Symmetrier, singulariteter och svarta hål

- från Einstein till matematikens djupaste mysterier -

Daniel Persson

Institutionen för matematiska vetenskaper

Chalmers tekniska högskola

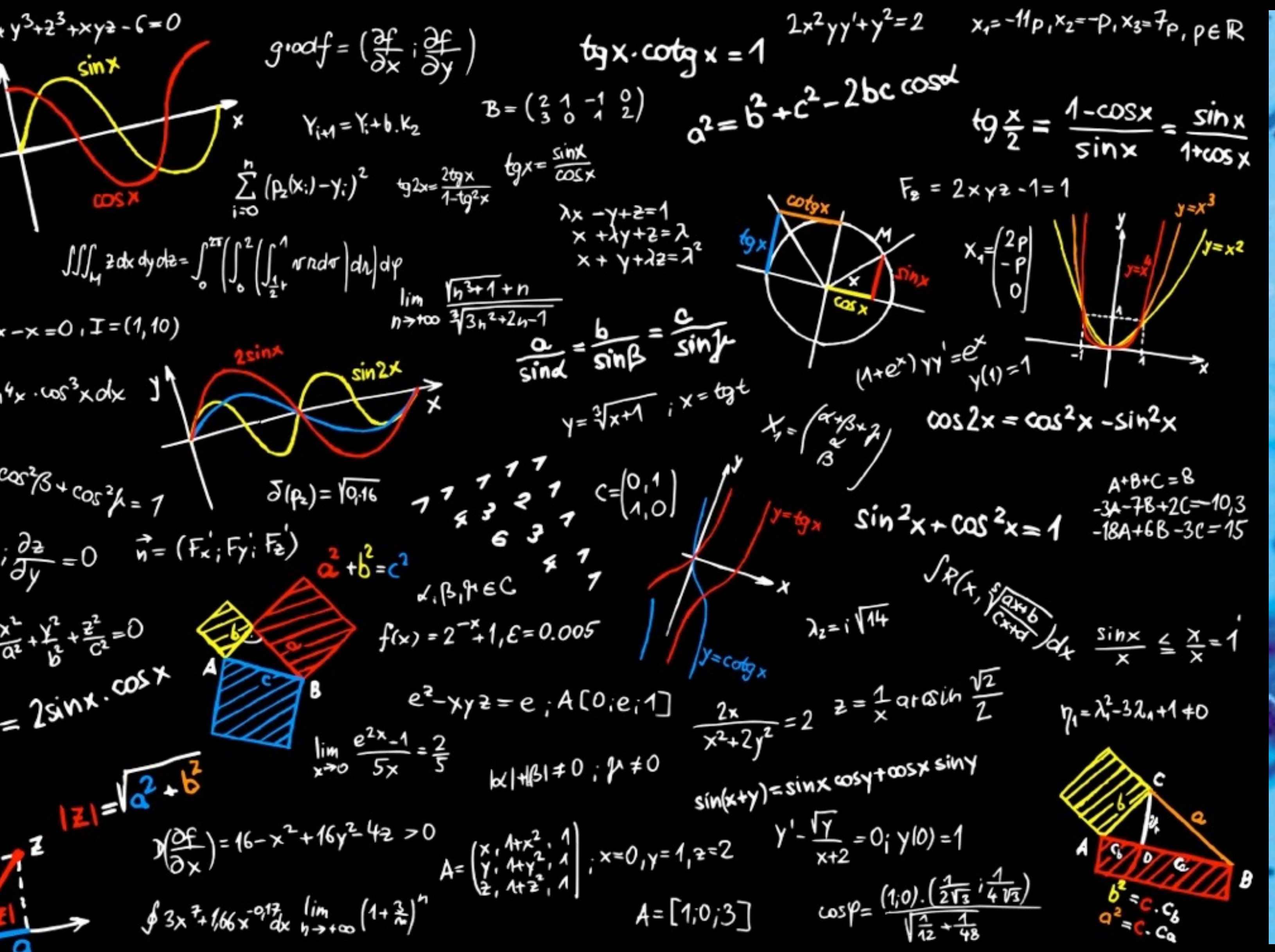
Rymdforskarskolan

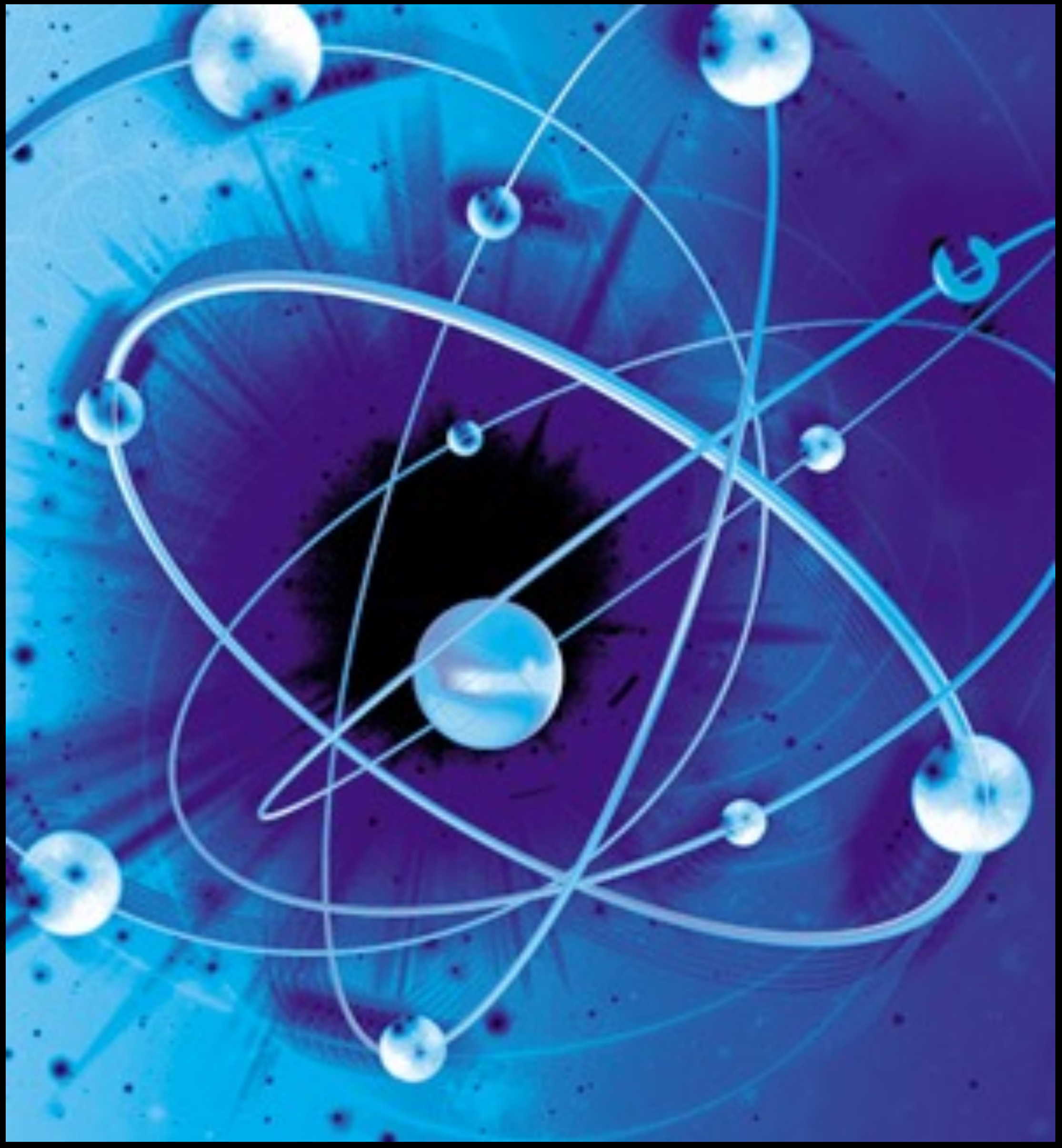
Astronomisk Ungdom

Chalmers

Tisdag 6 augusti 2019

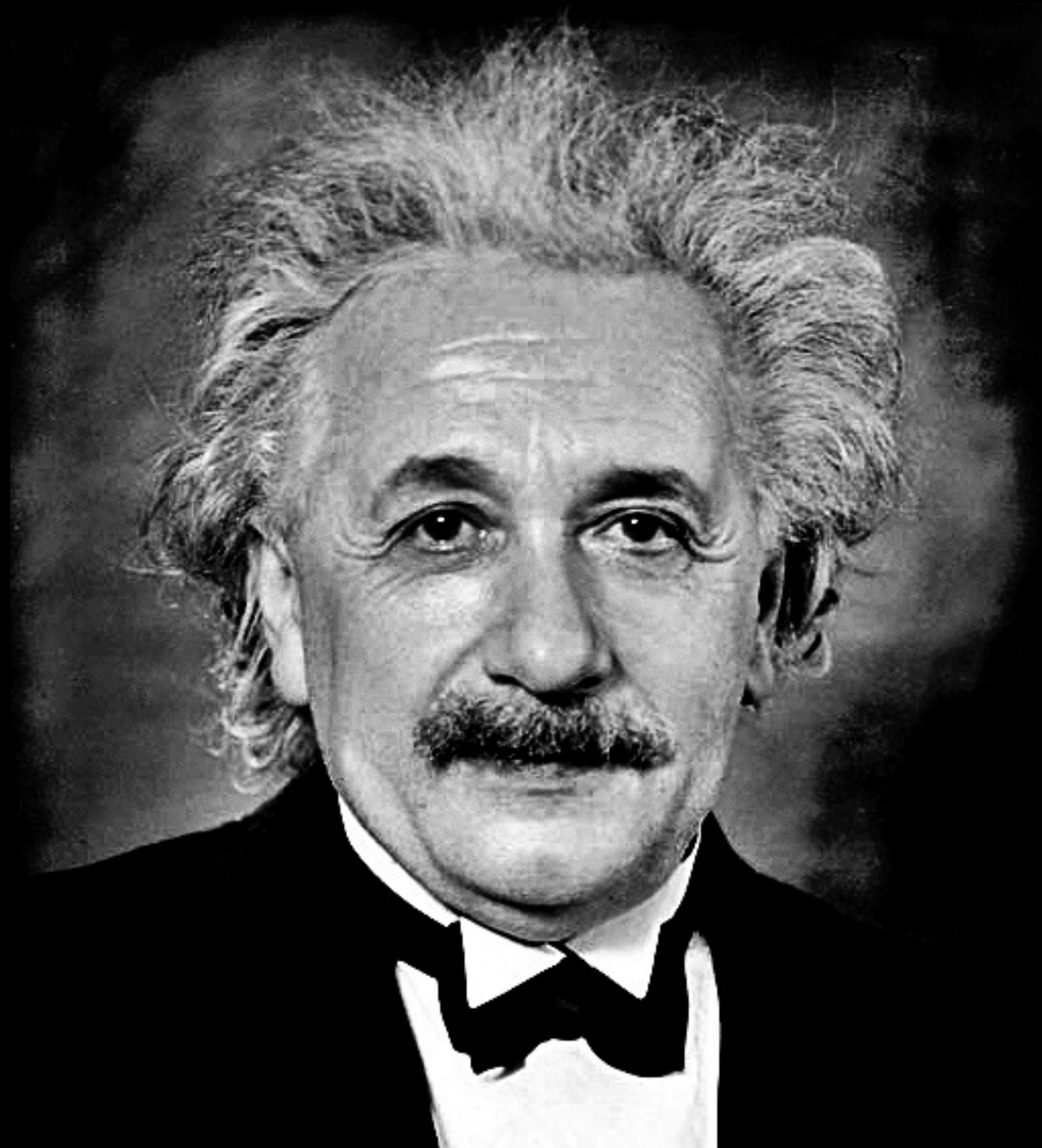
When we ask deep questions about Nature we are led deep into the world of mathematics



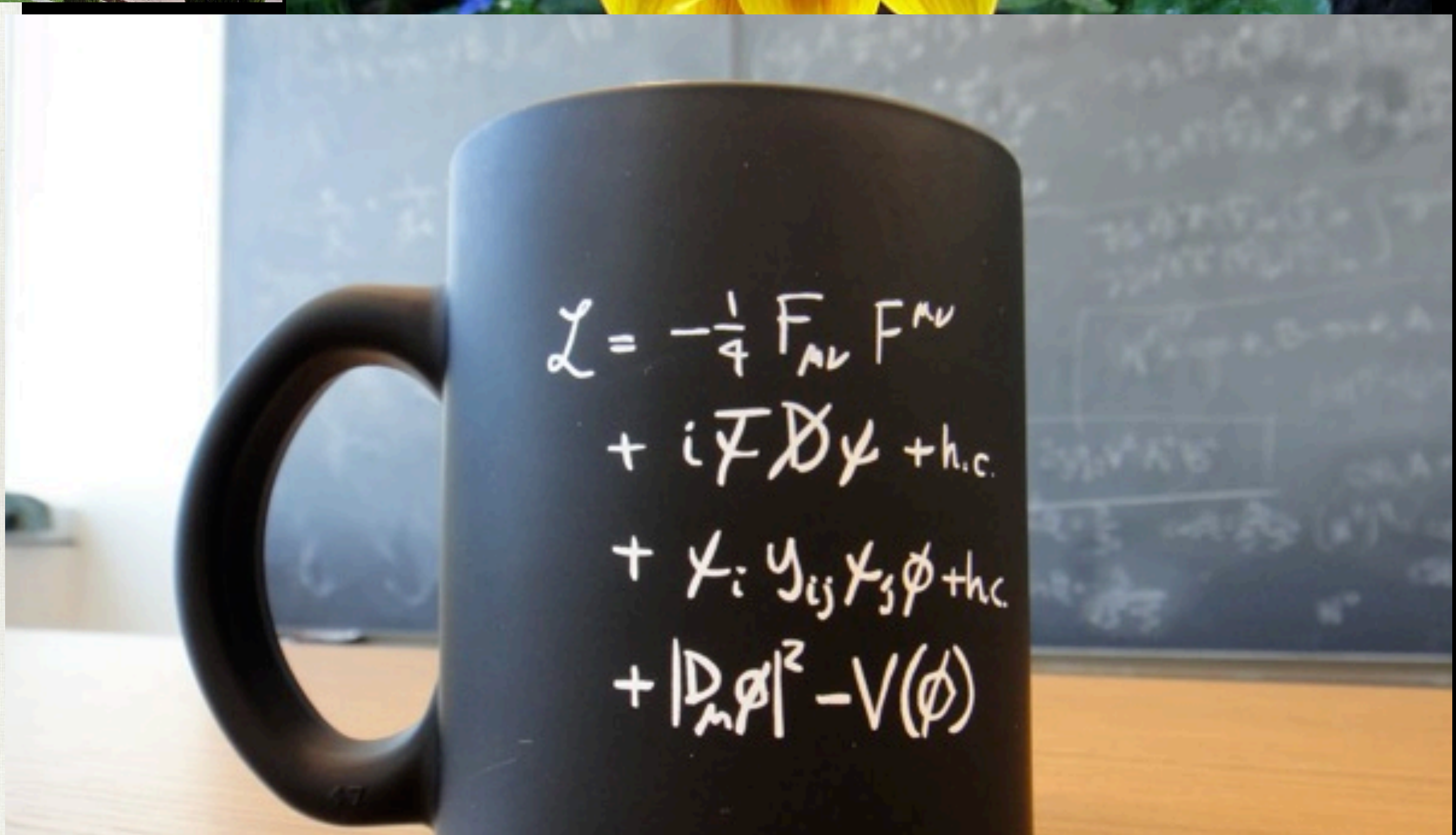


How can it be that mathematics, being after all a product of human thought which is independent of experience, is so admirably appropriate to the objects of reality?

- A. Einstein

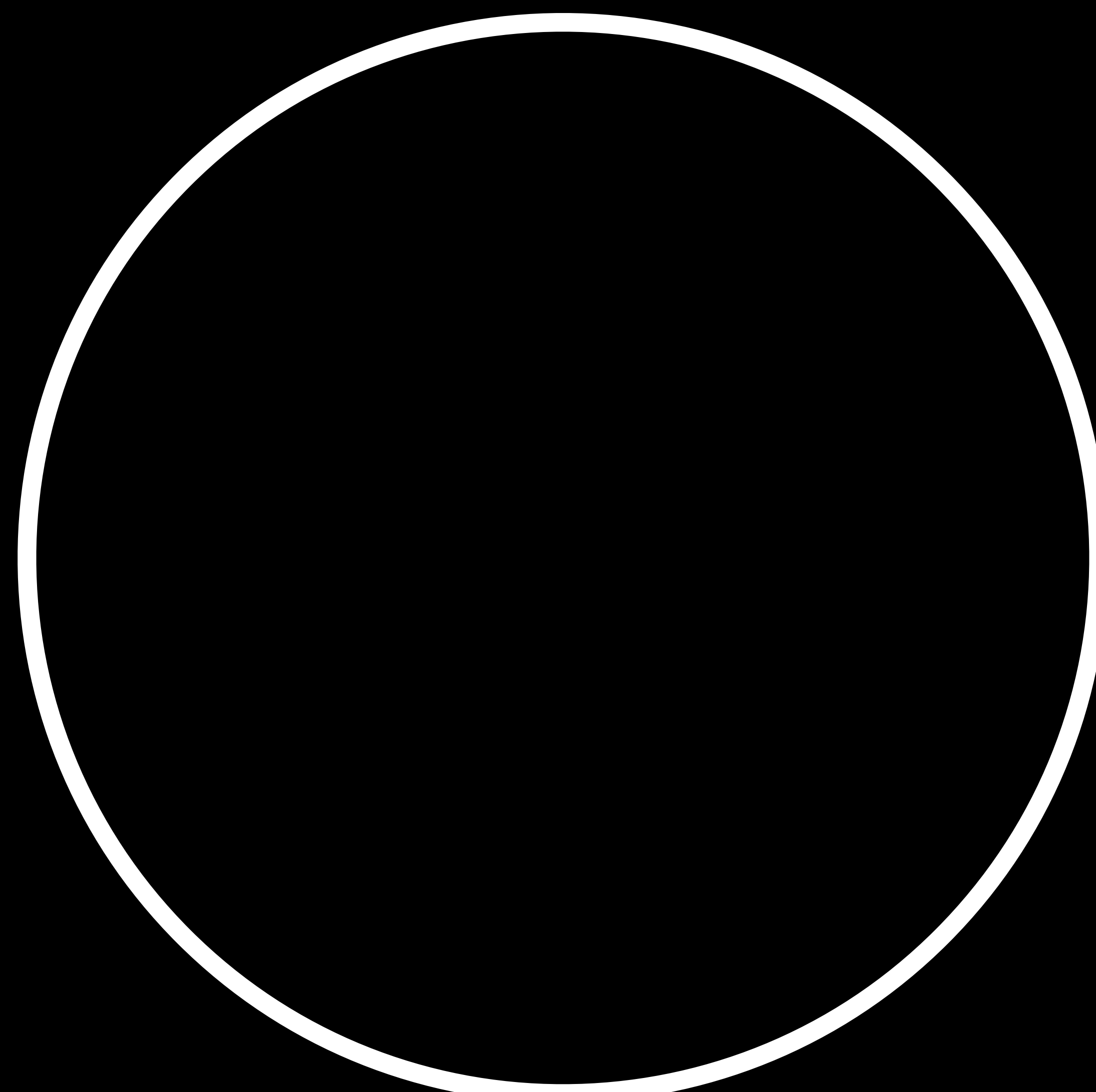


Nature strives for symmetry



Symmetries

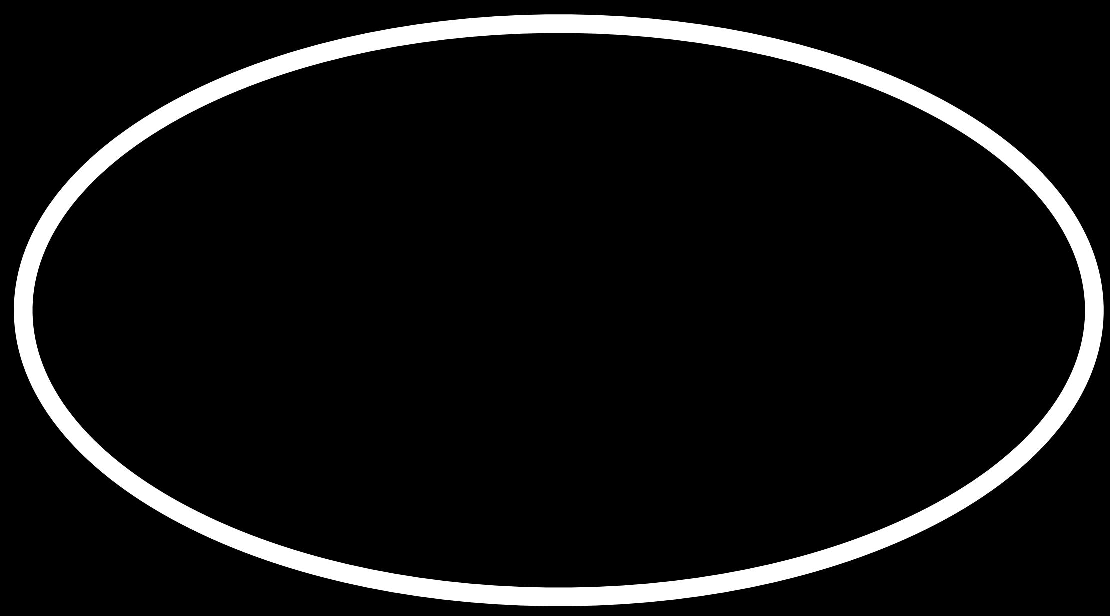
Transformations that leave a geometric object invariant



A circle has infinitely many rotational symmetries

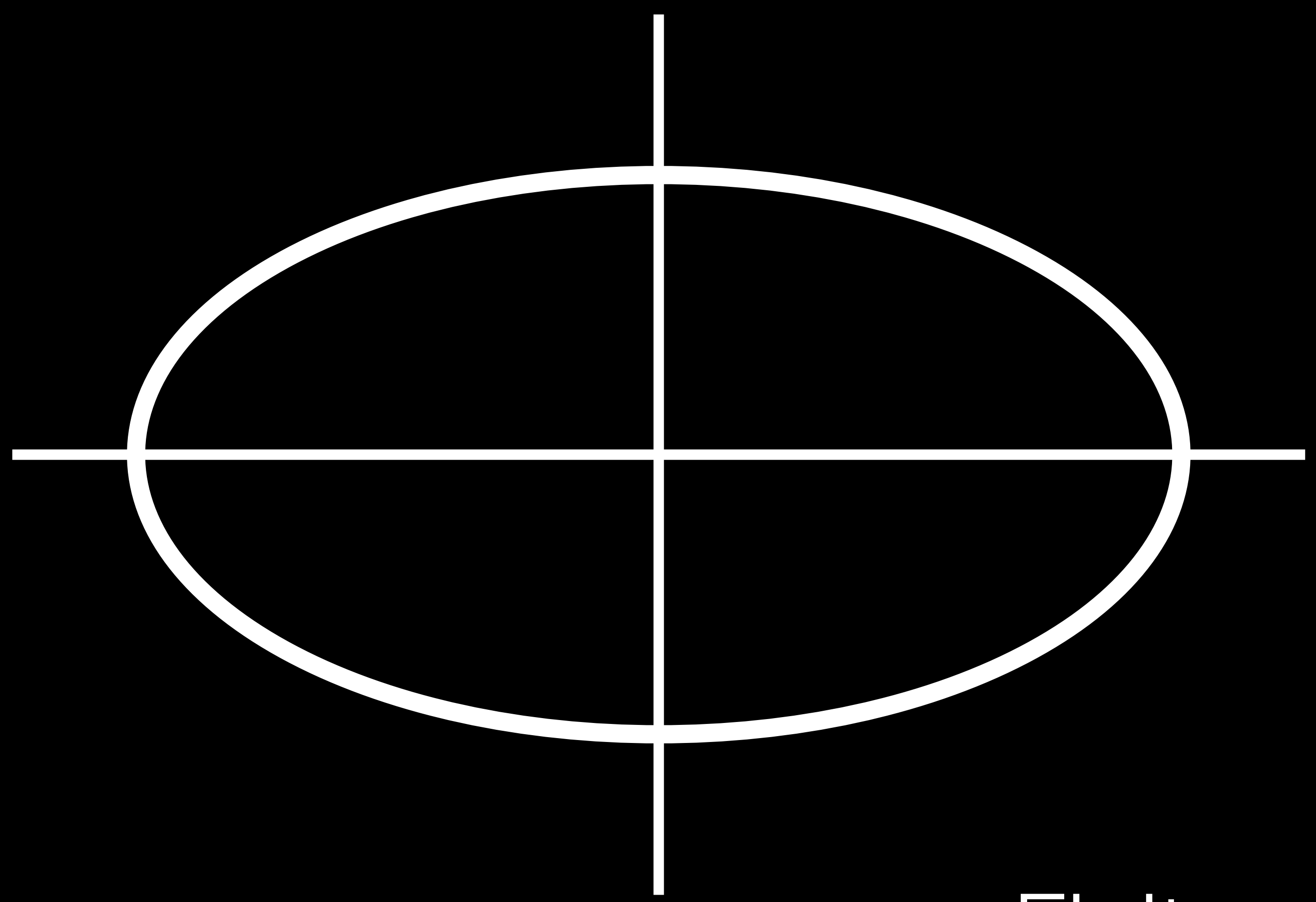
Symmetries

Transformations that leave a geometric object invariant



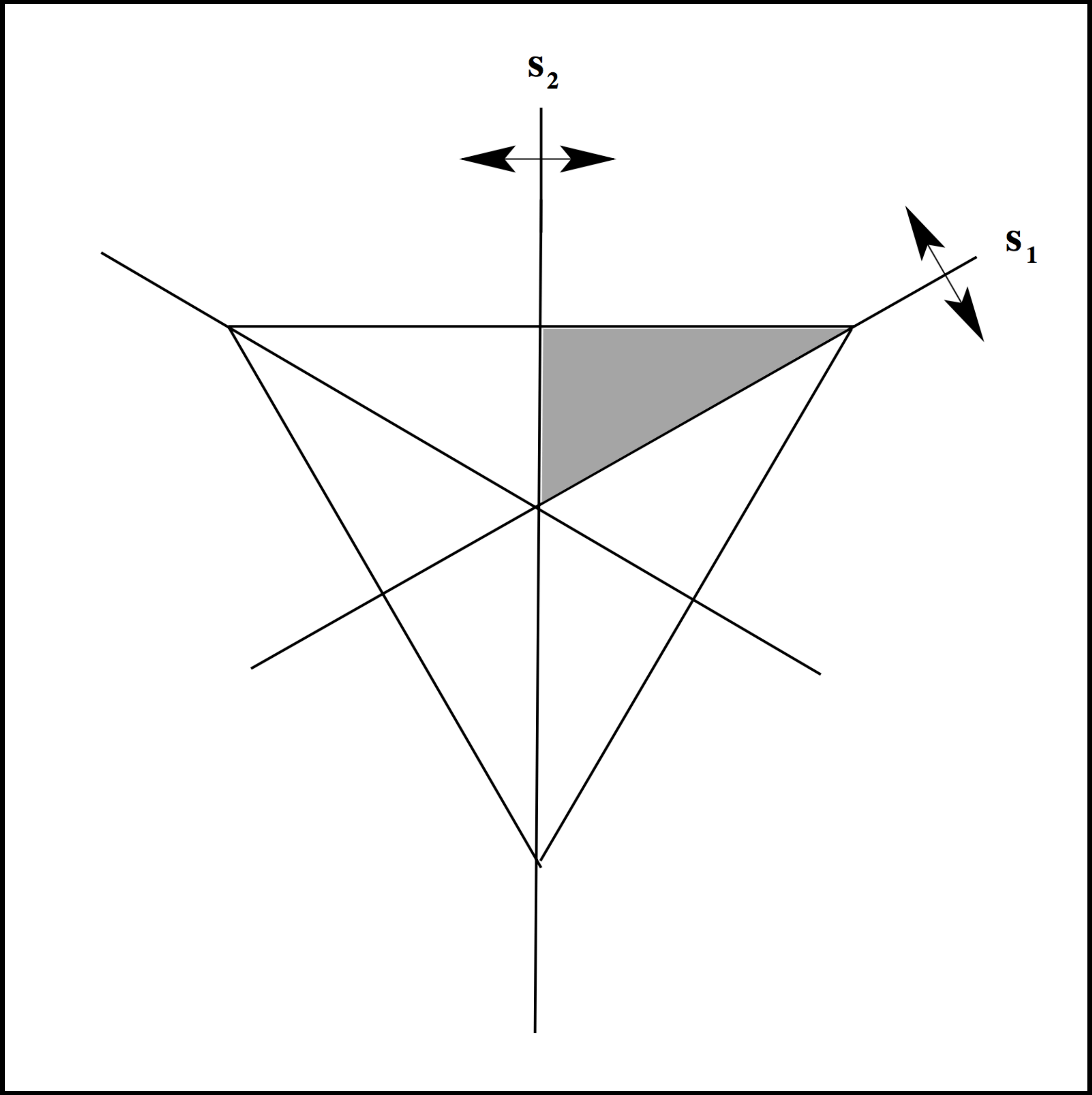
Symmetries

Transformations that leave a geometric object invariant



Finite **reflection** symmetry

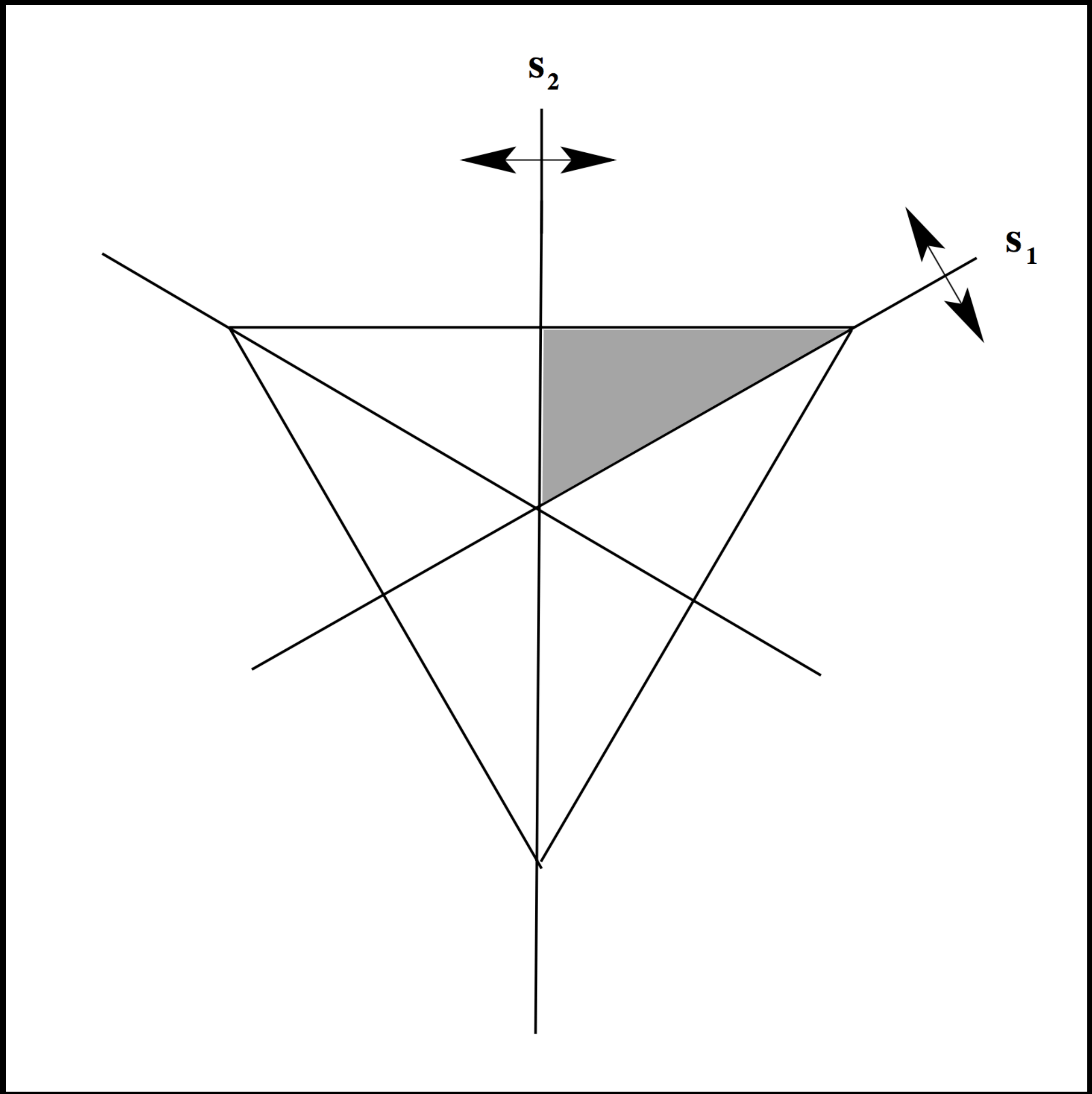
A triangle has 6 symmetries



3 rotations

3 reflections

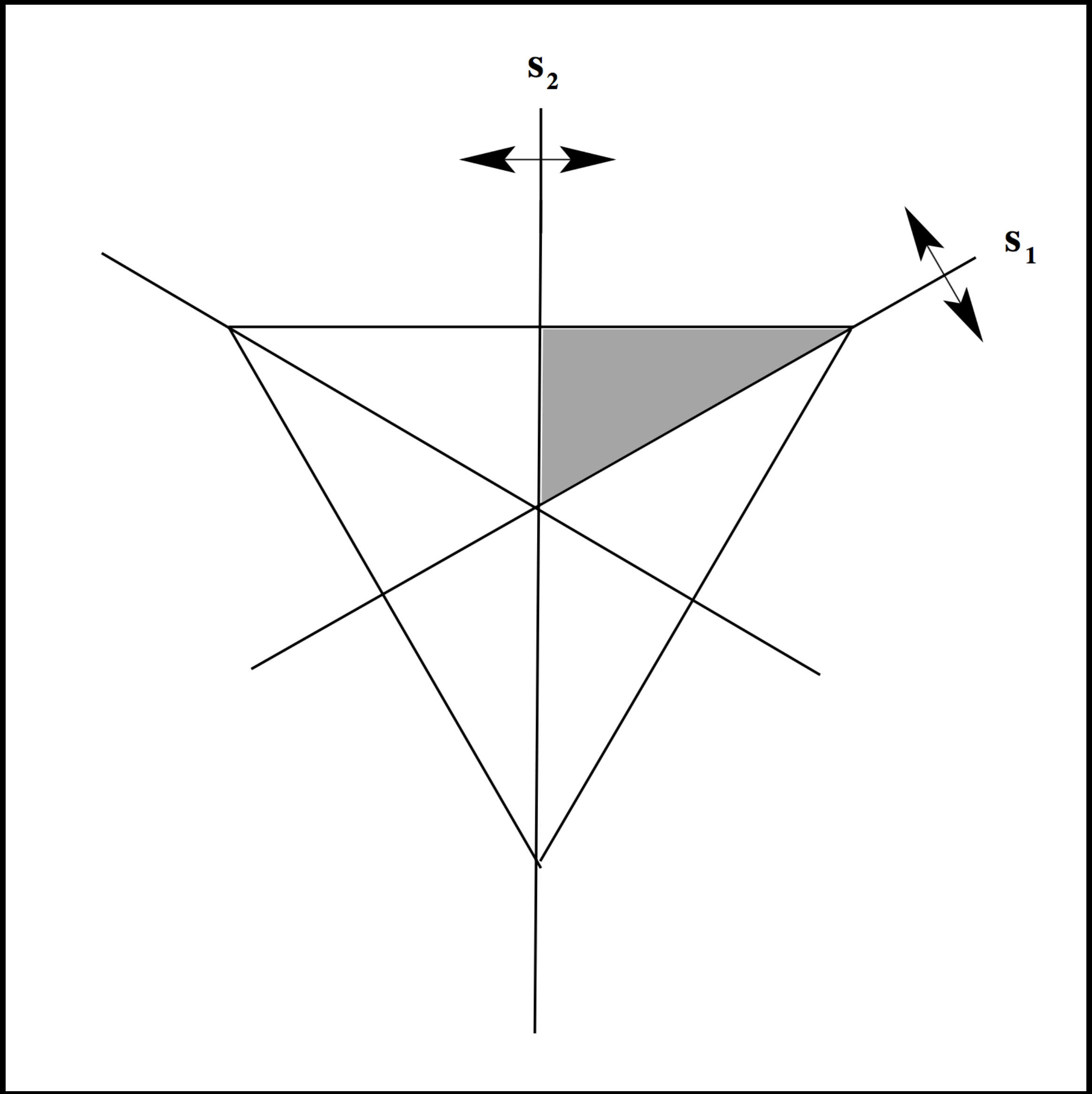
All these are generated by two reflections: S1 och S2



Symmetry transformations form a mathematical
object called a ***group***.

Key property: The combination of two transformations must also be a symmetry

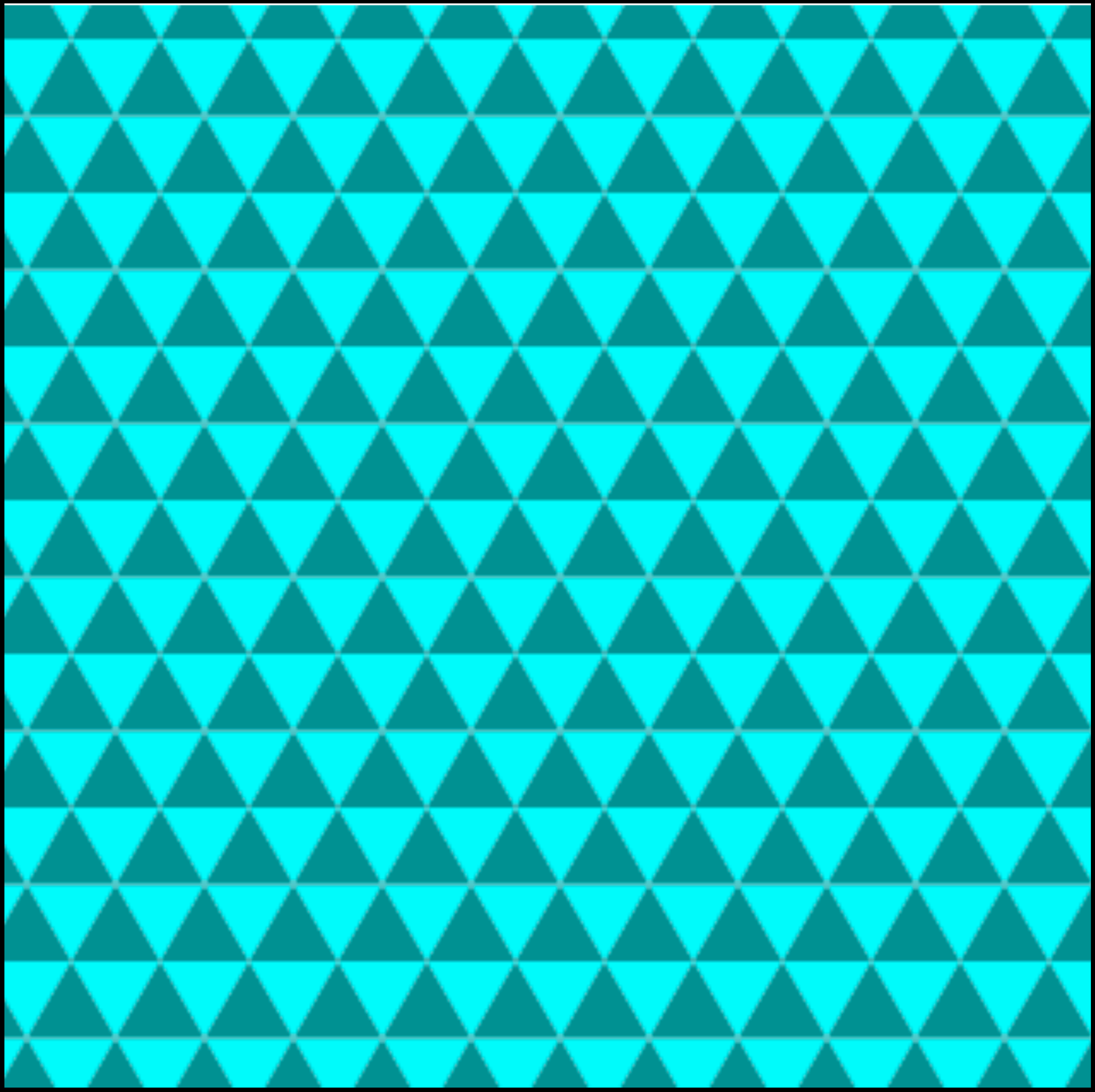
Symmetry transformations form a mathematical object called a ***group***.



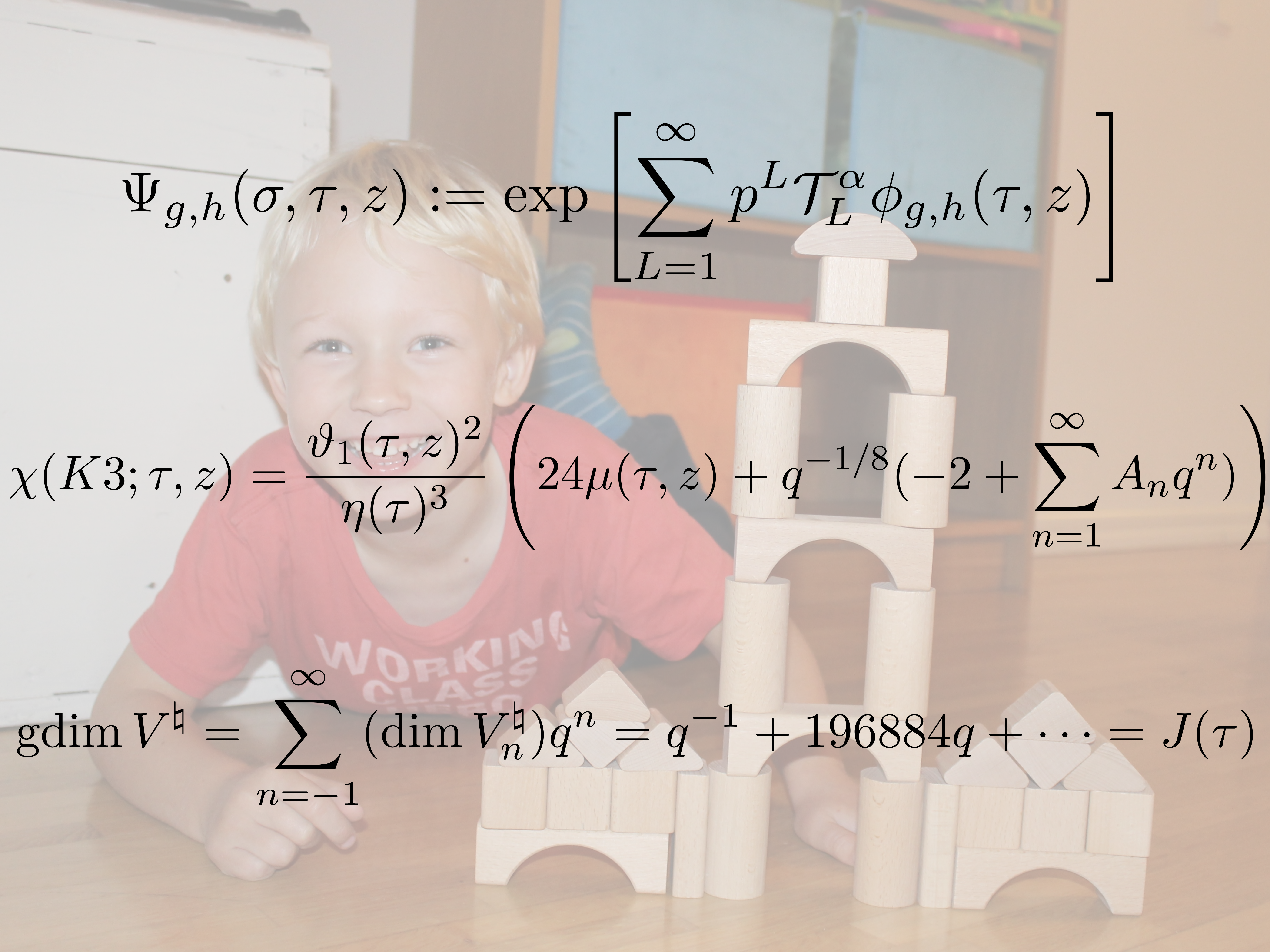
Key property: The combination of two transformations must also be a symmetry

Translation is not a symmetry of the triangle!

But a **tesselation** of the plane by triangles has translational symmetry!



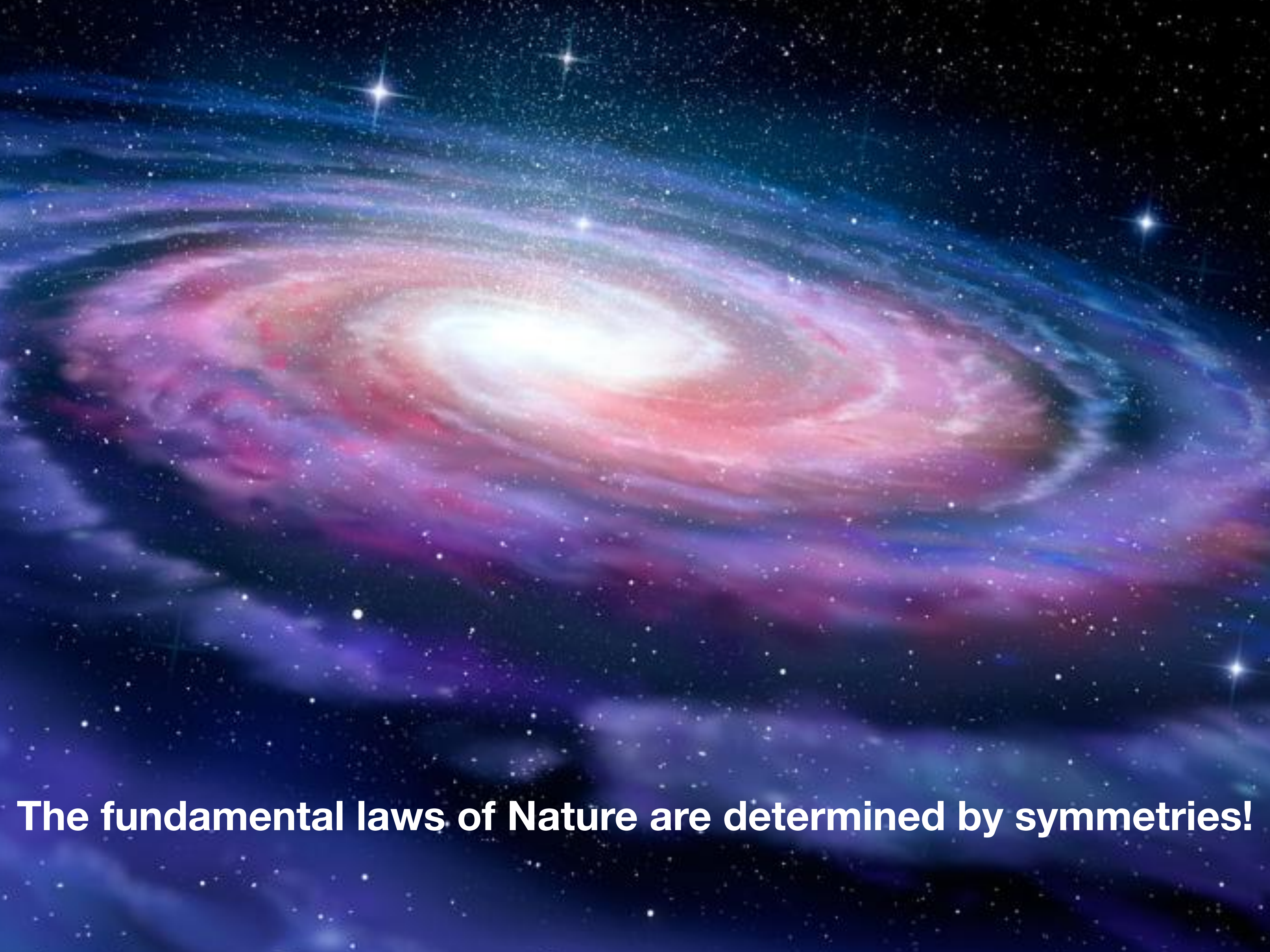

WORKING
CLASS



$$\Psi_{g,h}(\sigma,\tau,z) := \exp\left[\sum_{L=1}^{\infty} p^L \mathcal{T}_L^{\alpha}\phi_{g,h}(\tau,z)\right]$$

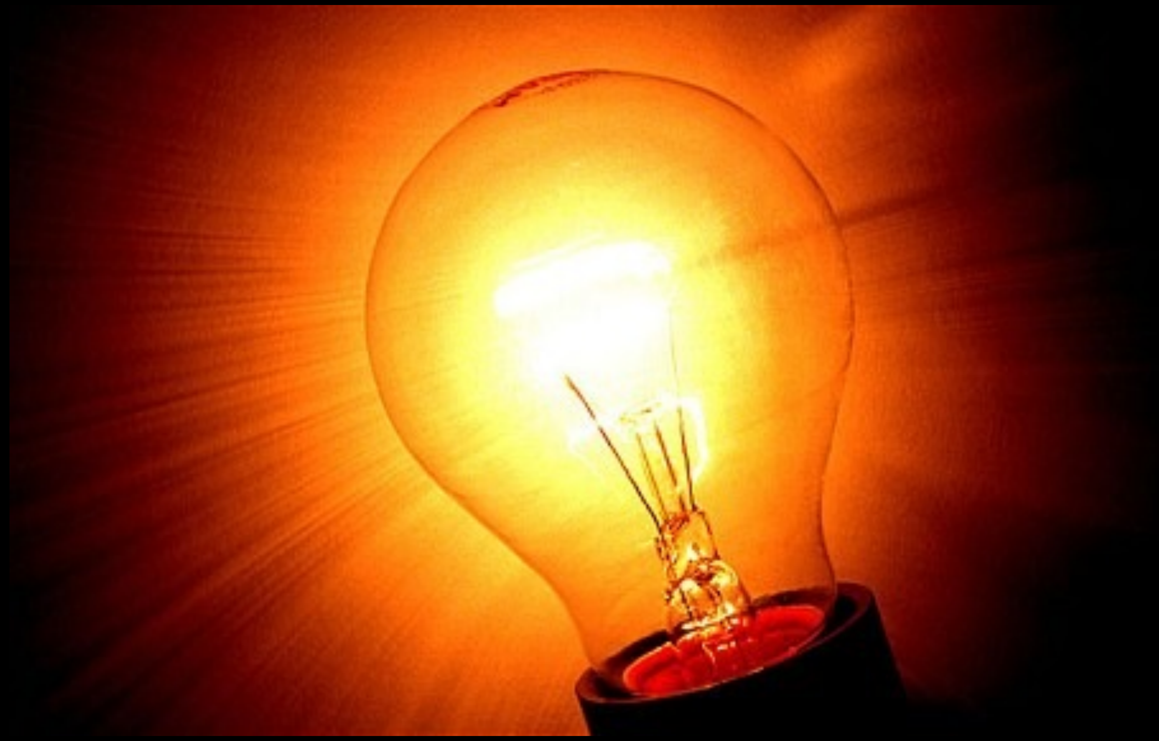
$$\chi(K3;\tau,z) = \frac{\vartheta_1(\tau,z)^2}{\eta(\tau)^3}\left(24\mu(\tau,z) + q^{-1/8}(-2+\sum_{n=1}^{\infty}A_n q^n)\right)$$

$$\operatorname{gdim} V^{\natural} = \sum_{n=-1}^{\infty}(\dim V_n^{\natural})q^n = q^{-1} + 196884q + \cdots = J(\tau)$$


The fundamental laws of Nature are determined by symmetries!

gravity

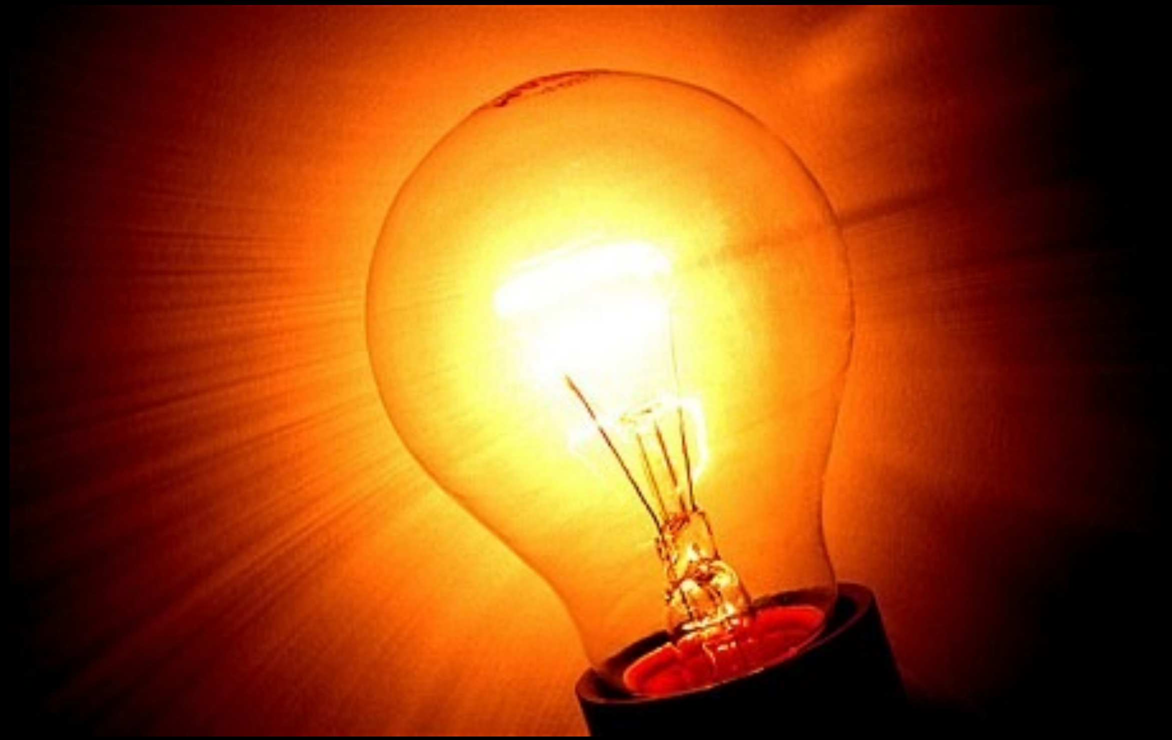
gravity



electromagnetism

strong nuclear force

gravity



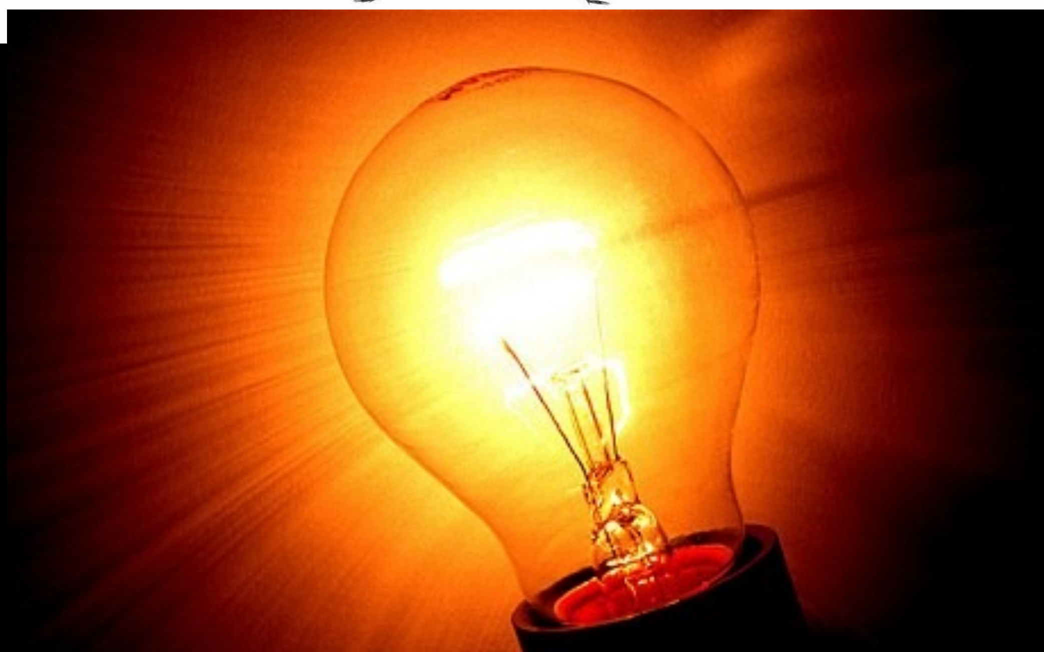
electromagnetism

strong nuclear force

gravity

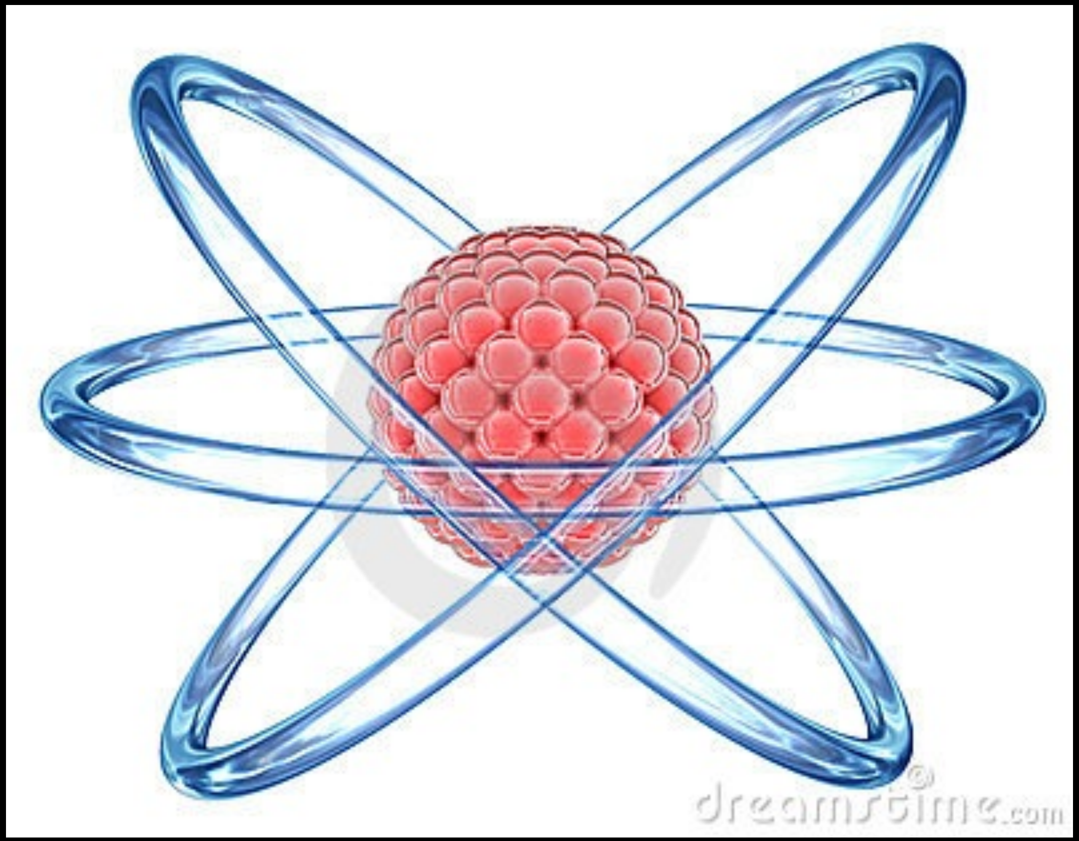
electromagnetism

weak nuclear force

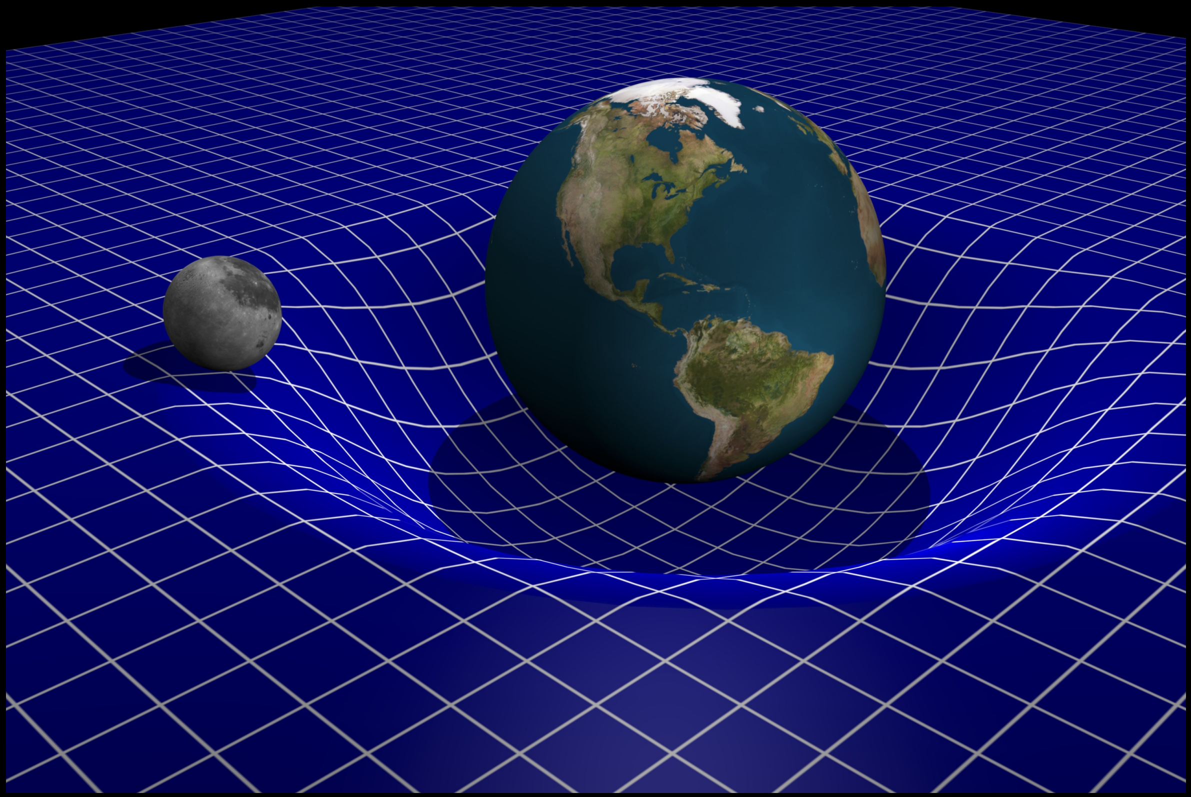



FAMILY GUY
THE FORCE IS WEAK WITH YOU, OLD MAN.

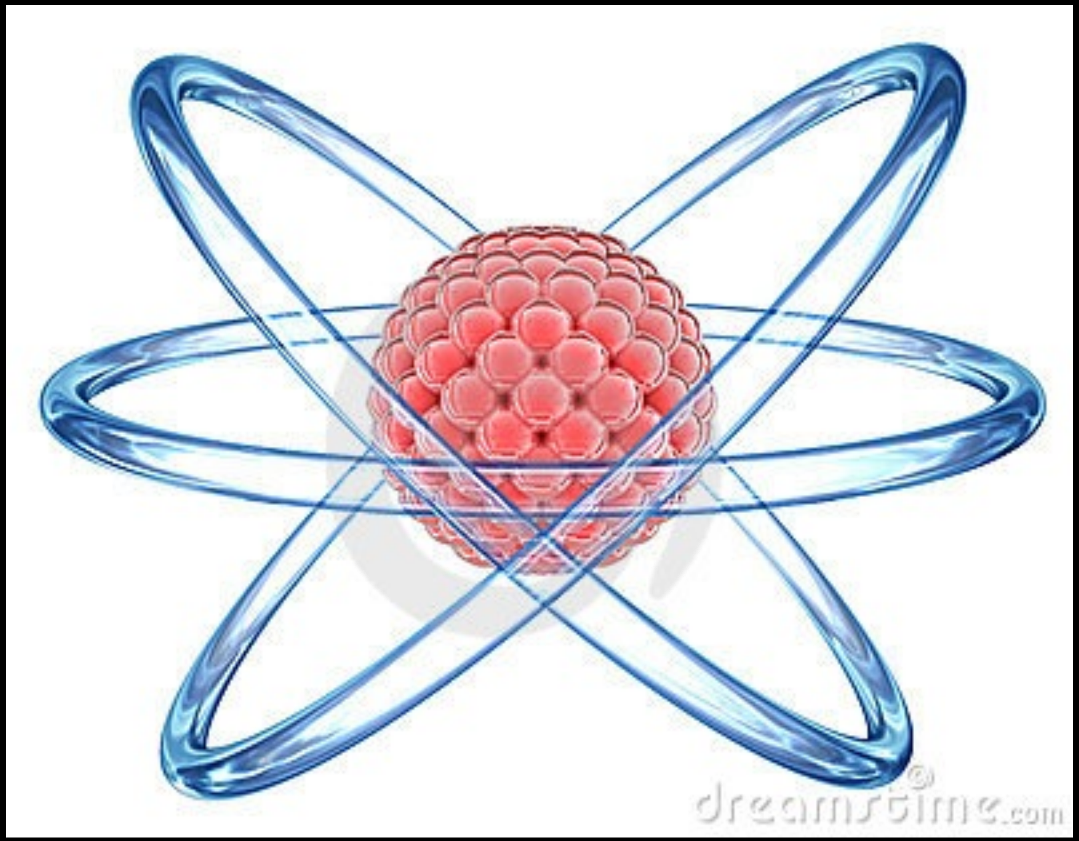

plonk
©Lana

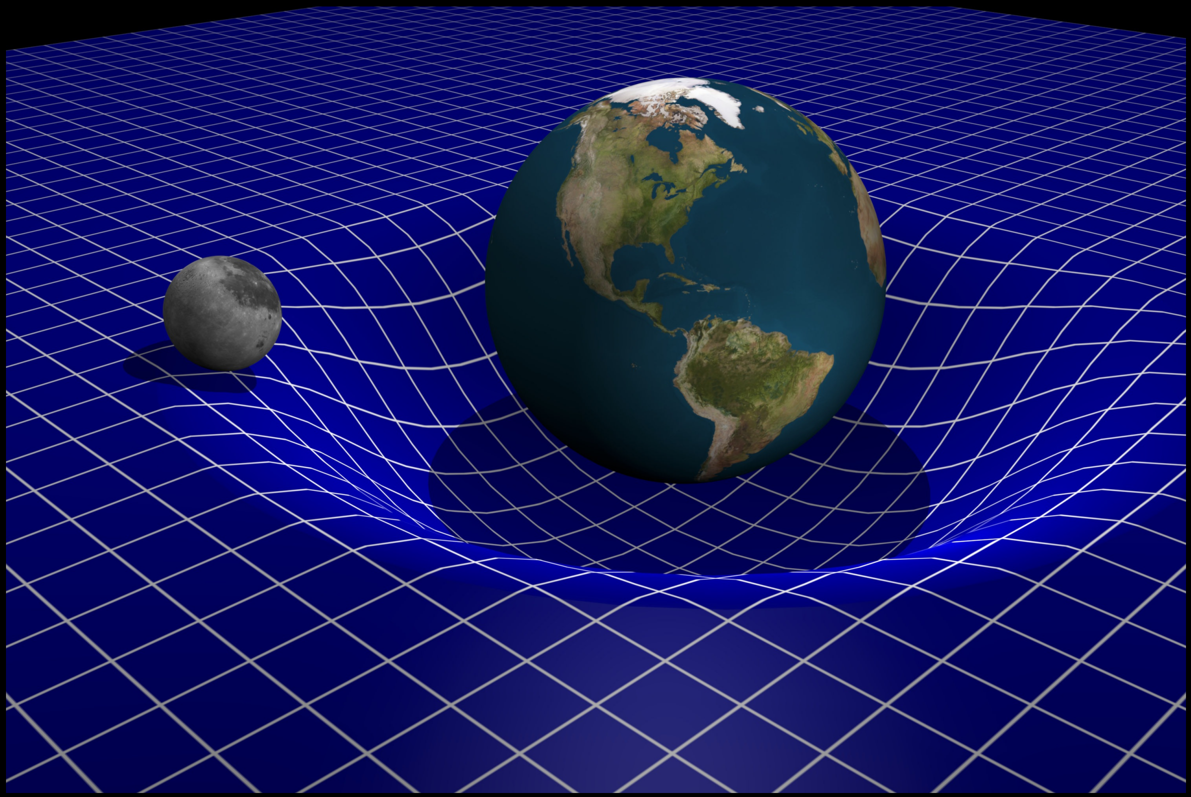


quantum mechanics



general relativity





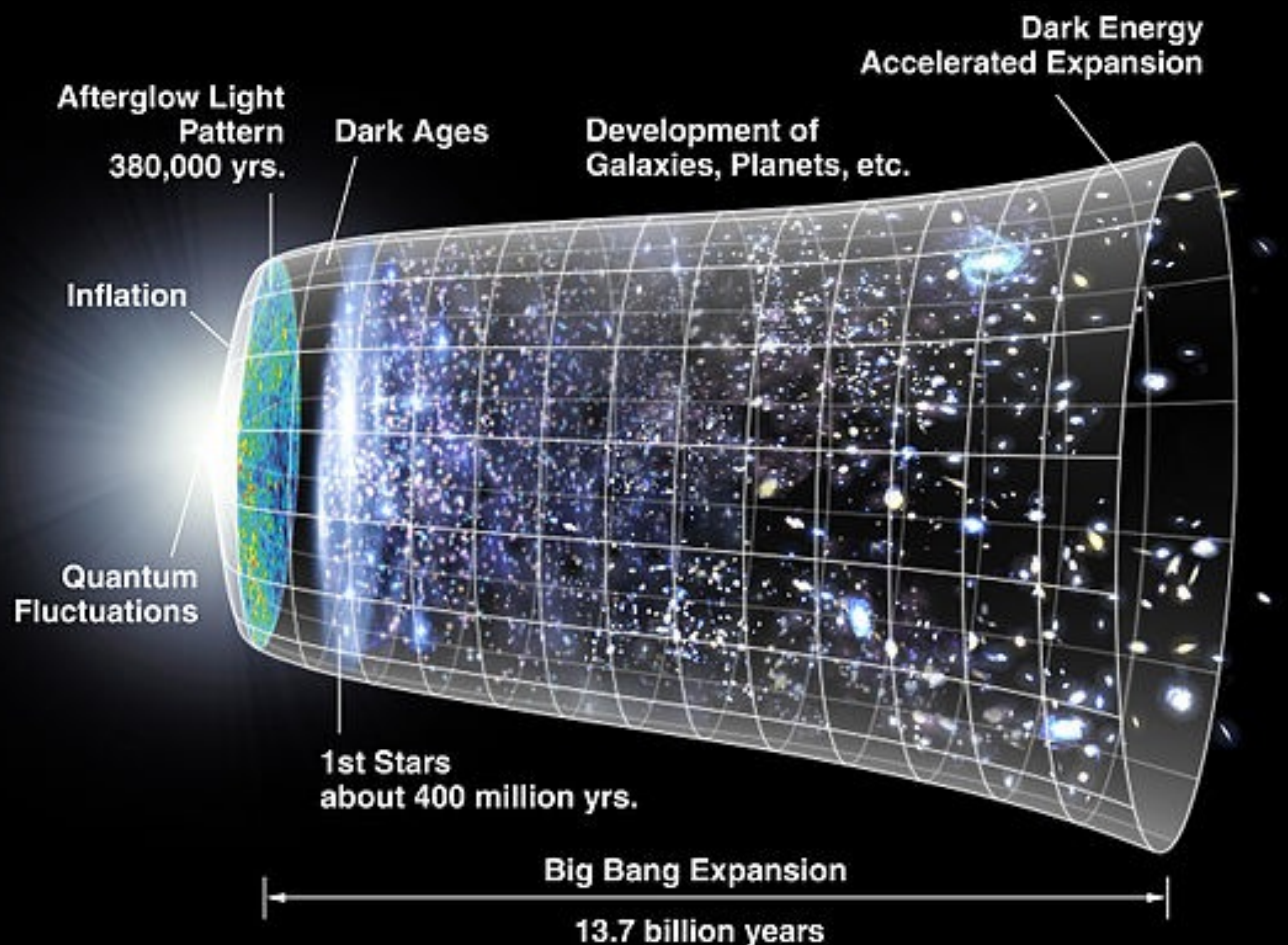
quantum mechanics

general relativity

microcosmos

macrocosmos

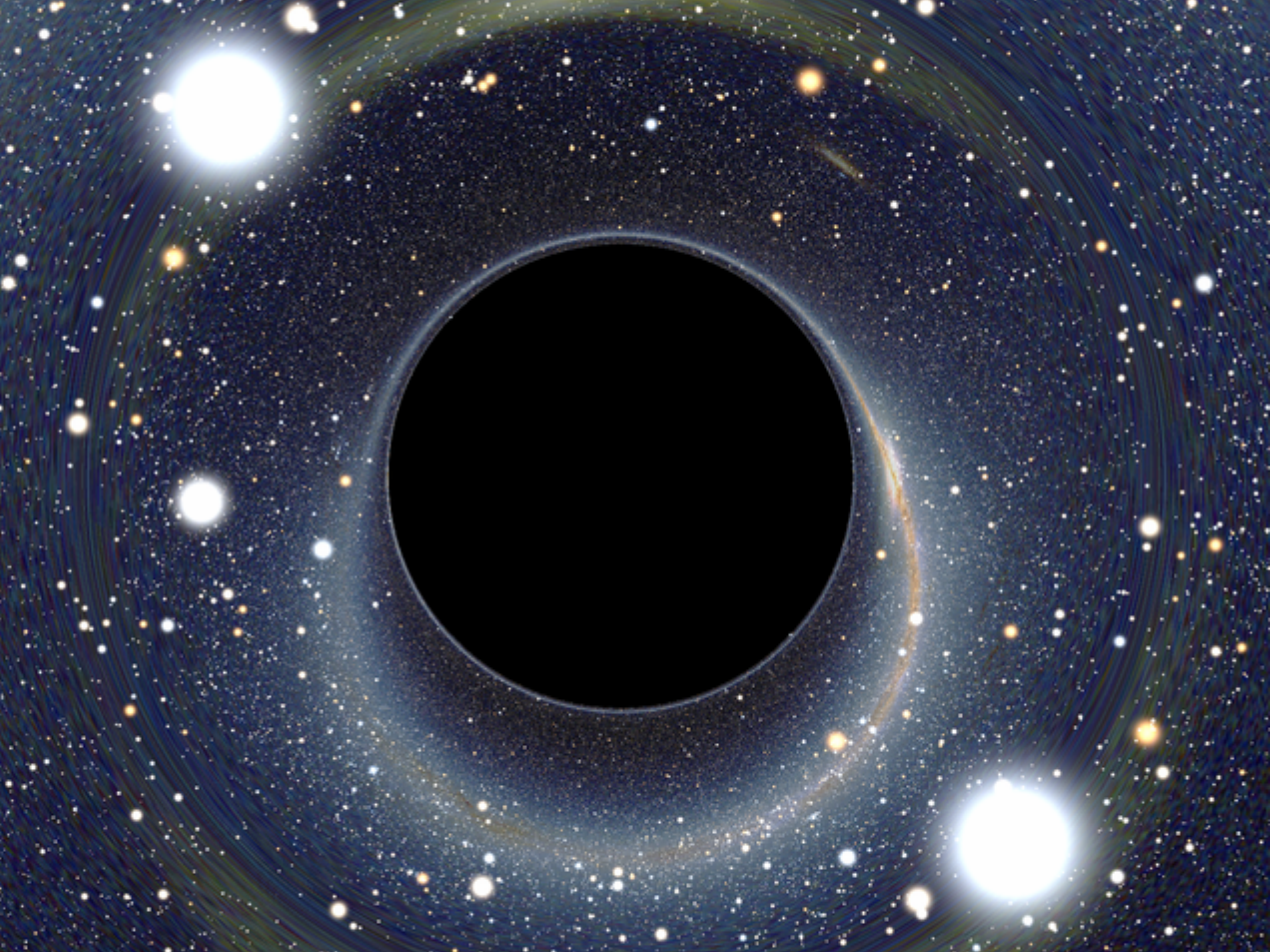
There are regions in the universe where QM and GR must **work together**!



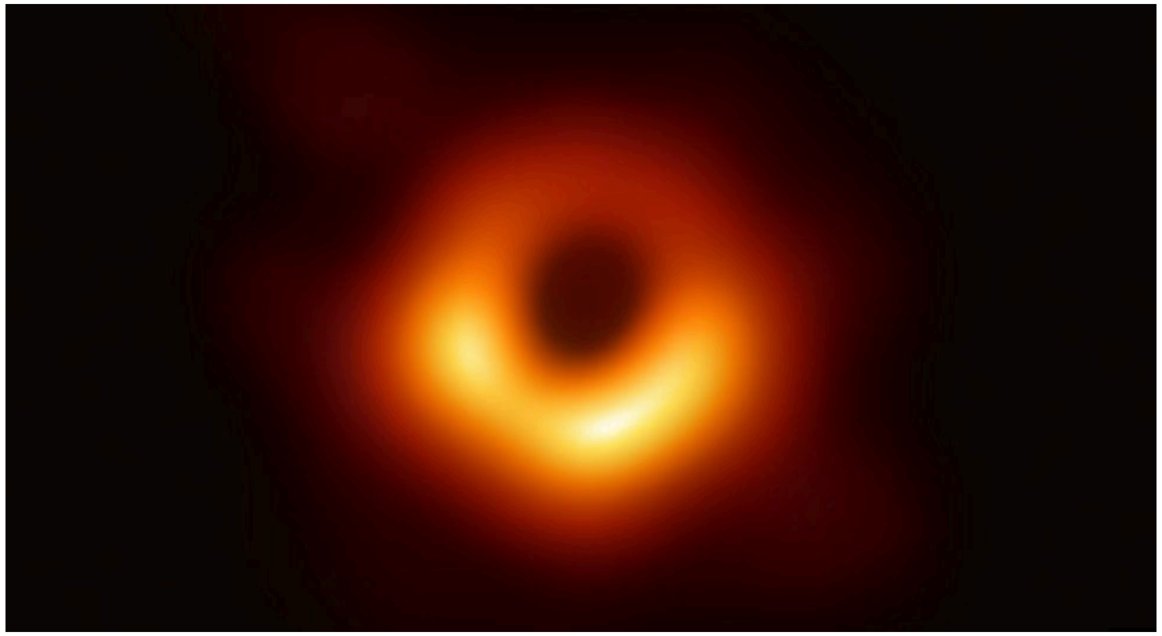
The Milky Way

DIGITAL IMAGE OF THE MILKY WAY BY PIKAIA IMAGING (WWW.PIKAIA-IMAGING.CO.UK)

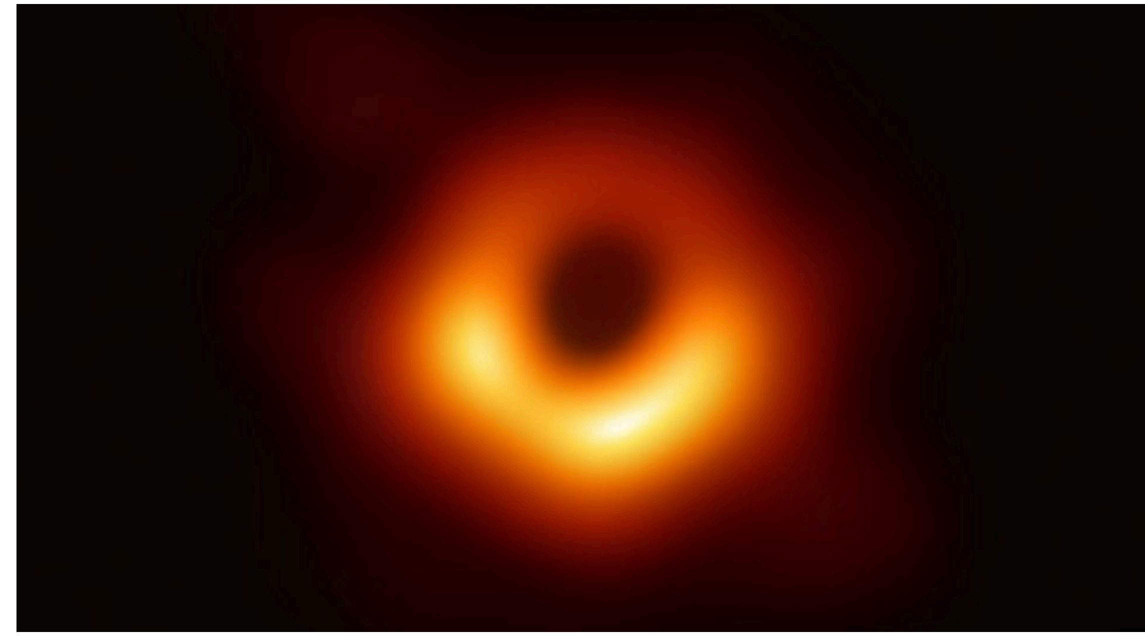
NewScientist



What is a black hole?



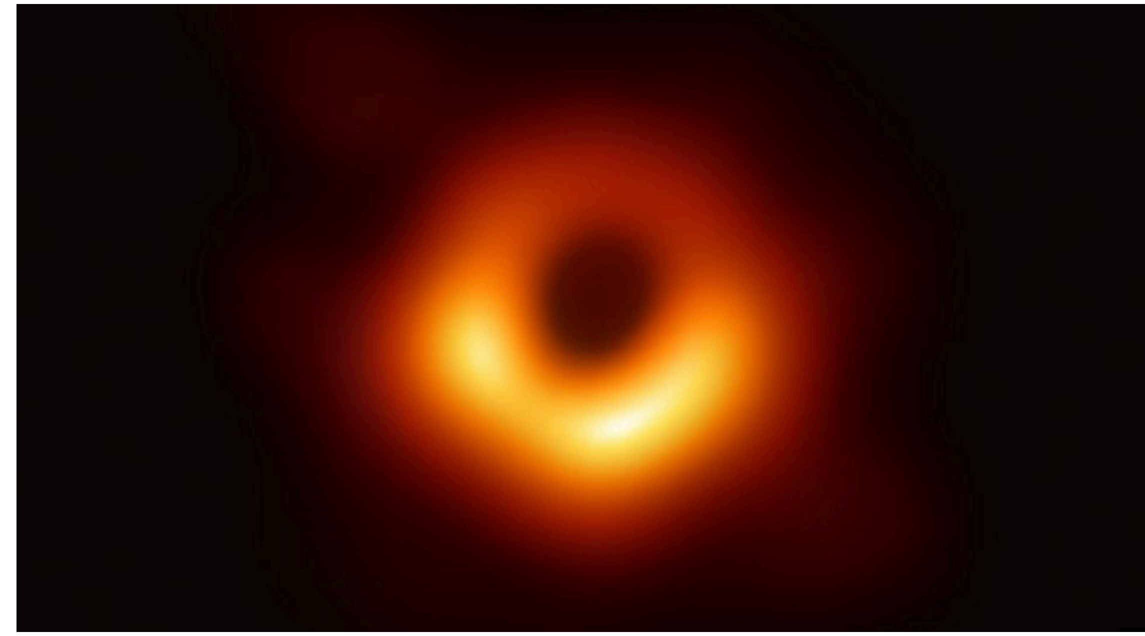
What is a black hole?



Einstein's equations describes the curvature of spacetime

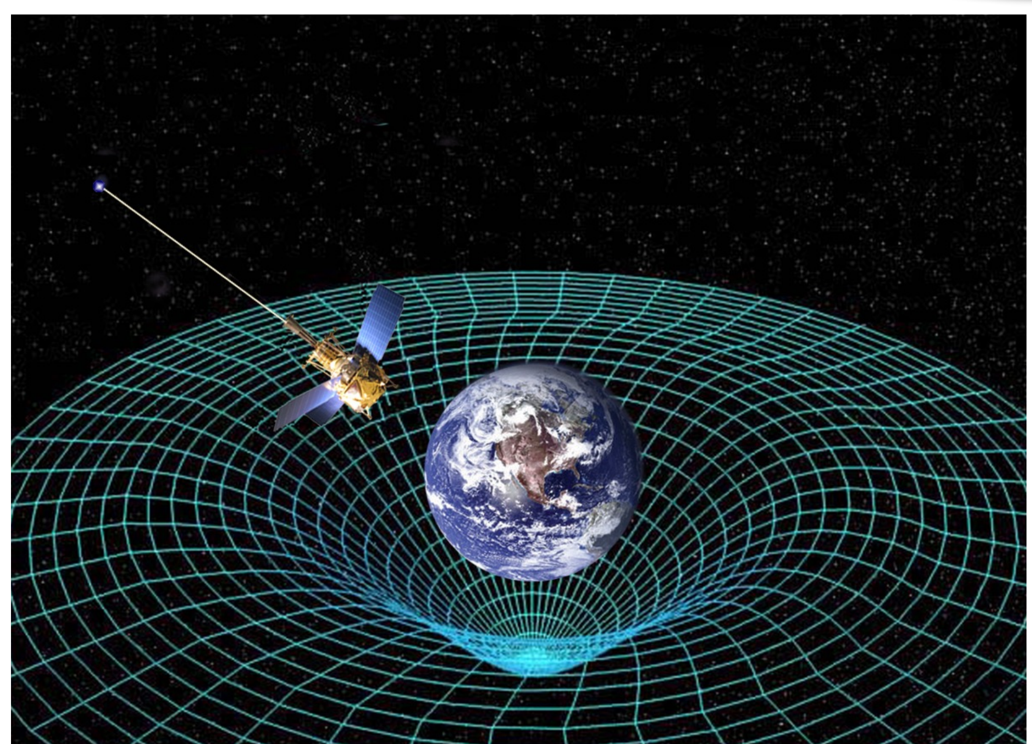
$$R_{\mu\nu}(g) - \frac{1}{2} g_{\mu\nu} R(g) = T_{\mu\nu}(g, F, \dots)$$

What is a black hole?



Einstein's equations describes the curvature of spacetime

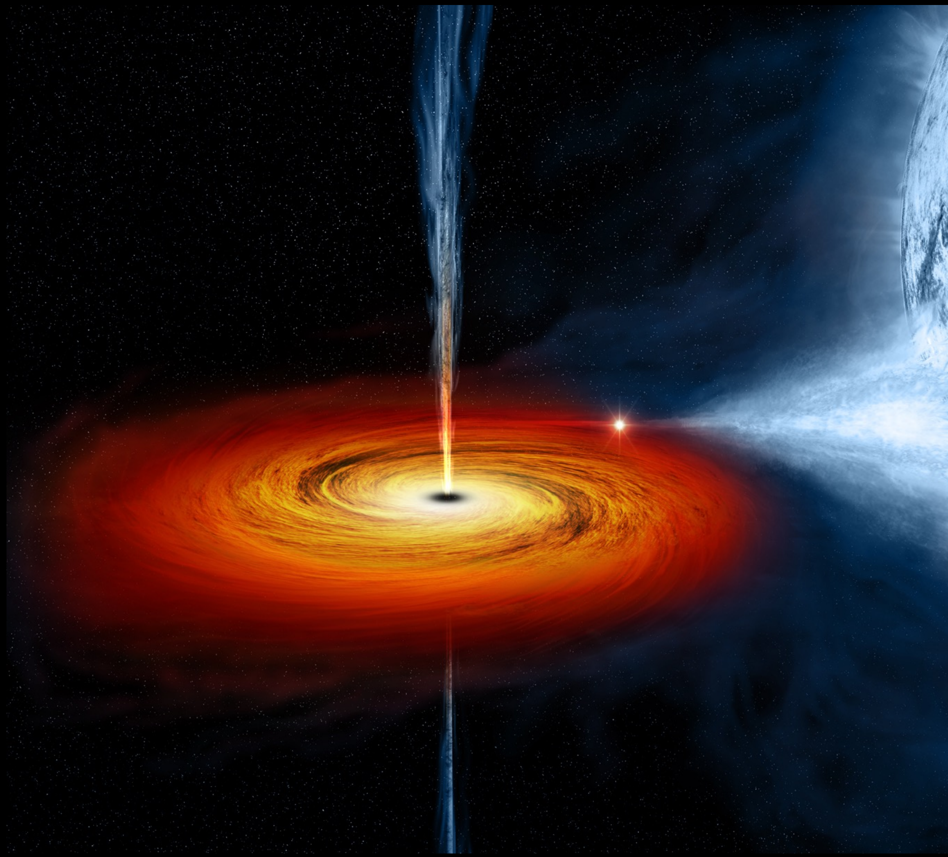
$$\underbrace{R_{\mu\nu}(g) - \frac{1}{2} g_{\mu\nu} R(g)}_{\text{geometry}} = \underbrace{T_{\mu\nu}(g, F, \dots)}_{\text{matter}}$$



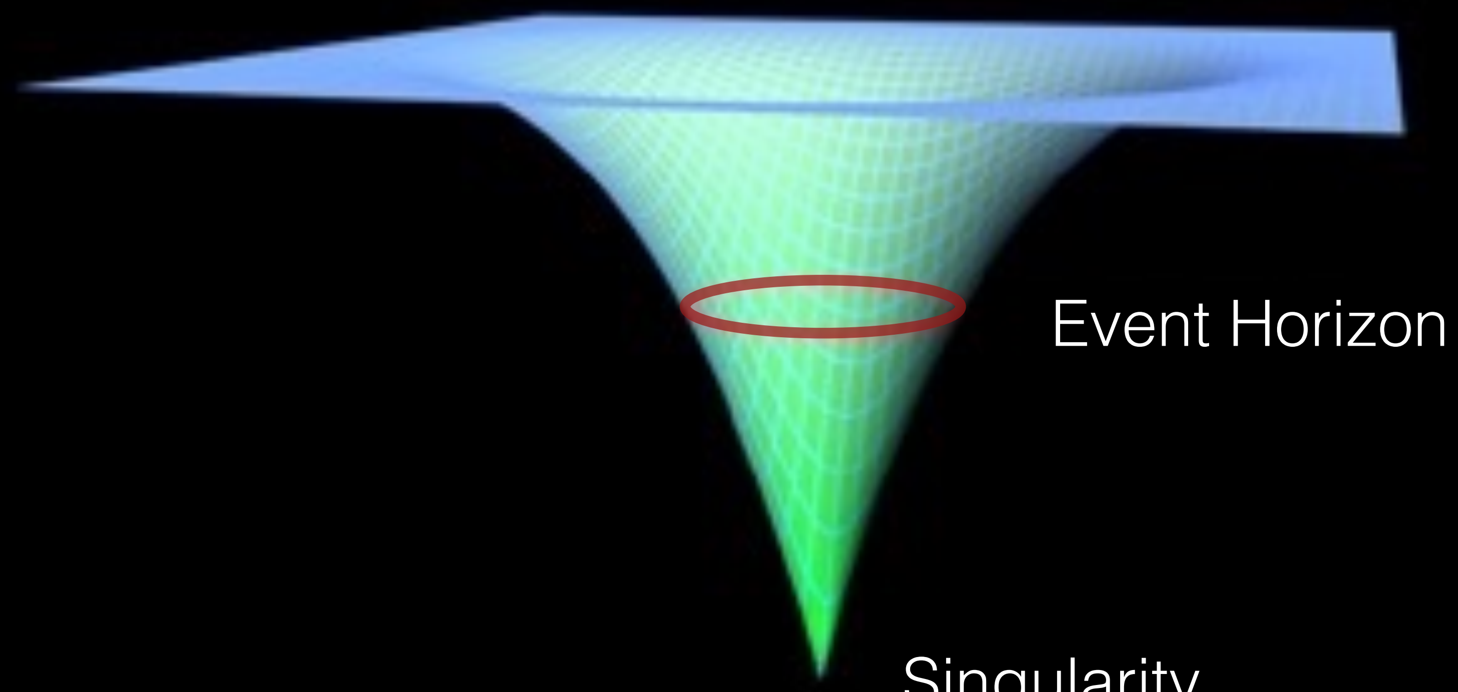
geometry

matter

A black hole forms in the final stage of the collapse of a sufficiently large star



At its centre space and time break down into a **singularity**



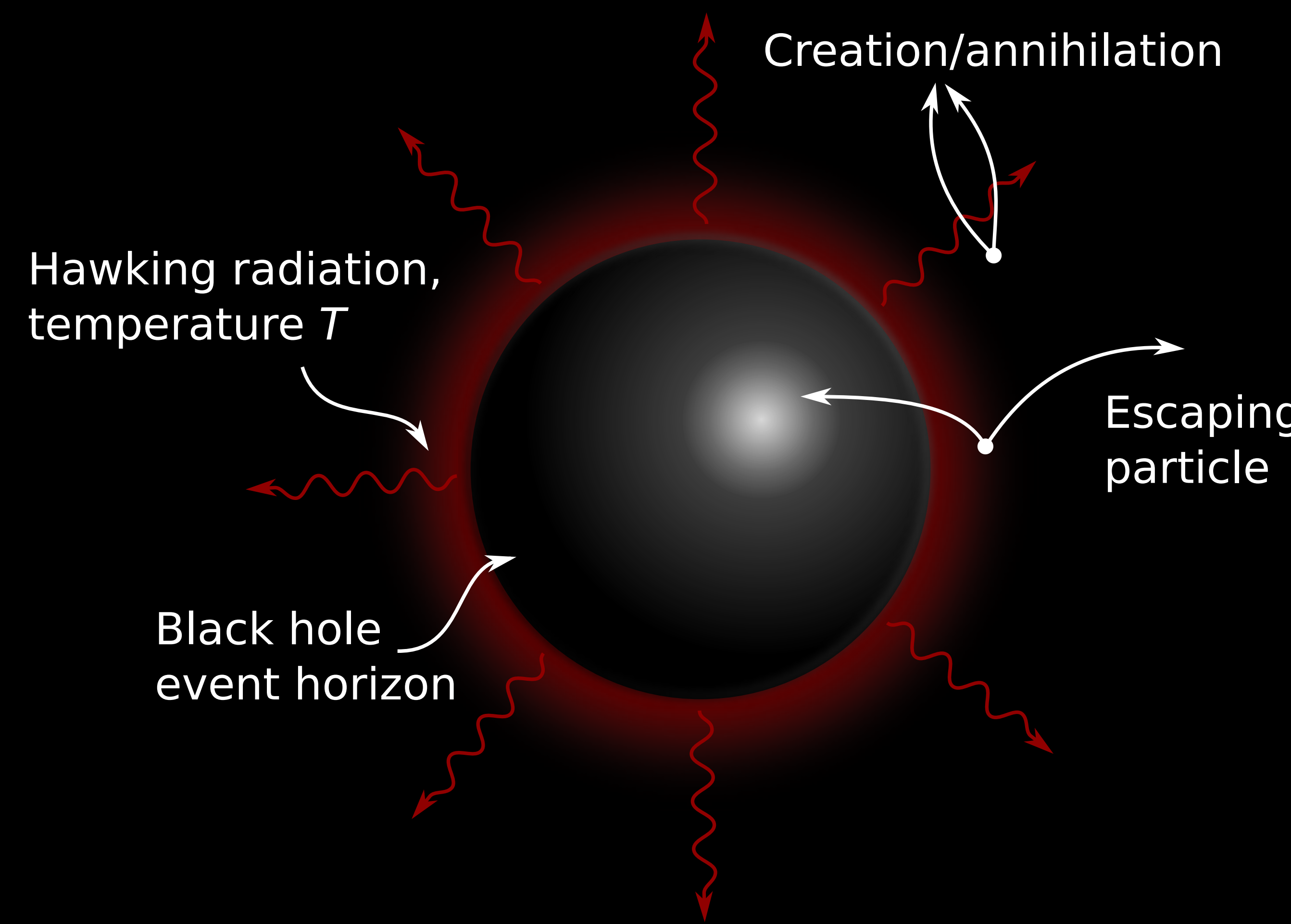
Do quantum effects resolve the singularity?

But black holes are not black!

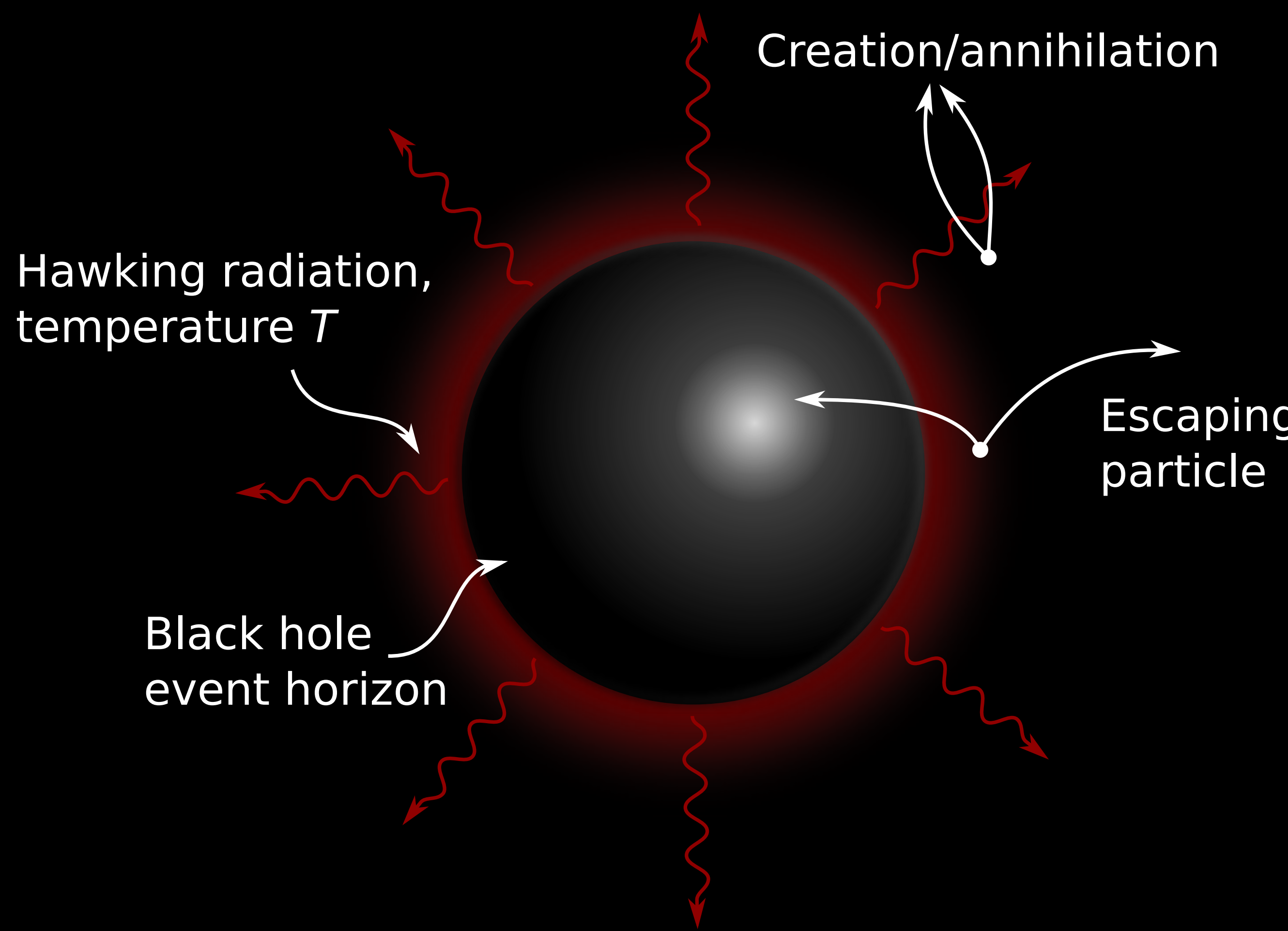
Hawking discovered that quantum effects close to the horion creates radiation

Quantum fluctuations outside the horizon cause pair creation

A particle with negative energy goes into the hole, while a positive energy particle escapes to infinity



Quantum fluctuations outside the horizon cause pair creation



A particle with negative energy goes into the hole, while a positive energy particle escapes to infinity

Hawking radiation!

Black holes behave like a black body with entropy

$$S = \frac{\mathrm{Area}}{4 G_N \hbar}$$

Bekenstein-Hawking
formula

Black holes behave like a black body with entropy

$$S = \frac{\text{Area}}{4G_N\hbar}$$

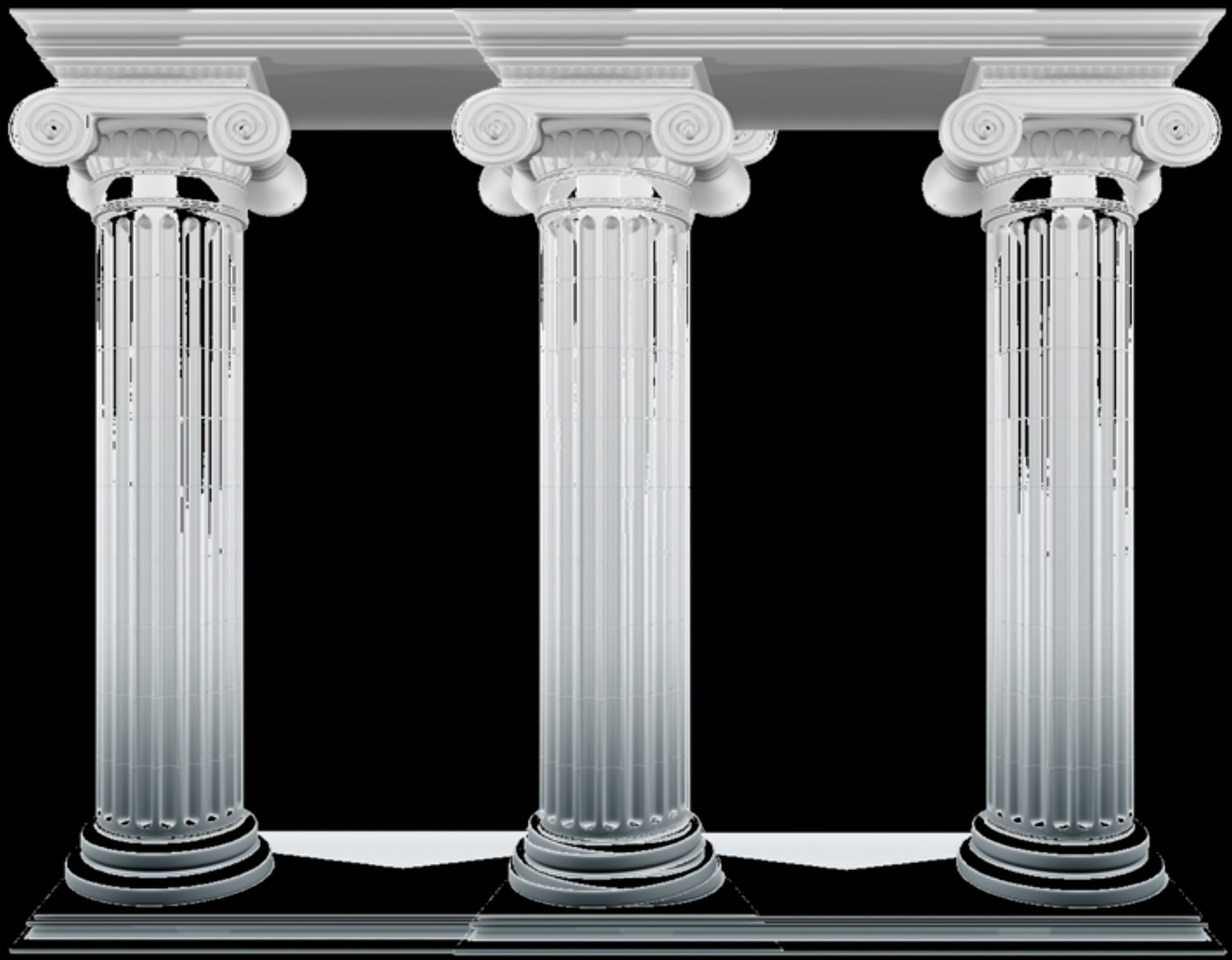
Bekenstein-Hawking formula

This is an amazing equation!

$$\text{Statistical Physics} = \frac{\text{Gravity}}{\text{Quantum Mechanics}}$$

$$\text{Statistical Physics} = \frac{\text{Gravity}}{\text{Quantum Mechanics}}$$

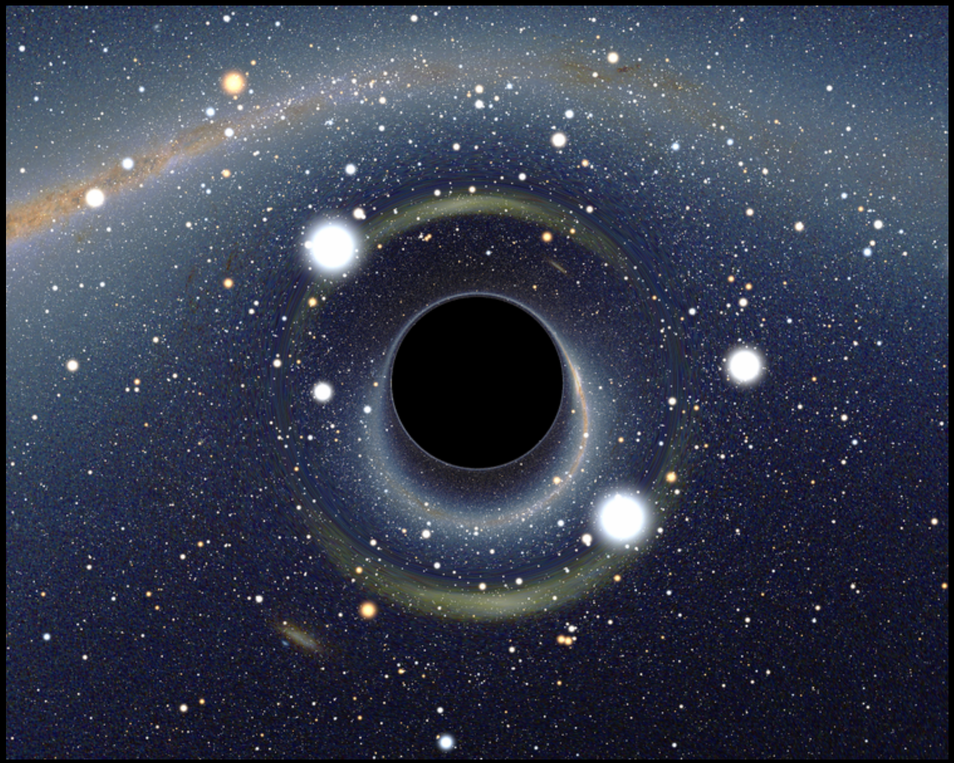
This relates three
pillars of theoretical physics!



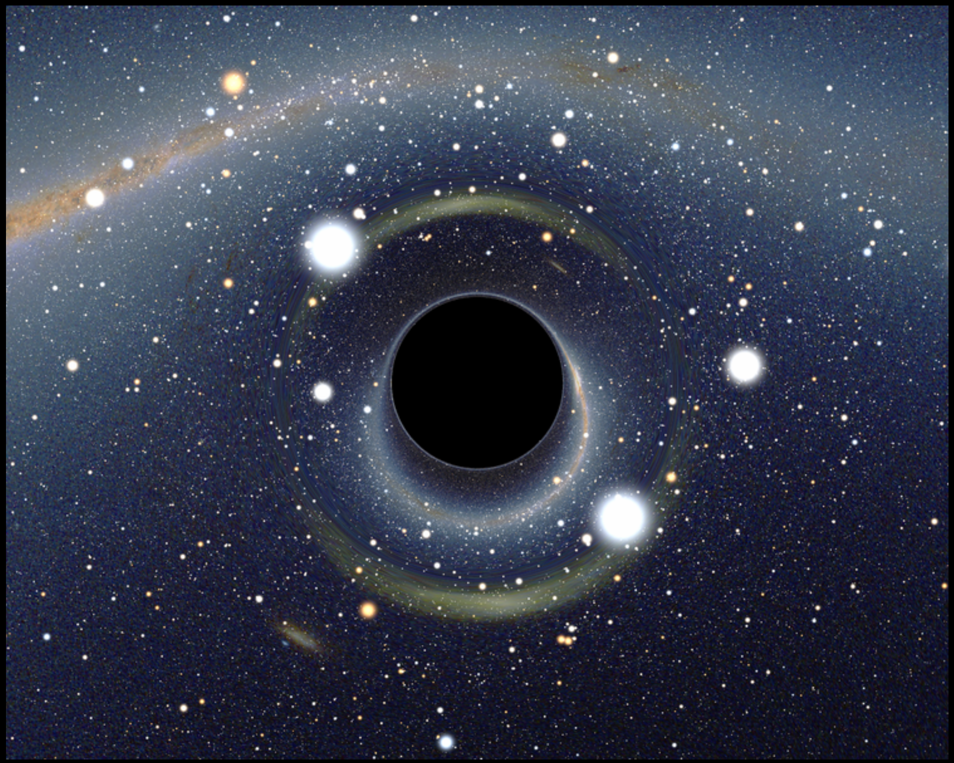
A **quantum theory of gravity** must be able to provide a microscopic account for the black hole entropy

$$S = \log\left(\sharp\,\text{microstates}\right)$$

But there is a fundamental problem with this picture:

$|\mathrm{pure}\rangle \xrightarrow{\text{IN}}$  $\xrightarrow{\text{OUT}}$ mixed state

But there is a fundamental problem with this picture:

$|\mathrm{pure}\rangle \xrightarrow{\text{IN}}$  $\xrightarrow{\text{OUT}}$ mixed state

Information is lost. Unitarity is violated.

INFORMATION PARADOX

INFORMATION PARADOX

Two parts:

INFORMATION PARADOX

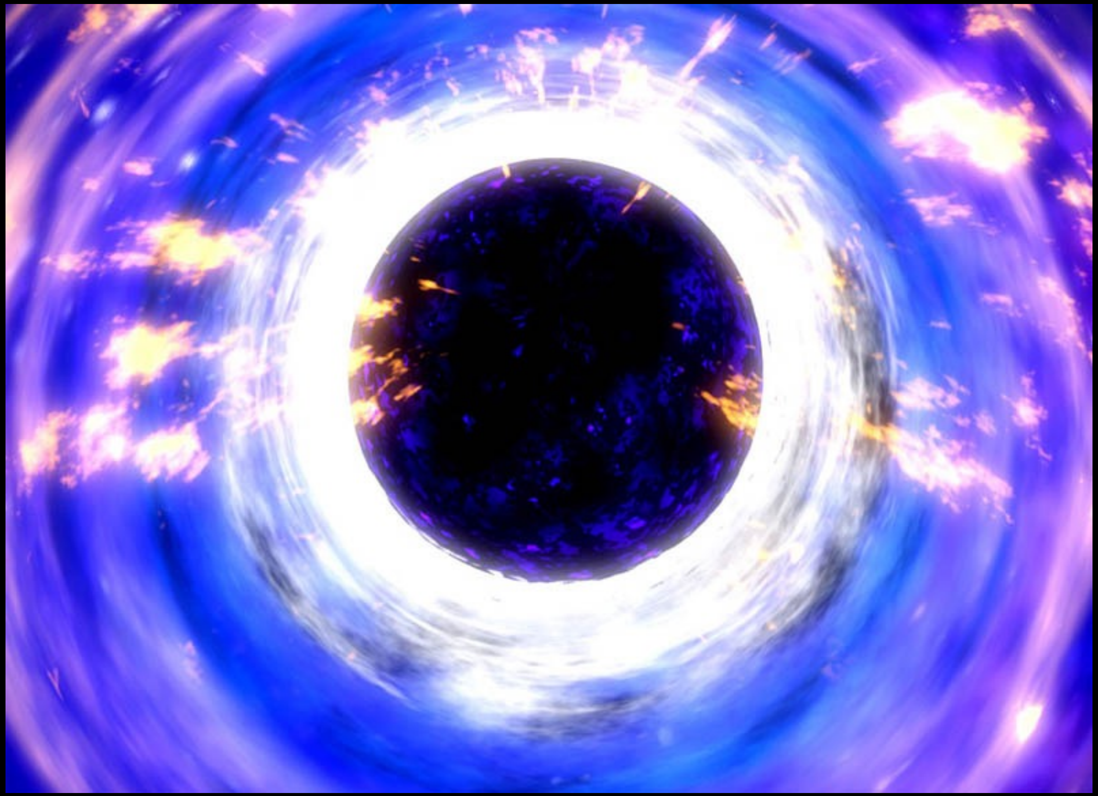
Two parts:

- The 'no hair theorem'

INFORMATION PARADOX

Two parts:

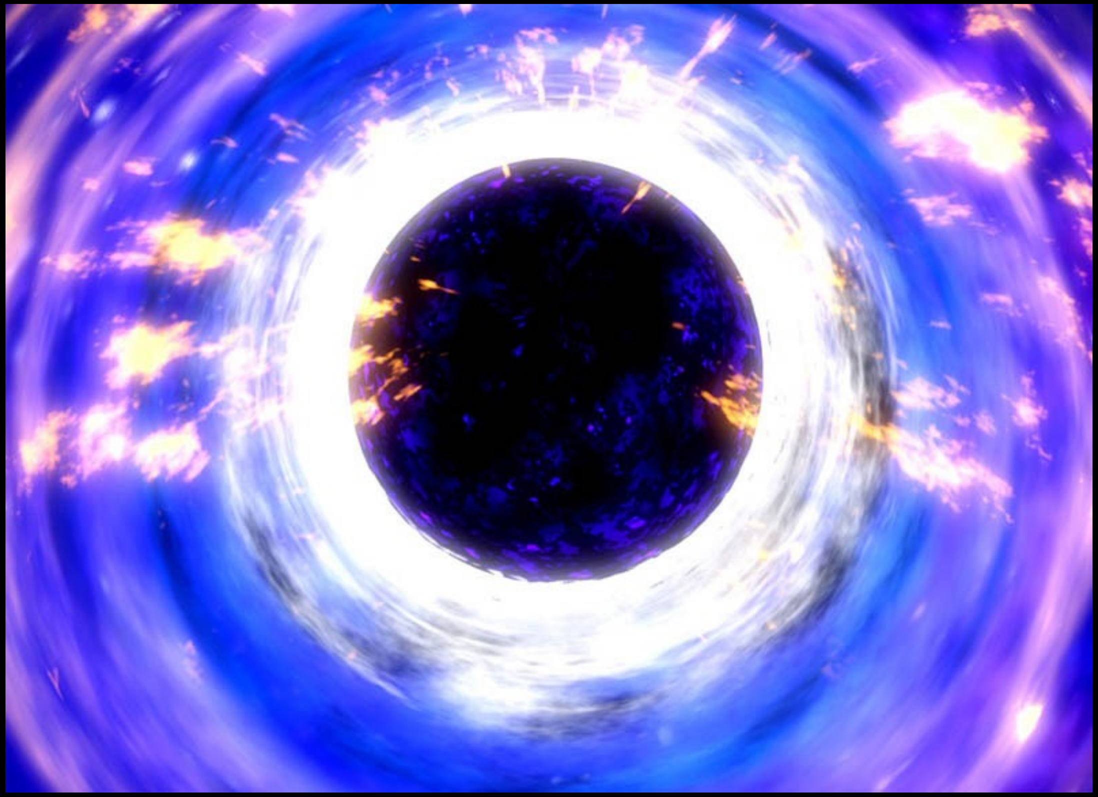
- The ‘no hair theorem’
- Thermal Hawking radiation



INFORMATION PARADOX

Two parts:

- The ‘no hair theorem’
- Thermal Hawking radiation



String theory is a candidate for a quantum theory of gravity. What does it have to say?

Crash course in string theory

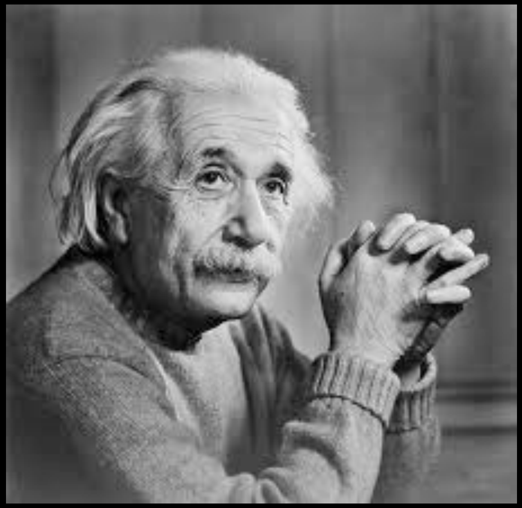
Elementary particles

↕

Vibrating strings of energy

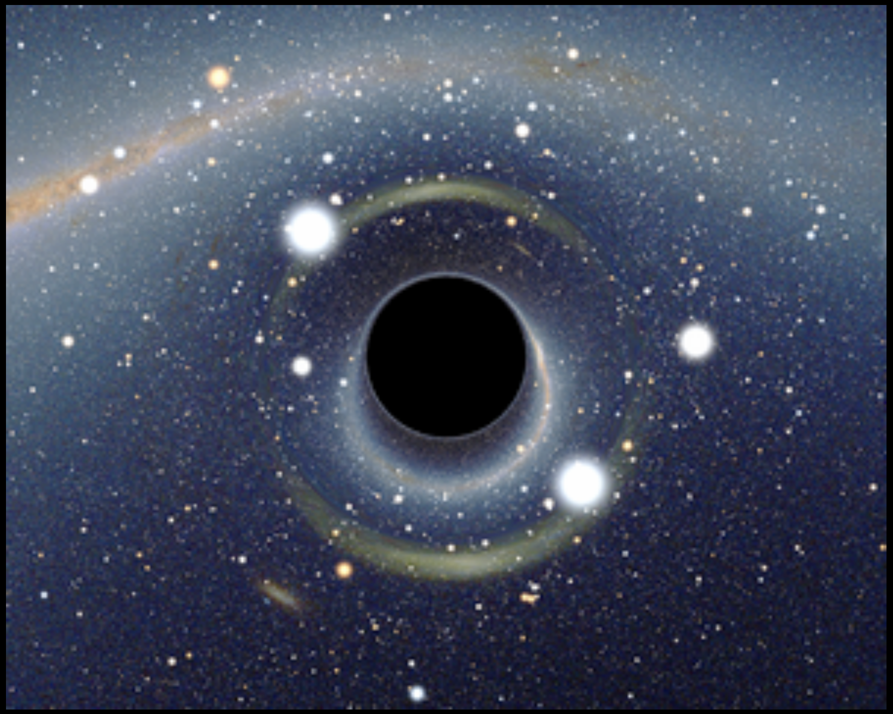
The vibrational modes of closed strings correspond to gravitons

String theory contains Einstein's general theory of relativity!



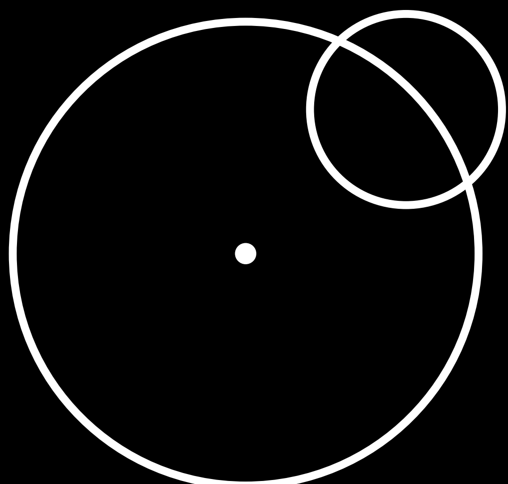
We can “build” black holes out of bounds states of strings and branes

When the system becomes sufficiently massive it gravitates and a black hole forms

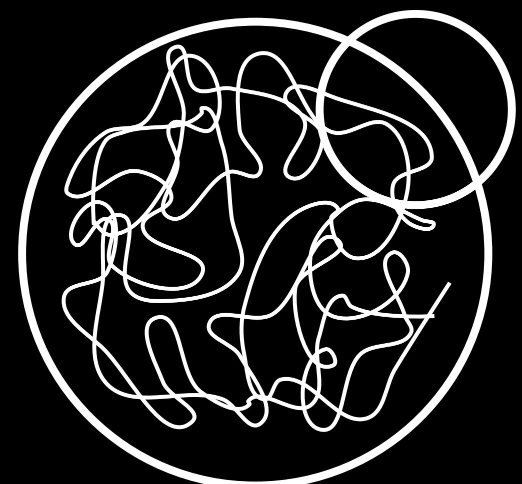


In string theory a “quantum black hole”
does not have a horizon in the traditional sense.

$|0\rangle$



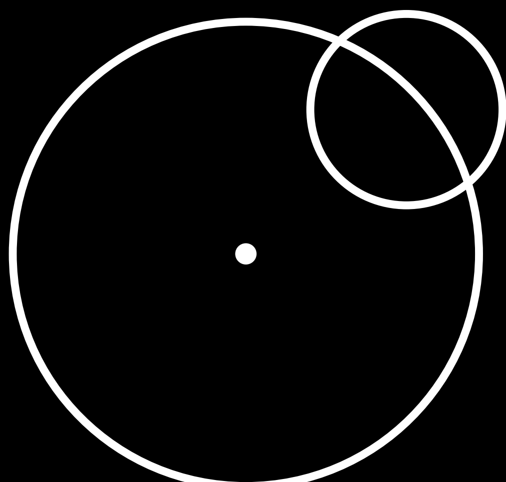
$|\Psi\rangle$



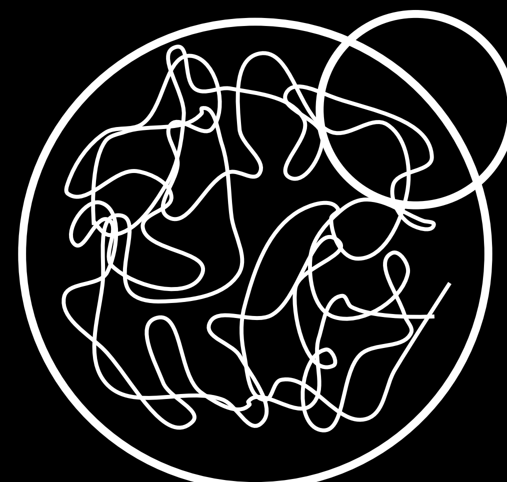
The horizon is **no longer in a vacuum state**

In string theory a “quantum black hole”
does not have a horizon in the traditional sense.

$|0\rangle$



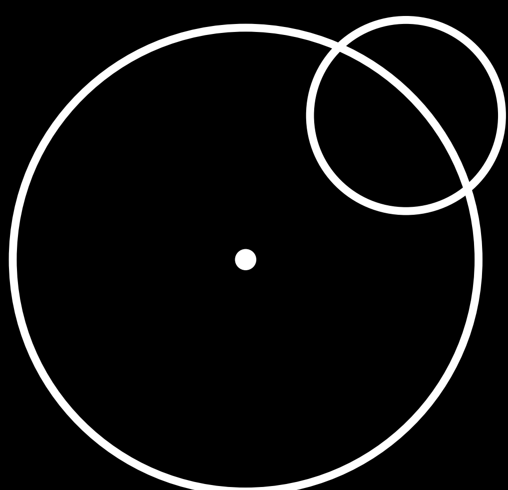
$|\Psi\rangle$



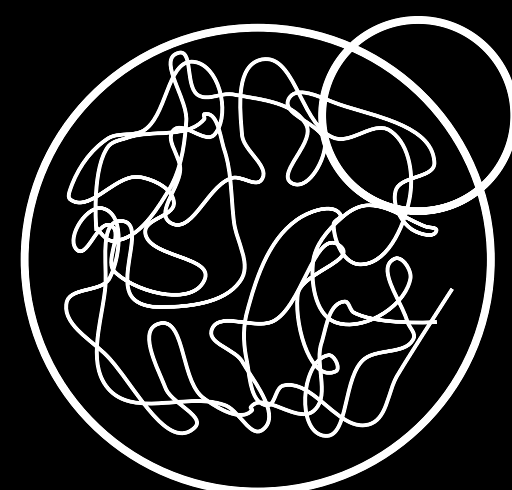
A stringy black hole has hair!

In string theory a “quantum black hole”
does not have a horizon in the traditional sense.

$|0\rangle$



$|\Psi\rangle$



~~INFORMATION PARADOX~~

How to test Boltzmann's formula?

$$S = \log\left(\sharp\,\text{microstates}\right)$$

The solution to our problem hides in the equations of Ramanujan!

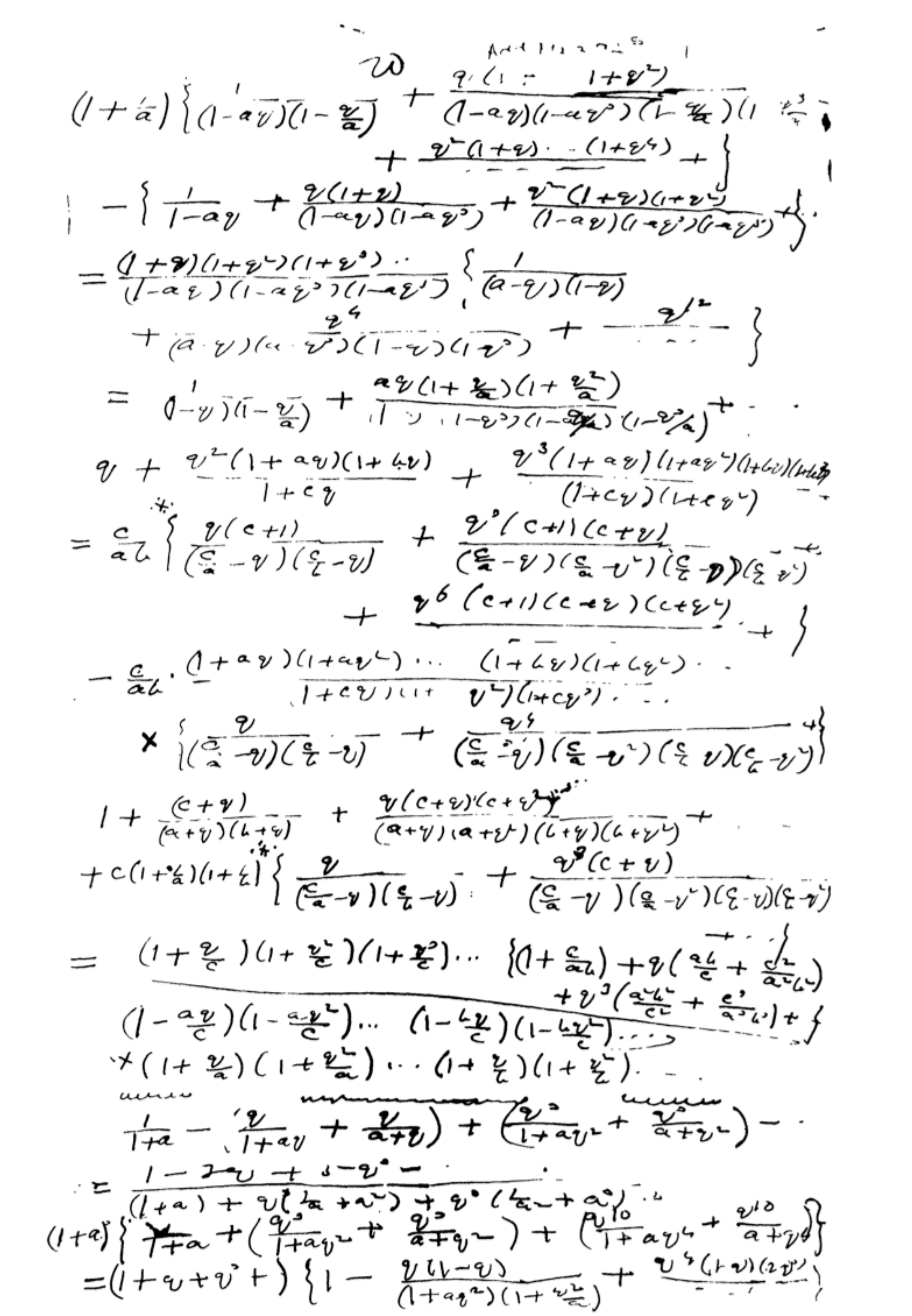
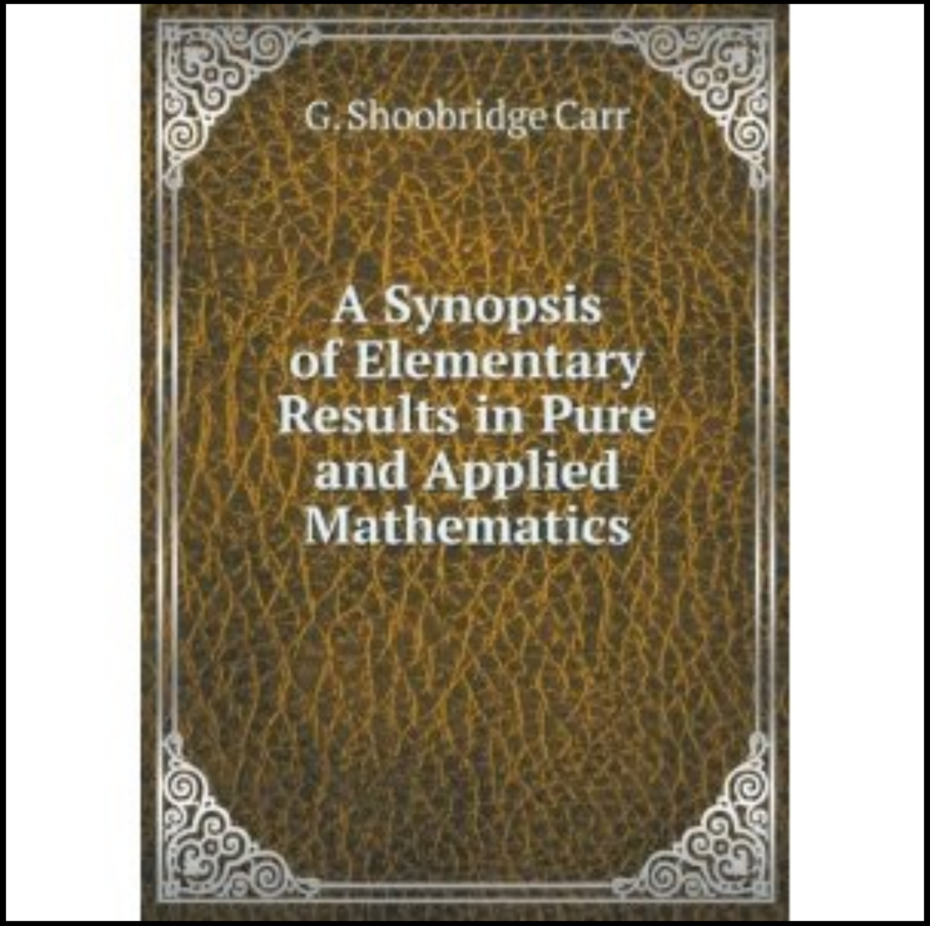


FIG. 1. Power series computations and comparisons from Ramanujan's "Lost" Notebook.

Srinivasa Ramanujan (1887-1920)

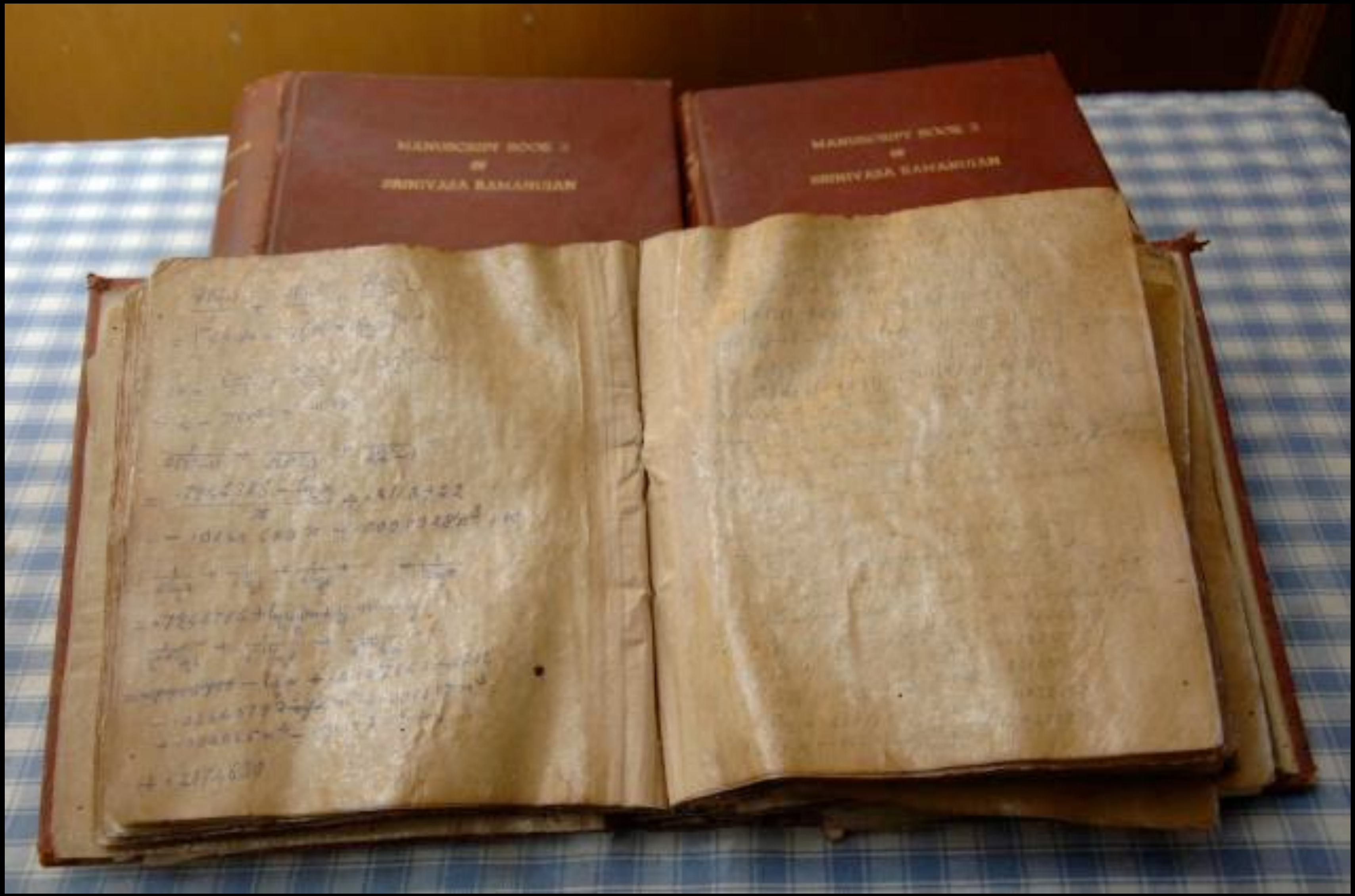
Born in a small village on the east coast of India

At the age of 11 he got his first mathematics book

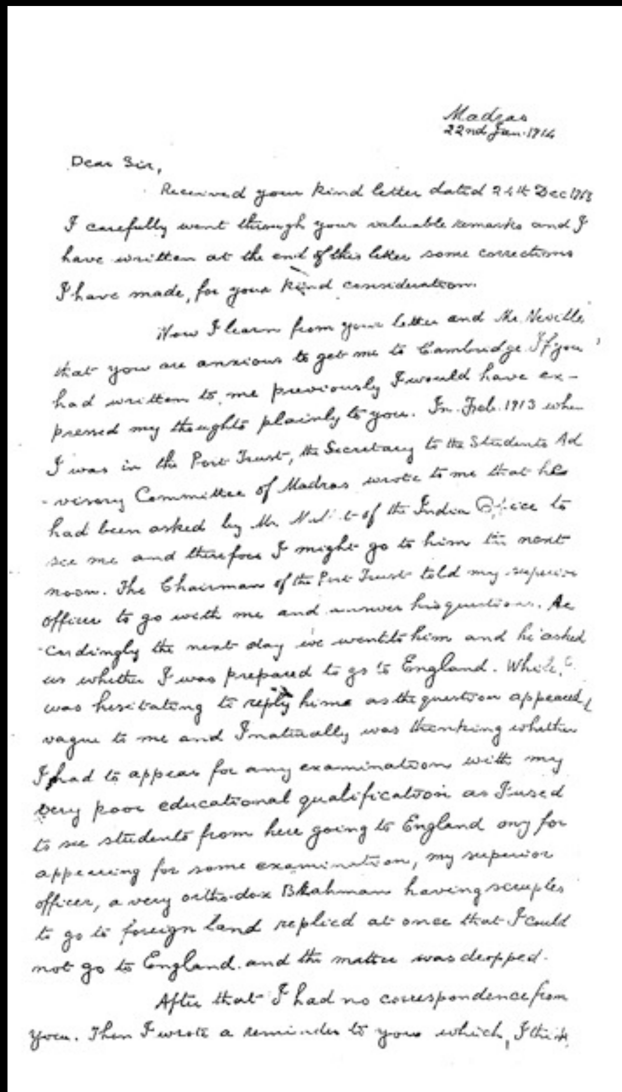


Ramanujan was absorbed by the world of mathematics and ignored all other subjects at school

He dropped out of school and pursued mathematics alone for many years



At the age of 25 he wrote letters to several british mathematicians, but for a long time received no replies


Madras
22nd Jan. 1914

Dear Sir,

Received your kind letter dated 24th Dec 1913
I carefully went through your valuable remarks and I
have written at the end of this letter some corrections
I have made, for your kind consideration.

Now I learn from your letter and Mr. Neville,
that you are anxious to get me to Cambridge. If you
had written to me previously I would have ex-
pressed my thoughts plainly to you. In Feb. 1913 when
I was in the Port Trust, the Secretary to the Students Ad-
-visory Committee of Madras wrote to me that he
had been asked by Mr. N[illegible] of the Indian Office to
see me and therefore I might go to him the next
noon. The Chairman of the Port Trust told my superior
officer to go with me and answer his questions. Ac-
-cordingly the next day we went to him and he asked
us whether I was prepared to go to England. While I
was hesitating to reply him as the question appeared
vague to me and I naturally was thinking whether
I had to appear for any examination with my
very poor educational qualification as I used
to see students from here going to England only for
appearing for some examination, my superior
officer, a very ortho-dox Brahman having scruples
to go to foreign land replied at once that I could
not go to England and the matter was dropped.

After that I had no correspondence from
you. Then I wrote a reminder to you which, I think

1913 he wrote a thick letter to the famous mathematician G. H. Hardy at Cambridge

"One morning in 1913, he (Hardy) found, among the letters on his breakfast table, a large untidy envelope decorated with Indian stamps. When he opened it…he found line after line of symbols. He glanced at them without enthusiasm. […] The script appeared to consist of theorems, most of them wild or fantastic… There were no proofs of any kind… A fraud or genius?"

After much deliberation Hardy decided to invite Ramanujan to Cambridge in 1914

Over a period of 4 years Ramanujan wrote over 30 articles, many jointly with Hardy

"What was to be done in the way of teaching him modern mathematics? The limitations of his knowledge were as startling as its profundity."

-Hardy

After much deliberation Hardy decided to invite Ramanujan to Cambridge in 1914

Recent movie about the life of Ramanujan!

“Taxi cab numbers”

$$1729$$

$$= 1^3 + 12^3$$

$$= 9^3 + 10^3$$

“Taxi cab numbers”

$$1729$$
$$= 1^3 + 12^3$$
$$= 9^3 + 10^3$$

Ramanujan was particularly interested in series that correspond to partitions of integers:

$$f(q) = \sum_{n=0}^{\infty} p(n) q^n = 1 + q + 2q^2 + 3q^3 + 5q^4 + \cdots$$

$$p(4) = 5 \quad \longleftrightarrow \quad 4 = 3 + 1 = 2 + 2 = 2 + 1 + 1 = 1 + 1 + 1 + 1$$

“Taxi cab numbers”

$$1729$$

$$= 1^3 + 12^3$$

$$= 9^3 + 10^3$$

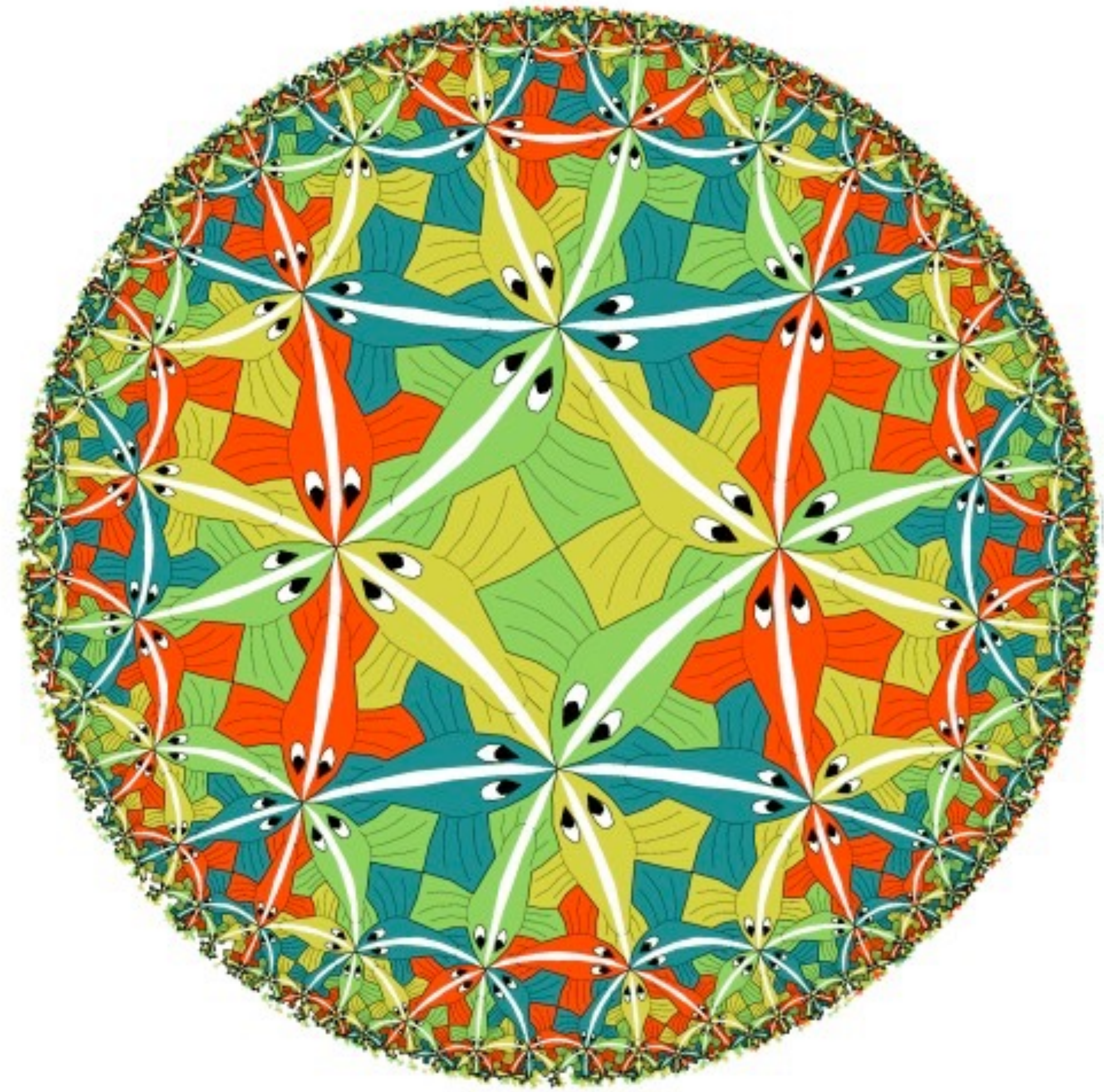
Ramanujan was particularly interested in series that correspond to partitions of integers:

$$f(q) = \sum_{n=0}^{\infty} p(n) q^n = 1 + q + 2q^2 + 3q^3 + 5q^4 + \cdots$$

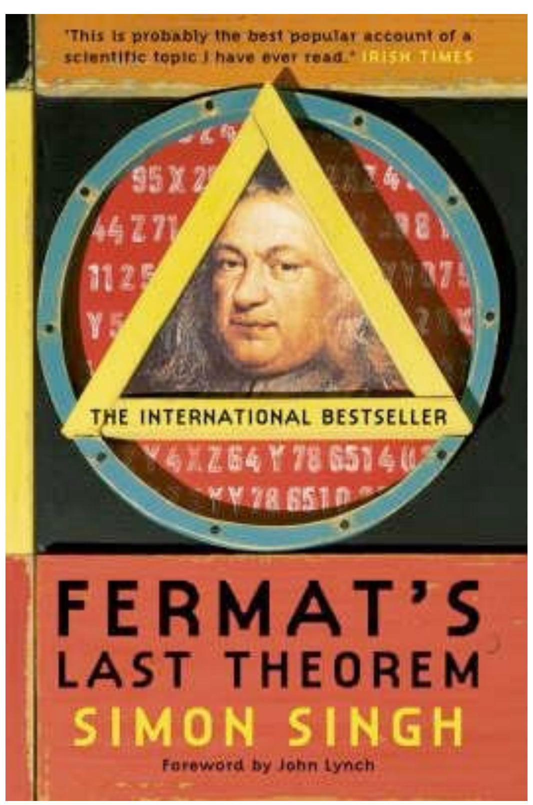
These often exhibit hidden symmetries, making them into *modular forms*

$$f(q)=\sum_{n=0}^{\infty}p(n)q^n = 1+q+2q^2+3q^3+5q^4+\cdots$$

These infinite series
often exhibit
hidden symmetries



“Modular Forms”



“Modular forms are some of the weirdest and most wonderful objects in mathematics.”

Ramanujans last letter

After 4 years in England Ramanujan became ill and returned to India in 1919. He died in April 1920, only 33 years old.

3 months before his death he sent a letter to Hardy listing 17 functions that he claimed had "amazing properties"

$$
\begin{aligned}
f(q) &= \sum_{n=0}^{\infty} \frac{q^{n^2}}{(1+q)^2 \cdots (1+q^n)^2}, \\
\phi(q) &= \sum_{n=0}^{\infty} \frac{(-q)^{n^2}}{(1+q^2)(1+q^4)\cdots(1+q^{2n})}, \\
\psi(q) &= \sum_{n=1}^{\infty} \frac{(-q)^{n^2}}{(1+q)(1+q^3)\cdots(1+q^{2n-1})}.
\end{aligned}
$$

Ramanujans last letter

After 4 years in England Ramanujan became ill and returned to India in 1919. He died in April 1920, only 33 years old.

3 months before his death he sent a letter to Hardy listing 17 functions that he claimed had “amazing properties”

$$f(q) = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(1+q)^2 \cdots (1+q^n)^2},$$

$$\phi(q) = \sum_{n=0}^{\infty} \frac{(-q)^{n^2}}{(1+q^2)(1+q^4)\cdots(1+q^{2n})},$$

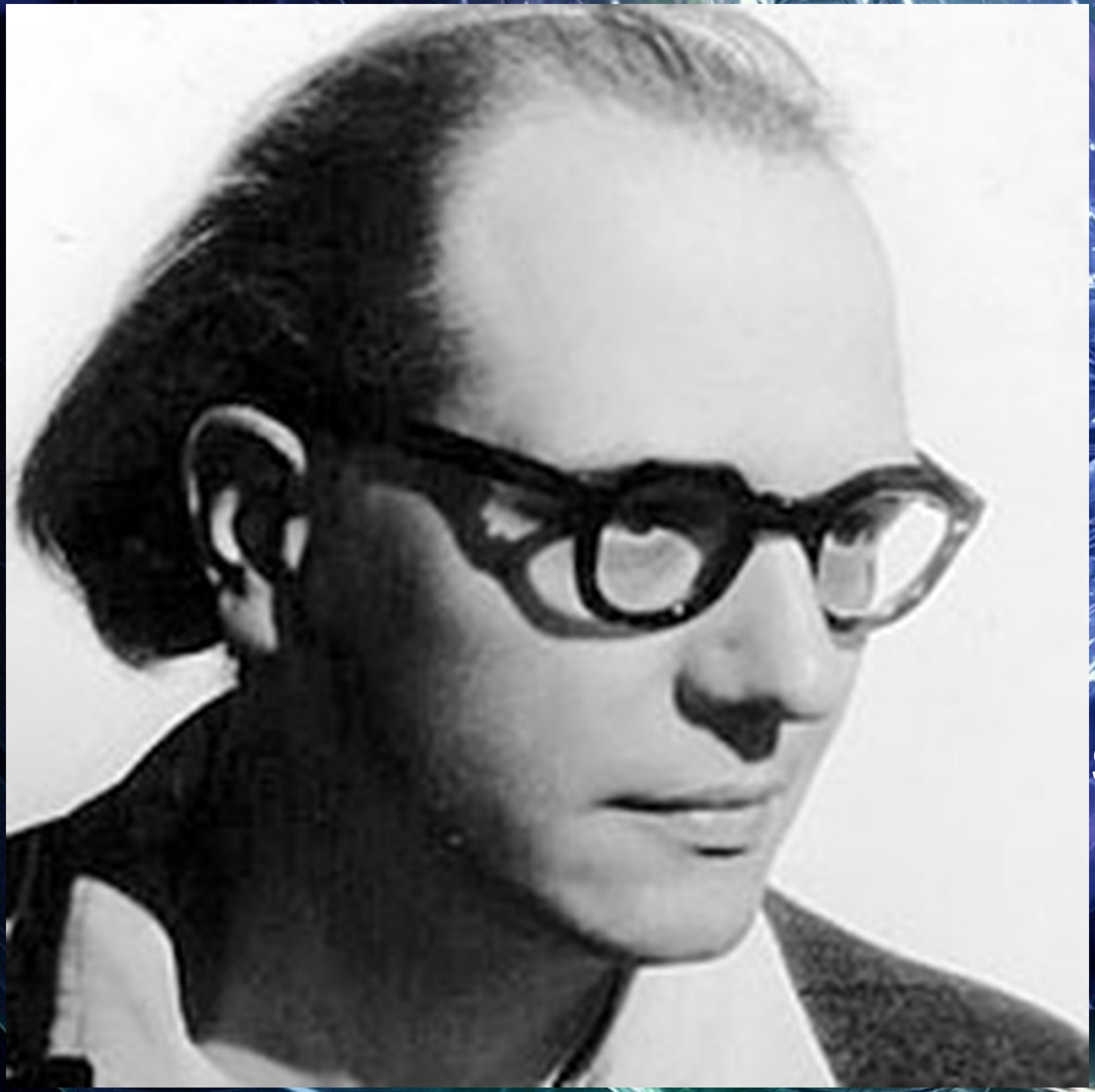
$$\psi(q) = \sum_{n=1}^{\infty} \frac{(-q)^{n^2}}{(1+q)(1+q^3)\cdots(1+q^{2n-1})}.$$

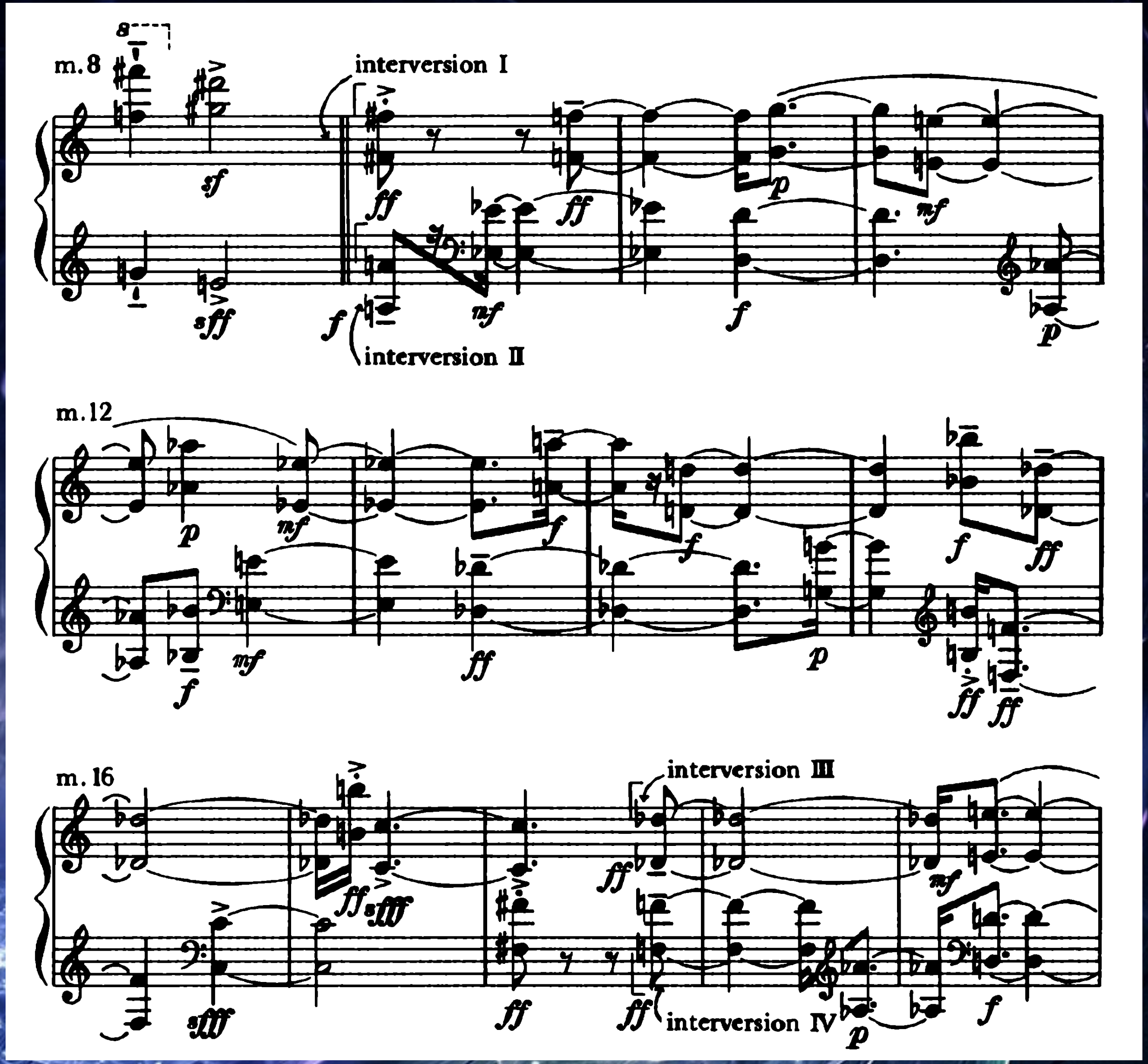
It took nearly 100 years and generations of mathematicians before the secrets of the last letter where finally unravelled…

The functions that Ramanujan wrote down turns out to have unexpected hidden symmetries that directly help to understand the quantum entropy of black holes!

“Ile de feu 2”
(Island of fire)

Olivier Messiaen 1949





This piece of music has the same symmetries as Ramanujan’s functions!

Let us bring the stories together

Goal: To calculate the black hole entropy

Entropi = Log (# states)

Let us bring the stories together

Goal: To calculate the black hole entropy

$$S(Q) = \log W(Q)$$

Entropy for a black hole with electric charge Q

Let us bring the stories together

Goal: To calculate the black hole entropy

$$S(Q) = \log W(Q)$$

Number of quantum states

Let us bring the stories together

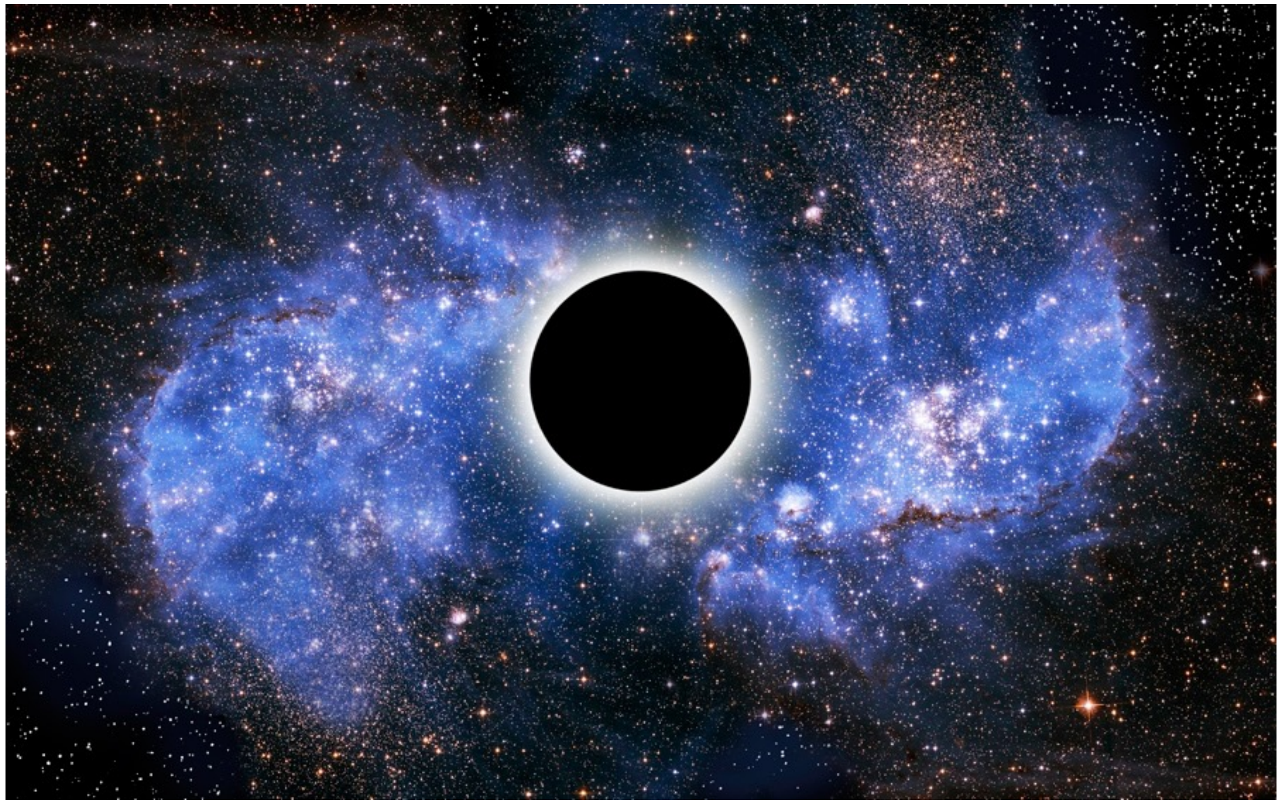
Goal: To calculate the black hole entropy

$$S(Q) = \log W(Q)$$

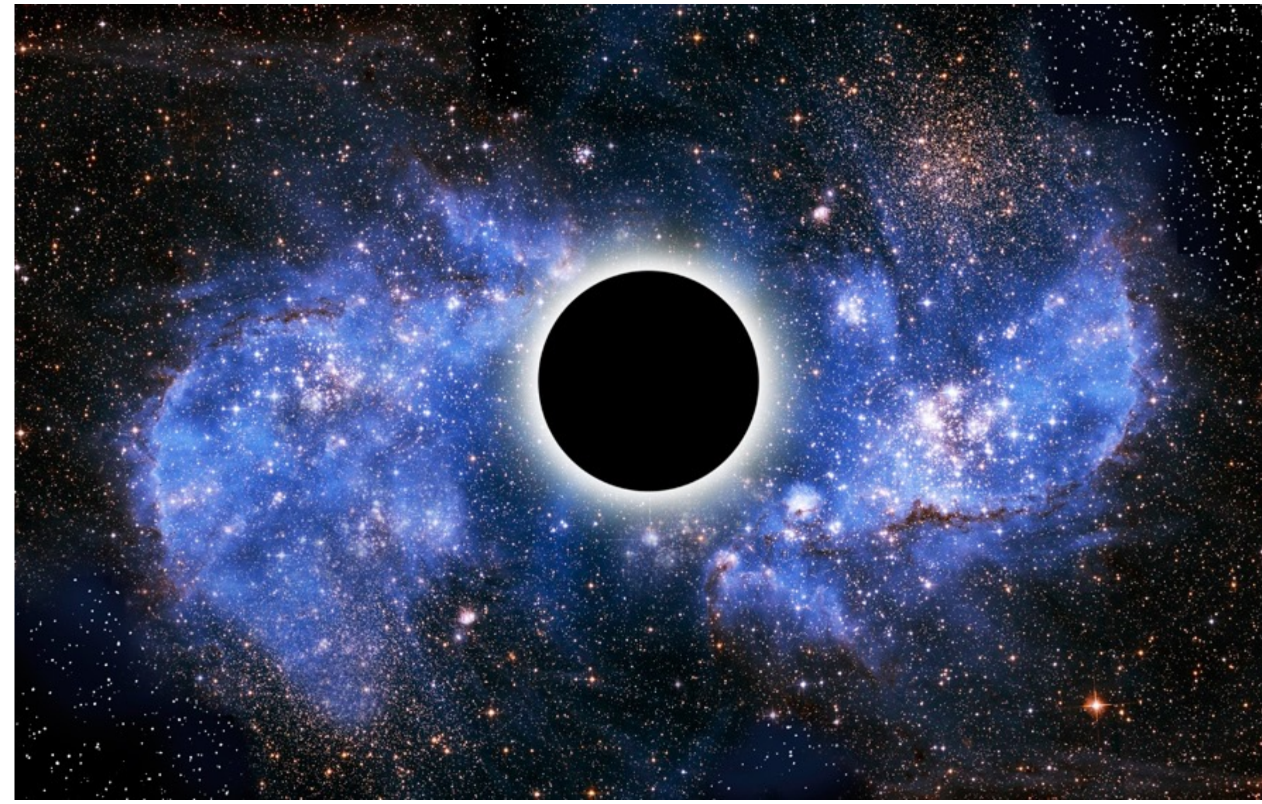
We now collect all this information in a partition function

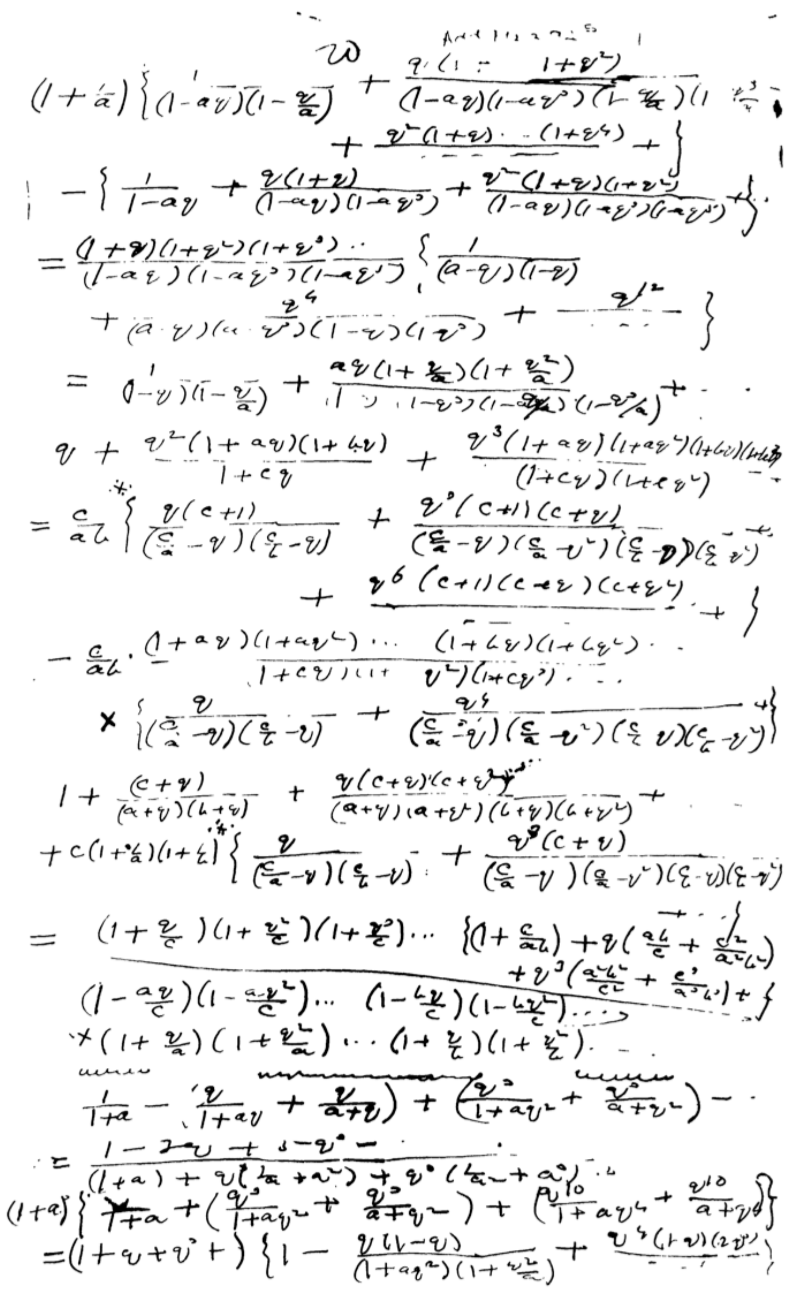
$$\Psi(q) = \sum_Q W(Q) q^Q$$

sum over all charges



$$\Psi(q) = \sum_{Q} W(Q) q^{Q}$$

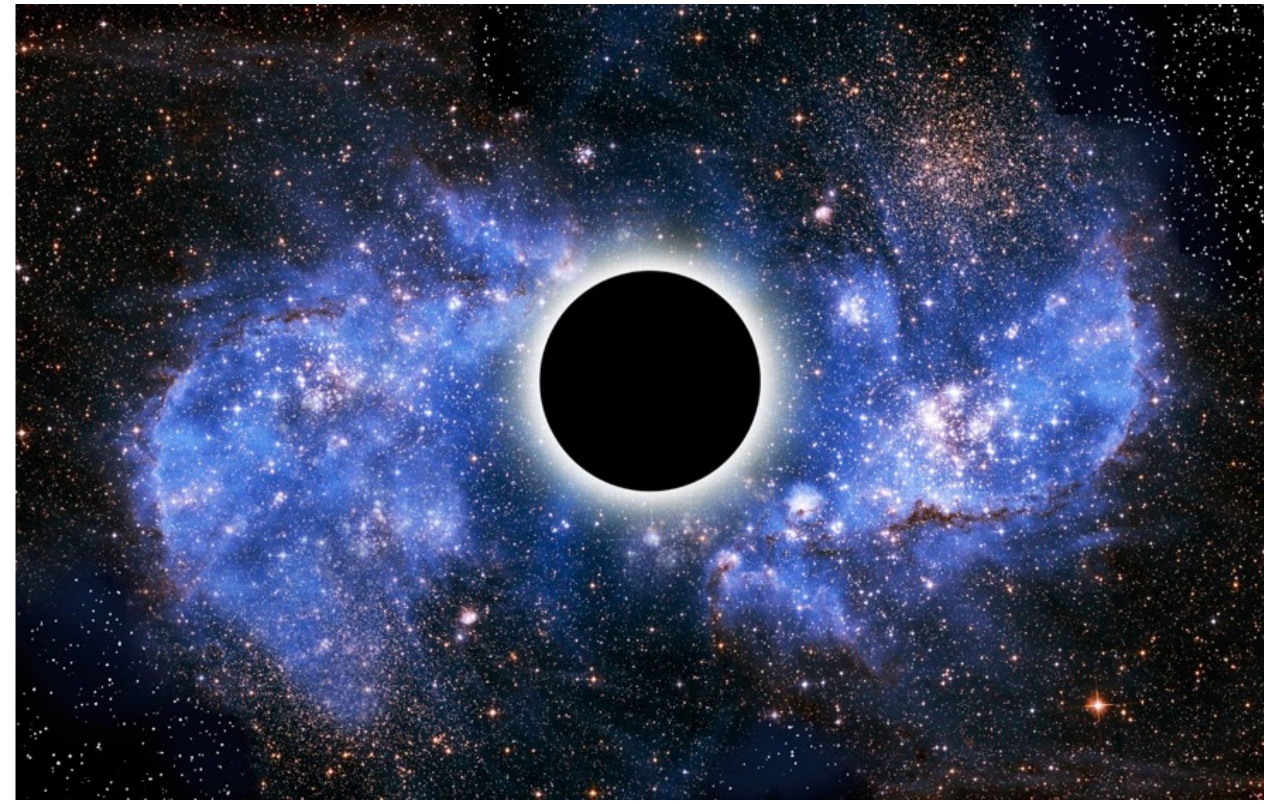




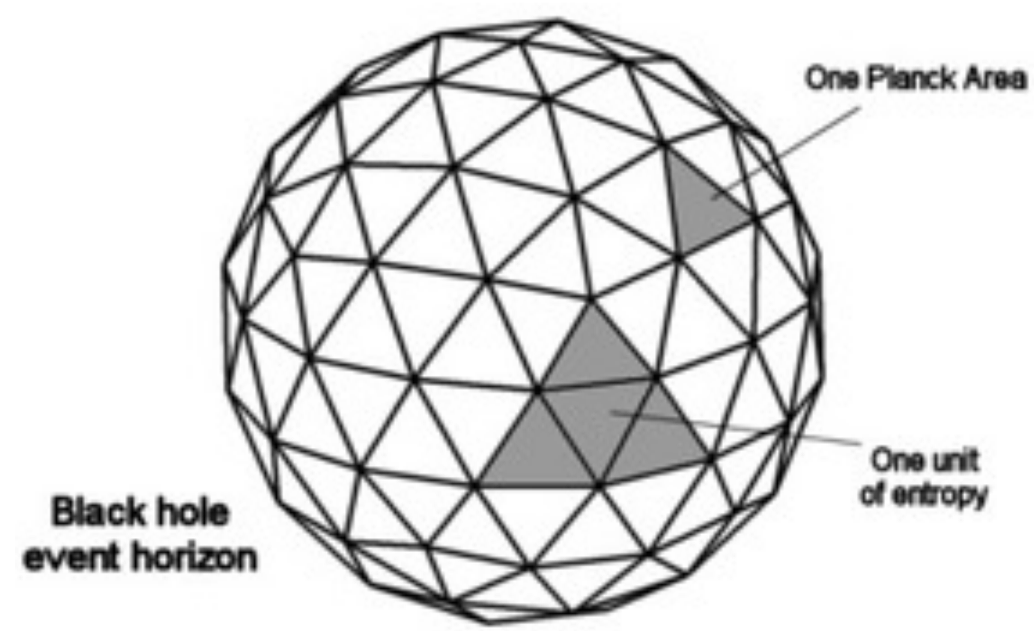
This turns out to be precisly one of the functions of Ramanujan's last letter

$$f(q) = \sum_{n=0}^{\infty} p(n) q^{n} = 1 + q + 2q^{2} + 3q^{3}$$

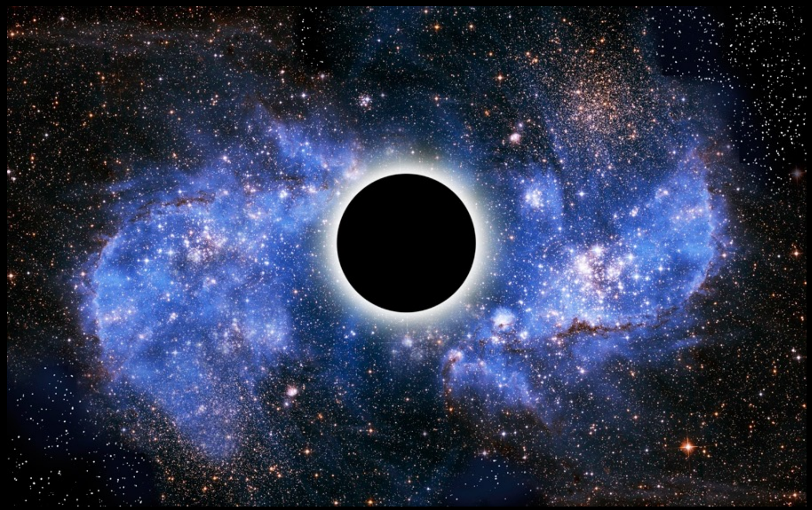
$$\Psi(q) = \sum_{Q} W(Q) q^{Q}$$



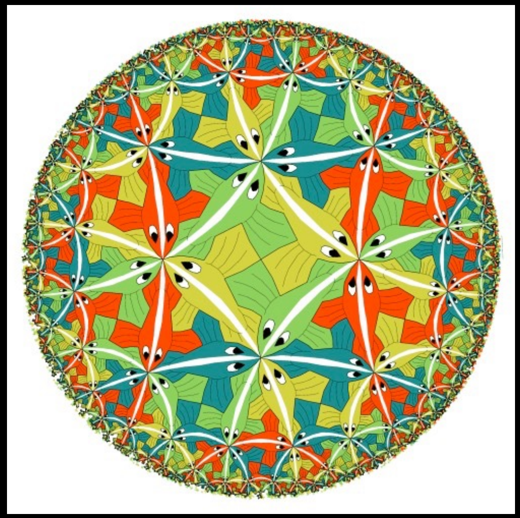
Black hole microstates organized into according to a huge symmetry



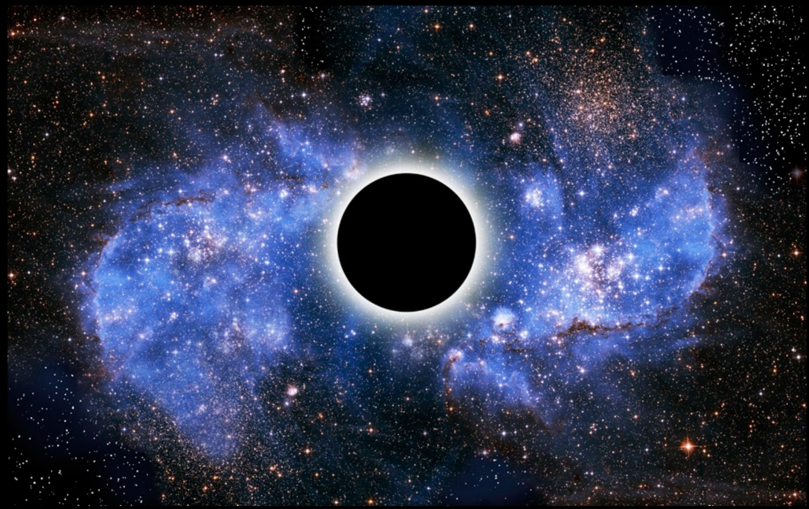
Quantum gravity



Modular forms



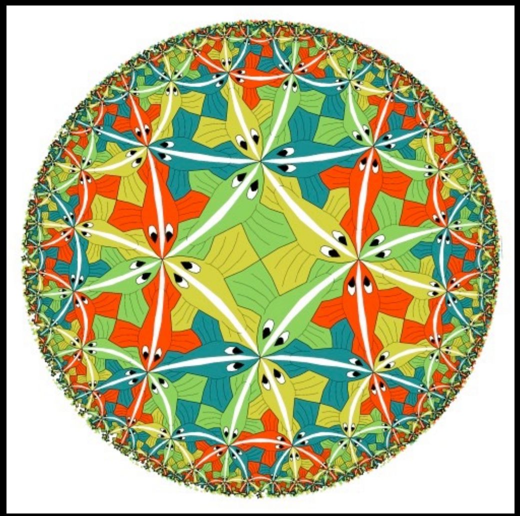
Quantum gravity



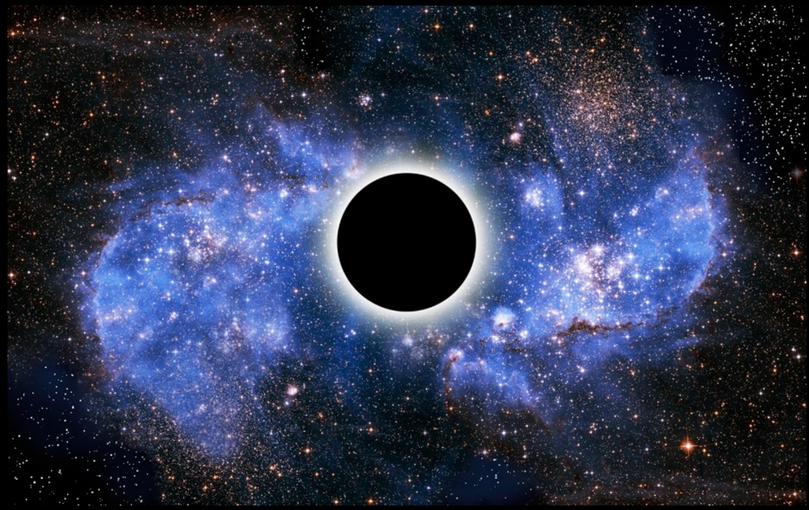
Number theory

1,2,3,4,5,….

Modular forms



Quantum gravity

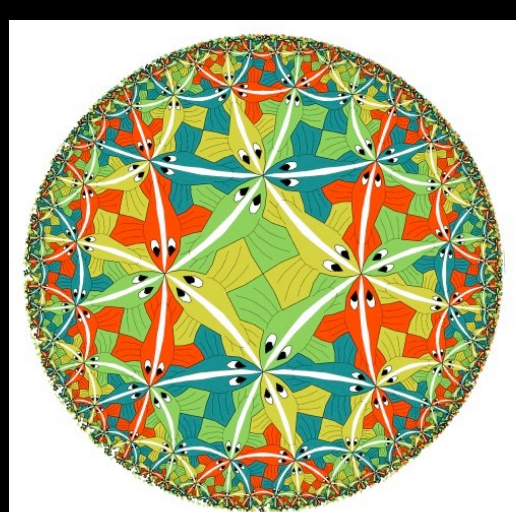


Number theory

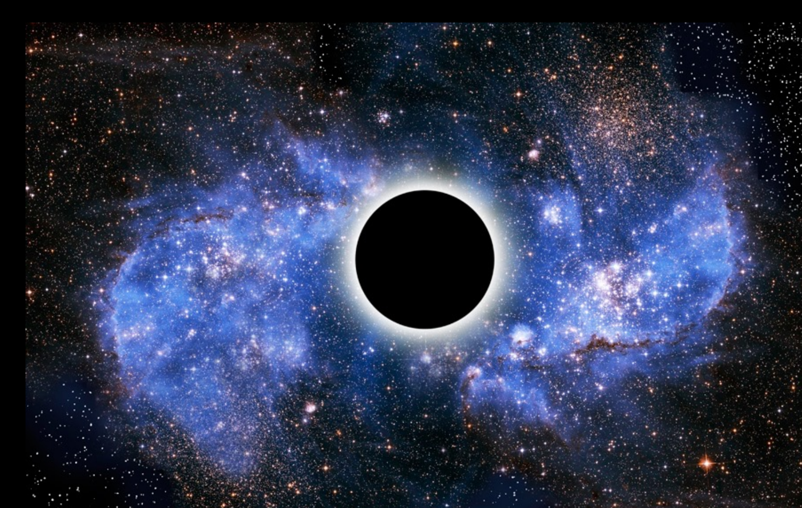
1,2,3,4,5,….

Symmetries

Modular forms



Quantum gravity



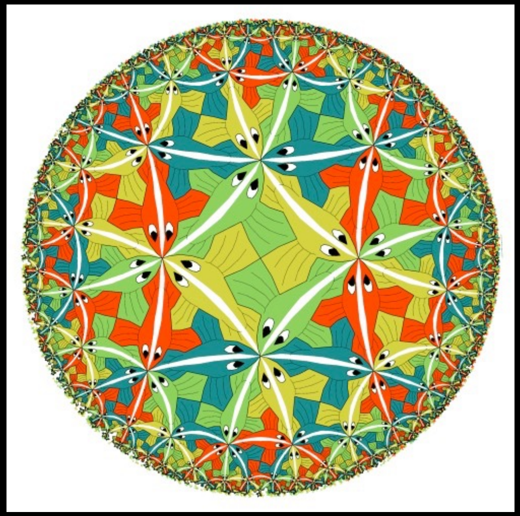
Number theory

1,2,3,4,5,….

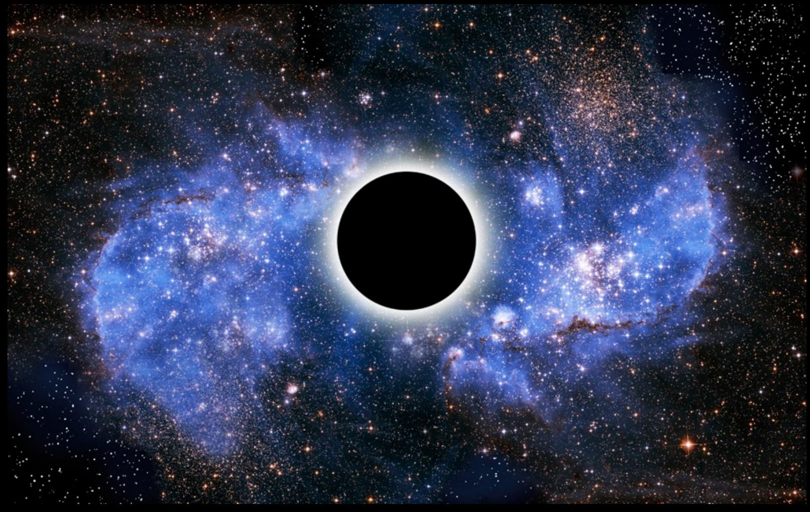
Symmetries

?

Modular forms



Quantum gravity

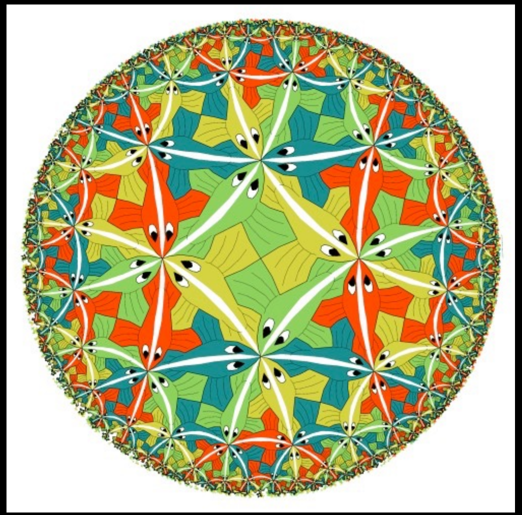


Number theory

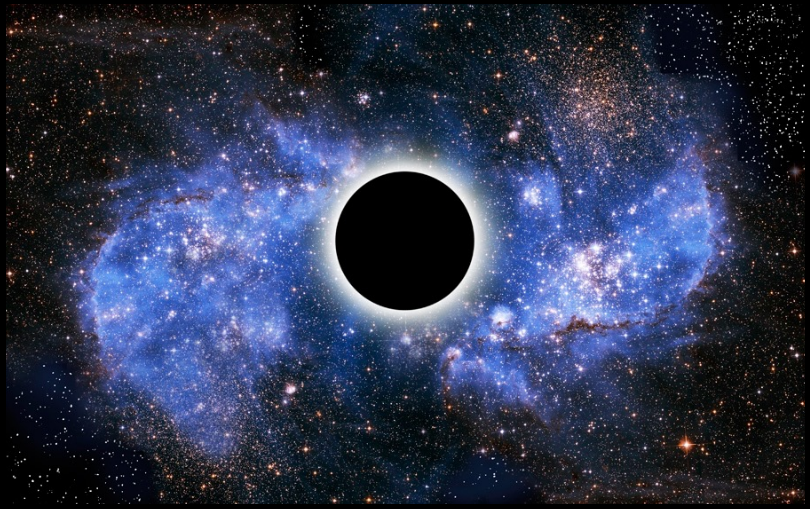
1,2,3,4,5,….

Symmetries

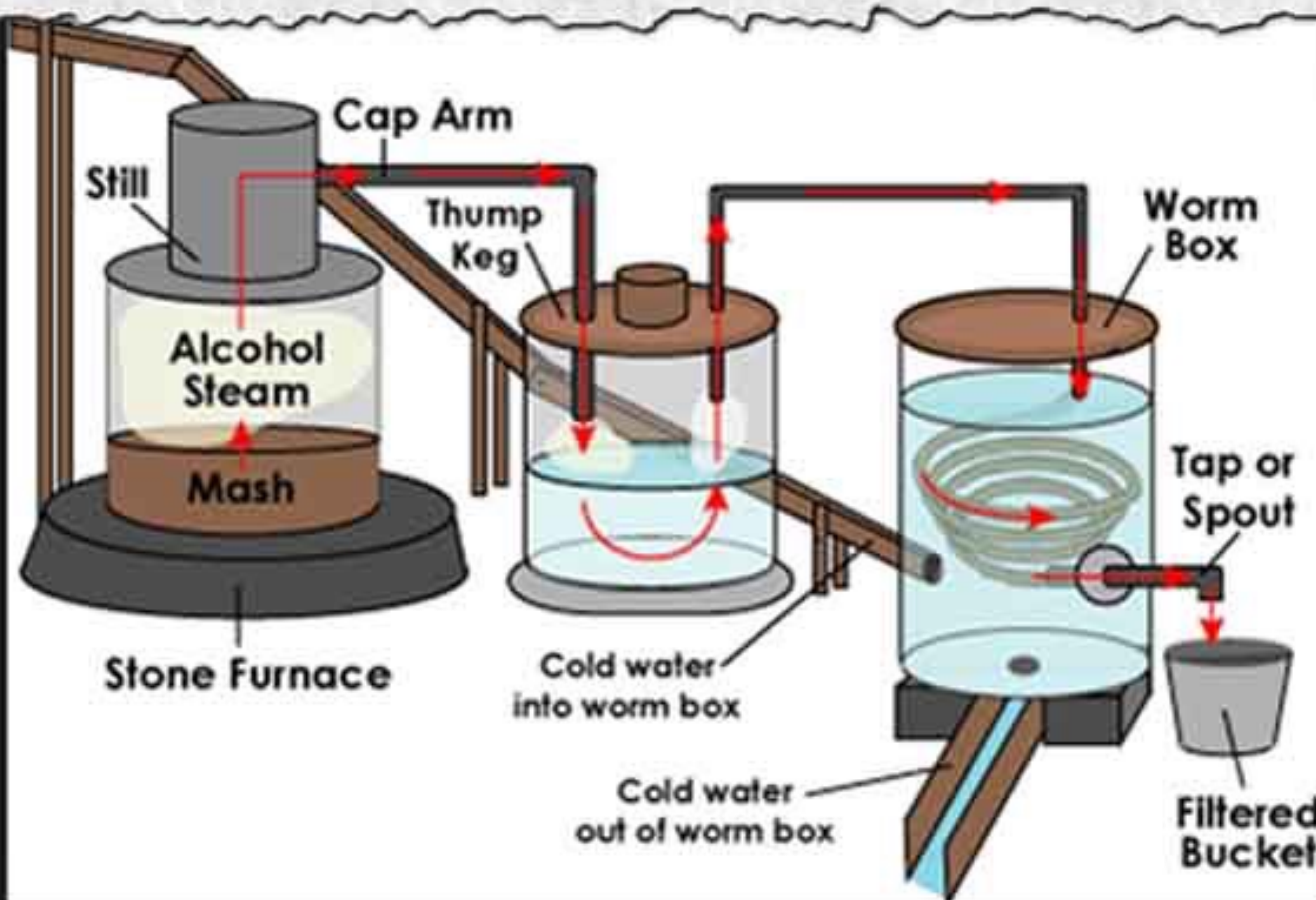
Modular forms

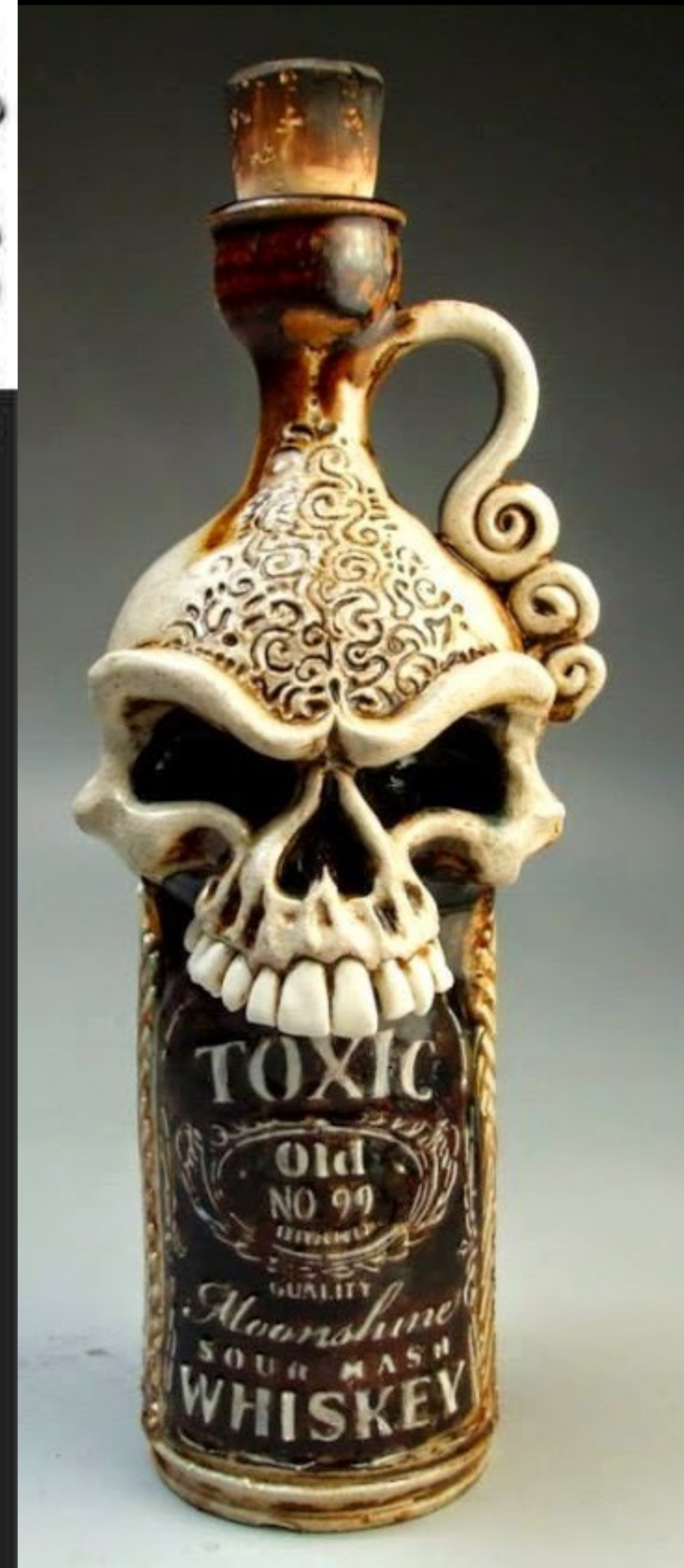


Quantum gravity



Step By Step Guide To Making Moonshine





Snapshots at jasonlove.com