Emergent geometry and quantum gravity

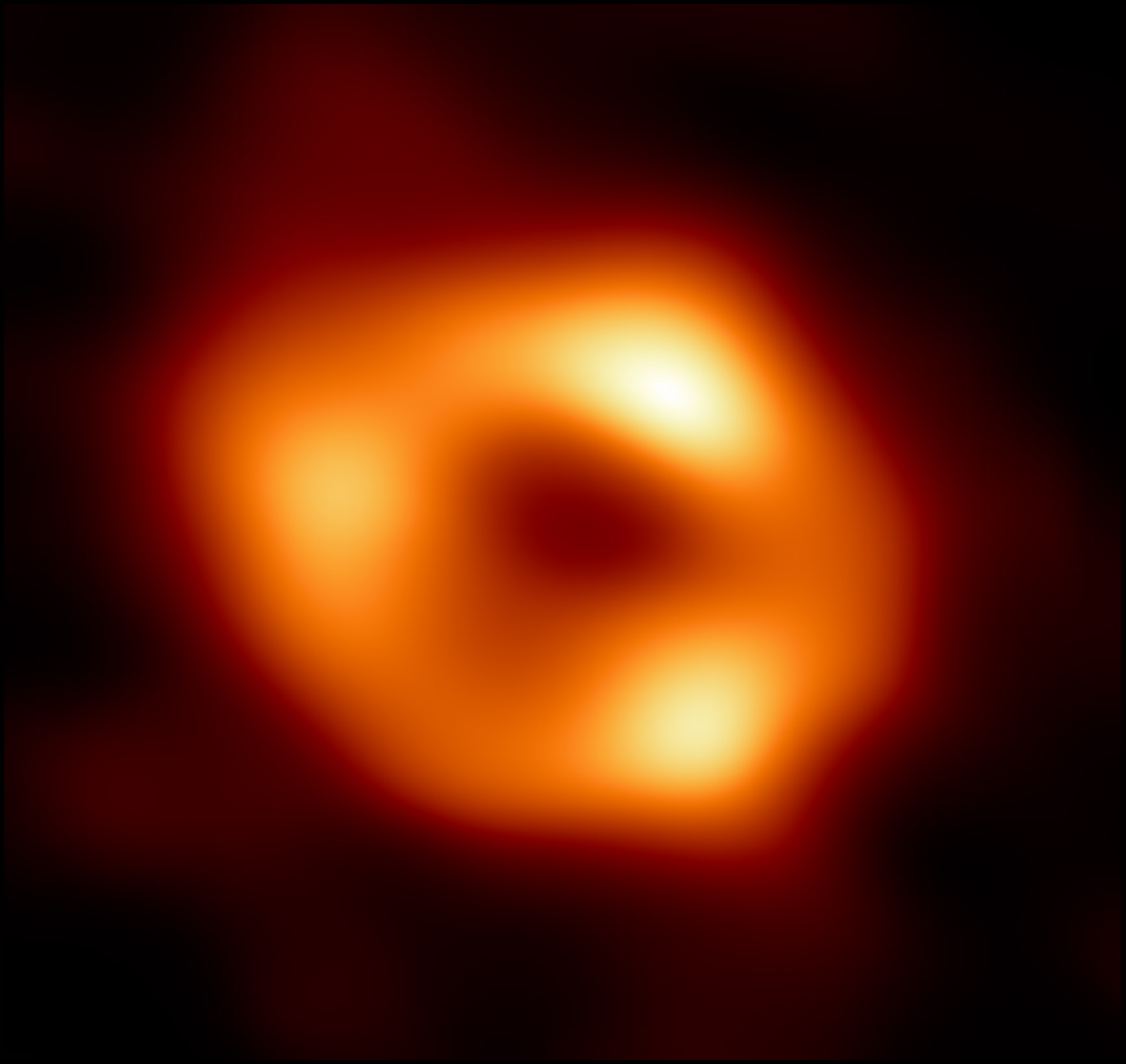
$$\Psi_{\det}(y_1,\ldots,y_{N_k}) := \sum_{\sigma\in S_{N_k}} (-1)^{|\sigma|}\psi_{\sigma(1)}(y_1)\cdots\psi_{\sigma(N_k)}(y_{N_k}).$$

$$\mathrm{dP}_{N,\beta} := \frac{1}{Z_{N,\beta}}\left|\Psi_{\det}(y_1,y_2,\ldots,y_N)^2\right|^{\frac{3\beta}{\lambda_k}}\mathrm{d}V_0^{\otimes N},$$

$$Z_{N,\beta} = C_N\int_{(S^2)^N}\prod_{1\leq i\neq j\leq N}\|x_i-x_j\|_{\mathbb{R}^3}^{\frac{2\beta}{N-1}}\,\mathrm{d}A^{\otimes N},$$

Daniel Persson
Department of Mathematical Sciences
Chalmers University of Technology
University of Gothenburg

Colloquium
May 16, 2022



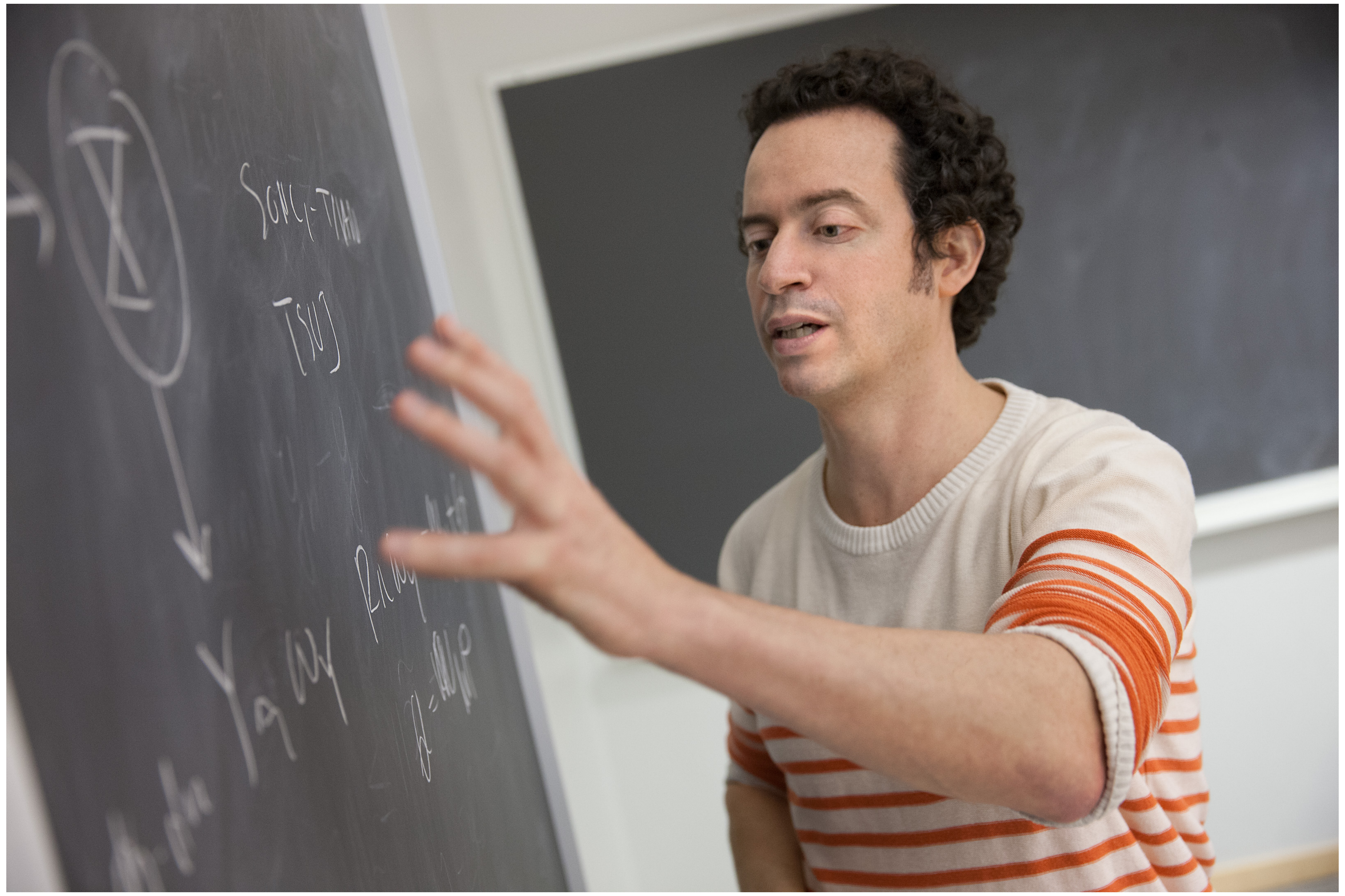
The talk is based on the paper

“Emergent Sasaki-Einstein Geometry and AdS/CFT”

Robert Berman, Tristan Collins, D.P.

Nature Communications, vol **13**, article number 365 (2022)

(and work in progress)



Phys-i-cal Math-e-ma-tics, n.

Pronunciation: Brit. /ˈfɪzᵻkl ˌmaθ(ə)ˈmatɪks / , U.S. /ˈfɪzək(ə)l ˌmæθ(ə)ˈmædɪks/

Physical mathematics is a fusion of mathematical and physical ideas, motivated by the dual, but equally central, goals of

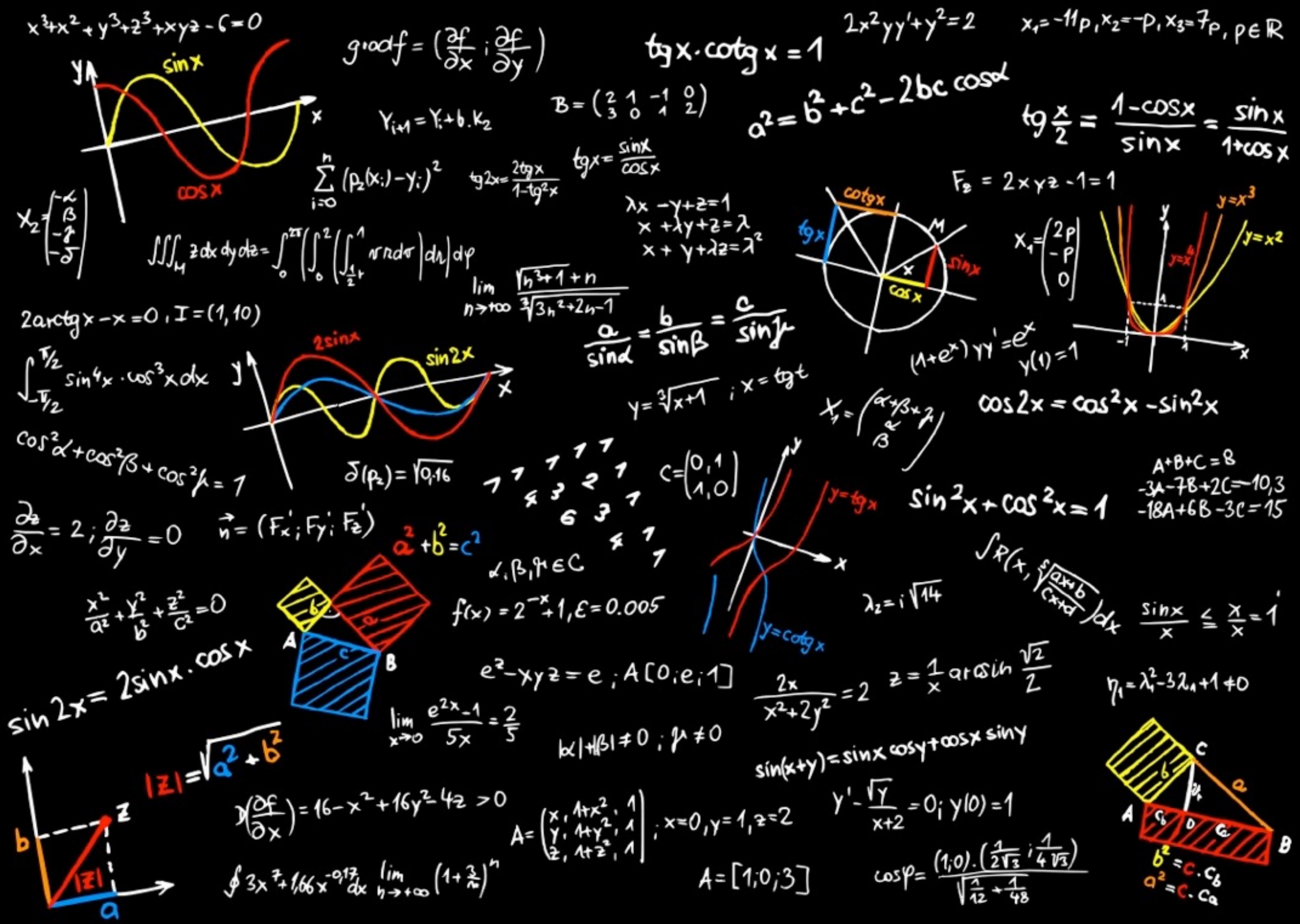
1. Elucidating the laws of nature at their most fundamental level,

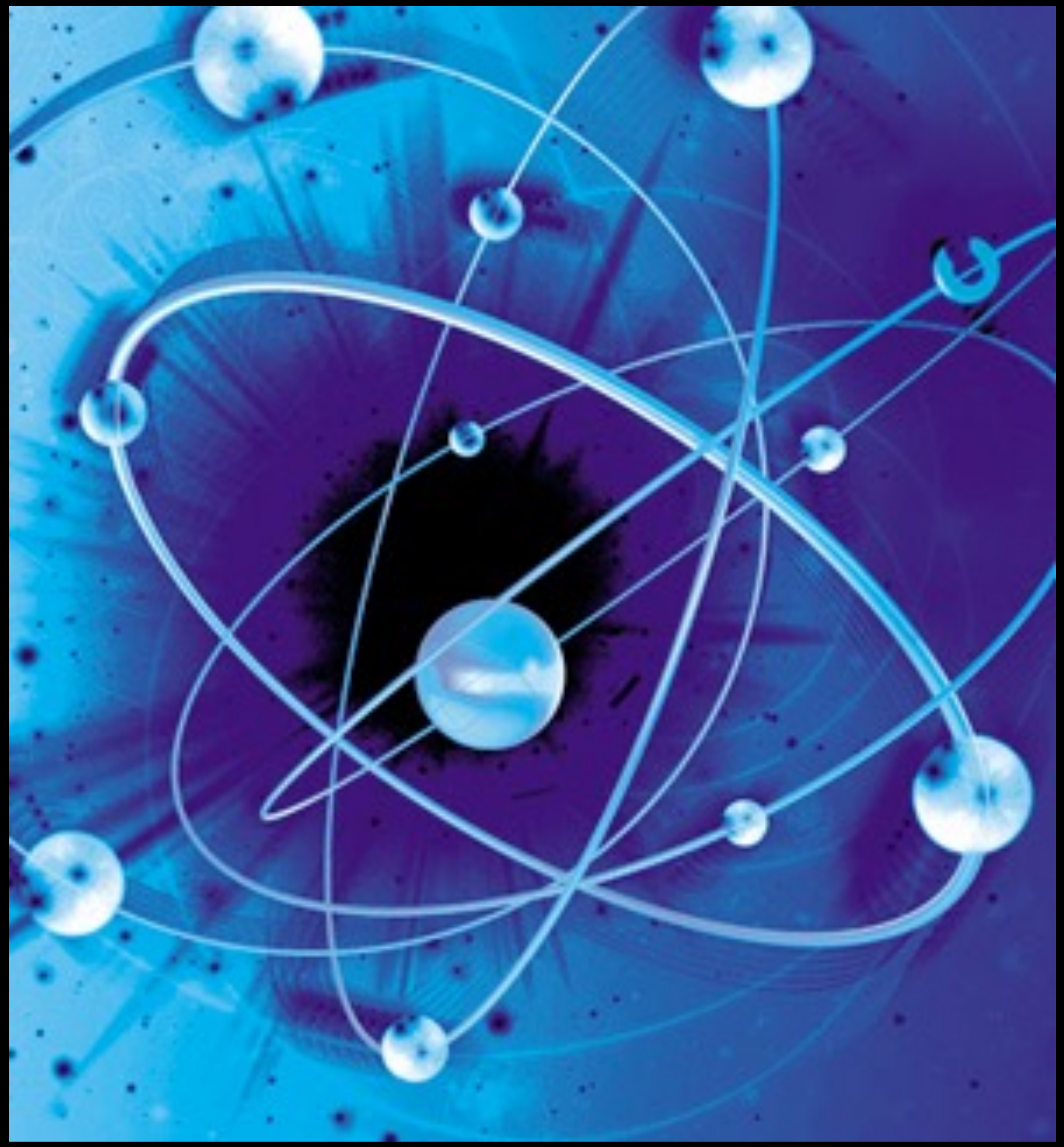
together with

2. Discovering deep mathematical truths.

[Greg Moore]

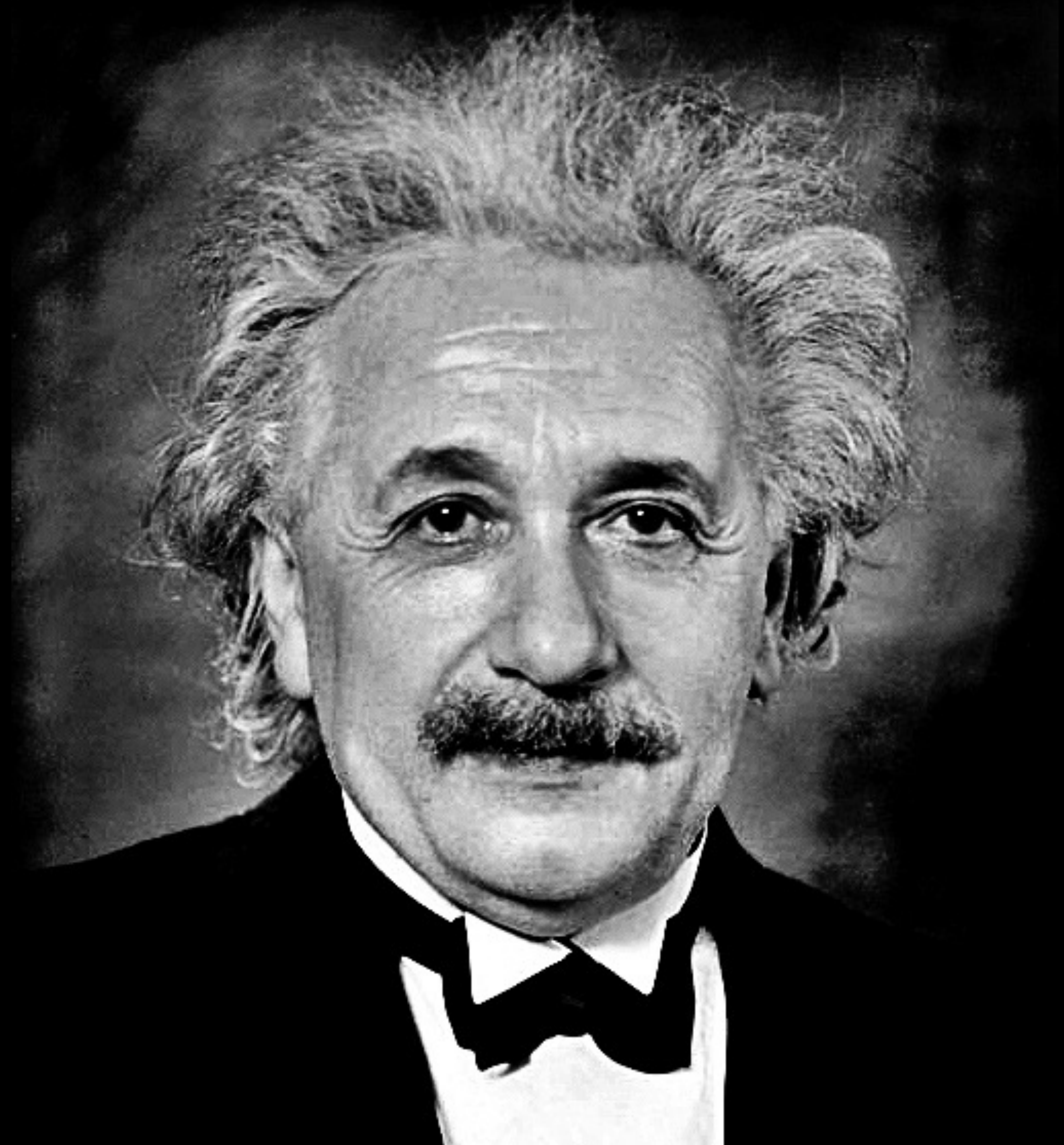
When we ask deep questions about Nature we are led deep into the world of mathematics





How can it be that mathematics, being after all a product of human thought which is independent of experience, is so admirably appropriate to the objects of reality?

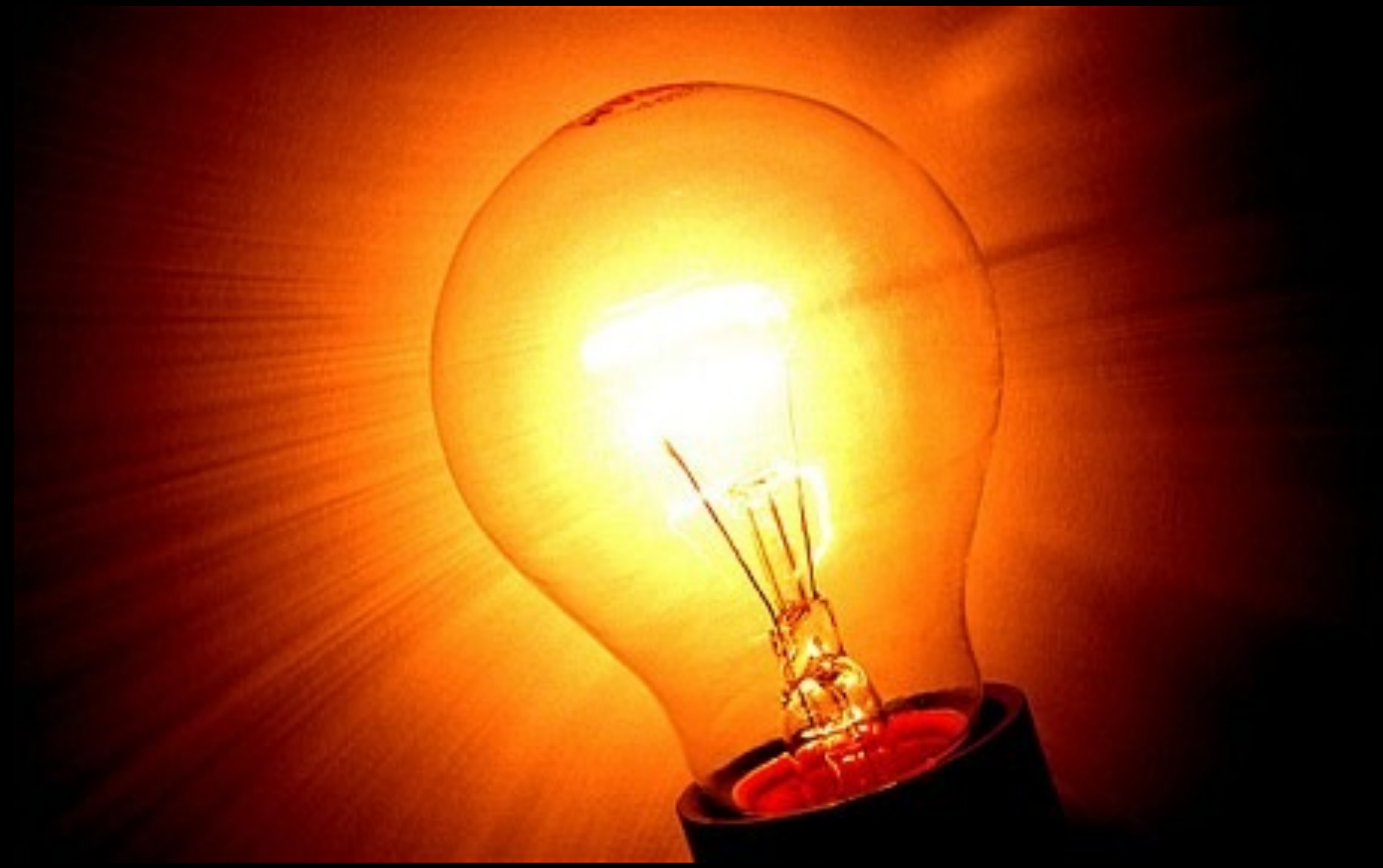
- A. Einstein



The fundamental laws of Nature are determined by symmetries!

strong nuclear force

gravity



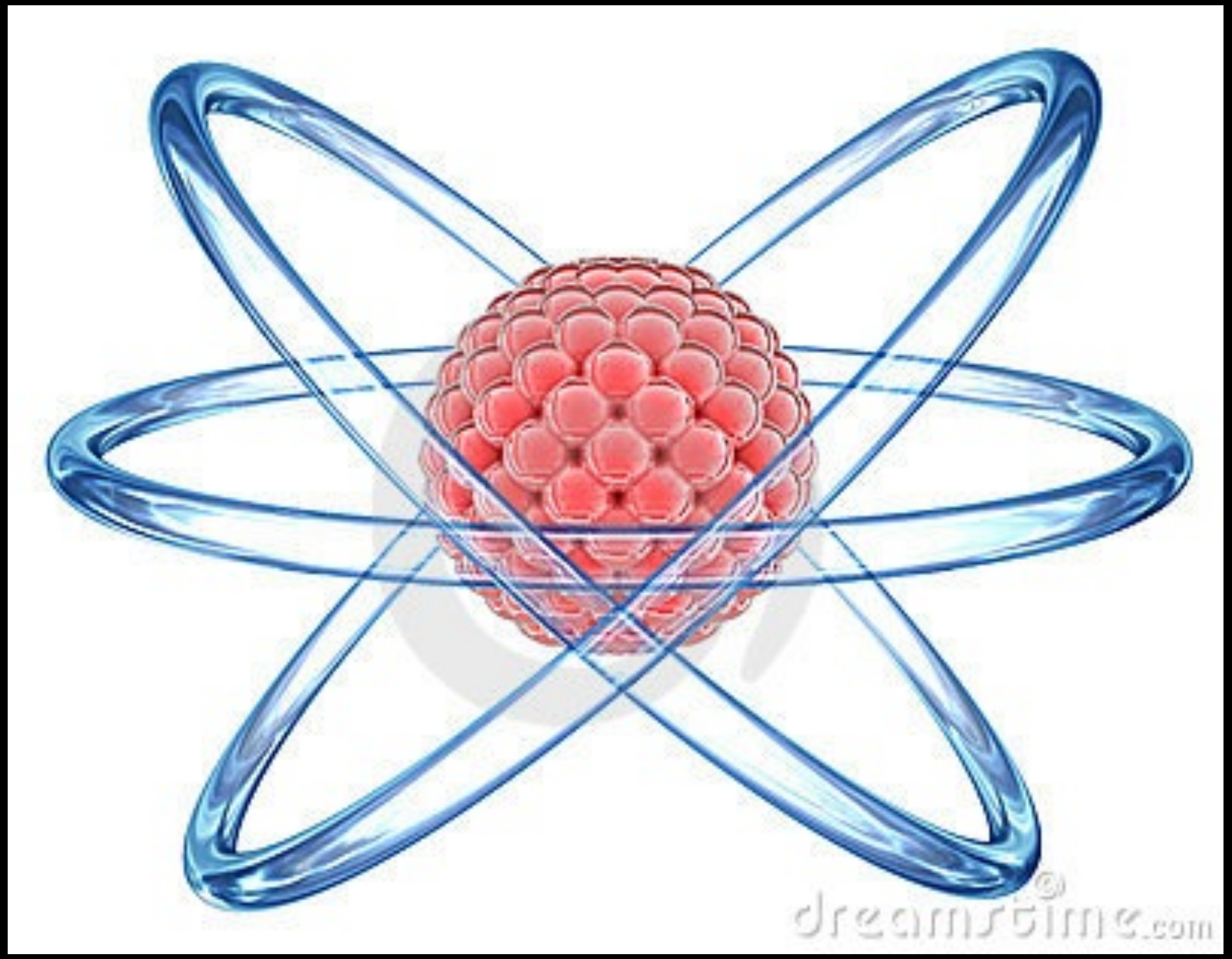
electromagnetism

weak nuclear force

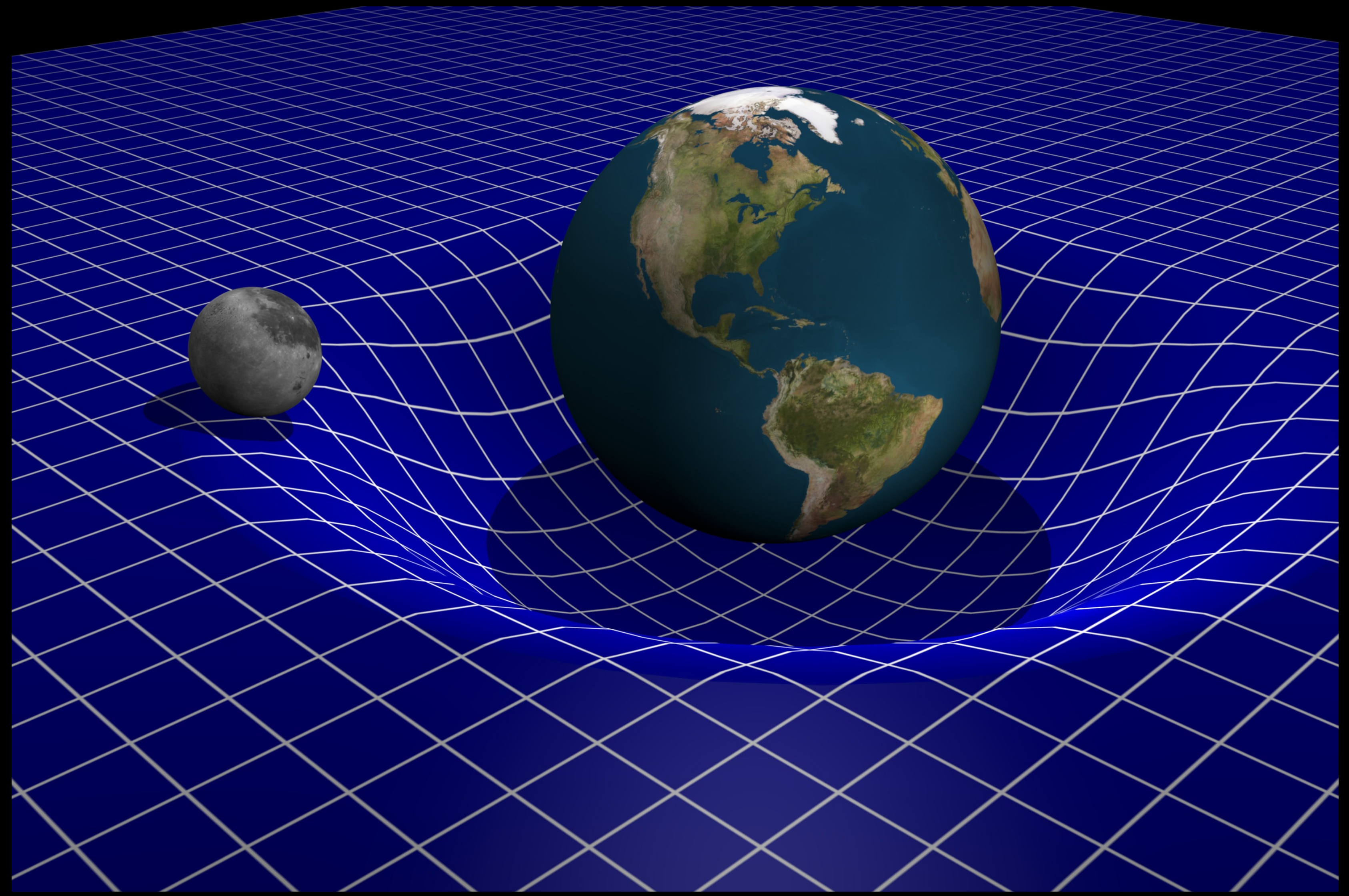




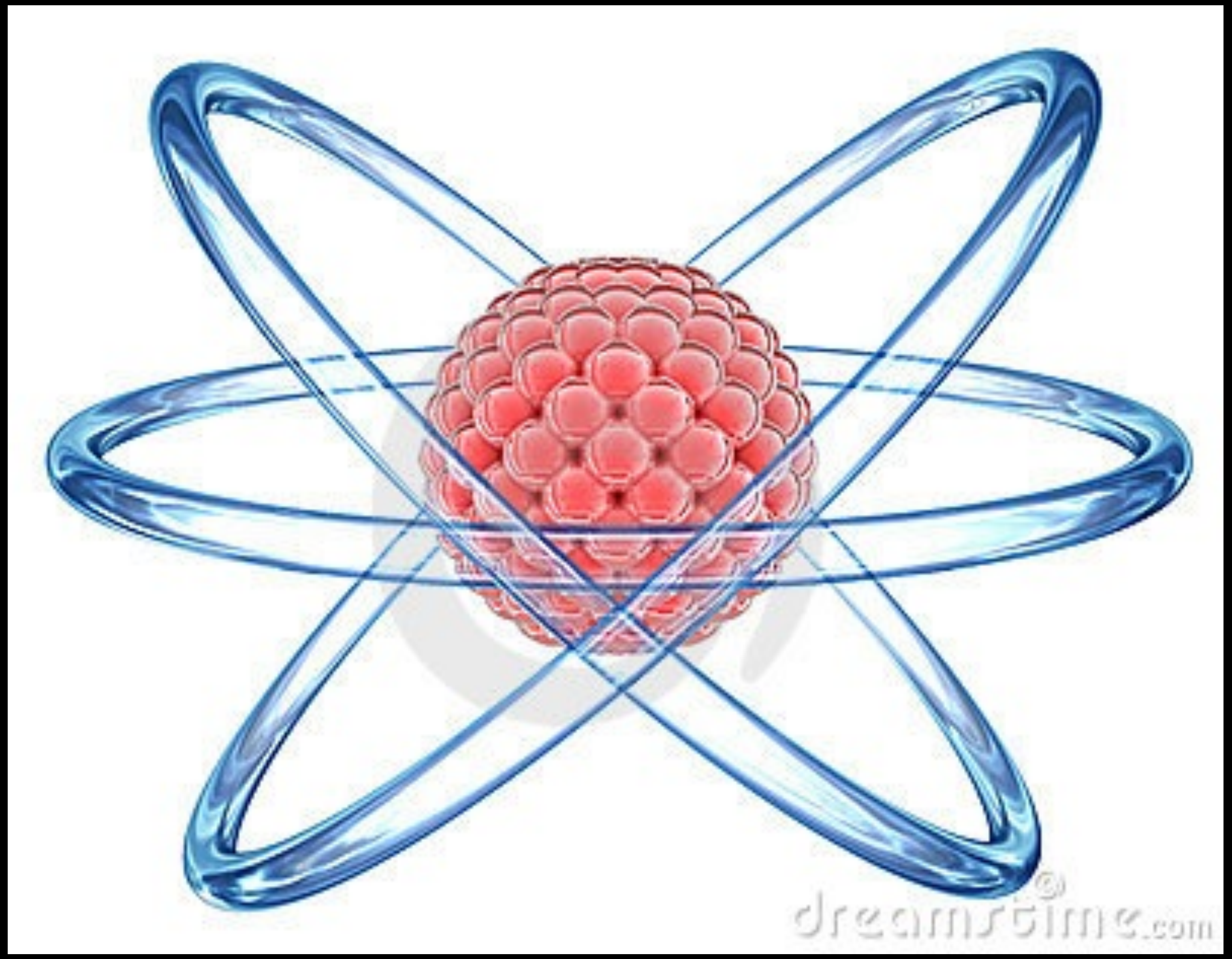
PlONK
©Lana



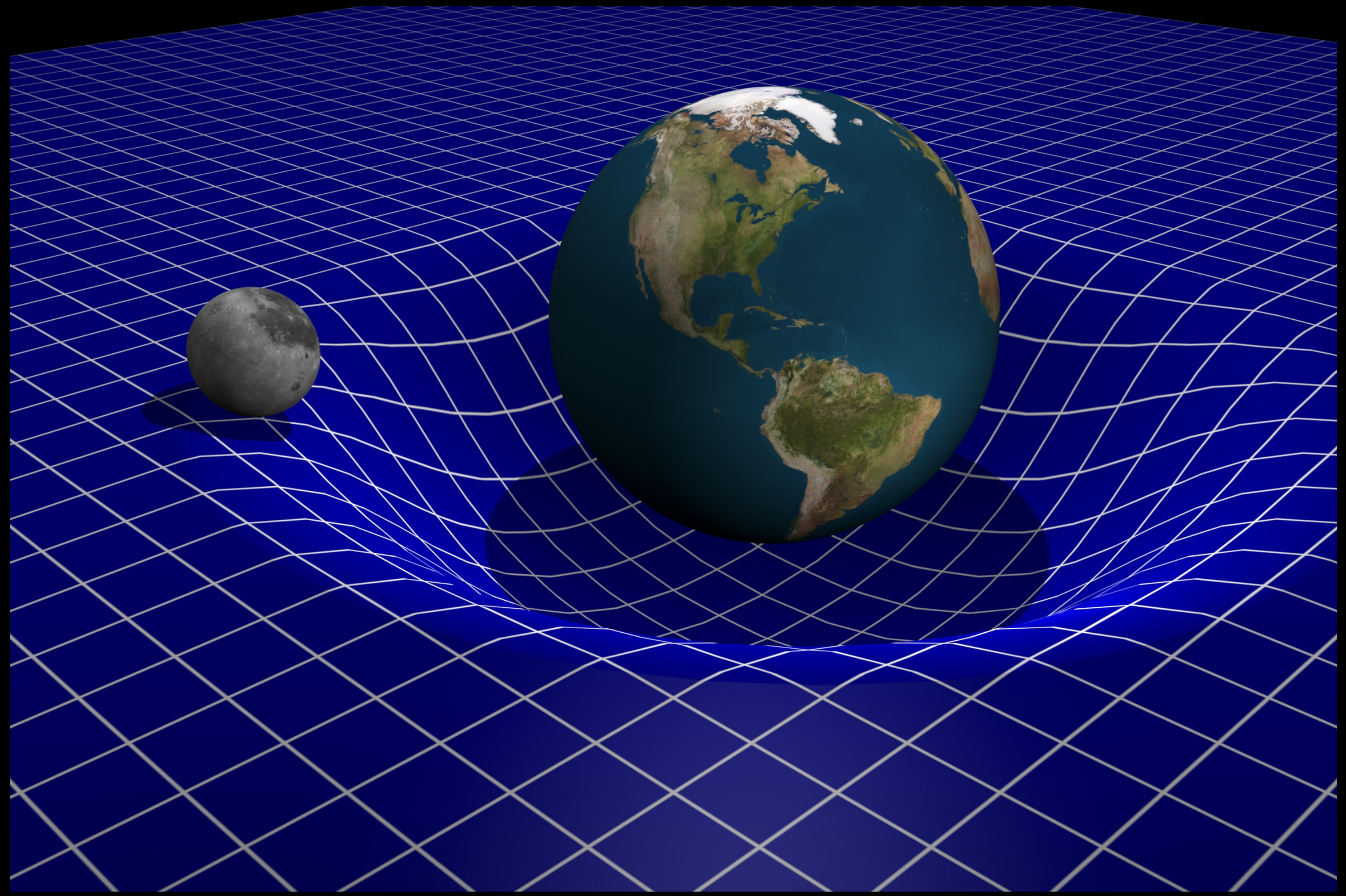
microcosmos



macrocosmos

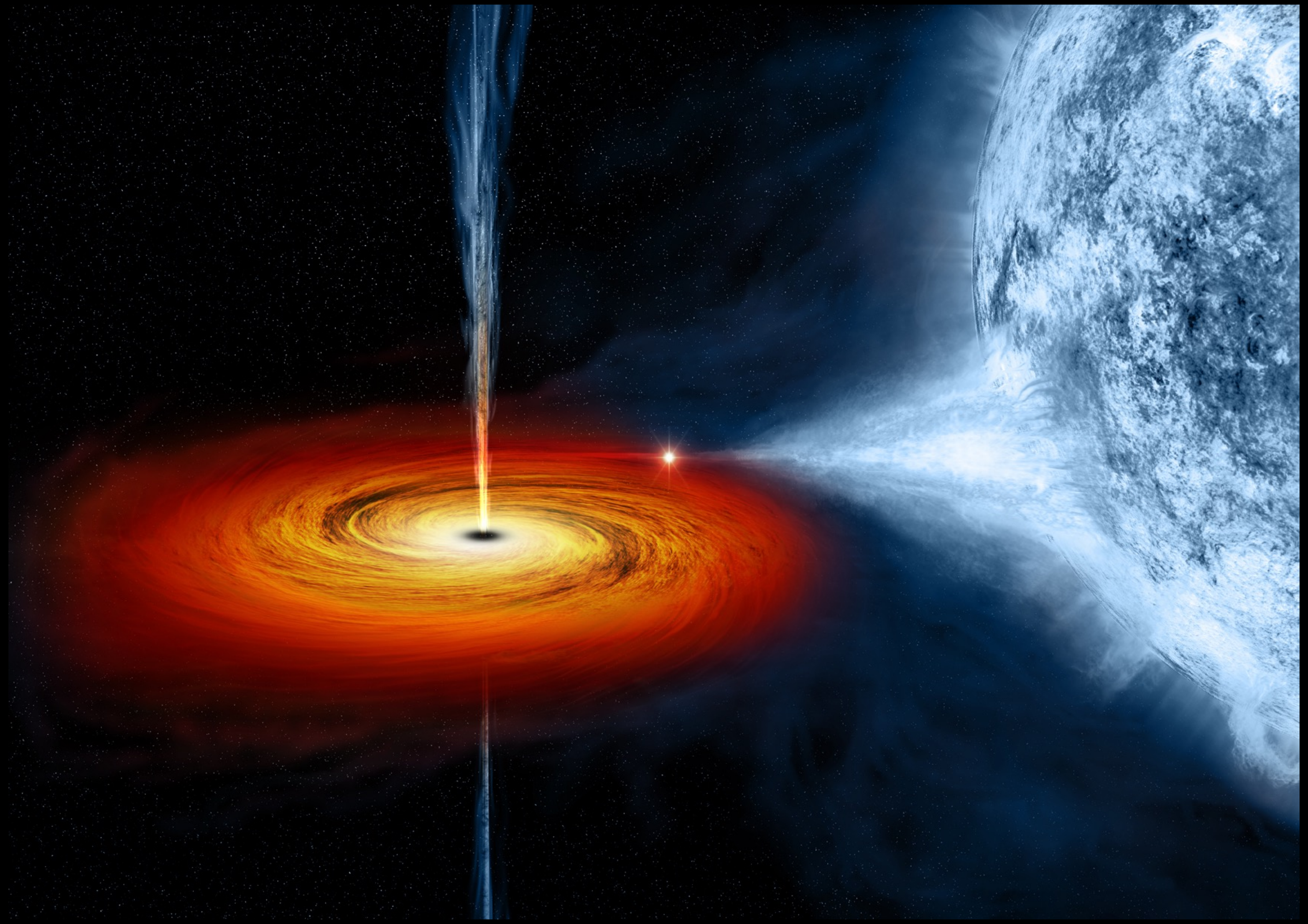


quantum field theory
(Yang-Mills gauge theory)

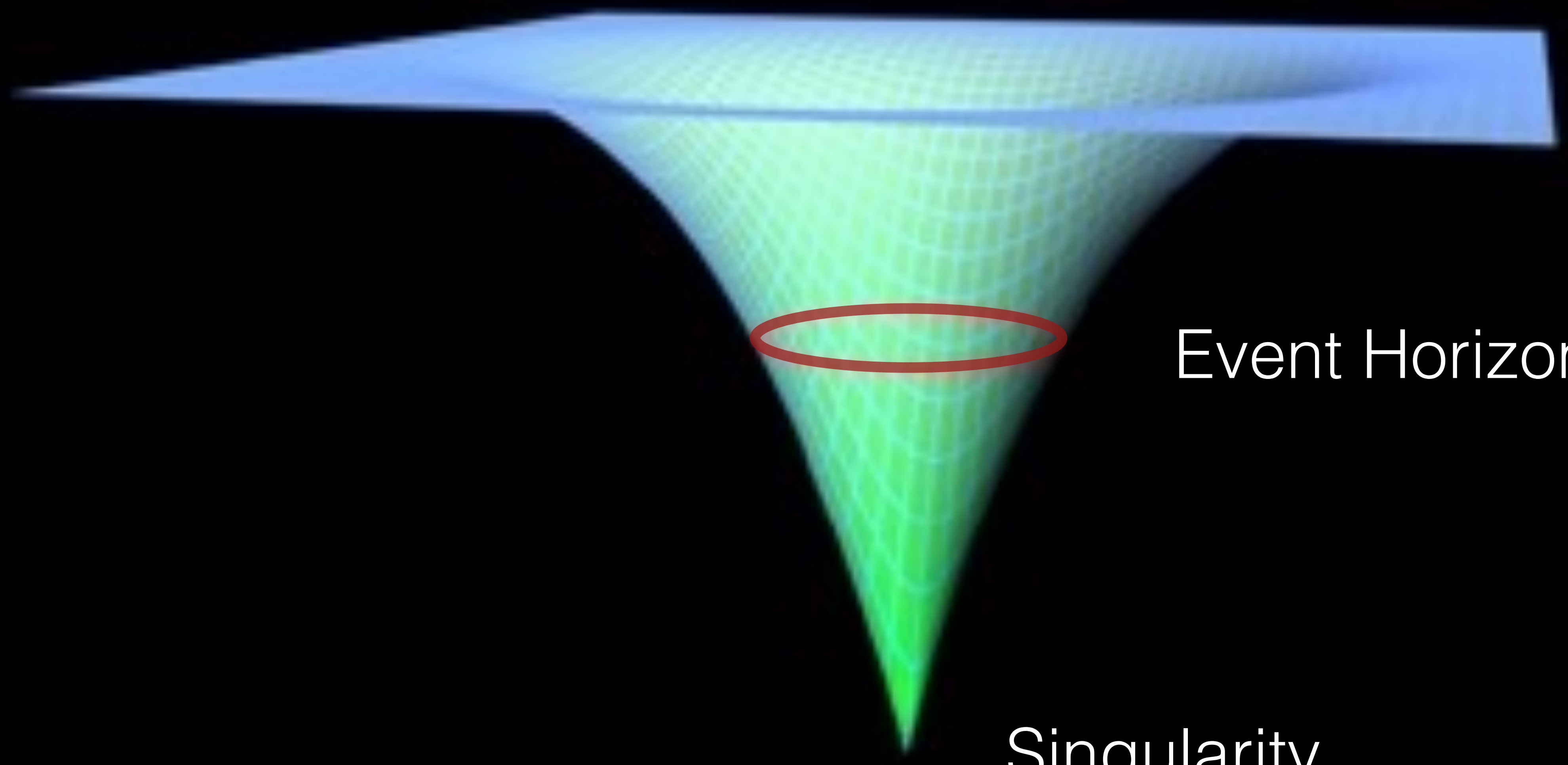


general relativity

A black hole forms in the final stage of the collapse of a sufficiently large star

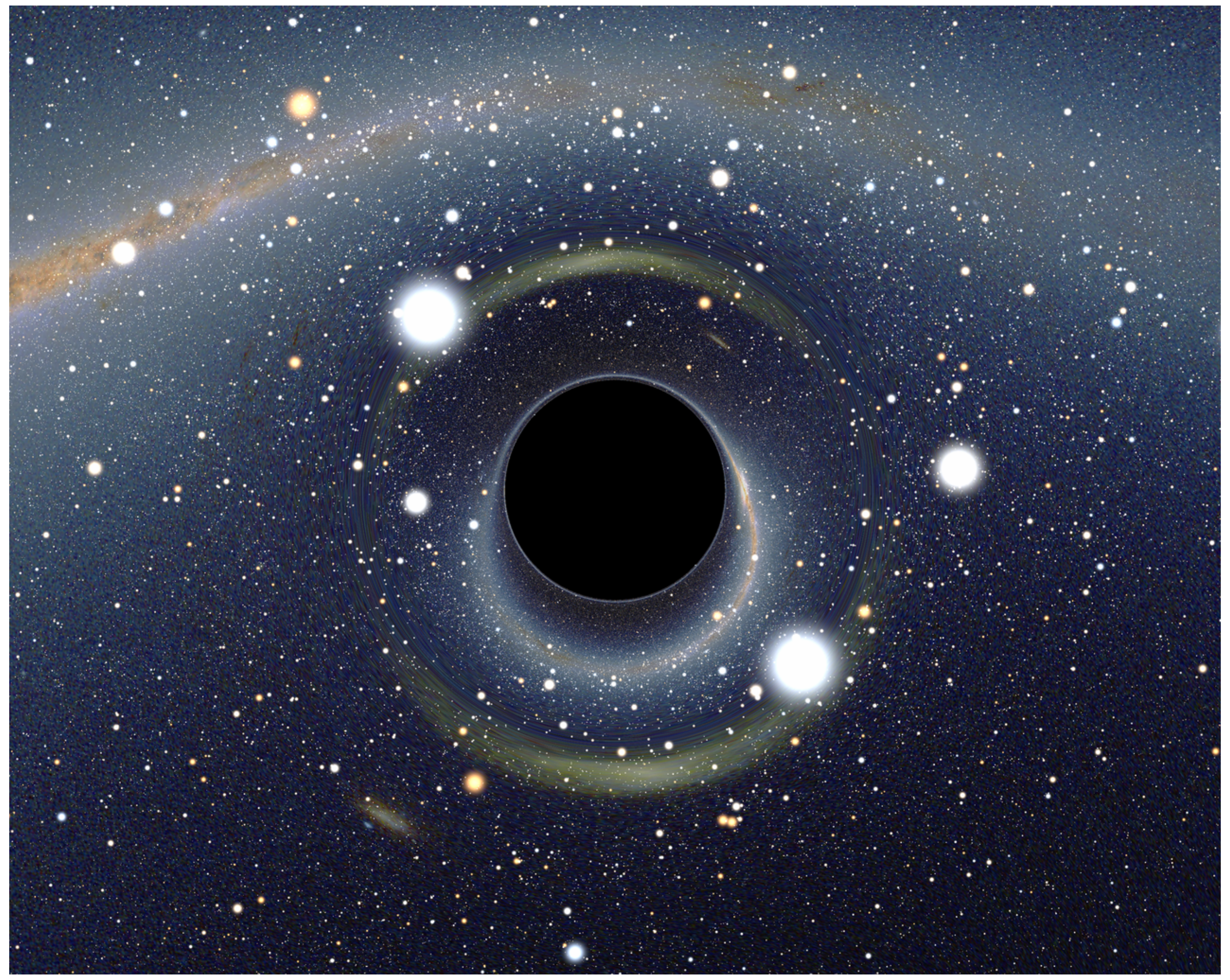


At its centre space and time break down into a **singularity**



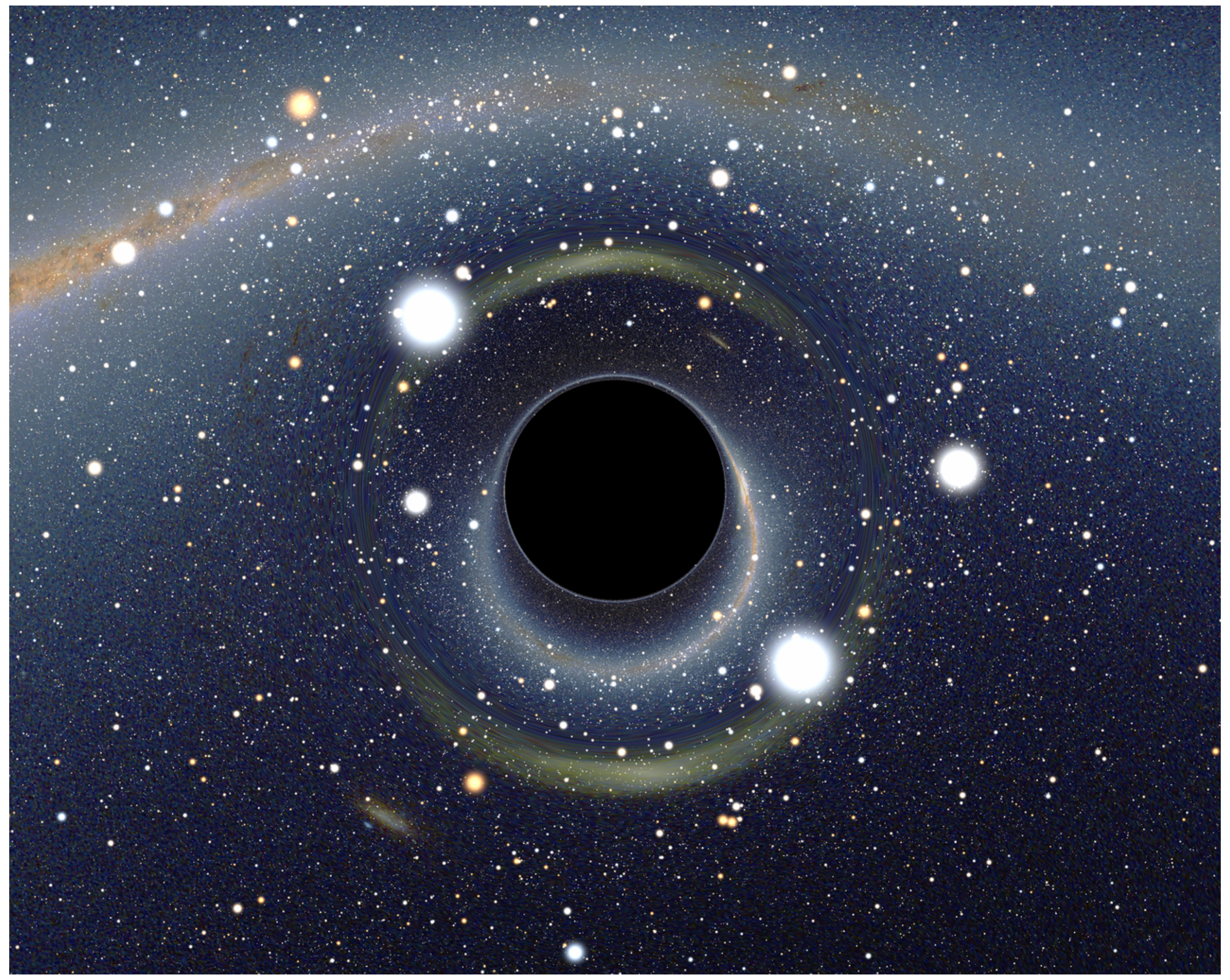
Do quantum effects resolve the singularity?

What is a black hole?



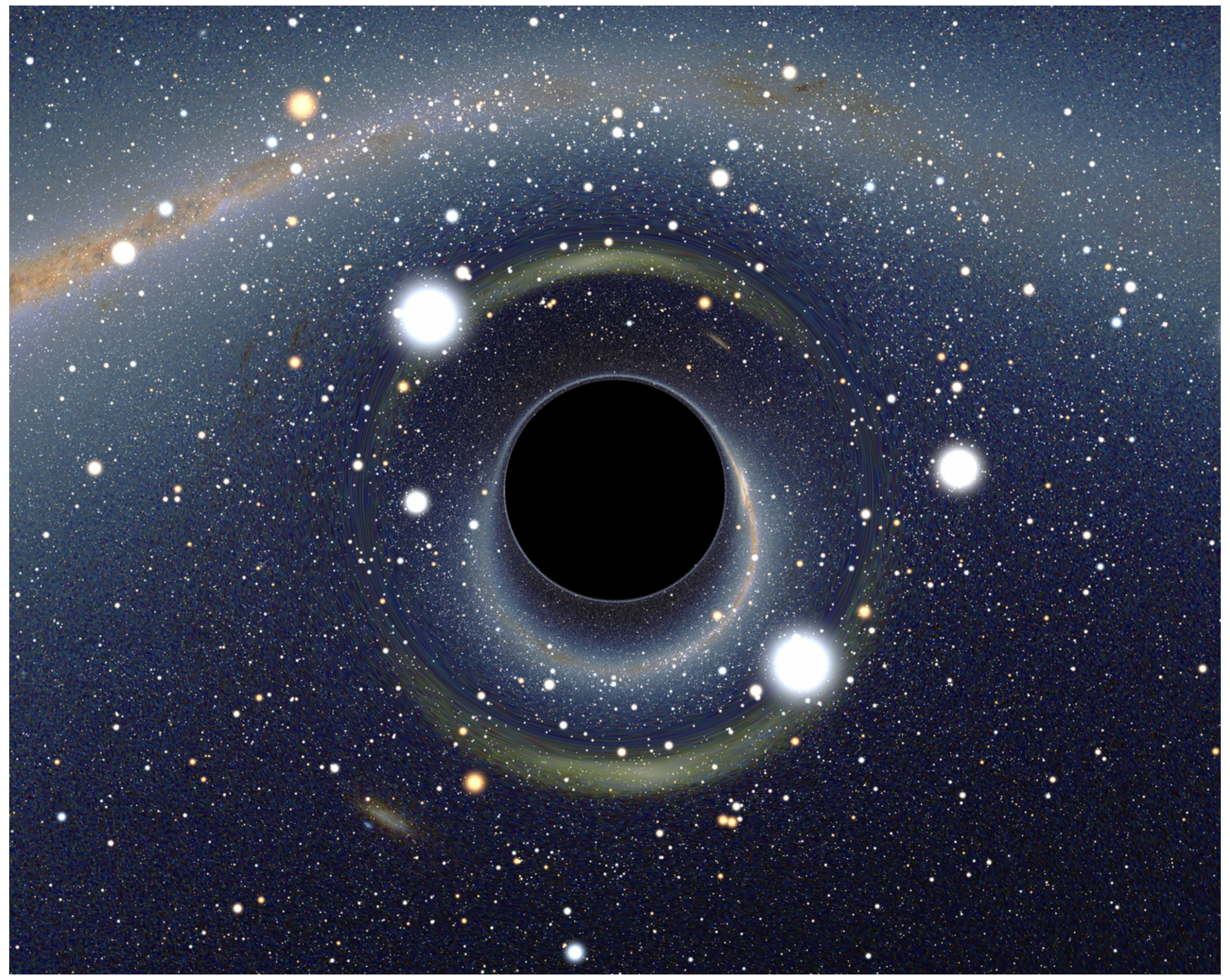
What is a black hole?

Einstein's equations describes the curvature of spacetime



$$R_{\mu\nu}(g) - \frac{1}{2} g_{\mu\nu} R(g) = T_{\mu\nu}(g, F, \dots)$$

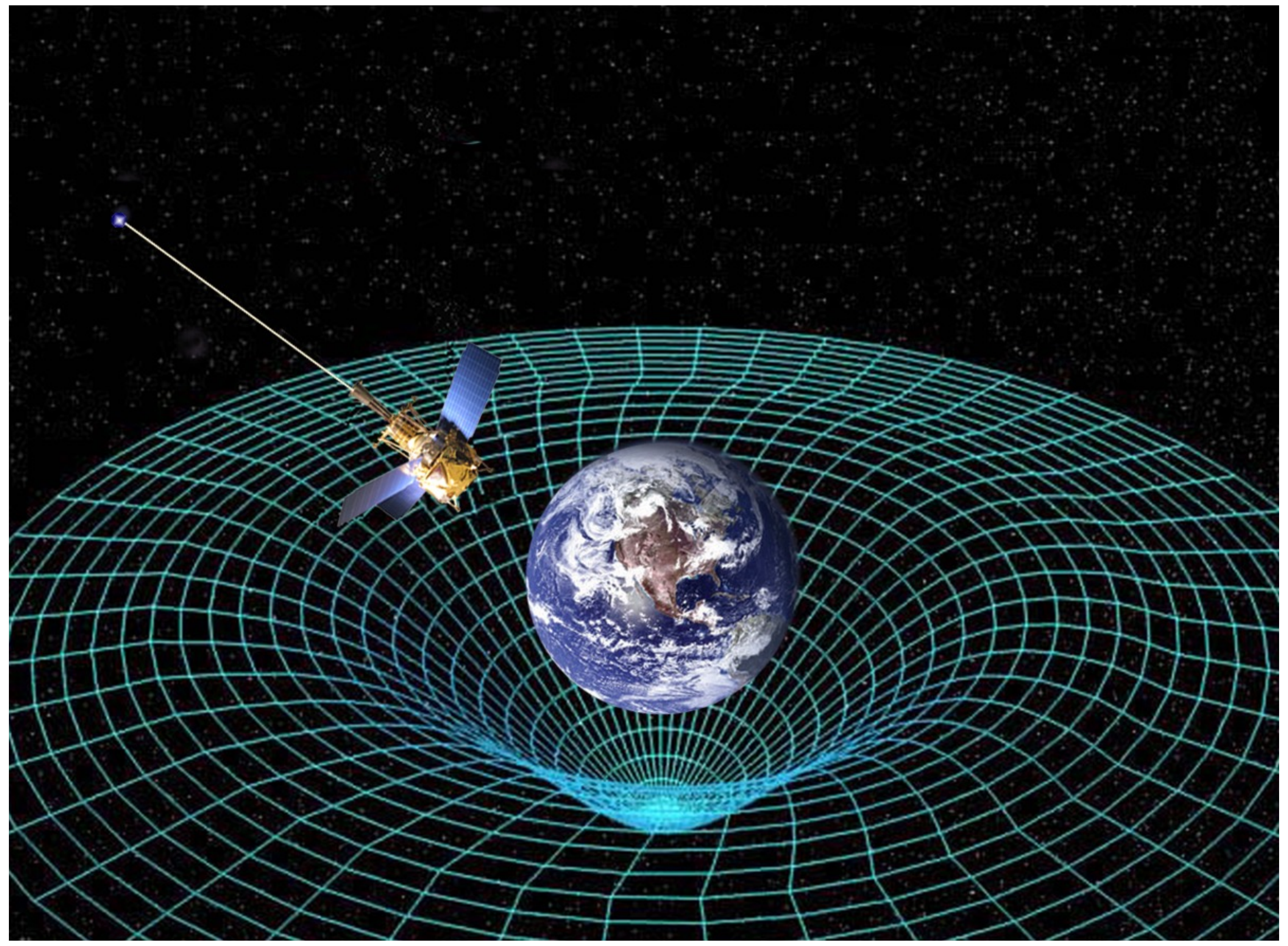
What is a black hole?



Einstein's equations describes the curvature of spacetime

$$R_{\mu\nu}(g) - \frac{1}{2} g_{\mu\nu} R(g) = T_{\mu\nu}(g, F, \ldots)$$

geometry
(Einstein tensor)

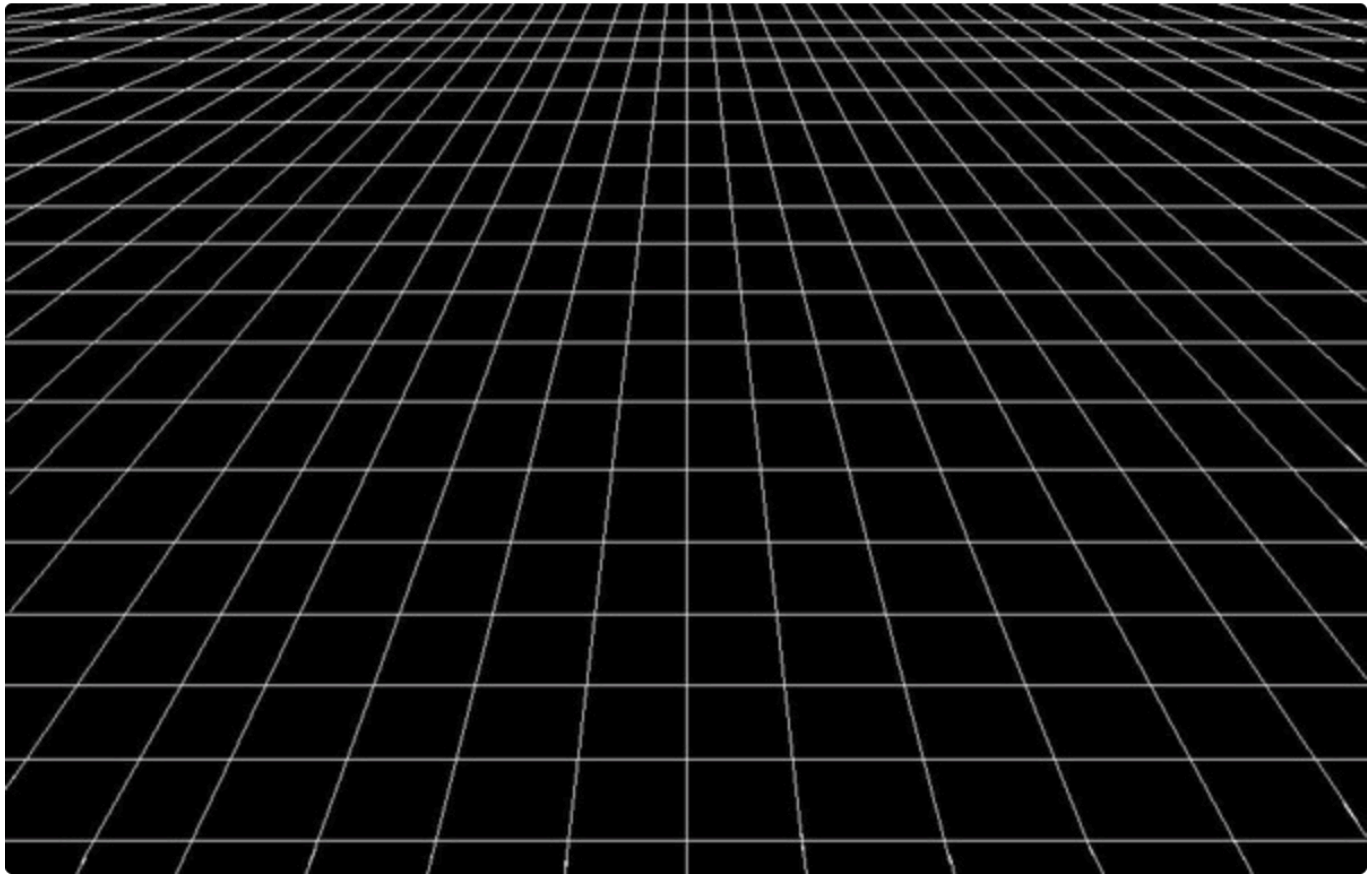


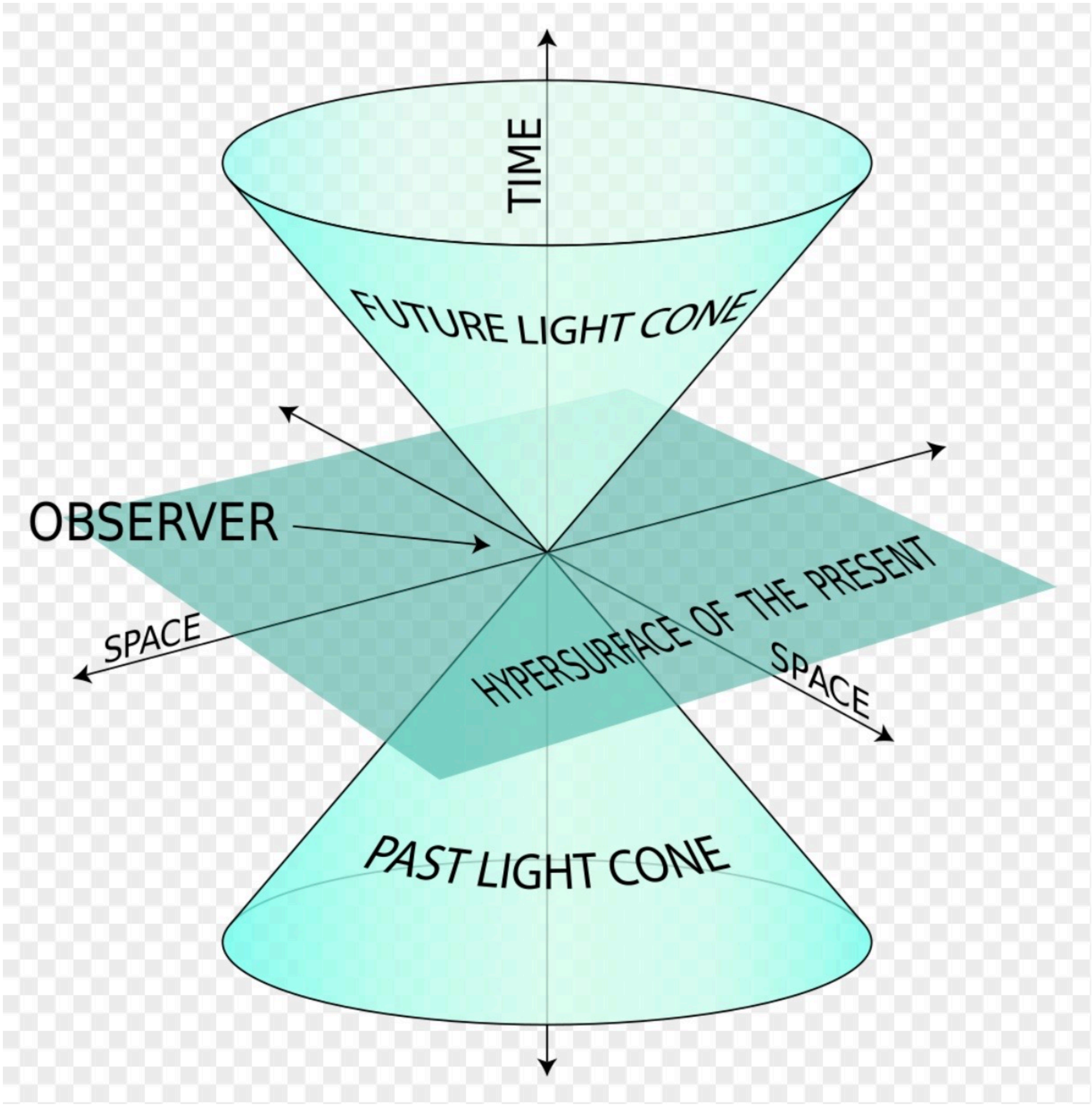
matter
(Stress-energy tensor)

Vacuum equations: $R_{\mu\nu}(g) = 0$ solutions are **Ricci flat:** $\mathrm{Ric}(g) = 0$

Simplest solution: Minkowski space

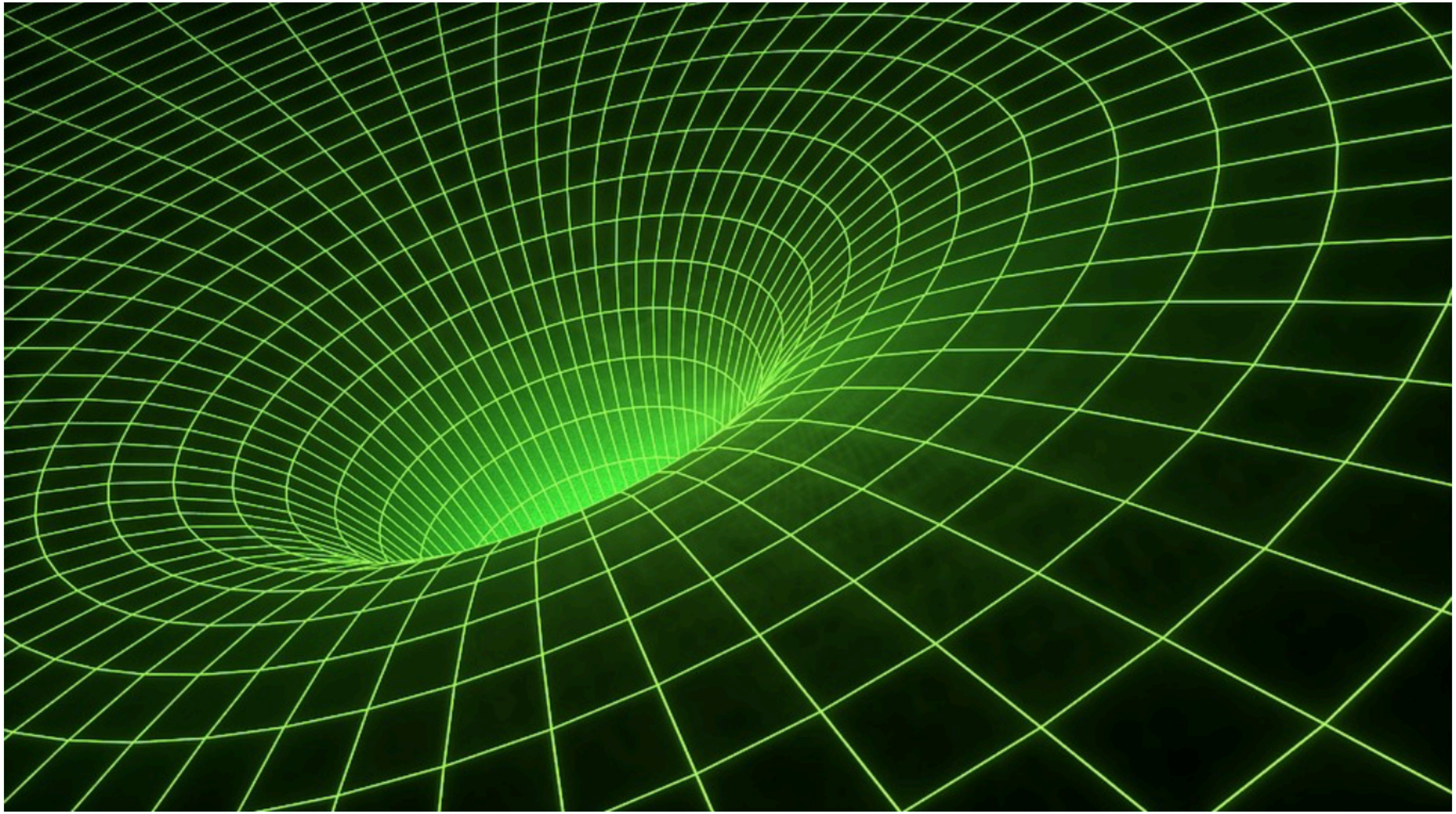
$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = -dt^2 + dx^2 + dy^2 + dz^2$$





Vacuum equations: $R_{\mu\nu}(g) = 0$ solutions are **Ricci flat:** $\mathrm{Ric}(g) = 0$

Schwarzschild black hole solution



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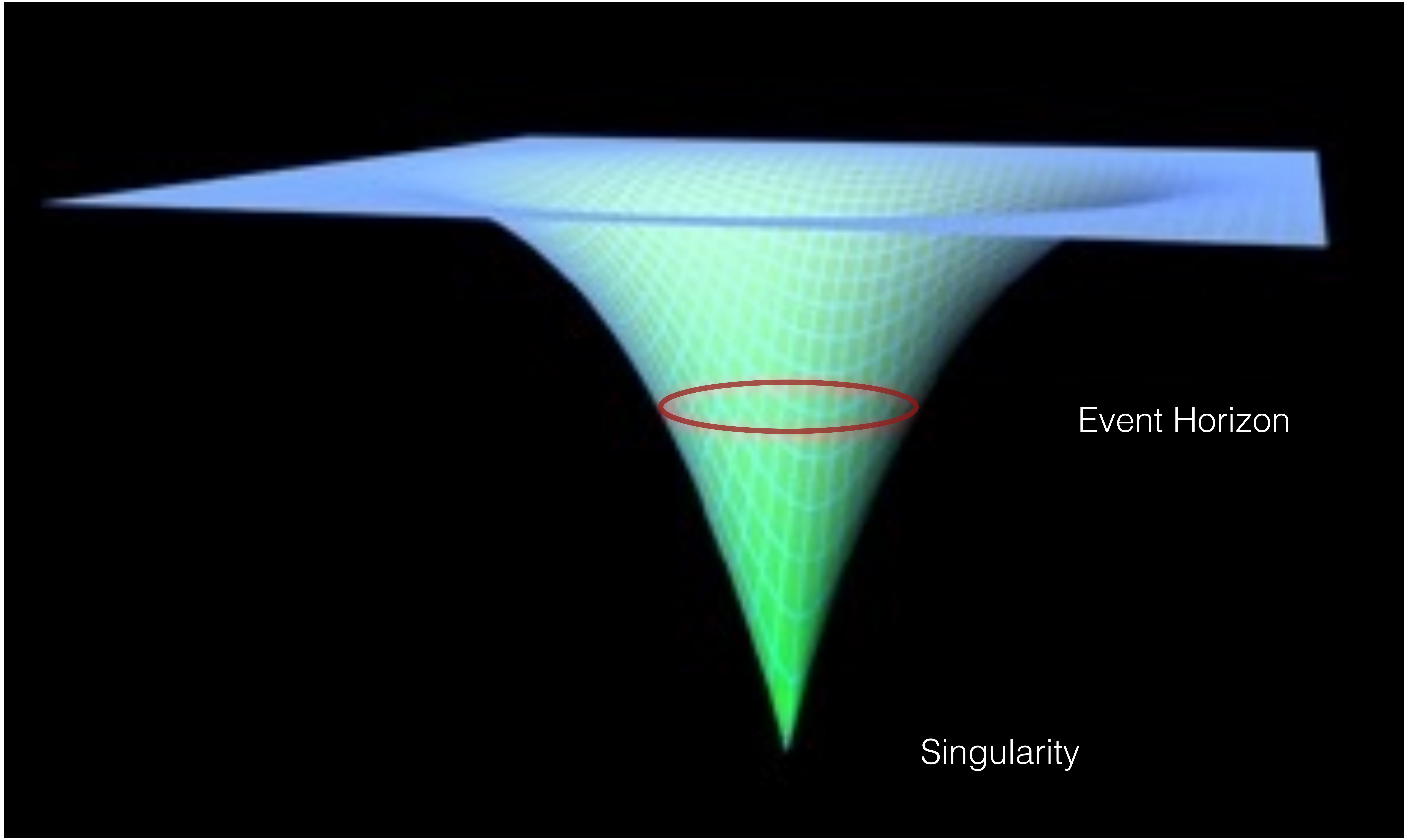
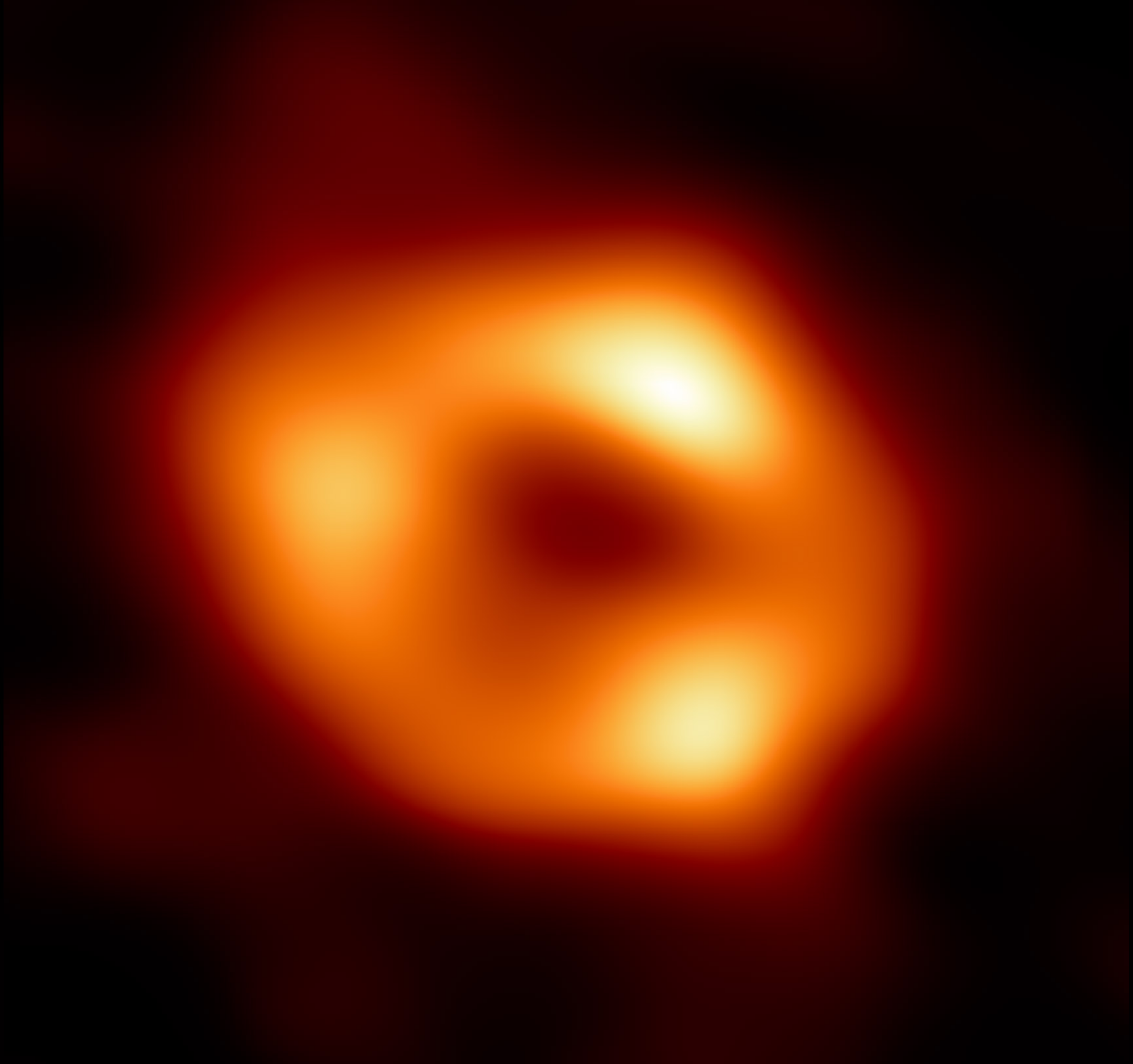


$r = 2M$

(coordinate singularity)

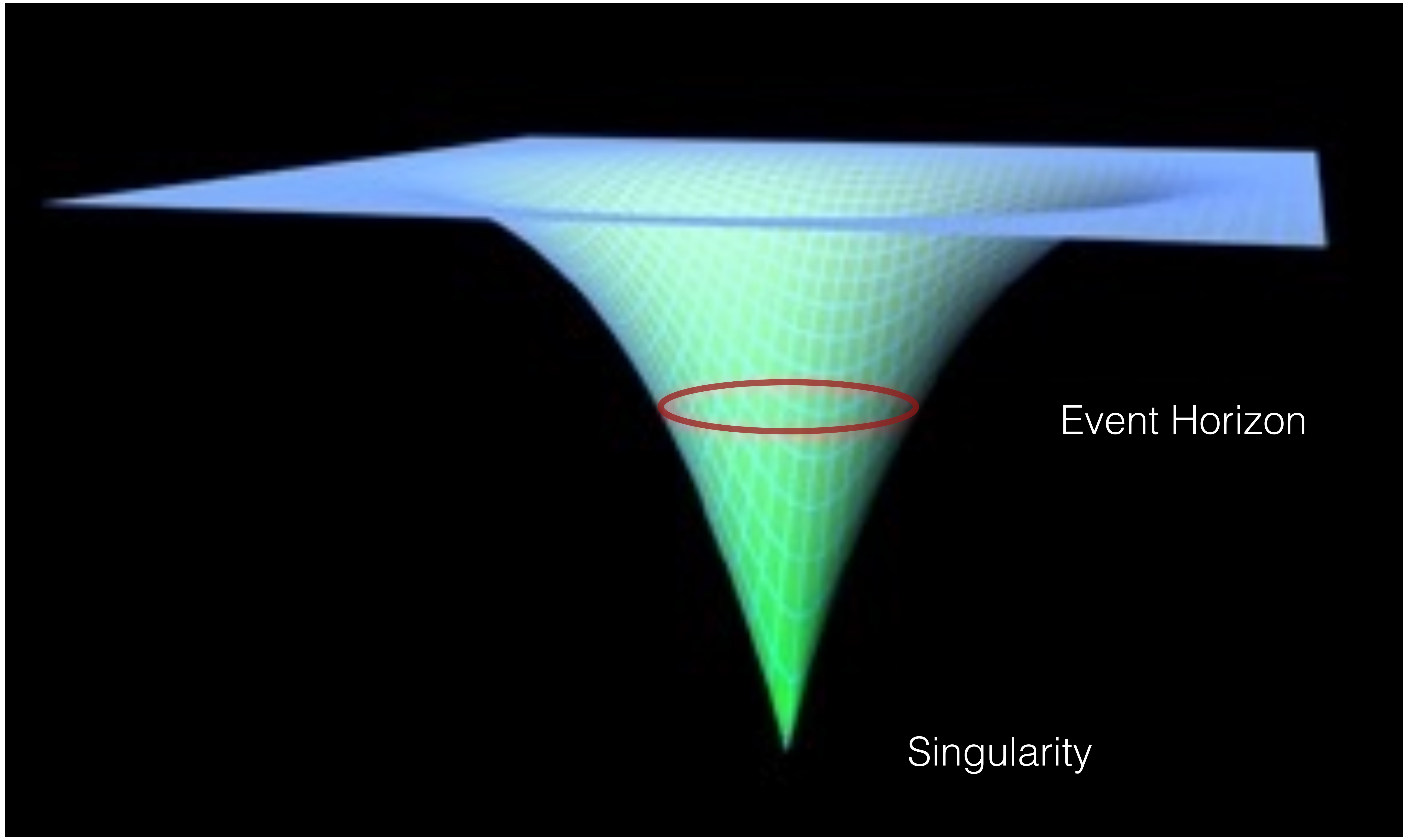
$r = 0$

(curvature singularity)




Event horizon

The existence of an event horizon suggests a **holographic** description of black holes



Can this be made more precise?

Vacuum equations: $R_{\mu\nu}(g) = 0$ (solutions are **Ricci flat**)

Schwarzschild black hole solution:

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = -\left(1 - \frac{2M}{r}\right)dt^2 + \left(1 - \frac{2M}{r}\right)^{-1}dr^2 + r^2 ds^2_{sphere}$$

Flips sign as we cross the horizon.

$r > 2M$ **outside**

$r < 2M$ **inside**

time t

$\updownarrow$

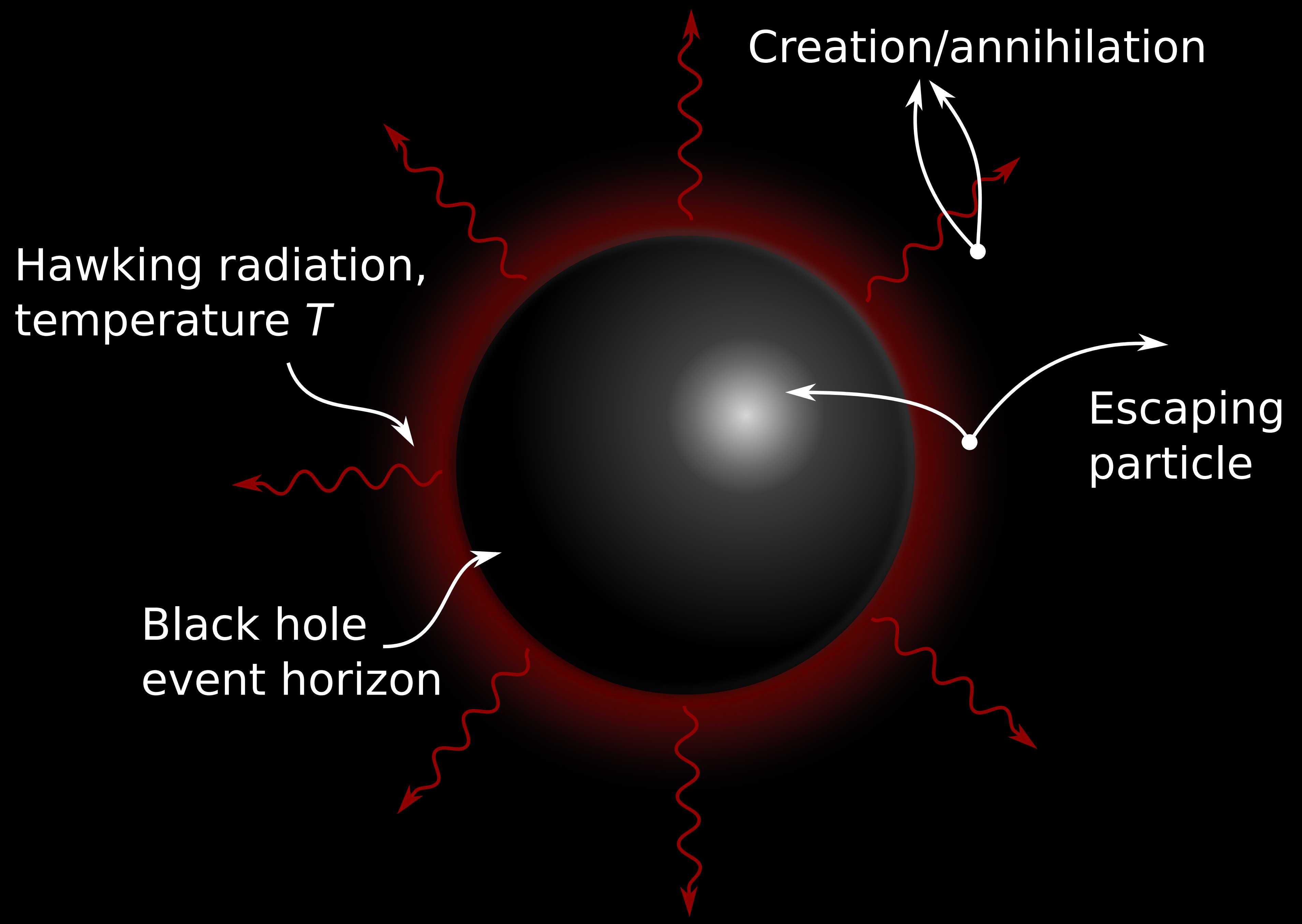
space r

But black holes are not black!

Hawking discovered that quantum effects close to the horion creates radiation

Quantum fluctuations outside the horizon cause pair creation

A particle with negative energy goes into the hole, while a positive energy particle escapes to infinity



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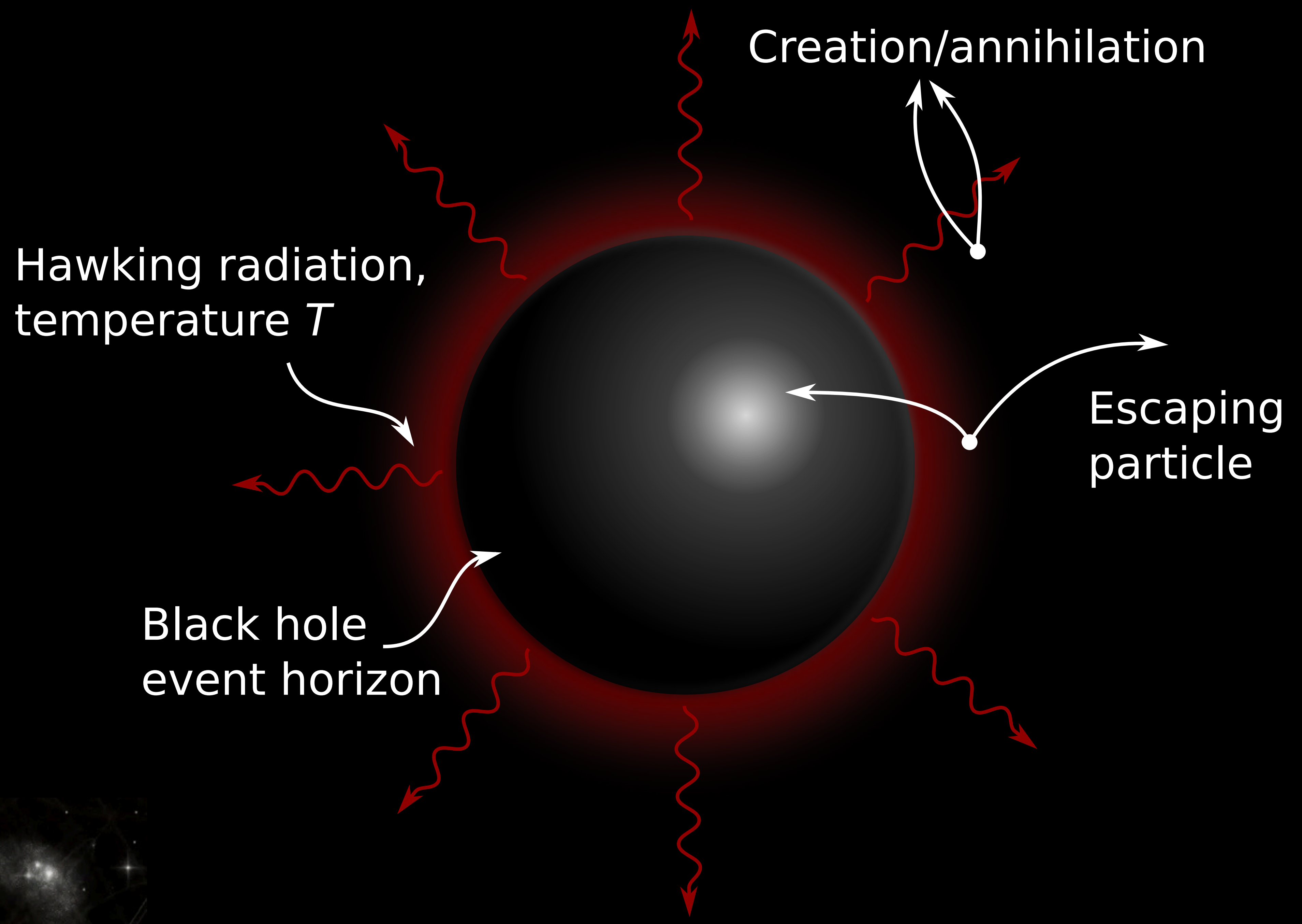
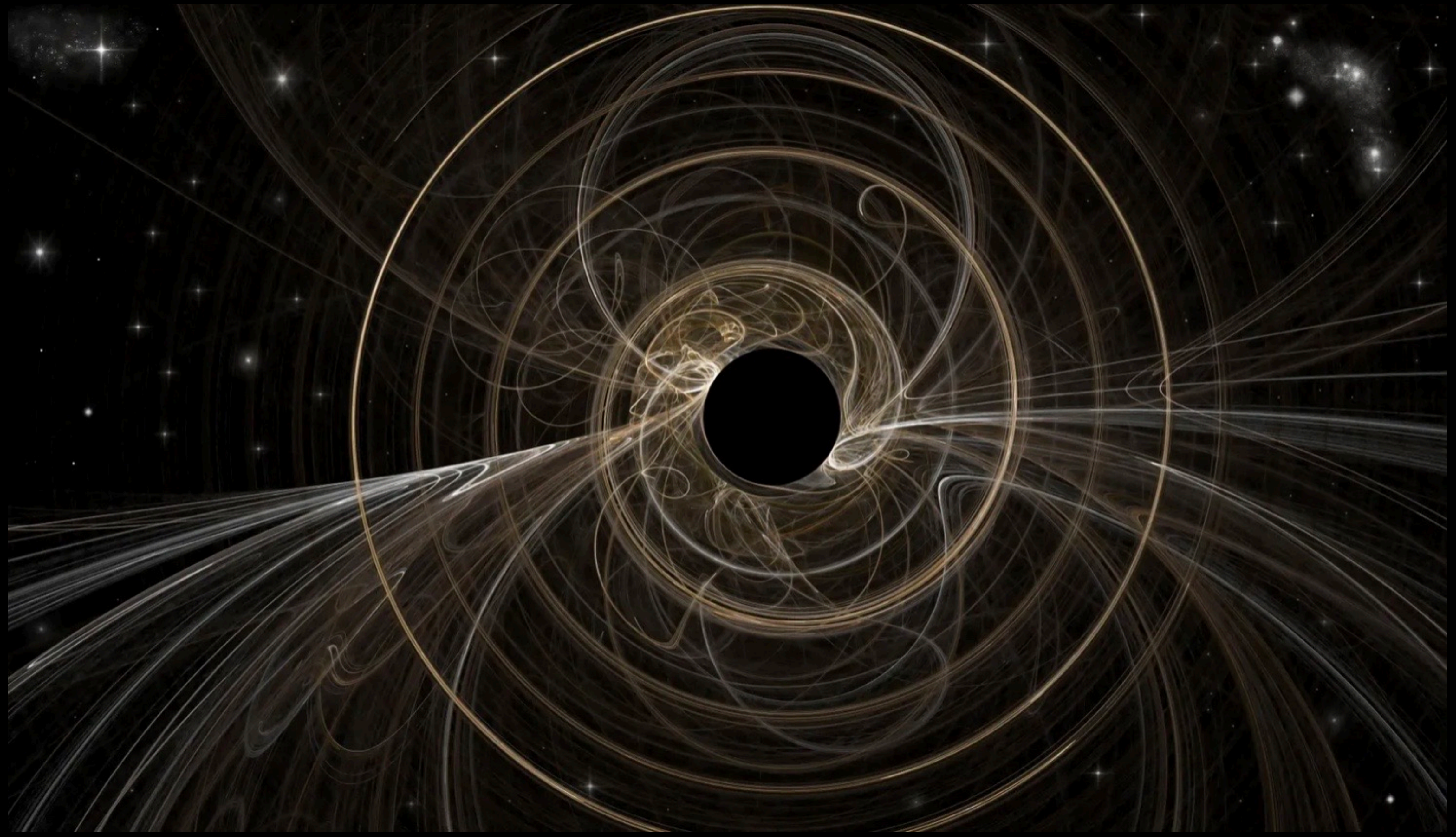




Hawking radiation!

Black holes behave like a black body with entropy

$$S = \frac{\text{Area}}{4G_N\hbar}$$

Bekenstein-Hawking
formula

Black holes behave like a black body with entropy

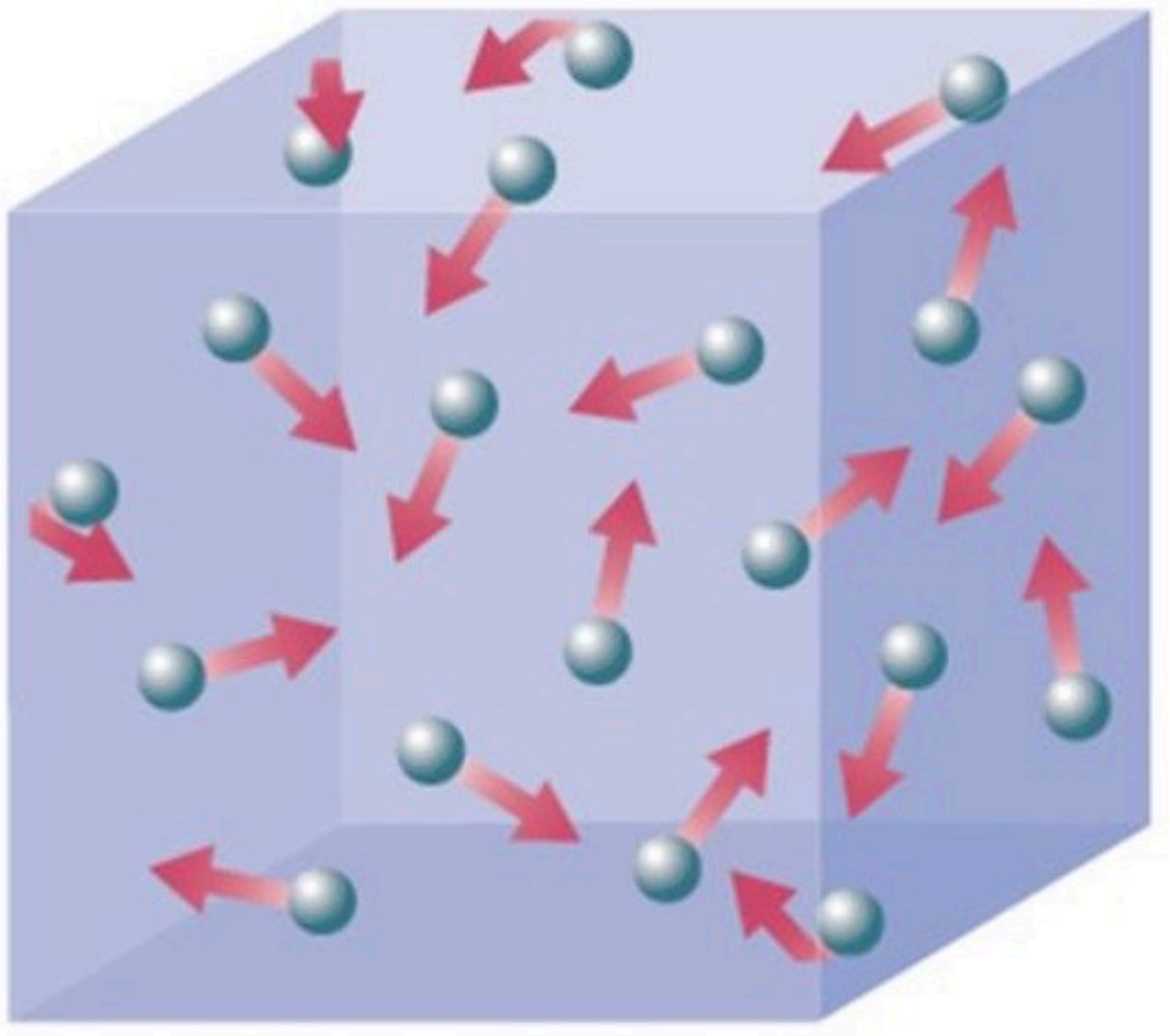
$$S = \frac{\text{Area}}{4 G_N \hbar}$$

Bekenstein-Hawking
formula

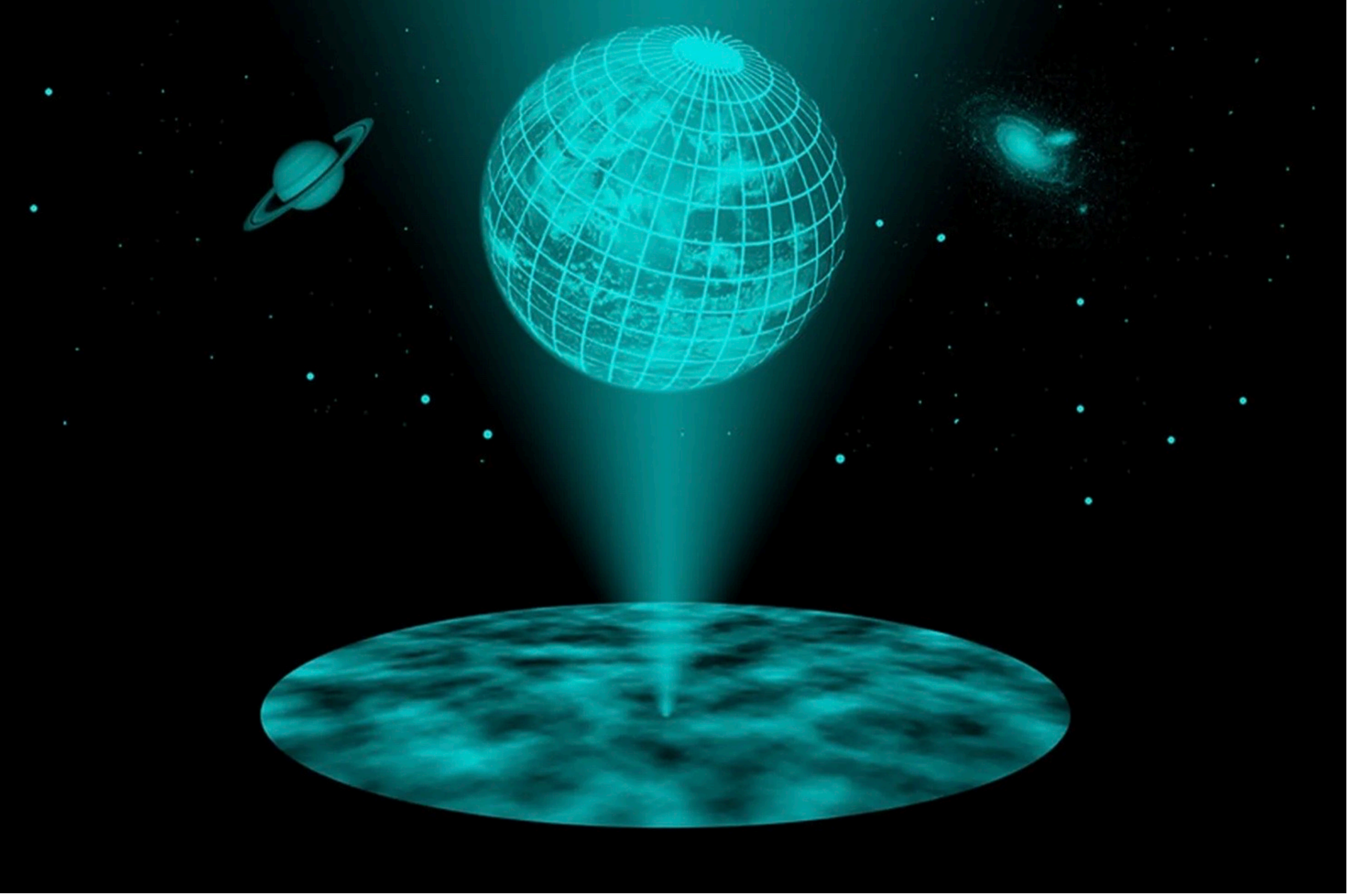
This is an amazing equation!

$$\text{Statistical Physics} = \frac{\text{Gravity}}{\text{Quantum Mechanics}}$$

Surprising that the entropy of a black hole is proportional to **area**



For ordinary matter systems entropy is proportional to the **volume**



Suggests a **holographic** quantum description of black holes

Holographic principle

But could the entire universe be a hologram?

AdS/CFT correspondence

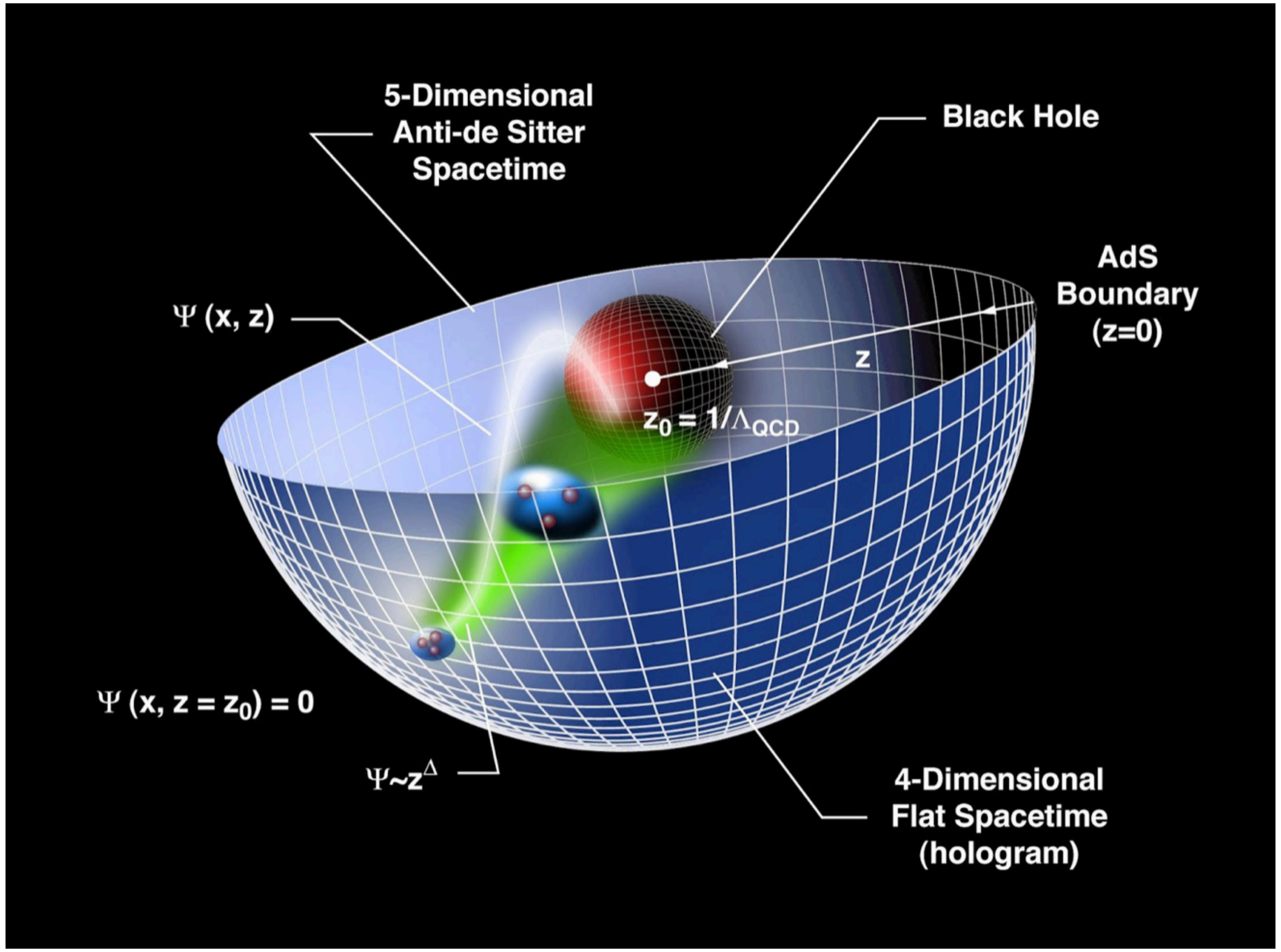
(Gauge/gravity correspondence)

General statement: There should be an exact **equivalence** between gravity in 5d and quantum field theory on its 4-dimensional flat boundary (Minkowski space)

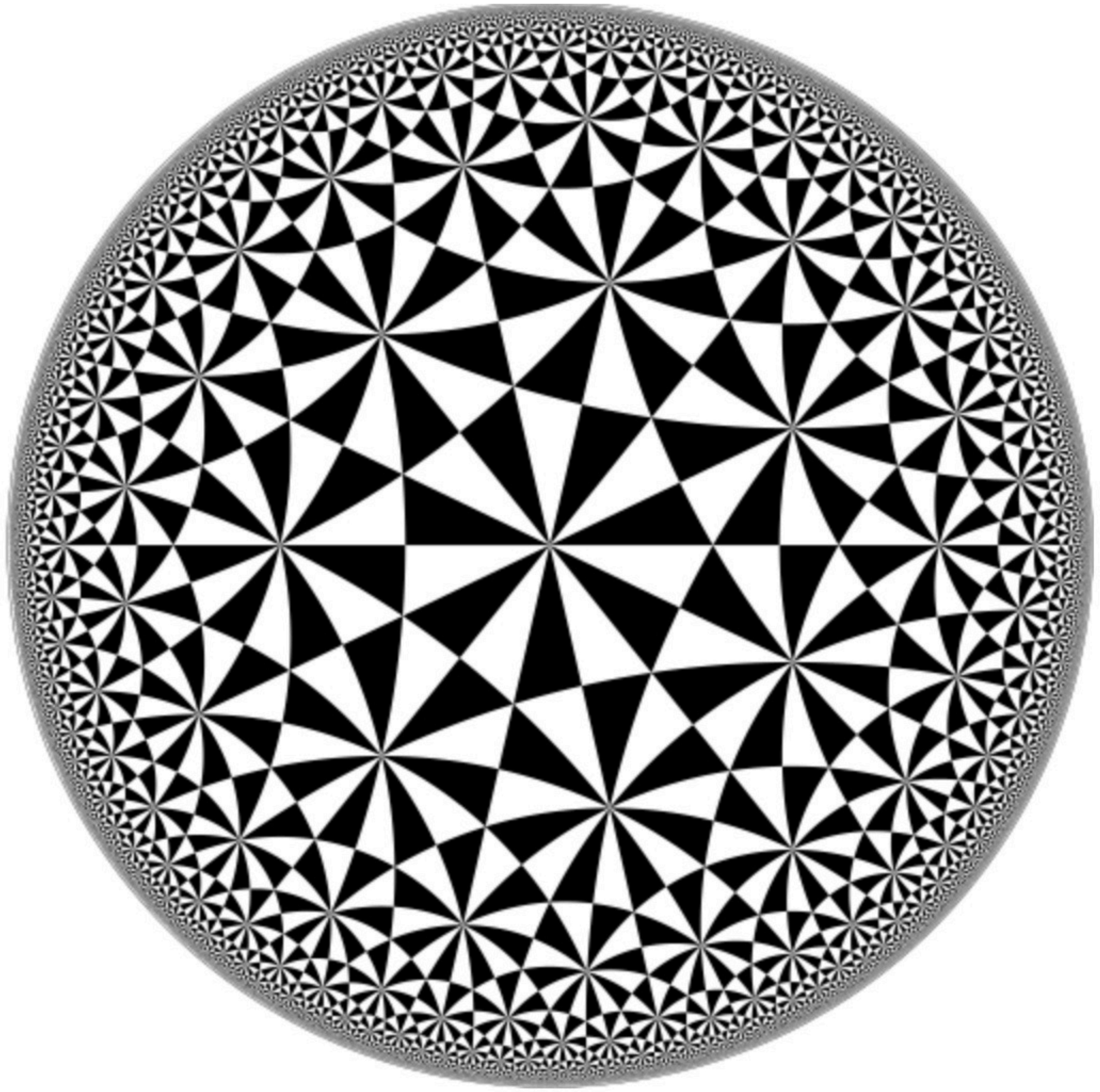
- Gravity in 5d anti-de Sitter space correspond to $SU(N)$ gauge theory (Yang-Mills theory) in Minkowski space

- Realization of the holographic principle

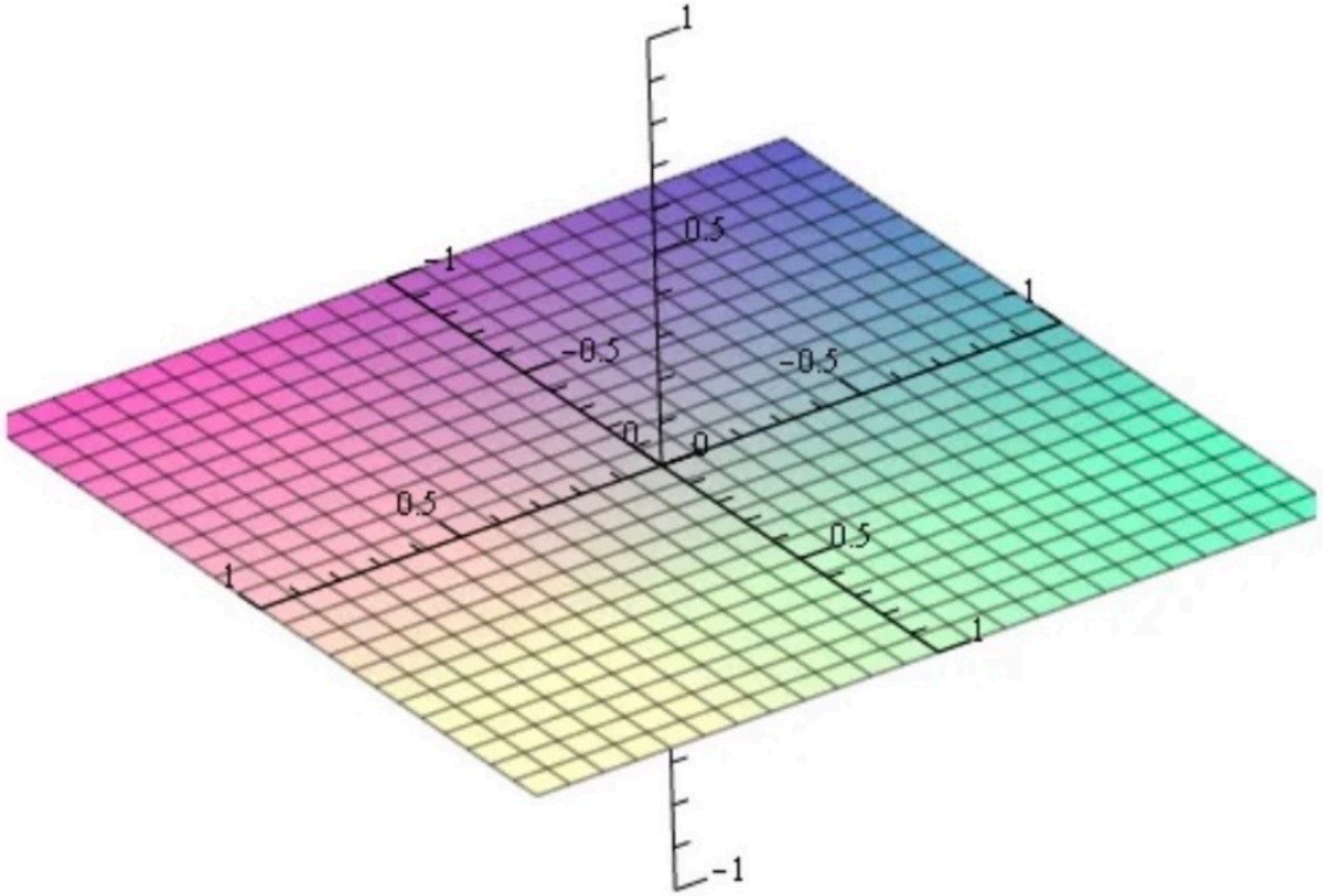
- Quantum description of gravity



Hyperbolic

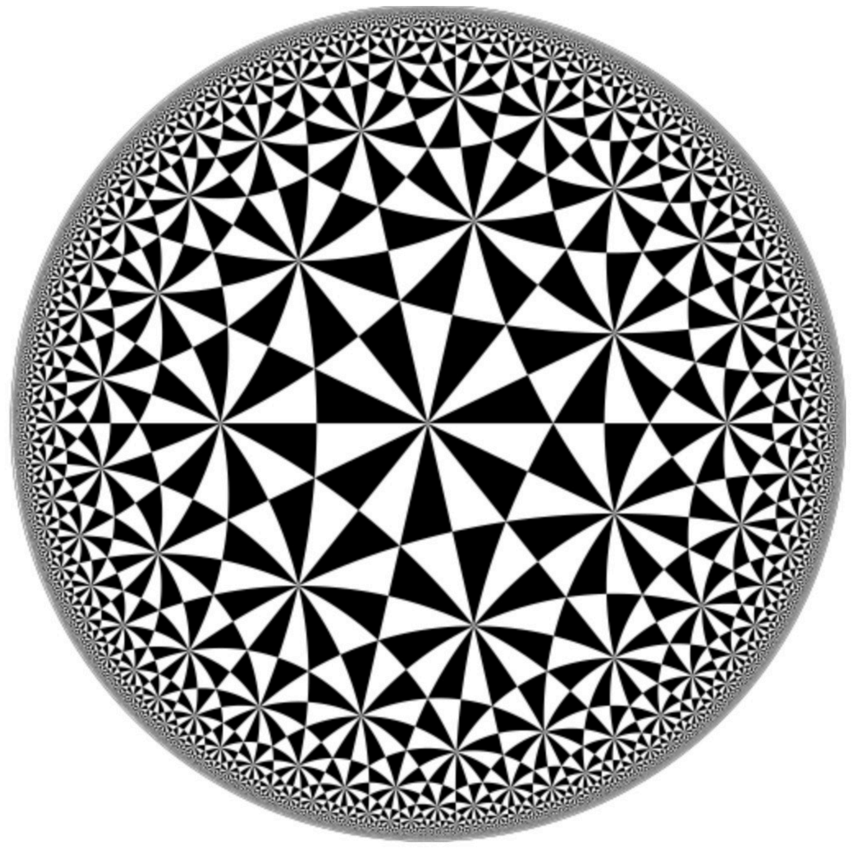


Flat



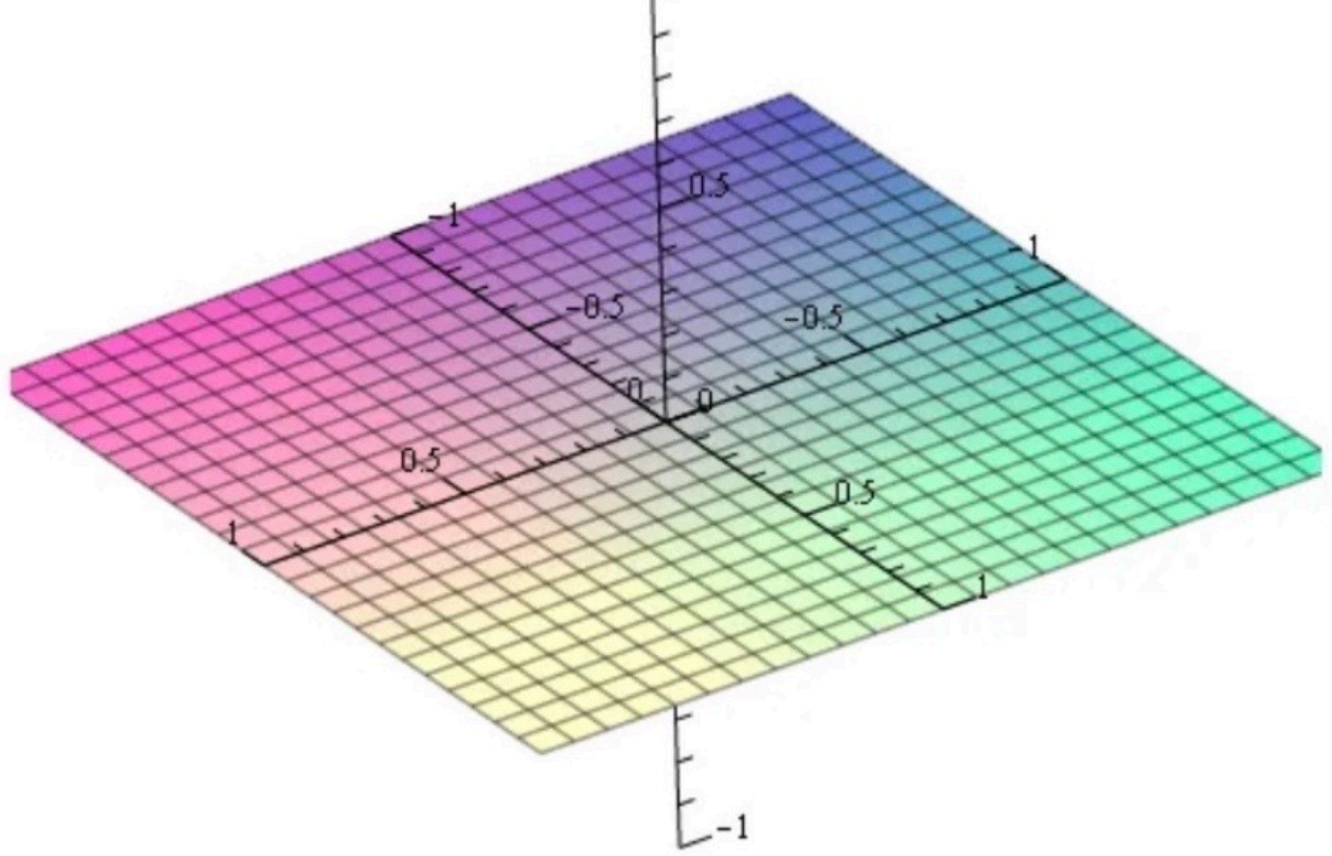
Anti-de Sitter space is like a Lorentzian version of hyperbolic space

5d gravity theory

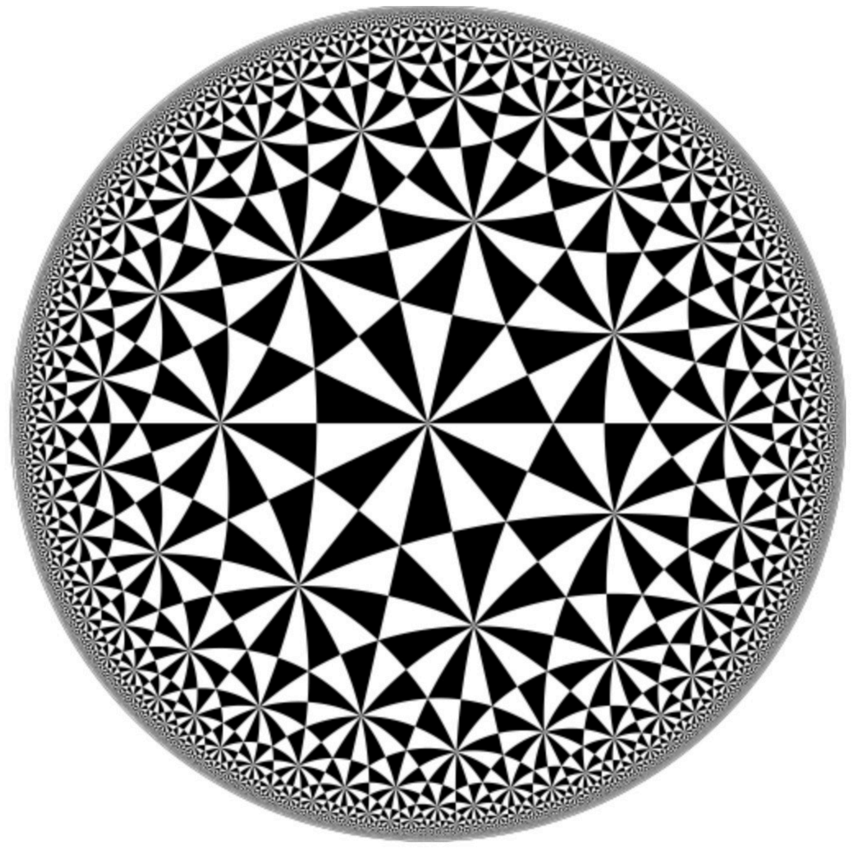


Equivalence

4d QFT



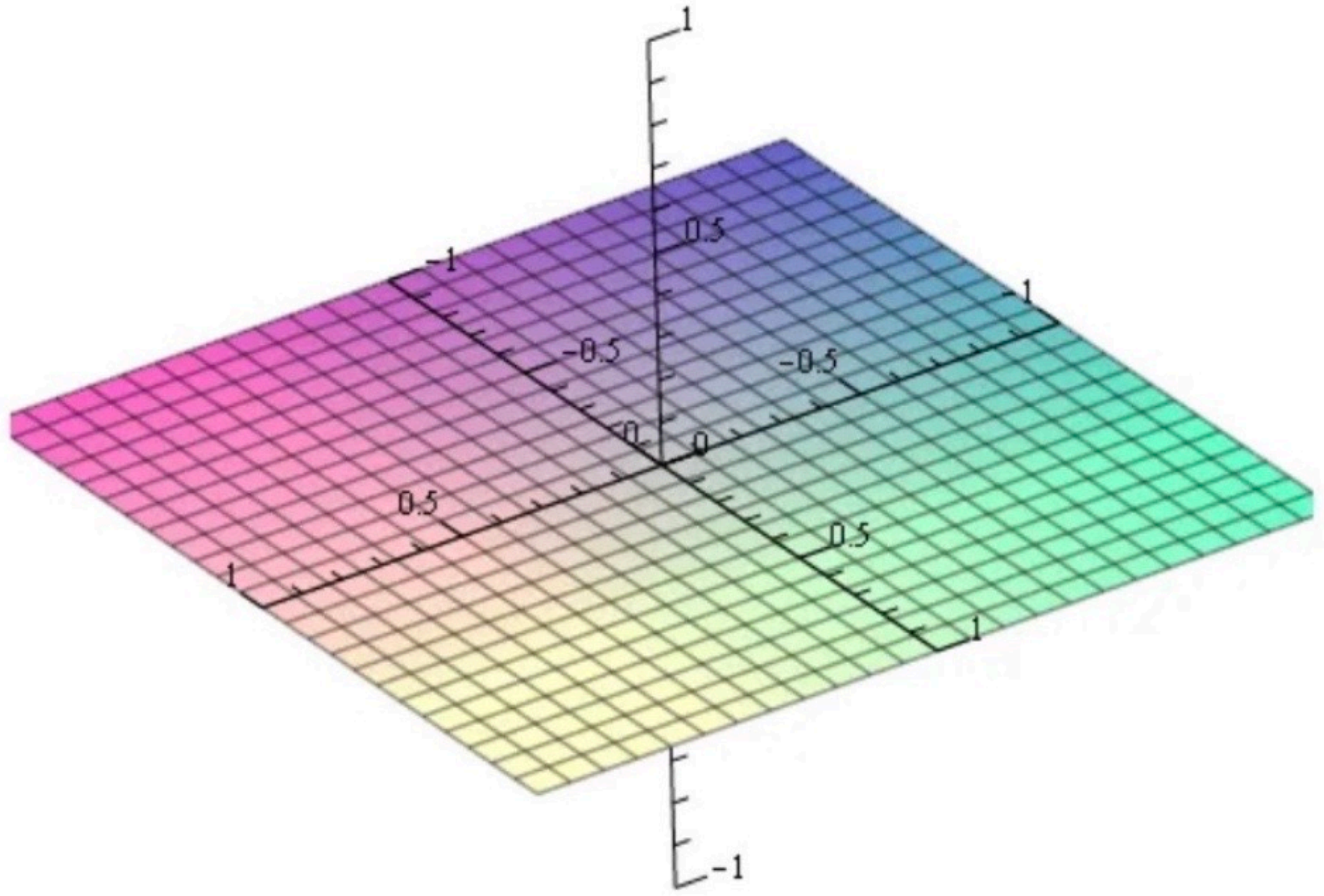
5d gravity theory



Equivalence

$\longleftrightarrow$

4d QFT

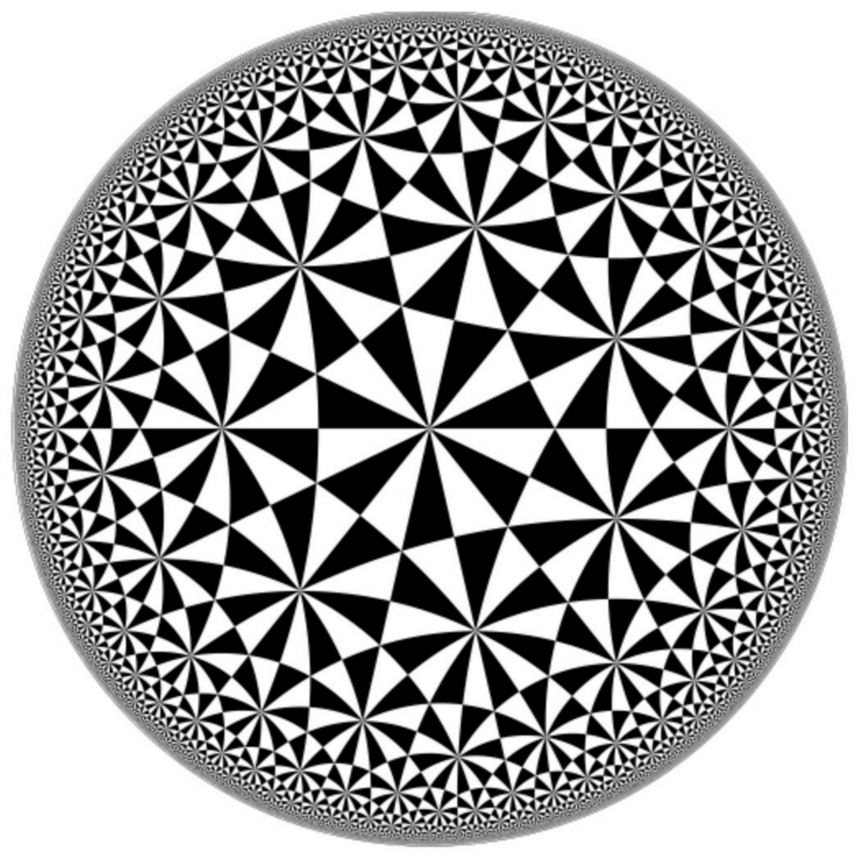


We also require an additional symmetry: **supersymmetry**

This puts constraints on the vacuum geometry to be of the form

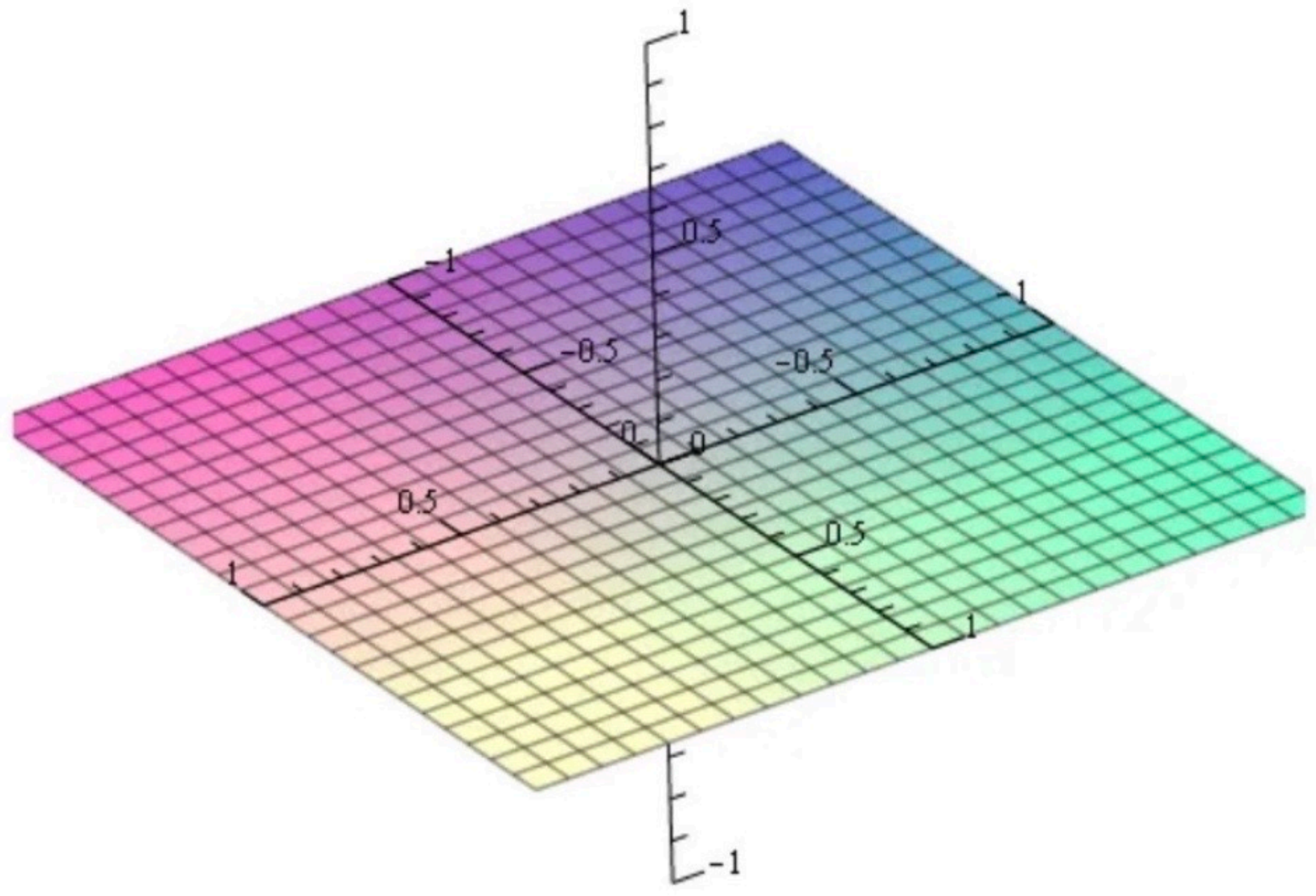
$$AdS_5 \times \mathcal{M}$$

5d gravity theory



Equivalence

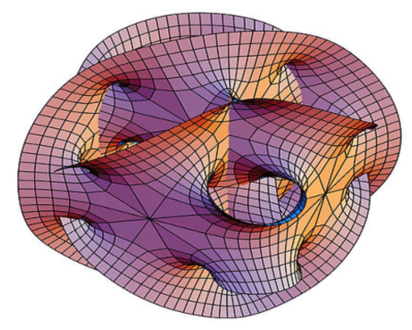
4d QFT



We also require an additional symmetry: **supersymmetry**

This puts constraints on the vacuum geometry to be of the form

External spacetime $\longrightarrow$ $AdS_5 \times \mathcal{M}$ $\longleftarrow$ **Internal compact space (Sasaki-Einstein)**



AdS/CFT originates from string theory

The gauge theory comes from string theory in 10 dimensional spacetime of the form

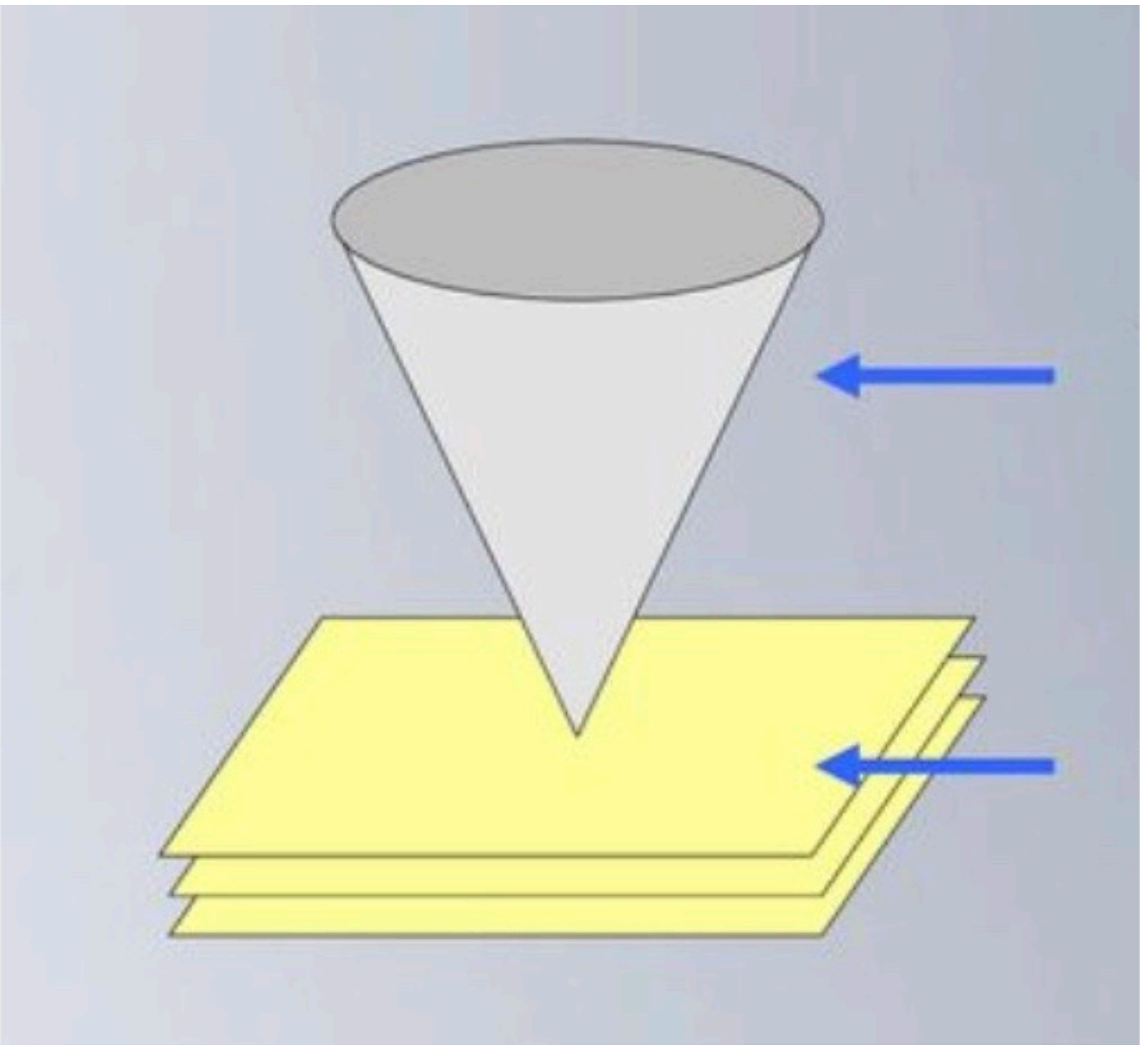
$$\mathbb{R}^4 \times Y$$

6d complex variety: Calabi-Yau cone

Y comes with a $\mathbb{C}^\times$- action fixing a unique point p which is singular (except for $Y = \mathbb{C}^3$)

Embedd stack of N coincident D3-branes at the tip of the cone

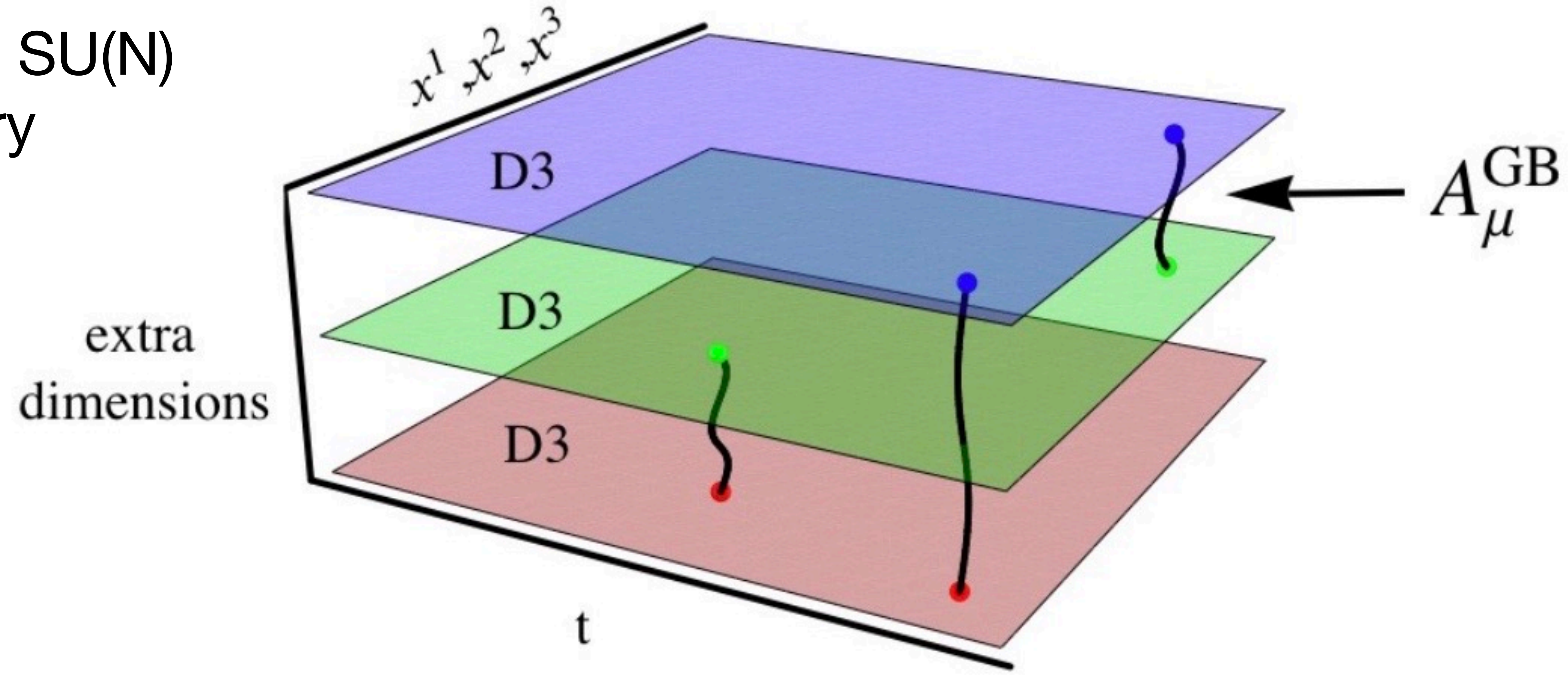
$$\mathbb{R}^4 \times \{p\} \subset \mathbb{R}^4 \times Y$$



Calabi-Yau Cone

stack of D3-branes

Open strings stretching between the branes gives rise to an SU(N)
Yang-Mills gauge theory with superconformal symmetry

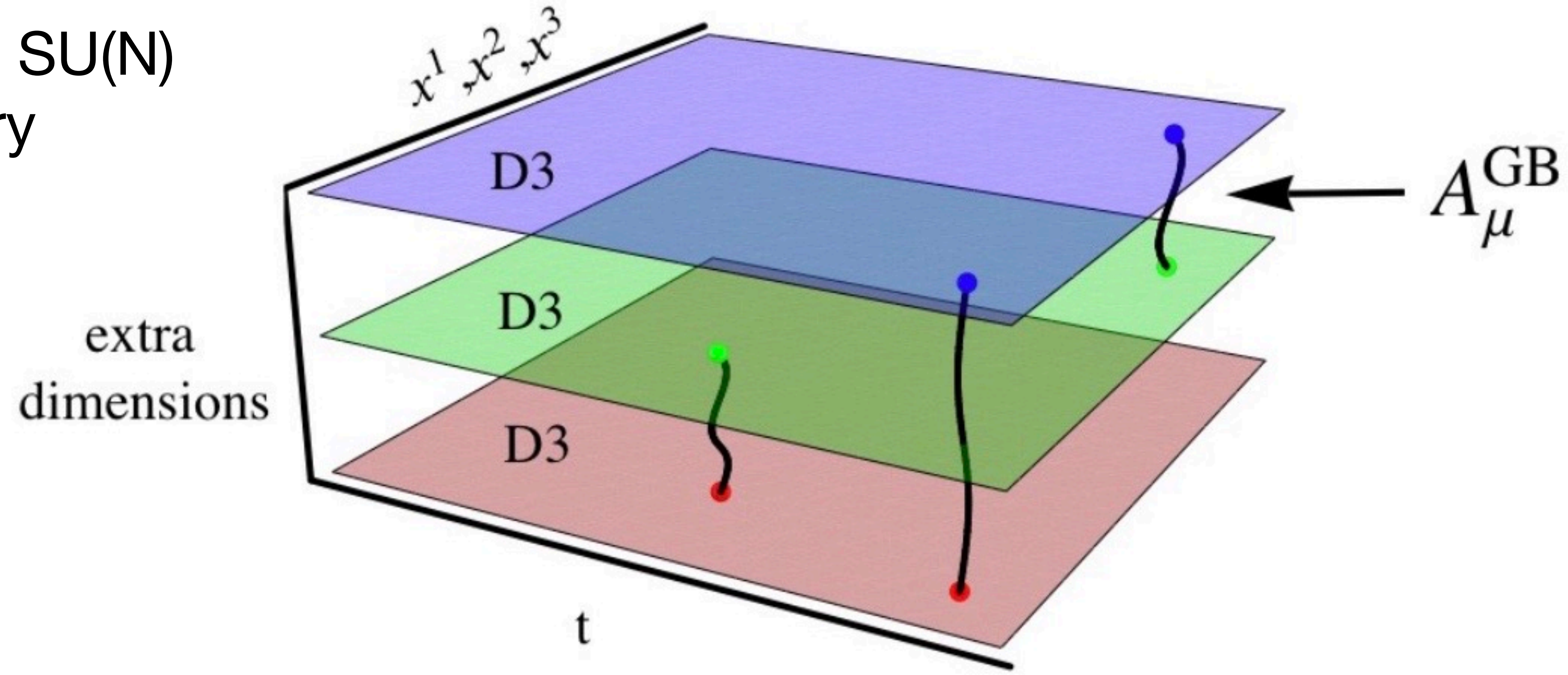


But D3-branes have an alternative description (at strong coupling):
a black hole solution with electric-magnetic charge N
and near-horizon geometry given by

$$AdS_5 \times \mathcal{M} \longrightarrow$$

Sasaki-Einstein manifold
(base of the cone)

Open strings stretching between the branes gives rise to an SU(N) Yang-Mills gauge theory with superconformal symmetry



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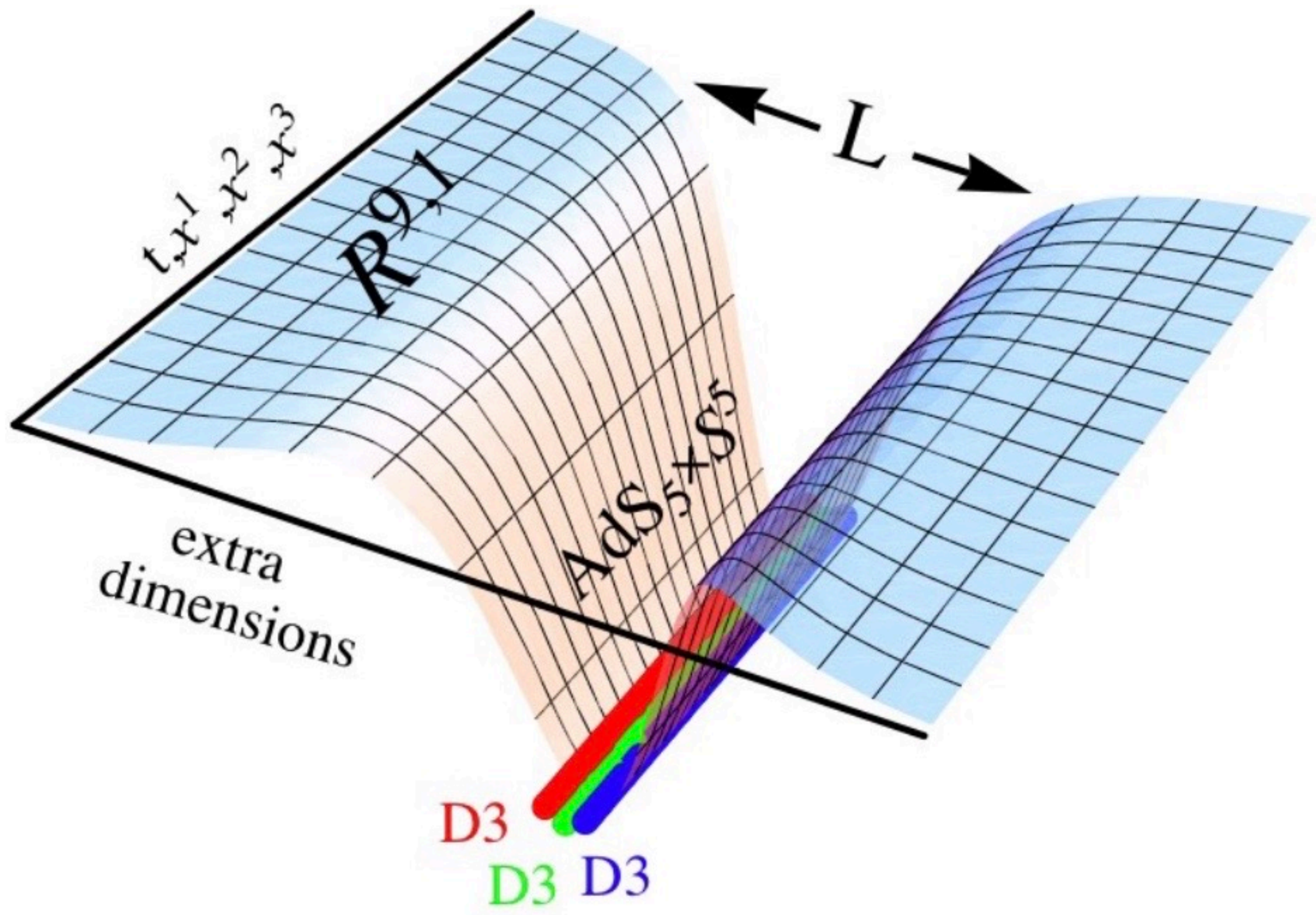
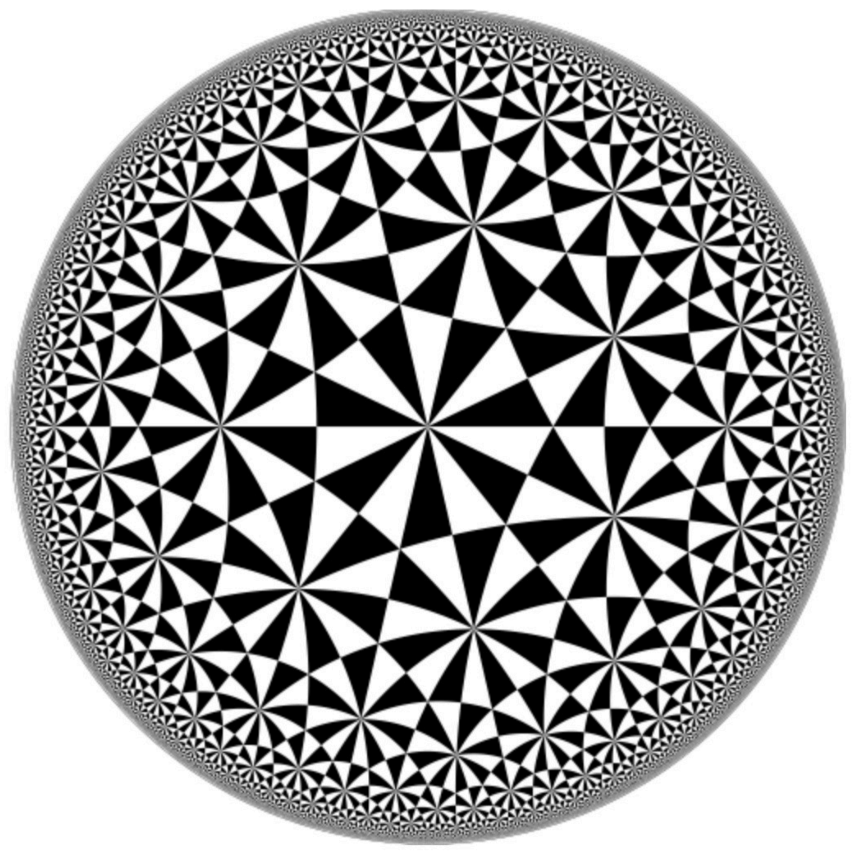
$$AdS_5 \times \mathcal{M}$$



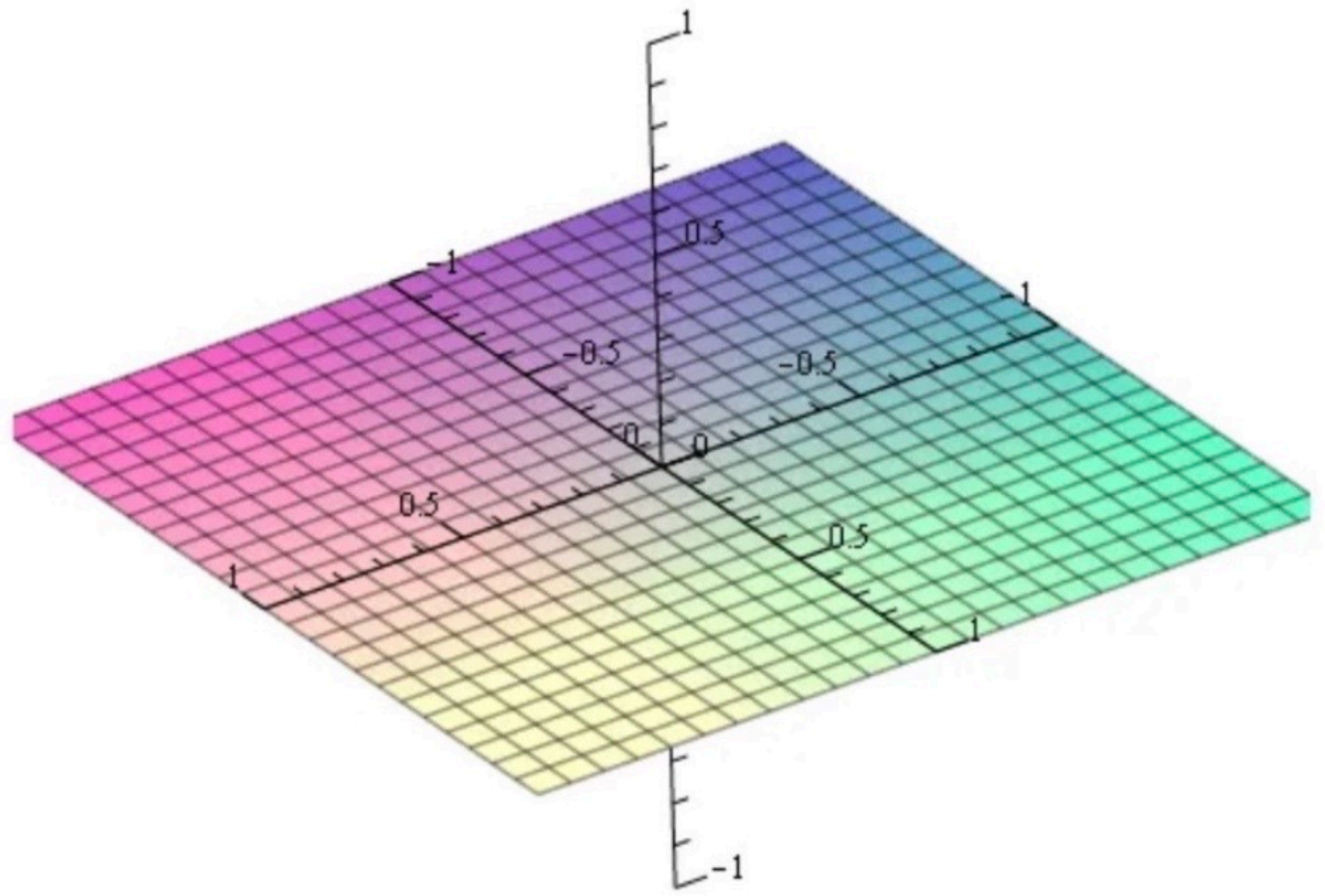
The basis for the AdS/CFT correspondence is that these two descriptions should be identified

5d gravity theory



Equivalence

4d QFT



Crucial expectation of the AdS/CFT correspondence:
The classical **gravity solution** (metric tensor) should ***emerge***
In the large N -limit from a particular quantum state of the dual QFT

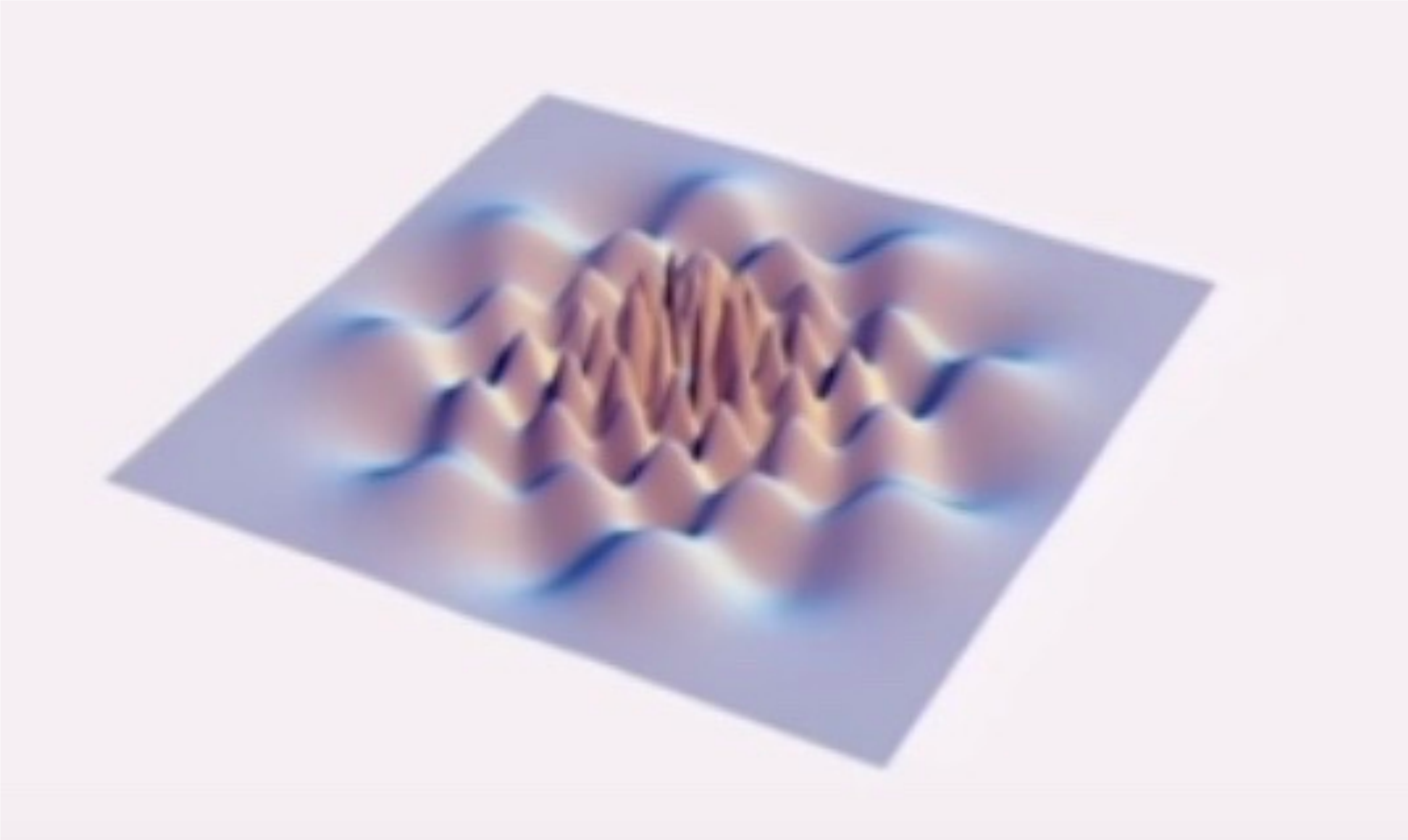
How to describe this emergent geometry?

Emergent geometry

Our aim: Make this precise by exhibiting a canonical (background independent) quantum state Ψ_N and show that its probability amplitude $||\Psi_N||^2$ reproduces the supergravity solution in the limit $N \to \infty$

How to find the state Ψ_N ?

Constrain it using our knowledge of the QFT (symmetries etc.)



Space of **vacua** in the QFT is Y^N/S_N

(These correspond to transversal degrees of freedom of the N branes on $\mathbb{R}^4 \times \{p\} \subset \mathbb{R}^4 \times Y$)

Chiral ring of the gauge theory (BPS-states) $\longleftrightarrow$ Holomorphic functions $\mathcal{O}(Y^N/S_N)$

3d complex variety: Calabi-Yau cone

Y

Space of **vacua** in the QFT is Y^N/S_N

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3d complex variety: Calabi-Yau cone

Y

Chiral ring of the gauge theory (BPS-states) $\longleftrightarrow$ Holomorphic functions $\mathcal{O}(Y^N/S_N)$

$\mathbb{R}_{>0}$ **conformal action**

We seek a holomorphic function Ψ_N such that $|\Psi_N|^2 \cdot [\text{measure}]$ is:

- $(\mathbb{R}_{>0})^N$-invariant
- S_N -invariant

Our proposal for the **canonical quantum state** is:

$$\boxed{\Psi_{N_k} := \Psi_{Slater}^{-1/k}}$$

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$$\boxed{\Psi_{N_k} := \Psi_{Slater}^{-1/k}}$$

- $N_k = \dim \mathcal{O}_k(Y) \qquad (N = N_k)$
- $\Psi_{Slater} = \sum_{\sigma \in S_{N_k}} (-1)^{|\sigma|} \psi_{\sigma(1)}(y_1) \cdots \psi_{\sigma(N_k)}(y_{N_k})$ (Slater determinant)
- $k \quad \mathbb{R}_{>0}$ -charge (**R-charge** in physics)
- $\psi_1, \ldots, \psi_{N_k}$ basis of $\mathcal{O}_k(Y)$

Our proposal for the **canonical quantum state** is:

$$\boxed{\Psi_{N_k} := \Psi_{Slater}^{-1/k}}$$

But the total integral of $|\Psi_{N_k}|^2 \cdot [\text{measure}]$ **diverges** since Y is non-compact

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$$\boxed{\Psi_{N_k} := \Psi_{Slater}^{-1/k}}$$

But the total integral of $|\Psi_{N_k}|^2 \cdot [\text{measure}]$ **diverges** since Y is non-compact

Resolution: Because of $\mathbb{R}_{>0}$ -invariance it reduces to a measure on the quotient

$$\mathcal{M} := (Y - \{0\})/\mathbb{R}_{>0}$$

compact
(internal space)

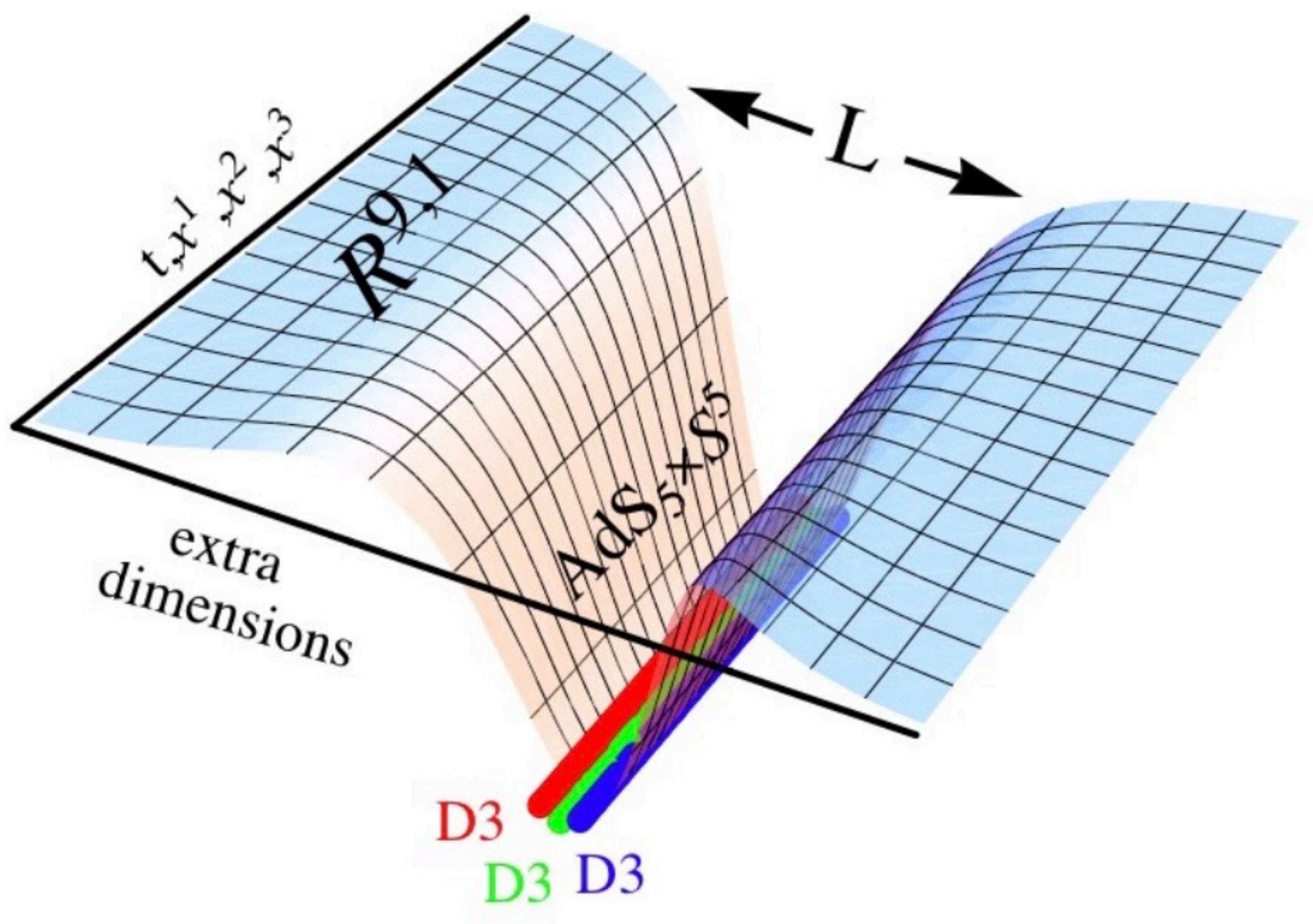
Metric is not given but we show that it **can be recovered from its volume form**.

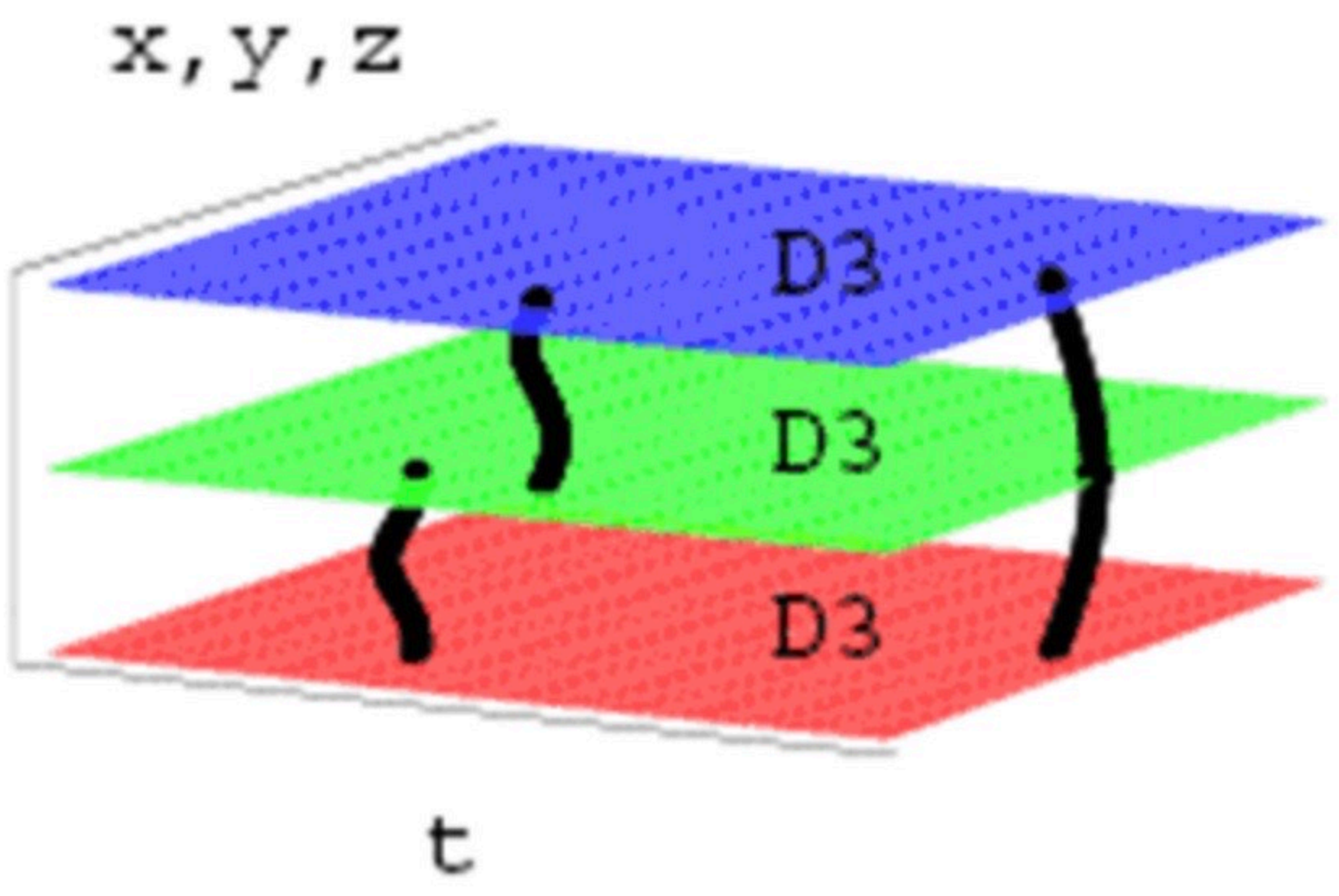
What is the physical interpretation of the state Ψ_N ?

Corresponds to a coherent state of N **dual giant gravitons**

These are D3-branes wrapping 3-cycles in AdS_5

(And thus points on the Sasaki-Einstein manifold $\mathcal{M}$)





Emergent geometry

$|\Psi_{N_k}|^2 \cdot [\text{measure}]$ yields a measure on $\mathcal{M}^{N_k}/S_{N_k}$

Emergent geometry

$\boxed{|\Psi_{N_k}|^2 \cdot [\text{measure}] \text{ yields a measure on } \mathcal{M}^{N_k}/S_{N_k}}$

Makes crucial use of the fact that the metric can be recovered from its volume form

$\downarrow$

[Can be inferred from previous results by Berman (2015)]

$\boxed{g_{N_k}\text{: canonical approximations to the Sasaki-Einstein metric } g \text{ on } \mathcal{M}}$

Emergent geometry

$|\Psi_{N_k}|^2 \cdot [\text{measure}]$ yields a measure on $\mathcal{M}^{N_k}/S_{N_k}$

Makes crucial use of the fact that the metric can be recovered from its volume form

[Can be inferred from previous results by Berman (2015)]

g_{N_k}: canonical approximations to the Sasaki-Einstein metric g on $\mathcal{M}$

Convergence still conjectural, but we prove it "modulo analytic continuation"

(using a β -deformation)

$$\lim_{N_k \to \infty} g_{N_k} = g$$

Recovering the metric

Probability measure

$$\left|\Psi_N(y_1, \ldots, y_N)\right|^2 \left(\Omega \wedge \bar{\Omega}\right)^{\otimes N}$$

holomorphic 3-form

Radial function ("potential")

$$r := \left(\frac{\iota_\delta \Omega \wedge \bar{\Omega}}{dV_{g_M}}\right)^{1/6}$$

Calabi-Yau equation

$$\left(dd^c(r^2)\right)^3 = \Omega \wedge \overline{\Omega},$$

$$d^c := J^* d$$

Metric

$$g_Y := dd^c(r^2)(\cdot, J\cdot)$$

Reduction to $\mathcal{M}$

$$d\mathbb{P}_{N_k} := \frac{1}{\mathcal{Z}_{N_k}} |\Psi_{N_k}|^2 \left(\iota_\delta(\Omega \wedge \overline{\Omega})\right)^{\otimes N_k}$$

$$\mathcal{Z}_{N_k} := \int_{M^{N_k}/S_{N_k}} |\Psi_{N_k}|^2 \left(\iota_\delta(\Omega \wedge \overline{\Omega})\right)^{\otimes N_k}$$

Probability measure on $\mathcal{M}$

$$d\mathbb{P}_N^{(1)}(y) = \frac{1}{\mathcal{Z}_N} \int_{M^{N-1}/S_{N-1}} |\Psi_{N_k}(y, y_2, ..., y_N)|^2 \times \left(\iota_\delta(\Omega \wedge \overline{\Omega})\right)^{\otimes N-1}.$$

Conjecture

$$\lim_{N\to\infty} d\mathbb{P}_N^{(1)} = dV_M$$

$$r_N := \left(\frac{d\mathbb{P}_N^{(1)}}{\iota_\delta(\Omega \wedge \overline{\Omega})}\right)^{-1/6}$$

$$\boxed{\lim_{N\to\infty} g_M^{(N)} = g_M}$$

β -deformation

$$d\mathbb{P}_{N,\beta} := \frac{1}{\mathcal{Z}_{N,\beta}} \left|\Psi_{\det}(y_1, y_2, ..., y_N)^2\right|^{\frac{3\beta}{\lambda_k}} dV_0^{\otimes N}$$

$$r_{N,\beta} := \left(\frac{d\mathbb{P}_{N,\beta}^{(1)}}{dV_0}\right)^{1/6\beta} r_0$$

Theorem B: For each $\beta > 0$, there exists

(i) a volume form μ_β on M such that

$$\lim_{N\to\infty} d\mathbb{P}^{(1)}_{N,\beta} = \mu_\beta,$$

(ii) a radial function r_β on Y such that

$$\lim_{N\to\infty} r_{N,\beta} = r_\beta,$$

(iii) and a Sasaki metric $g_{M,\beta}$ on M such that

$$\lim_{N\to\infty} g^{(N)}_{M,\beta} = g_{N,\beta}.$$

Sketch of proof

1. Circle bundle over Fano X

$$\begin{array}{ccc} M_0 & \hookrightarrow & L^*(= Y^*) \\ & \searrow & \downarrow \\ & & X \end{array}$$

$$M_0 = \{r_0 = 1\}$$

Sketch of proof

1. Circle bundle over Fano X

$$\begin{array}{ccc} M_0 & \hookrightarrow & L^*(=Y^*) \\ & \searrow & \downarrow \\ & & X \end{array}$$

$$M_0 = \{r_0 = 1\}$$

2. Express as a product measure: $\mu = \nu \otimes d\theta,$

$$d\mathbb{P}^{(1)}_{N,\beta} = \nu_{N,\beta} \otimes d\theta^{\otimes N},$$

measure on

X^N/S_N

Sketch of proof

1. Circle bundle over Fano X

$$\begin{tikzcd} M_0 \arrow[r, hook] \arrow[dr] & L^*(=Y^*) \arrow[d] \\ & X \end{tikzcd}$$

$$M_0 = \{r_0 = 1\}$$

2. Express as a product measure: $\mu = \nu \otimes d\theta$,

$$\Rightarrow \quad d\mathbb{P}'_{N,\beta} = \nu_{N,\beta} \otimes d\theta^{\otimes N},$$

3. Result follows now from [Berman (2015)]


Thank you!