Toward minimal automorphic representations of Kac-Moody groups

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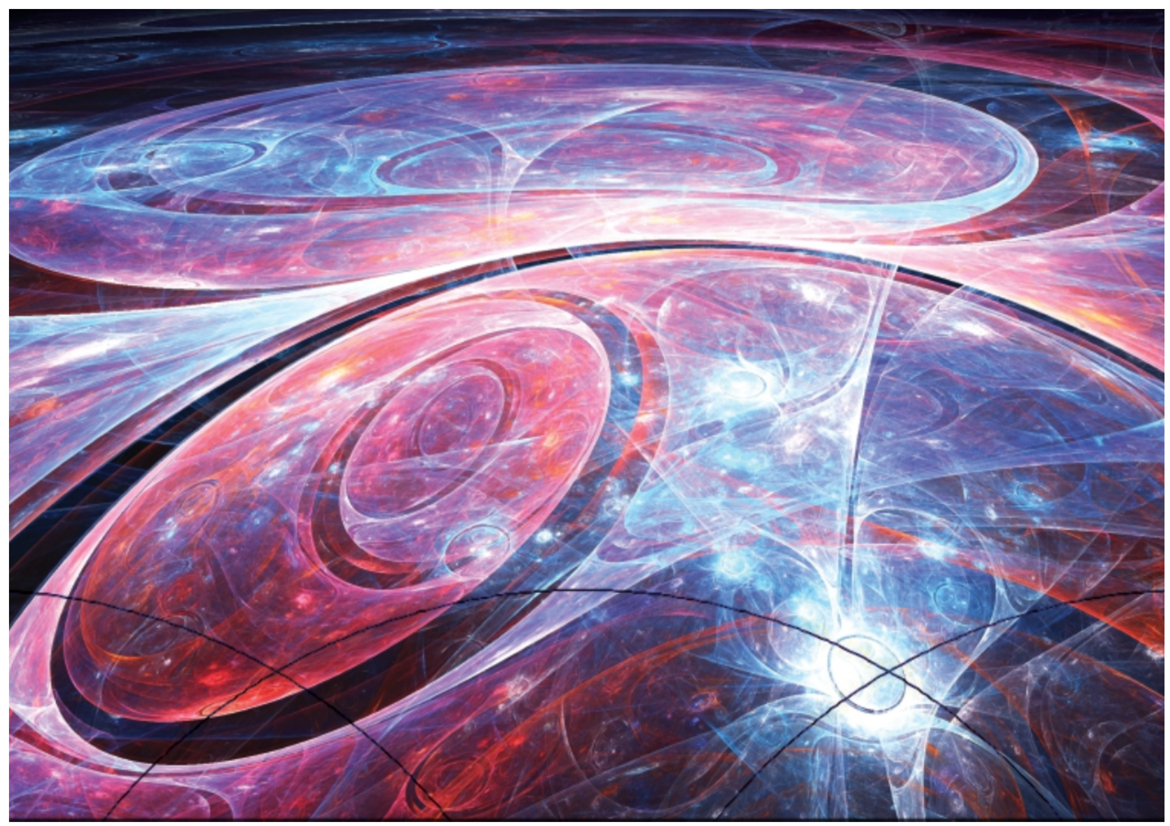


AG&NT seminar
Wednesday, November 30, 2022

Based on

Fourier expansion of Kac-Moody Eisenstein series and degenerate Whittaker vectors

w/ Fleig, Kleinschmidt; Commun. Num. Th. Phys. **08** (2014), 41-100

[arXiv:1707.08937]

Eisenstein series and automorphic representations

- with applications in string theory

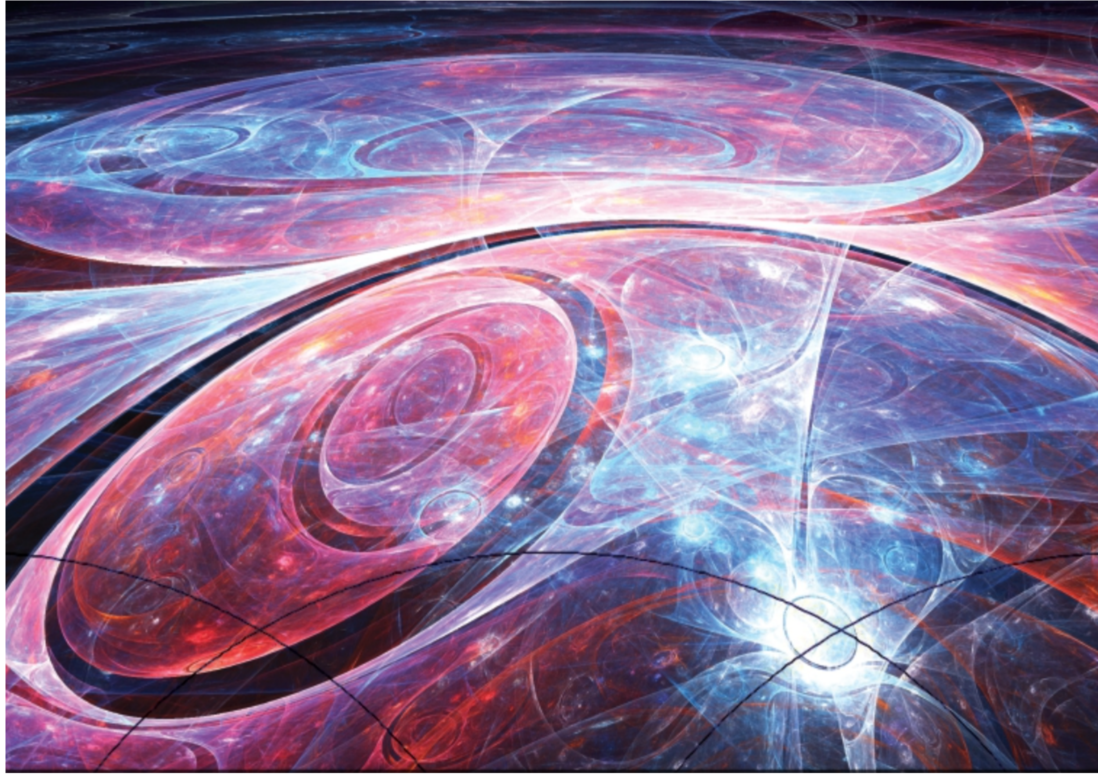
w/ Fleig, Gustafsson, Kleinschmidt; Cambridge University Press,

Cambridge studies in advanced mathematics, vol **176** (2018).

…and work in progress:

w/ Gourevitch, Kleinschmidt, Patnaik

Outline



1. Motivation

2. Kac-Moody algebras - a glimpse

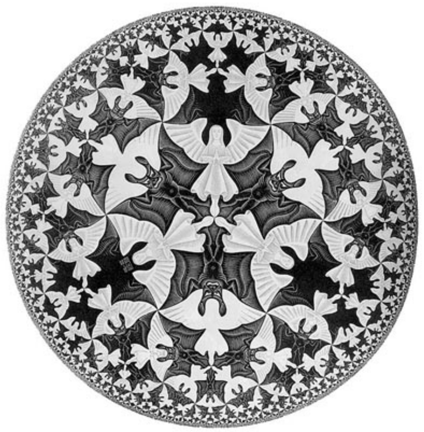
3. Automorphic forms and their Fourier coefficients

4. Eisenstein series on Kac-Moody groups

5. Outlook

1. Motivation

Fourier coefficients of automorphic forms



- Fourier coefficients of **classical modular forms** encode deep number-theoretic information (counting points on elliptic curves etc..)

- **Moonshine:** relations with finite sporadic groups and CFT/string theory

- **Enumerative geometry:** rational curves on K3, GW-theory…

- Higher rank groups: **Langlands program**
  (automorphic L-functions, functoriality…)

- The Fourier coefficients of Eisenstein series also encode **string theory scattering amplitudes**

Automorphic forms on Kac-Moody groups?

Kac-Moody groups are infinite-dimensional generalisations of Lie groups

Automorphic forms on Kac-Moody groups?

Kac-Moody groups are infinite-dimensional generalisations of Lie groups

- (Mid 90's): **Kapranov** constructed certain "geometric Eisenstein series on loop groups", motivated by Vafa-Witten theory
- (Late 90's): **Garland** constructed Langlands-type Eisenstein series on loop groups over number fields
- (2005): **Shahidi** suggested that Eisenstein series on Kac-Moody groups might provide a source of new L-functions
- (2008): **Patnaik** showed in his thesis that Kapranov's and Garland's constructions agree in the function field case
- (2013): **Fleig, DP, Kleinschmidt** calculate explicit Fourier expansion for Eisenstein serien on certain "exceptional" Kac-Moody groups (motivated by string theory)

1. Kac-Moody algebras - a glimpse

Kac-Moody algebras

Infinite-dimensional generalisations of simple Lie algebras

A rank r simple Lie algebra: $3r$ generators $\{e_i, f_i, h_i\}$

$$[e_i, f_i] = h_i, \qquad [h_i, e_i] = 2e_i, \qquad [h_i, f_i] = -2f_i.$$

SL_2 triple

Kac-Moody algebras

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SL_2
triple

$$\boxed{\begin{aligned} [e_i, f_j] &= \delta_{ij} h_j, \\ [h_i, e_j] &= A_{ij} e_j, \\ [h_i, f_j] &= -A_{ij} f_j, \\ [h_i, h_j] &= 0. \end{aligned}}$$

A

Cartan matrix

The Lie algebra $\mathfrak{g} = \mathfrak{g}(A)$ is the freely generated algebra on these generators subject to the Serre relations:

$$\mathrm{ad}_{e_i}^{1-A_{ij}}(e_j) = [e_i, [e_i, \ldots, [e_i, e_j] \cdots]] = 0,$$

$$\mathrm{ad}_{f_i}^{1-A_{ij}}(f_j) = [f_i, [f_i, \ldots, [f_i, f_j] \cdots]] = 0,$$

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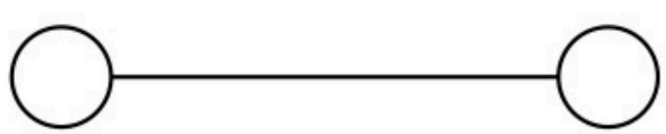
$$A[A_2] = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$$

Serre relation:

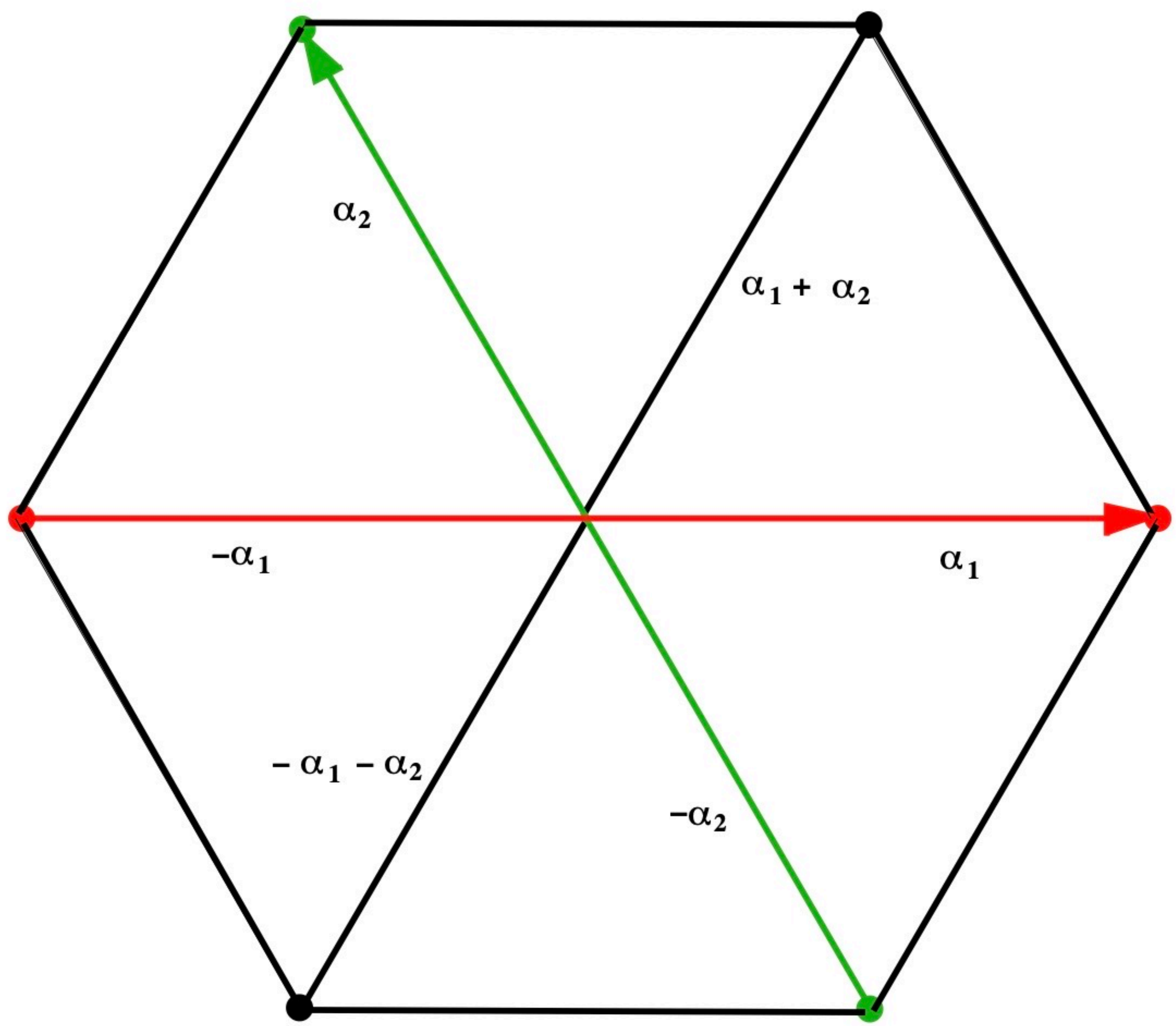
$$\mathrm{ad}_{e_1}^{1-A_{12}}(e_2) = [e_1, [e_1, e_2]] = 0$$

Dynkin diagram



$$\mathrm{ad}_h(e_i) = [h, e_i] = \alpha_i(h) e_i$$

simple roots



8-dimensional adjoint representation of

$$A_2 = SL_3$$

Classification

$$\boxed{\begin{aligned} A_{ii} &= 2, \quad i = 1, \ldots, r, \\ A_{ij} &= 0 \Leftrightarrow A_{ji} = 0, \\ A_{ij} &\in \mathbb{Z}_- \quad (i \neq j). \end{aligned}}$$

- If A is positive definite, the algebra $\mathfrak{g}(A)$ is finite-dimensional and falls under the Cartan-Killing classification, i.e., it is one of the finite simple Lie algebras A_n, B_n, C_n, D_n, G_2, F_4, E_6, E_7 or E_8.

- If A is positive-semidefinite, i.e., det $A = 0$ with one zero eigenvalue, the algebra is infinite-dimensional and is said to be an *affine* Kac-Moody algebra. All affine Kac-Moody algebras are classified

- If A is not part of the two classes above, the algebra $\mathfrak{g}(A)$ is infinite-dimensional and is generally called an *indefinite* Kac-Moody algebra, by virtue of the fact that A is of indefinite signature.

Kac-Moody algebras

- Simple and affine Lie algebras are a subclass

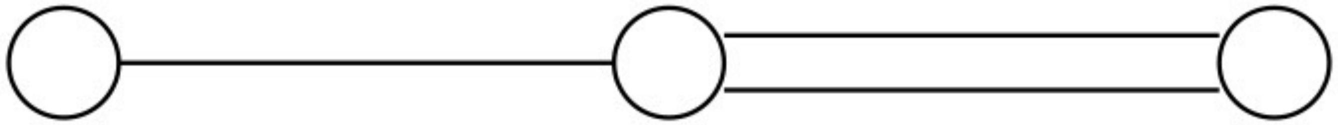
- Hyperbolic KM-algebras have been classified
  (For rank > 2 there are 238 of them) [Carbone et al]

- No classification of indefinite KM-algebras beyond hyperbolic.

- Root lattice is of indefinite signature so we have "imaginary roots", i.e. roots such that

$$(\alpha|\alpha) \leq 0$$

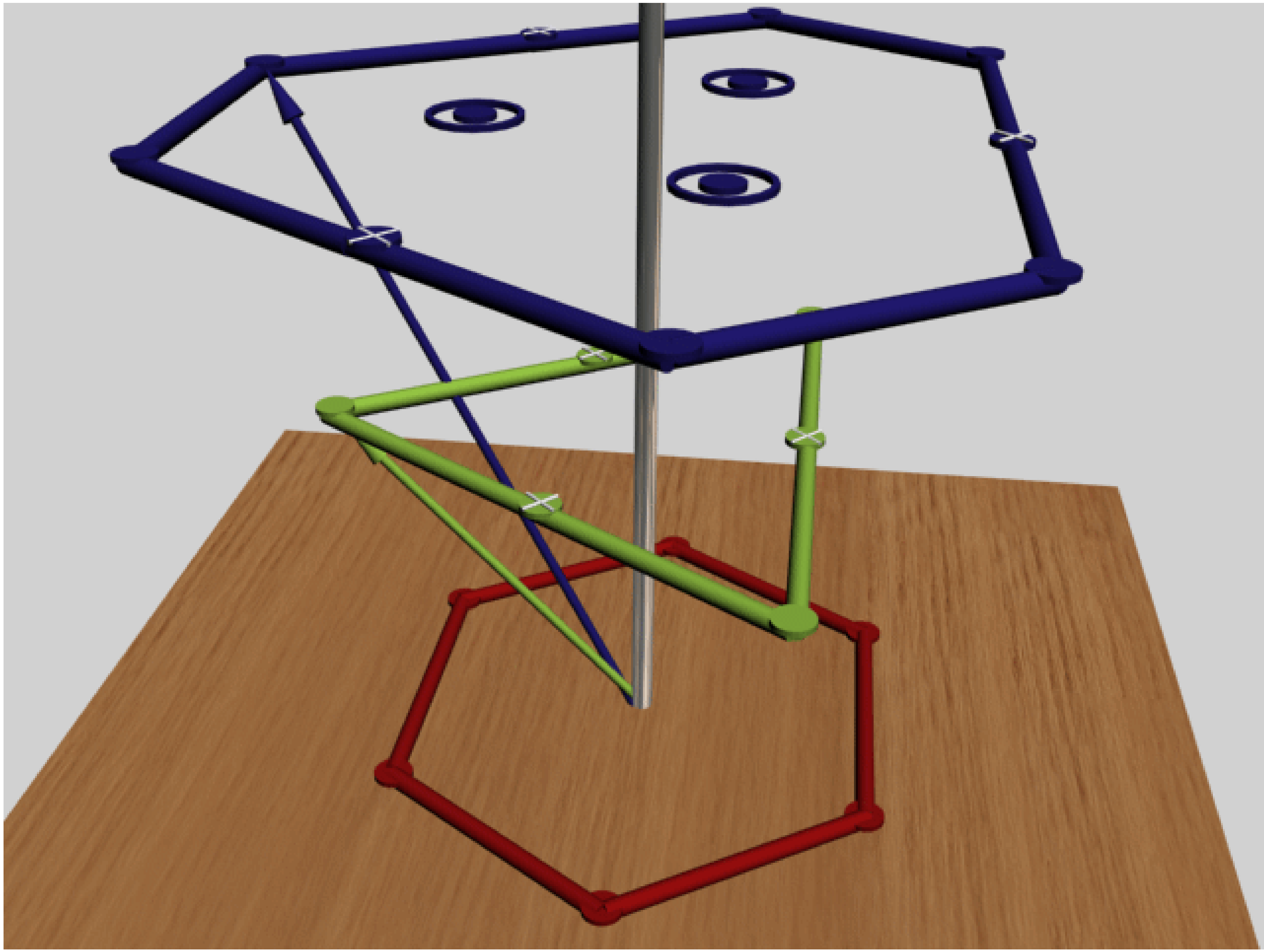
Example: Hyperbolic KM-algebra A_1^{++}

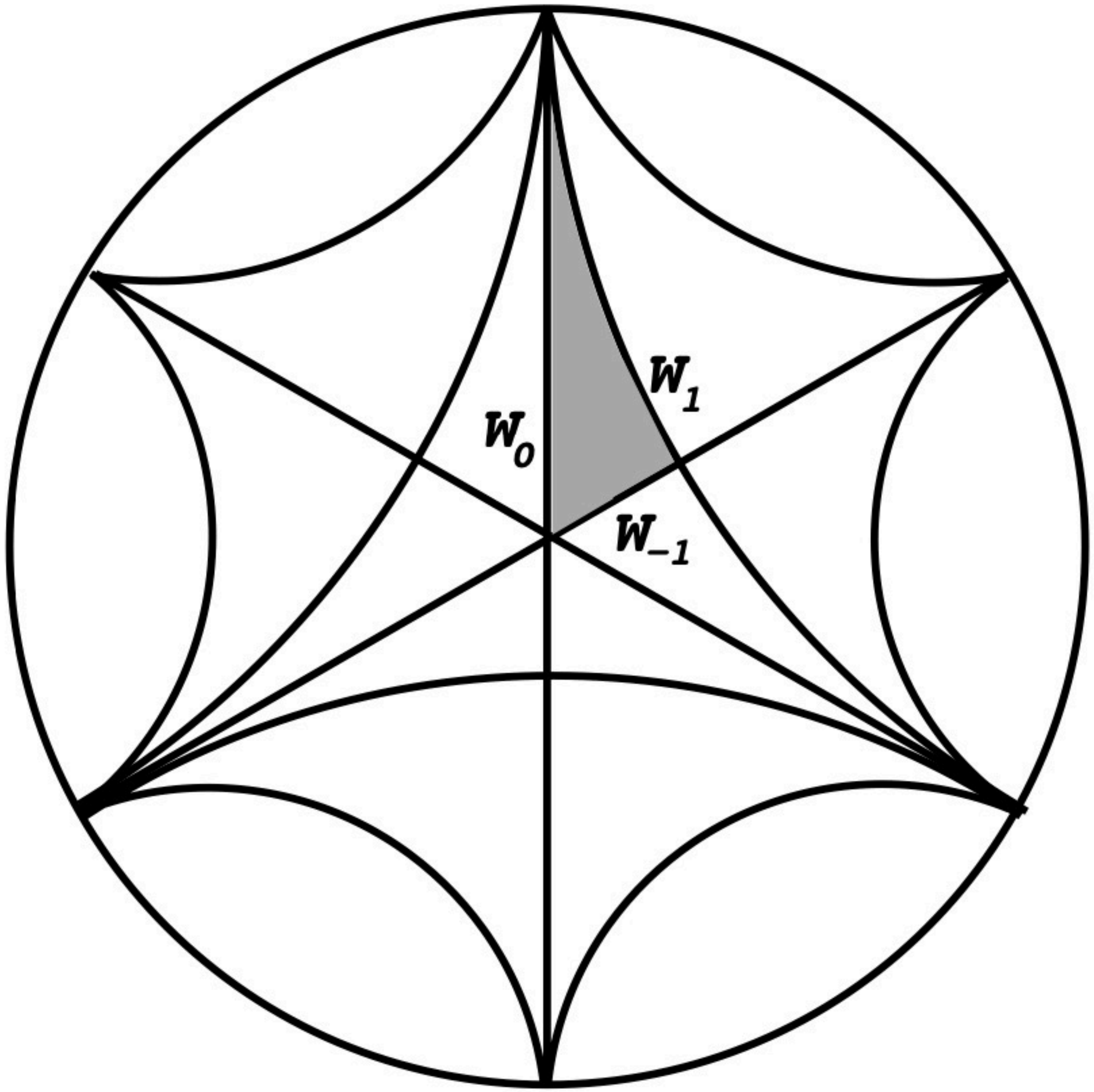


$$A(A_1^{++}) = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -2 \\ 0 & -2 & 2 \end{pmatrix}$$

Root lattice has **Lorentzian** signature

Roots with with crosses are **on the lightcone**





$$\mathcal{W}(A_1^{++}) \;\simeq\; PGL(2,\mathbb{Z}).$$

3. Automorphic forms and their Fourier coefficients

Data:

- $G(\mathbb{R})$ real simple Lie group (e.g. $SL(n,\mathbb{R})$)
- $G(\mathbb{Z}) \subset G$ arithmetic subgroup (e.g. $SL(n,\mathbb{Z})$)

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Definition:

An **automorphic form** is a smooth function $\varphi : G \longrightarrow \mathbb{C}$ satisfying

1. Automorphy: $\forall \gamma \in G(\mathbb{Z}), \;\; \varphi(\gamma g) = \varphi(g)$
2. φ is an eigenfunction of the ring of inv. diff. operators on G
3. φ has well-behaved growth conditions

Example: Eisenstein series on $SL(2,\mathbb{R})$

$$E(s,\tau) = \sum_{\substack{(m,n)\in\mathbb{Z}^2 \\ (m,n)\neq(0,0)}} \frac{y^s}{|m\tau+n|^{2s}} \qquad s\in\mathbb{C}$$

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→ a function on $\mathbb{H} = \{\tau = x+iy \in \mathbb{C}\,|\,y>0\}$

→ invariant under $\tau \mapsto \gamma\cdot\tau = \dfrac{a\tau+b}{c\tau+d}$

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z})$$

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$\longrightarrow$ a function on $\mathbb{H} = \{\tau = x+iy \in \mathbb{C}\,|\, y>0\}$

$\longrightarrow$ invariant under $\tau \mapsto \gamma\cdot\tau = \dfrac{a\tau+b}{c\tau+d}$

$\longrightarrow$ converges absolutely for $\Re s > 1$

$\longrightarrow$ $\Delta_{\mathbb{H}} E_s = s(s-1)E_s$

Eisenstein series on semi-simple Lie groups

The **Langlands Eisenstein series** on a semi-simple Lie group is defined by:

$$\boxed{E(\lambda, g) = \sum_{\gamma \in B(\mathbb{Z})\backslash G(\mathbb{Z})} e^{\langle \lambda + \rho | H(\gamma g) \rangle}}$$

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Iwasawa decomposition: $G = BK = NAK$

$$A \sim \begin{pmatrix} * & & \\ & * & \\ & & * \end{pmatrix} \qquad N \sim \begin{pmatrix} 1 & * & * \\ & 1 & * \\ & & 1 \end{pmatrix}$$

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Logarithm map: $H \ : \ G \to \mathfrak{h} = \mathrm{Lie}\, A$ $\qquad H(nak) = \log a$

Weight: $\lambda \in \mathfrak{h}^\star \otimes \mathbb{C}$ $\qquad$ Weyl vector: $\rho = \frac{1}{2} \sum_{\alpha > 0} \alpha$

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→ Converges absolutely on a subspace of $\mathfrak{h}^\star \otimes \mathbb{C}$

Godement's domain

$\{\lambda | \langle \lambda, \alpha \rangle > 1, \forall \alpha \in \Pi\}$

→ Can be continued to a meromorphic function on all of $\mathfrak{h}^\star \otimes \mathbb{C}$ [Langlands]

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- Can be continued to a meromorphic function on all of $\mathfrak{h}^\star \otimes \mathbb{C}$ [Langlands]
- Invariant: $E(\lambda, \gamma g k) = E(\lambda, g) \qquad \gamma \in G(\mathbb{Z}) \qquad k \in K$
- Eigenfunction of the Laplacian: $\Delta_{G/K} E(\lambda, g) = \frac{1}{2}(\langle \lambda | \lambda \rangle - \langle \rho | \rho \rangle) E(\lambda, g)$
- Functional relation: $E(\lambda, g) = M(w, \lambda) E(w\lambda, g), \quad \forall w \in W(\mathfrak{g})$

Fourier expansion

$$E(s,\tau) = \sum_{(c,d)=1} \frac{y^s}{|c\tau + d|^{2s}}$$

Periodicity:

$$E(s,\tau+1) = E(s,\tau)$$

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Periodicity:

$$\boxed{E(s,\tau+1) = E(s,\tau)}$$

$$= y^s + \frac{\xi(2s-1)}{\xi(2s)} y^{1-s}$$

$$+ \frac{2y^{1/2}}{\xi(2s)} \sum_{m\neq 0} |m|^{s-1/2} \sigma_{1-2s}(m) K_{s-1/2}(2\pi|m|y) e^{2\pi i m x}$$

$$\tau = x + iy$$

$$\sigma_s(m) = \sum_{d|m} d^s$$

Fourier coefficients

The periodicity $E(s,\tau+1)=E(s,\tau)$ generalises to

$$E(\lambda, ng) = E(\lambda, g) \qquad n \in N(\mathbb{Z})$$

Much more complicated since $N(\mathbb{Z})$ is **non-abelian**.

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General structure:

$$E(\lambda,g)=E^{\mathrm{const}}(\lambda,g)+\sum_{\psi}W_{\psi}(\lambda,g)+\cdots$$

constant term (zero-mode)
Calculated from **Langlands' constant term formula**

Fourier coefficients

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Whittaker coefficients

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$$W_\psi(g) = \int_{N(\mathbb{Z})\backslash N(\mathbb{R})} E(\lambda, ng)\overline{\psi(n)}dn$$

Whittaker coefficient

$$\psi\ :\ N(\mathbb{Z})\backslash N(\mathbb{R}) \to U(1)$$

unitary character on $N(\mathbb{R})$
trivial on $N(\mathbb{Z})$

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Whittaker coefficient

$$\psi \,:\, N(\mathbb{Z})\backslash N(\mathbb{R}) \to U(1)$$

unitary character on $N(\mathbb{R})$

$$\longrightarrow \quad \psi(n) = e^{2\pi i \sum_j m_j x_j}$$

(simple roots)

$$m_j \in \mathbb{Z}$$

$$x_j \in \mathbb{R}$$

if all $m_j \neq 0$ then ψ is **generic**

if some $m_j = 0$ then ψ is **degenerate**

Euler products

Before we proceed with the higher rank case we mention some further properties of the Fourier expansion, namely that it decomposes into Euler products

$$W_\psi(g) = W_\infty(g) \prod_{p<\infty} W_p(1)$$

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archimedean Whittaker coefficient

$$W_\infty(g) = \frac{2\pi^s}{\Gamma(s)} y|m|^{s-1/2} K_{s-1/2}(2\pi|m|y) e^{2\pi i m x}$$

$$g = nak = \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} \begin{pmatrix} y^{1/2} & \\ & y^{-1/2} \end{pmatrix} k \in SL(2,\mathbb{R})$$

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p -adic Whittaker coefficient

$$W_p(1) = (1 - p^{-2s}) \frac{1 - p^{-2s+1} |m|_p^{2s-1}}{1 - p^{-2s+1}}$$

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$1 \in SL(2,\mathbb{Q}_p)$

$$W_p(1) = (1-p^{-2s})\frac{1-p^{-2s+1}|m|_p^{2s-1}}{1-p^{-2s+1}}$$

This is the basis for the **adelic** formulation of automorphic forms

Theorem [Jacquet, Langlands]**:** The generic Whittaker coefficient is Eulerian

$$W_\psi(\lambda, g) = \int_{N(\mathbb{A})} \chi(w_0 n g)\, \overline{\psi(n)}\, dn = \prod_p W_{\psi_p}$$

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$$w_0 = \text{longest element of } W(\mathfrak{g})$$

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$$W_{\psi_\infty} = \int_{N(\mathbb{R})} \chi_\infty(w_0 n a_\infty) \overline{\psi_\infty(n)}\, dn$$

$$W_{\psi_p} = \int_{N(\mathbb{Q}_p)} \chi_p(w_0 n a_p) \overline{\psi_p(n)}\, dn$$

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Note: This can also be written as

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opposite unipotent (lower-triangular)

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Theorem [Shintani, Casselman-Shalika]**:** When $p < \infty$ we have

$$W_{\psi_p} = \text{Weyl character formula for } {}^L G$$

3. Automorphic forms on Kac-Moody groups

Eisenstein series can formally be defined for any **Kac-Moody group** G

$$E(\lambda, g) = \sum_{\gamma \in B(\mathbb{Z}) \backslash G(\mathbb{Z})} e^{\langle \lambda + \rho | H(\gamma g) \rangle}$$

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→ $G(\mathbb{Z}) \subset G(\mathbb{R})$ defined as a **Chevalley group** [Garland, Carbone…]
(through exponentiation of Lie algebra generators)

$$G(F) := \langle u_{\alpha_i}(s), u_{-\alpha_i}(t) \,|\, s, t \in F, i \in \{1, \ldots, r\}\rangle$$

→ **convergence** established by Garland in the affine case
and by Carbone, Lee, Liu for rank 2 hyperbolic. General case wide open!

→ generalization of Langlands **constant term formula** established by Garland in the (untwisted) affine case

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$$E(\lambda, g) = \sum_{\gamma \in B(\mathbb{Z})\backslash G(\mathbb{Z})} e^{\langle \lambda + \rho | H(\gamma g) \rangle}$$

$\longrightarrow$ **Function field analogue** studied by Braverman, Kazhdan, Patnaik.

$\longrightarrow$ **Cuspidal Eisenstein series** studied by Garland, Miller, Patnaik

$\longrightarrow$ **Fourier coefficients** studied by Lee, Liu, Patnaik, Fleig, Kleinschmidt, D.P. ...

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A **crucial difference** from the finite case:

$$B = B^{+} \quad \text{and} \quad B^{-} \quad \text{not conjugate!}$$

This is due to the **absence** of a longest Weyl word: w_0

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$$\boxed{\begin{array}{c}\textbf{\textit{Positive}}\\ \text{Eisenstein series}\end{array}} \quad \xleftrightarrow{\ ?\ } \quad \boxed{\begin{array}{c}\textbf{\textit{Negative}}\\ \text{Eisenstein series}\end{array}}$$

Over function fields these are related by a **functional equation** [Braverman, Kazhdan]

Non-constant terms

$$E(\lambda, g) = E^{\text{const}}(\lambda, g) + \sum_{\psi} W_{\psi}(\lambda, g) + \cdots$$

$$W_{\psi}(g) = \int_{N(\mathbb{Z})\backslash N(\mathbb{R})} E(\lambda, ng)\overline{\psi(n)}dn$$

Whittaker coefficient

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Whittaker coefficient

For **Kac-Moody groups**

$W_\psi = 0$ for ψ **generic**

due to the **lack of a longest Weyl word** in the Weyl group $W(\mathfrak{g})$

Recall **finite-dim** case: $\chi(g) = e^{\langle \lambda+\rho | H(g) \rangle}$

$$W_\psi(\lambda, g) = \int_{N(\mathbb{A})} \chi(w_0 n g)\, \overline{\psi(n)}\, dn = \prod_p W_{\psi_p}$$

$$W_{\psi_p} = \int_{N(\mathbb{Q}_p)} \chi_p(w_0 n a_p) \overline{\psi_p(n)}\, dn$$

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$$= \int_{N^-(\mathbb{Q}_p)} \chi_p(n a_p) \overline{\psi_p(n)} dn$$

opposite unipotent

Braverman-Kazhdan: Add this "by hand" in the Kac-Moody case

$$\int_{N^-(\mathbb{Q}_p)} \chi_p(na_p)\overline{\psi_p(n)}dn$$

Braverman-Kazhdan: Add this "by hand" in the Kac-Moody case

$$\int_{N^-(\mathbb{Q}_p)} \chi_p(na_p)\overline{\psi_p(n)}dn$$

- Affine generalization of **Casselman-Shalika formula** $(p < \infty)$ [Patnaik]
- Not known how to do this for $\mathbb{Q}_\infty = \mathbb{R}$
- Beyond the affine case not understood

Instead, we focus on **degenerate Whittaker coefficients**

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Theorem [Fleig, Kleinschmidt, D.P.] $(m_j = 0 \text{ for some } j)$

For **degenerate** characters (and assuming convergence) we have

$$\boxed{W_\psi(\lambda, a) = \sum_{ww_0' \in \mathcal{C}_\psi} e^{\langle (ww_0')^{-1}\lambda + \rho | a \rangle} M(w^{-1}, \lambda) W_\psi^{G'}(w^{-1}, \mathbb{I})}$$

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- Subgroup $G' \subset G$ selected by $m_j \neq 0$
- w_0' longest Weyl word of $W(\mathfrak{g}')$
- $W_\psi^{G'}$ generic Whittaker vector of G'
- $\mathcal{C}_\psi \subset W(\mathfrak{g})$

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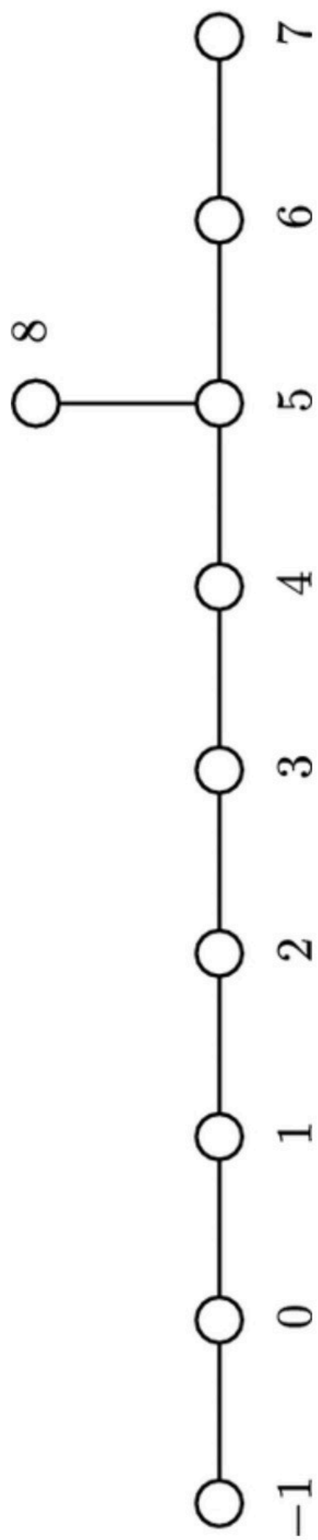
$$M(w, \lambda) = \prod_{\alpha > 0 \,|\, w\alpha < 0} \frac{\xi(\langle \lambda | \alpha \rangle)}{\xi(1 + \langle \lambda | \alpha \rangle)}$$

intertwiner

$$G = E_{10}(\mathbb{R})$$



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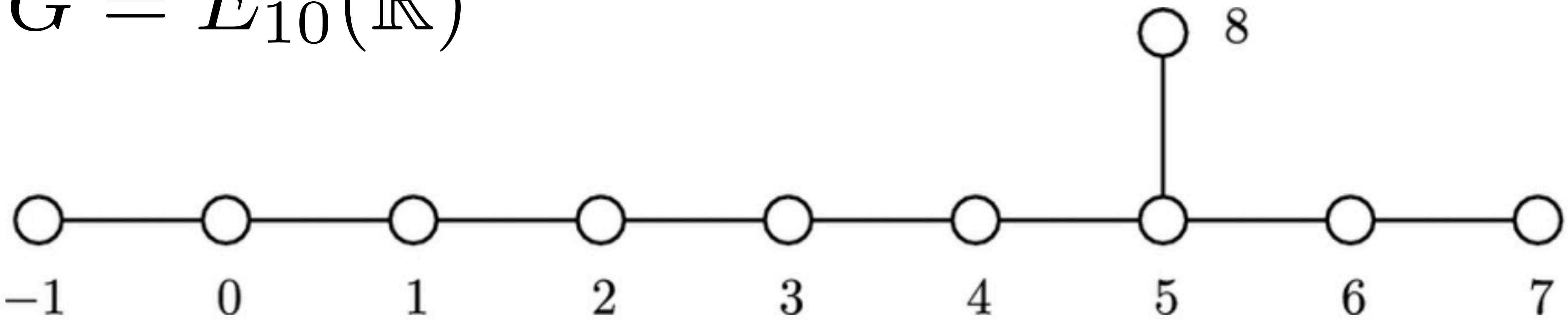

Eisenstein series: $$E(\lambda, g) = \sum_{\gamma \in B(\mathbb{Z})\backslash G(\mathbb{Z})} e^{\langle \lambda + \rho | H(\gamma g)\rangle}$$

Generically, this has an **infinite number** of Fourier coefficients for each character ψ

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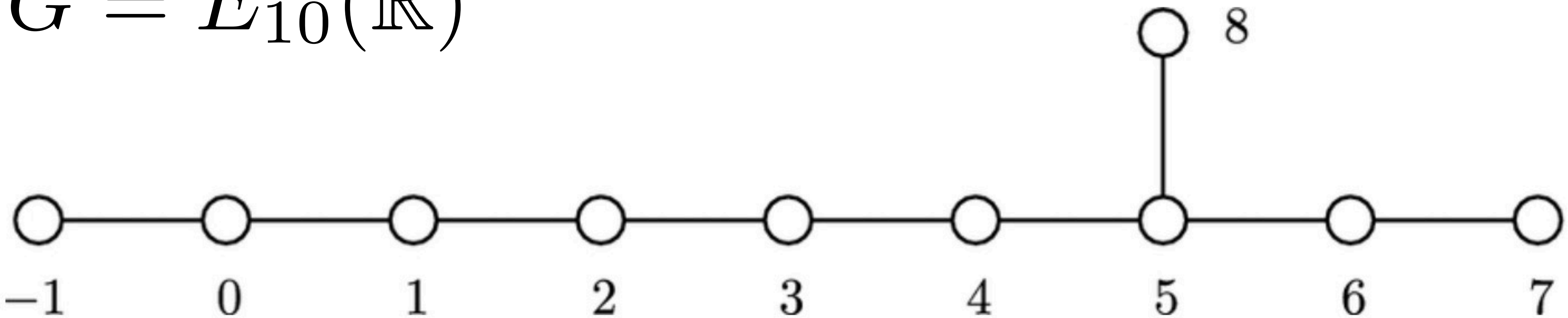

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Originally motivated by string amplitudes we **study the special value**:

$$s = 3/2$$

$$\lambda = 3\Lambda_1 - \rho$$

For the special value $\lambda = 3\Lambda_1 - \rho$ the sum **collapses**!

ψ	$W_\psi(\chi_{3/2}, a)$
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$$\boxed{B_{s,m}(a) \sim \Big(\sum_{d|m} d^s\Big) K_s(ma)}$$

This is due to the **vanishing** properties of the intertwiner

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$(0,0,0,0,0,m,0,0,0,0)$	$\xi(2) v_6^3 B_{-1/2}$ [illegible]
$(0,0,0,0,0,0,m,0,0$ [illegible]	[illegible]
$(0,0,0,0,0,0$ [illegible]	[illegible] $_{,m}\left(v_8^2 v_7^{-1} v_9^{-1}\right)$ / $\xi(3)$
$(0,0,0,0,0,0,$ [illegible]	$\frac{\xi(5) v_9^6 v_{10}^{-5} B_{-2,m}\left(v_9^2 v_8^{-1} v_{10}^{-1}\right)}{\xi(3)}$
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These are the only non-vanishing coefficients!

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Conjecture: *For* $G = E_9, E_{10}, E_{11}$ *the Eisenstein series* $E(3\Lambda_1 - \rho, g)$ *is attached to the* ***minimal representation*** π_{min}

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These representations have not yet been defined for Kac-Moody groups

(no notion of minimal nilpotent orbit etc…)

→ Attempts to prove this in progress with Kleinschmidt & Patnaik

Minimal automorphic representations

There exists special representations with smallest non-trivial **Gelfand-Kirillov** dimension

The GK-dimension gives a notion of the **size** of a representation on a function space

Example:

$$\mathrm{GKdim}\big(L^2(\mathbb{R}^n)\big) = n$$

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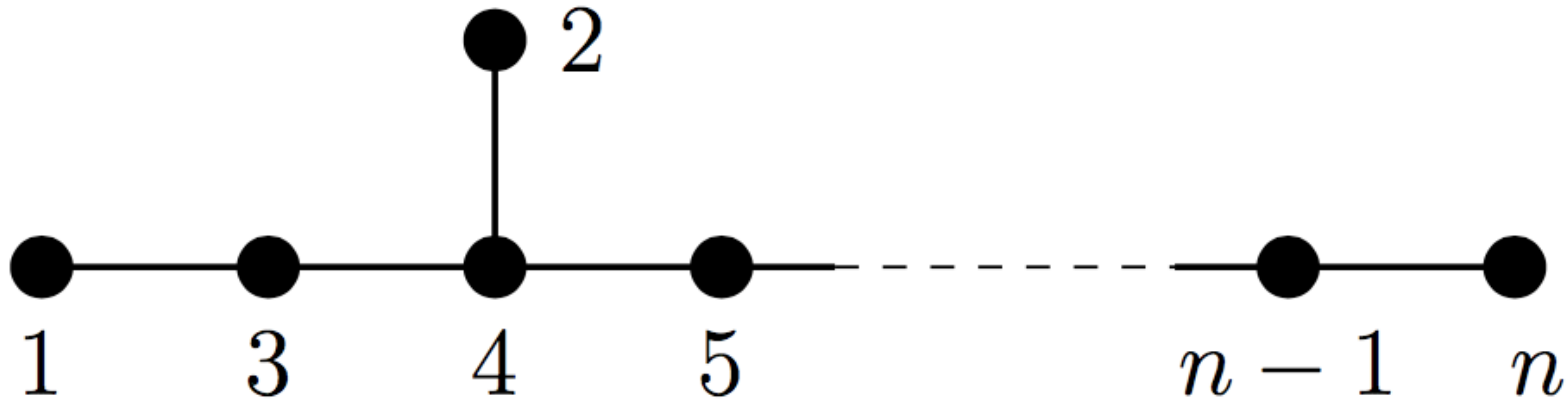
Minimal representations are generalizations of the **Weil representation**

→ **Theta series** $\theta(\tau) = \sum_{n\in\mathbb{Z}} e^{i\pi\tau n^2}$

In general, minimal automorphic forms are characterised by having **very few non-zero Fourier coefficients**

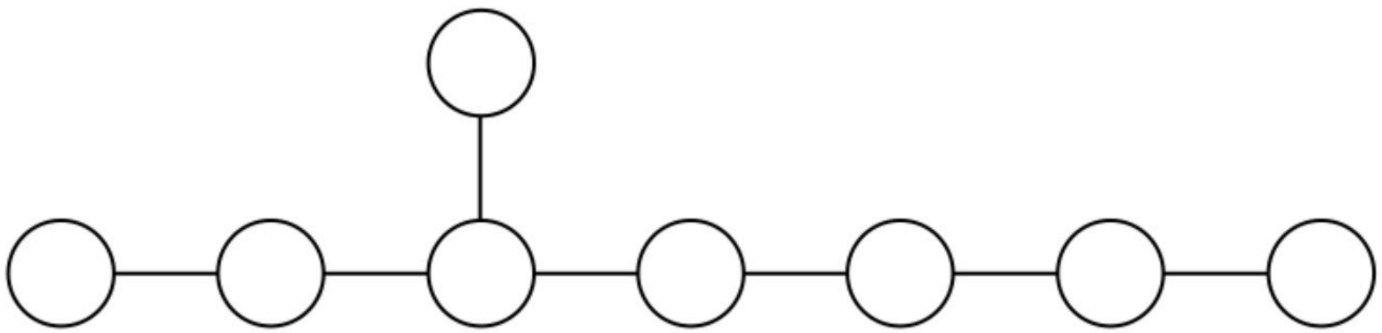
[Ginzburg, Rallis, Soudry]

Focus on exceptional groups



Functional dimension of minimal representations:

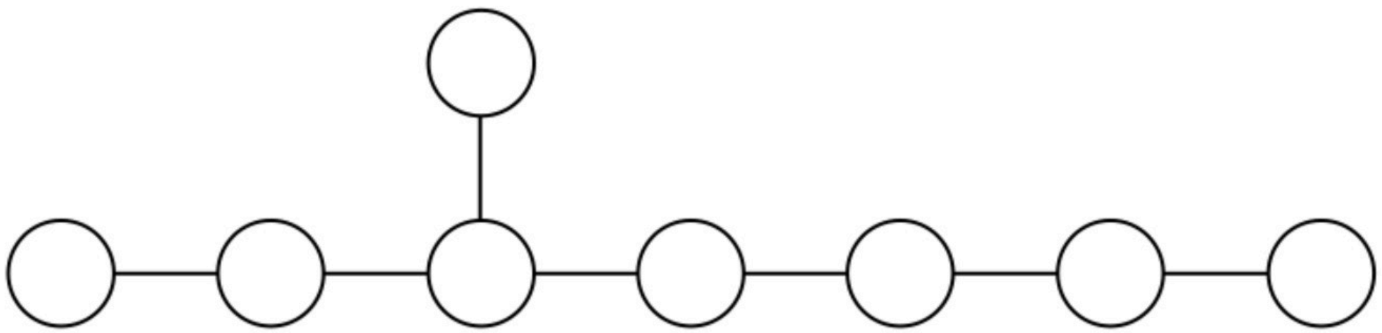
$$\mathrm{GKdim}\,\pi_{min} = \begin{cases} 11, & E_6 \\ 17, & E_7 \\ 29, & E_8 \end{cases}$$



$\mathrm{GKdim}(\pi_{min}) = 29$

(dim of generic rep: 120)

$$E_8 = \mathbf{1} \oplus \mathbf{56} \oplus (E_7 \oplus \mathbf{1}) \oplus \mathbf{56} \oplus \mathbf{1}$$



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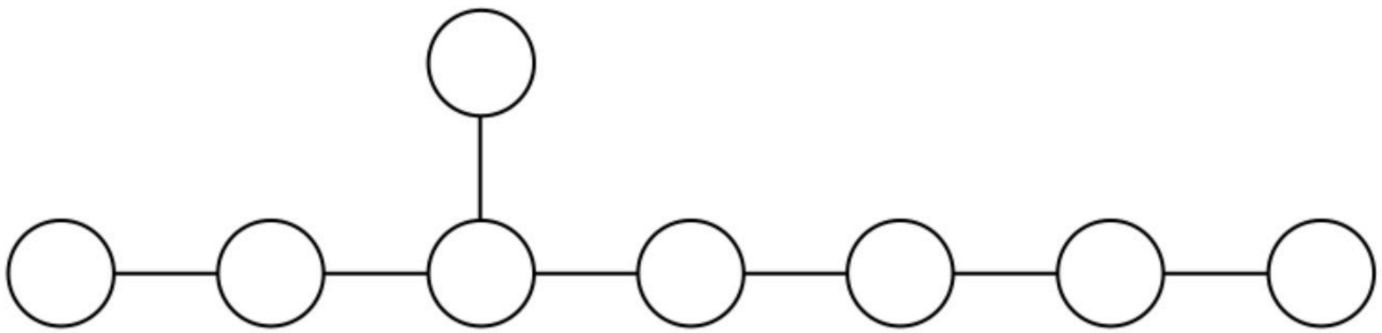

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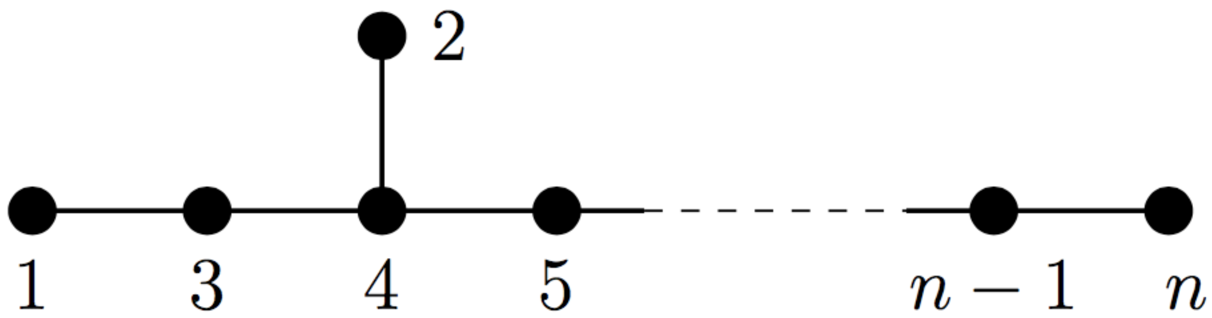
Minimal representation realized by the group action on functions of these 29 variables

Minimal representation of exceptional groups



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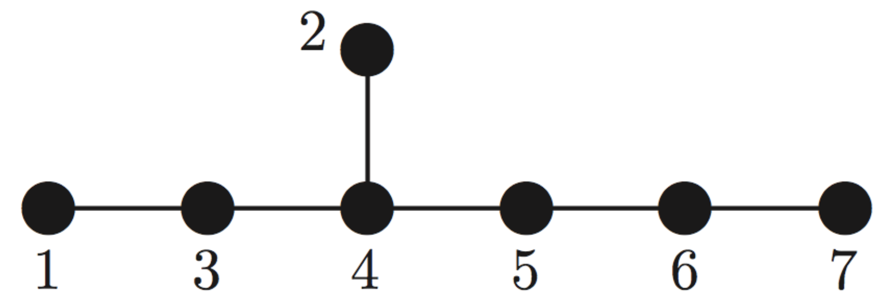
Minimal automorphic forms can be obtained as a **special values of Eisenstein series** [Ginzburg, Rallis, Soudry] (also [Green, Miller, Vanhove])

Theorem: $E(2s-\Lambda_1, g) \in \pi_{min} \quad \text{when} \quad s = 3/2$

Example: $G = E_7$



$$E_7 = \mathbf{27} \oplus (E_6 \oplus \mathbf{1}) \oplus \mathbf{27}$$

Theorem: $$\pi_{min} \ni \varphi(g) = \varphi_U + \sum_{\gamma \in \mathrm{Stab}_L(\psi_{\alpha_7}) \backslash L(\mathbb{Q})} F_{\psi_{\alpha_7}}(\gamma g)$$

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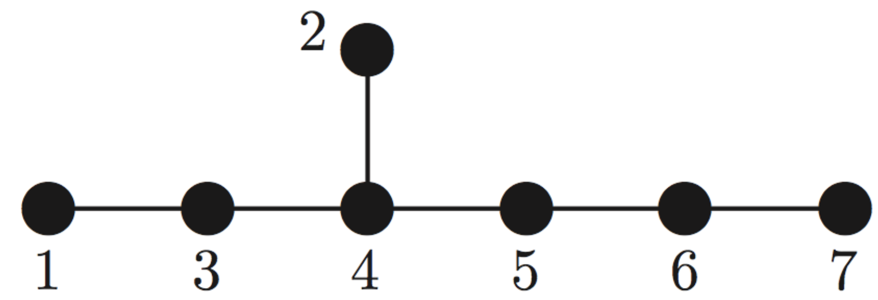

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The complete expansion is given in terms of the **smallest non-trivial** character variety orbit

$$\mathrm{Stab}_L(\psi_{\alpha_7})\backslash L \cong (SO(5,5) \times \mathbb{Q}^{16})\backslash E_6(\mathbb{Q})$$

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[Miller, Sahi]

dimension of the minimal representation!

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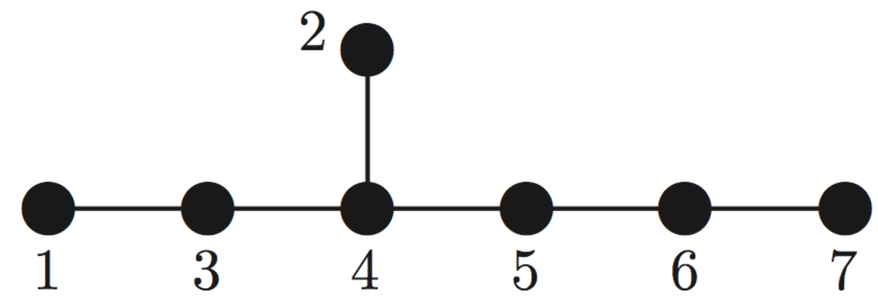

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This is the complete **Fourier expansion**!

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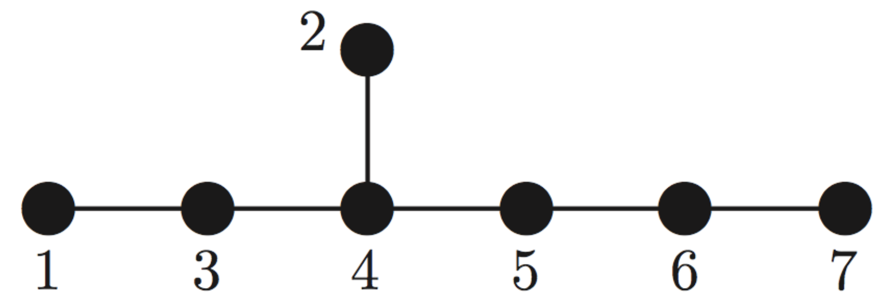

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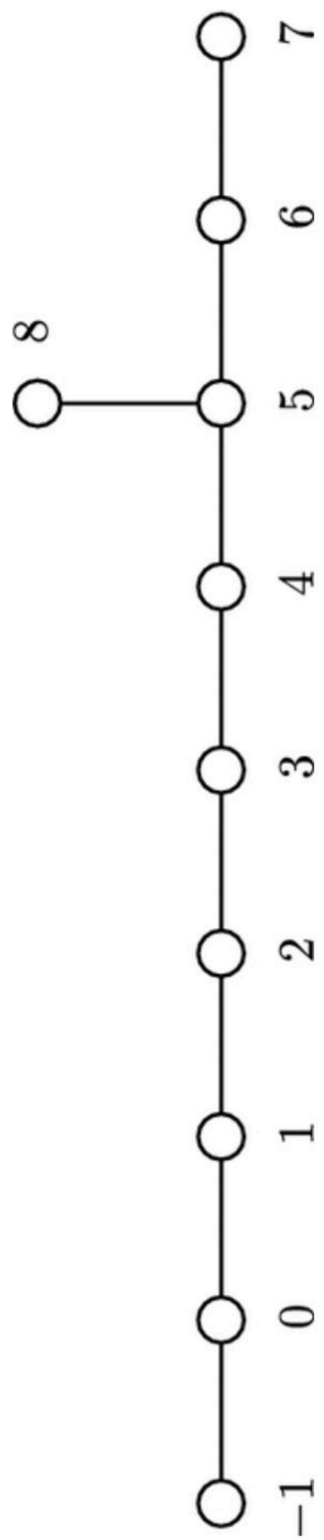
Theorem: Fourier coefficients of minimal automorphic forms are completely determined by maximally degenerate Whittaker coefficients W_{ψ_α}

[Ahlén, Gustafsson, Kleinschmidt, Liu, DP] [Gourevitch, Gustafsson, Kleinschmidt, Sahi, DP]

Back to E_{10}



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Implications for the Kac-Moody case

- The min rep is Eulerian $\pi_{min} = \bigotimes_p \pi_p$

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$$\int_{N(F)\backslash N(\mathbb{A})} E(s,ng)\psi_\alpha^{-1}(n)dn = F_\infty \times \prod_{p<\infty} F_p$$

and **Eulerian**

Open questions

- $p-$adic representation π_p : Related to reps of the **DAHA**? (Double Affine Hecke Algebra)
- archimedean representation π_∞ : analogue of the **Joseph ideal**?
- What is the **Gelfand-Kirillov** dimension of π_{min}?
- Minimal nilpotent co-adjoint orbit $\mathcal{O}_{min}$? $\boxed{G \cdot E_\alpha}$
- Theta correspondences for Kac-Moody groups? [Garland, Liu]
- Uniqueness?

Thank you!