An excursion to the moon:

- from finite groups to black holes via modular forms -

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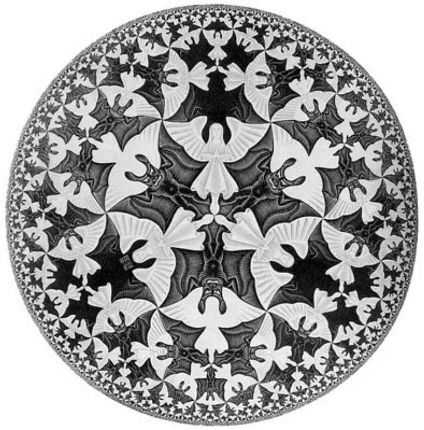
Fourier coefficients of automorphic forms



- Fourier coefficients of **classical modular forms** encode deep number-theoretic information (counting points on elliptic curves etc..)

- **Moonshine:** relations with finite sporadic groups and CFT/string theory

- **Enumerative geometry:** rational curves on K3, GW-theory…

- Higher rank groups: **Langlands program**
  (automorphic L-functions, functoriality…)

- The Fourier coefficients of Eisenstein series also encode **string theory scattering amplitudes**

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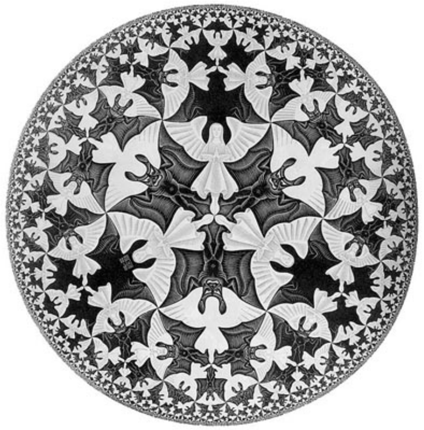

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Classification of finite simple groups

1832: Galois discovers the first infinite family (alternating groups)

1861-1873: Mathieu discovers $M_{11}, M_{12}, M_{22}, M_{23}, M_{24}$

1972-1983: Gorenstein program to classify all finite simple groups

The monster

Conjecture (Fischer & Griess, 1973):
There exists a huge finite simple group of order

The monster

$$2^{46} \cdot 3^{20} \cdot 5^{9} \cdot 7^{6} \cdot 11^{2} \cdot 13^{3} \cdot 17 \cdot 19 \cdot 23 \cdot 29 \cdot 31 \cdot 41 \cdot 47 \cdot 59 \cdot 71 \approx 8 \times 10^{53}$$

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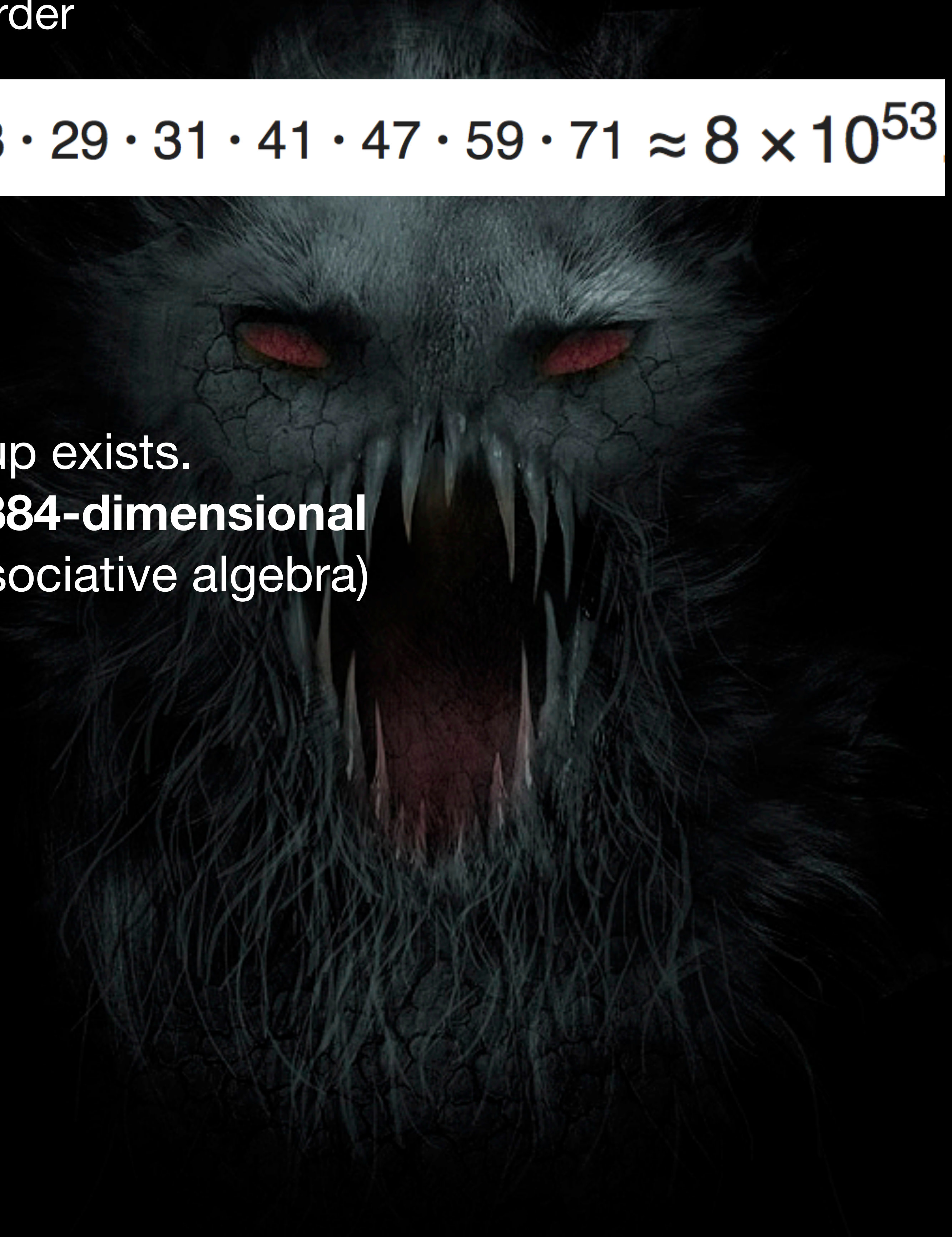
Theorem (Griess 1982): The monster group exists.
It is the **symmetry group** of a certain **196884-dimensional** algebraic structure. (commutative, non-associative algebra)



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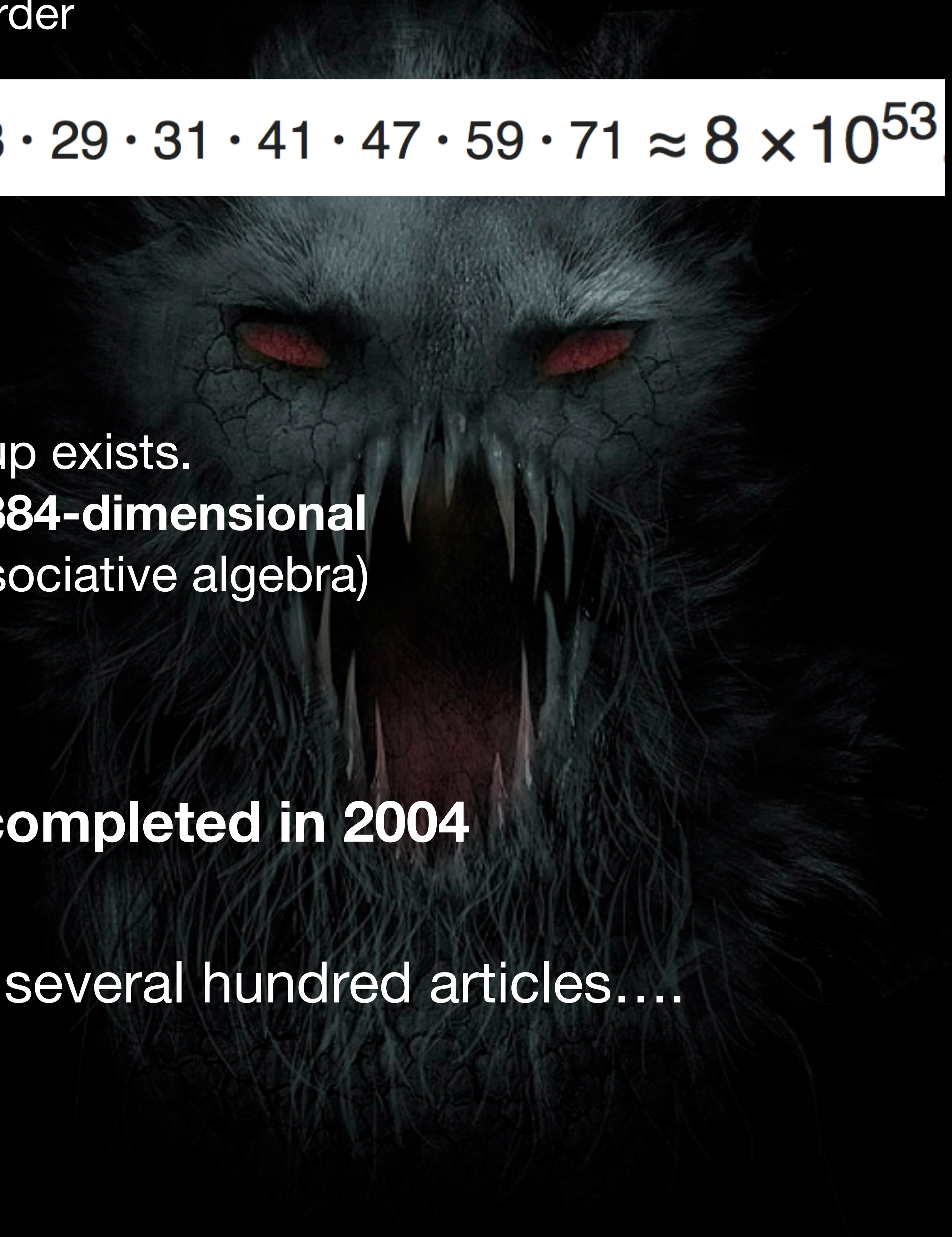

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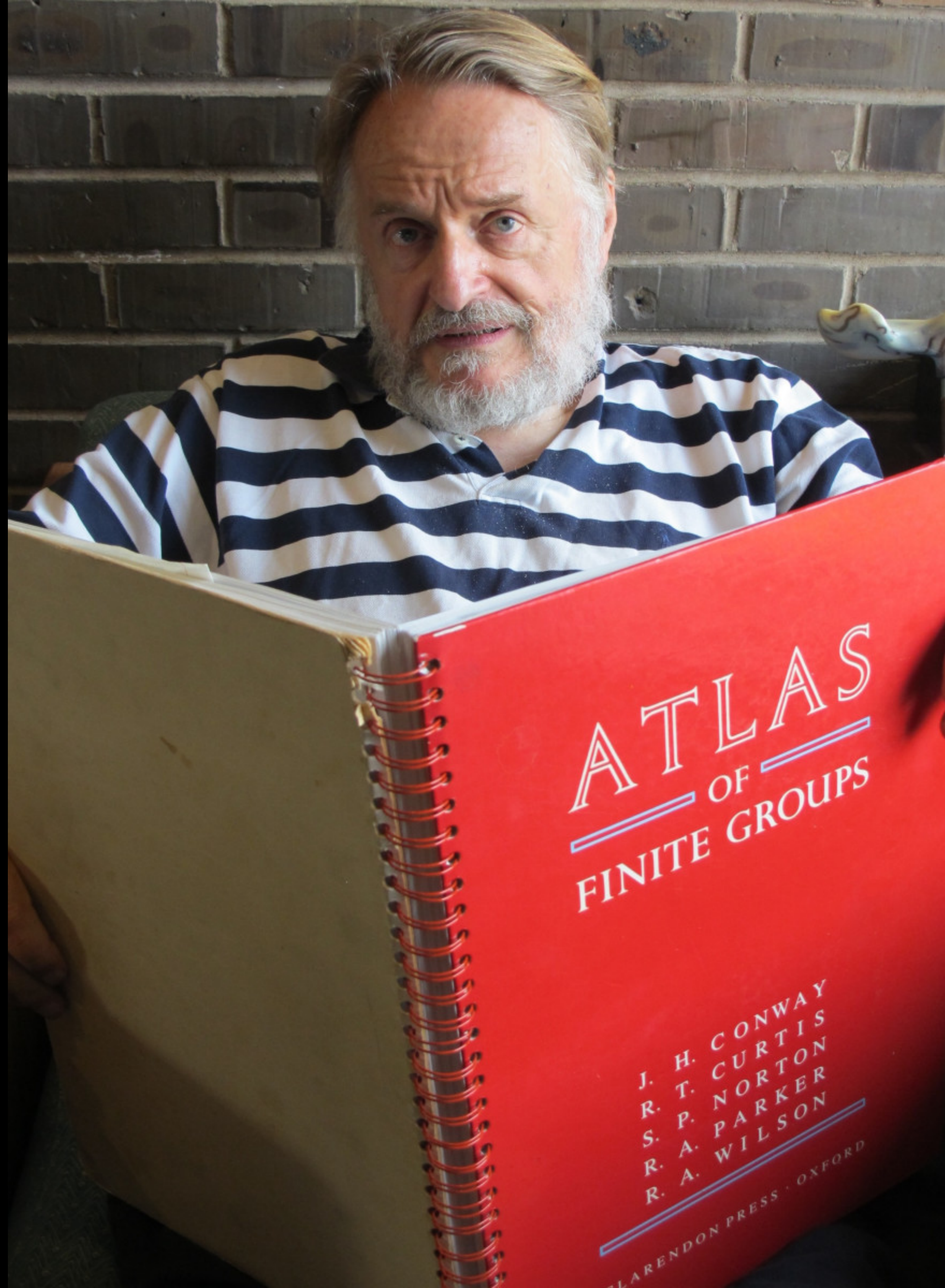
Full classification completed in 2004

Tens of thousands of pages, several hundred articles….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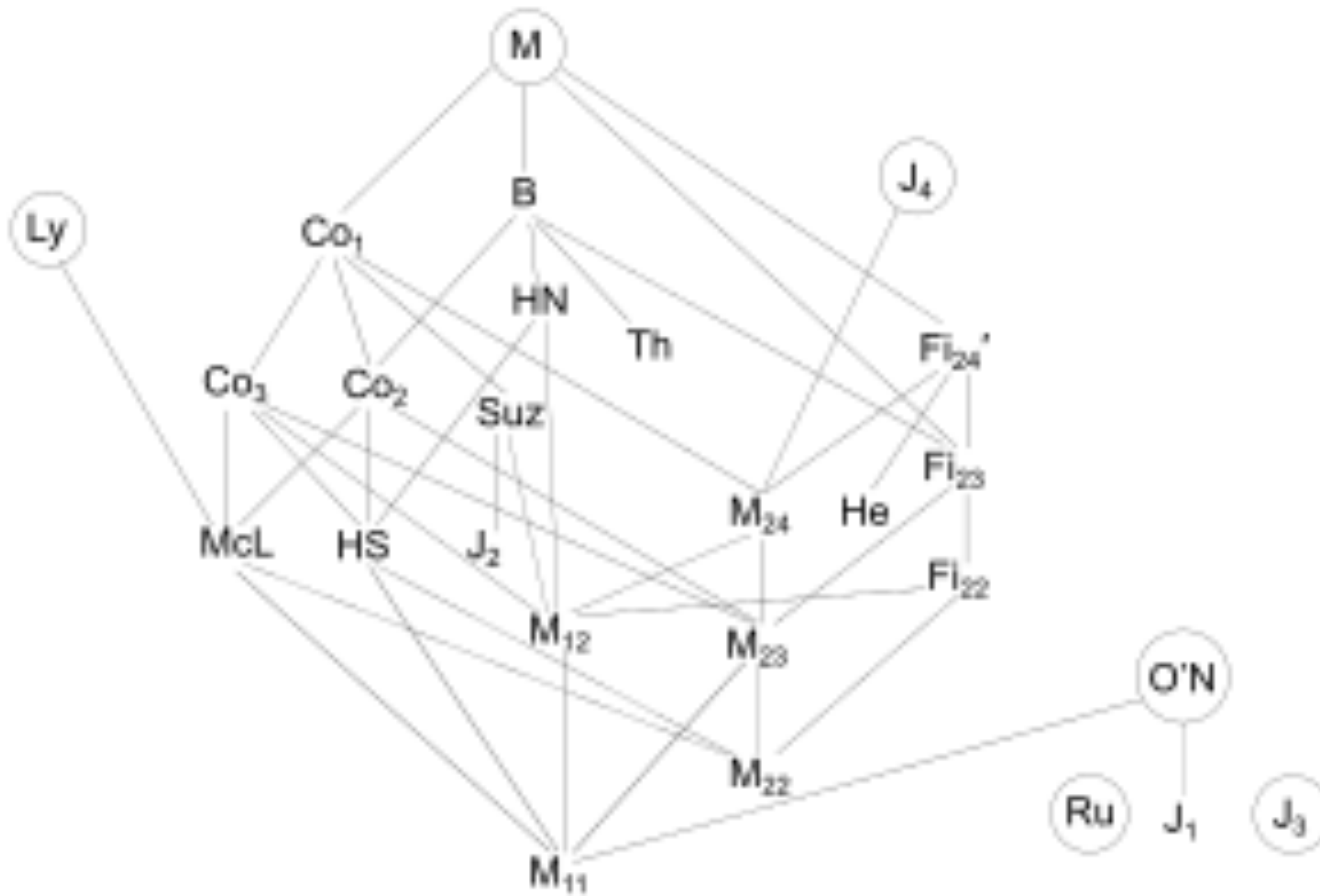
Classification

- **Several infinite families:** cyclic, alternating, Lie type
- **26 sporadic cases**



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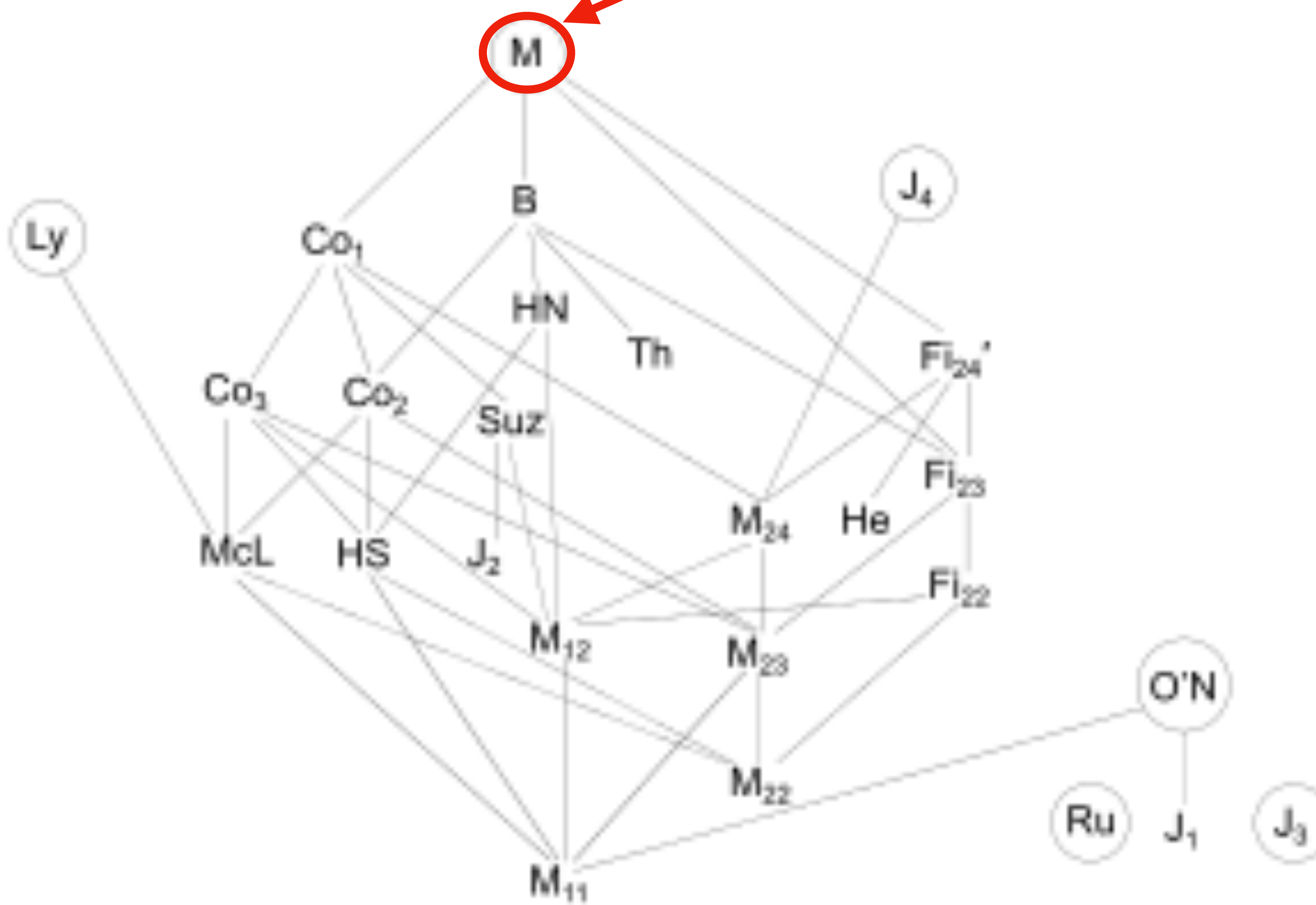
Monster $\mathbb{M}$



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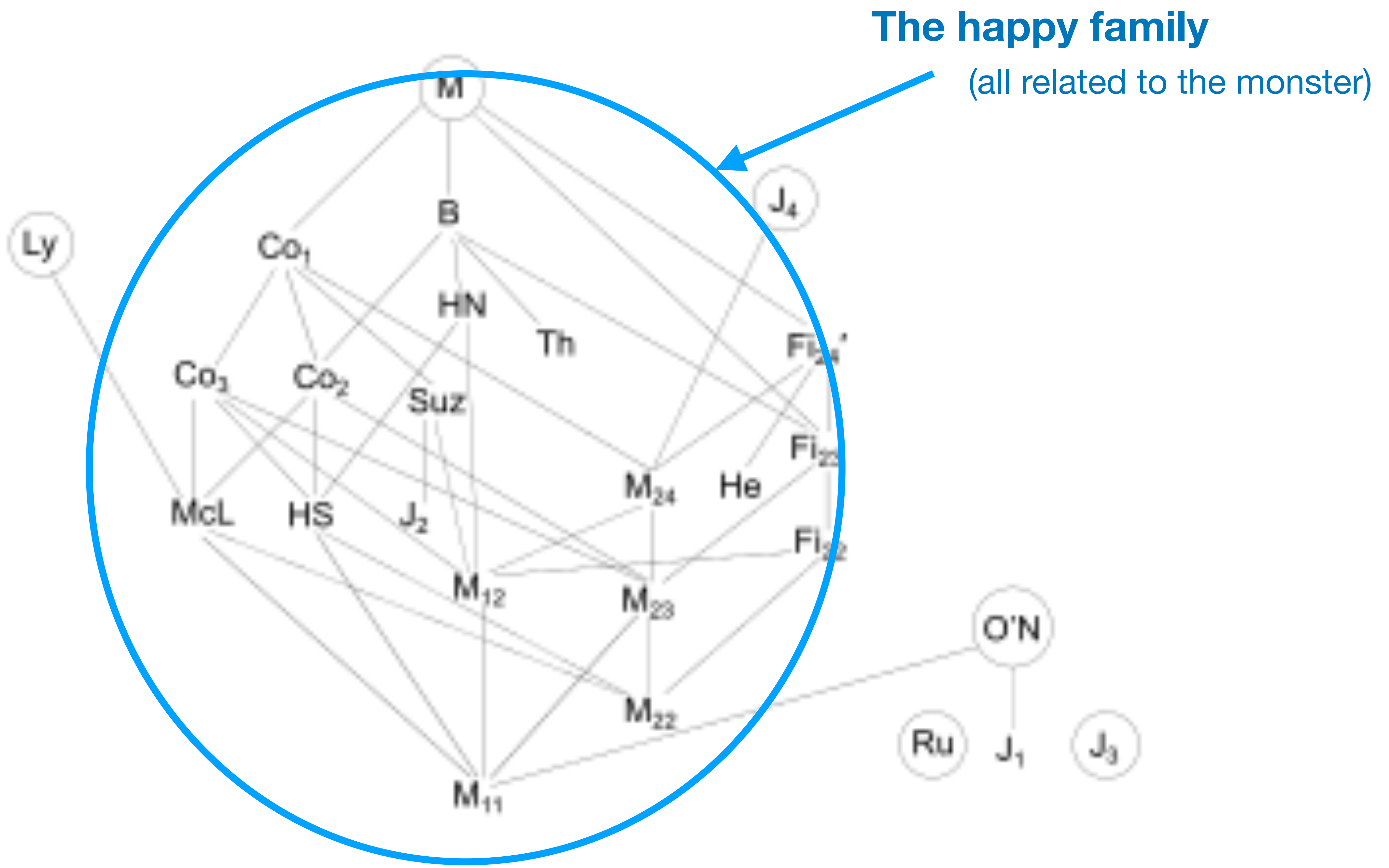

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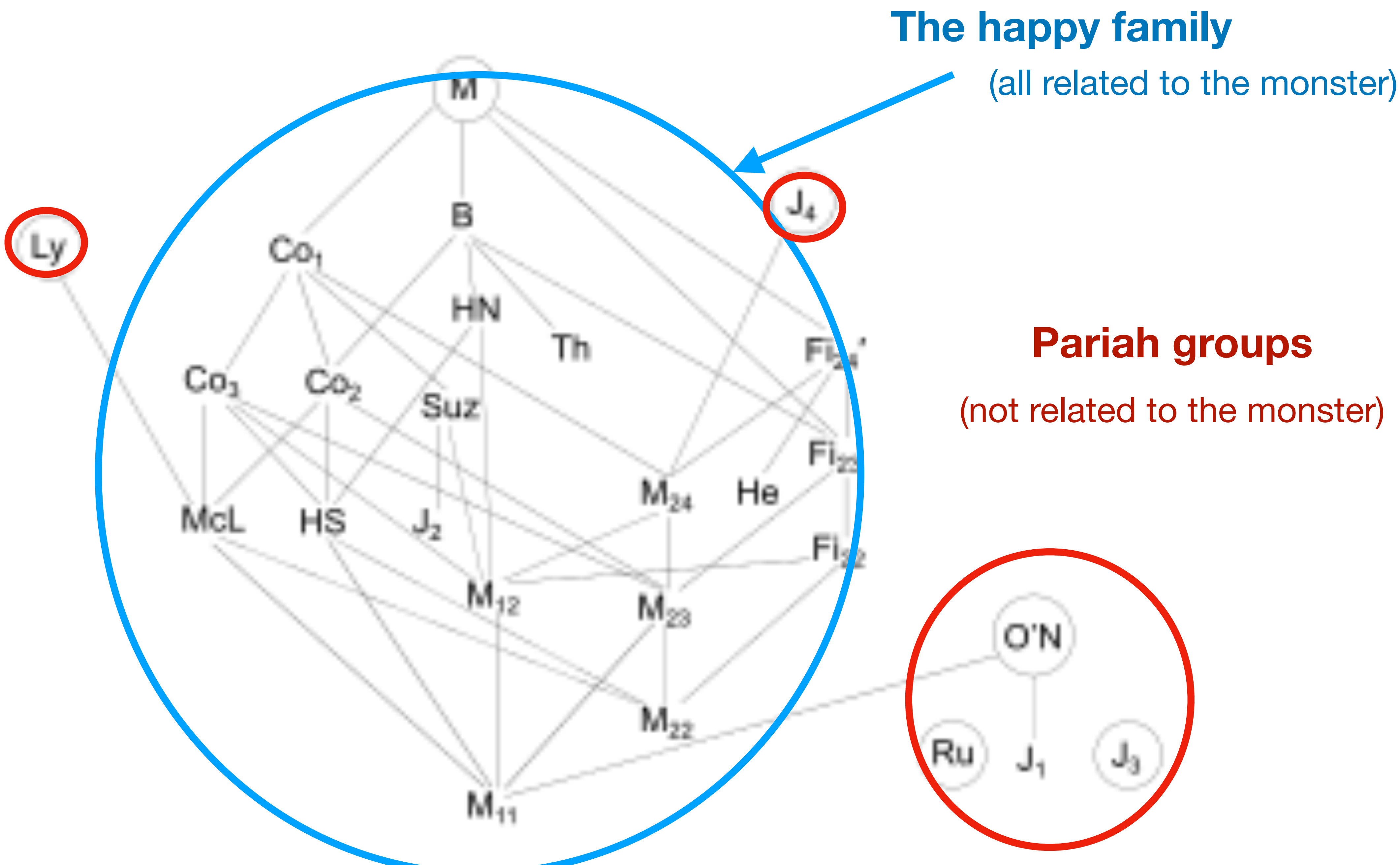

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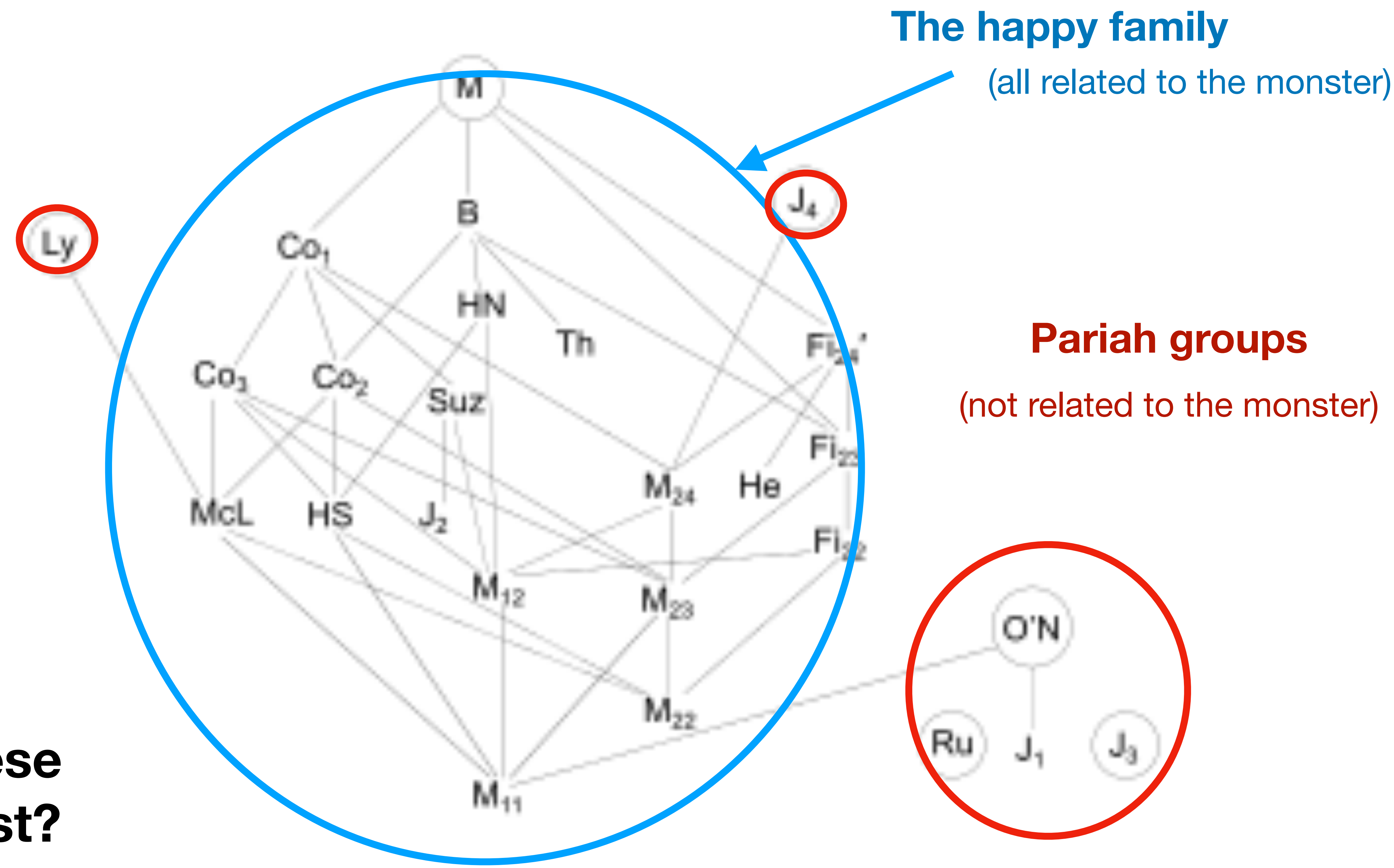

Why do these groups exist?

The first hints of moonshine…

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He then stumbled upon the following series expansion:

$$J(q) = \frac{1}{q} + 196884q + 21493760q^2 + 864299970q^3 + 20245856256q^4 + \cdots$$

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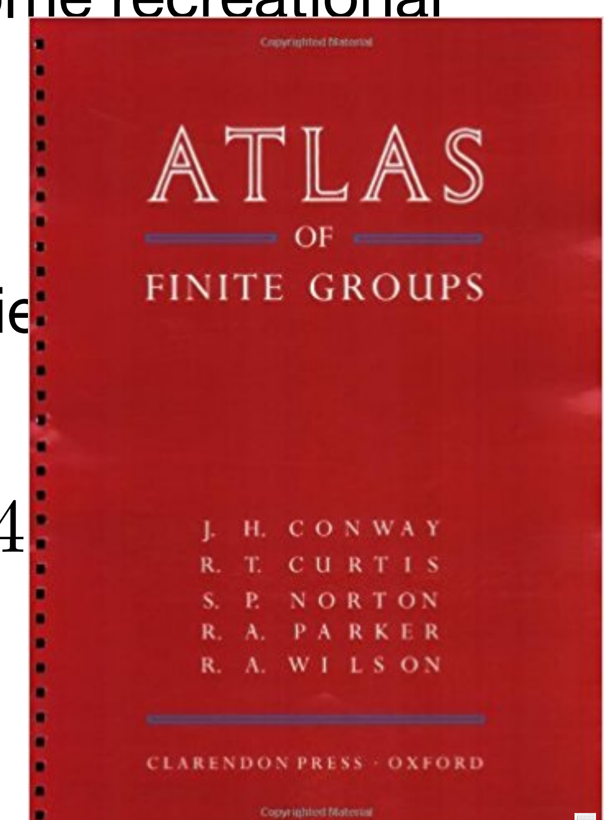
$$J(q) = \frac{1}{q} + 196884q + 21493760q^2 + 864299970q^3 + 2024$$

Being a finite group theorist he immediately opened up **the Atlas**…



;		@	@	@	@	@	@	@	@	@	@	@	@	@	@	@	@	@	@	@	@	@	@	@	@	@	@
		80801742479451287588645	83095629624528523	139511839126	376561712757	14296150	272237831	83175342	26489012	4870492	824432	1365154	9450	774741	269007	48157	13060	16124	2786	2821	847	7927	7785	1437	235	125	30
		9904961710757005754368000000000	82355161088000000	336328171520000	1985163878400	77540249600	663616000	73309696000	826931200	9136640	3942400	560000000	0000000	019852800	2985600	9499520	6940160	31360	91840	2710400	07280	23456	67680	69600	92960	82912	96576
		p power	A	A	A	A	A	B	A	B	B	A	A	AA	BB	AB	BA	BB	CB	A	A	C	A	B	C	C	D
		p' part	A	A	A	A	A	A	A	A	A	A	A	AA	BB	AB	BA	BB	CB	A	A	A	A	A	A	A	A
	ind	1A	2A	2B	3A	3B	3C	4A	4B	4C	4D	5A	5B	6A	6B	6C	6D	6E	6F	7A	7B	8A	8B	8C	8D	8E	8F
χ_{1}	+	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
χ_{2}	+	196883	4371	275	782	53	-1	275	51	19	-13	133	8	78	77	14	-3	5	-1	50	1	35	11	-1	-5	3	-1
χ_{3}	+	21296876	91884	-2324	7889	-130	248	1772	-52	-20	12	626	1	273	286	-47	30	-2	-8	153	6	92	-12	0	4	-4	0
χ_{4}	+	842609326	1139374	12974	55912	-221	-248	8878	782	-82	78	2451	-49	1000	923	104	-53	-13	8	426	-15	222	54	26	6	-2	-6
χ_{5}	+	18538750076	8507516	123004	249458	1598	248	28796	2652	380	156	6326	76	2354	1990	370	86	46	-8	798	14	364	132	0	20	12	0
χ_{6}	+	19360062527	9362495	-58305	297482	1508	-247	35903	-833	63	-65	8152	27	2378	2652	-246	-52	-12	9	1072	-6	511	-65	-25	-1	-1	7
χ_{7}	+	293553734298	53981850	98970	1055310	-3927	3876	94874	1274	-102	-454	17423	-77	5070	4785	78	129	-39	36	1700	-15	714	34	-26	-14	-22	6
χ_{8}	+	3879214937598	337044990	-690690	4751823	-4173	-3876	345598	-3874	-258	286	54473	98	12495	13299	-561	-141	51	-36	4249	35	1646	-154	26	-10	14	-6
χ_{9}	+	36173193327999	1354188159	2864511	12616074	18954	0	701823	20383	-897	351	91124	-126	20682	18954	1674	-54	-54	0	5355	-35	1807	375	27	-25	-17	27
χ_{10}	+	125510727015275	3215883115	1219435	24688454	-25375	248	1223531	19499	-661	-1365	145275	-350	33670	28105	70	217	-47	-8	7497	0	2379	219	-53	-5	11	-21
χ_{11}	+	190292345709543	2814161895	10249191	17144568	-8073	27000	734183	22151	-281	-1209	78793	168	12024	14103	2808	279	-9	120	3451	70	1079	351	-53	15	23	-21
χ_{12}	+	222879856734249	3864186921	-7196631	26057022	11310	34503	1187881	-39831	1577	1001	132374	-126	18174	24726	-1986	678	30	-57	6154	-20	1993	-583	25	25	9	-7
χ_{13}	+	1044868466775133	9223504989	-15756195	47292301	84448	31123	1762397	-46019	-2723	637	170258	133	11661	28392	-2163	888	-48	147	6272	0	1869	-507	25	21	-19	-7
χ_{14}	+	1109944460516150	9697078070	26155830	40851749	97733	61256	1379126	87126	4662	-1002	123025	525	43109	18381	3621	989	309	72	4200	35	966	846	26	-66	38	-6
χ_{15}	+	2374124840062976	22509162496	4100096	110509112	-1456	-61504	4100096	8192	4096	0	392976	476	62776	64064	56	-1376	-16	-64	14553	0	4096	0	0	0	0	0
χ_{16}	o	8980616927734375	-2720265625	39414375	1603525	200200	-116375	-9625	-20825	8295	1575	0	0	1925	0	-315	1520	-360	105	0	0	7	55	-25	-105	71	7
χ_{17}	o	8980616927734375	-2720265625	39414375	1603525	200200	-116375	-9625	-20825	8295	1575	0	0	1925	0	-315	1520	-360	105	0	0	7	55	-25	-105	71	7
χ_{18}	+	15178147608537368	72990279960	-29873896	255751145	33254	-30628	7620888	-44200	-2280	728	620368	-132	99177	88166	-3607	-858	38	92	17850	-84	4408	-440	0	40	-8	0
χ_{19}	+	39660520552077425	128459630705	47061105	367316348	-227227	30380	9660529	82705	-3215	6097	712425	-700	112892	93093	2940	869	-219	-84	17493	0	4033	617	53	25	-95	21
χ_{20}	+	60359800576579350	143552415510	71276310	371703759	-95718	120498	9004822	98294	-4074	1078	627475	1225	50895	79002	7119	1242	90	-78	13720	0	3206	638	274	14	38	-14
χ_{21}	+	251098487132187500	336140827500	339995500	599239034	251810	-147250	11393900	512396	10860	4940	653125	0	146874	67210	12922	-870	-110	-50	10879	50	1660	2212	376	20	-36	56
χ_{22}	+	290568421805921077	444043629365	45093685	948362611	36400	-147497	20222773	-122059	4917	-7371	1269827	-798	50675	147784	3955	-2056	256	-41	23324	0	5173	-331	-299	-75	53	21
χ_{23}	+	336041615485626050	536115345090	-288233790	1087588073	-68263	61256	22648514	-153374	-2622	-4446	1397925	1050	211497	155649	-13335	2193	201	72	25704	175	4818	-1254	-26	10	50	6
χ_{24}	+	2500435234254428856	1864421481144	319494840	2635262973	691236	116622	42900152	249016	-5448	-8008	2180231	-2394	245181	206052	6909	804	-156	-114	28322	0	4536	1144	-376	120	-72	-56
χ_{25}	+	2986480825407204125	1612726090525	-385717475	1933299500	-57343	779495	28797725	-204867	7709	7293	1358500	1000	121132	119977	-6356	1657	-335	-281	15078	-161	2541	-715	377	-155	77	57
χ_{26}	o	3503434660075044981	-89143381899	-755269515	0	0	767637	184437	161109	39285	18837	-8019	4356	0	0	0	0	0	405	153	-288	-27	-99	-351	45	-123	-63
χ_{27}	o	3503434660075044981	-89143381899	-755269515	0	0	767637	184437	161109	39285	18837	-8019	4356	0	0	0	0	0	405	153	-288	-27	-99	-351	45	-123	-63
χ_{28}	+	3605718753596953125	2239938073125	284185125	2906660052	189540	0	43954725	345253	-20955	4005	2131250	0	241812	189540	5076	-540	-540	0	25227	-155	3429	837	-27	5	-27	-27
χ_{29}	+	8456836343580310400	3618137095040	753384320	3560377184	-1117324	-30628	46447488	903552	-13440	11648	1954150	-2100	487136	149084	10592	-4372	-28	92	18852	-160	1664	1792	0	0	128	0
χ_{30}	+	8754193822112578125	3831845698125	-959677875	4307266557	-917865	27000	60123725	-883507	20045	9165	2678125	0	117117	228735	-19971	1935	135	120	26452	90	4109	-2739	53	-115	13	21
χ_{31}	+	28585990950721640625	7669080680625	-804759375	6880196379	1056405	-813750	82024625	-832559	6065	-11375	3200000	0	-13221	248325	-3045	-4635	-75	330	23324	0	3521	-1495	221	25	-95	-35
χ_{32}	+	30815545786259524745	3436802874505	1656343689	2715826568	2163434	809627	25226377	437801	-63607	-22295	842745	2870	-348920	66066	18312	3106	474	-357	2058	0	985	1441	-299	-175	57	21
χ_{33}	+	31569817307122699605	8003575389525	2741767509	6723221931	-821106	-895869	76528981	1062517	-7083	15925	2943480	2730	126699	211926	26283	-2010	318	387	21301	280	2533	2925	25	-35	69	-7
χ_{34}	+	47377503606648784400	7170718631440	6518654480	3475562246	3824405	-30628	26404368	2365584	98832	45968	702525	3150	695110	26741	38918	373	1685	92	3723	195	-944	3376	624	240	208	48
χ_{35}	+	49609712911192813665	4216672206945	5047513185	731909772	6751602	116622	3899489	1536001	50017	15041	-39710	4290	437580	858	12684	7146	1506	-114	-442	-246	49	1177	677	-55	273	133
χ_{36}	+	77316619273928125000	11487099325000	5193149000	7502151085	-1595360	988000	72944200	1281800	58440	-7800	2375000	0	-63635	150800	40109	10480	320	-160	11305	-210	1448	2520	0	120	-24	0
χ_{37}	+	130415350420342968750	19432400568750	-4691783250	13174268625	1409940	-27000	125829550	-1321970	11950	30030	4156250	0	-144495	263340	-33327	-2340	-540	-120	20825	0	1758	-2442	650	70	-2	42
χ_{38}	+	155943076739182582850	26382222060610	-786277310	16869395816	1510964	-30628	160795714	44642	19266	-46046	5251600	4725	960232	308308	-15512	-428	436	-164	29547	0	594	-550	-650	-54	-78	-42
χ_{39}	o	172399434201593354756	1932029655556	531831300	-441097657	-6179173	-1560013	2288132	-476476	-9212	-22204	-61369	1631	47047	3003	10311	-5765	987	819	0	0	-28	-396	-572	404	164	-28
χ_{40}	o	172399434201593354756	1932029655556	531831300	-441097657	-6179173	-1560013	2288132	-476476	-9212	-22204	-61369	1631	47047	3003	10311	-5765	987	819	0	0	-28	-396	-572	404	164	-28
χ_{41}	o	286243267692724486144	-53936390144	7514357760	-281380736	7326592	1646008	1835008	0	0	0	-30856	11144	-11648	0	-8064	-1280	1152	-840	0	0	0	0	0	0	0	0
χ_{42}	o	286243267692724486144	-53936390144	7514357760	-281380736	7326592	1646008	1835008	0	0	0	-30856	11144	-11648	0	-8064	-1280	1152	-840	0	0	0	0	0	0	0	0
χ_{43}	+	379913824694312370176	36886987145216	-2273050624	19084103168	-3939520	926744	150732800	0	0	0	4235176	-4824	206336	223808	-20992	4160	320	-232	14008	-496	0	0	0	0	0	0
χ_{44}	o	640558364167263622626	-1925003475486	9988806114	0	-3720087	0	26082	-584766	9954	-18942	39501	13251	0	-5103	0	6561	729	0	0	0	210	-54	78	-294	-142	-210
χ_{45}	o	640558364167263622626	-1925003475486	9988806114	0	-3720087	0	26082	-584766	9954	-18942	39501	13251	0	-5103	0	6561	729	0	0	0	210	-54	78	-294	-142	-210
χ_{46}	+	643356925889917747200	53859806085120	2614886400	25227138048	-460992	3411240	191102976	0	0	0	5042200	-2800	328704	251328	21504	12480	-960	552	13720	0	0	0	0	0	0	0
χ_{47}	o	691170144025469730622	9783256633150	-5972672706	-397261865	10656646	-4127123	491326	-1969058	-53698	63518	-40128	-10003	273559	6006	11991	-5162	1830	-147	-1029	0	78	-330	650	-346	46	-182
χ_{48}	o	691170144025469730622	9783256633150	-5972672706	-397261865	10656646	-4127123	491326	-1969058	-53698	63518	-40128	-10003	273559	6006	11991	-5162	1830	-147	-1029	0	78	-330	650	-346	46	-182
χ_{49}	+	776097192277137500000	29144516700000	11163740000	10339920710	6141590	5084000	66037600	106080	-20640	36960	1312500	0	-448890	70070	28742	11190	-490	-160	-1225	0	736	1056	0	160	-32	0
χ_{50}	+	918438233727730974720	31132239790080	-17203462144	8238100992	7656000	6757008	40370176	0	0	0	852720	-280	219648	13376	-60928	17472	-448	-880	1344	560	0	0	0	0	0	0
χ_{51}	+	1201241700908448332364	19824998517324	4387289676	184148811	-13607748	2597490	4917836	-1082900	-33460	98476	-251636	7364	531531	12012	10059	1548	-2532	882	0	0	28	-220	1352	-300	252	-56
χ_{52}	+	1353006807137391674268	95287105622940	-5268307044	40568500665	2360358	-2597490	285160348	-1527876	-77156	28028	7111643	-3857	602745	318318	-4935	-4674	726	-882	17493	0	-1364	-2156	376	4	12	56
χ_{53}	o	1480279477146615234375	90807234375	-1811005625	-593746725	-4751175	-5742750	2498375	-251225	154695	-64025	0	0	-10725	-3575	475	-1815	2905	610	-50	685	119	495	-325	-65	-233	251
χ_{54}	o	1480279477146615234375	90807234375	-1811005625	-593746725	-4751175	-5742750	2498375	-251225	154695	-64025	0	0	-10725	-3575	475	-1815	2905	610	-50	685	119	495	-325	-65	-233	251
χ_{55}	o	1768130802583126953125	-11890281046875	10838953125	1253956550	10610600	116375	-2646875	-1062075	157605	-20475	0	0	150150	0	-4410	-4560	-1800	-105	0	0	245	605	25	525	213	-7

;																											
		@	@	@	@	@	@	@	@	@	@	@	@	@	@	@	@	@	@	@	@	@	@	@	@	@	@
		80801742479451287588645	83095629624528523	139511839126	376561712757	14296150	272237831	83175342	26489012	4870492	824432	1365154	9450	774741	269007	48157	13060	16124	2786	2821	847	7927	7785	1437	235	125	30
		9904961710757005754368000000000	82355161088000000	336328171520000	1985163878400	77540249600	663616000	73309696000	826931200	9136640	3942400	560000000	0000000	019852800	2985600	9499520	6940160	31360	91840	2710400	07280	23456	67680	69600	92960	82912	96576
		p power	A	A	A	A	A	B	A	B	B	A	A	AA	BB	AB	BA	BB	CB	A	A	C	A	B	C	C	D
		p' part	A	A	A	A	A	A	A	A	A	A	A	AA	BB	AB	BA	BB	CB	A	A	A	A	A	A	A	A
ind		1A	2A	2B	3A	3B	3C	4A	4B	4C	4D	5A	5B	6A	6B	6C	6D	6E	6F	7A	7B	8A	8B	8C	8D	8E	8F
χ_{1}	+	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
χ_{2}	+	196883	4371	275	782	53	-1	275	51	19	-13	133	8	78	77	14	-3	5	-1	50	1	35	11	-1	-5	3	-1
χ_{3}	+	21296876	91884	-2324	7889	-130	248	1772	-52	-20	12	626	1	273	286	-47	30	-2	-8	153	6	92	-12	0	4	-4	0
χ_{4}	+	842609326	1139374	12974	55912	-221	-248	8878	782	-82	78	2451	-49	1000	923	104	-53	-13	8	426	-15	222	54	26	6	-2	-6
χ_{5}	+	18538750076	8507516	123004	249458	1598	248	28796	2652	380	156	6326	76	2354	1990	370	86	46	-8	798	14	364	132	0	20	12	0
χ_{6}	+	19360062527	9362495	-58305	297482	1508	-247	35903	-833	63	-65	8152	27	2378	2652	-246	-52	-12	9	1072	-6	511	-65	-25	-1	-1	7
χ_{7}	+	293553734298	53981850	98970	1055310	-3927	3876	94874	1274	-102	-454	17423	-77	5070	4785	78	129	-39	36	1700	-15	714	34	-26	-14	-22	6
χ_{8}	+	3879214937598	337044990	-690690	4751823	-4173	-3876	345598	-3874	-258	286	54473	98	12495	13299	-561	-141	51	-36	4249	35	1646	-154	26	-10	14	-6
χ_{9}	+	36173193327999	1354188159	2864511	12616074	18954	0	701823	20383	-897	351	91124	-126	20682	18954	1674	-54	-54	0	5355	-35	1807	375	27	-25	-17	27
χ_{10}	+	125510727015275	3215883115	1219435	24688454	-25375	248	1223531	19499	-661	-1365	145275	-350	33670	28105	70	217	-47	-8	7497	0	2379	219	-53	-5	11	-21
χ_{11}	+	190292345709543	2814161895	10249191	17144568	-8073	27000	734183	22151	-281	-1209	78793	168	12024	14103	2808	279	-9	120	3451	70	1079	351	-53	15	23	-21
χ_{12}	+	222879856734249	3864186921	-7196631	26057022	11310	34503	1187881	-39831	1577	1001	132374	-126	18174	24726	-1986	678	30	-57	6154	-20	1993	-583	25	25	9	-7
χ_{13}	+	1044868466775133	9223504989	-15756195	47292301	84448	31123	1762397	-46019	-2723	637	170258	133	11661	28392	-2163	888	-48	147	6272	0	1869	-507	25	21	-19	-7
χ_{14}	+	1109944460516150	9697078070	26155830	40851749	97733	61256	1379126	87126	4662	-1002	123025	525	43109	18381	3621	989	309	72	4200	35	966	846	26	-66	38	-6
χ_{15}	+	2374124840062976	22509162496	4100096	110509112	-1456	-61504	4100096	8192	4096	0	392976	476	62776	64064	56	-1376	-16	-64	14553	0	4096	0	0	0	0	0
χ_{16}	o	8980616927734375	-2720265625	39414375	1603525	200200	-116375	-9625	-20825	8295	1575	0	0	1925	0	-315	1520	-360	105	0	0	7	55	-25	-105	71	7
χ_{17}	o	8980616927734375	-2720265625	39414375	1603525	200200	-116375	-9625	-20825	8295	1575	0	0	1925	0	-315	1520	-360	105	0	0	7	55	-25	-105	71	7.
χ_{18}	+	15178147608537368	72990279960	-29873896	255751145	33254	-30628	7620888	-44200	-2280	728	620368	-132	99177	88166	-3607	-858	38	92	17850	-84	4408	-440	0	40	-8	0
χ_{19}	+	39660520552077425	128459630705	47061105	367316348	-227227	30380	9660529	82705	-3215	6097	712425	-700	112892	93093	2940	869	-219	-84	17493	0	4033	617	53	25	-95	21
χ_{20}	+	60359800576579350	143552415510	71276310	371703759	-95718	120498	9004822	98294	-4074	1078	627475	1225	50895	79002	7119	1242	90	-78	13720	0	3206	638	274	14	38	-14
χ_{21}	+	251098487132187500	336140827500	339995500	599239034	251810	-147250	11393900	512396	10860	4940	653125	0	146874	67210	12922	-870	-110	-50	10879	50	1660	2212	376	20	-36	56
χ_{22}	+	290568421805921077	444043629365	45093685	948362611	36400	-147497	20222773	-122059	4917	-7371	1269827	-798	50675	147784	3955	-2056	256	-41	23324	0	5173	-331	-299	-75	53	21
χ_{23}	+	336041615485626050	536115345090	-288233790	1087588073	-68263	61256	22648514	-153374	-2622	-4446	1397925	1050	211497	155649	-13335	2193	201	72	25704	175	4818	-1254	-26	10	50	6
χ_{24}	+	2500435234254428856	1864421481144	319494840	2635262973	691236	116622	42900152	249016	-5448	-8008	2180231	-2394	245181	206052	6909	804	-156	-114	28322	0	4536	1144	-376	120	-72	-56
χ_{25}	+	2986480825407204125	1612726090525	-385717475	1933299500	-57343	779495	28797725	-204867	7709	7293	1358500	1000	121132	119977	-6356	1657	-335	-281	15078	-161	2541	-715	377	-155	77	57
χ_{26}	o	3503434660075044981	-89143381899	-755269515	0	0	767637	184437	161109	39285	18837	-8019	4356	0	0	0	0	0	405	153	-288	-27	-99	-351	45	-123	-63
χ_{27}	o	3503434660075044981	-89143381899	-755269515	0	0	767637	184437	161109	39285	18837	-8019	4356	0	0	0	0	0	405	153	-288	-27	-99	-351	45	-123	-63
χ_{28}	+	3605718753596953125	2239938073125	284185125	2906660052	189540	0	43954725	345253	-20955	4005	2131250	0	241812	189540	5076	-540	-540	0	25227	-155	3429	837	-27	5	-27	-27
χ_{29}	+	8456836343580310400	3618137095040	753384320	3560377184	-1117324	-30628	46447488	903552	-13440	11648	1954150	-2100	487136	149084	10592	-4372	-28	92	18852	-160	1664	1792	0	0	128	0
χ_{30}	+	8754193822112578125	3831845698125	-959677875	4307266557	-917865	27000	60123725	-883507	20045	9165	2678125	0	117117	228735	-19971	1935	135	120	26452	90	4109	-2739	53	-115	13	21
χ_{31}	+	28585990950721640625	7669080680625	-804759375	6880196379	1056405	-813750	82024625	-832559	6065	-11375	3200000	0	-13221	248325	-3045	-4635	-75	330	23324	0	3521	-1495	221	25	-95	-35
χ_{32}	+	30815545786259524745	3436802874505	1656343689	2715826568	2163434	809627	25226377	437801	-63607	-22295	842745	2870	-348920	66066	18312	3106	474	-357	2058	0	985	1441	-299	-175	57	21
χ_{33}	+	31569817307122699605	8003575389525	2741767509	6723221931	-821106	-895869	76528981	1062517	-7083	15925	2943480	2730	126699	211926	26283	-2010	318	387	21301	280	2533	2925	25	-35	69	-7
χ_{34}	+	47377503606648784400	7170718631440	6518654480	3475562246	3824405	-30628	26404368	2365584	98832	45968	702525	3150	695110	26741	38918	373	1685	92	3723	195	-944	3376	624	240	208	48
χ_{35}	+	49609712911192813665	4216672206945	5047513185	731909772	6751602	116622	3899489	1536001	50017	15041	-39710	4290	437580	858	12684	7146	1506	-114	-442	-246	49	1177	677	-55	273	133
χ_{36}	+	77316619273928125000	11487099325000	5193149000	7502151085	-1595360	988000	72944200	1281800	58440	-7800	2375000	0	-63635	150800	40109	10480	320	-160	11305	-210	1448	2520	0	120	-24	0
χ_{37}	+	130415350420342968750	19432400568750	-4691783250	13174268625	1409940	-27000	125829550	-1321970	11950	30030	4156250	0	-144495	263340	-33327	-2340	-540	-120	20825	0	1758	-2442	650	70	-2	42
χ_{38}	+	155943076739182582850	26382222060610	-786277310	16869395816	1510964	-30628	160795714	44642	19266	-46046	5251600	4725	960232	308308	-15512	-428	436	-164	29547	0	594	-550	-650	-54	-78	-42
χ_{39}	o	172399434201593354756	1932029655556	531831300	-441097657	-6179173	-1560013	2288132	-476476	-9212	-22204	-61369	1631	47047	3003	10311	-5765	987	819	0	0	-28	-396	-572	404	164	-28
χ_{40}	o	172399434201593354756	1932029655556	531831300	-441097657	-6179173	-1560013	2288132	-476476	-9212	-22204	-61369	1631	47047	3003	10311	-5765	987	819	0	0	-28	-396	-572	404	164	-28
χ_{41}	o	286243267692724486144	-53936390144	7514357760	-281380736	7326592	1646008	1835008	0	0	0	-30856	11144	-11648	0	-8064	-1280	1152	-840	0	0	0	0	0	0	0	0
χ_{42}	o	286243267692724486144	-53936390144	7514357760	-281380736	7326592	1646008	1835008	0	0	0	-30856	11144	-11648	0	-8064	-1280	1152	-840	0	0	0	0	0	0	0	0
χ_{43}	+	379913824694312370176	36886987145216	-2273050624	19084103168	-3939520	926744	150732800	0	0	0	4235176	-4824	206336	223808	-20992	4160	320	-232	14008	-496	0	0	0	0	0	0
χ_{44}	o	640558364167263622626	-1925003475486	9988806114	0	-3720087	0	26082	-584766	9954	-18942	39501	13251	0	-5103	0	6561	729	0	0	0	210	-54	78	-294	-142	-210
χ_{45}	o	640558364167263622626	-1925003475486	9988806114	0	-3720087	0	26082	-584766	9954	-18942	39501	13251	0	-5103	0	6561	729	0	0	0	210	-54	78	-294	-142	-210
χ_{46}	+	643356925889917747200	53859806085120	2614886400	25227138048	-460992	3411240	191102976	0	0	0	5042200	-2800	328704	251328	21504	12480	-960	552	13720	0	0	0	0	0	0	0
χ_{47}	o	691170144025469730622	9783256633150	-5972672706	-397261865	10656646	-4127123	491326	-1969058	-53698	63518	-40128	-10003	273559	6006	11991	-5162	1830	-147	-1029	0	78	-330	650	-346	46	-182
χ_{48}	o	691170144025469730622	9783256633150	-5972672706	-397261865	10656646	-4127123	491326	-1969058	-53698	63518	-40128	-10003	273559	6006	11991	-5162	1830	-147	-1029	0	78	-330	650	-346	46	-182
χ_{49}	+	776097192277137500000	29144516700000	11163740000	10339920710	6141590	5084000	66037600	106080	-20640	36960	1312500	0	-448890	70070	28742	11190	-490	-160	-1225	0	736	1056	0	160	-32	0
χ_{50}	+	918438233727730974720	31132239790080	-17203462144	8238100992	7656000	6757008	40370176	0	0	0	852720	-280	219648	13376	-60928	17472	-448	-880	1344	560	0	0	0	0	0	0
χ_{51}	+	1201241700908448332364	19824998517324	4387289676	184148811	-13607748	2597490	4917836	-1082900	-33460	98476	-251636	7364	531531	12012	10059	1548	-2532	882	0	0	28	-220	1352	-300	252	-56
χ_{52}	+	1353006807137391674268	95287105622940	-5268307044	40568500665	2360358	-2597490	285160348	-1527876	-77156	28028	7111643	-3857	602745	318318	-4935	-4674	726	-882	17493	0	-1364	-2156	376	4	12	56
χ_{53}	o	1480279477146615234375	90807234375	-1811005625	-593746725	-4751175	-5742750	2498375	-251225	154695	-64025	0	0	-10725	-3575	475	-1815	2905	610	-50	685	119	495	-325	-65	-233	251
χ_{54}	o	1480279477146615234375	90807234375	-1811005625	-593746725	-4751175	-5742750	2498375	-251225	154695	-64025	0	0	-10725	-3575	475	-1815	2905	610	-50	685	119	495	-325	-65	-233	251
χ_{55}	o	1768130802583126953125	-11890281046875	10838953125	1253956550	10610600	116375	-2646875	-1062075	157605	-20475	0	0	150150	0	-4410	-4560	-1800	-105	0	0	245	605	25	525	213	-7

	ind	1A	2A	2B	3A
χ_1	+	1	1	1	1
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χ_{11}	+	190292345709543	2814161895	10249191	17144568
χ_{12}	+	222879856734249	3864186921	−7196631	26057022
χ_{13}	+	1044868466775133	9223504989	−15756195	47292301
χ_{14}	+	1109944460516150	9697078070	26155830	40851749
χ_{15}	+	2374124840062976	22509162496	4100096	110509112
χ_{16}	o	8980616927734375	−2720265625	39414375	1603525

$$J(\tau) = \frac{1}{q} + 196884q + 21493760q^2 + 864299970q^3 + 20245856256q^4 + \cdots$$

1

196883

$$196884 = 1 + 196883$$

McKay's equation

ind		1A	2A	2B	3A
χ_1	+	1	1	1	1
χ_2	+	196883	4371	275	782
χ_3	+	21296876	91884	−2324	7889
χ_4	+	842609326	1139374	12974	55912
χ_5	+	18538750076	8507516	123004	249458
χ_6	+	19360062527	9362495	−58305	297482
χ_7	+	293553734298	53981850	98970	1055310
χ_8	+	3879214937598	337044990	−690690	4751823
χ_9	+	36173193327999	1354188159	2864511	12616074
χ_{10}	+	125510727015275	3215883115	1219435	24688454
χ_{11}	+	190292345709543	2814161895	10249191	17144568
χ_{12}	+	222879856734249	3864186921	−7196631	26057022
χ_{13}	+	1044868466775133	9223504989	−15756195	47292301
χ_{14}	+	1109944460516150	9697078070	26155830	40851749
χ_{15}	+	2374124840062976	22509162496	4100096	110509112
χ_{16}	∘	8980616927734375	−2720265625	39414375	1603525

$$J(\tau) = \frac{1}{q} + 196884q + 21493760q^2 + 864299970q^3 + 20245856256q^4 + \cdots$$

$$\boxed{\begin{array}{r} 1 \\ 196883 \\ 21296876 \end{array}} \qquad \boxed{\begin{array}{r} 1 \\ 196883 \end{array}}$$

$$196884 = 1 + 196883$$ **McKay's equation**

$$21493760 = 1 + 196883 + 21296876$$ **Thompson's equation**

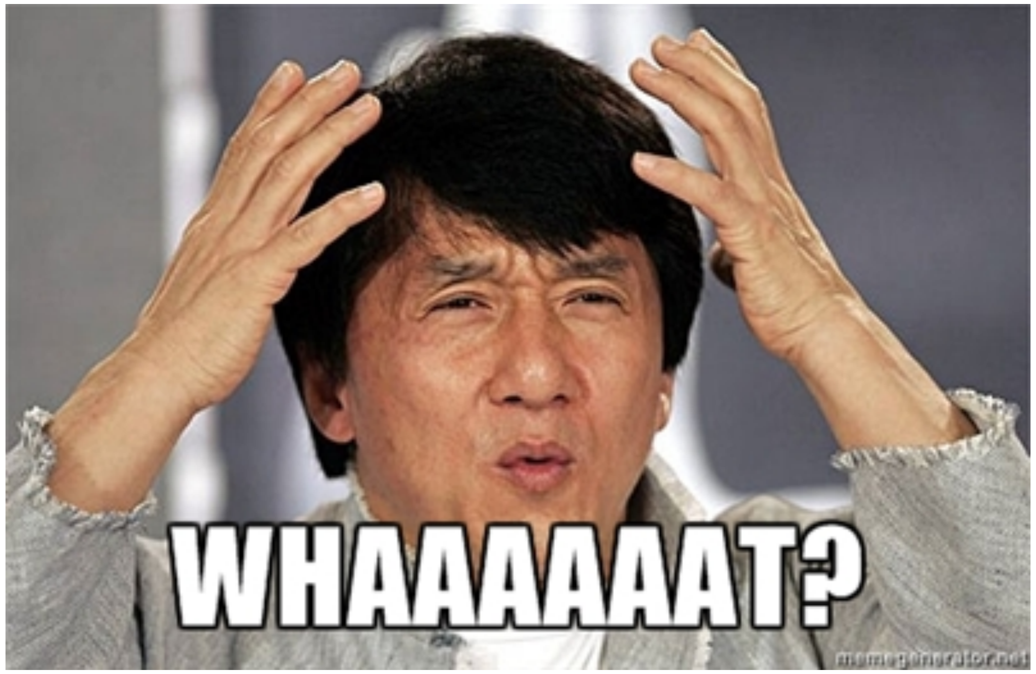
$$J(\tau) = \frac{1}{q} + 196884q + 21493760q^2 + 864299970q^3 + 20245856256q^4 + \cdots$$

$$\boxed{\begin{array}{r} 1 \\ 196883 \\ 21296876 \end{array}} \qquad \boxed{\begin{array}{r} 1 \\ 196883 \end{array}}$$

$$196884 = 1 + 196883$$ **McKay's equation**

$$21493760 = 1 + 196883 + 21296876$$ **Thompson's equation**

What does this really mean?



Fun with the monster

$$196884 = 1 + 196883 \qquad 21493760 = 1 + 196883 + 21296876$$

We can think of 196883 as the dimension of a **vector space** on which the monster acts

This is the smallest non-trivial **irreducible representation** of $\mathbb{M}$

The factor 1 then corresponds to the trivial reps.

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$$1 \longrightarrow V_1 \qquad \dim_{\mathbb{R}} V_1 = 1$$

$$196883 \longrightarrow V_2 \qquad \dim_{\mathbb{R}} V_2 = 196883$$

$$21296876 \longrightarrow V_3 \qquad \dim_{\mathbb{R}} V_3 = 21296876$$

Fun with the monster

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$$196883 \longrightarrow V_2 \qquad \dim_{\mathbb{R}} V_2 = 196883 \qquad = \mathrm{Tr}\left(e \middle| V_2\right)$$

$$21296876 \longrightarrow V_3 \qquad \dim_{\mathbb{R}} V_3 = 21296876 \qquad = \mathrm{Tr}\left(e \middle| V_3\right)$$

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$$196883 \longrightarrow V_2 \qquad \dim_{\mathbb{R}} V_2 = 196883 \qquad = \mathrm{Tr}\left(e \middle| V_2\right)$$

$$21296876 \longrightarrow V_3 \qquad \dim_{\mathbb{R}} V_3 = 21296876 \qquad = \mathrm{Tr}\left(e \middle| V_3\right)$$

characters of the identity element $e \in \mathbb{M}$

McKay and Thompson conjectured that there exists an **infinite-dimensional**, **graded monster module**, called $V^\natural$, such that

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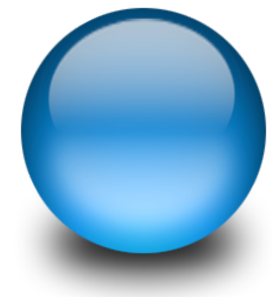
This suggests to also consider for each $g \in \mathbb{M}$ the **McKay-Thompson series**

$$T_g(q) = \sum_{n=-1}^{\infty} \mathrm{Tr}(g|V^{(n)})q^n$$

$$J(q) = \frac{1}{q} + 196884q + 21493760q^2 + 864299970q^3 + 20245856256q^4 + \cdots$$

- It generates the field of rational functions on the sphere (Hauptmodul)

$$SL(2,\mathbb{Z})\backslash\mathbb{H} \sim$$ 

$$\longrightarrow \qquad \text{any modular function} = \frac{\text{polynomial in } J(\tau)}{\text{polynomial in } J(\tau)}$$

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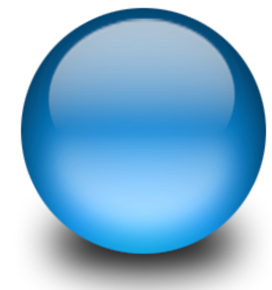
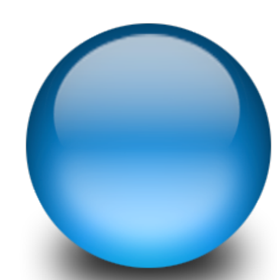
$$SL(2,\mathbb{Z})\backslash\mathbb{H} \sim$$ 

$$\longrightarrow \quad \text{any modular function} = \frac{\text{polynomial in } J(\tau)}{\text{polynomial in } J(\tau)}$$

Monstrous moonshine conjecture (Conway-Norton 1979):
For all elements $g \in \mathbb{M}$ The McKay-Thompson series $T_g(\tau)$
are hauptmoduls with respect to some genus zero $\Gamma_g \subset SL(2,\mathbb{R})$

$$T_g(q) = \sum_{n=-1}^{\infty} \mathrm{Tr}(g|V^{(n)})q^n$$

Γ_g genus zero $\longleftrightarrow$ $\Gamma_g\backslash\mathbb{H} \sim$ 

Monster group

$$\mathbb{M}$$

$$\updownarrow \; ????$$

Modular function

$$J(\tau)$$

Monster group

$\mathbb{M}$

Enter “physics”!

2d conformal field theory
(vertex operator algebra)

[Frenkel, Lepowsky, Meurman]

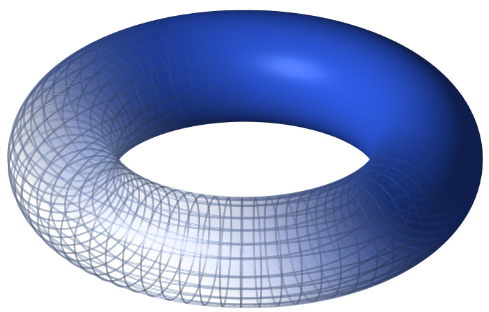
Modular function

$J(\tau)$

1988: Frenkel, Lepowsky, Meurman constructed the moonshine module $V^\natural$

It is a 2-dimensional **orbifold conformal field theory** (string theory),
a. k. a. vertex operator algebra, whose automorphism group is $\mathbb{M}$

A **quantum field theory** with **conformal symmetry defined** on a **Riemann surface**



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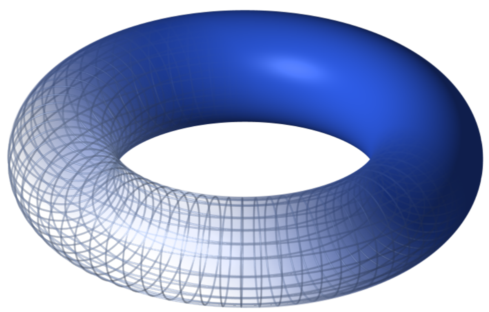

To explain moonshine we need a special class of CFT's:

holomorphic CFT = self-dual Vertex Operator Algebra (VOA)

In general a VOA describes only part of a CFT

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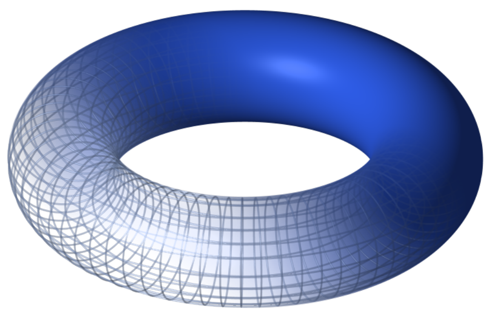

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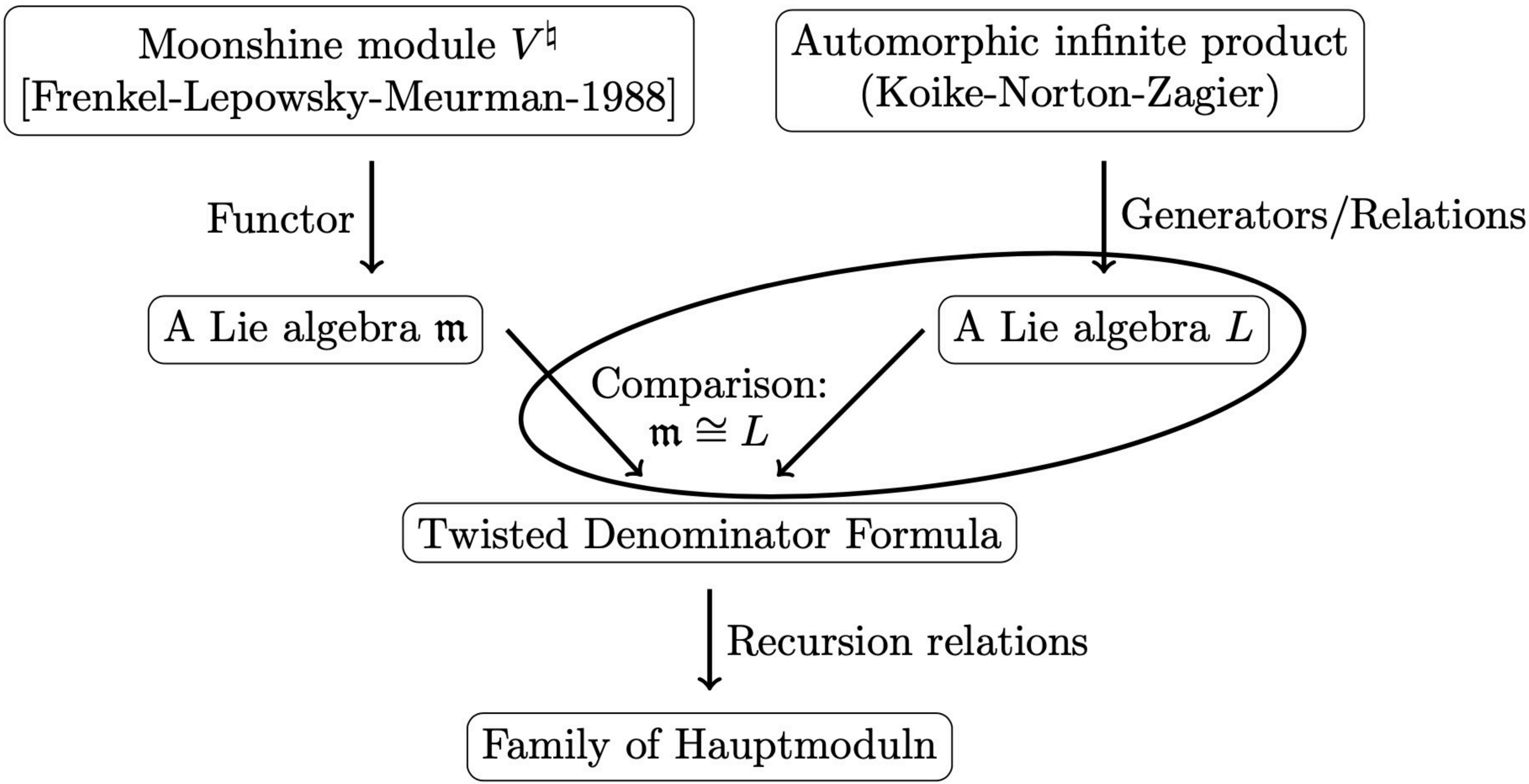
In general a VOA describes only part of a CFT

1992: Borcherds proved the full moonshine conjecture, earning him the Fields medal

[Tuite][Dong-Mason][Cummins-Gannon][Lian-Zuckerman][Jurisich]…

$$J(\sigma) - J(\tau) = p^{-1} \prod_{m>0, n\in\mathbb{Z}} (1 - p^m q^n)^{c(mn)}$$



[Figure from Carnahan]

Do we understand everything now?

- **Why genus zero?** Proven by Borcherds but no understanding of its origin.

2016: Explanation proposed by **Paquette-D.P.-Volpato** using T-duality in string theory.
(part of which is a mathematical proof using VOAs)

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The moonshine module $V^\natural$ is unique.

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- **Generalized moonshine conjecture by Norton in 1987:**

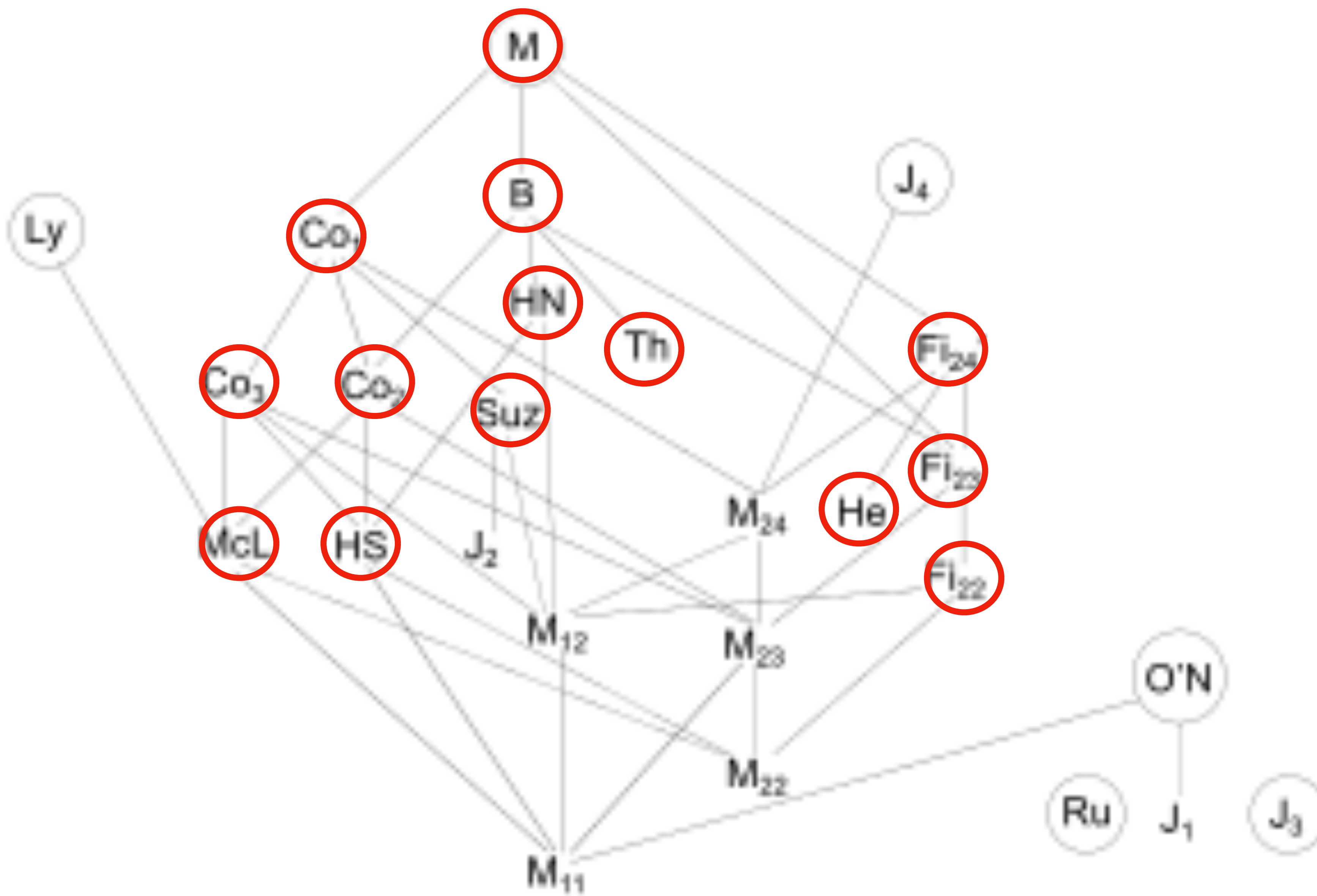
Moonshine for orbifolds of $V^\natural$

Given as a PhD project by Borcherds to his student **Carnahan** around 2005.

Complete proof announced by Carnahan in 2017.

$$T_{g,h}(\tau)$$
$$gh = hg$$

Generalized moonshine includes several more of the sporadic groups occurring as **centralizers of elements** in the monster.

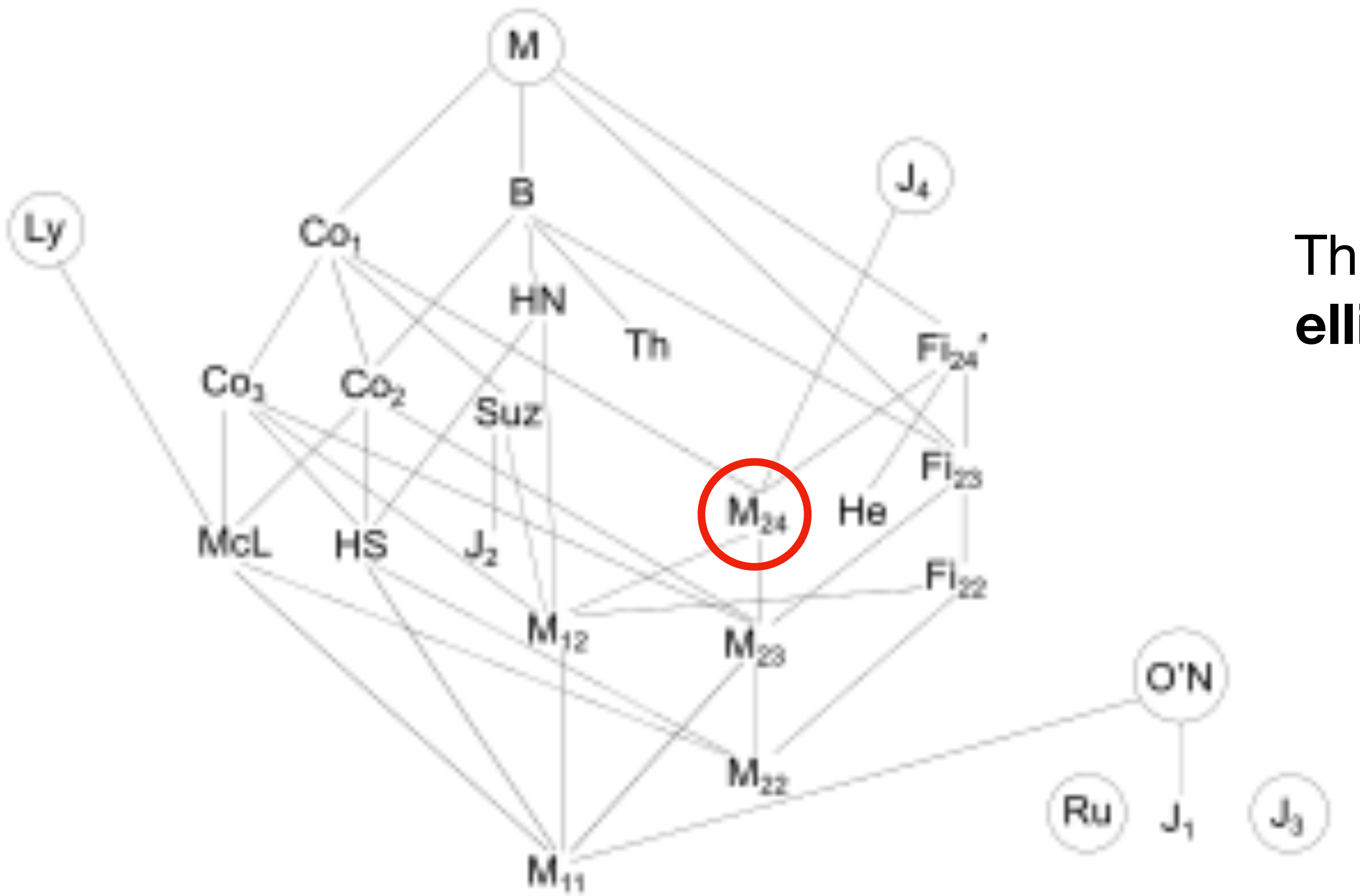


Frenkel-Lepowsky-Meurman: The sporadic groups exist because of moonshine.


Enter new moonshines!

“Moonshine revolution”

2010: New moonshine for M_{24} conjectured by Eguchi, Ooguri, Tachikawa



The role of the J-function is now played by the **elliptic genus** of string theory on a **K3-surface**

(weak Jacobi form)

The elliptic genus is a **topological invariant** of a K3-surface:

$$\phi_{K3}(\tau, z) = \int_X \mathrm{Td}(X)\, \mathrm{ch}(\mathbb{E}_{q,-y})$$

- $\phi_{K3}(\tau, z=0) = \chi(K3) = 24$ (Euler number)
- **Supersymmetric index** of **string theory on K3**
- It is a weak Jacobi form of weight 0, index 1

$$\phi_{K3} = \phi_{0,1} = 8\left(\frac{\vartheta_2(\tau,z)^2}{\vartheta_2(\tau,0)^2} + \frac{\vartheta_3(\tau,z)^2}{\vartheta_3(\tau,0)^2} + \frac{\vartheta_4(\tau,z)^2}{\vartheta_4(\tau,0)^2}\right)$$

[Eichler, Zagier][Eguchi, Ooguri, Taormina, Yang]

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2012: Proven by Gannon

But no analogue of the monster module is constructed!

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$$\phi_{g,h}(\tau, z)$$

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But why??

There is no string theory on K3 with Mathieu symmetry

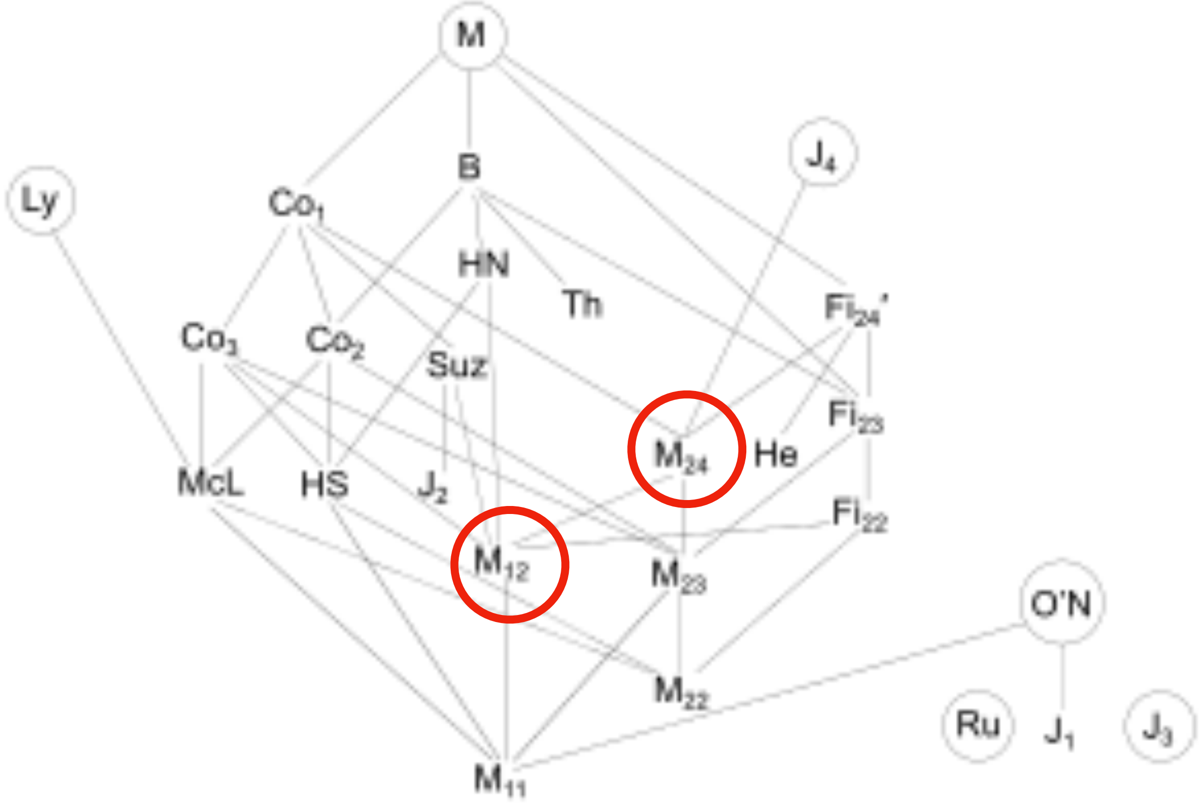
(Gaberdiel-Hohenegger-Volpato)

Also a mathematical formulation of no-go theorem by **Huybrechts** using autoequivalences of derived categories of coherent sheaves on K3-surfaces.

But it doesn’t stop there!

2012: Umbral moonshine proposed by **Cheng-Duncan-Harvey**

A class of 23 moonshines, including Mathieu moonshine.
Classified by Niemeier lattices. Involves also non-sporadic groups!



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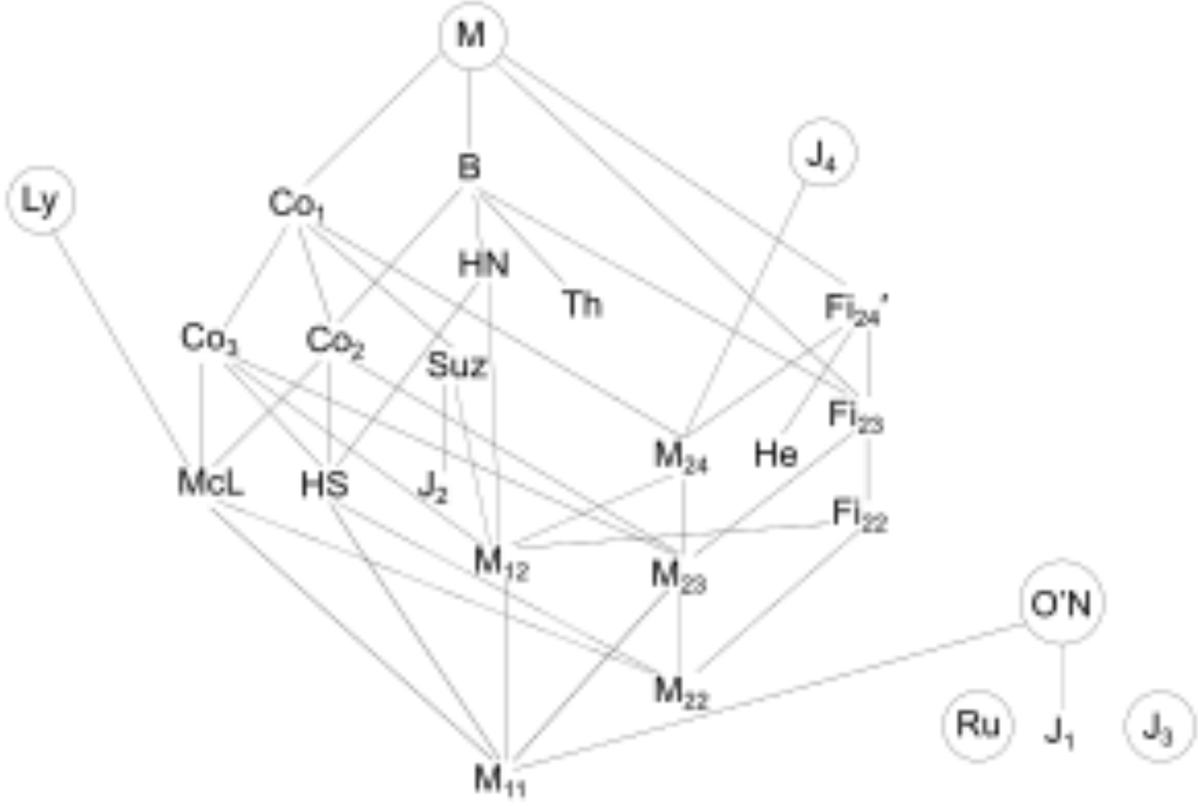

2015: Umbral moonshine conjecture proven by **Duncan-Griffin-Ono**

2016: Generalized Umbral moonshine established by **Cheng-de Lange-Whalen**

Naturally formulated using **Jacobi forms** or **mock modular forms**

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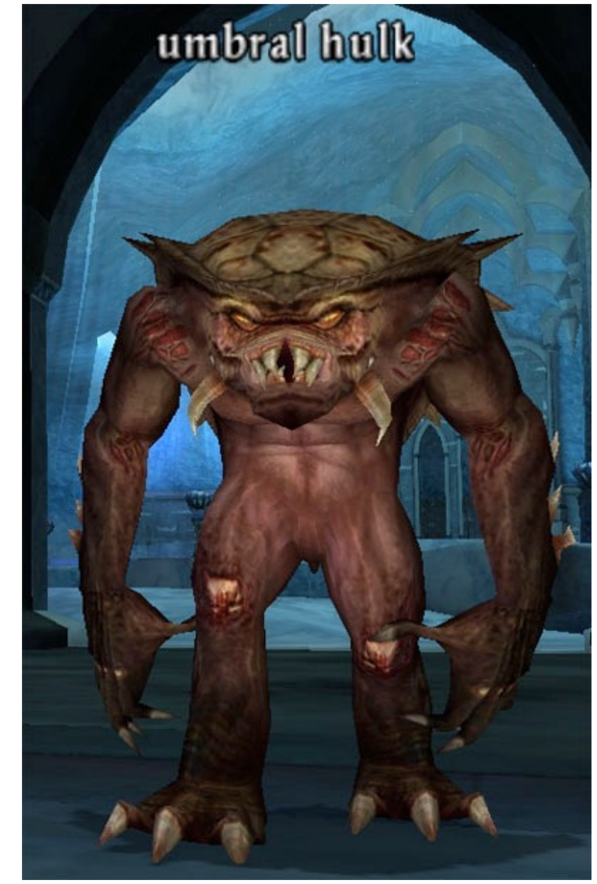
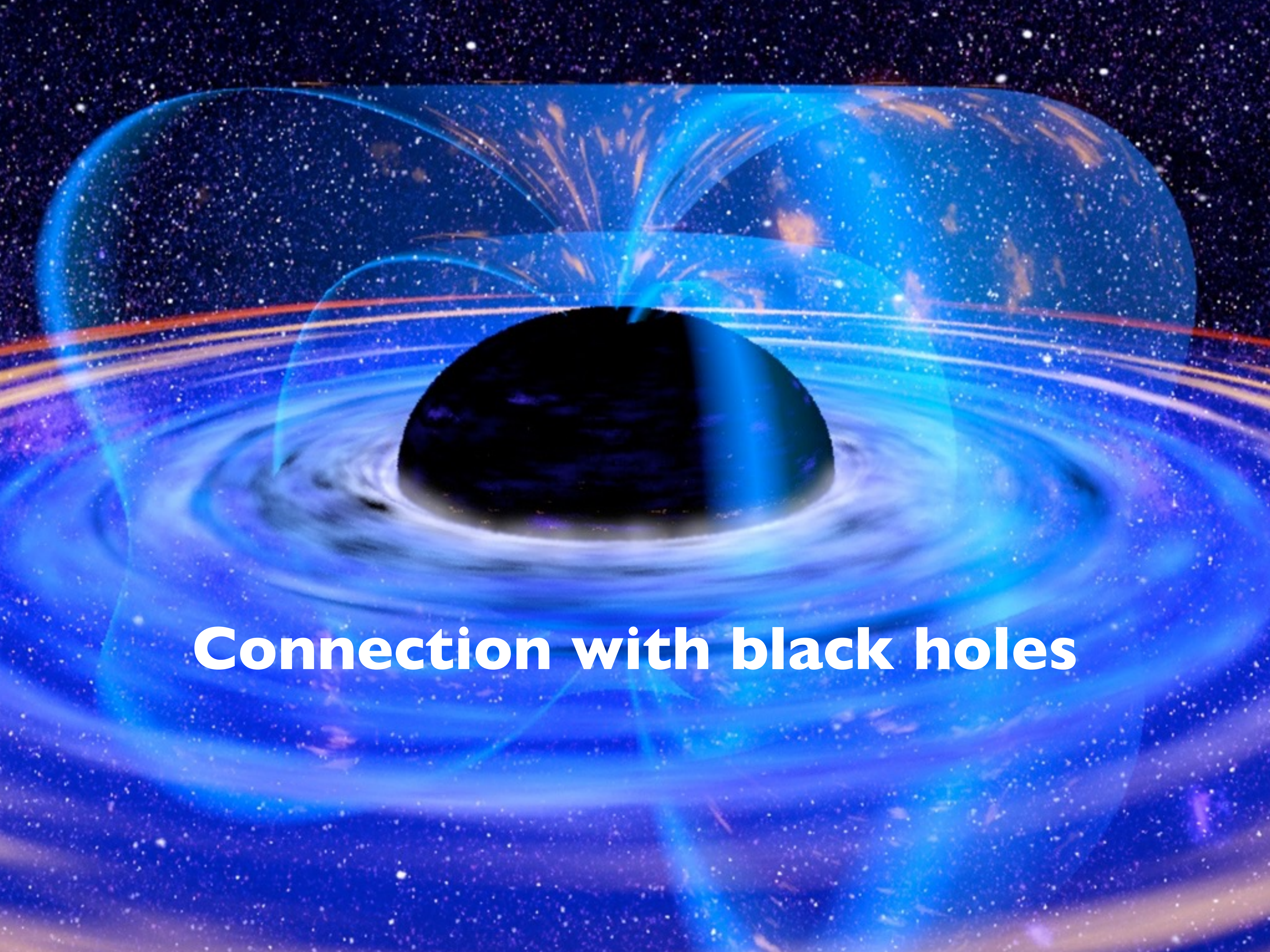


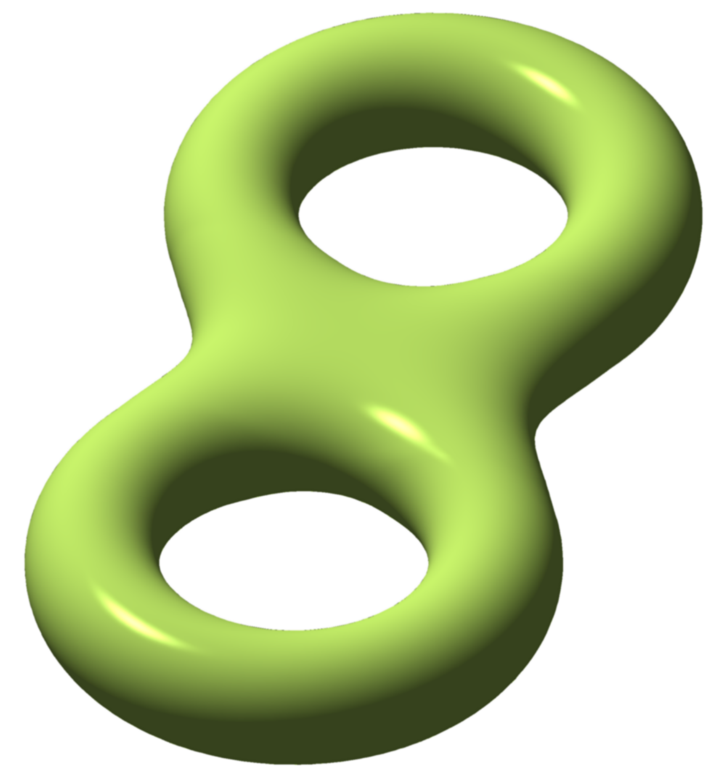
- **Thompson moonshine** weight 1/2 mock modular forms
  [Harvey, Rayhaun]
- **Pariah moonshine** weight 3/2 mock modular forms
  [Duncan, Mertens, Ono]

no genus zero property


Connection with black holes

With Volpato we defined a class of functions on the Siegel upper half plane:

$$\Phi_{g,h} : \mathbb{H}^{(2)} \to \mathbb{C}$$



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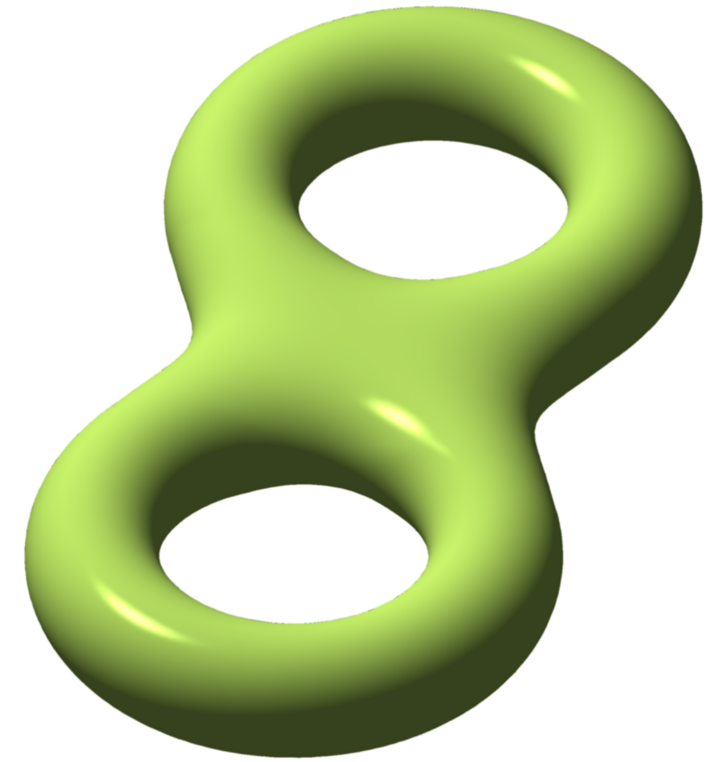

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Borchers lift: $\phi_{g,h} \longrightarrow \Phi_{g,h}$

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Theorem (D.P.-Volpato):

The functions $\Phi_{g,h}$ *are Siegel modular forms for certain subgroups* $\Gamma^{(2)}_{g,h} \subset Sp(4,\mathbb{R})$

(this generalises earlier works by [Cheng][Westerholt-Raum])

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What is the interpretation of these Siegel modular forms?

For $(g,h)=(1,1)$ this becomes $q=e^{2\pi i\tau}, y=e^{2\pi iz}$

$$\Phi_{1,1}=\Phi_{10}=pqy\prod_{(m,n,\ell)>0}(1-p^m q^n y^\ell)^{c(mn,\ell)}$$

Igusa cusp form of weight 10

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$$\phi(\tau,z)=\sum_{n\geq 0}\sum_{\ell\in\mathbb{Z}}c(n,\ell)q^n y^\ell$$ K3 elliptic genus

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The 'Igusa cusp form' Φ_{10} is the generating function of 1/4 BPS-states in N=4 string theory:

[Dijkgraaf, Verlinde, Verlinde][Shih, Strominger, Yin]

$$\frac{1}{\Phi_{10}}=\sum_{Q^2/2,P^2/2,P\cdot Q\in\mathbb{Z}} B_6(P,Q)p^{Q^2/2}q^{P^2/2}y^{P\cdot Q}$$

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electric-magnetic charges

$$(Q,P)\in\Gamma^{6,22}\oplus\Gamma^{6,22}$$

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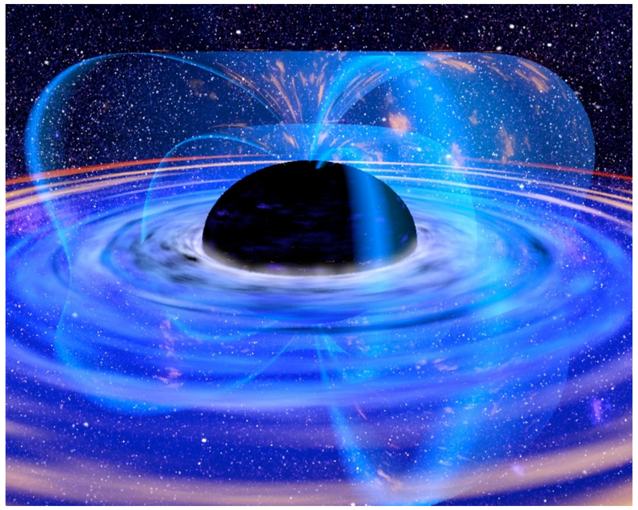
where the coefficients count certain microstates of black holes.




S = k. log W
LVDWIG
BOLTZMANN
1844 – 1906
Dr PHIL. PAULA
BOLTZMANN
GEB. CHIARI
1891 – 1977

Conjecture (Cheng, Govindarajan, DP-Volpato):

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The curious modular property $N = \mathcal{O}(g)$

$$\Phi_{g,h}(\sigma,\tau,z) = \Phi_{g,h'}\left(\tau/N, N\sigma, z\right)$$

then suggests a new '**electric-magnetic duality**' in CHL-models:

$$\begin{pmatrix} Q \\ P \end{pmatrix} \longmapsto \begin{pmatrix} \frac{1}{\sqrt{N}}P \\ -\sqrt{N}Q \end{pmatrix}$$

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This is a consequence of a novel **Fricke S-duality** in CHL-models!

[DP, Volpato]

[Jatkar, Sen][Sen][Shih, Strominger, Yin][Cheng, Verlinde][Dabholkar, Nampuri][Govindarajan]
[Bossard, Cosnier-Horeau, Pioline]

Monstrous CHL-models

(w/ Paquette & Volpato)

Monstrous CHL-Models

CHL-orbifold of: $\mathrm{Het}/(V^{\natural} \times \overline{V}^{s\natural} \times T^2)$

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moonshine module

$c = 24$

Duncan's super moonshine module for the Conway group

$c = 12$

These are supersymmetric models in 2 spacetime dimensions.

Space of BPS-states $\mathcal{H}_{BPS}$

(correspond to fermionic RR-states)

Supersymmetric indices and genus zero

We can define a spacetime **index** which **counts these BPS-states**

$$Z(T,U) := \mathrm{Tr}_{\mathcal{H}_{BPS}}\big((-1)^F e^{2\pi i T W} e^{2\pi i U M}\big)$$

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- $(T,U) \in \mathbb{H} \times \mathbb{H}$ are **Kähler** and **complex str.** moduli of T^2
- F is the **fermion number**
- (W,M) represent **winding** and **momenta** along $S^1 \subset T^2$

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We evaluate this and find

$$Z(T,U) = \Big(J(T) - J(U)\Big)^{24}$$

$$J(\tau) = \sum_{n=-1}^{\infty} c(n) q^n = q^{-1} + 196884q + 21493760q^2 + \cdots$$

In fact, for each commuting pair $(g, h) \in \mathbb{M} \times \mathbb{M}$ we define:

$$Z_{g,h}(T, U) := \mathrm{Tr}_{\mathcal{H}^g_{BPS}}\left(h(-1)^F e^{2\pi i T W} e^{2\pi i U M}\right)$$

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In particular, we find for the purely ‘twisted’ case:

$$Z_{g,\mathbb{1}}(T,U) = \left(e^{-2\pi i T} \prod_{\substack{n>0 \\ m\in\mathbb{Z}}} (1 - e^{2\pi i U \frac{m}{N}} e^{2\pi i T n})^{\hat{c}_{n,m}(\frac{mn}{N})}\right)^{24}$$

$$= \Big(T_{\mathbb{1},g}(T) - T_{g,\mathbb{1}}(U)\Big)^{24}$$

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Thus, we recover all **product formulas** for McKay-Thompson series!

$$Z_{g,\mathbb{1}}(T,U) = \Big(T_{\mathbb{1},g}(T) - T_{g,\mathbb{1}}(U)\Big)^{24}.$$

This yields a new **spacetime interpretation** of moonshine!

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Spacetime Fricke T-duality $\qquad N = \mathrm{ord}(N)$

$$T \to -\frac{1}{NT} \qquad\qquad U \to -\frac{1}{NU}$$

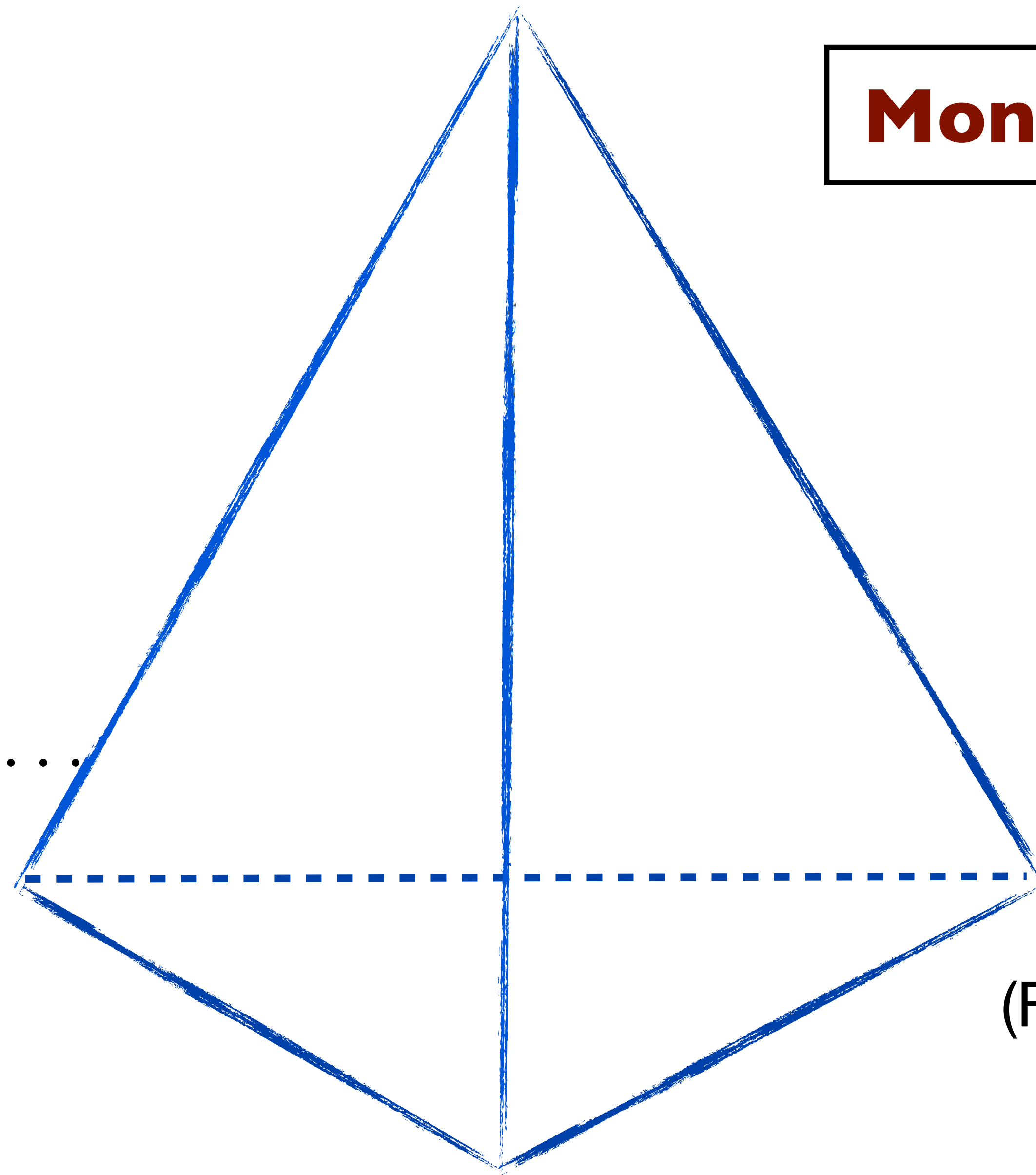
then yields a **physical explanation** for the elusive **genus zero property** of moonshine!

So what is moonshine, really?

Roles of vertex operator algebras/CFT?

Monstrous Moonshine

$\mathbb{M}$



$J(\tau) = q^{-1} + 196884q + \cdots$

modular function

holomorphic VOA $V^\natural$

(Frenkel-Lepowsky-Meurman monster module)

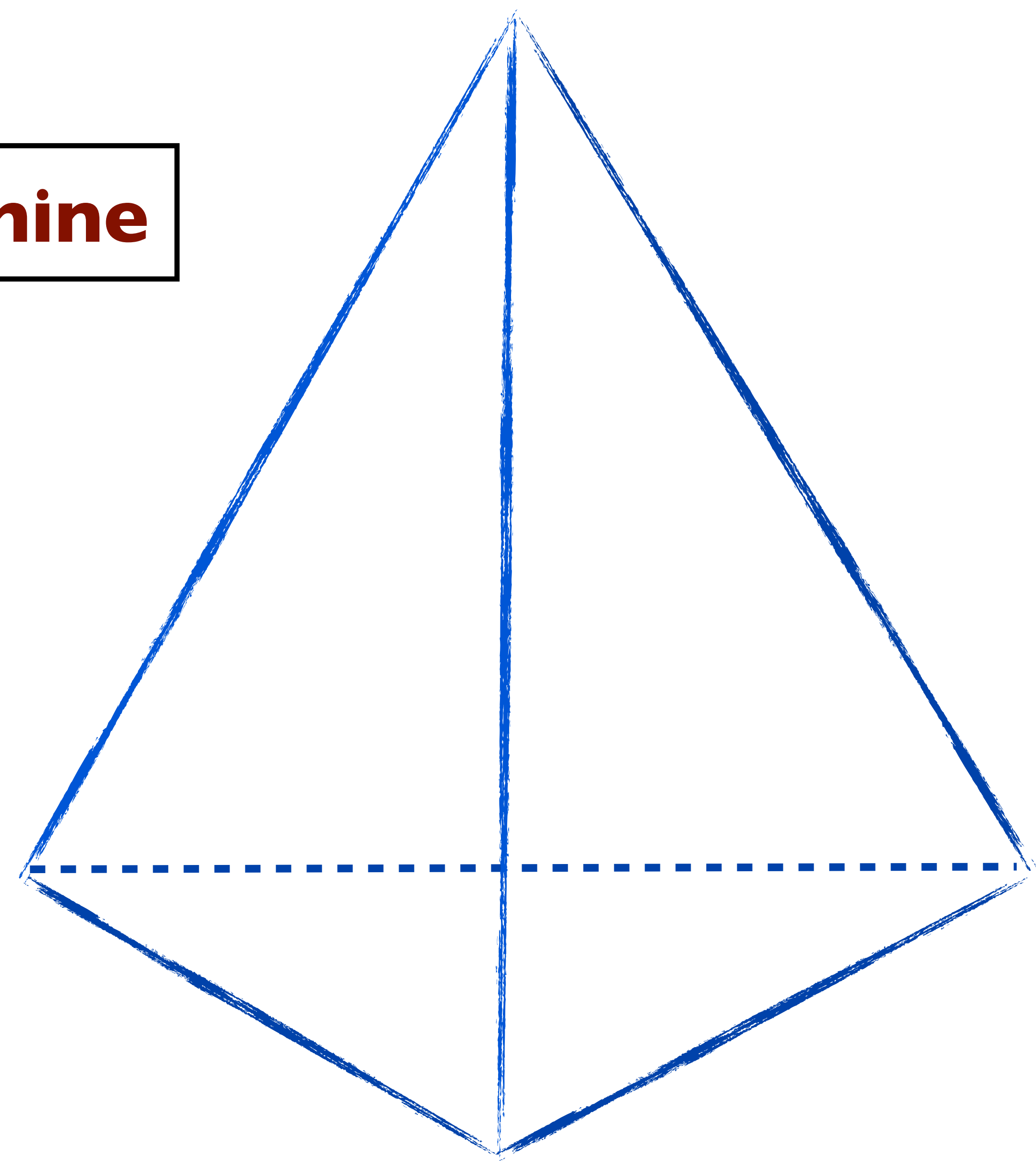
monster Lie algebra $\mathfrak{m}$

(Borcherds algebra)

Roles of vertex operator algebras/CFT?

Mathieu Moonshine

M_{24}



Jacobi forms/
mock modular forms

string theory on K3?
VOA?
Black holes?

chiral de Rham complex?

(Song)
(Wendland)

Borcherds Lie algebra…???


Thank you!

Theorem (D.P., Volpato):

The second quantized twisted twining genera satisfy the following properties

- *Infinite product formula*

$$\frac{1}{\Psi_{g,h}(\sigma,\tau,z)} = \prod_{d=1}^{\infty}\prod_{m=0}^{\infty}\prod_{\ell\in\mathbb{Z}}\prod_{t=0}^{M-1}(1-e^{\frac{2\pi i t}{M}}q^{\frac{m}{N\lambda}}y^{\ell}p^{d})^{\hat{c}_{g,h}(d,m,\ell,t)}$$

- *The ratio*

$$\Phi_{g,h}(\sigma,\tau,z) := \frac{A_{g,h}(\sigma,\tau,z)}{\Psi_{g,h}(\sigma,\tau,z)}$$

"Hodge anomaly"

$$\boxed{A_{g,h} = -p\frac{\vartheta_1(\tau,z)^2}{\eta(\tau)^6}\eta_{g,h}(\tau)}$$

Mason's generalized eta-products

is a Siegel modular form *for a subgroup* $\Gamma^{(2)}_{g,h} \subset Sp(4;\mathbb{R})$

For $g = e$ *this was conjectured by Cheng and proven by Raum.*

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The monster Lie algebra is a BPS-algebra!

This resolves a long-standing question, going back to Harvey and Moore.

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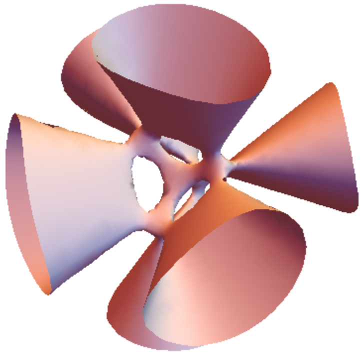
- denominator formula for a class of generalized Kac-Moody algebras
  [Cheng et al.]

- partition function of black holes in string theory
  [Sen et al.]



- generating function of equivariant Donaldson-Thomas invariants of K3-surfaces
  [Oberdieck et al.]