Transportation Research Part C 117 (2020) 102676

Contents lists available at ScienceDirect

Transportation Research Part C

journal homepage: www.elsevier.com/locate/trc

Parking sharing problem with spatially distributed parking supplies

Fangni Zhang[a], Wei Liu[b,c,*], Xiaolei Wang[d], Hai Yang[e]

[a] Department of Industrial and Manufacturing Systems Engineering, University of Hong Kong, Hong Kong, PR China
[b] School of Computer Science and Engineering, University of New South Wales, Sydney, NSW 2052, Australia
[c] Research Centre for Integrated Transport Innovation, School of Civil and Environmental Engineering, University of New South Wales, Sydney, NSW 2052, Australia
[d] School of Economics and Management, Tongji University, Shanghai, PR China
[e] Department of Civil and Environmental Engineering, The Hong Kong University of Science and Technology, Hong Kong, PR China

ARTICLE INFO

Keywords:
Parking sharing
Parking location
Parking choice equilibrium
Pricing strategy

ABSTRACT

This study models and manages the parking sharing problem in urban cities, where private parking owners can share their vacant spaces to parking users via a parking sharing platform. The proposed model takes into account the spatial dimension of parking, where clusters of curbside spaces and private shareable ones are distributed over different locations. On the supply side, private parking owners can "sell the right-of-use" of their spaces to the platform based on the rent they can receive and the inconvenience they would experience due to sharing. On the demand side, travelers make their parking choices of space type (curbside or shared) and location under given parking capacities and prices. The resulting parking choice equilibrium is formulated as a minimization problem and several properties of the equilibrium are identified and discussed. The platform operator's pricing strategy, i.e., rent paid to space owners and price charged on space users, can significantly affect the private parking owners' sharing decisions and the choice equilibrium of parking users. In this context, we examine the platform operator's optimal pricing strategies for revenue-maximization or social-cost-minimization. Numerical examples are also presented to illustrate the models and results and to provide further insights.

1. Introduction

Parking is a long-lasting problem in many cities. It has been documented both analytically and empirically that cruising for parking can create significant congestion and produce additional vehicular emission (Shoup, 2006; Arnott and Inci, 2006; Ayala et al., 2011; Van Ommeren et al., 2012; Liu and Geroliminis, 2016). Given the limited land supply in cities and the increasingly growing car ownership, the parking issue is particularly severe in cities where it is infeasible to build new parking facilities. Various parking management strategies have been proposed in the literature to improve parking and traffic efficiency such as parking pricing (Arnott et al., 1991; Zhang et al., 2005; Zhang et al., 2008; Qian et al., 2012; Qian and Rajagopal, 2014; Arnott et al., 2015; Liu, 2018; Zhang et al., 2019), parking reservation (Yang et al., 2013; Liu et al., 2014a; Chen et al., 2015; Chen et al., 2016), parking permit (Zhang et al., 2011; Liu et al., 2014b; Wang et al., 2020), and park-and-ride facilities (Wang et al., 2004; Liu et al., 2009; Liu and Geroliminis, 2017). A number of studies examined the interactions between parking supply and ride-sourcing/ride-sharing (Xiao et al., 2016; Xu et al., 2017; Su and Wang, 2019). For a recent review of economic studies of parking, one may refer to Inci (2015). Some recent studies also developed network models to optimize parking spot allocation or to incorporate the parking search process

* Corresponding author at: School of Computer Science and Engineering, University of New South Wales, Sydney, NSW 2052, Australia.
E-mail address: Wei.Liu@unsw.edu.au (W. Liu).

https://doi.org/10.1016/j.trc.2020.102676
Received 5 February 2019; Received in revised form 31 March 2020; Accepted 4 April 2020
Available online 02 July 2020
0968-090X/

as a part of the traffic equilibrium (Boyles et al., 2015; He et al., 2015; Liu et al., 2016; Zheng and Geroliminis, 2016; Du et al., 2019; Pi et al., 2019).

Recently, shared parking emerges as a new notion of making more efficient use of existing parking facilities. It uses existing gaps intended for parking cars when the owner is not using it. It is frequently the case that most parking spaces are only used part-time by a driver or owner who lives in one location and works in another. The utilization and availability patterns of parking generally follow predictable patterns. Given the fast development of information and communications technology, vacant private parking spaces now can be shared through online parking platforms, which is similar to Airbnb for house sharing and Didi Shunfengche or Uber Pool for ride-sharing. Parking sharing not only has the potential to help alleviate the shortage of parking spaces and reduce cruising for parking, but also can provide the private parking owners additional benefit through sharing their idle parking spaces. In this context, Guo et al. (2016) developed a simulation–optimization based decision method to delineate a parking operator's decision who re-purchases some private parking spots from private owners and sell them to public users during certain time of a day. Shao et al. (2016) further considered advanced booking and allocation of shared parking spots and optimized the allocation of parking requests to specific parking spots. Xu et al. (2016) addressed the money flow management problem underlying the shared parking matching by using the market design theory. More recently, Xiao et al. (2018) proposed truthful double auction mechanisms for shared parking management.

This study models the parking sharing problem in urban cities with a hybrid supply of curbside parking and shared parking, where the parking spaces are distributed around the city center. Three groups of decision-makers are modeled, i.e., travelers who need a parking space; residents or other parking owners who can supply a private parking space; and the operator who manages the parking sharing platform. Following Guo et al. (2016), we consider a parking sharing platform operator as a "reseller", who pays rent to private parking sharers to repurchase vacant parking spaces and charges parking fees on parking users (travelers). Given the parking rent paid to private parking owners, a shared parking supply equilibrium is achieved, at which no parking owners can improve his or her benefit by changing his or her sharing decisions. Given the curbside and shared parking supply (prices and capacities), the users' parking choice equilibrium is formulated as a minimization problem, at which no user can reduce his or her cost by unilaterally changing his or her parking choice. Several properties of this choice equilibrium have then been discussed. Furthermore, the pricing strategies of the parking sharing platform operator to maximize net revenue or reduce total social cost have been examined, which govern the parking choice equilibrium of users and the shared parking supply equilibrium.[1]

The rest of the paper is organized as follows. Section 2 introduces the parking sharing problem and the model setting. Section 3 formulates and analyzes parking choice equilibrium of travelers under given public curbside parking and private shared parking supplies. Section 4 examines the pricing strategies of the parking platform operator to achieve different objectives. Section 5 numerically illustrates the models and analysis. Finally, Section 6 concludes the paper and provides further discussions.

2. Problem description

This section introduces the commuting problem with an online platform for parking sharing. In the following, we describe and formulate the behaviors of the three groups of decision-makers that are involved: road users who need a parking space; private parking sharers who hold the potential to offer parking spaces; and the operator of the platform for parking sharing.

Every day there is a total number of d users traveling from home to city center. Each of them needs a parking space around the city center. Parking spaces are distributed at location $k \in \{1, 2, ..., K\}$ (or parking cluster k), where K is the total number of locations with parking supplies. The setting is illustrated in Fig. 1. The driving time from home to parking location k is t_k and the walking time from location k to work is w_k, which are both location-dependent. These parameters are assumed fixed and exogenously given in this study.[2]

Public curbside spaces and private shared parking spaces are distributed over the K locations. Travelers can choose either curbside spaces or shared spaces. We introduce subscript 'a' to represent the curbside parking option and subscript 'b' for shared parking. For location k, the number of curbside parking spaces is $n_{a,k} > 0$, and the curbside parking price is $\tau_{a,k}$. Travelers need to spend a cruising time of $t_{a,k}$ to find a vacant curbside parking space at location k if they choose curbside parking.

Besides the curbside parking spaces, we consider that there are m_k residents (or other land operators or owners) living at location k, who are the private parking owners and also the potential private parking suppliers or sharers. Note that m_k can be zero, indicating no potential parking sharers at location k. If appropriate rents are paid to these parking owners, they can offer their private spaces (which is vacant as, e.g., they may work elsewhere and leave their parking spots empty during daytime). We further denote the realized number of shared parking spaces at location k by $n_{b,k}$, where $n_{b,k} \leqslant m_k$ always holds. The realized number $n_{b,k}$ is depending

[1] The parking sharing problem exhibits similarities with the ride-sharing/sourcing problem. However, it has received much less attention than the ride-sharing/sourcing problem that studied by, e.g., Wang et al. (2016), Zha et al. (2016), Chen et al. (2017), Liu and Li (2017), Zhang et al. (2017), Wang et al. (2018).

[2] This study does not explicitly model the flow-dependent road congestion. This treatment allows us to focus on parking choices around city center and the specific externality of cruising-for-parking on other parking users, which has been adopted in existing parking studies, e.g., He et al. (2015). The driving time between home and a specific location near the city center is taken as a constant. It should be noted that the value of this constant can be affected by car travel demand and thus can reflect the road traffic conditions. Since road congestion is not explicitly modeled here, the negative impacts of cruising-for-parking on passing-by traffic cannot be captured (refer to Liu and Geroliminis, 2016). This means that the current study may underestimate the externality of cruising-for-parking. When interpreting the results obtained in relation to the externality of cruising-for-parking, one should keep this in mind while it does not affect the central idea in the paper.

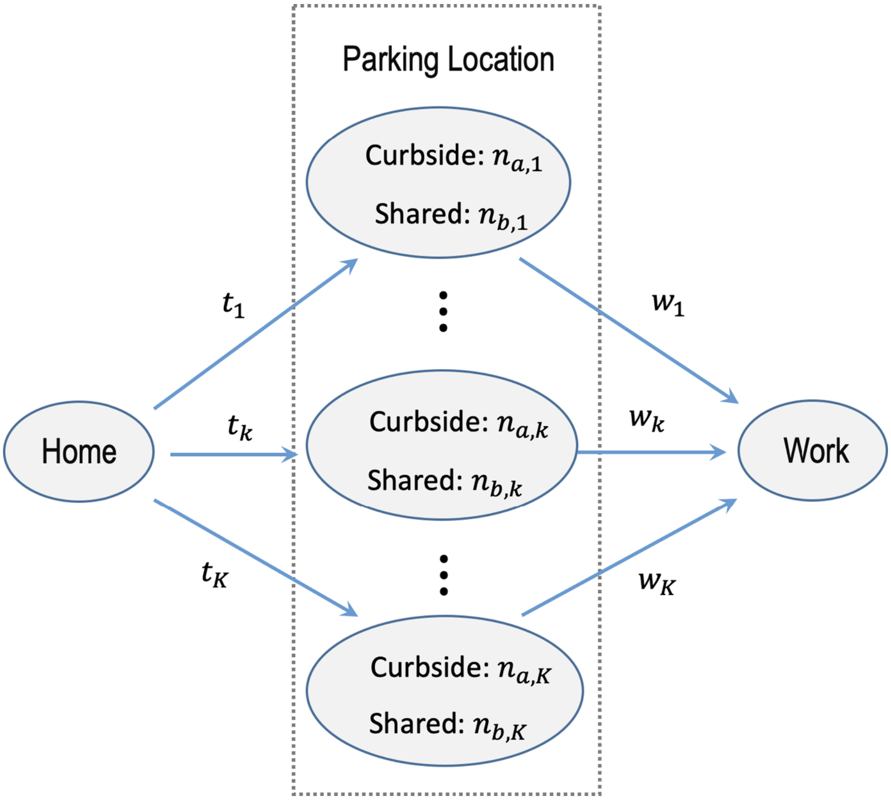

Fig. 1. Distributed parking supplies.

on the rent, which will be formulated in Section 2.2. Moreover, travelers have to spend a driving time of $t_{b,k}$ to access the shared parking at location k, where $k \in \{1, 2, ..., K\}$.

Below we summarize the main assumptions throughout this paper before proceeding to the detailed model formulations.

Assumption 1 (*Sufficient Parking*). It is assumed that $\sum_{k=1}^{K} n_{a,k} > d$.

Assumption 1 means that we always have sufficient parking supply for the d travelers even if there is no shared parking, which is to guarantee that the parking choice problem always has at least one feasible solution. In practice, in the long run, the number of travelers that drive to work will be bounded by the total parking supply as travelers can shift to other options (e.g., public transit, shared-ride) when parking is costly (either costly parking searching time or parking price). This might be modeled by introducing a third travel alternative or elastic parking demand, which is further discussed in Section 6.2.

Assumption 2 (*Complete Information*). Travelers have complete information regarding driving time, walking time, parking prices, expected curbside parking cruising time, and shared parking access time.

This paper examines recurrent parking choices where travelers know expected cruising time for each parking location from information provision and/or long-term experience. Travelers make rational parking choices based on the information mentioned in the above and minimize their own travel costs.

2.1. Travelers

Travelers compete against each other (in relation to parking) to minimize their own costs, which leads to a parking choice equilibrium state and no one can further reduce his or her cost by unilaterally changing his or her parking choice. The decision of the travelers entails two dimensions. In terms of parking type, each traveler chooses between (i) a curbside parking which incurs searching cost and a curbside parking price; and (ii) a shared and reserved/guaranteed parking space for a shared parking price and an access time. The other dimension is the parking location choice, which is mainly governed by the trade-off among driving time, walking time, and parking price.

When choosing curbside parking at location k, the traveler will drive to location k, search for a parking space there and then walk to the center (final destination). His or her individual travel cost is

$$C_{a,k} = \alpha \cdot (t_k + t_{a,k}) + c(w_k) + \tau_{a,k}. \tag{1}$$

where t_k is the driving time from home to parking location k, $t_{a,k}$ is the cruising-for-parking time to find a vacant space at location k, α is the value of driving time, $c(w_k)$ is the walking time cost, and $\tau_{a,k}$ is the curbside parking price at parking location k. It is assumed that the walking cost $c(\cdot)$ is strictly increasing and convex with respect to walking time. The above walking cost formulation reflects the nonlinearity when converting the walking time into a driving time equivalent.

Now we turn to formulate the cruising time for parking. For the curbside parking at location k, the parking occupancy rate experienced by travelers is

$$q_{a,k} = \frac{f_{a,k}}{n_{a,k}}, \tag{2}$$

where $f_{a,k}$ is the number of drivers choosing to park at the curbside parking spaces at location k and $n_{a,k}$ is the curbside parking capacity. The searching time at location k can be estimated as a function of parking occupancy rate at location k, i.e., $q_{a,k}$, which is

$$t_{a,k} = h(q_{a,k}), \tag{3}$$

where the searching time function $h(\cdot)$ is assumed to be strictly increasing and convex (Anderson and De Palma, 2004; Qian and Rajagopal, 2015; Arnott and Williams, 2017; Leclercq et al., 2017). Furthermore, we assume that $h(q_{a,k}) = +\infty$ when $q_{a,k} \geqslant 1$ and $h(q_{a,k}) < +\infty$ when $q_{a,k} < 1$.

The cost of choosing a curbside parking space at location k can be rewritten as a function of the numbers of users choosing this option, i.e.,

$$C_{a,k} = C_{a,k}(f_{a,k}) = \alpha\cdot\left(t_k + h\left(\frac{f_{a,k}}{n_{a,k}}\right)\right) + c(w_k) + \tau_{a,k}. \tag{4}$$

If a traveler chooses the shared parking at location k, he or she will also drive to location k, drive to (i.e., access time) and park at the shared (and reserved) parking space without searching, and pay a shared parking price. The travel cost for him or her then includes driving time, shared parking access time, walking time and parking price for the shared parking space, which is

$$C_{b,k} = \alpha\cdot(t_k + t_{b,k}) + c(w_k) + \tau_{b,k}, \tag{5}$$

where $\tau_{b,k}$ is the price paid to the platform operator for using the shared space at location k, $t_{b,k}$ is the access time, and other terms are defined before. In the above cost formulation, we consider additional driving time to access the shared parking. Similarly, one may also incorporate additional walking time in relation to the shared parking, which is omitted here for simplicity. It should be noted that the attractiveness of the shared parking option against the curbside parking also depends on the access time $t_{b,k}$ against the cruising for parking time $t_{a,k}$. If all shared parking are very difficult to access, i.e., $t_{b,k}$ can be extremely large, no one may choose shared parking.

Travelers will have to make choices regarding both parking types (curbside or shared) and parking locations to minimize their travel cost. Denote the numbers of travelers choosing the curbside parking and shared parking by d_a and d_b, respectively. We then have $d_a + d_b = d$. Furthermore, let the users choosing curbside space at location k be $f_{a,k}$, and those choosing shared space at location k be $f_{b,k}$. We have $\sum_{k=1}^{K} f_{i,k} = d_i$ for $i = \{a, b\}$. The total travel cost of all users can then be written as

$$TC = \sum_k \sum_i [C_{i,k}\cdot f_{i,k}]. \tag{6}$$

2.2. Parking owners (or sharers)

As mentioned earlier, at location k, there are m_k residents that may share a space. To share their spaces to others, they will encounter an inconvenience cost of δ. Parking owners may have different inconvenience costs. In particular, for parking owners at location k, we assume that δ is distributed over $[0, \delta_k]$ with a probability distribution function of $g_k(\delta)$ and a cumulative distribution function of $G_k(\delta)$. This treatment accounts for heterogeneity in individual parking suppliers (even if they are associated with the same location k). Moreover, a subscript k is added to the probability distribution function and the cumulative distribution function to indicate that they can also be location-dependent, i.e., owners from different locations are different as well. δ_k can approach infinity, which means that some private parking space owners might never share their spaces. To allow analytical tractability, we consider that $G_k(\delta)$ is strictly increasing and differentiable when $\delta \in [0, \delta_k]$.

If a rent (from the parking sharing platform operator) r_k is paid to the potential parking sharers, those with an inconvenience cost of $\delta \leqslant r_k$ will share or "sell the right-of-use" of their parking spaces to the platform operator. The parking sharer will earn a net benefit of $r_k - \delta \geqslant 0$. Those with $\delta > r_k$ will not share their spaces. This is termed as the "repurchase" strategy from the platform operator's point of view. The total number of realized shared parking spaces under a given rent r_k is then

$$n_{b,k} = G_k(r_k)\cdot m_k. \tag{7}$$

It is obvious that $n_{b,k} \leqslant m_k$ and the equality holds when $r_k = \delta_k$.

The total net benefit of all parking owners (sharers) can then be determined under a given r_k, which is equal to the difference between the total parking rent and the total inconvenience cost incurred, i.e.,

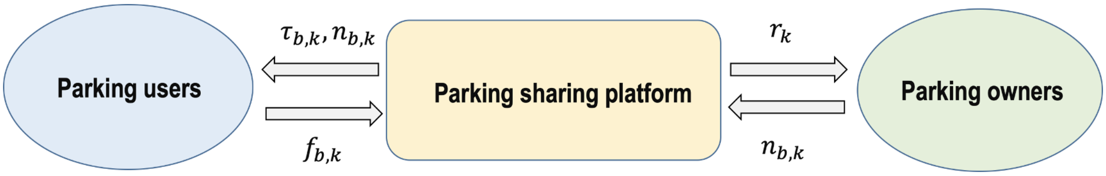

Fig. 2. The parking sharing platform and its interactions with parking users and owners.

$$\pi(\boldsymbol{r}) = \sum_{k=1}^{K} \left[n_{b,k} \cdot r_k - \int_0^{r_k} \delta \cdot g_k(\delta) \cdot m_k \cdot d\delta \right] \tag{8}$$

where $n_{b,k}$, as given in Eq. (7), will depend on r_k. It is obvious that the net benefit of parking owners π increases with r_k or at least does not decrease.

2.3. Parking sharing platform operator

For the parking sharing platform operator, as mentioned earlier, it charges the travelers (parking users) a fee of $\tau_{b,k}$ for a shared parking at location k, and compensates a private parking sharer at location k a rent of r_k. The interactions between the operator and the users or owners are briefly depicted in Fig. 2. As can be seen, the pricing strategies of the platform operator, i.e., the combinations of $\tau_{b,k}$ and r_k would significantly affect the demand and supply for the parking sharing platform, as well as its performance. This highlights the importance to explore the pricing strategies of the platform operator.

Given the curbside parking price $\tau_{a,k}$ (which might be set by the local government), the parking sharing platform operator can adjust its pricing strategy, i.e., $\tau_{b,k}$ and r_k, where $k \in \{1, 2, ..., K\}$, to maximize its net revenue subject to the users' parking choice equilibrium and the parking sharers' supply equilibrium. The net revenue is defined as the shared parking fee collected minus the shared parking rent paid, and then minus the operating cost of the platform, which is given as follows

$$\Pi\left(\tau_b, \boldsymbol{r}\right) = \sum_{k=1}^{K} [f_{b,k} \cdot \tau_{b,k}] - \sum_{k=1}^{K} [n_{b,k} \cdot r_k] - \phi(d_b), \tag{9}$$

where τ_b and $\boldsymbol{r}$ are the vectors for the shared parking prices and rents at different locations, and $\phi(\cdot)$ is the operating cost associated with the total number of shared parking users. It is assumed that $\phi(\cdot)$ is increasing and convex and $\phi(0) > 0$ (there is a fixed cost for operating the shared parking platform). It is noteworthy that at the users' parking choice equilibrium and the parking sharers' supply equilibrium, d_b depends on $f_{b,k}$ and $f_{b,k}$ further depends on $\tau_{b,k}$;$n_{b,k}$ depends on r_k; and furthermore $f_{b,k}$ also depends on $n_{b,k}$.

We also discusses a social-cost-minimizing parking sharing platform operator, which is to minimize the total social cost given as follows:

$$TSC = TC - \pi - \Pi - \Pi', \tag{10}$$

where TC is the total travel cost of parking users defined in Eq. (6), and π is the total net benefit of parking sharers defined in Eq. (8), and Π is the total net revenue of the platform operator defined in Eq. (9), and Π' is the total parking fee collected through the curbside parking side given by:

$$\Pi' = \sum_{k=1}^{K} [f_{a,k} \cdot \tau_{a,k}]. \tag{11}$$

The total social cost takes into account the parking users, parking sharers, and the platform operator, while excluding the money transfers among the three types of decision-makers.

In Section 4, we will discuss a public operator to reduce the total social cost in Eq. (10) and compare its performance with a private operator to maximize the net platform revenue defined in Eq. (9). Also, we will discuss the platform's pricing strategies under different curbside parking prices. Given the pricing strategy of the platform operator, the parking choice equilibrium of travelers is formulated and discussed in Section 3.

3. Parking choice equilibrium

This section formulates the parking choice equilibrium and examines the equilibrium characteristics. At the parking choice equilibrium state, no one can further reduce his or her cost by unilaterally changing his or her parking choice. A bi-section based algorithm for solving the choice equilibrium problem is also presented.

3.1. Formulation and properties

This section formulates the parking choice problem of users given the pricing strategy $(\tau_b, \boldsymbol{r})$ of the platform operator and the curbside parking capacities and prices. In particular, the parking choice equilibrium can be obtained by solving the following optimization problem

$$\min :Z\left(\boldsymbol{f}_{\boldsymbol{a}},\boldsymbol{f}_{\boldsymbol{b}}\right)=\sum_{k=1}^{K}\left[\int_{0}^{f_{a,k}}C_{a,k}(u)du+C_{b,k}\cdot f_{b,k}\right] \tag{12}$$

subject to

$$\sum_{k=1}^{K}\sum_{i\in\{a,b\}}f_{i,k}=d \tag{13a}$$

$$f_{b,k}\leqslant n_{b,k},\ \forall\ k\in\{1,2,\dots,K\} \tag{13b}$$

$$f_{i,k}\geqslant 0,\ \forall\ k\in\{1,2,\dots,K\},\ \forall\ i\in\{a,b\}, \tag{13c}$$

where $\boldsymbol{f}_{\boldsymbol{a}}$ and $\boldsymbol{f}_{\boldsymbol{b}}$ denote the two parking flow vectors $\{f_{a,k},\ k\in\{1,2,\dots,K\}\}$ and $\{f_{b,k},\ k\in\{1,2,\dots,K\}\}$, respectively. The optimization problem defined by Eqs. (12)-(13) follows the well-known Beckmann's formulation (Beckmann et al., 1956), with shared parking capacity constraints added, which is similar to the well-studied traffic assignment problems with road capacity constraints (e.g., Yang and Bell, 1997; Tong and Wong, 2000; Nie et al., 2004; Ryu et al., 2014). Note that it is not necessary to explicitly include the curbside parking space constraints $f_{a,k}\leqslant n_{a,k}$ into Eq. (13). This is explained below. When $f_{a,k}\geqslant n_{a,k}$, the parking occupancy rate $q_{a,k}\geqslant 1$, and the searching time for parking $h\to+\infty$. The travel cost $C_{a,k}\to+\infty$ according to Eq. (4). This will never occur at the equilibrium since we have Assumption 1 to guarantee the feasibility of the parking choice problem, where we can always find a flow pattern such that $C_{a,k}\ll+\infty$ for any k. In short, the parking capacity constraints for curbside spaces will never be binding at the equilibrium solution and the inequalities $f_{a,k}<n_{a,k}$ will be automatically satisfied after solving the problem in Eq. (12).

The feasible parking flow set defined by the conditions in Eq. (13) is non-empty, compact and convex. Furthermore, the objective function in Eq. (12) is continuous. The existence of solution(s) to the above problem is always guaranteed as per Weierstrass' theorem (Bazaraa et al., 1993). Furthermore, we denote the solution of the minimization problem in Eq. (12) by $(\boldsymbol{f}_{\boldsymbol{a}}^{*},\boldsymbol{f}_{\boldsymbol{b}}^{*})$ (the equilibrium flow solution).

The above problem is a parking traffic assignment problem with parking capacity constraints. We can write down the Lagrangian with respect to Eq. (12) as follows:

$$L\left(\boldsymbol{f}_{\boldsymbol{a}},\boldsymbol{f}_{\boldsymbol{b}},\eta,\lambda\right)=Z\left(\boldsymbol{f}_{\boldsymbol{a}},\boldsymbol{f}_{\boldsymbol{b}}\right)+\eta\cdot\left[d-\sum_{k=1}^{K}\sum_{i\in\{a,b\}}f_{i,k}\right]+\sum_{k=1}^{K}\lambda_k\cdot(f_{b,k}-n_{b,k}), \tag{14}$$

where η is the multiplier associated with the flow conservation constraint in Eq. (13a), and λ_k is the multiplier associated with the shared parking capacity constraint in Eq. (13b). The first-order optimality conditions can be derived as follows:

$$\frac{\partial L}{\partial f_{i,k}}\geqslant 0,\ f_{i,k}\geqslant 0,\ \frac{\partial L}{\partial f_{i,k}}\cdot f_{i,k}=0,\ \forall\ i,\ k \tag{15a}$$

$$\frac{\partial L}{\partial \lambda_k}\leqslant 0,\ \lambda_k\geqslant 0,\ \frac{\partial L}{\partial \lambda_k}\cdot\lambda_k=0,\ \forall\ k \tag{15b}$$

$$\frac{\partial L}{\partial \eta}=0 \tag{15c}$$

or in their detailed form as follows:

$$f_{a,k}\cdot(C_{a,k}-\eta)=0,\ \forall\ k \tag{16a}$$

$$C_{a,k}-\eta\geqslant 0,\ \forall\ k \tag{16b}$$

$$f_{b,k}\cdot(C_{b,k}-\eta+\lambda_k)=0,\ \forall\ k \tag{16c}$$

$$C_{b,k}-\eta+\lambda_k\geqslant 0,\ \forall\ k \tag{16d}$$

$$\lambda_k\cdot(n_{b,k}-f_{b,k})=0,\ \forall\ k \tag{16e}$$

$$n_{b,k}-f_{b,k}\geqslant 0,\ \forall\ k \tag{16f}$$

$$f_{i,k}\geqslant 0,\ \forall\ i,\ k \tag{16g}$$

$$\lambda_k\geqslant 0,\ \forall\ k \tag{16h}$$

$$\sum_{k=1}^{K}\sum_{i\in\{a,b\}}f_{i,k}=d \tag{16i}$$

Since the constraints in Eq. (13) are linear (satisfy a constraint qualification), the first-order optimality conditions in Eq. (16) are necessary. Furthermore, since the original objective function $Z(\boldsymbol{f}_a, \boldsymbol{f}_b)$ in Eq. (12) is convex, the first-order conditions are also sufficient. Similar Karush–Kuhn–Tucker (KKT) conditions can be found in Hearn (1980), Bazaraa et al. (1993), Larsson and Patriksson (1995), Nie et al. (2004), Wu et al. (2020).

Furthermore, we can define a generalized cost for those using shared parking as follows:

$$\overline{C}_{b,k} = C_{b,k} + \lambda_k \tag{17}$$

The corresponding conditions in Eq. (16) can be simplified as

$$f_{b,k}\cdot(\overline{C}_{b,k} - \eta) = 0, \ \forall\ k \tag{18a}$$

$$\overline{C}_{b,k} - \eta \geqslant 0, \ \forall\ k \tag{18b}$$

The physical meanings of the generalized costs for shared parking is as follows. $\overline{C}_{b,k}$ includes the direct and observable cost $C_{b,k}$ associated with the shared parking choice at location k, while λ_k reflects the additional indirect cost experienced by the users to ensure this chosen option (e.g., by booking this shared space in advance in the sharing platform). The advance booking cost has been discussed in Xu et al. (2018) in the context of high-speed railway network. While this is not the focus of the paper, a future study may examine the reservation schemes for parking sharing (Shao et al., 2016). Conditions in Eq. (16) lead to the following conditions:

$$\lambda_k > 0 \Rightarrow n_{b,k} - f_{b,k} = 0, \ \forall\ k \tag{19a}$$

$$\lambda_k = 0 \Rightarrow n_{b,k} - f_{b,k} \geqslant 0, \ \forall\ k \tag{19b}$$

With the generalized cost formulation for shared-parking options $\overline{C}_{b,k}$, Wardrop's first principle is then satisfied.

We now turn to the solution uniqueness issue of the optimization problem in Eq. (12). It is noted that the objective function in Eq. (12) is convex with respect to $f_{a,k}$ and $f_{b,k}$, but not strictly convex as $C_{b,k}$ is constant for potentially different values of $f_{b,k}$. This means that $(\boldsymbol{f}_a^*, \boldsymbol{f}_b^*)$ may be non-unique. We further explore the uniqueness/non-uniqueness of parking choice equilibrium solution in the following.

Proposition 3.1. *At the parking choice equilibrium, the total numbers of travelers choosing curbside and shared parking, i.e., d_a^* and d_b^*, are both unique.*

Proof. As discussed earlier, the solution $(\boldsymbol{f}_a^*, \boldsymbol{f}_b^*)$ may not be unique; otherwise, d_a^* and d_b^* are unique. Suppose there exists two different parking choice equilibrium solutions $(\boldsymbol{f}'_a, \boldsymbol{f}'_b)$ and $(\boldsymbol{f}''_a, \boldsymbol{f}''_b)$, where $d'_a = \sum_k f'_{a,k} \neq d''_a = \sum_k f''_{a,k}$, and accordingly $d'_b \neq d''_b$. Without loss of generality, we let $d'_a > d''_a$ and thus $d'_b < d''_b$. Note that superscripts are also added to other relevant notations to distinguish the two different solutions.

Since $d'_a > d''_a$, at least for some location k_1, we have $f'_{a,k_1} > f''_{a,k_1} \geqslant 0$. As C_{a,k_1} is strictly increasing with respect to f_{a,k_1}, then $\eta' = C'_{a,k_1} > C''_{a,k_1} \geqslant \eta''$ (and therefore $\eta' > \eta''$), where η' and η'' are the minimum travel costs defined by Eq. (16) under the two different solutions.

Since $d'_b < d''_b$, at least for some location k_2, we have $0 \leqslant f'_{b,k_2} < f''_{b,k_2} \leqslant n_{b,k_2}$. Based on Eq. (16) and the derived result $\eta' > \eta''$ in the above, we then have $C'_{b,k_2} + \lambda'_{k_2} \geqslant \eta' > \eta'' = C''_{b,k_2} + \lambda''_{k_2}$. Based on Eq. (5), it can be readily verified that $C'_{b,k_2} = C''_{b,k_2} = C_{b,k_2}$ (shared parking cost is constant). Therefore, we have $\lambda'_{k_2} > \lambda''_{k_2}$. Furthermore, $f'_{b,k_2} < f''_{b,k_2} \leqslant n_{b,k_2}$ (as assumed in the above) indicates that $\lambda'_{k_2} = 0$. We then can derive $0 = \lambda'_{k_2} > \lambda''_{k_2} \geqslant 0$, which is a contradiction. This completes the proof. □

Proposition 3.2. *The equilibrium number of travelers choosing curbside parking at any location $k \in \{1, 2, ..., K\}$ i.e., $f_{a,k}^*$, must be unique.*

Proof. Suppose the curbside parking flow solution to the problem defined in Eq. (12) is not unique, we can then find at least two solutions $\boldsymbol{f}'_a \neq \boldsymbol{f}''_a$. For at least one location k_1, we have $f'_{a,k_1} \neq f''_{a,k_1}$. Without loss of generality, let $f'_{a,k_1} > f''_{a,k_1} \geqslant 0$. Based on Proposition 3.1, we then must have $0 \leqslant f'_{a,k_2} < f''_{a,k_2}$ for at least one other location k_2.

Since $f'_{a,k_1} > f''_{a,k_1} \geqslant 0$, based on Eq. (16), we have $\eta' = C'_{a,k_1} > C''_{a,k_1} \geqslant \eta''$, where η' and η'' are the minimum travel costs defined in Eq. (16) under the two different choice equilibrium solutions. Similarly, based on $0 \leqslant f'_{a,k_2} < f''_{a,k_2}$, we have $\eta' \leqslant C'_{a,k_2} < C''_{a,k_2} = \eta''$. These two results conflict with each other. This completes the proof. □

Proposition 3.1 and Proposition 3.2 together say that even though the parking choice equilibrium solution $(\boldsymbol{f}_a^*, \boldsymbol{f}_b^*)$ to the optimization problem in Eq. (12) might not be unique, the aggregate split between curbside and shared parking usages is unique at the choice equilibrium, and furthermore, the parking flow pattern at the curbside parking side is unique at the choice equilibrium. This further means that the non-uniqueness only involves the shared parking flow $\boldsymbol{f}_b^*$.

We further introduce the following Assumption 3, which ensures the uniqueness of the shared parking flow solution $\boldsymbol{f}_b^*$ and the overall parking choice equilibrium.

Assumption 3. For two different locations $k_1, k_2 \in \{1, 2, ..., K\}$, (i) we should have $w_{k_1} \neq w_{k_2}$, and (ii) if $C_{b,k_1} = C_{b,k_2}$ and $w_{k_1} < w_{k_2}$, the shared parking at location k_1 is preferred by travelers.

Part (i) of Assumption 3 says that for parking clusters at two different locations, their walking times to the workplace are different. Part (ii) of Assumption 3 says that when costs associated with choosing shared parking at two parking locations are identical, the shared parking with a smaller walking time is preferred.

Proposition 3.3. *Under Assumption 3, the curbside and shared parking flows at the parking choice equilibrium is unique.*

Proof. From Proposition 3.1 and Proposition 3.2, we have unique $f^*_{a,k}$, d^*_a and d^*_b at equilibrium. We only need to show that the equilibrium $f^*_{b,k}$ is unique for any $k \in \{1, 2, ..., K\}$.

Suppose we have two different solutions $\boldsymbol{f}'_{\boldsymbol{b}} \neq \boldsymbol{f}''_{\boldsymbol{b}}$. Since $\boldsymbol{f}'_{\boldsymbol{b}}$ and $\boldsymbol{f}''_{\boldsymbol{b}}$ both minimize the objective function in Eq. (12) and we have unique $f^*_{a,k}$, d^*_a and d^*_b at equilibrium, we should have $\sum_k C_{b,k}f'_{b,k} = \sum_k C_{b,k}f''_{b,k}$, which is the minimum. If this equality does not hold, one of $\boldsymbol{f}'_{\boldsymbol{b}}$ and $\boldsymbol{f}''_{\boldsymbol{b}}$ cannot be the equilibrium solution.

Since $\boldsymbol{f}'_{\boldsymbol{b}} \neq \boldsymbol{f}''_{\boldsymbol{b}}$, for at least one location k_1, we have $f'_{b,k_1} \neq f''_{b,k_1}$. Without loss of generality, let $n_{b,k_1} \geqslant f'_{b,k_1} > f''_{b,k_1} \geqslant 0$. Since the total numbers of shared parking users are identical under the two solutions, we must have at least one location k_2 that $0 \leqslant f'_{b,k_2} < f''_{b,k_2} \leqslant n_{b,k_2}$.

We have two cases to consider, i.e., $C_{b,k_1} \neq C_{b,k_2}$ and $C_{b,k_1} = C_{b,k_2}$. For the first case, without loss of generality, we consider $C_{b,k_1} > C_{b,k_2}$. We then can construct a new solution $\boldsymbol{f}'''_{\boldsymbol{b}}$, where for $k \neq k_1$ and $k \neq k_2$, we let $f'''_{b,k} = f'_{b,k}$, and moreover, we let $f'''_{b,k_1} = \max\{0, f'_{b,k_1} - (n_{b,k_2} - f'_{b,k_1})\} < f'_{b,k_1}$ and $f'''_{b,k_2} = \min\{f'_{b,k_2} + f'_{b,k_1}, n_{b,k_2}\} > f'_{b,k_2}$. One then can readily verify that $\sum_k C_{b,k}f'_{b,k} > \sum_k C_{b,k}f'''_{b,k}$. This contradicts that $\boldsymbol{f}'_{\boldsymbol{b}}$ is a part of the parking choice equilibrium solution and $\sum_k C_{b,k}f'_{b,k}$ is the minimum.

We now discuss the second case with $C_{b,k_1} = C_{b,k_2}$. The Part (i) of Assumption 3 means that for $k_1 \neq k_2$, $w_{k_1} \neq w_{k_2}$. If $w_{k_1} > w_{k_2}$, $\boldsymbol{f}'_{\boldsymbol{b}}$ violates Part (ii) of Assumption 3 and if $w_{k_1} < w_{k_2}$, $\boldsymbol{f}''_{\boldsymbol{b}}$ violates Part (ii) of Assumption 3. This contradicts the statement that $\boldsymbol{f}'_{\boldsymbol{b}}$ and $\boldsymbol{f}''_{\boldsymbol{b}}$ are both equilibrium solutions. This completes the proof.

Later in this paper, we adopt Assumption 3 and consider that a closer parking (with a smaller walking time) is preferred by travelers. We discuss the computation of parking choice equilibrium solutions in the next subsection, which takes into account Assumption 3.

3.2. Solution approach for the parking choice equilibrium

We now discuss how to utilize the structure of the problem defined in Eq. (12) in order to solve it. The objective function in Eq. (12) can be rewritten as follows:

$$Z(\boldsymbol{f}_{\boldsymbol{a}}, \boldsymbol{f}_{\boldsymbol{b}}) = Z_a(\boldsymbol{f}_{\boldsymbol{a}}) + Z_b(\boldsymbol{f}_{\boldsymbol{b}})$$

where $Z_a(\boldsymbol{f}_{\boldsymbol{a}}) = \sum_{k=1}^{K} [\int_0^{f_{a,k}} C_{a,k}(u)du]$ and $Z_b(\boldsymbol{f}_{\boldsymbol{b}}) = \sum_{k=1}^{K} C_{b,k}\cdot f_{b,k}$. If d_a and d_b are given, the flow conservation constraint $d_a + d_b = d$ in Eq. (13) becomes redundant. To minimize $Z(\boldsymbol{f}_{\boldsymbol{a}}, \boldsymbol{f}_{\boldsymbol{b}})$ under given d_a and d_b, it is equivalent to solving the following problems for $\boldsymbol{f}_{\boldsymbol{a}}$ and $\boldsymbol{f}_{\boldsymbol{b}}$ separately as constraints for $f_{a,k}$ and $f_{b,k}$ are also independent, i.e., for $i \in \{a, b\}$

$$\min: Z_i(\boldsymbol{f}_{\boldsymbol{i}}) \tag{20}$$

subject to

$$\sum_{k=1}^{K} f_{i,k} = d_i \tag{21a}$$

$$f_{i,k} \leqslant n_{i,k}, \forall\ k = 1, 2, 3, ..., K \text{if } i = b \tag{21b}$$

$$f_{i,k} \geqslant 0, \forall\ k = 1, 2, 3, ..., K \tag{21c}$$

Our general idea is to utilize the special structure of the problem defined in Eq. (12), i.e., to solve the sub-problems defined by Eq. (20) for given d_a and d_b, and then utilize information from the solution to adjust d_a and d_b towards the equilibrium values. Once d_a and d_b are adjusted to the equilibrium values, then solving the above sub-problems would yield the choice equilibrium flow.

Firstly, we discuss how to minimize Z_a (sub-problem I). We have $t_{a,k} \to +\infty$ if $f_{a,k} \to n_{a,k}$, and $\sum_{i=1}^{K} n_{a,k} > d$. Therefore, at the choice equilibrium we should have $f_{a,k} < n_{a,k}$, and we can find a positive value ϵ that $1 - \frac{f_{a,k}}{n_{a,k}} \geqslant \epsilon > 0$. To avoid considering $+\infty$ in the computation and explicitly handling the curbside parking constraints, we can re-define $t_{a,k}$ as follows:

$$t_{a,k} = \begin{cases} h(q_{a,k}) & q_{a,k} \in [0, 1-\epsilon] \\ h(1-\epsilon) + \frac{dh(1-\epsilon)}{d(1-\epsilon)}\cdot(q_{a,k} - (1-\epsilon)) & q_{a,k} \in (1-\epsilon, +\infty) \end{cases}, \tag{22}$$

We can utilize the expanded definition of $t_{a,k}$ to minimize Z_a where we do not need to explicitly handle the constraints $f_{a,k} < n_{a,k}$ (see, e.g., Nie et al., 2004). Therefore, convex combination based algorithms such as the well-known Frank-Wolf algorithm or its variants can be directly adopted (Nguyen, 1974; Fukushima, 1984).

Now, we turn to discuss the minimization problem in relation to Z_b. It is evident that the problem in Eq. (20) for $i = b$ is a linear programming problem with linear constraints, which can be readily solved by the simplex method. However, as mentioned earlier, for $k_1 \neq k_2$, as $C_{b,k_1} = C_{b,k_2}$ might occur, solving the above linear programming problem may not give a solution that satisfies Assumption 3. We now discuss how to respect Assumption 3 when computing the solution. Since $C_{b,k}$ and w_k for any k are pre-determined and given, we can label all the locations by x_k, where $x_k \in \{1, 2, ..., K\}$, in the way that $x_{k_1} < x_{k_2}$ if $C_{b,k_1} < C_{b,k_2}$ or if $C_{b,k_1} = C_{b,k_2}$ and $w_{k_1} < w_{k_2}$. x_k indeed reflects users' relative preference over share parking location k against other locations, where a smaller x_k means that the shared parking at this location is more preferred. We then minimize $\sum_k x_k f_{b,k}$ instead of $\sum_k C_{b,k}f_{b,k}$, i.e., we minimize Z'_b (sub-problem II) as follows:

$$\min:Z_b'\left(\boldsymbol{f}_{\boldsymbol{b}}\right)=\sum_k x_k f_{b,k} \tag{23}$$

subject to

$$\sum_{k=1}^{K} f_{b,k}=d_b \tag{24a}$$

$$0\leqslant f_{b,k}\leqslant m_k,\ \forall\ k=1,2,3,\ ...,K \tag{24b}$$

The minimization problem in Eq. (23) means that we assign parking flows to more preferred shared parking locations by users, i.e., locations with a smaller x_k.

If the unique d_a^* and d_b^* at the parking choice equilibrium are given to us, following the above will obtain the choice equilibrium solution. The problem now reduces to how to find the equilibrium d_a^* and d_b^*. Before moving further, we define two values η_a and η_b, where η_a is the minimum travel cost for users using curbside parking space under a given d_a, and η_b is the maximum cost of users using shared parking under a given d_b. We will utilize η_a and η_b in order to compute the d_a^* and d_b^*.

We start with listing several results in relation to η_a and η_b. Firstly, η_a increases with d_a. This is explained as follows. Under a given d_a, η_a can be obtained by solving Eq. (20) for $i=a$, which is equal to $C_{a,k}$ where $f_{a,k}>0$. This is indeed an equilibrium curbside parking flow assignment problem given the total curbside parking flow. Since $C_{a,k}(\cdot)$ strictly increases with $f_{a,k}$, it can be readily verified that a larger d_a would result in a larger cost $\eta_a=C_{a,k}$ (with $f_{a,k}>0$), i.e., when the total demand (for curbside parking) is larger, the equilibrium cost (for curbside parking users) is larger. Secondly, η_b is non-decreasing over d_b. This can be readily verified by checking the problem defined by Eq. (23) and Eq. (24). When d_b increases, since shared parking capacities are all bounded, i.e., Eq. (24), the largest x_k with a positive flow $f_{b,k}>0$ is also larger or at least not smaller, which means that the largest cost $C_{b,k}$ with a positive flow $f_{b,k}>0$ is also larger or at least not smaller. It follows that η_b (the maximum cost of users using shared parking) is larger or at least not smaller.

Based on the above monotonic relationships between η_i and d_i, where $i\in\{a,b\}$, we propose the following bi-section procedure (in Table 1) to compute the d_a^* and d_b^* and thus compute the equilibrium parking flows, where η_a and η_b at each iteration give information on how to adjust d_a and d_b towards the equilibrium values d_a^* and d_b^*. Generally speaking, when $\eta_a>\eta_b$, curbside parking is more costly than shared parking, more flow should be shifted to shared parking if feasible; and when $\eta_a<\eta_b$, more flow should be shifted to curbside parking. Moreover, as discussed earlier, η_a increases with d_a and η_b is non-decreasing over d_b. This means that the bi-section procedure below ensures that the upper bound of d_a (i.e., d_a^u) decreases or at least does not increase over iterations during the computation process, and the lower bound of d_a (i.e., d_a^l) increases or at least does not decrease over iterations. The gap $d_a^u-d_a^l$ will be decreasing or at least non-increasing over iterations, which indeed gradually approaches zero over iterations.

4. Revenue maximization and system optimum

In this section, we discuss some general results for a revenue-maximizing platform operator, and then for a social-cost-minimizing operator and the system optimum parking flow pattern when all the three groups of decision-makers (parking users, parking owners, and parking operators) are considered.

4.1. Revenue Maximization (RM)

For a private revenue-maximizing parking sharing platform operator, the problem is to maximize the net revenue, i.e.,

$$\max:\Pi(\tau_b,\boldsymbol{r}) \tag{25}$$

subject to the parking choice equilibrium studied in Section 3 (with given curbside parking prices and capacities), and the parking owners' supplying equilibrium discussed in Section 2.2, where $\Pi(\tau_b,\boldsymbol{r})$ is defined in Eq. (9).

For a revenue-maximizing platform operator, we only need to consider $\tau_b\geqslant 0$ and $\boldsymbol{r}\geqslant 0$. It can be readily verified that $r_k<0$ is dominated by $r_k=0$;$\tau_{b,k}<0$ is dominated by $\tau_{b,k}=0$ in terms of maximizing net revenue of the parking sharing platform. Note that this does not hold for the system optimum discussed in Section 4.2.

Table 1
The procedure to solve the parking choice equilibrium.

Step 0 (Initialization): Set an initial upper bound for total curbside parking flow $d_a^u=d$ and a lower bound $d_a^l=d-\min\{d,\sum_k n_{b,k}\}$.
Step 1 (Update aggregate flow split): Let $d_a=\frac{d_a^u+d_a^l}{2}$ and $d_b=d-d_a$.
Step 2-1 (Curbside parking flow assignment): With d_a from Step 1, use Frank-Wolfe algorithm to solve the problem in Eq. (20) where $i=a$, and determine η_a.
Step 2-2 (Shared parking flow assignment): Given d_b from Step 1, use Simplex-method to solve the problem in Eq. (23), and determine η_b.
Step 3 (Convergence): If $\frac{d_a^u-d_b^l}{d}<\varepsilon$, stop and return the current solution. Otherwise, go to Step 4.
Step 4 (Update flow bounds d_a^u and d_b^l). If $\eta_a\geqslant\eta_b$, let $d_a^u=d_a$ and $d_a^l=d_a^l$; otherwise let $d_a^u=d_a^u$ and $d_a^l=d_a$. Then go to Step 1.

The revenue maximization problem given in Eq. (25) is an optimization problem with equilibrium constraints. It can be equivalently formulated as a bi-level optimization problem (Yang and Bell, 1998), where the upper-level is to maximize the net revenue and the lower-level is to find the parking choice equilibrium. The uniqueness of the solution to the above revenue maximization problem is not guaranteed. The non-uniqueness involves two dimensions. Firstly, the parking flow patterns and the shared parking supply at the revenue maximization may not be unique. It follows that the shared parking rents and prices may also be non-unique. Secondly, even if the parking flow patterns and the shared parking supply at the revenue maximization are unique, the shared parking prices may be non-unique. For example, if the optimal shared parking rent at location k is $r_k = 0$, i.e., no shared parking at location k, the shared parking price $\tau_{b,k}$ can be an arbitrary value (and no user can choose this option since the supply is zero).

The above bi-level problem (revenue maximization) is difficult to solve in general. The focus of this study is not to design algorithms for solving bi-level optimization problems. In the numerical studies, for simplicity, we adopt the sequential quadratic programming (SQP) method through the MATLAB platform to solve the upper-level problem, subject to the parking choice equilibrium problem solved by the algorithm presented in Section 3.2. It can be considered as a heuristic in the context of this paper. We tested different initial solutions, i.e., different combinations of $(\boldsymbol{\tau}_b, \boldsymbol{r})$, and took the solution with the highest net revenue as the final solution.

In the following, we mainly discuss some analytical properties in relation to the revenue-maximizing pricing strategies. Suppose $(\boldsymbol{\tau}_b^*, \boldsymbol{r}^*)$ is the solution to the problem in Eq. (25), we have the following results.

Proposition 4.1. *At the parking choice equilibrium under the pricing strategy* $(\boldsymbol{\tau}_b^*, \boldsymbol{r}^*)$, *we must have* $f_{b,k} = n_{b,k}$ *for every location* $k \in \{1, 2, ..., K\}$.

Proof. Since $f_{b,k} \leqslant n_{b,k}$ always holds, we only need to show that $f_{b,k} < n_{b,k}$ cannot occur when platform revenue Π is maximized. We have two cases to consider: $n_{b,k} = 0$ and $n_{b,k} > 0$.

For the case with $n_{b,k} = 0$, it is evident that $f_{b,k} = n_{b,k} = 0$ since $f_{b,k} \geqslant 0$. For the case with $n_{b,k} > 0$, we now show in the following that if $f_{b,k} < n_{b,k}$, we can adjust the rent r_k^* to further increase the net revenue Π, which contradicts that $(\boldsymbol{\tau}_b^*, \boldsymbol{r}^*)$ maximizes Π.

Since $f_{b,k} < n_{b,k}$, we can define a rent reduction $\Delta r > 0$, where $[G_k(r_k^*) - G_k(r_k^* - \Delta r)]\cdot m_k = n_{b,k} - f_{b,k}$. We can set a new rent $(r_k)^{new} = r_k^* - \Delta r$, which will yield a new shared parking sharing supply of $(n_{b,k})^{new} = G_k(r_k^* - \Delta r)\cdot m_k = f_{b,k}$.

At the same time, we still keep the same shared parking price $\tau_{b,k}^*$ at location k. Also, shared parking prices and rents for other locations remain unchanged. The curbside parking prices and capacities are also fixed. This means that the equilibrium parking choice flow pattern under the proposed rent change at location k will be identical to that with the original rent of r_k^*, since $(n_{b,k})^{new} \geqslant f_{b,k}$.

Reducing the rent as above, the operator will save the shared parking rent by the amount of $r_k^*\cdot n_{b,k} - (r_k^* - \Delta r)\cdot f_{b,k} > 0$, i.e., reduced rent per shared parking space ($r_k^* > r_k^* - \Delta r$) and reduced number of rented shared parking ($n_{b,k} > f_{b,k} = (n_{b,k})^{new}$). However, as the equilibrium parking choice flow pattern after the rent reduction remains unchanged and the shared parking prices remain unchanged, the operator will receive the same amount of shared parking fees from the same number of shared parking users and incur the same operating cost. However, the net revenue increases as the parking rent paid decreases. This completes the proof.

Proposition 4.1 means that the shared parking spaces should be fully utilized under a revenue-maximizing sharing platform operator. If the shared parking supply is not fully utilized, the operator can improve its net revenue by at least reducing shared parking rent to save its cost. This highlights the importance of the platform pricing strategies to manage and match shared parking demand and supply in a sufficient way.

Proposition 4.2. *At the parking choice equilibrium under the pricing strategy* $(\boldsymbol{\tau}_b^*, \boldsymbol{r}^*)$, *we must have* $r_k^* = G_k^{-1}\left(\frac{f_{b,k}}{m_k}\right)$ *for any location k with* $f_{b,k} > 0$, *where* $G_k^{-1}(\cdot)$ *is the inverse of the inconvenience cost distribution function* $G_k(\cdot)$ *and we define* $G_k^{-1}(1) = \delta_k$.

Proof. Based on Eq. (7), we know $n_{b,k} = G_k(r_k^*)\cdot m_k$. Further with Proposition 4.1, we have $f_{b,k} = G_k(r_k^*)\cdot m_k$. Therefore, $r_k^* = G_k^{-1}\left(\frac{f_{b,k}}{m_k}\right)$ should hold (note that $G_k^{-1}(1) = \delta_k$).

Proposition 4.2 means that, at the revenue-maximizing equilibrium, at location k with $f_{b,k} > 0$, the rent paid to parking sharers at location k is equal to the inconvenience cost experienced the $f_{b,k}$-th parking sharer from location k (if the sharers are ordered with an increasing inconvenience cost). Such a rent will yield a shared parking supply equal to the number of shared parking users, which is consistent with Proposition 4.1. Proposition 4.2 also means that the revenue-maximizing shared parking rent has a certain relationship with the number of shared parking users. If the target parking flow pattern under $(\boldsymbol{\tau}_b^*, \boldsymbol{r}^*)$ is known to us, we can calculate the optimal rent for shared parking directly.

Proposition 4.3. *At the parking choice equilibrium under the pricing strategy* $(\boldsymbol{\tau}_b^*, \boldsymbol{r}^*)$, *we must have* $\tau_{b,k}^* = \tau_{a,k} + \alpha\cdot(t_{a,k} - t_{b,k})$ *for any location k with* $f_{b,k} > 0$.

Proof. We need to show that if $f_{b,k} > 0$ under the revenue-maximizing equilibrium, the following two cases, i.e., $\tau_{b,k}^* > \tau_{a,k} + \alpha\cdot(t_{a,k} - t_{b,k})$ or $\tau_{b,k}^* < \tau_{a,k} + \alpha\cdot(t_{a,k} - t_{b,k})$, cannot occur. It suffices to show that if either of the two cases occurs, we end up with a contradiction or we can always change the shared parking price or rent to further improve the net revenue.

Firstly, suppose $\tau_{b,k}^* > \tau_{a,k} + \alpha\cdot(t_{a,k} - t_{b,k})$. Based on the travel cost formulations in Eq. (1) and Eq. (5), one can readily verify that $C_{b,k} > C_{a,k}$. This means that no users should choose shared parking spaces at location k, i.e., $f_{b,k} = 0$, which contradicts that $f_{b,k} > 0$.

Secondly, suppose $\tau_{b,k}^* < \tau_{a,k} + \alpha\cdot(t_{a,k} - t_{b,k})$. Similar to the above, we have $C_{b,k} < C_{a,k}$. Then the platform operator can increase the price to $\tau_{a,k} + \alpha\cdot(t_{a,k} - t_{b,k})$ such that $C_{b,k} = C_{a,k}$. Since the shared parking prices of other locations, shared parking capacities, and curbside parking prices and capacities remain unchanged, no users will change his or her parking choice. Note that shared parking users at location k will not change their choices since $C_{b,k} = C_{a,k}$ and supply $n_{b,k}$ remain unchanged. Therefore, the shared parking fees collected and the platform's operating cost both remain unchanged. However, an additional profit of $[\tau_{a,k} + \alpha\cdot(t_{a,k} - t_{b,k}) - \tau^*_{b,k}]\cdot f_{b,k} > 0$ can be gained, which is a contradiction to that the original pricing strategy $(\boldsymbol{\tau}^*_b, \boldsymbol{r}^*)$ already maximizes the net revenue.

Based on the above, we conclude that $\tau^*_{b,k} = \tau_{a,k} + \alpha\cdot(t_{a,k} - t_{b,k})$ when $f_{b,k} > 0$. This completes the proof.

Proposition 4.3 means that a revenue-maximizing parking sharing platform operator will set a shared parking price to the level that travelers will be indifferent between shared and curbside parking options at the same location. Over-priced shared parking will end up with zero users and thus zero shared parking fee, while under-priced shared parking does not fully utilize the profitability from the shared parking. Moreover, the access time to shared parking $t_{b,k}$ affects the profitability of the parking sharing platform, i.e., $\tau^*_{b,k} = \tau_{a,k} + \alpha\cdot(t_{a,k} - t_{b,k})$ (for $f_{b,k} > 0$). A larger $t_{b,k}$ implies a potentially smaller shared parking price if $t_{a,k}$ remains the same. It should also be noted that if $t_{b,k}$ approaches a very large value, no one will choose the shared parking at location k, and the operator can earn zero through shared parking at this location. Similar to Proposition 4.2 for shared parking rent, Proposition 4.3 further indicates that if the target parking flow pattern under $(\boldsymbol{\tau}^*_b, \boldsymbol{r}^*)$ is known to us, we can determine the optimal pricing for shared parking directly.

Furthermore, the two conditions $r^*_k = G_k^{-1}\left(\frac{f_{b,k}}{m_k}\right)$ and $\tau^*_{b,k} = \tau_{a,k} + \alpha\cdot(t_{a,k} - t_{b,k})$ from Proposition 4.2 and Proposition 4.3 also yield the following observation.

Observation 1. For location k with $f_{b,k} > 0$, $\frac{r^*_k}{\tau^*_{b,k}} = \frac{G_k^{-1}\left(\frac{f_{b,k}}{m_k}\right)}{\tau_{a,k} + \alpha\cdot(t_{a,k} - t_{b,k})}$. As $G_k^{-1}(\cdot)$ and $h(\cdot)$ (contained in $t_{a,k}$) are generally nonlinear functions, and m_k and $n_{a,k}$ may also vary in different ways over location k, the ratio $\frac{r^*_k}{\tau^*_{b,k}}$ generally will not be a constant for different locations. Therefore, it is generally a non-optimal pricing solution for the operator to adopt a constant ratio of $\frac{r_k}{\tau_{b,k}}$ for different locations. Moreover, the platform operator will obtain a positive net revenue only if $\frac{r^*_k}{\tau^*_{b,k}} = \frac{G_k^{-1}\left(\frac{f_{b,k}}{m_k}\right)}{\tau_{a,k} + \alpha\cdot(t_{a,k} - t_{b,k})} < 1$.

4.2. System Optimum (SO)

We now proceed to further discuss the social-cost-minimizing operator, and the system optimum parking choice and parking sharing supply pattern. The total social cost in Eq. (10) involves parking users, parking owners or sharers and the parking sharing platform, which can be rewritten as follows:

$$TSC = \sum_{k=1}^{K}\sum_{i=a}^{b}[C_{i,k} - \tau_{i,k}]\cdot f_{i,k} + \sum_{k=1}^{K}\int_0^{r_k}\delta\cdot g_k(\delta)\cdot m_k\cdot d\delta + \phi(d_b) \tag{26}$$

As can be seen, the total social cost contains three parts: the cost of travelers without the parking prices, and the inconvenience cost of the parking sharers who have supplied a private space, and the operating cost of the platform.

We now examine the system optimal parking flow pattern $(\boldsymbol{f}^{so}_{\boldsymbol{a}}, \boldsymbol{f}^{so}_{\boldsymbol{b}})$ and the corresponding shared parking supply $\boldsymbol{n_b}$ to achieve the minimal *TSC*. It should be noted that in order to achieve the system optimum, besides the shared parking prices and rents, i.e., $(\boldsymbol{\tau}_b, \boldsymbol{r})$, we also need to appropriately set the curbside parking prices $\boldsymbol{\tau}_a$, i.e., curbside parking should be priced based on the marginal cost.

Proposition 4.4. *At the system optimum, we must have* $f_{b,k} = n_{b,k}$ *for every location* $k \in \{1, 2, ..., K\}$.

Proof. Firstly, $n_{b,k} \geqslant f_{b,k}$ always holds (shared parking capacity constraint). Secondly, if $n_{b,k} > f_{b,k}$, by reducing the shared parking supply from $n_{b,k}$ to $(n_{b,k})^{new} = f_{b,k}$, travelers' parking choice will not change (similar to the proof for Proposition 4.1). The social cost related to parking users remain unchanged, i.e., the first term on the right-hand side of Eq. (26). The platform operating cost related to the number of shared parking users will remain unchanged, i.e., the third term on the right-hand side of Eq. (26). However, the parking owners' side can save the inconvenience cost for a total number of $n_{b,k} - f_{b,k} > 0$ parking owners. This saving amount is equal to $\int_{G_k^{-1}\left(\frac{f_{b,k}}{m_k}\right)}^{G_k^{-1}\left(\frac{n_{b,k}}{m_k}\right)}[\delta\cdot g_x(\delta)]d\delta$, i.e., a reduction in the second term on the right-hand side of Eq. (26). This means that we can always find a way to further decrease the social cost if $n_{b,k} > f_{b,k}$, which contradicts that $n_{b,k} > f_{b,k}$ occurs at the system optimum. This completes the proof.

Proposition 4.4 for the System Optimum is similar to Proposition 4.1 for revenue maximization. It means that to minimize the total social cost, the shared parking should be fully utilized and unnecessary inconvenience cost of parking sharers is avoided. Based on the proof of Proposition 4.4, one can see that if the realized shared parking supply is not fully utilized, we can always reduce the shared parking supply to save inconvenience cost of parking sharers, and thus reduce the total social cost.

Similar to Proposition 4.2 for the revenue maximization, at the system optimum, since $f_{b,k} = n_{b,k}$, r_k in Eq. (26) can be replaced by $r_k = G_k^{-1}\left(\frac{f_{b,k}}{m_k}\right)$, where $0 \leqslant f_{b,k} \leqslant m_k$ and thus $0 \leqslant r_k \leqslant \delta_k$. The total social cost can be rewritten as a function of $f_{a,k}$ and $f_{b,k}$ as follows:

$$TSC = \sum_{k=1}^{K}\sum_{i=a}^{b}[C_{i,k} - \tau_{i,k}]\cdot f_{i,k} + \sum_{k=1}^{K}\int_0^{G_k^{-1}\left(\frac{f_{b,k}}{m_k}\right)}\delta\cdot g_k(\delta)\cdot m_k\cdot d\delta + \phi\left(\sum_{k=1}^{K} f_{b,k}\right) \tag{27}$$

which is subject to $\sum_{k=1}^{K}\sum_{i\in\{a,b\}} f_{i,k} = d$ and $0 \leqslant f_{b,k} \leqslant m_k$. It can be readily verified that TSC is a convex function of $\boldsymbol{f_a}$ and $\boldsymbol{f_b}$. By minimizing Eq. (27) with respect to $\boldsymbol{f_a}$ and $\boldsymbol{f_b}$, we can identify the system optimal parking flow pattern, and the system optimal shared parking supply can be determined accordingly, i.e., $n_{b,k} = f_{b,k}$, as discussed in Proposition 4.4.

The Lagrangian with respect to Eq. (27) can be written as follows:

$$\begin{aligned} L^{so}(\boldsymbol{f_a}, \boldsymbol{f_b}, \eta^{so}, \boldsymbol{\lambda}^{so}) = & \sum_{k=1}^{K}\sum_{i=a}^{b}(C_{i,k} - \tau_{i,k})\cdot f_{i,k} + \sum_{k=1}^{K}\int_0^{G_k^{-1}\left(\frac{f_{b,k}}{m_k}\right)}\delta\cdot g_k(\delta)\cdot m_k\cdot d\delta \\ & + \phi\left(\sum_{k=1}^{K} f_{b,k}\right) + \eta^{so}\cdot\left[d - \sum_{k=1}^{K}\sum_{i\in\{a,b\}} f_{i,k}\right] + \sum_{k=1}^{K}\lambda_k^{so}\cdot[f_{b,k} - m_k] \end{aligned} \tag{28}$$

where η^{so} is the multiplier associated with the flow conservation constraint $\sum_{k=1}^{K}\sum_{i\in\{a,b\}} f_{i,k} = d$ and λ_k^{so} is the multiplier associated with the shared parking capacity constraint (i.e., $n_{b,k} = f_{b,k} \leqslant m_k$). Similar to those in Section 3.1, we can derive the first-order optimality conditions as follows:

$$f_{a,k}\cdot\left(C_{a,k} - \tau_{a,k} + f_{a,k}\frac{dC_{a,k}}{df_{a,k}} - \eta^{so}\right) = 0,\ \forall\ k \tag{29a}$$

$$C_{a,k} - \tau_{a,k} + f_{a,k}\frac{dC_{a,k}}{df_{a,k}} - \eta^{so} \geqslant 0,\ \forall\ k \tag{29b}$$

$$f_{b,k}\cdot\left(C_{b,k} - \tau_{b,k} + G_k^{-1}\left(\frac{f_{b,k}}{m_k}\right) + \frac{d\phi}{df_{b,k}} + \lambda_k^{so} - \eta^{so}\right) = 0,\ \forall\ k \tag{29c}$$

$$C_{b,k} - \tau_{b,k} + G_k^{-1}\left(\frac{f_{b,k}}{m_k}\right) + \frac{d\phi}{df_{b,k}} + \lambda_k^{so} - \eta^{so} \geqslant 0,\ \forall\ k \tag{29d}$$

$$\lambda_k^{so}\cdot(m_k - f_{b,k}) = 0,\ \forall\ k \tag{29e}$$

$$m_k - f_{b,k} \geqslant 0,\ \forall\ k \tag{29f}$$

$$f_{i,k} \geqslant 0,\ \forall\ i,\ k \tag{29g}$$

$$\lambda_k^{so} \geqslant 0,\ \forall\ k \tag{29h}$$

$$\sum_{k=1}^{K}\sum_{i\in\{a,b\}} f_{i,k} = d \tag{29i}$$

Again, since the constraints of the original social-cost-minimization problem (i.e., flow conservation and bounded flow) are linear, and the original objective function related to the Lagrangian in Eq. (28) is convex, the above first-order conditions are both necessary and sufficient. We further discuss the observations from the above optimality conditions in the following.

Observation 2. The marginal cost of an additional curbside parking user at location k is $C_{a,k} - \tau_{a,k} + f_{a,k}\frac{dC_{a,k}}{df_{a,k}}$, which includes the travel cost without the curbside parking price of the additional curbside parking user, i.e., $C_{a,k} - \tau_{a,k}$, and the externality caused, i.e., $f_{a,k}\frac{dC_{a,k}}{df_{a,k}}$. Differently, the marginal cost of an additional shared parking user at location k is $C_{b,k} - \tau_{b,k} + G_k^{-1}\left(\frac{f_{b,k}}{m_k}\right) + \frac{d\phi}{df_{b,k}}$, which includes the travel cost without the shared parking price of the additional shared parking user, i.e., $C_{b,k} - \tau_{b,k}$, the inconvenience cost of the parking owner that shares his or her vacant space for this additional shared parking user, i.e., $G_k^{-1}\left(\frac{f_{b,k}}{m_k}\right)$, and the marginal parking sharing operation cost caused, i.e., $\frac{d\phi}{df_{b,k}}$.

Observation 3. When $\lambda_k^{so} > 0$, based on Eq. (29e), we have $m_k = f_{b,k}$, i.e., all the potential shared parking supply from location k should be utilized. λ_k^{so} reflects the additional saving of the total social cost that can be achieved if there is an additional shared parking supply from location k.

The case in Observation 3, i.e., $m_k = f_{b,k}$ may not occur. This is because, the maximum inconvenience cost δ_k due to sharing can be very large, i.e., as a part of the marginal cost created by an additional shared parking user, $G_k^{-1}\left(\frac{m_k}{m_k}\right)$ can be very large. It is likely that it is socially preferable to avoid a very large inconvenience cost. In this case, $m_k > f_{b,k}$, and $\lambda_k^{so} = 0$.

Observation 4. For any location k, if the curbside parking flow at the system optimum is positive ($f_{a,k} > 0$), based on Eq. (29a), the marginal cost is $C_{a,k} - \tau_{a,k} + f_{a,k}\frac{dC_{a,k}}{df_{a,k}} = \eta^{so}$. This means that at the system optimum, the marginal cost of an additional curbside parking user should equalize over different parking locations with a positive curbside parking flow.

Observation 5. For any location k, if the shared parking flow at the system optimum is positive ($f_{b,k} > 0$), based on Eq. (29c), $C_{b,k} - \tau_{b,k} + G_k^{-1}\left(\frac{f_{b,k}}{m_k}\right) + \frac{d\phi}{df_{b,k}} + \lambda_k^{so} = \eta^{so}$. When $m_k = f_{b,k}$ and $\lambda_k^{so} > 0$, the marginal cost $C_{b,k} - \tau_{b,k} + G_k^{-1}\left(\frac{f_{b,k}}{m_k}\right) + \frac{d\phi}{df_{b,k}} < \eta_{so}$, i.e., the potential shared parking at location k should be fully utilized if the marginal cost is lower than that caused by an additional user of the other parking type or locations with a positive flow. When $m_k > f_{b,k}$ and $\lambda_k^{so} = 0$, the marginal cost $C_{b,k} - \tau_{b,k} + G_k^{-1}\left(\frac{f_{b,k}}{m_k}\right) + \frac{d\phi}{df_{b,k}} = \eta_{so}$, i.e., marginal cost of an additional shared parking user should equalize over different locations with a positive curbside parking flow and a non-binding shared parking constraint ($f_{b,k} < m_k$).

Observation 6. (**Marginal cost pricing**) By letting $\tau_{a,k} = f_{a,k}\frac{dC_{a,k}}{df_{a,k}}$ and $\tau_{b,k} = G_k^{-1}\left(\frac{f_{b,k}}{m_k}\right) + \frac{d\phi}{df_{b,k}}$, the system optimum conditions in Eq. (29a)-(29d) become similar to those in Eqs. (16a)-(16d) for parking choice equilibrium. This means that by implementing the marginal cost pricing, the system optimum can be supported as an equilibrium.

The results discussed in Observation 6 also indicate that both the curbside parking side and the shared parking side should be appropriately priced in order to achieve the system optimum. It is noteworthy that the shared parking rent should also be appropriately set such that $f_{b,k} = n_{b,k}$, as discussed in Proposition 4.4. This is to say, we need to control curbside parking prices τ_a, shared parking prices τ_b, and shared parking rents $\boldsymbol{r}$ in order to achieve the system optimum.

This paper does not include a third travel alternative such as public transit (the demand elasticity is not captured). Therefore, adding a same constant to the $\tau_{a,k}$ and $\tau_{b,k}$ given in Observation 6 will still yield the system optimum. However, if other travel alternatives and/or demand elasticity are considered, the pricing levels of all alternatives should be appropriately determined in order to achieve an overall system optimum. In this study, we will vary the constant added to $\tau_{a,k}$, and examine how it would affect the profitability of the platform (in the numerical studies).

It is noteworthy that the above results regarding system optimum are slightly different from those for system optimal road traffic assignment (Yang, 1999). This is because, for the parking sharing problem, the system or social cost includes those of parking users and sharers in the two-sided markets (i.e., both the demand and supply sides). The system optimal parking flow pattern to minimize the total social cost in Eq. (27) can be obtained by solving an equilibrium assignment problem similar to that for solving the parking choice equilibrium, where the travel cost should be replaced by the marginal cost given in Observation 2. This is similar to solving system optimal traffic assignment in the literature. We adopt this approach in the numerical studies. Details of the solution procedures are omitted to save space.

Since the marginal cost strictly increases with the parking flow (for both curbside parking side and shared parking side), one can readily verify that the total social cost in Eq. (27) is strictly convex with respect to the parking flows. Therefore, the system optimum parking flow pattern is unique (note that the feasible flow set is closed and convex). Once the system optimum parking flow pattern is determined, the curbside and shared parking prices and the shared parking rents can be determined accordingly.

We further discuss the profitability of the parking sharing platform at the system optimum in the following. To avoid tedious consideration and discussion of corner solutions, we focus on the case where at the system optimum we have $f_{a,k} > 0$, $f_{b,k} > 0$, and $f_{b,k} < m_k$, for $k \in \{1, 2, \ldots, K\}$.

Proposition 4.5. *Suppose we have an interior system optimum where $f_{a,k} > 0$, $f_{b,k} > 0$, and $f_{b,k} < m_k$ for any k. At the system optimum, the parking sharing platform operator's profitability depends on the curbside parking price: (i) if curbside parking is severely under-priced, i.e., $\tau_{a,k} \leqslant f_{a,k}\frac{dC_{a,k}}{df_{a,k}} - \frac{d\phi}{df_{b,k}}$ for any k, the platform will have a negative net revenue; (ii) if curbside parking is still under-priced but less severe, i.e., $f_{a,k}\frac{dC_{a,k}}{df_{a,k}} - \frac{d\phi}{df_{b,k}} < \tau_{a,k} \leqslant f_{a,k}\frac{dC_{a,k}}{df_{a,k}}$ for any k, the platform will have a negative net revenue if $\phi(\cdot)$ increases linearly, and the platform may have a positive net revenue if $\phi(\cdot)$ is strictly convex and increasing; (iii) when curbside parking is over-priced, i.e., $\tau_{a,k} > f_{a,k}\frac{dC_{a,k}}{df_{a,k}}$, the platform may have a positive profit.*

Proof. Firstly, since we consider an interior system optimum where $f_{a,k} > 0$, $f_{b,k} > 0$, and $f_{b,k} < m_k$ for any k, based on Observation 4 and Observation 5, we should have $C_{a,k} - \tau_{a,k} + f_{a,k}\frac{dC_{a,k}}{df_{a,k}} = C_{b,k} - \tau_{b,k} + G_k^{-1}\left(\frac{f_{b,k}}{m_k}\right) + \frac{d\phi}{df_{b,k}} = \eta_{so}$ at the system optimum (one can also refer to the first-order optimality conditions in Eq. (29)).

Since the interior system optimum in the above is supported as a user equilibrium, $C_{a,k}$ should be identical over different locations, and moreover, we have $C_{a,k} \geqslant C_{b,k}$. This can be readily derived based on the optimality conditions in Eq. (16).

With $C_{a,k} - \tau_{a,k} + f_{a,k}\frac{dC_{a,k}}{df_{a,k}} = C_{b,k} - \tau_{b,k} + G_k^{-1}\left(\frac{f_{b,k}}{m_k}\right) + \frac{d\phi}{df_{b,k}}$ and $C_{a,k} \geqslant C_{b,k}$, one can derive that

$$-\tau_{a,k} + f_{a,k}\frac{dC_{a,k}}{df_{a,k}} \leqslant -\tau_{b,k} + G_k^{-1}\left(\frac{f_{b,k}}{m_k}\right) + \frac{d\phi}{df_{b,k}}$$
$$\Leftrightarrow \tau_{b,k} - G_k^{-1}\left(\frac{f_{b,k}}{m_k}\right) \leqslant \tau_{a,k} - f_{a,k}\frac{dC_{a,k}}{df_{a,k}} + \frac{d\phi}{df_{b,k}}.$$

Furthermore, since at the system optimum $r_k = G_k^{-1}\left(\frac{f_{b,k}}{m_k}\right)$, we can further derive that

$$\tau_{b,k} - r_k \leqslant \tau_{a,k} - f_{a,k}\frac{dC_{a,k}}{df_{a,k}} + \frac{d\phi}{df_{b,k}}. \tag{30}$$

For Proposition 4.5(i), since $\tau_{a,k} \leqslant f_{a,k}\frac{dC_{a,k}}{df_{a,k}} - \frac{d\phi}{df_{b,k}}$ for any k, based on Eq. (30), one can derive that $\tau_{b,k} - r_k \leqslant 0$. This means that the net revenue defined in Eq. (9) is always negative since $\phi(\cdot) > 0$, i.e.,

$$\Pi = \sum_{k=1}^{K} [f_{b,k} \cdot \tau_{b,k}] - \sum_{k=1}^{K} [n_{b,k} \cdot r_k] - \phi(d_b) \leqslant 0 - \phi(d_b) < 0.$$

For Proposition 4.5(ii), since $f_{a,k}\frac{dC_{a,k}}{df_{a,k}} - \frac{d\phi}{df_{b,k}} < \tau_{a,k} \leqslant f_{a,k}\frac{dC_{a,k}}{df_{a,k}}$, based on Eq. (30), one can derive that $\tau_{b,k} - r_k \leqslant \frac{d\phi}{df_{b,k}}$. At the system optimum, $f_{b,k} = n_{b,k}$, further based on Eq. (9), we can derive the following

$$\Pi = \sum_{k=1}^{K} f_{b,k} \cdot [\tau_{b,k} - r_k] - \phi(d_b) \leqslant \sum_{k=1}^{K} f_{b,k} \cdot \frac{d\phi}{df_{b,k}} - \phi(d_b). \tag{31}$$

When $\phi(\cdot)$ increases linearly (i.e., $\frac{d\phi}{df_{b,k}}$ is constant), we have $\Pi \leqslant \sum_{k=1}^{K} f_{b,k} \cdot \frac{d\phi}{df_{b,k}} - \phi(d_b) = -\phi(0) < 0$. Differently, when $\phi(\cdot)$ is strictly convex and increasing, $\frac{d\phi}{df_{b,k}}$ can be relatively large at the system optimum, and the net revenue may be greater than zero. However, this is not guaranteed. This is because, as discussed earlier, $C_{a,k} \geqslant C_{b,k}$, which yields $\tau_{b,k} \leqslant \tau_{a,k} + \alpha \cdot (t_{a,k} - t_{b,k})$, i.e., $\tau_{b,k}$ is bounded from the above by another value, which may be lower than the bound established in Eq. (30).

For Proposition 4.5(iii), the reasoning is similar to that of Proposition 4.5(ii), which is omitted. This completes the proof. □

Proposition 4.5 means that at the system optimum, the parking sharing platform alone is likely to yield a negative net revenue, since the curbside parking is often under-priced. This affects how the local authority may regulate parking sharing platforms in order to achieve the system optimum or at least a status close to it, which is further discussed in Section 4.3.

4.3. Discussion on Implementing the SO

We now further discuss how to implement the SO discussed before. We focus on the cases where the curbside parking is controlled by local authorities (e.g., this is the case in many local councils in many cities) and the shared parking can be controlled by either local authority or profit-driven private operators.

As discussed in Observation 4 and Observation 6, to achieve the system optimum, the curbside parking, usually controlled by the local authorities, should be priced appropriately (i.e., equalized marginal costs for curbside parking over different locations with positive parking flows). Denote the curbside parking prices vector for the system optimum by $\boldsymbol{\tau}_a^{so}$. Besides, as discussed in Proposition 4.4, Observation 5 and Observation 6, the shared parking should be appropriately rented and priced in order to achieve the system optimum (i.e., we need to set the optimal prices $\boldsymbol{\tau}_b^{so}$ and rents $\boldsymbol{r}^{\boldsymbol{so}}$), which is further discussed in the following.

Firstly, we consider the case where the parking sharing platform is also controlled by the local authority or a regulated non-profit operator. Then, the SO pricing strategies similar to those discussed in Observation 6, denoted as $(\boldsymbol{\tau}_a^{so}, \boldsymbol{\tau}_b^{so}, \boldsymbol{r}^{\boldsymbol{so}})$, can be applied. However, in this case, the local authorities may have to subsidize the parking sharing platform operation to maintain break-even since the net revenue $\Pi(\boldsymbol{\tau}_b^{so}, \boldsymbol{r}^{\boldsymbol{so}})$ of the parking sharing platform may be negative (as discussed in Proposition 4.5). The compensation for the platform may come from the fees collected from the curbside parking.

Secondly, we consider the case where the parking sharing platform is controlled by a revenue-maximizing operator. The curbside parking should still be priced in an optimal way, i.e., we still adopt $\boldsymbol{\tau}_a^{so}$ for the curbside parking. Without any further intervention from the local authority, the private operator will set the shared parking rents and prices based on Revenue Maximization discussed in Section 4.1, which are denoted by $(\boldsymbol{\tau}_b^*, \boldsymbol{r}^*)$. The platform may gain a positive net profit of $\Pi(\boldsymbol{\tau}_b^*, \boldsymbol{r}^*)$. In order to achieve the system optimum, the local authority may require the platform to set its price strategy as $(\boldsymbol{\tau}_b^{so}, \boldsymbol{r}^{\boldsymbol{so}})$. However, the local authority will have to compensate the platform by the amount of $\Pi(\boldsymbol{\tau}_b^*, \boldsymbol{r}^*) - \Pi(\boldsymbol{\tau}_b^{so}, \boldsymbol{r}^{\boldsymbol{so}})$. Again, the compensation may come from the fees collected from the curbside parking.

Besides compensating the parking sharing platform to achieve the system optimum or a state close to that, the local authorities may set other quantitative regulations regarding shared parking prices and rents. For example, at the Revenue Maximization, the realized shared parking supply may be less than that under System Optimum (due to lower shared parking rents), while the shared parking prices are higher. The local authority may establish lower bounds for shared parking rents and upper bounds for shared parking prices on the parking sharing market. The local authority may also set Entry Standard that the platform should operate at least a certain number of shared parking spaces. This avoids the platform to charge heavily on users with a small amount of supply. This will be further discussed in the numerical analysis in Section 5.3.

As discussed in Section 4.2, this paper does not include a third travel alternative such as public transit or demand elasticity. Therefore, adding a constant to $\tau_{a,k}$ and $\tau_{b,k}$ given in Observation 6 will still yield the system optimum. If other travel alternatives and/or demand elasticity are considered, the pricing levels of all alternatives should be appropriately determined in order to achieve an overall system optimum. In this paper, we will vary the constant added to $\tau_{a,k}$ in the numerical studies and examine how it would affect the profitability of the platform and identify the subsidy required, i.e., $\Pi(\boldsymbol{\tau}_b^*, \boldsymbol{r}^*) - \Pi(\boldsymbol{\tau}_b^{so}, \boldsymbol{r}^{\boldsymbol{so}})$, in order to achieve the system optimum when we have a revenue-maximizing operator.

5. Numerical studies

This section presents some numerical studies to illustrate the proposed model and analysis. Specifically, we firstly illustrate the effectiveness of the developed solving algorithm, and then examine the parking flow pattern, platform revenue, owner net benefit, total curbside parking fee, total parking fee (including both curbside and shared parking), total social cost, and total user cost under different platform pricing strategies. We also vary the curbside parking pricing level (as mentioned in Section 4.2) and and the

Table 2
Summary of basic numerical settings.

Parameters or Functions	Specification
Total demand	$d = 4000$
Driving and walking speeds	$v = 25$(km/hr); $v_w = 5$(km/hr);
Value of driving time	$\alpha = 40$(AUD/hr);
Walking cost function	$c(w_k) = \alpha\cdot(0.616 + 1.94\cdot w_k + 0.053\cdot w_k^2)$(AUD);
Operating cost function	$\phi(d_b) = 300 + 0.5\cdot d_b$(AUD);

inconvenience cost distribution in the numerical analysis to generate further understanding.

5.1. Numerical settings

Table 2 summarizes the basic settings for the numerical studies. The driving speed takes into account, e.g., intersection delays and congestion effect, which is not an instantaneous speed. The walking cost is obtained by converting walking time into a driving time equivalent and then further converting it into monetary value. Besides, we adopt the following cruising time function: $h(q) = h_0 + h_1\cdot(h_2 + q)^{h_3}$, where $h_0 = 0.5$(min), $h_1 = 2$(min), $h_2 = 1$, and h_3 is a piece-wise linear and increasing function of parking occupancy rate q. Fig. 3 displays the cruising time against parking occupancy rate. When parking occupancy rate goes beyond 80%, the cruising time starts to increase sharply.

We consider that there are $K = 5$ locations in total. It should be noted that if we consider a single working destination, the number of parking locations with acceptable walking distances is often very limited (e.g., $K = 5$ in this paper). Under $K = 5$, the number of flow variables in the parking choice equilibrium problem defined in Section 3 is 10 (i.e., 5 for shared parking flow and 5 for curbside parking flow). This means that in the numerical study solving the parking choice equilibrium problem is very quick (please refer to the computation times in relation to Fig. 4 in Section 5.2).

The driving distance, walking distance, shared parking access time, curbside parking capacity, and potential shared parking capacity for the five locations are given in Table 3. Note that the driving time from to location k is $t_k = D_k/v$ and the walking time is $w_k = W_k/v_w$. The inconvenience cost for potential parking sharers at location k follows a uniform distribution, i.e., $\delta \sim \mathscr{U}[0, \delta_k]$. In the benchmark case, we adopt $\delta_k = 20$(AUD) for any location k. We also consider another two values, i.e., $\delta_k = 10$(AUD) and $\delta_k = 30$(AUD), for comparison purpose in Section 5.3.2.

5.2. Convergence of the bi-section based approach

To show the convergence of the bi-section based algorithm discussed in Section 3.2, we define the following term to measure the discrepancy for the convex combination method (Frank-Wolfe algorithm mentioned in Table 1).

$$e_1 = \sum_k \frac{f_{a,k}}{d_a}\cdot\left|C_{a,k} - \min_j\{C_{a,j}\}\right|. \tag{32}$$

where $k \in \{1, 2, ..., K\}$. Furthermore, we define the following gap for the bi-section based method to adjust the aggregate split between curbside parking and shared parking usage, i.e.,

$$e_2 = \frac{|d_a^u - d_a^l|}{d}. \tag{33}$$

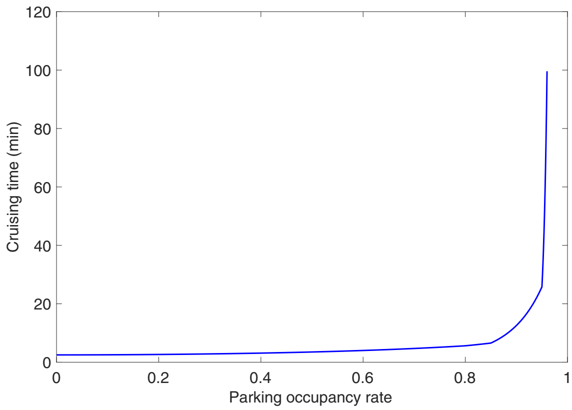

Fig. 3. The cruising time function: cruising time vs. occupancy rate.

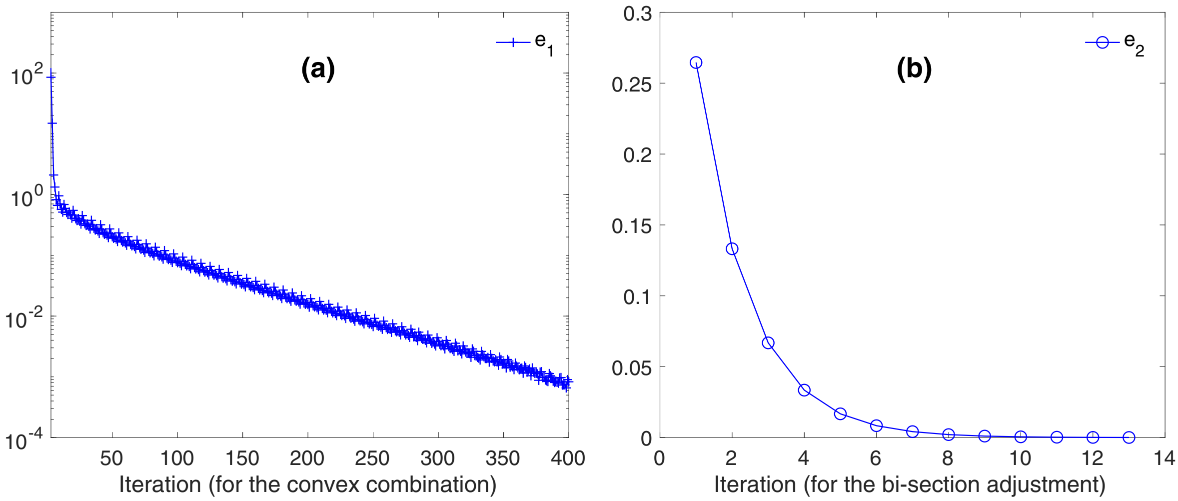

Fig. 4. The evolution of the two terms e_1 and e_2 over iterations.

Table 3
Summary of parking location-dependent numerical settings.

Location k	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$
Driving distance D_k(km)	12.0	11.5	11.0	11.0	10.5
Walking distance W_k(km)	0.1	0.5	0.8	1.0	1.2
Shared parking access time $t_{b,k}$(minute)	2	2	3	3	3
Curbside parking capacity $n_{a,k}$	500	750	1000	1250	2000
Potential shared parking capacity m_k	500	1000	1000	1250	1250

where d_a^u and d_a^l are the bounds mentioned in Table 1. If $e_1 \to 0$, it means that the total curbside parking flow d_a is equilibrated over locations, i.e., for locations with a positive flow, the travel cost approaches the minimum cost. If $e_2 \to 0$, it means that the aggregate choice split between curbside and shared parking converges to the equilibrium value.

Fig. 4 shows the evolution of the two terms e_1 and e_2 defined in the above. For illustration purpose, Fig. 4 only displays the evolution of e_1 under a given d_a. As can be seen, these terms approach zero over iterations and the solution converges. Note that for the results in Fig. 4, the parking prices and rents are set as those depicted in Fig. 5 for the Revenue Maximization (RM) case. The computation time for solving the curbside parking assignment problem once (i.e., running the Frank-Wolfe algorithm once, which corresponds to Fig. 4a) is around 0.024 s (the time varies slightly depending on the initial solution and the demand level). The computation time for solving the whole parking choice problem is 0.350 s (corresponds to the bi-section adjustment in Fig. 4b, where one iteration in Fig. 4b involves running the Frank-Wolfe algorithm once to solve the curbside parking assignment problem under given total curbside parking flow d_a). These computation times are based on coding via MATLAB (R2017b) platform on an iMac (Processor: 3.6 GHz Intel Core i7).

5.3. Revenue maximization, system optimum, and original user equilibrium

We have three cases for comparison, i.e., Revenue Maximization (RM) (to maximize operator revenue Π), and System Optimum (SO) (to minimize total social cost *TSC*), and the Original User Equilibrium (OUE) without parking sharing ($n_{b,k} = 0$ for any k). For the OUE, we further consider two sub-cases, i.e., (i) $\tau_{a,k}$ is optimally set (same as those to achieve SO), which is denoted by OUE, and (ii) $\tau_{a,k} = 0$, which is denoted by “OUE-0” later on.

5.3.1. Benchmark case and varying curbside parking price

This section starts with the benchmark case, where we fix the curbside parking price following the marginal cost pricing principle, i.e., $\tau_{a,k} = f_{a,k}\frac{dC_{a,k}}{df_{a,k}}$ (Observation 6), which is shown in Fig. 5d. We then compare RM, SO and OUE under the same curbside parking pricing. We also compare the three cases (RM, SO and OUE) with the ‘OUE-0’, where parking sharing is not introduced and $\tau_{a,k} = 0$. Note that later on we will vary the curbside parking pricing for comparison purpose, where the current $\tau_{a,k} = f_{a,k}\frac{dC_{a,k}}{df_{a,k}}$ is labeled as “Level-1” while there are another two pricing levels (‘Level-2’ and ‘Level-3’), as shown in Fig. 6a.

We now discuss and compare the RM, SO, and OUE in the benchmark case ($\tau_{a,k} = f_{a,k}\frac{dC_{a,k}}{df_{a,k}}$) and OUE-0 ($\tau_{a,k} = 0$). In particular, Fig. 5a displays the (curbside and shared) parking flows over different locations, Fig. 5b displays the curbside parking occupancy rate, i.e., $q_{a,k} = f_{a,k}/n_{a,k}$, and the shared parking utilization rate, i.e., $q_{b,k} = f_{b,k}/m_k = n_{b,k}/m_k$, Fig. 5c displays the location-dependent

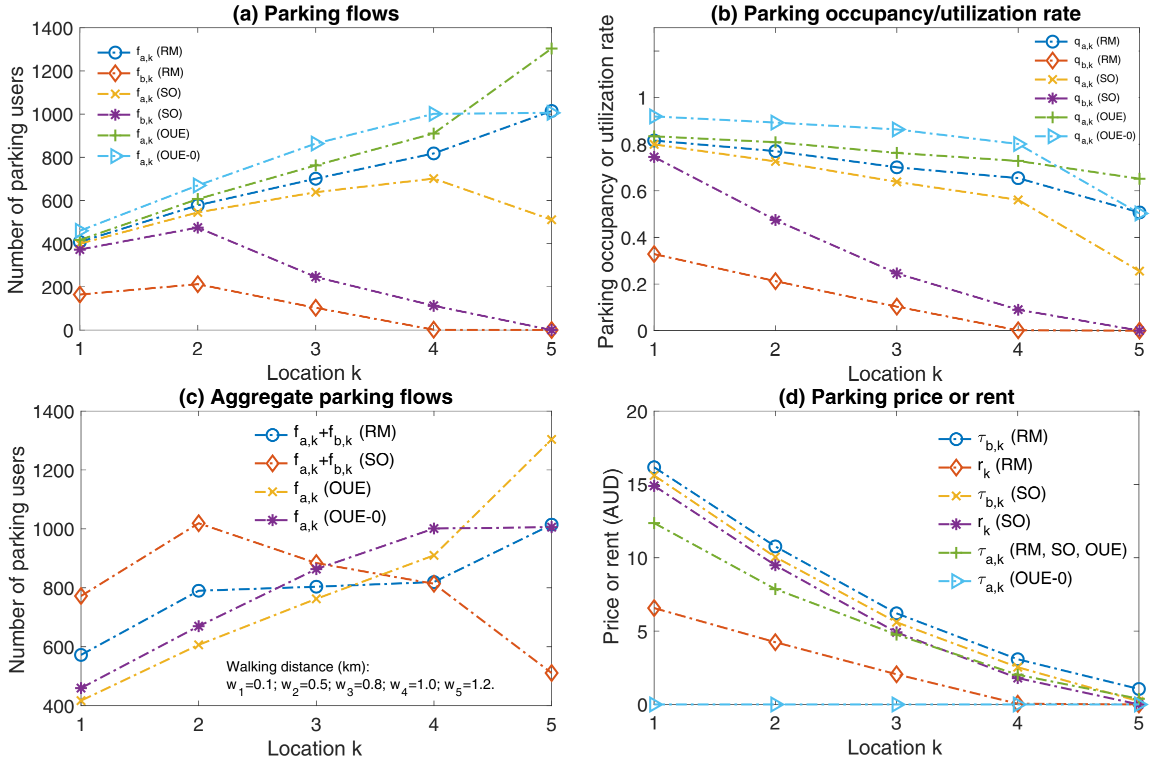

Fig. 5. Comparison of RM, SO, OUE, and OUE-0.

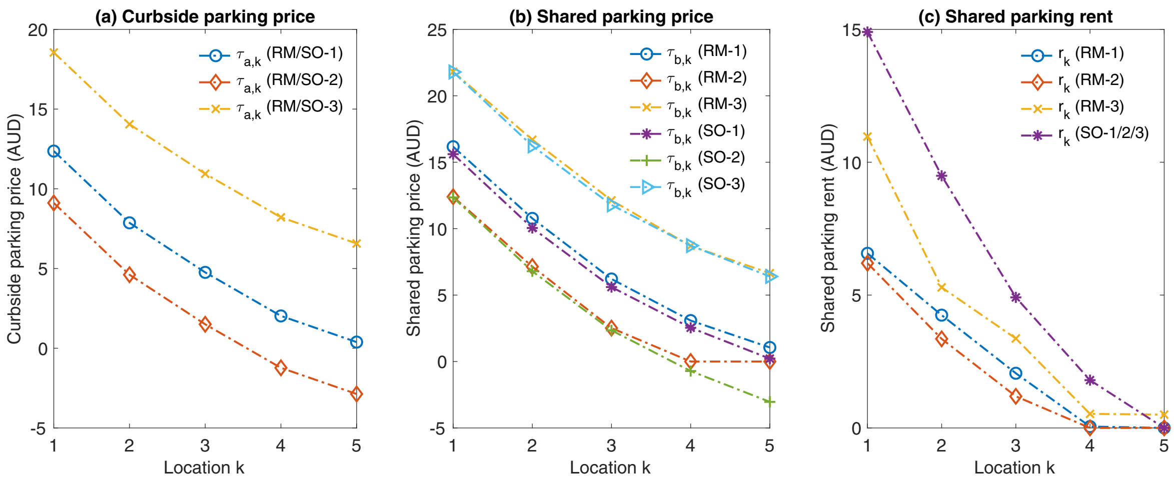

Fig. 6. Comparison of RM and SO under varying curbside parking pricing levels.

aggregate flows, and Fig. 5d displays the parking prices or rents. Besides, Table 4 summarizes the parking location-dependent cruising times, and Table 5 further summarizes seven efficiency metrics (i.e., total net revenue Π for parking sharing operator, total net benefit of parking sharers π, total curbside parking fee Π', total parking fee $\Pi + \Pi'$, total social cost TSC, total user cost TC, and market share of shared parking d_a/d).

We start with comparing OUE and OUE-0. There is no shared parking flow for OUE and OUE-0 since parking sharing is not introduced. As can be seen in Fig. 5a (curves labeled by OUE and OUE-0, respectively), the marginal-cost-based curbside parking pricing under OUE indeed shifts some of the parking flows at locations 1–4 to location 5 when compared to OUE-0. This indeed reduces over-competition for parking and costly cruising time at locations 1–4. These observations are consistent with the parking occupancy rates in Fig. 5b and the location-dependent cruising time summarized in Table 4.

As can be seen in Fig. 5b, under OUE-0, at location 1 with the smallest walking distance, the parking occupancy rate approaches 93%, and the cruising time approaches 20 min (refer to Table 4). It should be noted that usually parking closer to the city (with a

Table 4
Location-dependent cruising time (minute).

Location k	$k = 1$	$k = 2$	$k = 3$	$k = 4$	$k = 5$
RM	7.75	6.36	5.19	4.58	3.53
SO	6.96	5.57	4.40	3.79	2.74
OUE	8.77	7.38	6.21	5.61	4.56
OUE-0	20.56	14.05	9.38	6.99	3.51

smaller walking distance) is more expensive which limits the flow, and thus the cruising time at location 1 is not too long, e.g., at OUE, the cruising time at location 1 is only above 8 min. It follows that the total social cost at OUE (refer to OUE (Level-1) in Table 5) is smaller than that at OUE-0 (refer to OUE-0 ($\tau_{a,k} = 0$) in Table 5). It is noteworthy that OUE under "marginal cost pricing" in Observation 6 is indeed a "system optimum" (without parking sharing) compared to OUE-0. However, the total walking cost of users under OUE increases when compared to that under OUE-0 since more users have to park further away (at location 5) while doing so reduces the total social cost (refer to Fig. 5c for the walking distance dependent curbside parking flow). It is also noteworthy that the price $\tau_{a,k} = f_{a,k}\frac{dC_{a,k}}{df_{a,k}}$ for OUE decreases with k (refer to OUE in Fig. 5d). This implies that at OUE-0 ($\tau_{a,k} = 0$), the marginal cost of additional curbside parking flow at a location with a shorter walking distance is larger, where a smaller k indicates a shorter walking distance. This is because, users tend to park closer to final destination (with a shorter walking distance), and thus curbside parking at locations with a shorter walking distance is more crowed, which generates considerable cruising externalities.

We now further compare RM, SO, and OUE, under given optimal curbside parking prices of $\tau_{a,k} = f_{a,k}\frac{dC_{a,k}}{df_{a,k}}$. It is evident from Fig. 5a that in terms of the curbside parking flow $f_{a,k}$, we have OUE > RM > SO; in terms of the shared parking flow $f_{b,k}$: RM < SO (zero shared parking flow for OUE without parking sharing). This is because, in the RM regime to maximize net revenue, the operator tends to rent less shared parking (a lower shared parking rent) while sets a higher shared parking price when compared to the SO regime, as shown in Fig. 5d. For the SO regime, the operator tends to rent more shared parking spaces (especially at locations with a shorter walking distance) in order to reduce cruising for parking and walking distance of users. In fact, in terms of overall cruising time, SO < RM < OUE, as shown in Table 4. In terms of overall walking distance, SO < RM < OUE. This is shown in Fig. 5c, where in terms of parking flows at locations 1–3 with shorter walking distances, SO > RM > OUE, and in terms of parking flows at locations 4–5 with longer walking distances, we have SO < RM < OUE. These observations are also consistent with those in Fig. 5b, where the potential shared parking spaces at locations with a shorter walking distance are more heavily utilized, i.e., $q_{b,k}$ decreases with k for both RM and SO (no shared parking is utilized at OUE).

The above observations regarding RM and SO indicate that the local authorities may set lower bounds for shared parking rents, and/or upper bounds on shared parking prices, and/or minimum shared parking supply requirement as market entry standard, to drive the system to a state closer to SO even if we have a revenue-maximizing private operator, as discussed in Section 4.3.

We now move to compare the efficiency metrics for RM, SO, and OUE in the benchmark case (labeled as 'Level-1') and OUE-0 with $\tau_{a,k} = 0$ in Table 5. We summarize a few observations below. Firstly, SO yields a negative net revenue (-0.300×10^3) for the platform, while RM yields a positive net revenue (2.854×10^3). To persuade a profit-driven private operator to set SO prices and rents, the local authority has to compensate the platform by the amount of $[2.854 - (-3.000)] \times 10^3 = 3.154 \times 10^3$ (as discussed in Section 4.3). Doing so will yield a social cost reduction by $(2.328 - 2.300) \times 10^5$. Secondly, this amount of compensation may come from the total curbside parking fee. The total curbside parking fee is sufficient to cover the compensation given the current setting. Thirdly, both RM and SO reduce the total social cost when compared to OUE. This is mainly due to that shared parking with low inconvenience cost is utilized to reduce cruising for parking and walking distance as discussed before in Fig. 5. This can also be verified by checking the market share of shared parking d_a/d, where OUE (0.00%) < RM (12.02%) < SO (30.13%). Fourthly, as mentioned earlier, OUE with optimal curbside parking pricing reduces the total social cost when compared to OUE-0 (2.388 < 2.501). Last but not least, if we compare the RM with the OUE, total user cost is reduced (users "win"), parking sharing platform obtains a positive net revenue (the operator "wins"), and the parking sharers obtain a positive net benefit (parking sharers "win"). This is a Win–Win-Win situation.

Table 5 summarizes seven efficiency metrics for RM, SO, and OUE under three different levels of curbside parking pricing, i.e.,

Table 5
Performance Comparison: Revenue Maximization (RM), System Optimum (SO), and Original User Equilibrium (OUE) (monetary unit: AUD).

Cases	$\Pi(10^3)$	$\pi(10^3)$	$\Pi'(10^4)$	$\Pi + \Pi'(10^4)$	$TSC(10^5)$	$TC(10^5)$	d_b/d
RM (Level-1)	2.854	1.096	1.497	1.783	2.328	2.517	12.02%
RM (Level-2)	1.138	0.797	0.337	0.450	2.335	2.388	9.56%
RM (Level-3)	7.242	2.503	3.447	4.171	2.311	2.753	19.28%
SO (Level-1)	−0.300	5.732	1.332	1.302	2.300	2.487	30.13%
SO (Level-2)	−4.224	5.732	0.422	0.000	2.300	2.357	30.13%
SO (Level-3)	7.152	5.732	3.061	3.776	2.300	2.735	30.13%
OUE (Level-1)	0.000	0.000	1.590	1.590	2.388	2.544	0.00%
OUE (Level-2)	0.000	0.000	0.288	0.288	2.388	2.414	0.00%
OUE (Level-3)	0.000	0.000	4.064	4.064	2.388	2.792	0.00%
OUE-0 ($\tau_{a,k} = 0$)	0.000	0.000	0.000	0.000	2.501	2.501	0.00%

'Level-1' (benchmark case based on the SO parking flow): $\tau_{a,k} = f_{a,k}\frac{dC_{a,k}}{df_{a,k}}$, 'Level-2': $\tau_{a,k} = f_{a,k}\frac{dC_{a,k}}{df_{a,k}} - 3.255$, and 'Level-3': $\tau_{a,k} = f_{a,k}\frac{dC_{a,k}}{df_{a,k}} + 6.185$. Besides the efficiency metrics summarized in Table 5, Fig. 6 further displays the curbside parking prices, shared parking prices, and the shared parking rents for the three curbside parking pricing levels.

We now move to discuss the RM, SO, and OUE under different curbside parking pricing levels. We summarize a few observations in Fig. 6. Firstly, in terms of curbside parking price $\tau_{a,k}$, we have 'Level-3' > 'Level-1' > 'Level-2' (the benchmark case is in-between), as shown in Fig. 6a. To achieve the same SO parking flow pattern, the shared parking prices should be set accordingly, i.e., in terms of $\tau_{b,k}$ (SO), we have 'Level-3' > 'Level-1' > 'Level-2', as shown in Fig. 6b. Secondly, for the three different curbside pricing levels, the RM shared parking pricing is slightly higher than that of the SO shared parking pricing. It is also noted that for 'Level-2', the shared parking price for location 5 is bounded by zero, i.e., no potential spaces from location 5 should be shared to achieve RM. Moreover, under the three different curbside pricing levels, the RM rents are all lower than the SO rents. This indicates that RM tends to charge a slightly higher shared parking price than SO while provide much less shared parking supplies. Thirdly, for the three different curbside pricing levels, in terms of RM rents, 'Level-2' ⩽'Level-1' ⩽'Level-3'. This is because, when the curbside parking is more expensive, the shared parking price can also be higher (refer to Fig. 6b), and the RM operator tends to rent more shared parking. Fourthly, in terms of the compensation to persuade the RM operator to operate at SO, 'Level-2' > 'Level-1' > 'Level-3' (i.e., 1.138-(-4.224)>2.854-(-0.300)>7.242–7.152). It is because that when the curbside prices are higher, the RM parking sharing platform is more profitable, and the compensation needed is smaller.

We further discuss and compare the efficiency metrics for RM, SO, and OUE under the three different curbside pricing levels in Table 5. Since in the benchmark case, we have compared RM, SO and OUE, we now focus more on how the curbside parking pricing level could affect RM, SO or OUE. We start with OUE under different curbside pricing levels. In particular, the OUE parking flow patterns under the three different curbside pricing levels are identical, which yield identical total social costs. However, due to the difference in the curbside parking prices, the total cost of users are different, i.e., a higher curbside parking pricing level yields a larger total cost.

We now summarize a few observations in relation to SO in Table 5. Firstly, since the parking flow patterns at SO under the three different curbside pricing levels are identical (the optimal shared parking supply is also identical), the total social costs are identical. However, the total user cost increases with the curbside parking pricing level. Secondly, while the SO flows are identical, the three different pricing levels yield three different net revenue for the platform operator. In terms of the platform net revenue, we have 'Level-2' ⩽'Level-1' ⩽'Level-3', which is also consistent with the trend of pricing levels. Thirdly, under curbside parking pricing of 'Level-1', the net revenue is -0.300×10^3 at SO, which is equal to the fixed operating cost of the platform. This is indeed the case in Proposition 4.5(ii). When we reduce the curbside parking price, i.e., 'Level-2', the platform will lose more, while the total parking revenue from curbside and shared parking is zero. Differently, if the curbside parking price increases, i.e., 'Level-3', the platform can earn a positive net revenue even if its objective is to minimize total social cost.

We now summarize a few observations in relation to RM in Table 5. Firstly, while SO often yields a negative net revenue for the platform, the RM can always yield a positive net revenue, where in terms of the revenue we have 'Level-2' ⩽'Level-1' ⩽'Level-3' (similar to that for SO), which is consistent with the pricing levels. Secondly, since RM does not minimize social cost, the total social cost under RM is larger than that under SO (given the same curbside parking pricing level). Moreover, in terms of the total social cost, 'Level-2' > 'Level-1' > 'Level-3', i.e., a lower total social cost under a higher curbside parking pricing level. This is because, when curbside parking pricing level is higher, it is more profitable for the platform to rent more shared parking (refer to the d_b/d). Under the current setting, more shared parking is also beneficial in reducing total social cost. Thirdly, a higher pricing level yields a higher total user cost, a higher total curbside parking fee, a higher total parking fee, and a higher total net benefit of parking sharers at RM, which are similar to those at SO. Fourthly, the total user cost at RM is higher than that at SO (given the same curbside parking pricing level).

5.3.2. Varying sharing inconvenience cost

As mentioned in Section 5.1, we consider that the inconvenience cost for potential parking sharers at location k follows a uniform distribution, i.e., $\delta \sim \mathscr{U}[0, \delta_k]$, where $\delta_k = 20$(AUD) for any location k. We now vary δ_k to examine how the inconvenience cost distribution could affect the system. In particular, we consider another two values, i.e., $\delta_k = 10$(AUD) and $\delta_k = 30$(AUD).

It should be noted that since the inconvenience cost distribution changes, the SO parking flow pattern and shared parking supply also change. The SO curbside parking pricing is different under different inconvenience cost distributions. For comparison purpose, we adopt the marginal-cost-based curbside parking price $\tau_{a,k} = f_{a,k}\cdot\frac{dC_{a,k}}{df_{a,k}}$ in Observation 2 under different inconvenience cost distributions.

Fig. 7 displays curbside and shared parking prices, shared parking rents, curbside and shared parking flows under the three different inconvenience cost settings when curbside parking price is set as $\tau_{a,k} = f_{a,k}\cdot\frac{dC_{a,k}}{df_{a,k}}$. Table 6 further summarize the seven efficiency metrics mentioned before for RM and SO under the three different settings for inconvenience cost.

We now discuss the SO under three different inconvenience cost settings. Firstly, Fig. 7a shows that the curbside parking price based on the marginal cost principle is higher when δ_k increases, i.e., $10 \to 20 \to 30$. This is because, when δ_k increases, it is socially preferable to rent less shared parking spaces to avoid large inconvenience costs, which yields smaller shared parking flows (refer to SO flows in Fig. 7e) and larger curbside parking flows (refer to SO flows in Fig. 7d). This results in a larger marginal cost and thus a larger curbside price $\tau_{a,k} = f_{a,k}\cdot\frac{dC_{a,k}}{df_{a,k}}$. Secondly, since in terms of the curbside parking pricing, SO-10 (SO with $\delta_k = 10$) < SO-20 (SO with $\delta_k = 20$) < SO-30 (SO with $\delta_k = 30$), the corresponding SO shared parking price also decreases with δ_k, as can be seen in Fig. 7b.

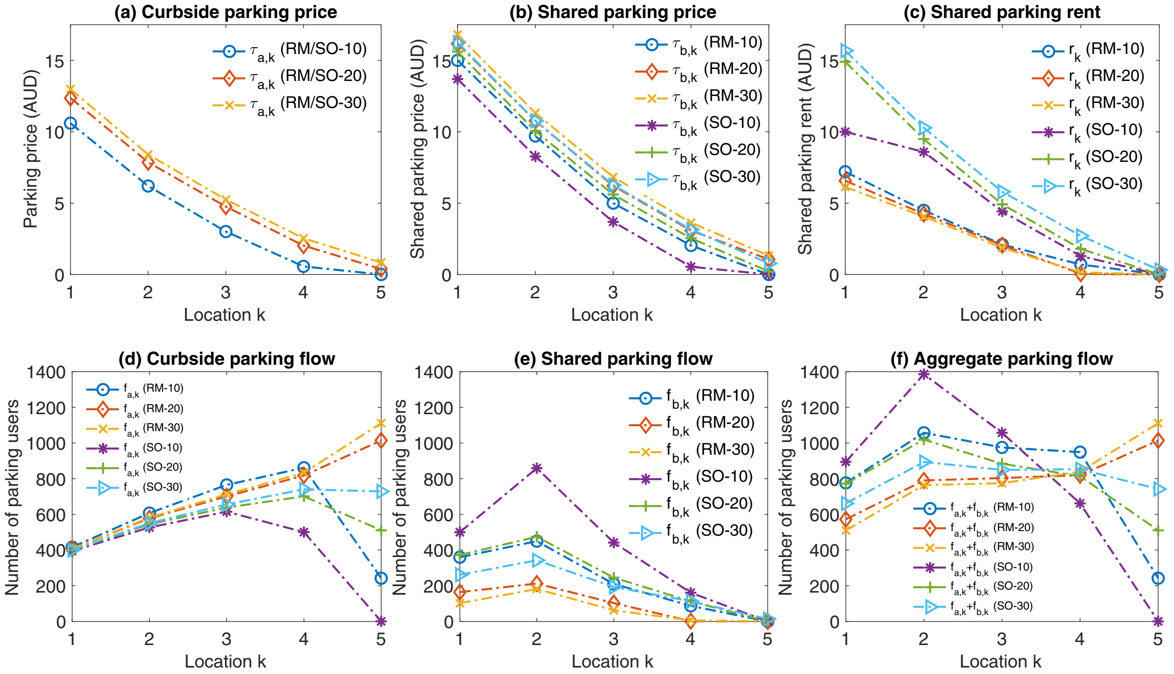

Fig. 7. Comparison of RM and SO under varying inconvenience cost distributions.

Table 6
Performance comparison under varying sharing inconvenience cost distribution (monetary unit: AUD).

Cases	$\Pi(10^3)$	$\pi(10^3)$	$\Pi'(10^4)$	$\Pi + \Pi'(10^4)$	$TSC(10^5)$	$TC(10^5)$	d_b/d
RM ($\delta_k = 10$)	5.022	2.560	1.107	1.609	2.288	2.474	27.69%
RM ($\delta_k = 20$)	2.854	1.096	1.497	1.783	2.328	2.517	12.02%
RM ($\delta_k = 30$)	2.012	0.869	1.699	1.900	2.343	2.541	8.82%
SO ($\delta_k = 10$)	−0.153	7.270	0.959	0.944	2.251	2.418	49.05%
SO ($\delta_k = 20$)	−0.300	5.732	1.332	1.302	2.300	2.487	30.13%
SO ($\delta_k = 30$)	−0.299	4.527	1.577	1.547	2.321	2.521	23.09%

Thirdly, it is noted that while the SO shared parking supply or flow is larger when δ_k is smaller, the SO rents under a smaller δ_k can be smaller (refer to Fig. 7c). This is because, when δ_k is smaller, more parking owners are willing to share their spaces given the same rent. Fourthly, it can be seen that given a smaller δ_k, SO tends to yield more users parked at locations closer to the final destination, i.e., a shorter walking distance, as can be seen in Fig. 7f.

We now further compare RM with SO under the three different inconvenience cost settings, where the curbside parking pricing remains the same for RM and SO under the same inconvenience cost setting. We have a few main observations. Firstly, as can be seen in Fig. 7b, the RM shared parking prices are higher than the SO shared parking prices since the RM operator aims to maximize revenue. When δ_k increases, the RM shared parking price increases. This is because, the curbside parking price is set based on the marginal cost, which is higher under a larger δ_k (as shown in Fig. 7a). The RM shared parking prices increase when curbside parking prices are higher. Secondly, the RM shared parking rents are lower than the SO shared parking rents. This is again due to the revenue-maximizing behavior of the operator. When δ_k increases, the rents decrease slightly. This means, RM operator reduces shared supply to save rents, which yields a higher cost of choosing curbside parking for users, and the RM operator can set a higher shared parking price accordingly. This is consistent with the parking flow observations in Fig. 7d and Fig. 7e, where the RM regime yields larger curbside parking flows and smaller shared parking flows than SO (given the same inconvenience cost distribution). Thirdly, when δ_k increases, the RM shared parking flow decreases, i.e., when shared parking is more costly, RM operator reduces shared parking supplies, as discussed earlier. Fourthly, Fig. 7f shows that when δ_k increases, more flows have a longer walking distance under both RM and SO. Moreover, RM yields a longer distance for more users than SO given the same inconvenience cost distribution.

We now further compare the efficiency metrics for RM and SO in Table 6 with varying inconvenience cost distributions. Firstly, it is obvious that given the same inconvenience cost setting, the RM yields a positive net revenue while the SO yields a negative revenue (refer to Π); the RM yields a larger total social cost (refer to TSC) and a larger total user cost (refer to TC) than SO; and the RM yields less shared parking flows (see d_a/d) and less net benefit for parking sharers (see π) than SO. Secondly, when δ_k increases, the net revenues of the platform at both RM and SO decrease. This is also linked to a decreasing shared parking flow, a decreasing net benefit of parking sharers, and an increasing total curbside parking fee collected, and an increasing total social cost and an increasing total user cost. These observations are consistent with those in Fig. 7. Thirdly, when δ_k increases, the compensation to persuade the RM operator to operate at SO decreases (i.e., 5.022-(-0.153)>2.854-(-0.300)>2.012-(-0.299)). It is because that when δ_k increases, the parking sharing platform is less profitable under the RM, and the compensation needed is smaller.

6. Conclusion and discussion

6.1. Conclusion

This paper formulates the parking problem in urban cities with a hybrid supply of curbside parking and shared parking, and examines the pricing strategies of the parking sharing platform operator. Particularly, three groups of decision-makers are modeled in the two-sided market for parking sharing, i.e., the travelers who need a parking space; the residents or other parking lot owners who can supply a private parking space; and the operator who manages the parking sharing platform. The interactions among the three parties jointly determine the user parking choice equilibrium and sharer (owner) parking supply equilibrium.

We find that the parking choice equilibrium can be formulated as a minimization problem and its solution might not be unique unless we assume the weakly preference over parking closer to city center, i.e., with a shorter walking distance (refer to Assumption 3). However, without this assumption, even if the equilibrium parking flow at the shared parking side might not be unique, the parking flow at the curbside parking side, and the overall split between curbside and shared parking usage can be uniquely determined. A bi-section based approach is proposed to solve the parking choice equilibrium.

We then further analyze the pricing strategies of the parking sharing platform operator under revenue-maximizing and social-cost-minimizing objectives. We find that in both cases, shared parking supply should be fully utilized. Moreover, we found that a private revenue-maximizing operator tends to rent less shared parking and set a higher shared parking price, while a public social-cost-minimizing operator tends to rent more shared parking to avoid costly cruising for parking and thus reduce total social cost. To persuade a private revenue-maximizing operator to operate under the SO shared parking rents and prices, the authorities have to subsidize the operator. Alternatively, the local authority may set quantitative regulations regarding shared parking prices, rents and/or supplies such that the system may be closer to SO. We also discuss how a number of variations, including curbside parking pricing and inconvenience cost distribution, could affect the profitability of the parking sharing platform.

6.2. Discussion

(**Elastic Demand**) This paper considers that the total parking demand d is fixed. In the long run, the parking demand may react to traffic and parking supply conditions. In this case, we may consider total parking demand to be elastic. This can be achieved by introducing parking demand as a function of travel cost, i.e., $d = d(C)$, where C is the travel cost of driving. It is expected that $d(\cdot)$ will be decreasing, i.e., demand decreases with cost. In this case, the above demand function has to be combined with the parking choice equilibrium model in this paper in order to establish the equilibrium with elastic demand.

While some properties of the parking choice equilibrium in this paper may still hold, the demand elasticity will introduce several points to be carefully considered. Firstly, the sensitivity of demand with respect to cost governs how parking pricing would affect demand, which can substantially affect the profitability of the parking sharing platform. If a shared parking price increase results in sharper demand reduction, the optimal shared parking price might be lower. Secondly, after introducing the elastic parking demand, to achieve the System Optimum, we need to maximize the social welfare rather than minimizing total social cost. In this context, an optimal parking demand level will be established. It is also noteworthy that road congestion has to be incorporated to fully capture the impacts of the demand, which depends on the parking demand level.

(**Endogenous boundary for parking**) This study considers that there are K locations for parking around the city center, of which the curbside parking capacities are sufficient to serve the total parking demand (Assumption 1). An underlying expectation is that the K locations have acceptable walking distances for travelers. When we consider elastic demand, we do not have to restrict the K locations to be within acceptable walking distance to the city center. Moreover, it is not necessary to have Assumption 1. We may include as many locations as possible in the city, and solving the elastic demand parking choice equilibrium will endogenously determine the furthest parking location from the final destination that is used by some travelers, i.e, the boundary of parking near the city center.

(**Heterogeneous travelers**) This paper consider homogeneous travelers or parking users, i.e., cost formulations and preferences of travelers are identical. In practice, travelers are generally heterogeneous. For example, they may have different values of driving time. Also, travelers may value walking time cost differently. This indeed can be further incorporated into the modeling framework in the current paper. How to optimally match heterogeneous travelers to spatially differentiated parking supplies and parking types (curbside or shared) to achieve either the revenue maximization or social cost minimization will be an interesting question to be investigated. Auction-based mechanism may be needed to help heterogeneous traveler to truthfully report their preferences and valuations (Liu et al., 2014b).

Besides the above points, this paper can be further extended in several other ways. Firstly, in a future study, a general parking network combined with a road network such as those in Boyles et al. (2015) can be adopted, where road congestion and interactions between vehicles cruising for parking and vehicles passing by can be captured. Following this, the spillover effect of the emerging parking sharing business on the traffic congestion with induced demand may also considered in future studies. Secondly, the current study does not consider other alternatives for travelers. Future research can integrate the parking sharing problem into the multimodal system, especially when the transit service is responsive to roadway conditions (Zhang et al., 2014; Zhang et al., 2016; Wang et al., 2017; Zhang et al., 2018). Thirdly, the current study considers the decision of a monopolistic parking sharing platform operator. The behavior of multiple operators where competition exists is worthy of further investigation.

CRediT authorship contribution statement

Fangni Zhang: Conceptualization, Formal analysis, Investigation, Methodology, Writing - original draft. **Wei Liu:** Conceptualization, Formal analysis, Funding acquisition, Investigation, Methodology, Project administration, Writing - review & editing. **Xiaolei Wang:** Conceptualization, Funding acquisition, Methodology, Writing - review & editing. **Hai Yang:** Conceptualization, Funding acquisition, Writing - review & editing.

Acknowledgment

We would like to thank the anonymous referees for their constructive comments, which have helped improve both the technical quality and exposition of this paper substantially. This research was partly supported by the National Natural Science Foundation of China (No. 71974146), and the Australian Research Council (DE200101793), and the Research Grants Council of Hong Kong (HKUST16206317).

References

Anderson, S.P., De Palma, A., 2004. The economics of pricing parking. J. Urban Econ. 55 (1), 1–20.
Arnott, R., De Palma, A., Lindsey, R., 1991. A temporal and spatial equilibrium analysis of commuter parking. J. Public Econ. 45 (3), 301–335.
Arnott, R., Inci, E., 2006. An integrated model of downtown parking and traffic congestion. J. Urban Econ. 60 (3), 418–442.
Arnott, R., Inci, E., Rowse, J., 2015. Downtown curbside parking capacity. J. Urban Econ. 86, 83–97.
Arnott, R., Williams, P., 2017. Cruising for parking around a circle. Transport. Res. Part B: Methodol. 104, 357–375.
Ayala, D., Wolfson, O., Xu, B., Dasgupta, B., Lin, J., 2011. Parking slot assignment games. In: Proceedings of the 19th ACM SIGSPATIAL International Conference on Advances in Geographic Information Systems. ACM, pp. 299–308.
Bazaraa, M.S., Sherali, H.D., Shetty, C., 1993. Nonlinear Programming: Theory and Algorithms. John Wiley & Sons.
Beckmann, M., McGuire, C.B., Winsten, C.B., 1956. Studies in the Economics of Transportation. Yale University Press, New Haven, CT.
Boyles, S.D., Tang, S., Unnikrishnan, A., 2015. Parking search equilibrium on a network. Transport. Res. Part B: Methodol. 81, 390–409.
Chen, X.M., Zahiri, M., Zhang, S., 2017. Understanding ridesplitting behavior of on-demand ride services: An ensemble learning approach. Transport. Res. Part C: Emerg. Technol. 76, 51–70.
Chen, Z., Xu, Z., Zangui, M., Yin, Y., 2016. Analysis of advanced management of curbside parking. Transport. Res. Rec.: J. Transport. Res. Board 2567, 57–66.
Chen, Z., Yin, Y., He, F., Lin, J.L., 2015. Parking reservation for managing downtown curbside parking. Transport. Res. Rec.: J. Transport. Res. Board 2498, 12–18.
Du, Y., Yu, S., Meng, Q., Jiang, S., 2019. Allocation of street parking facilities in a capacitated network with equilibrium constraints on drivers' traveling and cruising for parking. Transport. Res. Part C: Emerg. Technol. 101, 181–207.
Fukushima, M., 1984. A modified frank-wolfe algorithm for solving the traffic assignment problem. Transport. Res. Part B: Methodol. 18 (2), 169–177.
Guo, W., Zhang, Y., Xu, M., Zhang, Z., Li, L., 2016. Parking spaces repurchase strategy design via simulation optimization. J. Intell. Transport. Syst. 20 (3), 255–269.
He, F., Yin, Y., Chen, Z., Zhou, J., 2015. Pricing of parking games with atomic players. Transport. Res. Part B: Methodol. 73, 1–12.
Hearn, D., 1980. Bounding flows in traffic assignment models. Res. Report 80–84.
Inci, E., 2015. A review of the economics of parking. Econ. Transport. 4 (1), 50–63.
Larsson, T., Patriksson, M., 1995. An augmented lagrangean dual algorithm for link capacity side constrained traffic assignment problems. Transport. Res. Part B: Methodol. 29 (6), 433–455.
Leclercq, L., Sénécat, A., Mariotte, G., 2017. Dynamic macroscopic simulation of on-street parking search: A trip-based approach. Transport. Res. Part B: Methodol. 101, 268–282.
Liu, T.-L., Huang, H.-J., Yang, H., Zhang, X., 2009. Continuum modeling of park-and-ride services in a linear monocentric city with deterministic mode choice. Transport. Res. Part B: Methodol. 43 (6), 692–707.
Liu, W., 2018. An equilibrium analysis of commuter parking in the era of autonomous vehicles. Transport. Res. Part C: Emerg. Technol. 92, 191–207.
Liu, W., Geroliminis, N., 2016. Modeling the morning commute for urban networks with cruising-for-parking: an MFD approach. Transport. Res. Part B: Methodol. 93, 470–494.
Liu, W., Geroliminis, N., 2017. Doubly dynamics for multi-modal networks with park-and-ride and adaptive pricing. Transport. Res. Part B: Methodol. 102, 162–179.
Liu, W., Yang, H., Yin, Y., 2014a. Expirable parking reservations for managing morning commute with parking space constraints. Transport. Res. Part C: Emerg. Technol. 44, 185–201.
Liu, W., Yang, H., Yin, Y., Zhang, F., 2014b. A novel permit scheme for managing parking competition and bottleneck congestion. Transport. Res. Part C: Emerg. Technol. 44, 265–281.
Liu, W., Zhang, F., Yang, H., 2016. Managing morning commute with parking space constraints in the case of a bi-modal many-to-one network. Transportmetrica A: Transport Sci. 12 (2), 116–141.
Liu, Y., Li, Y., 2017. Pricing scheme design of ridesharing program in morning commute problem. Transport. Res. Part C: Emerg. Technol. 79, 156–177.
Nguyen, S., 1974. An algorithm for the traffic assignment problem. Transport. Sci. 8 (3), 203–216.
Nie, Y., Zhang, H., Lee, D.-H., 2004. Models and algorithms for the traffic assignment problem with link capacity constraints. Transport. Res. Part B: Methodol. 38 (4), 285–312.
Pi, X., Ma, W., Qian, Z.S., 2019. A general formulation for multi-modal dynamic traffic assignment considering multi-class vehicles, public transit and parking. Transport. Res. Part C: Emerg. Technol. 104, 369–389.
Qian, Z.S., Rajagopal, R., 2014. Optimal dynamic parking pricing for morning commute considering expected cruising time. Transport. Res. Part C: Emerg. Technol. 48, 468–490.
Qian, Z.S., Rajagopal, R., 2015. Optimal dynamic pricing for morning commute parking. Transportmetrica A: Transport Sci. 11 (4), 291–316.
Qian, Z.S., Xiao, F.E., Zhang, H., 2012. Managing morning commute traffic with parking. Transport. Res. Part B: Methodol. 46 (7), 894–916.
Ryu, S., Chen, A., Xu, X., Choi, K., 2014. A dual approach for solving the combined distribution and assignment problem with link capacity constraints. Networks Spatial Econ. 14 (2), 245–270.
Shao, C., Yang, H., Zhang, Y., Ke, J., 2016. A simple reservation and allocation model of shared parking lots. Transport. Res. Part C: Emerg. Technol. 71, 303–312.
Shoup, D.C., 2006. Cruising for parking. Transp. Policy 13 (6), 479–486.
Su, Q., Wang, D.Z., 2019. Morning commute problem with supply management considering parking and ride-sourcing. Transport. Res. Part C: Emerg. Technol. 105, 626–647.
Tong, C., Wong, S., 2000. A predictive dynamic traffic assignment model in congested capacity-constrained road networks. Transport. Res. Part B: Methodol. 34 (8),

625–644.
Van Ommeren, J.N., Wentink, D., Rietveld, P., 2012. Empirical evidence on cruising for parking. Transport. Res. Part A: Policy Practice 46 (1), 123–130.
Wang, J., Wang, H., Zhang, X., 2020. A hybrid management scheme with parking pricing and parking permit for a many-to-one park and ride network. Transport. Res. Part C: Emerg. Technol. 112, 153–179.
Wang, J.Y., Yang, H., Lindsey, R., 2004. Locating and pricing park-and-ride facilities in a linear monocentric city with deterministic mode choice. Transport. Res. Part B: Methodol. 38 (8), 709–731.
Wang, W.W., Wang, D.Z., Zhang, F., Sun, H., Zhang, W., Wu, J., 2017. Overcoming the downs-thomson paradox by transit subsidy policies. Transport. Res. Part A: Policy Practice 95, 126–147.
Wang, X., He, F., Yang, H., Gao, H.O., 2016. Pricing strategies for a taxi-hailing platform. Transport. Res. Part E: Logist. Transport. Rev. 93, 212–231.
Wang, X., Yang, H., Zhu, D., 2018. Driver-rider cost-sharing strategies and equilibria in a ridesharing program. Transport. Sci. 52 (4), 868–881.
Wu, W., Zhang, F., Liu, W., Lodewijks, G., 2020. Modelling the traffic in a mixed network with autonomous-driving expressways and non-autonomous local streets. Transport. Res. Part E: Logist. Transport. Rev. 134, 101855.
Xiao, H., Xu, M., Gao, Z., 2018. Shared parking problem: A novel truthful double auction mechanism approach. Transport. Res. Part B: Methodol. 109, 40–69.
Xiao, L.-L., Liu, T.-L., Huang, H.-J., 2016. On the morning commute problem with carpooling behavior under parking space constraint. Transport. Res. Part B: Methodol. 91, 383–407.
Xu, G., Yang, H., Liu, W., Shi, F., 2018. Itinerary choice and advance ticket booking for high-speed-railway network services. Transport. Res. Part C: Emerg. Technol. 95, 82–104.
Xu, S.X., Cheng, M., Kong, X.T., Yang, H., Huang, G.Q., 2016. Private parking slot sharing. Transport. Res. Part B: Methodol. 93, 596–617.
Xu, Z., Yin, Y., Zha, L., 2017. Optimal parking provision for ride-sourcing services. Transport. Res. Part B: Methodol. 105, 559–578.
Yang, H., 1999. System optimum, stochastic user equilibrium, and optimal link tolls. Transport. Sci. 33 (4), 354–360.
Yang, H., Bell, M.G., 1997. Traffic restraint, road pricing and network equilibrium. Transport. Res. Part B: Methodol. 31 (4), 303–314.
Yang, H., Bell, M.G.H., 1998. Models and algorithms for road network design: a review and some new developments. Transport Rev. 18 (3), 257–278.
Yang, H., Liu, W., Wang, X., Zhang, X.N., 2013. On the morning commute problem with bottleneck congestion and parking space constraints. Transport. Res. Part B: Methodol. 58, 106–118.
Zha, L., Yin, Y., Yang, H., 2016. Economic analysis of ride-sourcing markets. Transport. Res. Part C: Emerg. Technol. 71, 249–266.
Zhang, F., Lindsey, R., Yang, H., 2016. The Downs-Thomson paradox with imperfect mode substitutes and alternative transit administration regimes. Transport. Res. Part B: Methodol. 86, 104–127.
Zhang, F., Liu, W., Wang, X., Yang, H., 2017. A new look at the morning commute with household shared-ride: How does school location play a role? Transport. Res. Part E: Logist. Transport. Rev. 103, 198–217.
Zhang, F., Yang, H., Liu, W., 2014. The Downs-Thomson Paradox with responsive transit service. Transport. Res. Part A: Policy Practice 70, 244–263.
Zhang, F., Zheng, N., Yang, H., Geroliminis, N., 2018. A systematic analysis of multimodal transport systems with road space distribution and responsive bus service. Transport. Res. Part C: Emerg. Technol. 96, 208–230.
Zhang, X., Liu, W., Waller, S.T., Yin, Y., 2019. Modelling and managing the integrated morning-evening commuting and parking patterns under the fully autonomous vehicle environment. Transport. Res. Part B: Methodol. 128, 380–407.
Zhang, X.N., Huang, H.-J., Zhang, H., 2008. Integrated daily commuting patterns and optimal road tolls and parking fees in a linear city. Transport. Res. Part B: Methodol. 42 (1), 38–56.
Zhang, X.N., Yang, H., Huang, H.-J., 2011. Improving travel efficiency by parking permits distribution and trading. Transport. Res. Part B: Methodol. 45 (7), 1018–1034.
Zhang, X.N., Yang, H., Huang, H.-J., Zhang, H.M., 2005. Integrated scheduling of daily work activities and morning–evening commutes with bottleneck congestion. Transport. Res. Part A: Policy Practice 39 (1), 41–60.
Zheng, N., Geroliminis, N., 2016. Modeling and optimization of multimodal urban networks with limited parking and dynamic pricing. Transport. Res. Part B: Methodol. 83, 36–58.