mathcal{L}_{Z'}$

$$\Upsilon(t) = e^{-2\pi i \alpha(t)}$$

which is non-zero on each twistor line, i.e. the fibers $\pi^{-1}(t)$.
Hence, by the Atiyah-Ward twistor correspondence this descends to a hyperholomorphic line bundle $\mathcal{L} \to M'$ with connection

$$\lambda = \frac{1}{4}\left(\bar{\partial}^{(t)}\alpha + \partial^{(t)}\bar{\alpha}\right)$$

On the dual side a contact transformation V_γ reduces to a symplectomorphism on Z' and a "gauge transf" along the fibers of $\mathcal{L}_{Z'} \to Z'$.

We should then have a lift of the KS-formula to the total space $\mathcal{L}_{Z'}$:

$$\prod_{\substack{\gamma = M\gamma_1 + N\gamma_2 \\ M \geq 0,\, N \geq 0}}^{\curvearrowleft} V_\gamma(z_+) = \prod_{\substack{\gamma = M\gamma_1 + N\gamma_2 \\ M \geq 0,\, N \geq 0}}^{\curvearrowright} V_\gamma(z_-)$$

This formula could fail at most by a shift

$$\hat{\alpha} \to \hat{\alpha} + \Delta\hat{\alpha}$$

along the fibers.

Global existence requires

$$\Delta\hat{\alpha} \in 2\mathbb{Z}$$

To check this, we rewrite the wall-crossing formula as an operator identity

$$\prod_s V_{\gamma_s}^{\epsilon_s} = 1$$

($\epsilon_s \in \pm 1$).

When acting on the section $\tilde{\Upsilon} = e^{i\pi\hat{\alpha}}$ this yields a total effect:

$$\prod_s V_{\gamma_s}^{\epsilon_s} \cdot \tilde{\Upsilon} = e^{-\frac{1}{2\pi i}\sum_s \epsilon_s \Omega(\gamma_s) L_{\lambda_D(\gamma_s)}\left(\lambda_D(\gamma_s)\chi_{\gamma_s}\right)} \times \tilde{\Upsilon}$$

i.e.

$$\Delta\hat{\alpha} = \frac{1}{2\pi^2}\sum_s \epsilon_s \Omega(\gamma_s) L_{\lambda_D(\gamma_s)}\left(\lambda_D(\gamma_s)\chi_{\gamma_s}(u)\right)$$

where

$$\mathcal{X}_{\gamma_s}(s) = U_{\gamma_{s-1}} \circ U_{\gamma_{s-2}} \circ \cdots \circ U_{\gamma_1} \cdot \mathcal{X}_{\gamma_s}$$

Hence, smoothness of the contact structure on $\mathcal{Z}$ reduces to a non-trivial identity for Rogers dilog:

$$\sum_s \epsilon_s \Omega(\gamma_s) L_{\lambda_D(\gamma_s)}\left(\lambda_D(\gamma_s) \mathcal{X}_{\gamma_s}(s)\right)$$

$$= 0 \mod 4\pi^2.$$

This formula can be shown to follow from the quasi-classical limit $q^{1/2} \to -1$ of the motivic KS-formula.

[Alexandrov, D.P., Pioline]

4. NS5-instantons I:

The partition function

In addition to D-instanton effects the metric on $\mathcal{M}_H$ receives corrections of order e^{-1/g_s^2} coming from Euclidean NS5-branes wrapping X.

For k NS5-branes the schematic form of such a correction is

$$g_{NS5}^{(k)} \sim e^{-2\pi|k| V(X)/g_s^2 - \pi i k \sigma} Z_k$$

where $V(X) = \mathrm{Vol}(X)$ and Z_k is the partition function of the degrees of freedom localized on the NS5-brane.

[These effects are of interest for a variety of reasons:

- should resolve the singularity of the perturbative metric.
- improve the large order behavior of the D-instanton series caused by the exponential growth $\Omega(\gamma) \sim e^{c\|\gamma\|^2}$ for $c \to \infty$.
- Restore the S-duality symmetry which is broken by D5-instantons.]

The mathematical interpretation of these effects is far from clear.

Is there a description of NS5-branes via some extension of homological mirror symmetry?

We will give indications that NS5-branes are related to DT theory via the quantization of integrable systems & S-duality.

Preliminaires

The main complication is that the worldvolume theory is not a standard gauge theory but is a rather mysterious 6-dimensional superconformal field theory ("(2,0)-theory").
[For a discussion see Greg Moore's recent Klein lectures in Bonn.]

In particular, the worldvolume ~~the~~ supports a chiral 2-form potential B with imaginary self-dual field strength $\mathcal{H}=dB$:

$$*_X \mathcal{H} = i\,\mathcal{H}$$

Witten: The partition function of the fivebrane is not ~~gauge~~ gauge invariant, and hence corresponds to a section of a line bundle over the space of background fields $(\in H^3(X,\mathbb{R}))$.

4.1 The NS5-partition function

In general the partition function depends on the moduli:

$$\mathcal{Z}_k = \mathcal{Z}_k(\underset{\text{dilaton}}{\phi}, \underset{\mathcal{M}_c(X)}{z^a}, \underset{C \in H^3(X,\mathbb{R})/H^3(X,\mathbb{Z})}{C})$$

The dependence on C puts strong constraints due to Heisenberg symmetry of $\mathcal{M}_H$:

$$\mathrm{Heis}_{\mathbb{Z}} : \mathcal{M}_H \to \mathcal{M}_H$$

$$(C, \sigma) \longmapsto \left(C + H, \sigma + 2k + \langle C, H \rangle + 2c(H)\right)$$

with $H \in H^3(X, \mathbb{Z})$, $k \in \mathbb{Z}$.

$$\lambda(H) = (-1)^{2c(H)} = e^{-i\pi m_\Lambda n^\Lambda + 2\pi i (m_\Lambda \theta^\Lambda - n^\Lambda \phi_\Lambda)}$$

characteristics: $\Theta = (\theta^\Lambda, \phi_\Lambda)$

Hence, Heisenberg invariance requires that Z_k is periodic:

$$Z_k(C+H) = [\lambda(H)]^k \mathbb{E}\left(\tfrac{k}{2}\langle C, H\rangle\right) Z_k(C)$$

This implies that Z_k is a section of the line bundle $(\mathcal{L}_\Theta)^{\otimes k}$ where

$$\mathcal{L}_\Theta \to H^3(X,\mathbb{R}) / H^3(X,\mathbb{Z}) = \mathcal{T}$$

is the "theta line bundle" with

$$c_1(\mathcal{L}_\Theta) = \omega_\mathcal{T} = \underbrace{d\xi_\wedge \wedge d\xi^\wedge}_{\text{"principal polarization"}}$$

The index of $\bar\partial$ on $\mathcal{T}$ is

$$\text{index}(\bar\partial) = \sum_{i=0}^{b_3(X)} (-1)^i \dim H^i(\mathcal{T}, \mathcal{L}_\Theta)$$

$$= \underline{1} \qquad \text{(Witten)}$$

So $\dim H^0(\mathcal{T}, \mathcal{L}_\theta) = 1$ and hence $\mathcal{L}_\theta$ admits a unique holomorphic section which is $Z_{k=1}$.

More generally, $(\mathcal{L}_\theta)^{\otimes k}$ admits $|k|^{b_3(X)/2}$ holomorphic sections, correspond to the Siegel theta series

$$\Theta_{k,\mu}(C) = \sum_{n \in \Gamma_m + \mu + \theta} \mathbb{E}\left(\frac{k}{2}(y^\Lambda - n^\Lambda)\bar{N}_{\Lambda\Sigma}(y^\Sigma - n^\Sigma)\right)$$

$$\times\, \mathbb{E}\left(k(\zeta_\Lambda - \phi_\Lambda)n^\Lambda + \frac{k}{2}(\theta^\Lambda \phi_\Lambda - y^\Lambda \zeta_\Lambda)\right)$$

labelled by $|k|^{b_3(X)/2}$ vectors $\mu \in \Gamma_m / (|k|\Gamma_m)$ where $\Gamma_m \subset H^3(X, \mathbb{Z})$ is a Lagrangian sublattice.

The sum over n^Λ corresponds to a sum over D2-instantons bound to the NS5-brane.

[The condition of holomorphy is equivalent to

$$\bar{D}^\Lambda \Theta_{k,\mu} = 0$$

where $\bar{D}^\Lambda$ is the anti-holomorphic covariant derivative in the ~~wey~~ Weil complex structure:

$$\bar{D}^\Lambda = \frac{\partial}{\partial \bar{w}_\Lambda} - \frac{\pi k}{2} \operatorname{Im} N^{\Lambda\Sigma} w_\Sigma$$

where $w_\Lambda = \xi_\Lambda - \bar{N}_{\Lambda\Sigma} \zeta^\Sigma$]

~~[illegible]~~

The vector-valued Siegel theta series $\Theta_{k,\mu}$ is only the Gaussian, weak-coupling approximation of the full NS5-partition function.

In general Z_k is non-Gaussian and the periodicity constraint requires:

$$Z_k(C,N) = \sum_{\mu \in \Gamma_m/(|k|\Gamma_m)} \sum_{n \in \Gamma_m + \mu + \theta}$$

$$\underbrace{\Psi_{k,\mu}(\zeta^\wedge - n^\wedge; N)}_{\text{NS5-wave function}} \, \mathbb{E}\left(k(\zeta_\wedge - \phi_\wedge)n^\wedge + \frac{k}{2}\left(\theta^\wedge \phi_\wedge - \zeta^\wedge \hat{\zeta}_\wedge\right)\right)$$

Metric dependence

The non-Gaussian Z_k does not extend to a holomorphic section of $\mathcal{L}_\Theta$, but should rather correspond to a section of the unit circle bundle $\mathcal{C}_\Theta \subset \mathcal{L}_\Theta$.

Z_k is also a function on $\mathcal{M}_C$. This dependence is constrained by anomaly cancellation.

Consider again the coupling

$$e^{-i\pi k\sigma} Z_k(C, N)$$

Recall that $e^{i\pi\sigma}$ transform like a section of a circle bundle $\mathcal{C}_\sigma \to Jw(X)$

$[\, A(C) = \zeta_\wedge d\zeta^\wedge - \zeta^\wedge d\zeta_\wedge + 8C A_K \,]$

So for the coupling to be well-defined Z_k must be a section of $(\mathcal{L}_\sigma)^{\otimes k}$ with

$$c_1(\mathcal{L}_\sigma) = \underbrace{c_1(\mathcal{L}_\Theta)}_{\omega_\tau} + \frac{\chi(X)}{24} \underbrace{c_1(\mathcal{L}_X)}_{\omega_C}$$

Large gauge transf. Monodromies

Under a monodromy transformation $M \in Sp(b_3(X); \mathbb{Z})$ in $\mathcal{M}_C(X)$ one has

$$\Omega_3^{3,0} \mapsto e^{f_M} \Omega^{3,0}$$

$$\sigma \mapsto \sigma + \frac{\chi(X)}{24\pi} \operatorname{Im} f_M + 2K(M)$$

[This follows from invariance of $A(c)$ under

$$K \mapsto K - f_M - \bar{f}_M$$

$$A_K \mapsto A_K + d \operatorname{Im} f_M$$

]

where $K = \log \Psi$ with

$$\Psi : Sp(b_3(X)) \to U(1)$$

is a unitary character of the monodromy group.

So under monodromies the Z_k transform as

$$Z_k \mapsto e^{ik\frac{\chi(X)}{24}\,\mathrm{Im}\, f_M}\, \Psi(M)\, Z_k$$

These are formally the transf. properties of a section of

$$\mathcal{L}_X^{\chi(X)/24}$$

However, unless $\frac{\chi}{24} \in \mathbb{Z}$ then is no canonical definition of this bundle.

In fact

$$\frac{\chi(X)}{12}\,\omega_c = c_1(D_X) \in H^2(M_c, \mathbb{Z})$$

is the first chern class of the so called BCOV determinant line bundle D_X.

So the ambiguity in choosing the character $K(M)$ corresponds to a choice of square root $\sqrt{D_X}$.

This seem to be related to the choice of orientation data in the KS theory of DT-invariants.

<u>Remark</u>: First hint of a connection with topological strings

$$\psi_{BCOV} \in \Gamma(\mathcal{L}_X^{\chi/24 - 1})$$

5: NS5 - Instantons II: S-duality and topological strings.

So far we have mainly worked in the type II sector. Now we move to the type IIB side where we expect

$$\hat{\mathcal{M}}_H(\hat{X}) \circlearrowleft SL(2,\mathbb{Z})$$

This action mixes D5- and NS5-effects.

Idea: Start from the D-instanton corrected twistor space and enforce S-duality to generate the effects of NS5-branes

---

A general prediction of S-duality is also that the partition function of a single NS5-brane on X should be governed by the ordinary DT-invariants with $p^0 = \text{rank} = 1$ (i.e. ideal sheaves). Therefore, by the DT/GW correspondence, the NS5-partition function should be related to the topological string.

We will see an explicit realization of these expectations.

5.1 Brief excursion into top. strings

* B-model: knows about complex structure of X
  $\Rightarrow$ relevant for type IIA hypers!

* A-model: knows about Kähler structure of $\hat{X}$.
  $\Rightarrow$ relevant for type IIB hypers!

B-~~A~~model perspective (á la BCOV)

The basic object is the generating function of genus g amplitudes:

$\longrightarrow$

$$W_g(x; z, \bar{z}) = \left\langle \exp \sum_a x^a \int_{\Sigma_g} \mathcal{O}_a \right\rangle_{z,\bar{z}}$$

Here, $z^a \in \mathcal{M}_c(X)$ and x^a are formal couplings for the vertex operators $\mathcal{O}_a$.

Introduce the topological string partition function:

$$\Psi_{top}(x, \lambda; z, \bar{z}) = \lambda^{\frac{\chi(X)}{24} - 1} \exp \sum_g \lambda^{2g-2} W_g$$

where λ is the coupling constant. With respect to monodromies we have:

$$\Psi_{top} \in \Gamma\left(\mathcal{L}_x^{\frac{\chi}{24} - 1} \to \mathcal{M}_c(X)\right)$$

The dependence of Ψ_{top} on $z^a \in \mathcal{M}_c(X)$ is further controlled by the hol. anomaly eq.'s:

$$\frac{\partial}{\partial \bar{z}^{\bar{a}}} \Psi_{top} = [\quad] \Psi_{top}$$

As realized by Verlinde, it is illuminating to rescale:

$$\Psi_G(z, \bar{z}; \lambda^{-1}, X) = e^{f_1(z)} \Psi_{top}(z, \bar{z}; \lambda, X)$$

where G is for Griffiths and $f_1(z)$ is the hol. part of the one-loop vacuum amplitude.

$$F_1 = \log \left[e^{f_1(z) + \bar{f}_1(\bar{z})} M(z, \bar{z})^{-1/2} \right]$$

$$M(z,\bar{z}) = |G| \, e^{-2\left(\frac{b_3(X)}{4} - \frac{\chi(X)}{24} + 1 \right) K}$$

$$\left(G = \det(\partial_a \partial_{\bar{b}} K) \right)$$

[This implies that

$$e^{f_1(z)} \in \Gamma\left(\mathcal{L}_x^{1-\frac{\chi}{24}+\frac{b_3}{4}} \otimes K_c^{1/2}\right)$$

where K_c is the canonical bundle of $\mathcal{M}_c$, locally trivialized by

$$dz^1 \wedge \dots \wedge dz^{h_{2,1}}$$

So under $z \to z'(z)$ &

$(\lambda^{-1}, X^a) \to e^{-f}(\lambda^{-1}, X^a)$ we have:

$$f_1 \mapsto f_1 + \left(\frac{b_3}{4} - \frac{\chi(X)}{24} + 1\right) f + \frac{1}{2}\log\left|\frac{\partial z}{\partial z'}\right|$$

This implies that

$$\Psi_a \in \Gamma\left(\mathcal{L}_x^{\frac{b_3}{4}} \otimes K_c^{1/2}\right)$$

$$\Psi_a \mapsto \sqrt{\left|\frac{\partial z}{\partial z'}\right|}\, e^{\frac{b_3(X)}{4} f}\, \Psi_a$$

]

For Ψ_G the hol. anomaly equations take a nicer form

$$\left[\frac{\partial}{\partial \bar{z}^{\bar{a}}} - \frac{1}{2} e^{2K} \bar{C}_{\bar{a}\bar{b}\bar{c}} G^{b\bar{b}} G^{c\bar{c}} \frac{\partial^2}{\partial x^b \partial x^c} - G_{\bar{a}b} x^b \frac{\partial}{\partial \lambda^{-1}}\right] \Psi_G = 0$$

$$\left[\nabla_a - \Gamma_{ab}^c x^b \frac{\partial}{\partial x^c} - \frac{1}{2} \partial_a \log|G| - \lambda^{-1} \frac{\partial}{\partial x^a} + \frac{1}{2} C_{abc} x^a x^b x^c\right] \Psi_G = 0$$

where

$$\nabla_a = \frac{\partial}{\partial z^a} + \partial_a K \left(x^b \frac{\partial}{\partial x^b} + \lambda^{-1} \frac{\partial}{\partial \lambda^{-1}} + \frac{b_3}{4} \right)$$

Quantization of $H^3(X, \mathbb{R})$

$H^3(X, \mathbb{R})$ has a natural symplectic structure

$$\omega(\alpha, \beta) = \int_X \alpha \wedge \beta$$

The hol. anomaly equations for Ψ_a can be interpreted in terms of the quantization of $H^3(X, \mathbb{R})$ in the Griffiths complex structure:

$$H^3(X, \mathbb{R}) = H^{1,2}(X) \oplus H^{0,3}(X)$$

Roughly, the space of solutions to the hol. anomaly equation is identified with the space of wave functions ~~ea~~ in $L^2(H^3(X, \mathbb{R}))$ in the Griffiths polarization.

There is a state $|\Psi_{top}\rangle \in \mathcal{H}$ such that Ψ_a is the overlap

$$\Psi_a(z,\bar{z};\lambda^{-1},x) = {}_{(z,\bar{z})}\langle \lambda^{-1};x \,|\, \Psi_{top}\rangle$$

where ${}_{(z,\bar{z})}\langle \lambda^{-1};x|$ is a basis of coherent states diagonalizing the action of the operators $\hat{\lambda}^{-1}, \hat{x}^a$.

The holomorphic anomaly equation reflect the unitary transf. undergone by coherent states when changin complex structure.

[Alternatively, one can diagonalize $\bar{\lambda}^{-1}$ & x^a which leads to the Weil polarization. The associated wave functions Ψ_W and Ψ_a are related by Fourier transform.]

Real polarization

If we fix a symplectic basis $(\alpha_\Lambda, \beta^\Lambda)$ of $H^3(X,\mathbb{R})$ we can perform a "background independent" quantization, i.e. independent of the complex structure (at least locally).

Expand

$$C = \xi^\Lambda \alpha_\Lambda - \tilde{\xi}_\Lambda \beta^\Lambda \in H^3(X,\mathbb{R})$$

and express the state $|\Psi_{top}\rangle \in \mathcal{H}$ on a basis of states diagonalizing either ξ^Λ or $\tilde{\xi}_\Lambda$:

This yields the real polarized wave function

$$\Psi_{\mathbb{R}}(\xi^\Lambda) = \langle \xi^\Lambda | \Psi_{top} \rangle$$

This is locally independent of $z^a \in \mathcal{M}_c(X)$ but transform in the metaplectic reps ρ_m under monodromies, i.e change of symplectic basis:

$$\rho_m \begin{pmatrix} A^{-T} & 0 \\ 0 & A \end{pmatrix} \cdot \Psi_{\mathbb{R}}(\xi^\Lambda) = \Psi_{\mathbb{R}}(A^T \xi^\Lambda)$$

$$\rho_m \begin{pmatrix} 1 & 0 \\ B & 1 \end{pmatrix} \cdot \Psi_{\mathbb{R}}(\xi^\Lambda) = \mathbb{E}\left(\tfrac{1}{2} B_{\Lambda\Sigma} \xi^\Lambda \xi^\Sigma\right) \Psi_{\mathbb{R}}(\xi^\Lambda)$$

$$\rho_m \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \cdot \Psi_{\mathbb{R}}(\xi^\Lambda) = \int \mathbb{E}(-\xi^\Lambda \xi_\Lambda) \Psi_{\mathbb{R}}(\xi_\Lambda) d\xi_\Lambda$$

Holomorphic limit

We now want to relate the real polarized wave function $\Psi_{\mathbb{R}}(y^\Lambda)$ to the holomorphic limit

$$\bar{z}^a \to \infty \quad , \quad z^a \text{ fixed}$$

of the BCOV wave function $\Psi_{top}(z,\bar{z},\lambda,x)$

In this limit the period matrix

$$\bar{\tau}_{\Lambda\Sigma} \to \infty .$$

Use the relation

$$X_\Lambda = \hat{\zeta}_\Lambda - \bar{\tau}_{\Lambda\Sigma} y^\Sigma$$

$$= \frac{2i}{\sqrt{2\pi}} \mathrm{Im}\, \tau_{\Lambda\Sigma} \left(\lambda^{-1} X^\Sigma + x^a D_a X^\Sigma\right)$$

$$\to - \bar{\tau}_{\Lambda\Sigma} y^\Sigma$$

At the locus $x^a = 0$, λ fixed this limit implies that $\zeta^\wedge$ becomes complexified and equal to $X^\wedge / \lambda\sqrt{2\pi}$. We denote this complexified variable by $\xi^\wedge$.

We now define the holomorphic topological partition function

$$e^{F_{hol}(z,\lambda)} \equiv e^{f_1(z)} \lim_{\bar{z}\to\infty} \left[\lambda^{1-\frac{\chi}{24}} \psi_{top}(z,\bar{z};\lambda) \right]$$

One has the relation

$$\boxed{e^{F_{hol}(z,\lambda)} = (\xi^0)^{\frac{\chi}{24}-1} \psi_{\mathbb{R}}(\xi^\wedge)}$$

with $\lambda \sim 1/\xi^0$, $z^a = \dfrac{\xi^a}{\xi^0}$.

This relation will play a key role in connection with NS5-branes.

Mirror version: A-model

B-model on X equivalent to A-model on $\hat{X}$ after replacing:

$$\chi(X) \to -\chi(\hat{X})$$

The A-model encodes the complexified Kähler deformation of $\hat{X}$. The A-model holomorphic wave function $e^{F_{hol}(z,\lambda)}$ therefore depends on

$$z^i = \int_{\gamma_i} B + iJ \in \mathcal{M}_K(\hat{X})$$

This is related to the GW-partition function by:

$$e^{F_{hol}(z,\lambda) - F_{pol}(z,\lambda)} = Z_{GW}$$

where the "polar part" is

$$F_{pol} = -\frac{(2\pi i)^3}{\lambda^2}\left(\frac{1}{6}K_{abc}z^a z^b z^c - \frac{1}{2}A_{\Lambda\Sigma}z^\Lambda z^\Sigma\right)$$
$$-\frac{2\pi i}{24}c_{2,a}z^a.$$

We have also

$$Z_{GW} = Z^0_{GW} Z'_{GW}$$

where the degree zero part is

$$Z^0_{GW} = \underbrace{M(e^{-\lambda})}_{\text{MacMahon function}}{}^{\chi(X)/2}$$

while Z'_{GW} is related to the generating function of rank 1 DT-inv:

$$Z'_{GW} = \left[M(e^{-\lambda})\right]^{-\chi(X)} Z_{DT}$$

with

$$Z_{DT} = \sum_{Q_a, J} (-1)^{2J} N_{DT}(Q_a, 2J)$$

$$e^{-2\lambda J + 2\pi i Q_a z^a}$$

This counts bound states of one D6-brane with $2J$ D0-branes and Q_a D2-branes on $\gamma^a \in H_2(\hat{X}, \mathbb{Z})$.

Combining we find

$$e^{F_{hol}(z,\lambda)} = \left[M(e^{-\lambda}) \right]^{-\chi(\hat{X})/2} e^{F_{pol}(z,\lambda)}$$

$$\times Z_{DT}(z, \lambda)$$

Using the relation between F_{hol} and $\Psi_{\mathbb{R}}$ we then find

$$\Psi_{\mathbb{R}}(\xi^\Lambda) = (\xi^0)^{1+\frac{\chi(\hat{X})}{24}} \left[M\left(e^{+2\pi i/\xi^0}\right) \right]^{\chi(\hat{X})/2} e^{F_{pol}(\xi^\Lambda)} Z_{DT}(\xi^\Lambda).$$

5.2 S-duality in twistor space ~~...~~

Across a BPS-ray ~~...~~ ℓ_γ the Darboux coordinates ~~...~~ are discontinuous, with contact transformation generated by:

$$H_\gamma(\Sigma) = \frac{\overline{\Omega}(\gamma)}{(2\pi)^2} \lambda_D(\gamma)\, \mathbb{E}(\langle \gamma, \Sigma \rangle)$$

This is like an infinitesimal ~~version~~ version of the KS-symplectomorphism $U_\gamma^{\Omega(\gamma)}$.

Here $\gamma \in H^{even}(\hat{X}, \mathbb{Z}) = \hat{\Gamma}$

$$= (p^0, p^a, q_a, q_0)$$
$$D5 \quad D3 \quad D1 \quad D(-1)$$

$\overline{\Omega}(\gamma) : \hat{\Gamma} \to \mathbb{Q}$ rational BPS-invariants

Interlude: charge quantization in IIB

Bound states of D(-1)-D1-D3-D5 instantons are described by a coherent sheaf $\mathcal{E}$ on $\hat{X}$ of rank p^0 and Mukai vector.

$$\gamma' : K^0(\hat{X}) \to H^{\text{even}}(\hat{X}, \mathbb{Q})$$

with $\gamma'(\mathcal{E}) = ch(\mathcal{E})\sqrt{Td\,\hat{X}}$

$$= p^0 + p^a \omega_a - q'_a \omega^a + q'_0 \omega_{\hat{X}}$$

$$\left[\begin{array}{l} ch = rk + c_1 + \left(\frac{1}{2}c_1^2 - c_2\right) + \cdots \\ Td = 1 + \frac{1}{2}c_1 + \frac{1}{12}\left(c_1^2 + c_2\right) + \cdots \end{array}\right]$$

[ω_a basis of 2-form in $H^2(\hat{X}, \mathbb{R})$
ω^a basis 4-form $H^4(\hat{X}, \mathbb{R})$
$\omega_{\hat{X}}$ volume form on $\hat{X}$]

We have the charges

$$p^a = \int_{\gamma^a} c_1(\mathcal{E}) \qquad \text{D3-brane}$$

$$q_a' = -\int_{\gamma_a} ch_2(\mathcal{E}) + \frac{p^0}{24}\, c_{2,a} \qquad \text{D1-branes}$$

$$q_0' = \int_{\hat{X}} \left(ch_3(\mathcal{E}) + \frac{1}{24}\, c_1(\mathcal{E})\, c_2(\hat{X}) \right) \quad \text{D(-1)}$$

The primes indicate that these charges are not integral.

To establish the mirror map to

$$\gamma \in H^3(X, \mathbb{Z})$$

one should match the central charges

$$e^{-K/2} \int_\gamma \Omega^{3,0} = \int_{\hat{X}} e^{B+iJ}\, ch(\mathcal{E}) \sqrt{\mathrm{Td}\,\hat{X}}$$

This leads to the conclusion that the integer charges which are mirror dual to $\gamma = (p^\Lambda, q_\Lambda) \in H^3(X, \mathbb{Z})$ are related to γ' via a symplectic transf:

$$q_\Lambda = q'_\Lambda + A_{\Lambda\Sigma} p^\Sigma$$

where $A_{\Lambda\Sigma}$ is a constant, real symmetric matrix.

The q_Λ so obtained are integers provided $A_{\Lambda\Sigma}$ satisfies certain "quantization conditions". [Alexandrov, P.P. Piolé]

Note that if we define

$$\tilde{\xi}'_\Lambda \equiv \tilde{\xi}_\Lambda - A_{\Lambda\Sigma}\zeta^\Sigma$$

then $\langle \gamma', C' \rangle = \langle \gamma, C \rangle$.

5.2. S-duality in twistor space

Across a BPS-ray ℓ_γ the Darboux coordinates $(\xi^\Lambda, \tilde{\xi}_\Lambda)$ ~~[illegible]~~ discontinuous with the associated contact transf. generated by:

$$H_\gamma(\Sigma) = \frac{\Omega(\gamma)}{(2\pi)^2}\, \mathrm{Li}_2\left(\lambda_D(\gamma)\, \mathcal{X}_\gamma\right)$$

[This is the infinitesimal version of the KS transf. $U_\gamma^{\Omega(\gamma)}$.]

Since the BPS-rays ℓ_γ & $\ell_{n\gamma}$ are identical one can dispose of the dilog and instead work with the transition function:

$$\bar{H}_\gamma(\Sigma) = \frac{\bar{\Omega}(\gamma)}{(2\pi)^2}\, \lambda_D(\gamma)\, \mathbb{E}\left(p^\Lambda \tilde{\xi}_\Lambda - q_\Lambda \xi^\Lambda\right)$$

S-duality in twistor space

The action of S-duality

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z})$$

on Darboux coordinates is

$$\xi^0 \mapsto \frac{a\xi^0 + b}{c\xi^0 + d}, \qquad \xi^a \mapsto \frac{\xi^a}{c\xi^0 + d}$$

$$\tilde{\xi}'_a \mapsto \tilde{\xi}'_a + \frac{c}{2(c\xi^0 + d)} K_{abc}\xi^b\xi^c - c_{2,a}\,\varepsilon(g)$$

$$\left[\begin{array}{l} c_{2,a} = \int_{\gamma_a} c_2(X) \\[2ex] \eta\left(\frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^{-1/2}\, E(\varepsilon(g))\,\eta(\tau) \end{array}\right]$$

$$\begin{pmatrix} \tilde{\xi}'_0 \\ \alpha' \end{pmatrix} \mapsto \begin{pmatrix} d & -c \\ -b & a \end{pmatrix} \begin{pmatrix} \tilde{\xi}'_0 \\ \alpha' \end{pmatrix} + \text{stuff}$$

[Alexandrov, Saueressig, Pioline, Vandoren] [Alexandrov, D.P., Pioline]

The Kähler potential on Z transf. according to.

$$K_Z \mapsto K_Z - \log(|c\xi^0 + d|)$$

with $K_Z = \log \frac{1+t\bar{t}}{|t|} + \mathrm{Re}\,\Phi$

$\Phi : Z \to \mathbb{C}$ is called the "contact potential".

If we choose

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ -k/p^0 & p/p^0 \end{pmatrix}$$

such that $p^0 = \gcd(k,p)$ with k the NS5-charge and p the D5-charge, then this maps the D5-brane into a (k,p)-fivebrane.

we wish to construct an $SL(2,\mathbb{Z})$-invariant contact structure on $\mathcal{Z}$ by adding all images of

$$\left(H_\gamma(\Sigma), l_\gamma\right)$$

under $\Gamma_\infty \backslash SL(2,\mathbb{Z})$, where $\Gamma_\infty = \left\{ \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \,\middle|\, n \in \mathbb{Z} \right\}$.

we obtain new transition functions on BPS-rays:

$$\begin{cases} g \cdot H_\gamma = H_{k,p,\hat{\gamma}} \\ g \cdot l_\gamma = l_{k,p,\hat{\gamma}} \end{cases}$$

where $\hat{\gamma} = (p^a, q_a, \tilde{q}_0)$

The $H_{k,p,\hat{\gamma}}$ is a generating function of contact transformation across $\ell_{p,k,\hat{\gamma}}$ in the presence of NS5-branes. Explicitly, we have

$$H_{p,k,\hat{\gamma}} = -\frac{\overline{\Omega}(\gamma)}{(2\pi)^2}\frac{k}{p^0}\left(\xi^0 - n^0\right)\lambda_D(\gamma)$$

$$\times\, \mathbb{E}\left(k S_\alpha + \frac{p^0\left(k q_a(\xi^a - n^a) + p^0 q_0\right)}{k^2(\xi^0 - n^0)}\right)$$

$$\times\, \mathbb{E}\left(-c_{2,a}\, p^a\, \epsilon(g)\right).$$

where

$$S_\alpha = \alpha + n^\Lambda \tilde{\xi}_\Lambda + F(\xi^\Lambda - n^\Lambda) - \frac{1}{2}A_{\Lambda\Sigma}n^\Lambda n^\Sigma$$

with $(n^0, n^a) = \left(\frac{p}{k}, \frac{p^a}{k}\right) \in \mathbb{Z}/k$

[Note: $\ell_{k,p,\hat{\gamma}} \subset \mathbb{P}^1$ joins the two roots of $\xi^0(t) - n^0 = 0$ where $H_{k,p,\hat{\gamma}}$ has essential singularities]

5.3. The NS5-partition function in twistor space

Now consider the formal sum over all charges; this is the twistor space version of the NS5-partition function:

$$H^{(k)}_{NS5}(\xi^\Lambda,\tilde\xi_\Lambda,\hat\alpha) = \sum_{(p,p^0,q_\Lambda)\in\Gamma} H_{k,p,\hat\gamma}(\xi^\Lambda,\tilde\xi_\Lambda,\hat\alpha)$$

By Heisenberg invariance, this can be recast into a linear combination of wave functions:

$$H^{(k)}_{NS5}(\xi^\Lambda,\tilde\xi_\Lambda,\hat\alpha) = \frac{1}{4\pi^2}\sum_{\mu\in\Gamma_m/(|k|\Gamma_m)}\ \sum_{n\in\Gamma_m+\mu+\theta}$$

$$H^{(k,\mu)}_{NS5}(\xi^\Lambda-n^\Lambda)\, E\left(k n^\Lambda(\tilde\xi_\Lambda-\phi_\Lambda)-\frac{k}{2}(\hat\alpha+\xi^\Lambda\tilde\xi_\Lambda)\right)$$

For simplicity, we now restrict to $k=1$ and set $(\theta,\phi)=(0,0)$ (which appears to be required for the compatibility between S-duality and $\mathrm{Heis}_{\mathbb{Z}}$).

This yields

$$H^{(1)}_{NS5} = \frac{1}{4\pi^2} \sum_{n^\Lambda \in \Gamma_m} \mathcal{H}^{(1,0)}_{NS5}\left(\xi^\Lambda - n^\Lambda\right) \mathbb{E}\left(n^\Lambda \tilde{\xi}_\Lambda - \tfrac{1}{2}\left(\tilde{\alpha} + \xi^\Lambda \tilde{\xi}_\Lambda\right)\right)$$

with

$$\mathcal{H}^{(1,0)}_{NS5}(\xi^\Lambda) = \sum_{q_a, q_0} \bar{\Omega}(\gamma)(-1)^{q_0} \times$$

$$\times\, \mathbb{E}\left(-\frac{N(\xi^a)}{\xi^0} + \frac{q_a \xi^a + q_0}{\xi^0} + \frac{1}{2} A_{\Lambda\Sigma} \xi^\Lambda \xi^\Sigma \right)$$

Now rewrite this as follows

$$\mathcal{Z}^{(1,0)}_{NS5}(\xi^\Lambda) = \mathbb{E}\left(-\frac{N(\xi^a)}{\xi^0} + \frac{1}{2}A_{\Lambda\Sigma}\xi^\Lambda\xi^\Sigma - \frac{c_{2,a}}{24}\frac{\xi^a}{\xi^0}\right)$$

$$\times \sum_{q_a, q_0} \bar{\Omega}(\gamma)(-1)^{q_0}\,\mathbb{E}\left(\left(q_a + \frac{c_{2,a}}{24}\right)\frac{\xi^a}{\xi^0} + \frac{q_0}{\xi^0}\right)$$

Identifying

$$\begin{cases} q_a = Q_a - \dfrac{c_{2,a}}{24} \\ q_0 = 2J \\ \bar{\Omega}(\gamma) = N_{DT}(Q_a, 2J) \\ \lambda = -2\pi i/\xi^0 \end{cases}$$

this becomes

$$\mathcal{Z}^{(1,0)}_{NS5}(\xi^\Lambda) = e^{F_{hol}(\xi^\Lambda)}\, Z_{DT}(\xi^\Lambda)$$

$$= (\xi^0)^{-1-\frac{\chi(\hat{X})}{24}}\left[M(e^{2\pi i/\xi^0})\right]^{-\chi(\hat{X})/2}\Psi_{IR}(\xi^\Lambda)$$

So the NS5-wave function in type IIB is proportional to the A-model topological string wave function in the real polarization, considered as a section on twistor space:

$$\Psi_{\mathbb{R}}(\xi^{\wedge}) \in H^1(Z, \mathcal{O}(2))$$

This clearly calls for a more precise, mathematical explanation...

The contribution from $k > 1$ NS5-branes should then be given by the generating function of rank $r = \gcd(k, p) > 1$ DT-invariants.

Formally, we get contribution to the contact structure in terms of a non-abelian Fourier expansion along Heis$_\mathbb{Z}$:

$$H_{NS5}(\xi^\Lambda, \tilde{\xi}_\Lambda, \tilde{\alpha}) = \sum_{k \neq 0} \sum_{\mu \in \Gamma_m / (k)\Gamma_m} \sum_{n \in \Gamma_m + \mu}$$

$$\underbrace{\Psi_{\mathbb{R}}^{(k,\mu)}(\xi^\Lambda - n^\Lambda)}_{\substack{\text{vector-valued 1-parameter} \\ \text{extension of the topological} \\ \text{string wave function [illegible].}}} \, E\left(k n^\Lambda \tilde{\xi}_\Lambda - \frac{k}{2}(\tilde{\alpha} + \xi^\Lambda \tilde{\xi}_\Lambda)\right)$$

Penrose transform

We can project $H_{NS5}(\hat{\xi}_\Lambda, \hat{\xi}^\Lambda, \hat{\alpha})$ onto a function on the base $\mathcal{M}_H$ using Penrose transform along the $\mathbb{P}^1$-fiber.

This gives a candidate expression for the full non-Gaussian partition function of the NS5-brane.

Let us do this explicitly for $k=1$:

$$Z_1(C,N) \equiv \int \frac{dt}{2\pi i t} H^{(1)}_{NS5}\left(\xi^\Lambda(t), \xi_\Lambda(t), \alpha(t)\right)$$

$$\underset{\text{saddle point approx}}{\sim} e^{f_1(z)} \sum_{n\in\Gamma_m} \left[(\xi^\Lambda - n^\Lambda) z_\Lambda\right]^{\frac{\chi}{24}-1} e^{-S_{Gauss}(N,C)}$$

This reproduces the expected structure of the Gaussian partition function with an additional insertion of $(\mathcal{J}^\wedge - n^\wedge)$ and an explicit formula for the overall normalization in terms of the holomorphic part $e^{f_1(z)}$ of the one-loop vacuum amplitude

$$F_1 = \log\left[e^{f_1(z) + \bar{f}_1(\bar{z})} M(z,\bar{z})^{-1/2}\right]$$

Note that this can be rewritten in terms of a particular product of Ray-Singer torsions:

→

$$F_1 = \frac{(\det{}'\Delta^{0,0})^{9/2}\,(\det{}'\Delta^{1,1})^{1/2}}{(\det{}'\Delta)^{3}}$$

where $\det{}'\Delta^{p,q}$ is the regularized determinant of the Laplacian acting on (p,q)-forms on X.

[Bershadsky, Cecotti, Ooguri, Vafa]
[Fang, Lu, Yoshikawa]

Thus the normalization of the NS5-partition function is given by the holomorphic square root

$$e^{f_1(z)} = \frac{(\det{}'\Delta^{0,0})^{9/4}\,(\det{}'\Delta^{1,1})^{1/4}}{(\det{}'\Delta)^{3/2}}$$

This is in accordance with earlier speculation about the normalization factor. [Witten] [Belov, Moore]

This also confirms our previous results suggesting that Z_1 should be a section of $\sqrt{D_x}$ where D_x is the BCOV determinant line bundle where F_1 is valued.

6. NS5-branes & quantization of integrable systems

DT-invariants are intimately related with integrable systems:

* $\mathcal{M}'_H \to B$ complex integrable system (Hitchin system)

  (recent work of KS)

* Integral equation for Darboux coordinates $\mathcal{X}_\gamma$ on Z has the form of a TBA.

* Close relation between DT-theory and cluster algebras (varieties).

We will now provide some evidence that NS5-branes can be understood in terms of the quantization of these integrable systems.

6.1 Cluster varieties

Roughly, a cluster variety is a collection of complex tori $(\mathbb{C}^*)^N$ glued together into an algebraic variety using cluster transformations.

Basic definitions from Fock-Goncharov:

Def: A seed i is a pair (I, B) where I is a finite set & B is a matrix with components $B_{ij} = -B_{ji}$, $i, j \in I$. To each seed we associate a torus $X_i = (\mathbb{C}^*)^I$ with coordinates $\{x_i \mid i \in I\}$ and equipped with a Poisson structure

$$\{x_i, x_j\} = B_{ij} x_i x_j$$

Let $\mathbf{i}$ and $\mathbf{i}'$ be two seeds.
A mutation in direction $k \in I$
is an isomorphism

$$\mu_k : I \to I'$$

such that:

$$B'_{\mu_k(i)\mu_k(j)} = \begin{cases} -B_{ij} & \text{if } i=k \text{ or } j=k \\ B_{ij} & \text{if } B_{ik}B_{kj} \leq 0 \\ B_{ij} + |B_{ik}|B_{kj} & \text{if } B_{ik}B_{kj} > 0 \end{cases}$$

An automorphism σ of $\mathbf{i}$
is an automorphism of I preserving B.

Automorphism and mutations induce
rational maps between the cluster
seed tori $\mathcal{X}_{\mathbf{i}}$ acting on coordinates
by the formula

$$\sigma^* X_{\sigma(i)} = X_i$$

and

$$\mu_k^* X_{\mu_k(i)} = \begin{cases} X_k^{-1} & i = k \\ X_i \left(1 + X_k^{-\operatorname{sign}(B_{ik})}\right)^{-B_{ik}} & i \neq k. \end{cases}$$

A cluster transformation is a composition

$$\mu_k \circ \sigma$$

Def: A cluster $\mathcal{X}$-variety is (a scheme over $\mathbb{Z}$) a variety obtained by gluing the seed tori $\mathcal{X}_{\mathbf{i}}$ using the cluster transformations.

Every seed $\mathcal{X}_{\mathbf{i}}$ gives a local rational coordinate system $\{X_i \mid i \in I\}$ on $\mathcal{X}$ called cluster coordinates.

Def: A cluster $\mathcal{A}$-variety is an algebraic variety obtained by gluing seed tori $\mathcal{A}_{\mathbf{i}} = (\mathbb{C}^*)^I$ equipped with coordinates $\{a_i \mid i \in I\}$ and carrying a canonical (pre-)symplectic structure

$$\Omega_{\mathbf{i}} = \sum_{i,j \in I} B_{ij}\, d\log a_i \wedge d\log a_j$$

For the cluster $\mathcal{A}$-variety the gluing conditions are

$$\mu_k^* a_{\mu_k(i)} = \begin{cases} a_i & i \neq k \\ a_k^{-1}\left(\prod_{i \mid B_{ki}>0} a_i^{B_{ki}} + \prod_{i \mid B_{ki}<0} a_i^{-B_{ki}}\right) & i = k. \end{cases}$$

Remark: There is a map

$$p : \mathcal{A} \to \mathcal{X}$$

given in any cluster coordinate system by the formula

$$p^* X_k = \prod_{i \in I} a_i^{B_{ki}}$$

.2 Quantization of $\mathcal{X}$

Def: The quantum cluster variety $\mathcal{X}_q$ is a canonical non-commutative q-deformation of $\mathcal{X}$.

Start from a fixed quantum torus algebra $\mathbb{T}_i^q$ defined by

$$X_i X_j = q^{2B_{ij}} X_j X_i$$

where $q = e^{\pi i \hbar}$, $\hbar \in \mathbb{Q}$, is a root of unity.

The gluing condition between different seed quantum tori $\mathbb{T}_{i}^{q}$ are described by

$$\hat{\mu}_k : X_i \mapsto \boldsymbol{\Phi}_{q_k}(X_k)\, X_i\, \boldsymbol{\Phi}_{q_k}(X_k)^{-1}$$

where $\boldsymbol{\Phi}_q(x)$ is the q-dilog:

$$\boldsymbol{\Phi}_q(x) = \prod_{n=0}^{\infty} (1 + q^{2n+1} x)^{-1}$$

Using functional relation for $\boldsymbol{\Phi}_q(x)$ one can rewrite the transform as

$$\hat{\mu}_k : X_i \mapsto X_i \prod_{m=1}^{|B_{ik}|} \left(1 + q^{(2m-1)\,\mathrm{sign}\, B_{ik}} X_k\right)^{-\mathrm{sign}\, B_{ik}}$$

6.3 Geometric quantization of A

Fock & Goncharov have recently analyzed the geometric quantization of the A-cluster variety. (unpublished).

The geometric quantization of the symplectic structure Ω_{ii} on a seed A_{ii}-torus produces a pre-quantum vector bundle

$$V_\hbar \to A$$

depending on a quantization param

$$\hbar = s/r \in \mathbb{Q}$$

we have

$$\begin{cases} c_1(V_\hbar) = s\,\Omega_{ii} \\ \operatorname{rank}(V_\hbar) = r^{(\operatorname{rank} B_{ij})/2} \end{cases}$$

For simplicity, we first restrict to $\hbar = 1$ which give a line bundle $\mathcal{V}_1 \to A$ with

$$c_1(\mathcal{V}_1) = \Omega_{ii} = d\beta$$

with connect. $\beta = \sum_{i,j} B_{ij} \log a_i \, d\log a_j$.
Let $I = \{1, \dots, N\}$. Sections of $\mathcal{V}_1$ are multi-valued functions $F(a_1, \dots, a_N)$ on $(\mathbb{C}^*)^N$ which transform under monodromies according to

$$F(a_1, \dots, e^{2\pi i} a_i, \dots, a_N) = \prod_j a_j^{B_{ij}/2} F(a_1, \dots, a_N)$$

$$= \left(\frac{A_i^+}{A_i^-}\right)^{1/2} F(a_1, \dots, a_N)$$

To further simplify the story restrict to $N = 2$ such that $A_{ii} = (\mathbb{C}^*) \times (\mathbb{C}^*)$.
with coordinates $(a_1, a_2) \equiv (a, b)$

we then have

$$F(a_* e^{2\pi i}, b_*) = b_*^{1/2} F(a_*, b_*)$$

$$F(a_*, b_* e^{2\pi i}) = a^{-1/2} F(a, b)$$

~~By geometric quantization By a choice of polarization one may restrict the functions on $\mathbb{C}^* \times \mathbb{C}^*$ to holomorphic functions $\Psi(a)$ on $\mathbb{C}^*$.~~

To construct the bundle $V_1 \to A$ globally we must consider transition functions between local bases A_i.

For example the cluster transformation μ_2 acts on A_i according to

$$\mu_2 : \{a, b, B_{ij}\} \mapsto \left\{a, \frac{(1+a)}{b}, -B_{ij}\right\}$$

$$\sigma : (a, b) \mapsto (b, a)$$

sections $F(a,b) \in \Gamma(\mathcal{V}_1)$ transform according to

$$\sigma \circ \mu_2 : F(a,b) \mapsto F\left(\frac{1+a}{b}, a\right) f(a,b)$$

where $f(a,b)$ is a transition function. This is determined by demanding that the image of $\sigma \circ \mu$ satisfies the same monodromy relation as $F(a,b)$.

Under this transform the connection β transforms according to

$$\beta \longmapsto \beta - dL(-a)$$

where $L(x)$ is Rogers dilog.

Hence:

$$\sigma \circ \mu_2 : F(a,b) \mapsto F\left(\frac{1+a}{b}, a\right) e^{-\frac{L(-a)}{2\pi i}}$$

Pentagon relation for $L(x)$ ensures that

$$(\sigma \circ \mu_2)^5 = \mathrm{Id}.$$

Thus we see that the line bundle $\mathcal{V}_1 \to A$ ~~$\$$~~ is determined by the same gluing conditions as for our holomorphic bundle

$$\mathcal{L}_{Z'} \to Z'$$

This suggests that we may view $\mathcal{L}_{Z'}$ ~~as~~ in terms of the geometric quantization of Z' with connection

$$\beta = h \langle \Sigma , d\Sigma \rangle$$

with $\Sigma = (\xi^A, \xi_A)$.

(This is in accordance with recent results of Hitchin)

6.4 Connection with NS5-brane

To explain the connection with NS5-branes, note that by a choice of polarization one may restrict from functions on $\mathbb{C}^* \times \mathbb{C}^*$ to holomorphic functions $\Psi(a)$ on $\mathbb{C}^*$.

Fock & Goncharov implement this via Fourier expansion along $\mathbb{C}^*$:

$$F(a,b) = \sum_{n\in\mathbb{Z}} \Psi(\log a - 2\pi i n)\, e^{-\frac{\log a \log b}{4\pi i}}\, b^n$$

If we identify

$$a = e^{2\pi i \xi} \qquad b = e^{2\pi i \xi}$$

this takes the same form as the NS5-partition function in twistor space:

$$F(\xi,\tilde{\xi}) = \sum_{n\in\mathbb{Z}} \Psi(\xi-n)\,\mathbb{E}\left(\frac{1}{2}\xi\tilde{\xi} + n\tilde{\xi}\right)$$

So, the twistor space partition function of a single NS5-brane may be viewed as a section of a pre-quantum line bundle over a cluster $\mathcal{A}$-variety.

<u>What about more generally for $\hbar \neq 1$?</u>

In this case sections of $V_\hbar$ can be represented by vector valued function $\Psi_{\hbar^{-1},l}(\xi)$ on $\mathbb{C}^*$. For $s=1$ and $r = 1/\hbar \in \mathbb{Z}_+$ one has explicitly:

$$F_\hbar(\xi,\tilde{\xi}) = \sum_{l\in\mathbb{Z}/(\mathbb{Z}\,k^{-1})} \sum_{n\in\mathbb{Z}+l\hbar} \Psi_{\hbar^{-1},l}(\xi-n)\,\mathbb{E}\left(-\frac{\xi\tilde{\xi}}{2\hbar} + \frac{n\tilde{\xi}}{\hbar}\right)$$

This may be recognized as the non-abelian Fourier expansion associated with k NS5-branes provided we identify

$$\hbar = 1/k.$$

So the NS5-charge becomes interpreted as a quantization parameter!

Moreover, under cluster transformations the wave-function $\Psi_{k,l}(\xi)$ transforms by a convolution with the q-dilog:

$$\cancel{\Psi_{k,l}(\xi)}$$

$$\Psi_{k,l}(\xi) \longmapsto \int \Psi_{k,l}(\xi)\, \Phi_{e^{\pi i/k}}(\xi) \times e^{-2\pi i k \xi \tilde{\xi}}\, d\xi$$

Hence this might give a framework in which to understand wall-crossing in the presence of NS5-branes.

6.5 Speculative conclusion

We seem to arrive at two different perspectives on NS5-branes.

(1) S-duality suggests that NS5-branes are controlled by the (generalized) DT-invariants $\Omega(\gamma)$ upon reinterpreting $p^0 \to p = \gcd(k, p^0)$.

(2) The connection with quantization of cluster varieties and the q-dilog, on the other hand suggest some relation with the motivic (or quantum) DT-invariants.

In general we expect new invariants attached to NS5-branes, correspond to functions

$$\Omega_{NS5} : \Gamma \times \mathbb{Z} \to \mathbb{Z}$$

Formally, we might hope to identify these with motivic DT-invariants

$$\Omega_{NS5}(\gamma; k) = \Omega_{mot}(\gamma, q)$$

under $q = e^{i\pi/k}$.

~~...~~ Compatibility with S-duality then suggests an intriguing relation

$$\Omega_{mot}(\gamma, q) \xleftrightarrow{\text{S-duality}} \Omega(\gamma).$$