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Battery-electric transit vehicle scheduling with optimal number of stationary chargers

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ABSTRACT

Because of zero emissions and other social and economic benefits, electric vehicles (EVs) are currently being introduced in more and more transit agencies around the world. One of the most challenging tasks involves efficiently scheduling a set of EVs considering the limited driving range and charging requirement constraints. This study examines the battery-electric transit vehicle scheduling problem (BET-VSP) with stationary battery chargers installed at transit terminal stations. Two equivalent versions of mathematical formulations of the problem are provided. The first formulation is based on the deficit function theory, and the second formulation is an equivalent bi-objective integer programming model. The first objective of the math-programming optimization is to minimize the total number of EVs required, while the second objective is to minimize the total number of battery chargers required. To solve this bi-objective BET-VSP, two solution methods are developed. First, a lexicographic method-based two-stage construction-and-optimization solution procedure is proposed. Second, an adjusted max-flow solution method is developed. Three numerical examples are used as an expository device to illustrate the solution methods, together with a real-life case study in Singapore. The results demonstrate that the proposed math-programming models and solution methods are effective and have the potential to be applied in solving large-scale real-world BET-VSPs.

1. Introduction

1.1. Background and motivation

New technologies are emerging and becoming part of our lives at an increased pace. The smartphone revolution, and the evolution of motor vehicles and new energy technologies have created a technological breakaway from traditional procedures and thinking. This affects all aspects of daily life, including public-transit (PT) service planning and operations (Häll et al., 2019). Nowadays, electric vehicles (EVs) are used by an increased number of PT agencies because of their reduced emissions and other social and economic benefits. In 2016, the total number of battery-powered electric buses in use around the world was about 345,000, an increase of about 100% from 2015 (International Energy Agency, 2017). In recent years, we have observed an increased number of

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Table 1
List of the costs of some available electric transit vehicles and associated fast battery chargers.

Company	Ebus USA	Proterra USA	Hengtong China
Vehicle	$295 K	$1.2 M	¥1.7 M
Battery charger	$58 K	$1M	¥500-600 K

Data source: Li (2016).

EV-related research—for example: EV charging infrastructure planning and design (He et al., 2013; Mak et al., 2013; Nie and Ghamami, 2013; Liu and Song, 2017; Liu and Wang, 2017; Chen et al., 2017, 2018; Xylia et al., 2017; He et al., 2018; Wei et al., 2018; An et al., 2019; Lin et al., 2019; Wang et al., 2019; An 2020), battery EV scheduling (Li, 2013; Wang et al., 2017; Rogge et al., 2018), EV battery charging and purchasing optimization (Schneider et al., 2017), and EV-based car-sharing system optimization (Boyacı et al., 2017; Brendel et al., 2018; Xu et al., 2018). One of the most interesting and challenging problems is the battery-electric transit/PT vehicle scheduling problem (BET-VSP). To that end, Table 1 lists the high costs of some known electric transit vehicles and associated battery chargers. To achieve the efficient use of EVs, the main task of PT schedulers is to minimize (i) the total number of electric transit vehicles and (ii) the total number of battery chargers required to perform a given set of scheduled services.

Currently, there are mainly three charging strategies for PT vehicles: (i) battery swapping, (ii) regular charging, and (iii) fast charging (Li, 2016). In terms of charging policies, there are two types: full and partial (Montoya et al., 2017). In this study we consider the popular use of the fast charging strategy, coupled with both the full and partial charging policies, for the BET-VSP with stationary battery chargers installed at PT terminal stations. In this way, we are able to take into account the high mileage and low standstill times of PT vehicles.

1.2. Literature review

1.2.1. Literature review on transit vehicle scheduling problem

The transit vehicle scheduling problem (VSP) refers to the problem of determining the optimal allocation of vehicles to carry out all the trips in a given transit timetable. A chain of trips is assigned to each vehicle, although some of them may be deadheading (DH) or empty trips in order to reach optimality. The major objective of the transit VSP is to minimize fleet size or, correspondingly, total operation cost, which includes both fixed and variable costs. The number of feasible solutions to this problem is extremely high, especially in the case in which the vehicles are based at multiple depots. Much of the focus in the literature on scheduling procedures is, therefore, on computational issues. Over the past few years, a rich literature on VSPs has emerged. An early survey by Bodin et al. (1983) provides detailed classifications, mathematical formulations, and solution algorithms of the VSPs. A review of network models and methods for the VSPs can be found in Carraresi and Gallo (1984). Daduna and Paixão (1995) reviewed mathematical models for VSPs in transit systems. Recent detailed reviews on model formulations, solution approaches, and algorithm developments can be found in Desaulniers and Hickman (2007), Bunte and Kliewer (2009), Ibarra-Rojas et al. (2015), Ceder (2016), and Liu and Ceder (2017). For small and medium-size VSPs, some existing optimization techniques can provide exact optimal solutions in a short computation time. For large-scale multi-depot VSPs, due to the computational complexity, heuristic or meta-heuristic algorithms are usually used to solve them with approximated good solutions in an acceptable computation time.

1.2.2. Literature review on electric vehicle scheduling problem

The electric vehicle scheduling problem (E-VSP) is an extension of the VSP by considering the limited driving range and charging requirement of EVs. The VSP with driving range constraints has been addressed in some previous studies—e.g., Freling and Paixão (1995), Haghani and Banihashemi (2002), and Wang and Shen (2007). Desrosiers et al. (1995) provides an earlier overview of optimization methods for the time-constrained vehicle routing and scheduling problem. Recently, Adler and Mirchandani (2016) studied a similar problem by considering the possibility of allowing vehicles to refuel at some places, which is similar to allowing vehicles to recharge at some charging stations. Zhu and Chen (2013) proposed a Non-dominated Sorting Genetic Algorithm (NSGA-II) to optimize battery electric bus transit vehicle scheduling with battery exchanging. Wen et al. (2016) proposed a mixed integer programming (MIP) model for the E-VSP with the aim of minimizing both the total number of vehicles required and the total travel distance. An adaptive large neighborhood search heuristic algorithm was developed to solve the problem. Wang et al. (2017) developed a mixed integer linear programming model for the electric bus recharging scheduling problem with the objective of minimizing the annual total operating costs of the electric bus recharging system. The math-programming model was solved using CPLEX. van Kooten Niekerk et al. (2017) developed two math-programming models for the E-VSP based on different charging processes. For small and medium-size problems, the two models can be solved using commercial optimization solvers. For large-size problems, two solution methods that are based on column generation and Lagrangian relaxation are developed. Sassi and Oulamara (2017) developed a MIP model considering optimal charging. CPLEX was used to solve small and medium problems, and two heuristics were developed to solve large-size problems. Rogge et al. (2018) studied the EV scheduling fleet size and mix problem with optimization of charging infrastructure. A solution framework that is based on a grouping genetic algorithm, coupled with a mixed-integer charger optimization, was developed to minimize the total cost of EV fleet ownership. Tang et al. (2019) developed both static and dynamic models within a branch-and-price solution framework for the robust scheduling of electric buses under stochastic road traffic conditions. Li et al. (2019) developed an integer linear programing model using time-space network concept for the multi-depot and multi-vehicle-type electric bus scheduling problem considering battery range and refueling constraints. Pelletier et al. (2019) proposed an integer linear programing model for the electric bus fleet transition problem. Some computational experiments based on several scenarios were conducted. Yao et al. (2020) developed a genetic algorithm based heuristic procedure for solving the PT electric vehicle scheduling problem with multiple vehicle types.

1.2.3. Literature review on deficit function theory

The concept of deficit function (DF) was first proposed by Linis and Maksim (1967) for scheduling aircrafts and later formally defined and described by Gertsbach and Gurevich (1977) and Ceder and Stern (1981). Ceder and Stern (1981), Ceder (2002), and Ceder (2016) applied the DF modeling approach for assigning the minimum number of vehicles for a given timetable considering both fixed and variable schedules. Ceder (2016) developed the DF theory further with applications in other PT operation planning activities, such as transit network design, crew scheduling, and vehicle parking management. A DF is a step function associated with a transportation terminal that increases by one at the time of each vehicle trip departure and decreases by one at the time of each vehicle trip arrival. To construct a set of DFs, the only information needed is a timetable of required trips. The main advantage of the DF is its visual nature. For a detailed description of the basic theory and major developments of the DF modeling and applications, over the past 50 years, readers are referred to Liu and Ceder (2017).

1.3. Research gaps, objectives and contributions

The above literature review clearly indicates that the BET-VSP is a new and important research problem from theoretical and practical perspectives. However, there are remarkable differences between the BET-VSP defined in our work and the electric bus scheduling problems of previous work. First, the BET-VSP considers stationary chargers installed at transit terminals. Second, in the BET-VSP the objective is not only to minimize the total number of electric transit vehicles required, but also to minimize the total number of battery chargers required to perform a given set of scheduled services. This is important and totally different from previous studies. Thus, the solution methods proposed by others cannot be directly applied to solve the BET-VSP. To bridge these research gaps, our study proposes new and innovative math-formulation and solution methods for the BET-VSP.

The theoretical contributions of this study are fourfold. First, it offers a novel modeling method based on the DF theory, together with a related surplus function (SF), for the BET-VSP with optimal number of stationary chargers; to the best of our knowledge, this is the first time that the DF theory is applied to solve the BET-VSP. Second, we provide two different mathematical formulations of the BET-VSP. The first formulation is based on the DF theory, and the second formulation is based on an equivalent bi-objective integer programming model. Third, a powerful and practical lexicographic method-based two-stage construction-and-optimization solution procedure is developed to solve this BET-VSP. Fourth, an adjusted max-flow solution method is developed to serve as an alternative for the DF-based method. Three numerical examples are used as an expository device to illustrate the solution procedure developed, together with a real-life case study from Singapore. The results demonstrate that the proposed mathematical programming models and solution methods are very effective in solving BET-VSPs.

This work comprises seven sections including this introductory section. Section 2 provides a formal description of the BET-VSP with a small numerical example to explain the problem and the DF-based solution approach. Section 3 presents the DF formulations and the equivalent bi-objective integer programming formulations. The lexicographic method-based two-stage construction-and-optimization solution procedure, along with a detailed illustrative example, is presented in Section 4. In Section 5, the second solution method based on max-flow solution approach is presented with a detailed example. Section 6 presents the results of a real-life case study conducted in Singapore. Section 7 concludes our work and discusses limitations, as well as possible directions for future research.

2. Problem description

2.1. Formal description

Given a set of required vehicle trips $I = \{i: i = 1, \ldots, n\}$ that are conducted between a set of PT terminals $K = \{k: k = 1, \ldots, q\}$, each trip is to be serviced by a single vehicle, and each vehicle is able to service any trip. Each trip i can be represented as a 4-tuple (p^i, t_s^i, q^i, t_e^i), in which the ordered elements denote departure terminal, departure (start) time, arrival terminal, and arrival (end) time, respectively. It is assumed that each trip i lies within a schedule horizon $[T_1, T_2]$—i.e., $T_1 \leqslant t_s^i \leqslant t_e^i \leqslant T_2$. The set of all trips $S = \{(p^i, t_s^i, q^i, t_e^i): p^i, q^i \in K, i \in I\}$ constitutes the timetable. The BET-VSP represents the problem of creating chains of vehicle trips; each chain is referred to as a vehicle schedule according to a given timetable. This chaining process is often called vehicle blocking (a block is a sequence of revenue and non-revenue activities for an individual vehicle). A trip can involve either transporting passengers along the given route or carrying out a DH trip in order to connect two service trips efficiently. The travel times of DH trips between different transit terminals are given. The scheduler's task is to list all the daily chains of vehicle trips, including some DH trips, for each vehicle so as to ensure that operation requirements, such as recharging, maintenance, and parking, are fulfilled.

In this study, it is assumed that PT EVs can only be charged at PT terminal stations at which stationary battery chargers are installed. Both full and partial charging policies are allowed. That is, EVs can be charged at any charging station to obtain any amount of energy up to the full battery capacity. The charging time is a function of the battery energy gained and the charging rate. It is assumed that the charging rate and battery capacity are given. The energy consumptions of all vehicle and DH trips are also assumed to be known. Because of the need to recharge every night, EVs must start from the same terminal daily. This repetition between

Table 2
Trip schedule for Example 1.

Trip number i	Departure terminal p^i	Depature time t_s^i	Arrival terminal q^i	Arrival time t_e^i	Energy consumption c^i
1	*a*	0	*b*	2	4
2	*a*	1	*b*	4	7
3	*b*	4	*a*	5	6
4	*b*	6	*a*	8	5

different scheduling periods calls for a balanced individual vehicle schedule, which is a special case of the balanced vehicle schedule proposed and studied by Stern and Ceder (1983). A vehicle schedule is considered balanced if for every terminal *k*, the number of vehicles starting their schedules from *k* is equal to the number of vehicles ending their schedules at *k* (not necessarily the same vehicles). A vehicle schedule is said to be a balanced individual vehicle schedule if for every vehicle chain/block, the departure terminal is the same as the arrival terminal; otherwise, the schedule is an unbalanced individual vehicle schedule. In practice, PT agencies seek this balanced condition at terminals with overnight parking and charging requirements. This means that the BET-VSP considered in this study can be formulated as special case of the balanced schedule depot constraint VSP, with the consideration of vehicle charging requirement. The major objective, from a cost perspective, of the BET-VSP is to minimize the total number of EVs required. The second objective is to minimize the total number of battery chargers required.

2.2. Illustrative example

Example 1:. A simple numerical example, including two terminals and four vehicle trips, is used to illustrate the BET-VSP. The departure terminal, departure time, arrival terminal, and arrival time, together with the associated energy consumption, of each trip, are given in Table 2. The DH trip travel time and the associated energy consumption are given in Table 3. The vehicle battery capacity is assumed to be 10 units of energy, and the battery-recharging rate is assumed to be one per unit of time. It is assumed that at least one charger is located at terminal *a* so that all vehicles can be fully charged during night. The scheduling horizon is set as [0, 24]. Fig. 1(a) shows the schedule of the four vehicle trips. Without considering the battery capacity limitation, the corresponding DFs of terminal *a* and terminal *b* can be constructed as shown in Fig. 1(b). The maximum value of the DF of terminal *a* is $D(a) = \max d(a, t) = 2$. For terminal *b*, it is $D(b) = \max d(b, t) = 0$. According to the DF theory, a minimum number of $D(a) + D(b) = 2$ vehicles are required to carry out the four vehicle trips. By applying the vehicle chain extraction procedure (Gertsbach and Gurevich, 1977; Liu and Ceder, 2017), two solution sets of vehicle trip chains can be obtained: Solution Set 1, $S_1 = \{[1, 3]; [2, 4]\}$ and Solution Set 2, $S_2 = \{[1, 4]; [2, 3]\}$. Chains [1, 3] and [1, 4] can be carried out by a vehicle without battery recharging because the energy consumption is no more than the battery capacity. Chain [2, 4] can also be carried out by a vehicle if the vehicle can be recharged between the two trips. However, chain [2, 3] cannot be carried out by a vehicle because the total unit of energy consumed is $7 + 6 = 13$ units, which is more than the battery capacity, and because there is no time to recharge the battery between the two trips. Thus, two vehicles are needed to perform trips 2 and 3, separately. Fig. 1(c) shows the SFs, the mirror image of the DFs (Ceder, 2016; Liu and Ceder, 2017), of solution S_1, where V_1 denotes the potential recharging duration of chain [1, 3] and V_2 denotes the potential recharging duration of chain [2, 4], at each terminal. The maximum values of the SFs, i.e., $R(a) = 2$ and $R(b) = 1$, provide information on the upper-bound number of chargers required at each terminal. By minimizing these upper bounds, we can move in the direction of minimizing the number of chargers required. That is, for this example, the scheduler can choose to recharge vehicle V_1 at terminal *a* with a recharging time period of (5, 15] and vehicle V_2 at terminal *a* with recharging time periods of (15, 24] and (0, 1] as well as at terminal *b* with a recharging time period of (4, 6], such that both vehicles can be fully recharged. By doing so, the scheduler can reduce the upper bound of $R(a)$ from 2 to 1, with $R(b)$ kept as 1. These upper bounds cannot be reduced any further due to the recharging constraints. Therefore, Solution Set 1 results in a solution S_1 with a minimum number of two vehicles and two chargers required—that is, one vehicle for chain $[R_a, 1, 3]$ and the other vehicle for chain $[R_a, 2, R_b, 4]$, where R_a and R_b represent the vehicles being at standstill for recharging at terminal *a* and terminal *b*, respectively. Using the same optimization process, Solution Set 2 results in a solution S_2 with a minimum number of three vehicles and only one charger required—that is, one vehicle for chain $[R_a, 1, 4]$, the second vehicle for chain $[R_a, 2, DH_{ba}]$, and the third vehicle for chain $[R_a, DH_{ab}, 3]$, where DH_{ba} and DH_{ab} represent a DH trip from terminal *b* to terminal *a* and a DH trip from terminal *a* to terminal *b*, respectively. Thus, following the consideration of battery capacity and the recharging, the two updated feasible solution sets are: $S_1 = \{[R_a, 1, 3]; [R_a, 2, R_b, 4]\}$ and $S_2 = \{[R_a, 1, 4]; [R_a, 2, DH_{ba}]; [R_a, DH_{ab}, 3]\}$. These two solutions are graphically displayed in a two-dimensional (2D) space in Fig. 1(b), to show a

Table 3
DH trip travel times (DH trip energy consumption) for Example 1.

	Arrival terminal	
Departure terminal	*a*	*b*
a	0 (0)	2 (3)
b	1 (2)	0 (0)

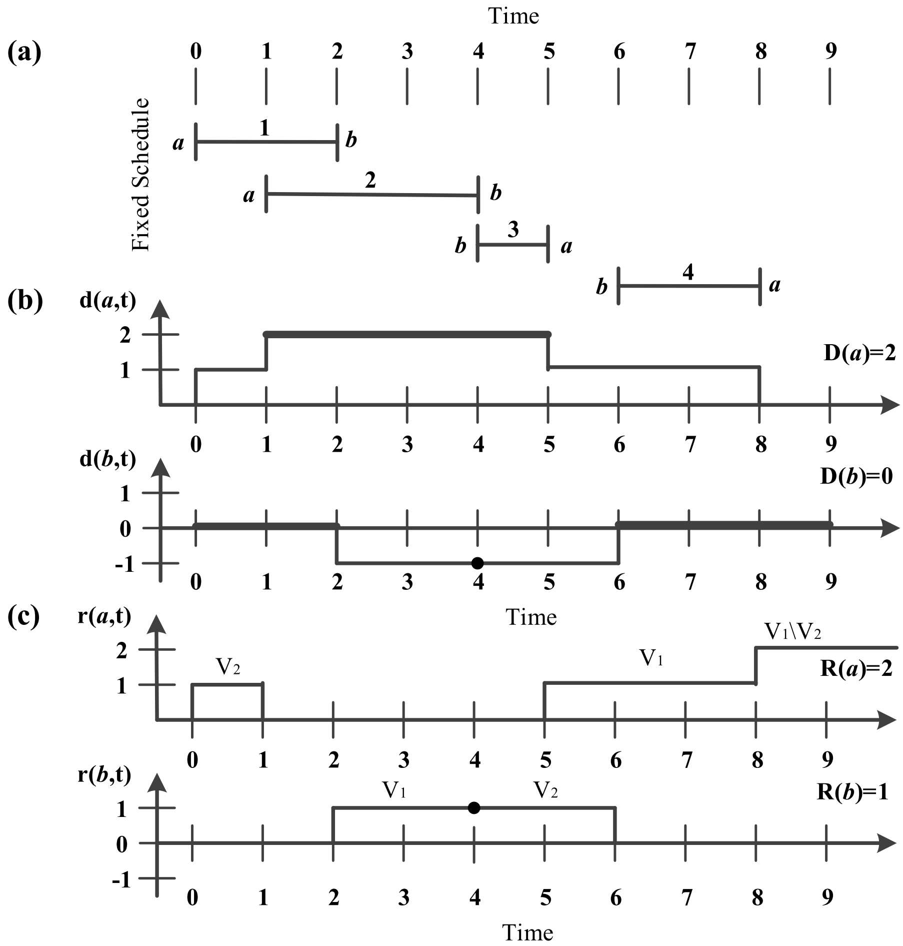

Fig. 1. Example 1: (a) vehicle trip schedule; (b) DFs of the example problem; (c) SFs of solution S_1 of the example problem.

trade-off between the number of vehicles required N_v and the number of chargers required N_c.

If the vehicle cost and charger cost are known, the decision-maker can easily make a final choice from these Pareto-efficient solutions. For example, assume that the electric vehicle and battery charger produced by Ebus USA are used for this sample problem, with their costs shown in Table 1. Based on this cost information, the total cost of solution S_1 is $C(S_1) = 2 \times 295 + 2 \times 58 = \$706$ K. For solution S_2, the total cost is $C(S_2) = 3 \times 295 + 1 \times 58 = \$943$ K. Thus, S_1 is preferable to S_2 unless other non-quantitative constraints/considerations are taken into account.

From Fig. 2, we can further observe that $d(a, t) = 0$ and $d(b, t) = 0$, when $t \geq 8$. According to Stern and Ceder (1983), this implies that a balanced vehicle schedule can be created for this sample problem. The solution set $S_1 = \{[R_a, 1, 3]; [R_a, 2, R_b, 4]\}$ proves this

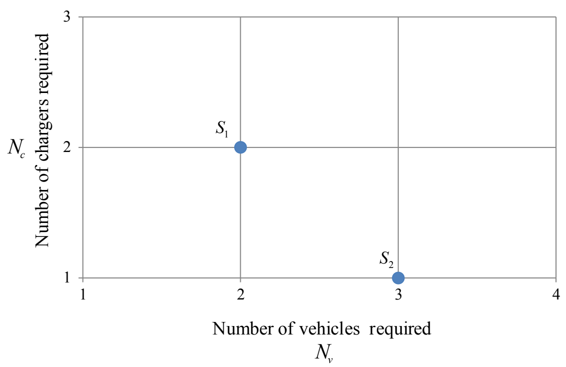

Fig. 2. Pareto-efficient solutions of the example problem displayed in a 2D space.

implication.

3. Mathematical formulations

In this section, two versions of mathematical formulations of the BET-VSP are provided. The first version is based on the DF theory; the second version is an equivalent bi-objective integer programming model.

3.1. DF-based mathematical formulations

Let d(k,t,S) denote the DF for terminal k at time t for schedule S. The value of d(k,t,S) represents the total number of departures minus the total number of trip arrivals at terminal k, up to and including time t. The maximum value of d(k,t,S) over the schedule horizon [T_1, T_2], designated D(k,S), depicts the deficit number of vehicles required at k. Let r(k,t,S) denote the SF for terminal k at time t for schedule S. The value of r(k,t,S) represents the possible total number of chargers required at terminal k, up to and including time t. The maximum value of r(k,t,S) over the schedule horizon [T_1, T_2], designated R(k,S), gives an upper bound of the total number of chargers required. Note that S will be deleted when it is clear which underlying schedule is being considered. The relationship between the DF and SF of a terminal k can be described as follows:

$$r(k, t) = D(k) - d(k, t) \tag{1}$$

Two vehicle trips i, j may be serviced sequentially (feasibly joined) by the same vehicle if and only if:

$$t_e^i \leqslant t_s^j \tag{2}$$

$$q^i = p^j \tag{3}$$

$$B_e^i - B^j \geqslant 0 \tag{4}$$

where B_e^i denotes the remaining battery energy at the end of vehicle trip i; B^j is the battery energy required for vehicle trip j.

If trip i is feasibly joined to trip j, then trip i is said to be the predecessor of trip j, and trip j the successor of trip i. A sequence of trips i_1, i_2,…, i_w ordered in such a way that each adjacent pair of trips satisfies (2), (3), and (4) is called a vehicle chain or block. It follows that a vehicle chain is a set of trips that can be serviced by a single vehicle. A set of chains in which each trip i is included in I exactly once is said to constitute a vehicle schedule. The problem of finding the minimum number of chains for a fixed schedule S is known as the minimum fleet-size problem (Levin, 1971; Ceder, 2016).

Let us define a DH trip as an empty trip from some terminal p to some terminal q in time $\tau(p,q)$. If introducing DH trips into the schedule is permitted, then conditions (2), (3), and (4) for the feasible joining of two trips, i, j, may be replaced by the following:

$$t_e^i + \tau(q^i, p^j) \leqslant t_s^j \tag{5}$$

$$B_e^i - B(q^i, p^j) + B^{ij} - B^j \geqslant 0 \tag{6}$$

where $B(q^i, p^j)$ is the energy consumption of DH trip from the end of trip i to the beginning of trip j; B^{ij} is the energy gained from battery recharging during the time period between trip i and trip j.

Proposition 1.. The vehicle schedule S of the BET-VSP is a balanced vehicle schedule—that is, the value of each DF at the end of the schedule horizon is zero:

$$d(k, T_2) = 0, \;\; k \in K \tag{7}$$

Proof.. According to the definition of the BET-VSP, the departure terminal of each vehicle chain is the same as the arrival terminal. That is, the number of vehicles starting their schedules from k is equal to the number of vehicles ending their schedules at$k$. According to the definition of $d(k, T_2)$, it is therefore obvious that $d(k, T_2) = 0$.□

For a given vehicle schedule S, if $d(k, T_2) = 0$, then this schedule is a balanced vehicle schedule. However, it may not be a balanced individual vehicle schedule S', as that is an infeasible solution to the BET-VSP. To change it into a balanced individual vehicle schedule S', one can add new DH trips into the schedule or rearrange the existing DH trips. The relationships between the value of DF and balanced vehicle schedule/balanced individual vehicle schedule are graphically illustrated in Fig. 3. For details

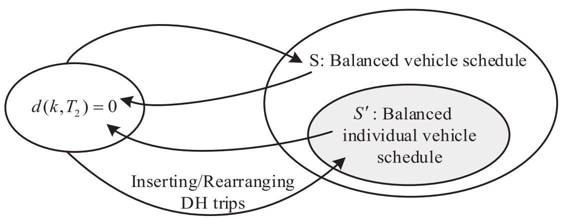

Fig. 3. Relationships between the value of DF and balanced vehicle schedule/balanced individual vehicle schedule.

regarding inserting/rearranging DH trips, readers are referred to Ceder and Stern (1981), Stern and Ceder (1983), and Ceder (2016).

Thus, by inserting as many DH trips as necessary into the schedule of required trips, the DF-based formulation of the BET-VSP is as follows:

$$(\mathbf{P1})\quad Min\ Z_1 = \sum_{k\in K} D(k) = \sum_{k\in K} \underset{t\in[T_1,T_2]}{Max}\ d(k,t) \tag{8}$$

$$Min\ Z_2 = \sum_{k\in K} R(k) = \sum_{k\in K} \underset{t\in[T_1,T_2]}{Max}\ r(k,t) \tag{9}$$

s.t.

$$d(k,T_2) = 0,\ \ k\in K \tag{10}$$

$$D(k) \leqslant Q(k),\ \ k\in K \tag{11}$$

$$R(k) \leqslant C(k),\ \ k\in K \tag{12}$$

where $Q(k)$ denotes the maximum parking facilities available at terminal k; $C(k)$ is the maximum number of chargers at terminal k. The objective (8) is to minimize the total number of vehicles. The objective (9) is to minimize the total number of chargers. Constraints (10) guarantee that the schedule is a balanced vehicle schedule. If it is not a balanced individual vehicle schedule, DH trips are inserted so as to transform it into one. Constraints (11) ensure that the terminal parking capacity is satisfied. Constraints (12) guarantee that the number of chargers required is no more than the maximum number of battery chargers at terminal k.

3.2. Bi-objective integer programming model

The BET-VSP can also be formulated as a bi-objective integer programming model in the form of maximum network flow problems (Ahuja et al., 1993). Given a set of required vehicle trips I and a set of required vehicle charging events R, a directed graph $G = (N, A)$ can be constructed. The set of nodes N is defined as the set of required vehicle trips and charging events. The set of arcs is defined as the set of arcs connecting vehicle trips A^I and arcs connecting charging events A^R, i.e., $A{:} = A^I \cup A^R$. A trip-joining array for the BET-VSP may be constructed by associating the i^{th} row with the arrival event of the i^{th} trip and the j^{th} column with the departure event of the j^{th} trip. Cell (i,j) will be admissible if i and j can feasibly be joined. Otherwise, cell (i,j) will be an inadmissible cell. Let x_{ij} be a 0–1 variable associated with cell (i,j). If vehicle trips i and j are joined, then $x_{ij} = 1$; otherwise, $x_{ij} = 0$. By doing the same, a 0–1 variable y_{lm} can be defined to describe the connection between different charging events. That is, if vehicle charging events l and m are connected within the same vehicle chain, then $y_{lm} = 1$; otherwise, $y_{lm} = 0$.

In order to include the balanced individual vehicle schedule constraint, a number of dummy overnight trips may be defined that travel backwards in time. For each terminal k, let $n + k$ represent an undisclosed number of trips of the form $(n+k,\ t_s^{n+k},\ n+k,\ t_e^{n+k})$, such that $t_e^{n+k} \leqslant T_1 < T_2 \leqslant t_s^{n+k}$. The times t_e^{n+k} and t_s^{n+k} represent the latest time at which a vehicle can arrive at depot k after servicing its last required trip and the earliest time at which a vehicle can leave depot k to service its first required trip, respectively. Such times may be stipulated in a driver–union contract. The following additional variables are now introduced:

$z_{n+k,j} = 1$ if a vehicle departs for depot k (after completing dummy trip $n + k$) to service j, its first required trip; otherwise, equals 0.

$z_{i,n+k} = 1$ if a vehicle services its last required trip as trip i before arriving at depot k to park for the night (start dummy trip $n + k$); otherwise equals 0.

q_{kk} = an integer variable whose value represents the number of unused spaces at depot k ($0 \le q_{kk} \le Q(k)$, $k \in K$).

Let the travel time of the morning and evening DH trips from and to depot k be defined as $(n + k, p^j)$ and $(q^i, n + k)$, respectively. The joining $(n + k, j)$ will be considered admissible if $t_e^{n+k} + \tau(n+k,\ p^j) \leqslant t_s^{n+k}$ for $j \in I,\ \ k\in K$. Otherwise, joining $(n + k, j)$ will be inadmissible, and $z_{n+k,j} = 0$. The joining $(i, n + k)$ is admissible if $t_e^i + \tau(q^i,\ n+k) \leqslant t_s^{n+k}$ for $i \in I,\ \ k \in K$. Otherwise, joining $(i, n + k)$ is inadmissible, and $z_{i,n+k} = 0$. Note that joinings $(n + k, j)$ and $(i, n + k)$ are always admissible if $p^j = k$ and $q^i = k$, respectively. Let A^I represent the set of admissible joinings of the above type vehicle service trips in addition to those between required trips (i,j) as described earlier, and A^R represents the set of admissible joinings of vehicle charging events. The mathematical programming version of this BET-VSP with DH trips insertion may now be described as follows:

$$(\mathbf{P2})\quad Max\ Z_3 = \sum_{i\in I}\sum_{j\in I} x_{ij} \tag{13}$$

$$Max\ Z_4 = \sum_{l\in R}\sum_{m\in R} y_{lm} \tag{14}$$

s.t.

$$\sum_{j\in I} x_{ij} + \sum_{k\in K} z_{i,n+k} = 1,\ i\in I \tag{15}$$

$$\sum_{i \in I} x_{ij} + \sum_{k \in K} z_{n+k,j} = 1, j \in I \tag{16}$$

$$\sum_{j \in I} z_{n+k,j} + q_{kk} = Q(k), k \in K \tag{17}$$

$$\sum_{i \in I} z_{i,n+k} + q_{kk} = Q(k), k \in K \tag{18}$$

$$\sum_{m \in R} y_{lm} \leqslant 1, l \in R \tag{19}$$

$$\sum_{l \in R} y_{lm} \leqslant 1, m \in R \tag{20}$$

$$q_{kk} \in \{0, 1, ..., Q(k)\}, \ \ k \in K \tag{21}$$

$$x_{ij} \in \{0, 1\}, (i, j) \in A^I \tag{22}$$

$$x_{ij} = 0, (i, j) \notin A^I \tag{23}$$

$$y_{lm} \in \{0, 1\}, (l, m) \in A^R \tag{24}$$

$$y_{lm} = 0, (l, m) \notin A^R \tag{25}$$

$$z_{i,n+k} \in \{0, 1\}, (i, n + k) \in A^I \tag{26}$$

$$z_{i,n+k} = 0, (i, n + k) \notin A^I \tag{27}$$

$$z_{n+k,j} \in \{0, 1\}, (n + k, j) \in A^I \tag{28}$$

$$z_{n+k,j} = 0, (n + k, j) \notin A^I \tag{29}$$

The objective functions (13) and (14) maximize the number of joinings among vehicle service trips and vehicle charging events, which are equivalent to minimizing the total number of vehicles and the total number of required chargers, respectively. Constraint (15) ensures that each required trip i is joined to exactly one successor trip. The successor trip may be either another required trip j or a dummy trip n + k. The latter implies that trip i is the last trip serviced by a vehicle before returning to depot k. Constraint (16) ensures that each required trip j is joined to exactly one predecessor trip. The predecessor trip may be either a required trip i or a dummy trip n + k. If joined to the latter trip, j is the first trip serviced by a vehicle departing from depot k. Constraints (17) and (18) are the depot parking space constraints. Constraint (19) ensures that each vehicle charging event is joined to at most one successor vehicle charging event. Similarly, constraint (20) indicates that each vehicle charging event may be joined to at most one predecessor vehicle charging event. Constraints (21)-(29) are the decision variable constraints.

Theorem 1.. P2 is equivalent to P1. The P2 solution minimizes the total number of vehicles required and the total number of chargers required, subject to depot capacity and balanced individual vehicle schedule constraints.

Proof. (*(i)*). Summing up Eq. (16) for all j, we can get

$$Max \sum_{j \in I} \sum_{i \in I} x_{ij} = n - Max \sum_{j \in I} \sum_{k \in K} z_{n+k,j}. \tag{30}$$

According to the definition of $z_{n+k,j}$ and definition of DF, we can get

$$Max \sum_{j \in I} \sum_{k \in K} z_{n+k,j} = \sum_{k \in K} \underset{t \in [T_1, T_2]}{Max} d(k, t). \tag{31}$$

Thus, we have

$$Min\ Z_1 = \sum_{k \in K} D(k) = \sum_{k \in K} \underset{t \in [T_1, T_2]}{Max} d(k, t) = Max \sum_{j \in I} \sum_{k \in K} z_{n+k,j} = n - Max \sum_{j \in I} \sum_{i \in I} x_{ij} \tag{32}$$

This shows that objective (13) is equivalent to objective (8).

(ii) Given a set of vehicle charging events R, the assigning of each charging event separately to individual chargers leads to a number of $|R|$ chargers required. If $y_{lm} = 1$, then charging event m can be conducted after charging event l by the same charger. Thus, the charger assigned to charging event m can be saved. The required number of chargers thus can be reduced from $|R|$ to $|R| - 1$. Similarly, the value of max-flow Z_4 means Z_4 chargers can be saved by performing feasible linking of charging events. This proves that objective (14) is equivalent to objective (9).

The constraints of P2 satisfy the depot-capacity constraint as well as the balanced individual vehicle schedule constraint. Thus, P2 is equivalent to P1 through combining (i) and (ii).□

A special case of P2 with only objective (13) is known as the capacitated transportation problem, which can be transformed into a

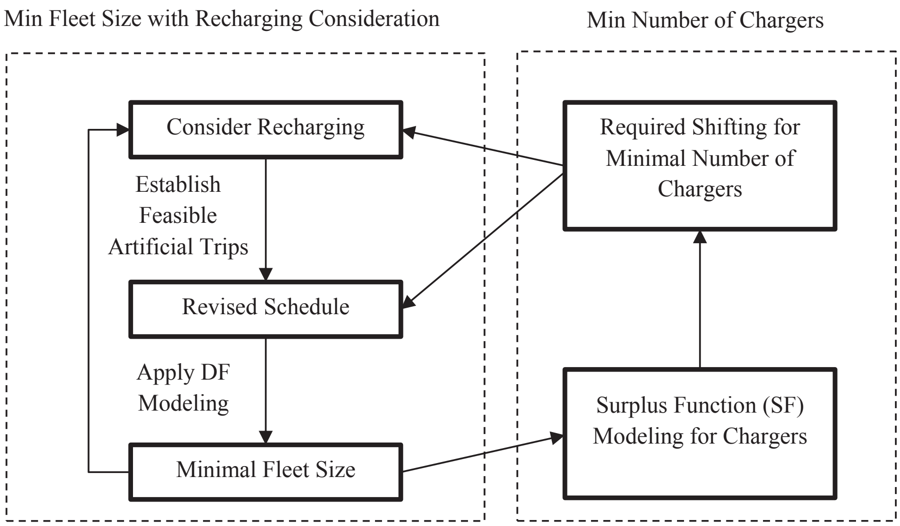

Fig. 4. DF-based analysis framework for minimizing the fleet size and number of chargers.

unit-capacity bipartite network with solution-time's complexity of $O(n^{1/2}(|A^I| + |A^R|))$ for n nodes (departure times) and $|A^I| + |A^R|$ arcs (Ceder, 2016). Because of the complexity involved in the bi-objective integer programming formulation and the need to respond to PT operations planning changes, it is advantageous to introduce an interactive man-machine scheduling solution method as well as a max-flow-based solution method.

4. DF-based solution method

This section provides a practical and graphical two-stage construction-and-optimization solution method for solving the BET-VSP. The solution method is explained using a detailed numerical example.

4.1. Solution framework

Problems P1 and P2 are bi-objective optimization problems associated with the group of multi-objective optimization or multiple criteria decision-making problems. Operations research (OR) studies cover more than a few solution methods for this group of problems, such as the weighted sum method, ε−constraint method, interactive method, and lexicographic method. For a detailed description of these methods, readers are referred, for instance, to Ehrgott (2005). This study adopts the lexicographic method to solve the bi-objective BET-VSP because of the higher cost of an EV than the cost of an EV battery charger as shown in Table 1.

The overall optimization framework for the schematic analysis of the BET-VSP is depicted in Fig. 4. It is comprised of two basic stages: stage (i) finds the minimum fleet size of PT vehicles with recharging consideration, and stage (ii) finds the minimum number of stationary chargers required. Initially, we'll assume that each terminal, defined as an endpoint of a vehicle trip, has enough chargers to accommodate the recharging process during the idle time-window between trip's ending and starting times at the terminal.

Based on the DF use-of-hollow concept (Ceder, 2016; Liu and Ceder, 2017) the link between the end of trip i, t_e^i and the start of trip j, t_s^j, for any i, j within a hollow is possible. However, with the consideration of recharging requirements, this possible i-j link may not be feasible, and a check must be performed. Consequently, the methodology will include the possibility of shifting the trip's end time forward so as to create a feasible set of trips with their extension—that is, to make it feasible from the perspective of the recharging requirements.

The working principle of the proposed two-stage construction-and-optimization solution procedure uses four operators, namely (i) trip swapping, (ii) inserting DH trips, (iii) shifting trip departure time within given tolerances, and (iv) adding another vehicle. This is to guarantee the feasibility of the initial solution generated. According to the DF theory, this initial feasible solution has the minimal number of vehicles required to perform the schedule, i.e., minimal fleet size Z_1^*. Based on this initial solution, at the second stage, the SF modeling and optimization are utilized to minimize the number of stationary chargers required at each terminal. This optimization process includes the determination of the optimal recharging location, recharging time and recharging duration. This can be done using the SF-based modeling and optimization.

It's clear that the two-stage construction-and-optimization solution procedure follows the working principles of the lexicographic method. That is, the original bi-objective problem P1 can be decomposed to sequentially solve two single objective optimization problems described mathematically as follows:

$$Min\ Z_2 = \sum_{k\in K} R(k) = \sum_{k\in K} \underset{t\in[T_1,T_2]}{Max}\ r(k,t) \tag{33}$$

s.t.

$$Z_1 \leqslant Z_1^* \tag{34}$$

Eqs: (10)–(12)

where Z_1^* is the optimal value of the below single objective optimization problem

$$Min\ Z_1 = \sum_{k \in K} D(k) = \sum_{k \in K} \underset{t \in [T_1, T_2]}{Max} d(k, t) \tag{35}$$

s.t.

Eqs: (10)-(12)

Proposition 2.. The sum of the maximal values of SFs of all terminals provides an upper bound of the minimal number of required chargers, i.e., $Z_2 \leqslant \sum_{k \in K} R(k)$.

Proof.. This is in the worst-case scenario, in which the maximal values of SFs cannot be reduced, i.e., the actual minimum number of chargers required at each terminal k is $R(k)$. Thus, the total number of chargers required is $\sum_{k \in K} R(k)$.□

Proposition 3.. The minimum number of chargers required is one; this provides a lower bound of the minimal number of required chargers, i.e., $Z_2 \geqslant 1$.

Proof.. This is the best-case scenario, in which all vehicles can be charged at the same terminal without overlapped charging times. □

4.2. Heuristic BET-VSP solution procedure

Given PT timetables and the flexible adjustments of inserting DH trips and shifting trip's departure time within an acceptable range of tolerances, the following procedure is proposed. Based on the information of Table 1 the priority is given to minimize the number of vehicles required over the number of chargers required. The procedure is as follows:

Algorithm 1.. A DF theory-based heuristic algorithm for the BET-VSP.

Step 1:	Create the DFs for all terminals based on the input vehicle trip data (Liu and Ceder, 2017).
Step 2:	Apply the unit reduction deadheading chain (URDHC) insertion subroutine (Ceder and Stern, 1981; Ceder, 2016) and the shifting departure time (SHIFT) subroutine (Ceder, 2016) to attain the minimum number of vehicles required$N_v^{\min}$.
Step 3:	Map all possible chain-based solutions to construct the set K = {k| k = 1,2,…K^}, where each k is comprised of $N_v^{\min}$ vehicles, or more if $N_v^{\min}$ is an infeasible solution with respect to the state of charge (SoC); the mapping process is carried out simultaneously with a feasibility check of SoC. Let k: = 1 for the FIFO chain solution; if the SoC < 0, then the solution is infeasible – STOP; otherwise, let k: = 2, go to Step 4.***Note:*** Assume a battery charger at the end terminal of each chain in a given solution with a DH required from the end to the starting terminal of each chain unless: (i) both terminals are same; (ii) the SoC at the end terminal is enough to allow for a DH trip to the starting terminal, in which case the starting terminal will have the charger. The DH trip (if needed) from end-to-start terminals is performed outside the span of the chain time to balance the schedule. If the charger is at the end terminal, the vehicle will leave with SoC = 100% this terminal, but will arrive with SoC < 100% to the starting terminal.
Step 4:	Create the k^{th} solution with SoC feasibility check; if SoC < 0, then the solution is infeasible – STOP; otherwise, set k: = k + 1 and go to Step 4. If k > K^, go to Step 5.
Step 5:	Arrange all feasible chain-based solutions in a set K* = {k| k = 1,2,…K$^+$}; go to Step 6.
Step 6:	Find for k, k∈K*, the minimum number of chargers required; set k: = k + 1 and start Step 6 again; if k > K$^+$, go to Step 7.
Step 7:	Arrange all k, k∈K*, in increasing order of the number of chargers required; for multiple solutions use a secondary list of increasing DH trip travel times, and as a thirdly stage (if needed) use increasing shifting times, to minimize operator's cost. Go to Step 8.
Step 8:	Select the first solution of the list established in Step 7, as the optimal solution.

4.3. Explanatory example

Example 2:. An explanatory example that comprises three terminals (*a, b, c*) and five vehicle trips is provided to comprehend the underlying principles of the DF theory-based heuristic algorithm for solving the BET-VSP. The departure terminal, departure time, arrival terminal, and arrival time, together with the associated energy consumption, of each trip, are given in Table 4. Table 5

Table 4
Trip schedule for Example 2.

Trip number i	Departure terminal p^i	Depature time t_s^i	Arrival terminal q^i	Arrival time t_e^i	Energy consumption c^i
1	*b*	0	*a*	4	4
2	*a*	2	*b*	7	5
3	*b*	7	*c*	10	3
4	*c*	13	*b*	17	3
5	*a*	15	*b*	20	5

Table 5
DH trip travel times (DH trip engergy consumption) for Example 2.

	Arrival terminal		
Departure termnal	*a*	*b*	*c*
a	0 (0)	3 (2)	5 (4)
b	3 (2)	0 (0)	6 (5)
c	5 (4)	6 (5)	0 (0)

provides the data on travel times and associated energy consumption of DH trips. The shifting tolerances—i.e., the maximum advance of the scheduled departure time (early departure) and the maximum delay allowed (late departure)—are both set as 0.5 time unit. The battery capacity is assumed to be 10 units of energy. Unlike Example 1, which assumes a fixed recharing rate, i.e., a linear charging function, a more realistic nonlinear charging function is used for this Example 2. The nonlinear charging function is shown in Fig. 5, which is based on the suggestions of Montoya et al. (2017). That is, when SoC is less than 80%, the battery-charge level is a linear function of the charging time; when SoC is more than 80%, the relationship is nonlinear.

What follows is a demonstration of using the heuristic algorithm to solve this example problem.

Step 1: The input vehicle trip schedule is displayed in Fig. 6(a). Based on the input data, three DFs are constructed for the three terminals, as shown in Fig. 6(b).

Step 2: After checking the three DFs, it can be seen that the maximum values of each DF cannot be reduced by applying the URDHC insertion and SHIFT subroutines; that is, neither DH trip insertion nor shifting departure times can reduce the minimum fleet size. Thus, the minimum number of vehicles required is $N_v^{\min} = \mathrm{D}(a) + \mathrm{D}(b) + \mathrm{D}(c) = 1 + 1 + 0 = 2$.

Step 3: After performing the mapping process of constructing vehicle chains, four possible chain-based solutions are found. That is, $\mathrm{K} = \{\mathrm{S}_1, \mathrm{S}_2, \mathrm{S}_3, \mathrm{S}_4\}$, where $\mathrm{S}_1 = \{[1, 5]; [2, 3, 4]\}$, $\mathrm{S}_2 = \{[1, 3, 4]; [2, 5]\}$, $\mathrm{S}_3 = \{[2, 3, 5]; [1, 4]\}$, and $\mathrm{S}_4 = \{[1, 3, 5]; [2, 4]\}$.

Step 4: The feasibility-checking process of all the four solutions are based on the SoC curves of each vehicle chain. The detailed checking process is outlined in the below Table 6:

Step 5: This step of the algorithm, applied to the example, results with two feasible solutions: $\mathrm{S}_1 = \{[2, 3, 4]; [1, 5]\}$ and $\mathrm{S}_2 = \{[1, 3, 4]; [2, 5]\}$; that is, $\mathrm{K}^* = \{\mathrm{S}_1, \mathrm{S}_2\}$. Fig. 6(c) shows the changes of SoC curves of the two solutions S_1 and S_2, with each contains two vehicles V_1 and V_2. Using Fig. 6(c) and Table 6 together allows for the comprehension of Steps 3 and 4 including the indication of one required DH trip for Solution S_1 and three required DH trips for Solution S_2.

Step 6: Based on the vehicle trip chains of Solution S_1, the SFs can be constructed as shown in Fig. 6(d). According to the SoC curves shown in Fig. 6(c), the R(*a*) can be reduced from 1 to 0, R(*b*) can be reduced from 2 to 1, and R(*c*) cannot be reduced. Thus, for Solution S_1 a minimum of two chargers are required, i.e., one charger at terminal *b* and the other one at terminal *c*. Performing the same SF modeling and optimization, the same results can be found for Solution S_2 can be found; in other words, for Solution S_2 a minimum of two chargers is required as well, i.e., one charger at terminal *b* and the other one at terminal *c*.

Step 7: The comparison of the two feasible solutions with respect to the number of vehicles required, the number of chargers required and the DH trip travel times required is outlined in Table 7 below. We can see that both these solutions need two vehicles and two chargers at terminals *b* and *c*. However, Solution S_1 needs one DH trip with 3 units of DH travel times, and Solution S_2 needs three DH trips with 9 units of DH travel times. Thus, Solution S_1 is listed before Solution S_2.

Step 8: The decision-maker determines Solution S_1 to be the final optimal solution of this problem.

The performance of Algorithm 1 in solving Example 2 is compared with an enumeration method. The computation results are summarized in Table 8. It shows that both methods can obtain the Pareto-efficient solutions in terms of the number of vehicles required, the number of chargers required and the units of DH trips travel times. Nonetheless, Algorithm 1 is more efficient with less computational time than of the enumeration method.

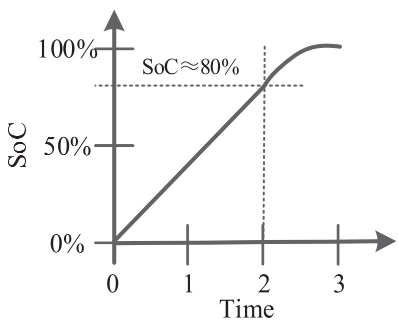

Fig. 5. The battery charging curve for Example 2.

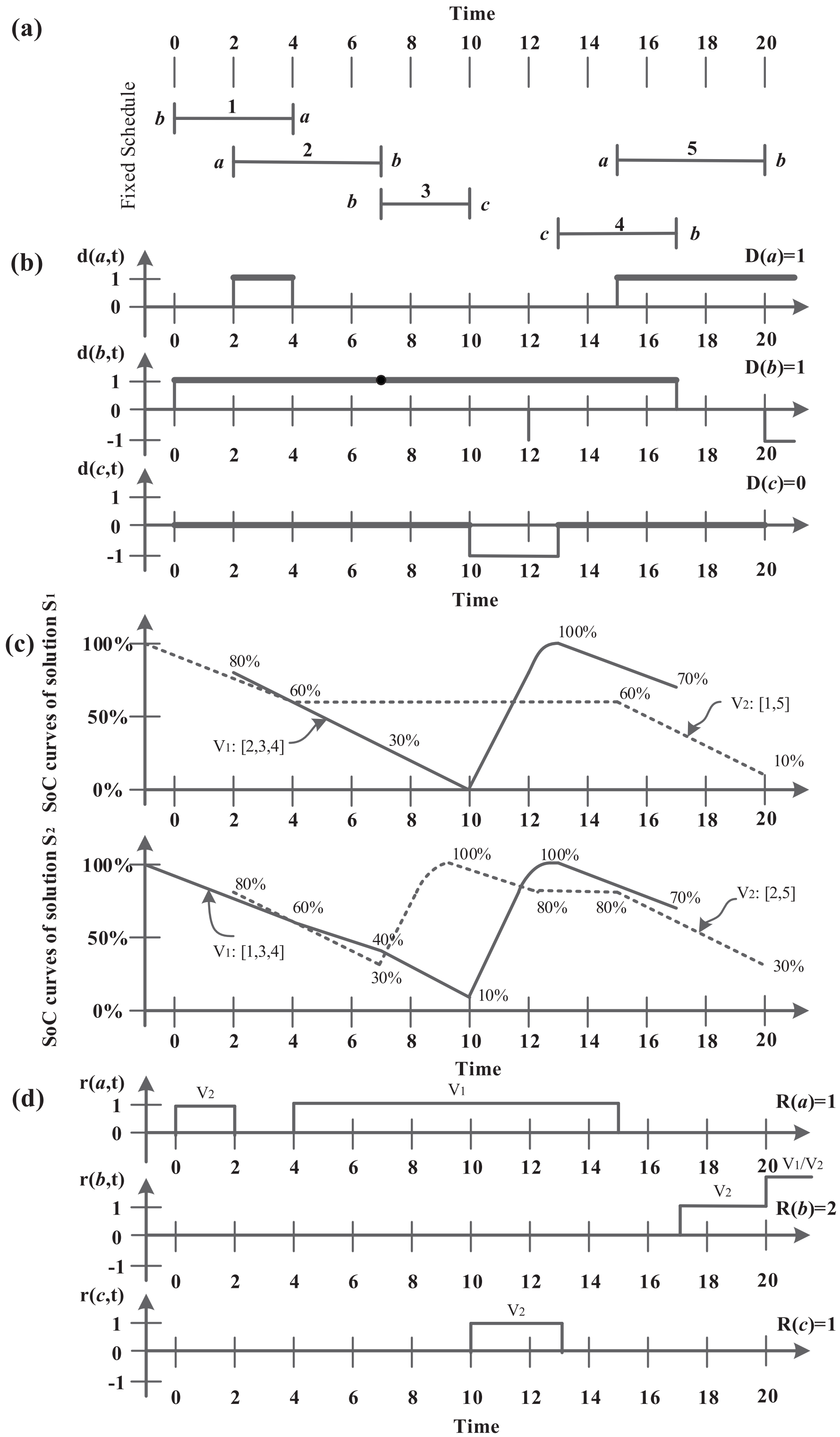

Fig. 6. The explanatory Example 2 after undergoing Steps 1 and 2 of the Algorithm 1, including part of Steps 3 and 4: (a) schedule of trips; (b) DFs of the three terminals; (c) SoC curves of the two solutions S_1 and S_2; (d) SFs of Solution S_1.

Table 6
The detailed checking process of *Step 4* of the heuristic algorithm.

Possible solution and its vehicle chains	Vehicle chain considered	Checking process
S_1 = {[1, 5]; [2, 3, 4]}	[1, 5]	Can be made with a charger at terminal *b* (see Fig. 6)
	[2, 3, 4]	Can be made with chargers at terminals *b* and *c* (see Fig. 6); starts at terminal *a* with 80% full battery capacity after inserting a DH trip (DH_1) from terminal *b* to terminal *a*
S_2 = {[1, 3, 4]; [2, 5]}	[1, 3, 4]	Can be made with chargers at terminals *b* and *c*, and with a DH trip (DH_2) between Trips 1 and 3 (see Fig. 6)
	[2, 5]	Can be made with a charger at terminal *b*, and with two DH trips (DH_3, DH_4) from terminal *b* to terminal *a*, with a charging time between t = 7 and t = 9.25 at terminal *b*
S_3 = {[2, 3, 5]; [1, 4]}	[2, 3, 5]	Infeasible with a DH trip between Trips 3 and 5; that is, SoC < 0%
	[1, 4]	Not applicable (N/A)
S_4 = {[1, 3, 5]; [2, 4]}	[1, 3, 5]	Infeasible with DH trips between Trips 1 and 3, and between Trips 3 anf 5; that is, SoC < 0%
	[2, 4]	Not applicable (N/A)

Table 7
Performance comparison of the two feasible solutions S_1 and S_2.

Solution	Number of vehicles required	Number of chargers required	Units of DH trip travel times required
S_1	2	2	3
S_2	2	2	9

Table 8
Comparison between Algorithm 1 and Enumeration method in solving Example 2.

Solution method	Number of vehicles required	Number of chargers required	Units of DH trip travel times required	Computation time (s)
Algorithm 1	2	2	3	< 60
Enumeration method	2	2	3	> 300

5. Adjusted max-flow solution method

In this section, we provide an alternative solution method to the DF-based solution method. Ceder (2016) shows that the procedure for solving the minimum fleet-size problem has its roots in the classic max-flow work of Ford and Fulkerson (1962). The approach described uses network-flow techniques well known in the OR field.

5.1. Formulation

Using the same notations of the first paragraph of Section 2.1, let SoC_v represent the battery charge level of vehicle chain *v* in %, $0\% \le SoC_v \le 100\%$, $\%B_i$ is battery consumption for trip *i*, where $i \in I$, $[u]_k$ means charging u% battery at terminal *k*, and $(h)_{pq}$ means consumption of h% battery of a DH trip between terminals *p* and *q*, where $p, q \in K$.

The two trips *i*, *j* may be serviced sequentially (feasibly joined) by the same vehicle if and only if

(a) $t_e^j \le t_s^i + t(p,q)$, if trip *i* ends at *p* and trip *j* starts at *q*; if $p = q$, then $t(p,q) = 0$.
(b) $100\text{-}\%B_i\text{-}(h)_{pq}\text{-}\%B_j + [u1]_p + [u2]_q \ge 0\%$, if there is a charger at the departure terminal of *i*; if there is a charger at *p*, and there is an opportunity to charge at *p* then $[u1]_p > 0\%$. Similarly $[u2]_q > 0\%$ if there is a charger at *q*, and there is an opportunity to charge at *q*; otherwise these components are nil.
(c) $100\text{-}(h)_{uv}\text{-}\%B_i\text{-}(h)_{pq}\text{-}\%B_j + [u1]_p + [u2]_q \ge 0\%$, if the departure terminal of *i* is *v*, and the vehicle is charged at *u* and deadhead to *v*. The rest of the conditions are same as in (b).

Every chain must satisfy (a) and (b) or (c). A trip-joining array for *S* may be constructed by associating the i^{th} row with the arrival event of the i^{th} trip, and the j^{th} column with the departure event of the j^{th} trip. Cell (*i*, *j*) will be admissible if *i* and *j* can feasibly be joined. Otherwise, cell (*i*, *j*) will be an inadmissible cell. Let x_{ij} be a 0–1 variable associated with cell (*i*, *j*) and I be the set of required trips; consider, then, the following problem:

$$(\mathbf{P3}) \quad Max\ Z_5 = \sum_{i \in I} \sum_{j \in I} x_{ij} \tag{36}$$

$$\text{s.t.} \sum_{j \in I} x_{ij} \leqslant 1,\ i \in I \tag{37}$$

$$\sum_{i \in I} x_{ij} \leqslant 1, j \in I \tag{38}$$

$$\left.\begin{array}{l} x_{ij} \in \{0, 1\}, \text{all } (i, j) \text{ admissible} \\ x_{ij} = 0, \ \text{ all } (i, j) \text{ inadmissible} \end{array}\right\} \tag{39}$$

A solution with $x_{ij} = 1$ indicates that trips i and j are joined. The objective function maximizes the number of such a joining. Constraint (37) ensures that each trip is joined with at most one successor trip. Similarly, constraint (38) indicates that each trip may be joined with at most one predecessor trip. The following theorem states that maximizing (36) in P3 is tantamount to minimizing the number of chains for a trip schedule of size n.

Theorem 2:. Let N and |I|denote the number of chains and trips, respectively. Then, Min N = |I| - Max Z_5.

Proof:. Can be found in Ford and Fulkerson (1962).

The problem P3 is equivalent to a special arrangement of the maximum-flow (max-flow) problem, appearing in an extensive treatment in the classic book by Ford and Fulkerson (1962). However, with the constraints of the battery of the electric vehicle, there is a need to adjust the max-flow problem and construct a different algorithm. For our new problem, the network exhibited in P3 can be transformed to a unit-capacity bipartite network, in which the solution time has the complexity of more than $O(n^{1/2}m)$ with n nodes (departure times) and m arcs. This solution time, first shown by Even and Tarjan (1975), was explicated by Ahuja et al. (1993).

5.2. Adjusted max-flow (AMF) algorithm

In the max-flow problem, one considers a directed graph (network) $G = \{N,A\}$, with a single node s as *source* and another node t as a *sink*. Each arc *(i,j)* has a capacity *c(i,j)* = the maximum flow that can be traversed from *i* to *j* and the flow *f(i,j)*. In our case *c (i,j)* = 1. An *s-t* flow is the amount of the flow that leaves *s* and arrives at *t*, provided that:

$0 \leqslant f(i, j) \leqslant c(i, j) \ \ \forall \ (i, j) \in A$, and flow conservation exists, $\sum_{j|(i,j)\in A} f(i, j) = \sum_{k|(k,i)\in A} f(k, i), \ \ \forall \ i \in N, \ \ i \neq s, t$, where $f(i, j)$ is the arc *(i, j)*'s flow.

The original max-flow problem was first solved by Ford and Fulkerson (1962) using the augmenting-path algorithm. The following AMF algorithm is based on the augmenting-path algorithm, but it is adjusted and extended to accommodate the electric-vehicle scheduling problem.

Algorithm 2.. Adjusted max-flow (AMF) algorithm for the BET-VSP.

Step 1: Construct a unit-capacity bipartite network with arrival nodes as the subset of nodes connected to *s*, and departure nodes as the other subset of nodes connected to *t*, where each node of each subset represents a trip *i* of the given schedule, *i*∈I. Each node contains the time, location, and $\%B_i$ for each *i*. Construct the arcs *(i, j)* between the arrival and departure subset of nodes, for the possibility that trips *i, j* be serviced sequentially based strictly on the three conditions (a) and (b) or (c) listed above.

Step 2: Construct a *residual network* $G(f) = \{N, A(f)\}$ where each arc is labeled by $\alpha(i, j)$ as follows:

- for $f(i, j) < c(i, j)$, *then* $(i, j) \in A(f)$and $\alpha(i, j) = c(i, j) - f(i, j)$; *(i, j)* is called a *forward arc;*
- for $f(i, j) > 0$, *then* $(j, i) \in A(f)$and $\alpha(j, i) = f(i, j)$; *(j, i)* is called a *reverse arc.*

Step 3: If there is no path from *s* to *t* in *G(f),* then go to ***Step 6***; *f* is the upper bound of the *s-t* flow.

Step 4: Let P be the shortest path (least number of arcs), and let $\alpha = \min_{(i,j)\in P} \alpha(i, j)$. Define a new flow f' on G:

(a) if *(i, j)* is a forward arc on P, let$f'(i, j) = f(i, j) + \alpha$

(b) if *(i, j)* is a reverse arc on P, let$f'(i, j) = f(i, j) - \alpha$

(c) if *(i, j)* does not belong to P, let$f'(i, j) = f(i, j)$

Step 5: Let $f: =f'$; go to ***Step 2***.

Step 6: Construct all feasible chains of trips using a feasible unit of *s-t* flow and considering condition (b) or (c) above, mentioned in ***Step 1***; start with *s-i-j-t* path, then check *s-j-k-t* path and subsequent paths to cover all trips *i, i*∈I. Mark a successful set of chains covering all trips *i, i*∈I, as a solution set S_k, k = 1,2,… Select the desired solution based on cost criteria of the cost of vehicles and chargers, including the consideration of DH trip cost. If the minimum number of vehicles across all S_k is the number of scheduled trips minus the max-flow, then the optimal number of vehicles and its lower bound coincide (see Theorem 3).

Definition 1 (*(s-t cut)*.). If in a directed graph G = {N,A} the set of nodes N is partitioned into two sets$X$ and $\bar{X}$ (i.e., $X \cap \bar{X} = \varnothing$ and $X \cup \bar{X} = N$), such that $s \in X$ and $t \in \bar{X}$, then $(X, \bar{X})$= the set of all arcs in A connecting nodes in X to nodes in $\bar{X}$, is called the *s-t cut.* The capacity of the cut is defined as $c(X, \bar{X})$.

Theorem 3:. is the lower bound of the number of electric vehicles required to serve the schedule, where |I| is the number of trips in the schedule.

Proof.. Appears in Appendix A.

5.3. Example as an explanatory tool

Example 3:. Following is the five-trip example problem, with its data appearing in Tables 9–11 and with a shifting tolerance of ± one time unit. This example problem has undergone the AMF algorithm in Fig. 7 and Table 12. In Fig. 7, the nodes refer to the five trips

Table 9
Trip Schedule S for the example problem.

Trip Number i	Departure Terminal p^i	Departure Time t_e^i	Arrival Terminal q^i	Arrival Time t_s^i	$\%B_i$
1	*b*	t = 0	*a*	t = 4	40%
2	*a*	t = 2	*b*	t = 7	50%
3	*b*	t = 7	*c*	t = 10	30%
4	*c*	t = 13	*b*	t = 17	30%
5	*a*	t = 15	*b*	t = 20	50%

Table 10
Average DH travel-time (in time units) matrix for the example of Table 9.

		Arrival terminal		
		a	*b*	*c*
Departure terminal	*a*	0	3	5
	b	3	0	6
	c	5	6	0

Table 11
The charge percentage of a full battery, $[u]_k$, and the consumption percent of DH trip, $(h)_{pq}$, for given time units t.

t	2	3	4	5	6	7	8	9	10	11
$[u]_k$ or $(h)_{pq}$ for terminals *a,b,c*	10%	20%	30%	40%	50%	60%	70%	80%	90%	100%

per the data in Table 9. In addition, the charging requirements of some of the arcs between the arrival and departure nodes are as follows: for Trips 1–3→$(-20\%)_{ab}$ meaning that the DH trip between *a* (arrival of Trip 1) and *b* (departure of Trip 3) requires 20% consumption of the full battery based on Table 11; for Trips 1–4→$(-40\%)_{ac}$ and (+30%) charging at *a* or *c*; for Trips 1–5→(+100%) at *a*; for Trips 2–4→$(-50\%)_{bc}$; for Trips 2–5→$(-20\%)_{ba}$ and (+40%) charging at *a* or *b*; and for Trips 3–5→$(-40\%)_{ca}$.

The two solution sets of the AMF algorithm are: S_1 = {[1–3–4], [2–5]} and S_2 = {[2–3–4], [1–5]}, with different possibilities of the locations of chargers in S_2. We observe that for both S_1 and S_2 the number of vehicles required is two, same as $|I| - \min_{s-t\ cuts\ (X,\bar{X})} c(X, \bar{X})$ of Theorem 3. That is, the minimum number of vehicles is based on the max-flow of the AMF algorithm and the same as the lower bound of Theorem 3; thus, it is the optimal solution in terms of number of vehicles. In terms of the minimum number of chargers required, again both S_1 and S_2 have the same solution, with two chargers for each. The third possible criterion is the deadheading (DH) time of each solution. In S_1 the chargers are at *b* and *c*, with total DH time of 60 time units (=20 + 20 + 20). In S_2 the chargers are either at *a* and *c* with total DH time of 40 time units (=20 + 20), or at *b* and *c* with total DH time of 0 time units (no DH trips), but with a shift of Trip 4 by one time unit for a late departure. All in all, S_2 is preferred to S_1 and the decision-maker will have to choose between having two chargers at *a* and *c* with DH time of 40 and no shifting of Trip 4, or two chargers at *b* and *c* with no DH time, but with a shifting of Trip 4.

Comparison between Algorithm 2 and the enumeration method is exhibited in Table 13 for Example 3. It shows that both methods can obtain the Pareto-efficient solutions. However, Algorithm 2 is more efficient than the enumeration method. In addition, Algorithm 2 has the flexibility of allowing for shifting trip departure times, which can further optimize the solutions by reducing the DH trip travel times.

6. Case study

The DF-based solution method was applied to a real-life case study using real-world bus service data collected in Singapore. Being a well-organized island country, Singapore is the ideal place to develop an electrified transportation system, especially an electrified multimodal PT system (Massier et al., 2018). Three major bus interchanges in Singapore were selected, and for each pair of interchanges, one bus line operating between them was chosen. As shown in Fig. 8, line 157 operates between Boon Lay Int and Toa Payoh Int, line 139 between Toa Payoh Int and Bt Merah Int, and line 198 between Bt Merah Int and Boon Lay Int. The solid and dashed lines indicate the two different directions of each line. Some operation characteristics of these three bus lines are summarized in Table 14. These statistical data were adapted from a three-month Contactless e-Purse Application (CEPAS) data set. The CEPAS data set, which is based on a smart card system, includes the detailed information of each vehicle trip as well as rider and service information. The average energy required per trip was estimated using an energy-demand model developed by Massier et al. (2018). It is assumed that the buses are fully charged just before the first trip of the day and that during the day they use the layover (recovery) time between two consecutive trips to recharge opportunistically at the terminals at which they finished their previous trip.

Due to the variances of vehicle travel times and the associated energy consumption for different cross sections of a day, PT

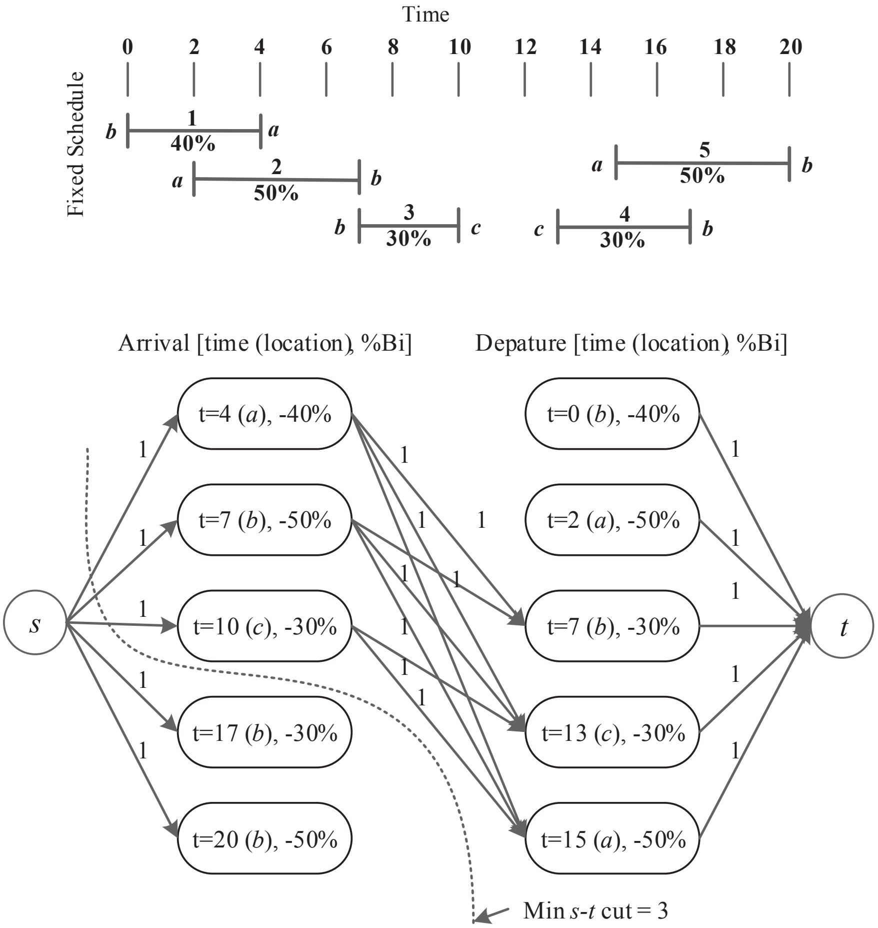

Fig. 7. Example problem for the AMF algorithm including the fixed schedule with trip number and $\%B_i$, and the max-flow network.

Table 12
Materializing *Step 6* of the algorithm.

Solution #	Chain #	Checked Chain	Time Feasibility	BFL=	Chargers' allocation
S_1	1	*s*-1–3-*t*	Feasible in time (FIT)	$[100]_b$-40-$(20)_{ab}$-30 = 10%	OK if at *b*
		s-3–4-*t*	FIT	10 + $[20]_c$-30 = 0%	OK if at *b* and *c*
		s-4…[1,3,4] done	Not feasible		
	2	*s*-2–5-*t* [2,5] done	FIT	$[100]_b$-$(20)_{ab}$-50-$(20)_{ab}$ + $[40]_b$-50 = 0%	OK if at *b*
–	1	*s*-1–4-*t*	FIT	$[100]_a$-$(20)_{ab}$-40 + $[30]_a$-$(40)_{ac}$ = 30%	OK if at *a*
	2	*s*-4…	Not feasible		
S_2	1	*s*-2–3-*t*	FIT	$[100]_a$-50-30 = 20% $[100]_b$-$(20)_{ba}$-50–30 = 0%	→ OK if at *a* → OK if at *b*
		s-3–4-*t*	FIT	20 + $[20]_c$-30 = 10% 0 + $[30]_c$-30 = 10%	→ OK if at *b* and *c* → OK if at *b* and *c*, and Trip 4 is shifted 1 to right
		s-4…[2,3,4] done	Not feasible		
	2	*s*-1–5-*t*	FIT	$[100]_a$-$(20)_{ab}$-40 + $[60]_a$-50-$(20)_{ba}$ = 30% $[100]_b$-40–50 = 10%	→ OK if at *a* → OK if at *b*

schedulers, in practice, usually divide the overall scheduling horizon into a set of different scheduling periods. For each scheduling period, the average vehicle travel time and average energy consumption are used for planning a vehicle schedule. By doing so, it can significantly reduce the complexity of the vehicle scheduling problem. In this study, we adopted the same approach, and use the

Table 13
Comparison between Algorithm 2 and Enumeration method in solving Example 3.

Solution method	Number of vehicles required	Number of chargers required	Units of DH trip travel times required	Computation time (s)
Algorithm 2 without shifting	2	2	40	< 110
Algorithm 2 with shifting	2	2	0	< 150
Enumeration method	2	2	40	> 420

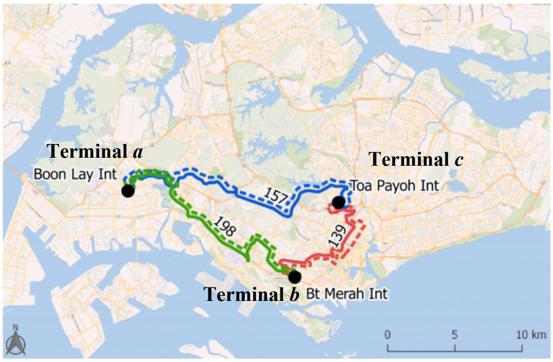

Fig. 8. Route configuration of the three bus lines in the case study area in Singapore.

Table 14
Operational characteristics of the routes selected for the case study.

Bus line	Line direction	Number of vehicle trips per day	Line length [km]	Average trip duration [min]	Average energy required per trip [kWh]
139	0	92	17.2	61	30.7
139	1	93	18.3	67	33.7
157	0	100	24.4	73	44.2
157	1	112	24.3	68	44.9
198	0	127	20.5	60	43.0
198	1	145	19.5	63	45.0

morning peak-hour period as the scheduling period, i.e., $[T_1, T_2]$ = [7:00 am, 10:00 am], to test the solution method. Other input data includes the available charging power at each charging terminal, which is assumed to be 50 kW, and the battery capacity of buses, which is set as 100 kWh. The average travel times, together with the average energy consumptions, of the DH trips among the three terminals are shown in Table 15. The allowable left and right shifting times for all the trips are set as two minutes. The setting-up times required at terminal charging stations for all the vehicles are set as 10 min. The capacity of each terminal is assumed to be sufficient large to accommodate all the vehicles considered in the scheduling period.

The implementation of Algorithm 1 was conducted in a Microsoft Excel Worksheet on a personal computer (PC) with an Intel Core TM i7-7600U CPU @ 2.80 GHz, 8.00 GB RAM, and a 64-bit Windows 10 operating system. By implementing the first step of Algorithm 1, the original DFs of the three terminals can be constructed in just a few seconds, as shown in Fig. 9. From it, we can see the initial total number of vehicles required $N_v^{\min}$ for the case study is $D(a) + D(b) + D(c) = 13 + 11 + 13 = 37$. After implementing the SHIFT subroute of the second step of Algorithm 1, the departure and arrival times of nine trips are systematically changed following the left-shifting and right-shifting time tolerance. The new and updated DFs of the three terminals are shown in Fig. 9 using dotted lines. We see that the maximum values of DFs of terminal *a* and *c* can be reduced from 13 to 12. This suggests that two vehicles can be removed, and the fleet size can be reduced from 37 vehicles to 35 vehicles. Checking the DFs of the three terminals further, we can see the condition of implementing the URDHC subroute is not met. Thus, the fleet size cannot be reduced further by inserting DH trips. This finalizes the first stage of Algorithm 1; the optimal (minimal) fleet size Z_1^* obtained by performing

Table 15
Average DH trip travel times (min) and average energy consumption (kWh) for the case study.

Departure termnal	Arrival terminal *a*	*b*	*c*
a	0 (0)	30 (9)	35(10)
b	30(9)	0 (0)	28 (7)
c	35 (10)	28 (7)	0 (0)

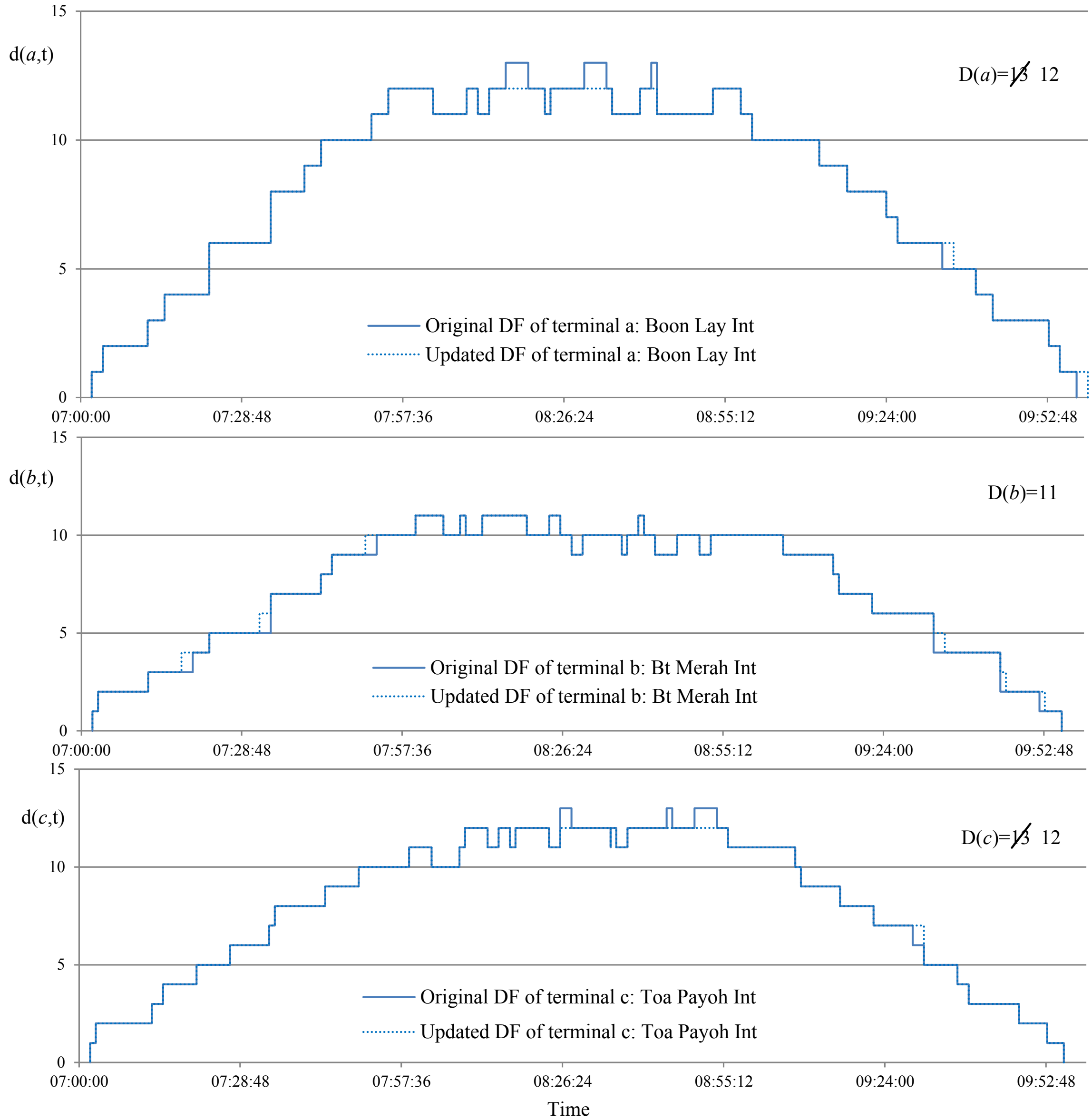

Fig. 9. DFs of the three terminals of the case study.

DF modeling and optimization is 35 vehicles.

The vehicle chains of the 35 vehicles can be easily construced using the FIFO rule or the chain extraction procedure developed by Gertsbach and Gurevich (1977). After performing the SoC feasibility check of the fourth step of Algorithm 1, a subset of feasible vehicle chains is obtained, which complies with the SoC and charging constraints. An examination of the number of chargers required for each solution set using the solution procedure suggests that the minimum number of chargers required is 6, with 2 chargers for

Table 16
Sumarry of DH trips required for the optimal vehicle schedule.

DH trips	Number of DH trips	Total number of DH trips	Total DH trip travel time (min)	Total DH trip engergy consumption (kWh)
DH_{ab}	4	25	799	222
DH_{ba}	0			
DH_{ac}	5			
DH_{ca}	8			
DH_{bc}	6			
DH_{cb}	2			

Table 17
Different instances and the related parameter values.

Instance	Number of terminals	Number of lines	Number of trips	Charging function
I1	2	2	98	linear
I2	2	4	164	nonlinear
I3	3	6	272	nonlinear

each terminal. Thus, by applying the solution method to the case study problem, an optimal solution with a minimum of 35 vehicles and a minimum of 6 chargers was obtained.

In order to maintain the schedule of the optimal solution as a balanced individual vehicle schedule, a set of DH trips are added into the existing schedule. Table 16 summarizes the DH trips required for the optimal vehicle schedule obtained. It shows that in total 25 DH trips are required, with 4 DH trips from terminal a to terminal b, 5 DH trips from terminal a to terminal c, 8 DH trips from terminal c to terminal a, 6 DH trips from terminal b to terminal c, and 2 DH trips from terminal c to terminal b, to balanced each individual vehicle chain. The total travel time of these 25 DH trips is 799 min and the associated total energy consumption is 222 kWh. The total travel time and associated energy consumption of all required DH trips can be further used to evaluate the quality of the vehicle schedule obtained.

Because of the large size of the case study problem it is very time-consuming to solve it by an enumeration method, and thus the latter was not applied for. Instead, the AMF algorithm (Algorithm 2), coded in MATLAB using its built-in integer programming solver, was applied to solve the case study problem. We tested the effectiveness and efficiency of Algorithm 2 with the same PC used for Algorithm 1. The first part of Algorithm 2 involves solving a max-flow problem by using MATLAB linear programming solver enabling the case study solution in just a few minutes. The initial vehicle chains were obtained by the linear programming solution. The second part of Algorithm 2 was to check the feasibility of vehicle chains concerning the driving range and battery charging constraints. This was done similarly to Step 4 of Algorithm 1. The computation results show that Algorithm 2 can generate the same solutions as of Algorithm 1. However, Algorithm 2 takes less time to attain the solutions, because of using the built-in integer programming solver of MATLAB.

To further evaluate the efficacy and efficiency of the solution algorithms, different problem instances, under different scenarios, are randomly generated based on the case study problem. These problem instances and their related parameter values are given in Table 17. Instance I1 considers vehicle trips conducted in the morning period from 7:00 to 10:00 with two terminals, two lines, and a linear battery charging function. Instance I2 considers vehicle trips conducted in the afternoon period from 13:00 to 17:00 with two terminals, four lines, and a nonlinear battery charging function. Instance I3 considers vehicle trips conducted in the evening and night period from 17:00 to 23:00 with three terminals, six lines, and a nonlinear battery charging function. The related vehicle trips are taken or modified from the bus schedule of the existing study area. The average DH trip travel times and average energy consumption of DH trips are assumed to be same as of the case study problem.

The three different problem instances are solved by using Algorithms 1 and 2, which are coded and implemented in the same PC using Microsoft Excel Worksheet and MATLAB linear programming solver. For Algorithm 2, two scenarios, with small vehicle departure time shifting and without vehicle departure time shifting, are considered. The computational results of the solution algorithms for the three problem instances are summaried in Table 18. It shows that for a given problem instance Algorithm 1 and Algorithm 2 without shifting, can generate the same number of Pareto-efficient solutions with the same DH trip energy comsumption. The type of battery charging function, linear or nonlinear, does not have a significant impact on the computational results. Algorithm 2 without shifting requires less computational time compared with Algorithm 1. Algorithm 2 with shifting can generate more Pareto-efficient solutions because of allowing for shifting vehicle departure times; however, it consumes more DH trip energy and takes longer computational times than with no use of shifting. Thus, the decision-maker should make a trade-off between the number of Pareto-efficient solutions and the DH trip energy comsumption and computational time.

The results of the case study suggest that the proposed two-stage construction-and-optimization heuristic solution method and the AMF algorithm can also be used to solve real-world BET-VSPs, ensuring an efficient allocation of PT vehicles to service trips while reducing the total number of chargers required to a minimum level. In fact, the graphical interactive optimization feature of the DF and SF models can provide some flexibility in finding the optimal solutions. That is, at the second stage of the solution method, while following the minimum fleet size constraint, the scheduler can make a trade-off between minimizing only the total number of chargers and minimizing a combination of the total travel time of DH trips and the total number of chargers. The computational results show that the increase of the number of terminals and lines will increase the computational times.

7. Conclusions

With the advantages of reduced emissions, quieter engines and fewer vibrations, and increased comfort for riders, electric transit vehicles are currently being introduced in more and more public transit (PT) agencies around the world. Undoubtedly, the introduction of electric PT vehicles will have a significant impact on traditional PT operations planning activities, including network design, timetable development, vehicle and crew scheduling, and real-time control. One of the most challenging tasks is how to efficiently schedule a set of electric PT vehicles considering their limited driving range and charging requirements.

This study examines the battery-electric transit vehicle scheduling problem (BET-VSP) with stationary battery chargers installed

Table 18
Comparison between various computational results of the three problem instances using different solution algorithms.

	I1			I2			I3		
	Number of Pareto-efficient solutions	DH trip energy consumption (kWh)	Time (s)	Number of Pareto-efficient solutions	DH trip energy consumption (kWh)	Time (s)	Number of Pareto-efficient solutions	DH trip energy consumption (kWh)	Time (s)
Algorithm 1	1	16	38	1	31	73	2	58	112
Algorithm 2 without shifting	1	16	29	1	31	62	2	58	101
Algorithm 2 with shifting	2	28	45	2	40	96	3	72	143

at PT terminal stations. We consider the popular use of fast charging strategy, together with full and partial charging policies, for the BET-VSP. Two different mathematical formulations of the BET-VSP are provided. The first formulation is based on the deficit function (DF) theory, and the second formulation is an equivalent bi-objective integer programming model. The first objective of the math-programming optimization is to minimize the total number of electric vehicles required, while the second objective is to minimize the total number of battery chargers needed to carry out a given set of scheduled services. To solve this bi-objective BET-VSP, two solution methods are developed. First, a lexicographic method-based two-stage construction-and-optimization solution procedure using the DF theory is proposed. Second, an adjusted max-flow solution method is developed. To the best of our knowledge, this is the first time that the DF theory, together with the related surplus function, has been applied to solve the BET-VSP along with an adjusted max-flow (AMF) based approach. One of the main advantages of the DF-based solution approach is its graphical visual nature that provides schedulers with some flexibility in terms of interjecting their own practical scheduling considerations. Three numerical examples are used as an expository device to illustrate the solution procedures developed, together with a real-life case study from Singapore. The results demonstrate that the proposed mathematical programming models and solution methods are effective in practice and have the potential to be applied in solving large-scale and realistic BET-VSPs.

One limitation in the results of this study is the need of constructing and checking the feasibility of all vehicle chains following the charging and balanced individual schedule constraints. However, checking for the DF-based approach can be done using the hollow concept of the DF theory and the DH trip insertion technique. For the AMF solution approach, the checking can be done during the construction of the max-flow network. Nonetheless, additional intelligent and automated ways can be developed to facilitate these processes.

Subsequent and future research may include: (i) comparison between the two solution methods developed; (ii) further extension of the mathematical models to other electric transit vehicle scheduling problems, considering different charging policies, such as regular charging, battery swapping and wireless charging (Abdolmaleki et al., 2019); (iii) increasing the automation of the solution procedure by incorporating integer programming solvers with the existing DF-based graphical interactive optimization and AMF methods; (iv) testing the models with large-scale real-world electric transit vehicle scheduling problems taking into consideration of various practical modeling features, such as multi-depot, multi-period, and multi-vehicle type; and (v) integrating electric transit vehicle scheduling with crew scheduling.

8. Authors' statement

All authors certify that they have participated sufficiently in the work to take public responsibility for the content, including participation in the concept, design, analysis, writing, or revision of the manuscript. Furthermore, each author certifies that this material or similar material has not been and will not be submitted to or published in any other publication before its appearance in the *Transportation Research Part C: Emerging Technologies*.

9. Author contribution statement

The authors confirm contribution to the paper as follows: Study conception and design: A. Ceder, T. Liu; Data collection: T. Liu; Analysis and interpretation of results: T. Liu, A. Ceder; Manuscript preparation: T. Liu, A. Ceder. All authors reviewed the results and approved the final version of the manuscript.

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Appendix A. .

Lemma 1:.

$$V(f) = f(X, \bar{X}) - f(\bar{X}, X)$$

Proof:.

$$V(f) = f(s, N) - f(N, s) + \sum_{\substack{i \in X \\ i \neq s}} [f(i, N) - f(N, i)] = f(X, N) - f(N, X)$$
$$= [f(X, \bar{X}) + f(X, X)] - [f(\bar{X}, X) + f(X, X)] = f(X, \bar{X}) - f(\bar{X}, X)$$

Lemma 2:. For any *s-t* flow f and any *s-t* cut $(X, \bar{X})$: $V(f) \leqslant c(X, \bar{X})$.

Proof:. For every f and$(X, \bar{X})$: $f(X, \bar{X}) \leqslant c(X, \bar{X})$ and $f(X, \bar{X}) = 0$. Therefore, by *Lemma 1* $\rightarrow V(f) = f(X, \bar{X}) - f(\bar{X}, X) \leqslant c(X, \bar{X})$.

Proof of Theorem 3: Let Y be all the nodes belonging to the paths from s to i in the last augmented network (certainly $i \neq t$). Also, $\bar{Y} = N - Y$; hence, $t \in \bar{Y}$. Therefore, for all $(i, j) \in A$ such that $i \in Y$ and $j \in \bar{Y}$, $f(i, j) = c(i, j)$ (otherwise, we would have an augmented path between Y and $\bar{Y}$), and for all $(i, j) \in A$ such that $i \in Y$ and $j \in \bar{Y}$, $f(i, j) = 0$. Hence, $f(Y, \bar{Y}) = c(Y, \bar{Y})$ and $f(\bar{Y}, Y) = 0$. Now, based on *Lemma 2* $V(f) \leqslant c(X, \bar{X})$ and thus if the algorithm is terminated with a flow f^*(see ***Step 3***), then $V(f^*) = \min_{s-t \ cuts \ (X, \bar{X})} c(X, \bar{X})$ and f^* has a maximum value. This could follow the known max-flow = min-cut theorem of Ford and Fulkerson (1962), but because of ***Step 1*** there is a possibility that although its condition (a) may exist, i.e., $t_e^j \leq t_s^i + t(p, q)$, condition (b) or (c) may be violated; for instance $\%B_i + \%B_j > 100\%$. Thus, only *Lemma 2* holds, $V(f) \leqslant c(X, \bar{X})$, and $\min_{s-t \ cuts \ (X, \bar{X})} c(X, \bar{X})$ is becomes the upper bound of $V(f)$. Using Theorem 2 we arrive at the proof that $|I| - \min_{s-t \ cuts \ (X, \bar{X})} c(X, \bar{X})$ is the lower bound for the number of vehicles.

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