Discrete Mathematics Daily Quiz Questions For GATE 2024

GATE And Tech
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1 Mathematical Logic

1.1 Propositional Logic

1. Which of these sentences are not propositions? What are the truth values of those that are propositions? (Mark all the correct options)

   A. Boston is the capital of Massachusetts.

   B. Miami is the capital of Florida.

   C. $2 + 3 = 5$.

   D. $5 + 7 = 10$.

   E. $x + 2 = 11$.

   F. Answer this question.

   Solution:

   A. It is a proposition and the truth value of this proposition is TRUE.

   B. It is a proposition and the truth value of this proposition is FALSE ($\because$ Tallahassee is the capital of Florida).

   C. It is a proposition and the truth value of this proposition is TRUE.

   D. It is a proposition and the truth value of this proposition is FALSE ($\because$ $5 + 7 = 12$).

   E. It is not a proposition ($\because$ the given statement is neither TRUE nor FALSE unless we assign a value for the variable x).

   F. It is not a proposition. ($\because$ this is an imperative sentence, not a declarative sentence).

   Correct Answer: E; F

2. The following propositional statement is: $\neg((\neg P) \land Q) \leftrightarrow (P \to Q)$

   A. Tautology

   B. Contradiction

   C. Contingent

   D. More than one of these

   Solution: Apply the negation and solve them

$$\neg((\neg P) \land Q) \leftrightarrow (P \to Q) \equiv (P \lor \neg Q) \leftrightarrow (\neg P \lor Q) \equiv (P \leftrightarrow Q) \equiv P \odot Q$$

P	Q	$(P \lor \neg Q) \leftrightarrow (\neg P \lor Q)$
F	F	T
F	T	F
T	F	F
T	T	T

A proposition is contingent **if and only if** it is neither a contradiction nor a tautology is called **Contingency**.

Correct Answer: C

3. The number of propositional functions on n variables?

   A. 2^n

   B. 2^{2^n}

   C. n^{2^n}

   D. n^{n^2}

   Solution:

   An n variable propositional is a mapping from $\{T, F\}^n \rightarrow \{T, F\}$

p_1 p_2 p_3 ……… p_n	f
T T T ……….. F	
T T T ……….. F	
.. ….. … …. F F F ……….. F	

2^n

   Each propositional function is mapping these 2^n rows to $\{T, F\}$.so that each row can be mapped to either T or F Forming propositional function is nothing but mapping each row with 2 options (T or F) The number of different mappings for 2n rows $= 2 \times 2 \times 2 \times \ldots\ldots 2^n$ times $= 2^{2^n}$

   Correct Answer: B

4. The proposition $p \wedge (\sim p \vee q)$ is logically equivalent to _____

   A. tautology

   B. logically equivalent to $p \wedge q$

   C. logically equivalent to $p \vee q$

   D. none

   Solution: $p \wedge (\sim p \vee q) \Leftrightarrow (p \wedge \sim p) \vee (p \wedge q)$ (distributive law)

   $$\Leftrightarrow F \vee (p \wedge q) \quad [p \wedge \sim p \Leftrightarrow F]$$
   $$\Leftrightarrow p \wedge q$$

   Therefore given proposition is logically equivalent to $p \wedge q$.

   Correct Answer: B

5. The binary operation $\square$ is defined as follows

p	q	p □ q
T	T	T
T	F	T
F	T	F
F	F	T

   Which one of the following is equivalent to $p \vee q$?

   A. $\sim q \square \sim p$ B. $p \square \sim q$ C. $\sim p \square q$ D. $\sim p \square \sim q$

   Solution:

$$p \square q \Leftrightarrow q \rightarrow p$$
$$\Leftrightarrow \sim q \vee p$$
$$p \vee q \Leftrightarrow q \vee p \Leftrightarrow \sim(\sim q) \vee p \Leftrightarrow p \square \sim q$$

Correct Answer: B

6. **(TIFR 2023)** Consider the following two statements:

(P) The current population of Bhutan is greater than the current population of India.

(Q) The Moon is smaller than the Earth. Clearly, (P) is false, while (Q) is true. Which of the following logical statements evaluates to true?

A. $\neg P \Rightarrow Q$

B. $Q \Rightarrow P$

C. $\neg(P \Rightarrow Q)$

D. $P \Leftrightarrow Q$

Solution: Truth Value of P: False

Truth Value of Q: True

A. $(P \Rightarrow Q) \equiv (F \Rightarrow T) \equiv (T \Rightarrow T) \equiv$ True

B. $(Q \Rightarrow P) \equiv (T \Rightarrow F) \equiv$ False

C. $(P \Rightarrow Q) \equiv (F \Rightarrow T) \equiv T \equiv$ False

D. $(P \Leftrightarrow Q) \equiv (F \Leftrightarrow T) \equiv$ False

Correct Answer: A

7. **(UGC NET 2017)** In propositional logic if $(P \rightarrow Q) \wedge (R \rightarrow S)$ and $(P \vee R)$ are two premises such that

$$\begin{array}{c} (P \rightarrow Q) \wedge (R \rightarrow S) \\ P \vee R \\ \hline Y \\ \hline \end{array}$$

Y is the premise : (Mark all the correct options)

A. $P \vee R$

B. $P \vee S$

C. $Q \vee R$

D. $Q \vee S$

Solution: Each option is a Valid Conclusion of the given premises.

A. $P \vee R$ trivially follows from $P \vee R$.

B. $P \vee R$ is given. If R then due to $R \rightarrow S$, S will follow, hence, $P \vee S$ will follow. If P then $P \vee S$ will follow. So, in every case, $P \vee S$ will follow.

C. $P \vee R$ is given. If P then due to $P \rightarrow Q$, Q will follow, hence, $Q \vee R$ will follow. If R then $Q \vee R$ will follow. So, in every case, $Q \vee R$ will follow.

D. $P \vee R$ is given. If P then due to $P \rightarrow Q$, Q will follow, hence, $Q \vee S$ will follow. If R then due to $R \rightarrow S$, S will follow, hence, $Q \vee S$ will follow.

Correct Answer: A; B; C; D

8. **(UGC NET 2015)** In propositional logic, given P and $P \rightarrow Q$, we can infer _______ (Mark all the correct options)

A. $\sim Q$

B. Q

C. $P \wedge Q$

D. $\sim P \wedge Q$

Solution: Given premises: $P, P \rightarrow Q$

Valid Conclusion: Q (By Modus Ponens rule of inference)

Transitively another Valid Conclusion: $P \wedge Q$ (By Conjunction rule of inference).

Correct Answer: B; C

9. Which statement is/are true?

S1: A formula is valid iff its complement is not satisfiable

S2: A formula is satisfiable iff its complement is not valid.

A. Only S1

B. Only S2

C. Both S1 and S2

D. None

Solution:

S1: A formula is valid iff its complement is not satisfiable. This statement is true because a valid formula is one that is true in all possible interpretations, which means there is no interpretation in which its complement is true (since its complement would be false in all interpretations).

S2: A formula is satisfiable iff its complement is not valid. This statement is also true because a formula is satisfiable if and only if there exists an interpretation in which it is true, which means its complement (negation) is not valid because a valid formula is true in all interpretations, and in this case, there exists an interpretation where the complement is false.

Correct Answer: C

10. If S_d is a dual of S then $(S_d)_d \equiv$

A. $\sim S$

B. S

C. T

D. F

Solution: Let's suppose S= A + B

Dual of S is $S_d = A \cdot B$

Dual of S_d is $(S_d)_d = A + B = S$

If S_d is a dual of S then $(S_d)_d \Leftrightarrow S$

NOTE: The dual of the compound proposition that contains only the logical operators $\wedge, \vee, \sim$ is the proposition obtained by replacing each $\vee, \wedge$, by each $\wedge, \vee$, each T by F and each F by T. But negation remains unchanged.

Correct Answer: B

11. How many n-variable propositional functions (distinct) are there such that their truth tables have equal numbers of true and false?

A. $2^{2^{n-1}}$

B. $2^{2^{n/2}}$

C. ${}^{2^n}C_{2^{n-1}}$

D. None

Solution: With n variables, the number of entries in the table = 2^n

For an equal number of true and false values, you can choose any half values out of 2^n as true, and the rest others will be automatically false.

So, from 2^n places/values, we choose $\dfrac{2^n}{2} = 2^{n-1}$ places for 'true'.

Correct Answer: C.

12. Which of the following statements is the contrapositive of the statement, ”You win the game if you know the rules but are not overconfident.”

A. If you lose the game then you don’t know the rules or you are overconfident.

B. A sufficient condition that you win the game is that you know the rules or you are not overconfident.

C. If you don’t know the rules or are overconfident you lose the game.

D. A necessary condition that you know the rules or you are not overconfident is that you win the game.

Solution: The contrapositive of a statement ”If P, then Q” is ”If not Q, then not P.” In this case, the original statement is ”You win the game if you know the rules but are not overconfident.” Let’s break it down into the form ”If P, then Q,” where P represents ”you know the rules” and Q represents ”you win the game.”

Original Statement: If P (you know the rules) and not Q (you are not overconfident), then Q (you win the game).

The contrapositive will be: If not Q (you don’t win the game), then not P (you don’t know the rules or you are overconfident).

So, the contrapositive is:

A. If you lose the game then you don’t know the rules or you are overconfident.

Therefore, the correct answer is A.

13. A sufficient condition that a triangle T be a right triangle is that $a^2+b^2=c^2$. An equivalent statement is _____ (Mark all the correct options)

A. If T is a right triangle then $a^2+b^2=c^2$.

B. If $a^2+b^2=c^2$ then T is a right triangle.

C. T is a right triangle only if $a^2+b^2=c^2$.

D. T is a right triangle unless $a^2+b^2=c^2$.

Solution: Original Statement: ”A sufficient condition that a triangle T be a right triangle is that $a^2+b^2=c^2$.”

A. ”If T is a right triangle, then $a^2+b^2=c^2$.” - This statement means that if T is a right triangle, then it is guaranteed that $a^2+b^2=c^2$ is true. In other words, being a right triangle is a sufficient condition for $a^2+b^2=c^2$.

B. ”If $a^2+b^2=c^2$, then T is a right triangle.” - This statement means that if the Pythagorean theorem, $a^2+b^2=c^2$, is true for the sides of a triangle, then that triangle (T) is a right triangle. This statement also aligns with the original statement because it states that the Pythagorean theorem being true is sufficient for T to be a right triangle.

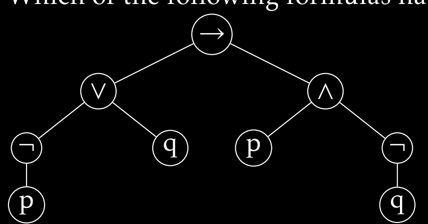
C. ”T is a right triangle only if $a^2+b^2=c^2$.” - This statement is not equivalent to the original statement. It says that T can only be a right triangle if $a^2+b^2=c^2$, which is a necessary condition, not a sufficient condition.

D. ”T is a right triangle unless $a^2+b^2=c^2$.” - This statement is also not equivalent to the original statement. It suggests that T is a right triangle unless $a^2+b^2=c^2$ is true, which is a different condition and not equivalent to saying that $a^2+b^2=c^2$ is a sufficient condition for T to be a right triangle.

So, options A and B are equivalent to the original statement, while options C and D do not express the same idea and are not equivalent.

Correct Answer: A;B

14. Which of the following formulas has the parse tree:



A.$(p \wedge \neg q) \rightarrow (p \vee \neg q)$

B.$(\neg p \vee q) \rightarrow (p \vee \neg q)$

C.$(\neg p \vee q) \rightarrow (p \wedge \neg q)$

D.$(p \vee \neg q) \rightarrow (p \wedge \neg q)$

Solution: Traverse the parse tree from top to bottom and left to right, highest precedence operator should be evaluated first.

$$(\neg p \vee q) \rightarrow (p \wedge \neg q)$$

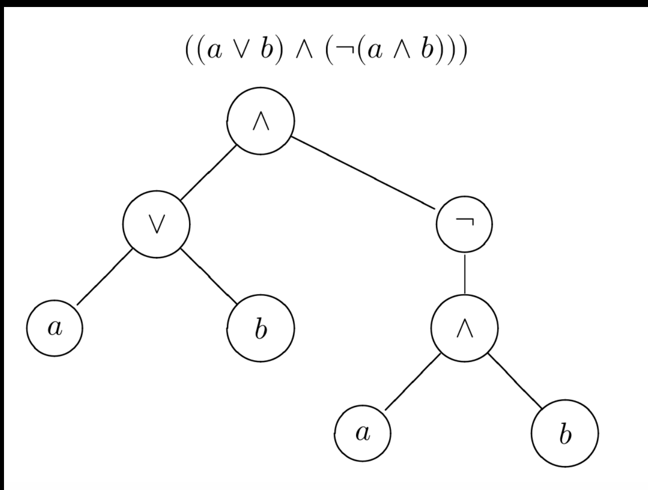
Correct Answer: C

The parse tree of a well-formed formula:

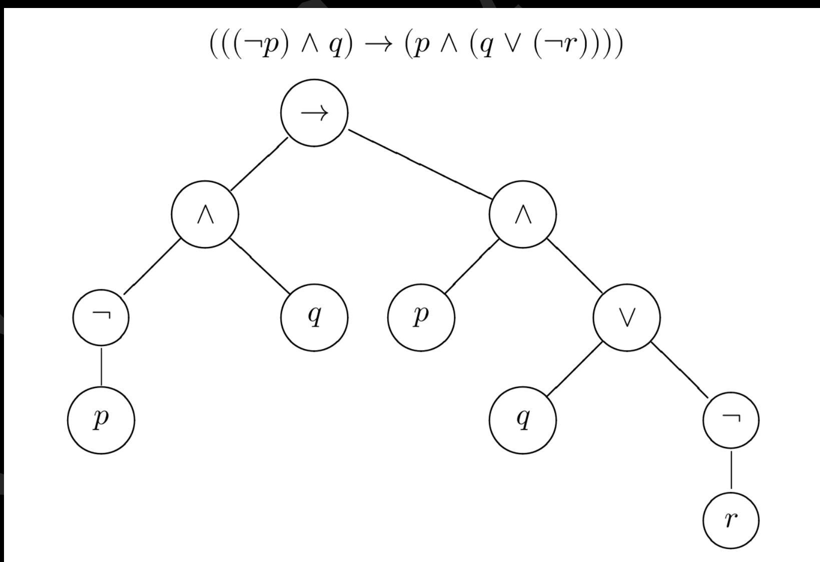
A parse tree is another way to represent a well-formed formula. The parse tree makes the structure of the formula explicit.

The parse tree of the following well-formed formulas:

1. $((a \vee b) \wedge (\neg(a \wedge b)))$



2. $(((\neg p) \wedge q) \rightarrow (p \wedge (q \vee (\neg r))))$



Parse Tree Notes:

1. The leaves are proposition symbols (these are valid trees). 2. All non-leaves are connectives.

3. Negation has one child (a tree)

4. Binary connectives have two children (each is another tree)

5. Can form the tree by starting with the inner most bracket, resolving the bracket then moving upward.

15. Which of the following arguments is valid? (Mark all the correct options)

A. It is snowing or it is raining. It is snowing, therefore it is raining.
B. If there is snow, I will go snowboarding. If I go snowboarding, I will skip the class. There is snow, therefore I will skip the class.
C. I am rich or I don't have to work. I am not rich or I like playing hockey.
Therefore I have to work or I like playing hockey.
D. If you are blonde then you are not smart. You are smart
therefore you are blonde.
Solution:

A.
p: It is snowing
q: It is raining
$p \lor q$
p

$\therefore q$
Using the inference rule, we can't get q.
It is an Invalid argument.
B.
p: There is snow
q: I will go snowboarding
r: I will skip the class
$p \to q$
$q \to r$
p

$\therefore r$
Using the inference rule, we can get r.
It is a valid argument.
C.
p: I am rich
q: I have to work
r: I like playing hockey
$p \lor \neg q$
$\neg q \lor r$

$\therefore q \lor r$
Using inference rule, we can't get $q \lor r$.
Actually we can get $p \lor r$.
It is an Invalid argument.
D.
p: You are blonde
q: You are smart
$p \to \neg q$
q

$\therefore p$
Using the inference rule, we can't get p.
It is an Invalid argument.
Correct Answer: B

1.2 First Order Logic

16. Let P and Q be two predicates.

Which of the following is correct?
i.$\forall x P(x) \lor \forall x Q(x) \implies \forall x[P(x) \lor Q(x)]$
ii.$\forall x[P(x) \lor Q(x)] \implies \forall x P(x) \lor \forall x Q(x)$

iii.$\exists x[P(x) \wedge Q(x)] \Longleftrightarrow \exists x P(x) \wedge \exists x Q(x)$
iv.$\exists x[P(x) \vee Q(x)] \Longleftrightarrow \exists x P(x) \vee \exists x Q(x)$
A. Only i, iii
B. Only i, iv
C. Only i, ii, iv
D. All of i, ii, iii and iv
Solution: Let say in our domain there is x_1, x_2.
Convert the propositional symbol to equivalent digital logic symbol.
$\neg \equiv '$
$\wedge \equiv \cdot$
$\vee \equiv +$
$\forall x P(x) \equiv P(x_1) \wedge P(x_2) \equiv P_1 \wedge P_2 \equiv P_1 \cdot P_2$
$\exists x P(x) \equiv P(x_1) \vee P(x_2) \equiv P_1 \vee P_2 \equiv P_1 + P_2$
i. $\forall x P(x) \vee \forall x Q(x) \Longrightarrow \forall x[P(x) \vee Q(x)]$
$P_1P_2 + Q_1Q_2 \implies (P_1 + Q_1)(P_2 + Q_2)$
$P_1P_2 + Q_1Q_2 \implies P_1P_2 + P_1Q_2 + P_2Q_1 + Q_1Q_2$
Here we can't get $1 \implies 0$.
It is correct statement.
ii. $\forall x[P(x) \vee Q(x)] \Longrightarrow \forall x P(x) \vee \forall x Q(x)$
$(P_1 + Q_1)(P_2 + Q_2) \implies P_1P_2 + Q_1Q_2$
$P_1P_2 + P_1Q_2 + P_2Q_1 + Q_1Q_2 \implies P_1P_2 + Q_1Q_2$
Here we can get $1 \implies 0$.
It is not a correct statement.
iii. $\exists x[P(x) \wedge Q(x)] \Longleftrightarrow \exists x P(x) \wedge \exists x Q(x)$
$P_1Q_1 + P_2Q_2 \Longleftrightarrow (P_1 + P_2)(Q_1 + Q_2)$
$P_1Q_1 + P_2Q_2 \Longleftrightarrow P_1Q_1 + P_1Q_2 + P_2Q_1 + P_2Q_2$
Here we can get $0 \Longleftrightarrow 1$ (or) $1 \Longleftrightarrow 0$.
It is not a correct statement.
iv. $\exists x[P(x) \vee Q(x)] \Longleftrightarrow \exists x P(x) \vee \exists x Q(x)$
$P_1 + Q_1 + P_2 + Q_2 \Longleftrightarrow P_1 + P_2 + Q_1 + Q_2$
$P_1 + P_2 + Q_1 + Q_2 \Longleftrightarrow P_1 + P_2 + Q_1 + Q_2$
Here we can't get $0 \Longleftrightarrow 1$ (or) $1 \Longleftrightarrow 0$.
It is a correct statement.

Correct Answer: B

17. Which of the following formulas does not express that there is at most one element of D that has property $P(x)$?

    A. $\forall x_1, x_2 \in D\,[P(x_1) \wedge P(x_2) \rightarrow x_1 = x_2]$

    B. $\forall x_1, x_2 \in D\,[x_1 \neq x_2 \rightarrow \neg P(x_1) \vee \neg P(x_2)]$

    C. $\neg\exists x_1, x_2 \in D\,[x_1 \neq x_2 \wedge P(x_1) \wedge P(x_2)]$

    D. $\neg\exists x_1, x_2 \in D\,[P(x_1) \wedge x_1 \neq x_2 \rightarrow \neg P(x_2)]$

    Solution: Option A: it is saying that for all x1, x2 in domain D if x1 satisfies property P and x2 satisfies Property P then x1 = x2 means if any two elements satisfy property P then both are the same. So it can easily be inferred that there is at most one element of D that has the property P(x). It is TRUE.

    Option B:- It is saying that for all x1, and x2 in Domain D if x1 and x2 are not equal then either x1 not satisfy Property P or x2 not satisfy Property P or both will not satisfy the Property So it is also True.

    Option C:- There does not exist x1, x2 in Domain D such that both are not same and Both will satisfy Property P. Which is the negation OF option B only. It is also TRUE.

    Option D:- There does not exist x1, x2 in Domain D such that if x1 satisfies Property P and both are not same then x2 does not satisfy Property P which is wrong because it is saying there does not exist x1, x2.

18. We use the dictionary:

M	domain of men
s	Sharon
$N(x)$	x is nice
$L(x,y)$	x loves y

Which of the following formulas corresponds to the sentence:

There is a nice man who loves Sharon.

A. $\exists x \in M[N(x) \wedge L(x,s)]$

B. $\exists x \in M[N(x) \rightarrow L(x,s)]$

C. $\forall x \in M[N(x) \wedge L(x,s)]$

D. $\forall x \in M[N(x) \rightarrow L(x,s)]$

Solution: Here's a concise explanation for each option:

A. $\exists x \in M[N(x) \wedge L(x,s)]$: There exists at least one man (x) who is nice and loves Sharon. This correctly represents the sentence.

B. $\exists x \in M[N(x) \rightarrow L(x,s)]$: There exists at least one man (x) such that if he is nice, he loves Sharon. This does not accurately represent the sentence.

C. $\forall x \in M[N(x) \wedge L(x,s)]$: All men are nice and love Sharon. This is not an accurate representation of the sentence.

D. $\forall x \in M[N(x) \rightarrow L(x,s)]$: All men are such that if they are nice, they love Sharon. This also does not accurately represent the sentence.

Correct Answer: A

19. Consider the following statements

$$\text{S1} : \forall \text{x}(\text{P}(\text{x}) \vee \text{A}) \equiv (\forall \text{xP}(\text{x})) \vee \text{A}$$
$$\text{S2} : \forall \text{x}(\text{P}(\text{x}) \wedge \text{A}) \equiv \forall \text{xP}(\text{X}) \wedge \text{A}$$

Which of the following is valid?

A. S1 only

B. S2 only

C. S1 and S2 both

D. Neither S1 nor S2

Solution: S1: Assume universe of discourse contains elements $\{x1, x2, x3\}$.

$$\forall \text{x}(\text{P}(\text{x})) \equiv \text{p}(\text{x1}) \wedge \text{p}(\text{x2}) \wedge \text{p}(\text{x3})$$
$$\forall \text{x}[\text{p}(\text{x}) \vee \text{A}] = [\text{P}(\text{x1}) \vee \text{A}] \wedge [\text{P}(\text{x2}) \vee \text{A}] \wedge [\text{p}(\text{x3}) \vee \text{A}]$$

$\Leftrightarrow$ According to distributive law

$$\text{P} \vee (\text{Q} \wedge \text{R}) \Leftrightarrow (\text{P} \vee \text{Q}) \wedge (\text{P} \vee \text{R})$$
$$\Leftrightarrow \text{p}(\text{x1}) \wedge \text{P}(\text{x2}) \wedge \text{P}(\text{x3}) \vee \text{A}$$
$$\Leftrightarrow \forall \text{xP}(\text{x}) \vee \text{A}$$

$\text{S2} : \forall \text{x}(\text{P}(\text{x}) \wedge \text{A}) \equiv \forall \text{x}P(\text{X}) \wedge \text{A}$

$$\begin{aligned}\forall \text{x}(\text{P}(\text{x}) \wedge \text{A}) &\equiv (\text{P}(\text{x1}) \wedge \text{A}) \wedge (\text{p}(\text{x2}) \wedge \text{A}) \wedge (\text{P}(\text{x3}) \vee \text{A}) \\ &\equiv \text{P}(\text{x1}) \wedge \text{P}(\text{x2}) \wedge \text{P}(\text{x3}) \wedge \text{A} \wedge \text{A} \wedge \text{A} \\ &\equiv \text{P}(\text{x1}) \wedge \text{P}(\text{x2}) \wedge \text{P}(\text{x3}) \wedge \text{A} \\ &\equiv \forall \text{x}(\text{P}(\text{x}) \wedge \text{A})\end{aligned}$$

Both S1 and S2 are true.

Correct Answer: C

20. Consider the following statements

$$S1: \forall x[P(x) \vee Q(x)] \Leftrightarrow \forall x P(x) \vee \forall x Q(x)$$
$$S2: \exists x[P(x) \wedge Q(x)] \Leftrightarrow \exists x P(x) \wedge \exists x Q(x)$$

Which of the following is true?

A. S1 only

B. S2 only

C. S1 and S2 both

D. Neither S1 nor S2

Solution: Assume that P(x) : x has passed physics exam $Q(x)$: x has passed chemistry exam.

S1: $\forall \mathbf{x}[\mathbf{P}(\mathbf{x}) \vee \mathbf{Q}(\mathbf{x})]$: The description of the above formula is ”every student has either passed physics exam or chemistry exam.

$$\forall \mathbf{x}\mathbf{P}(\mathbf{x}) \vee \forall \mathbf{x}\mathbf{Q}(\mathbf{x})$$

The description of the above formula is ”every student has passed the physics exam or every student has passed the chemistry exam. Whenever LHS is true then RHS need not be true but if RHS is true then definitely LHS is true. Hence $\forall x[P(x) \vee Q(x)] \Rightarrow \forall x P(x) \vee \forall x Q(x)$ But $\forall x P(x) \vee \forall x Q(x) \Longrightarrow \forall x[P(x) \vee Q(x)]$ is false

S2: $\exists x[P(x) \wedge Q(x)] \Leftrightarrow \exists x P(x) \wedge \exists x \mathbf{Q}(x)$ $\exists \mathbf{x}[\mathbf{P}(\mathbf{x}) \wedge \mathbf{Q}(\mathbf{x})]$: There is at least one student who has passed both physics and chemistry exams. $\exists \mathbf{x}\mathbf{P}(\mathbf{x}) \wedge \exists \mathbf{x}\mathbf{Q}(\mathbf{x})$: some student passed in physics and some students has passed in chemistry. When every RHS is true then some student has passed the physics exam and some student has passed the chemistry exam. Then one student may not exist who passed both exams. Hence we can conclude that $\exists x P(x) \wedge \exists x Q(x) \Rightarrow \exists x[P(x) \wedge Q(x)]$ is not true $\exists x[P(x) \wedge Q(x)] \Rightarrow \exists x P(x) \wedge \exists x Q(x)$ is true. Hence S2 is false.

Correct Answer: D

21. $(\forall x P(x) \vee \exists y P(y))$ is equivalent to _____

A. $\exists x(P(x))$

B. $(\forall x P(x))$

C. $\neg(\forall x P(x))$

D. $\neg(\exists x P(x))$

Solution: Original Expression: $(\forall x P(x) \vee \exists y P(y))$

This expression combines two statements:

$\forall x P(x)$: This part means ”For all x, P(x) is true.” It asserts that a certain property P(x) holds true for every element x in the domain.

$\exists y P(y)$: This part means ”There exists some y such that P(y) is true.” It asserts that there is at least one element y in the domain for which the property P(y) is true.

The entire original expression $(\forall x P(x) \vee \exists y P(y))$ states that either ”For all x, P(x) is true” or ”There exists some y such that P(y) is true.” In other words, it means that the property P(x) is true for at least one element in the domain.

This can be simplified as ”There exists an x or a y for which P(x) or P(y) is true.” In logical terms, it means that P(x) is true for at least one element in the domain.

Let’s explore each of the options:

Option A: $\exists x(P(x))$ - This expression means ”There exists an x such that P(x) is true.” It asserts that there is at least one element x in the domain for which the property P(x) is true. Option A is equivalent to the original expression because it covers the case where ”For all x, P(x) is true” as well as the case ”There exists some y such that P(y) is true,” which is what the original expression is saying.

Option B: $(\forall x P(x))$ - This expression means ”For all x, P(x) is true.” It asserts that P(x) is true for every element x in the domain. Option B is not equivalent to the original expression because it doesn’t account for the ”there exists some y” part of the original expression.

Option C: $\neg(\forall x P(x))$ - This expression means "It is not the case that for all x, P(x) is true." It negates the statement that P(x) is true for every element x in the domain. It also does not account for the "there exists some y" part of the original expression, so it's not equivalent.

Option D: $\neg(\exists x P(x))$ - This expression means "It is not the case that there exists an x such that P(x) is true." It negates the statement that there exists at least one element x for which P(x) is true. This is not equivalent to the original expression because it fails to capture both aspects of the original expression, which allows for both "For all x" and "There exists some y."

Correct Answer: A

22. Which of the following is/are correct?

There is a student who is loved by every other student.

A. $\exists$x(Student(x) $\wedge$ $\forall$y(Student(y) $\wedge$ $\neg$(x = y) $\wedge$ Loves(y, x)))

B. $\exists$x(Student(x) $\rightarrow$ $\forall$y(Student(y) $\wedge$ $\neg$(x = y) $\wedge$ Loves(y, x)))

C. $\exists$x(Student(x) $\wedge$ $\forall$y(Student(y) $\wedge$ $\neg$(x = y) $\vee$ Loves(y, x)))

D. $\exists$x(Student(x) $\wedge$ $\forall$y(Student(y) $\wedge$ $\neg$(x = y) $\rightarrow$ Loves(y, x)))

Solution: Let's explore each of the options:

A. $\exists$x(Student(x)$\wedge$$\forall$y(Student(y)$\wedge$$\neg$(x = y)$\wedge$Loves(y, x))): There exists a specific student (x) whom all other students (y) love, excluding themselves. This accurately represents the statement.

B. $\exists$x(Student(x) $\rightarrow$ $\forall$y(Student(y) $\wedge$ $\neg$(x = y) $\wedge$ Loves(y, x))): There exists a student (x), and if that student exists, then all other students (y), excluding themselves, love that student. This option doesn't accurately represent the statement.

C. $\exists$x(Student(x) $\wedge$ $\forall$y(Student(y) $\wedge$ $\neg$(x = y) $\vee$ Loves(y, x))): There exists a specific student (x) who is either loved or not loved by all other students (y). This option doesn't accurately represent the statement.

D. $\exists$x(Student(x) $\wedge$ $\forall$y(Student(y) $\wedge$ $\neg$(x = y) $\rightarrow$ Loves(y,x)): There exists a specific student (x) whom every other student (y) loves, excluding themselves. This accurately represents the statement, and it's the correct choice.

Correct Answer: D

23. Which of the following statements is/are False?

A. $\exists x(P(x) \rightarrow Q(x)) \equiv \forall x P(x) \rightarrow \exists x Q(x)$

B. $\exists x(P(x) \vee Q(x)) \equiv \exists x(P(x)) \vee \exists x(Q(x))$

C. $\forall x(P(x) \wedge Q(x)) \equiv \forall x(P(x)) \wedge \forall x(Q(x))$

D. $\exists x(P(x) \wedge Q(x)) \equiv \exists x(P(x)) \wedge \exists x(Q(x))$

Solution: Let's analyze each statement:

A. $\exists x(P(x) \rightarrow Q(x)) \equiv \forall x P(x) \rightarrow \exists x Q(x)$

This statement is true. It represents the logical equivalence between the existential quantifier and the universal quantifier in the context of conditional statements. It's known as the exportation law and is a valid logical equivalence.

B. $\exists x(P(x) \vee Q(x)) \equiv \exists x(P(x)) \vee \exists x(Q(x))$

This statement is true. It represents the logical equivalence of distributing the existential quantifier over a disjunction (OR) in both directions, which is a valid equivalence.

C. $\forall x(P(x) \wedge Q(x)) \equiv \forall x(P(x)) \wedge \forall x(Q(x))$

This statement is true. It represents the logical equivalence of distributing the universal quantifier over a conjunction (AND) in both directions, which is a valid equivalence.

D. $\exists x(P(x) \wedge Q(x)) \equiv \exists x(P(x)) \wedge \exists x(Q(x))$

This statement is false. It represents an incorrect application of quantifier distribution. The correct form of distribution in this case is the one presented in statement B, where the existential quantifier is distributed over a disjunction (OR), not over a conjunction (AND).

So, the false statement is D, and the other statements are true.

Correct Answer: D

24. Which of the following first-order logic statements are equivalent?

A. $\sim \forall \mathbf{x}\{\mathbf{P}(\mathbf{x}) \vee \exists \mathbf{y}[\mathbf{Q}(\mathbf{y}) \wedge \mathbf{P}(\mathbf{y})]\} \equiv \exists \mathbf{x}\{\sim \mathbf{P}(\mathbf{x}) \wedge \forall \mathbf{y}[(\mathbf{P}(\mathbf{y}) \rightarrow \sim Q(y)) \vee (\mathbf{Q}(\mathbf{y}) \rightarrow \sim P(\mathbf{y}))]\}$

B. $\sim \forall \mathbf{x}\{\mathbf{P}(\mathbf{x}) \vee \exists \mathrm{y}[\mathrm{Q}(\mathrm{y}) \wedge \mathbf{P}(\mathbf{y})]\} \equiv \exists \mathbf{x}\{\sim \mathbf{P}(\mathbf{x}) \wedge \forall \mathbf{y}[\sim (\mathbf{P}(\mathbf{y}) \rightarrow \sim Q(y)) \vee (\sim \mathrm{Q}(\mathrm{y}) \rightarrow \sim \mathrm{P}(\mathrm{y}))]\}$

C. $\sim \forall \mathbf{x}\{\mathbf{P}(\mathbf{x}) \vee \exists \mathbf{y}[\mathbf{Q}(\mathbf{y}) \wedge \mathbf{P}(\mathbf{y})]\} \equiv \exists \mathbf{x}\{\sim \mathbf{P}(\mathbf{x}) \wedge \forall \mathbf{y}[(\mathbf{P}(\mathbf{y}) \rightarrow \sim Q(y)) \vee (\sim \mathbf{Q}(\mathbf{y}) \rightarrow \sim \mathbf{P}(\mathbf{y}))]\}$

D. $\sim \forall \mathbf{x}\{\mathbf{P}(\mathbf{x}) \vee \exists \mathbf{y}[\mathbf{Q}(\mathbf{y}) \wedge \mathbf{P}(\mathbf{y})]\} \equiv \exists \mathbf{x}\{\sim \mathbf{P}(\mathbf{x}) \wedge \forall \mathbf{y}[\sim (\mathbf{P}(\mathbf{y}) \rightarrow \sim Q(y)) \vee (\mathbf{Q}(\mathbf{y}) \rightarrow \sim \mathbf{P}(\mathbf{y}))]\}$

Solution:

Correct Answer: A

25. Which of the following formulas is a formalization of the sentence: "There is a computer which is not used by any student"

A. $\exists \mathrm{x}(\mathrm{Computer}(\mathrm{x}) \wedge \forall \mathrm{y}(\neg \mathrm{Student}(\mathrm{y}) \wedge \neg \mathrm{Uses}(\mathrm{y}, \mathrm{x})))$

B. $\exists \mathrm{x}(\mathrm{Computer}(\mathrm{x}) \rightarrow \forall \mathrm{y}(\mathrm{Student}(\mathrm{y}) \rightarrow \neg \mathrm{Uses}(\mathrm{y}, \mathrm{x})))$

C. $\exists \mathrm{x}(\mathrm{Computer}(\mathrm{x}) \wedge \forall \mathrm{y}(\mathrm{Student}(\mathrm{y}) \rightarrow \neg U \,\mathrm{Uses}(\mathrm{y}, \mathrm{x})))$

D. $\exists \mathrm{x}(\mathrm{Computer}(\mathrm{x}) \rightarrow \forall \mathrm{y}(\neg \mathrm{Student}(\mathrm{y}) \wedge \neg \mathrm{Uses}(\mathrm{y}, \mathrm{x})))$

Solution:

Correct Answer: C

26. Consider the following statement

$$\exists x \exists y(\text{parent}(x, \text{Ramu}) \wedge \text{parent}(y, \text{Ramu}))$$

where parent (x, y) means x is a parent of y. Which of the following statements is true about the above first-order logic statement?

A. Ramu has at least one parent

B. Ramu has at least two parents

C. Ramu has at most one parent

D. Ramu has at most two parents

Solution:

Correct Answer: A

27.
$$S_1 : \forall_x (P(x) \rightarrow A) \equiv \exists_x P(x) \rightarrow A$$
$$S_2 : \exists_x (P(x) \rightarrow A) \equiv \forall_x P(x) \rightarrow A$$

Which of the following statements is true?

A. Only S1

B. Only S2

C. Both S1 and S2

D. Neither S1 nor S2

Solution:

Correct Answer: C

28. Consider a domain $S = \{1, 2, 3, 4\}$

$$\mathrm{P}(\mathrm{x}, \mathrm{y}) : \mathrm{x} * \mathrm{y} \geq 2$$

Which of the following statements is true? (Mark all the correct options)

A. $\forall \mathrm{x} \forall \mathrm{y} P(\mathrm{x}, \mathrm{y})$

B. $\forall \mathrm{x} \exists \mathrm{y} P(\mathrm{x}, \mathrm{y})$

C. $\exists \mathrm{x} \forall \mathrm{y} P(\mathrm{x}, \mathrm{y})$

D. $\exists \mathrm{x} \exists \mathrm{y} P(\mathrm{x}, \mathrm{y})$

Solution:

A. $\forall x \forall y P(x,y)$:

$$\forall x P(x) = P(e1) \wedge P(e2) \wedge \ldots\ldots P(en)$$

Where e1, e2, en are all the elements in the domain Similarly

$$\begin{aligned} \forall x \forall y P(x,y) = & P(1,1) \wedge P(1,2) \wedge P(1,3) \wedge P(1,4) \wedge P(2,1) \wedge P(2,2) \wedge P(2,3) \\ & P(2,4) \wedge P(3,1) \ldots\ldots P(4,4) \\ P(1,1) = & 1 * 1 \geq 2 \Rightarrow \text{ false} \\ = & F \wedge P(1,2) \wedge P(1,3) \ldots\ldots P(4,4) \\ = & F \end{aligned}$$

B. $\forall x \exists y P(x,y) \Leftrightarrow \exists y P(1,y) \wedge \exists y P(2,y) \wedge \exists y P(3,y) \wedge \exists y P(4,y)$ $\exists y p(1,y) \Leftrightarrow P(1,1) \vee P(1,2) \vee P(1,3) \vee P(1,4)$ Since $P(1,4) = 1 * 4 \geq 2$, which is true total formula $\exists y P(1,y)$ is true.

$(P(1,1) \vee P(1,2) \vee P(1,3) \vee T \Leftrightarrow T$

Similarly we can prove that $\exists y P(2,y) \Leftrightarrow T$ $\exists y P(3,y) \Leftrightarrow T$ $\exists y P(4,y) \Leftrightarrow T$ Hence $\forall x \exists y P(x,y) \Leftrightarrow T$

C. $\exists x \forall y P(x,y)$; is there any x so that P(x,y) true for all y ? x cannot be 1 since $1^* y \geq 2$ cannot be true for all y x can be 2 since $2x1 \geq 2$

$$\begin{aligned} 2 \times 2 \geq 2 \\ 2 \times 3 \geq 2 \\ 2 \times 4 \geq 2 \end{aligned}$$

That means P(2, y) is true for all y in the domain hence $\exists x \forall y P(x,y) \Leftrightarrow T$

IV. $\exists x \forall y P(x,y)$; can you show atleast one x, atleast one y so that p(x, y) is true. Yes when x = 4 y = 4 $x^* y \geq 2$ is true.

Hence it is true.

Correct Answer: B; C; D

29. Which of the following FOL statements is incorrect? (Mark all the correct options)

i. $\forall n \in \mathbb{N}. \exists m \in \mathbb{N}. n < m$

ii. $\exists n \in \mathbb{N}. \forall m \in \mathbb{N}. n < m$

iii. $\forall n \in \mathbb{N}. \forall m \in \mathbb{N}. (n < m \rightarrow \exists p \in \mathbb{N}. (n < p \wedge p < m))$

iv. $\forall n \in \mathbb{R}. \forall m \in \mathbb{R}. (n < m \rightarrow \exists p \in \mathbb{R}. (n < p \wedge p < m))$

v. $\forall n \in \mathbb{N}. \forall m \in \mathbb{N}. \exists p \in \mathbb{N}. (n = p \cdot m)$

vi. $\forall \mathrm{n} \in \mathbb{R}. \forall \mathrm{m} \in \mathbb{R}. \exists p \in \mathbb{R}. (n = p \cdot m)$

A. i, ii, iii

B. ii, iii

C. ii, v, vi

D. ii, iii, vi

Solution: i. $\forall n \in \mathbb{N}. \exists m \in \mathbb{N}. n < m$

True. This says "for every natural number, there's a larger natural number."

ii. $\exists n \in \mathbb{N}. \forall m \in \mathbb{N}. n < m$

False. This says "there is a natural number that's smaller than all natural numbers." No matter what you pick for n, if you pick $m = n$, then you'll have $n \geq m$. Remember that quantifiers can talk about the same object at the same time!

iii. $\forall n \in \mathbb{N}. \forall m \in \mathbb{N}. (n < m \rightarrow \exists p \in \mathbb{N}. (n < p \wedge p < m))$

False. This says "there is a natural number between any two natural numbers." If you pick $n = 0$ and $m = 1$, you cannot find a natural number p where $0 < p$ and $p < 1$.

iv. $\forall n \in \mathbb{R}. \forall m \in \mathbb{R}. (n < m \rightarrow \exists p \in \mathbb{R}. (n < p \wedge p < m))$

True. This says "there is a real number between any two real numbers." Given two different real numbers n and m, the real number $(n+m)/2$ is between n and m.

v. $\forall n \in \mathbb{N}.\forall m \in \mathbb{N}.\exists p \in \mathbb{N}.(n = p \cdot m)$

False. This says "for any two natural numbers, the first is a multiple of the second." Try picking $n = 5$ and $m = 3$; there's no choice of p that works.

vi. $\forall \mathrm{n} \in \mathbb{R}.\forall \mathrm{m} \in \mathbb{R}.\exists p \in \mathbb{R}.(n = p \cdot m)$

False. This says "for any two real numbers, there is a real number you can multiply the second number by to get the first." Try picking $n = 1$ and $m = 0$.

Correct Answer: B;C;D

30. Which of the following is/are valid? (Mark all the appropriate answers)

A. $P(A) \Rightarrow \forall x P(x)$

B. $P(A) \Rightarrow \forall x \neg P(x)$

C. $P(A) \Rightarrow \exists x P(x)$

D. $P(A) \Rightarrow \exists x \neg P(x)$

Solution: For each of the following sentences in first-order logic, specify whether it is valid, satisfiable, and/or unsatisfiable:

A. $P(A) \Rightarrow \forall x P(x)$

Satisfiable but not valid.

B. $P(A) \Rightarrow \forall x \neg P(x)$

Satisfiable but not valid.

C. $P(A) \Rightarrow \exists x P(x)$

Valid.

D. $P(A) \Rightarrow \exists x \neg P(x)$

Satisfiable but not valid.

Correct Answer: C

2 Set Theory & Algebra

2.1 Set Theory

31. Which of the following statements is FALSE?

A. $\{2,3,4\} \subseteq A$ implies that $2 \in A$ and $\{3,4\} \subseteq A$.

B. $\{2,3,4\} \in A$ and $\{2,3\} \in B$ implies that $\{4\} \subseteq A - B$.

C. $A \cap B \supseteq \{2,3,4\}$ implies that $\{2,3,4\} \subseteq A$ and $\{2,3,4\} \subseteq B$.

D. $A - B \supseteq \{3,4\}$ and $\{1,2\} \subseteq B$ implies that $\{1,2,3,4\} \subseteq A \cup B$.

Solution:

Let's explore each option:

A. $\{2,3,4\} \subseteq A$ mean $2,3,4$ are in A for sure and some extra elements i.e elements $2,3,4 \in A$. Since elements $3,4 \in A$ we can say $3,4 \subseteq A$

B. $\{2,3,4\} \in A$ mean $\{2,3,4\}$ is an element in A. Take $A = \{1,\{2,3,4\}\}$ and $B = \{1,\{2,3\}\}$. $A - B = \{\{2,3,4\}\}$ which doesn't have 4 as an element Therefore 4 not $\subseteq A - B$

C. $\{2,3,4\} \subseteq A \cap B$ means both A and B have $2,3,4$ elements in common for sure and they can have more elements in common too. So we can say $\{2,3,4\} \subseteq A$ and $\{2,3,4\} \subseteq B$

D. $\{3,4\} \subseteq A - B$ that means A has $3,4$ for sure and some extra elements which are not in common with B $\{1,2\} \subseteq B$ mean $1,2$ are in B for sure and some extra elements. So, $A \cup B$ will have $1,2,3,4$ for sure. So we can say $\{1,2,3,4\} \subseteq A \cup B$

Correct Answer: B

32. Let $A = \{0,1\} \times \{0,1\}$ and $B = \{a,b,c\}$. Suppose A is listed in lexicographic order based on $0 < 1$ and B is in alphabetic order. If $A \times B \times A$ is listed in lexicographic order, then the next element after $((1,0),c,(1,1))$ is ______.

A. $((1,0),a,(0,0))$

B. $((1,1),c,(0,0))$

C. $((1,1),a,(0,0))$

D. $((1,1),b,(1,1))$

Solution: The ordered triplets A are in the order:

$$A = \{(0,0),(0,1),(1,0),(1,1)\}$$

After $(1,0)$, the whole $(1,0)$ part got completed. So, now the $(1,1)$ part will start at the first position in ordered triplets. The sequence becomes:

$$((1,0),c,(1,1))$$
$$((1,1),a,(0,0))$$

Correct Answer: C

33. Which of the following statements is TRUE? (Mark all the appropriate choices)

A. For all sets A, B, and $C, A-(B-C) = (A-B)-C$.

B. For all sets A, B, and $C, (A-B) \cap (C-B) = (A \cap C) - B$.

C. For all sets A, B, and $C, (A-B) \cap (C-B) = A - (B \cup C)$.

D. For all sets A, B, and C, if $A \cap C = B \cap C$ then $A = B$.

Solution:

Correct Answer: B

34. Consider the true theorem, "For all sets A and B, if $A \subseteq B$ then $A \cap B^c = \emptyset$." Which of the following statements is NOT equivalent to this statement: (Mark all the appropriate choices)

A. For all sets A^c and B, if $A \subseteq B$ then $A^c \cap B^c = \emptyset$.

B. For all sets A and B, if $A^c \subseteq B$ then $A^c \cap B^c = \emptyset$.

C. For all sets A^c and B^c, if $A^c \subseteq B^c$ then $A^c \cap B = \emptyset$.

D. For all sets A and B, if $A^c \supseteq B$ then $A \cap B = \emptyset$.

Solution:

Correct Answer: A

35. The power set $\mathcal{P}((A \times B) \cup (B \times A))$ has the same number of elements as the power set $\mathcal{P}((A \times B) \cup (A \times B))$ if and only if

A. $A = \emptyset$ or $B = \emptyset$

B. $B = \emptyset$ or $A = B$

C. $A = \emptyset$ or $B = \emptyset$ or $A = B$

D. $A = \emptyset$ or $B = \emptyset$ or $A \cap B = \emptyset$

Solution:

Correct Answer: C

36. Let $\mathcal{P}(A)$ denote the power set of A. If $\mathcal{P}(A) \subseteq B$ then

A. $2^{|A|} \leq |B|$

B. $2^{|A|} \geq |B|$

C. $2|A| < |B|$

D. $2^{|A|} \geq 2^{|B|}$

Solution:

Correct Answer: A

2.2 Functions

37. Let $f : \{1,2,3,4,5,6,7,8,9\} \to \{a,b,c,d,e\}$. In one-line notation, $f = (e,a,b,b,a,c,c,a,c)$ (use number order on the domain).

 Which is correct? (Mark all the appropriate choices)

 A. Image $(f) = \{a,b,c,d,e\}$, Coimage $(f) = \{\{6,7,9\},\{2,5,8\},\{3,4\},\{1\}\}$

 B. Image $(f) = \{a,b,c,e\}$, Coimage $(f) = \{\{6,7,9\},\{2,5,8\},\{3,4\}\}$

 C. Image $(f) = \{a,b,c,e\}$, Coimage $(f) = \{\{6,7,9\},\{2,5,8\},\{3,4\},\{1\}\}$

 D. Image $(f) = \{a,b,c,d,e\}$, Coimage $(f) = \{\{1\},\{3,4\},\{2,5,8\},\{6,7,9\}\}$

 Solution:

 Correct Answer: C

38. The number of partitions of $\{1,2,3,4,5\}$ into three blocks is $S(5,3) = 25$. The total number of functions $f : \{1,2,3,4,5\} \to \{1,2,3,4\}$ with
 | Image(f) |= 3 is _______.

 A. 4×25

 B. 25×6

 C. $4 \times 25 \times 6$

 D. $3 \times 25 \times 6$

 Solution:

 Correct Answer: C

39. Let $f : X \to Y$ and $g : Y \to Z$. Let $h = g \circ f : X \to Z$. Suppose g is one-to-one and onto. Which of the following is/are FALSE? (Mark all the appropriate choices)

 A. If f is one-to-one then h is one-to-one and onto.

 B. If f is not onto then h is not onto.

 C. If f is not one-to-one then h is not one-to-one.

 D. If f is one-to-one then h is one-to-one.

 Solution:

 Correct Answer: A

40. Define $f(n) = \frac{n}{2} + \frac{1-(-1)^n}{4}$ for all $n \in \mathbb{Z}$. Thus, $f : \mathbb{Z} \to \mathbb{Z}, \mathbb{Z}$ the set of all integers. Which is/are correct? (Mark all the appropriate choices)

 A. f is not a function from $\mathbb{Z} \to \mathbb{Z}$ because $\frac{n}{2} \notin \mathbb{Z}$.

 B. f is a function and is onto and one-to-one.

 C. f is a function and is not onto and not one-to-one

 D. f is a function and is onto but not one-to-one.

 Solution:

 Correct Answer: D

41. Let $f : X \to Y$. Consider the statement, "For all subsets C and D of $Y, f^{-1}(C \cap D^c) = f^{-1}(C) \cap \left[f^{-1}(D)\right]^c$. This statement is

 A. True and equivalent to: For all subsets C and D of $Y, f^{-1}(C-D) = f^{-1}(C) - f^{-1}(D)$.

 B. False and equivalent to: For all subsets C and D of $Y, f^{-1}(C-D) = f^{-1}(C) - f^{-1}(D)$.

 C. True and equivalent to: For all subsets C and D of $Y, f^{-1}(C-D) = f^{-1}(C) - \left[f^{-1}(D)\right]^c$.

 D. False and equivalent to: For all subsets C and D of $Y, f^{-1}(C-D) = f^{-1}(C) - \left[f^{-1}(D)\right]^c$.

 Solution:

 Correct Answer: A

42. Which of the following is/are correct? (Mark all the appropriate choices)

A. If $f : X \to Y$ is one-to-one, then $|X| \le |Y|$, and if f is onto, then $|X| \ge |Y|$, so if f is both, $|X| = |Y|$.

B. If $f : X \to Y$ is a one-to-one and onto, then f^{-1} is a one-to-one and onto function.

C. The number of injective functions from a set with three elements to a set with four elements is 24.

D. The number of surjective functions from a set with four elements to a set with three elements is 36.

Solution:

C. Suppose $X = \{a, b, c\}$ and $Y = \{u, v, w, x\}$ and suppose $f : X \to Y$ is a function. Then we can define f by simply specifying the images of a, b, and c, and to make it injective, we need to make sure none of the images are the same. But this will simply be equal to the number of 3 permutations from the set Y, and thus there will be a total of $4 \cdot 3 \cdot 2 = 24$ total different injective functions from X to Y.

Next note that if X has four elements and Y has three elements, no function from X to Y will be injective since at least two elements from X must map to the same element in Y.

D. Suppose $f : X \to Y$ is a function. If Y has four elements and X has three elements, then no function from X to Y will be surjective since there are at most three images under f (since there are only three elements in X.

Now suppose $X = \{a, b, c, d\}$ and $Y = \{u, v, w\}$. Then we can define f by simply specifying the images of a, b, c, and d, and to make it surjective, we need to make sure every element in the codomain is mapped to by at least one element. Note that if f is onto, then one of u, v and w will have two elements in its preimage and the others will have a single element. Therefore, we shall break up the set of surjective maps into three subsets - those where $u \in Y$ has two preimages, those where $v \in Y$ has two preimages, and those where $w \in Y$ has two preimages. Note that by symmetry, each of these sets will have the same size, so we shall just consider the first. Suppose that $f : X \to Y$ is a surjective map and $u \in Y$ has two preimages. We can think of all the different ways of constructing f as a counting problem. Specifically, for the first step, to determine $f^{-1}(u)$, we need to choose 2 objects from 4 , so there are $\begin{pmatrix} 4 \\ 2 \end{pmatrix}$ possibilities. Next, for $f^{-1}(v)$, there are two remaining possibilities, and for $f^{-1}(w)$, one last possibility. Thus the total number of possibilities will be $\begin{pmatrix} 4 \\ 2 \end{pmatrix} \cdot 2 \cdot 1 = 12$. Since we will get equal numbers if we assume either v or w have two preimages, it follows that there are a total of $3 \cdot 12 = 36$ surjective maps.

Correct Answer: A;B;C;D

2.3 Relation

43. If the number of binary relations on a set A, where $|A| = 5$ is α. Then the value of $\alpha^0 + \alpha$ is ______. (Numerical Answer Type)

Solution: The number of binary relations on a set A, where $|A| = n$ is: 2^{n^2}

Proof: If $|A| = n$ then the cardinality of the Cartesian product $|A \times A| = n^2$. R is a binary relation on A if $R \subseteq A \times A$ (that is, R is a subset of $A \times A$).

The number of subsets of a set with k elements: 2^k

The number of subsets of $A \times A$ is : $2^{|A \times A|} = 2^{n^2}$

$\therefore \alpha = 2^{5^2} = 2^{25} = 33554432$

Now, the value of $\alpha^0 + \alpha = 1 + 33554432 = 33554433$.

Correct Answer: 33554433

44. Let $\mathbb{N}^+$denote the nonzero natural numbers. Define a binary relation R on $\mathbb{N}^+ \times \mathbb{N}^+$ by $(m, n)R(s, t)$ if $\gcd(m, n) = \gcd(s, t)$. The binary relation R is ________.

A. Reflexive, Not Symmetric, Transitive

B. Reflexive, Symmetric, Transitive

C. Reflexive, Symmetric, Not Transitive

D. Reflexive, Not Symmetric, Not Transitive

Solution:

Correct Answer: B

45. Given that $A = \{1, 2, 3, 4\}$ and R is the relation defined by $(x, y) \in R$ if $3x + 2y \leq 11$
Which of the following statements is true?

A. It is reflexive

B. It is symmetric

C. It is transitive

D. None of the above

Solution:

Correct Answer: D

46. Let $\mathbb{N}_2^+$denote the natural numbers greater than or equal to 2. Let mRn if $\gcd(m, n) > 1$. The binary relation R on $\mathbb{N}_2$ is ________.

A. Reflexive, Symmetric, Not Transitive

B. Reflexive, Not Symmetric, Transitive

C. Reflexive, Not Symmetric, Not Transitive

D. Not Reflexive, Symmetric, Not Transitive

Solution:

Correct Answer: A

47. Let R and S be binary relations on a set A. Suppose that R is reflexive, symmetric, and transitive and that S is symmetric, and transitive but is not reflexive. Which statement is always true for any such R and S?

A. $R \cup S$ is symmetric but not reflexive and not transitive.

B. $R \cup S$ is transitive and symmetric but not reflexive.

C. $R \cup S$ is reflexive and symmetric.

D. $R \cup S$ is symmetric but not transitive.

Solution:

Correct Answer: C

48. Define an equivalence relation R on the positive integers $A = \{2, 3, 4, \ldots, 20\}$ by mRn if the largest prime divisor of m is the same as the largest prime divisor of n. The number of equivalence classes of R is ______. (Numerical Answer Type)

Solution:

Correct Answer: 8

2.4 Partial Orders and Lattices

49. Let $R = \{(a, a), (a, b), (b, b), (a, c), (c, c)\}$ be a partial order relation on $\Sigma = \{a, b, c\}$. Let $\preceq$ be the corresponding lexicographic order on Σ^*. Which of the following is/are true? (Mark all the appropriate choices)

A. $bc \preceq ba$

B. $abbaaacc \preceq abbaab$

C. $abbac \preceq abbab$

D. $abbac \preceq abbaac$

Solution:

Correct Answer: B

50. Consider the divides relation, $m \mid n$, on the set $A = \{2, 3, 4, 5, 6, 7, 8, 9, 10\}$. The cardinality of the covering relation for this partial order relation (i.e., the number of edges in the Hasse diagram) is _______.

A. 4

B. 6

C. 5

D. 7

Solution:

Correct Answer: D

51. Consider the divides relation, $m \mid n$, on the set $A = \{2, 3, 4, 5, 6, 7, 8, 9, 10\}$. Which of the following permutations of A is not a topological sort of this partial order relation?

A. $2, 3, 7, 6, 9, 5, 4, 10, 8$

B. $2, 6, 3, 9, 5, 7, 4, 10, 8$

C. $3, 7, 2, 9, 5, 4, 10, 8, 6$

D. $3, 2, 6, 9, 5, 7, 4, 10, 8$

Solution:

Correct Answer: B

52. Let $A = \{2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16\}$ and consider the divides relation on A. Let C denote the length of the maximal chain, M the number of maximal elements, and m the number of minimal elements. Which is true?

A. $C = 3, M = 8, m = 6$

B. $C = 4, M = 8, m = 6$

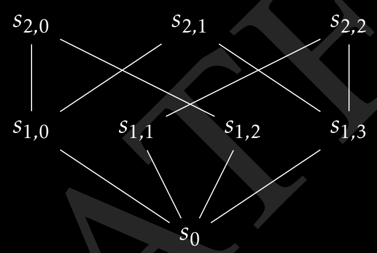
C. $C = 3, M = 6, m = 6$

D. $C = 4, M = 6, m = 4$

Solution:

Correct Answer: A

53. Given the following Hasse diagram, which of the following is correct?



A. Join semi lattice
B. Meet semi lattice
C. Lattice
D. Not a lattice
Solution: If $'a'$ is an upper bound for $'S'$ which is related to all other upper bounds then it is the least upper bound, denoted lub(S).
If $'a'$ is a lower bound for S and all other lower bounds which are related to $'a'$ then it is the greatest lower bound, denoted glb(S).
A post S is called meet semi lattice if $\forall a, b \in L$ glb(a, b) is not empty.
A poset S is called a join semi lattice if $\forall a, b \in L$ lub(a, b) is not empty.
A poset is a lattice if every pair of elements has a lub and a glb.
LUB of $\{ s_{2,0}, s_{2,1} \}$ = does not exist.

So, it is not a join semi-lattice.

But it meets semi-lattice because every pair of elements glb exists.

Correct Answer: B

54. Which of the following is/are correct? (Mark all the appropriate choices)

A. If there is a distributive lattice, then every element should have at most one compliment.

B. If there is a complemented lattice, then every element should have at least one complement.

C. If there is a Boolean lattice, then every element has exactly one complement.

D. None of the above

Solution:

Correct Answer: D

2.5 Group Theory

55. Which of the following is/are correct? (Mark all the appropriate choices)

A. If $(G,*)$ is a group and $a \in G$, then $a * a = a$ implies $a = e$.

B. A group $(G,*)$ is abelian if $a * b = b * a$ for all elements $a, b \in G$.

C. If G is a group, H and K are two subgroups of G, then $H \cap K$ is also a subgroup of G.

D. Let $G = G_1 \times G_2$ be a finite group with $\gcd(|G_1|, |G_2|) = 1$. Then every subgroup H of G is of the form $H = H_1 \times H_2$ where H_i is a subgroup of G_i for $i = 1, 2$.

Solution:

A. Suppose $a \in G$ satisfies $a*a = a$ and let $b \in G$ be such that $b*a = e$. Then $b*(a*a) = b*a$ and thus

$$a = e * a = (b * a) * a = b * (a * a) = b * a = e$$

B.

C. This is true. It is enough to check that $xy^{-1} \in H \cap K$ for $x, y \in H \cap K$. But since $x, y \in H$, we have $xy^{-1} \in H$ since H is a subgroup, and likewise, $xy^{-1} \in K$ for $x, y \in K$ since K is a subgroup.

D. Let H be a subgroup of G. Let π_i be the natural projection from G to G_i. Then the restriction of π_i to H gives homomorphisms from H to G_i for $i = 1, 2$. Let $H_i = \pi_i(H)$ for $i = 1, 2$. Then clearly $H \le H_1 \times H_2$ and $H_i \le G_i$ for $i = 1, 2$. Then $H/\mathrm{Ker}(\pi_1) \cong H_1$ implies that $|H_1| | |H|$ similarly $|H_2| | |H|$. But $\gcd(|H_1|, |H_2|) = 1$ implies that $|H_1||H_2| | |H|$. So $H = H_1 \times H_2$.

Correct Answer: A;B;C;D

56. Let G be a finite group on 243 elements. The size of the largest possible proper subgroup of G is ______. (Numerical Answer Type)

Solution: Lagrange's theorem, states that for any finite group G, the order (number of elements) of every subgroup H of G divides the order of G.
Divisors of the Positive Integer 243 : $1, 3, 9, 27, 81, 243$ All are possible proper subgroups excluding the size of 1 and 243 because they are trivial subgroups but here the question is asking, the Largest possible proper subgroup possible is 81.
Correct Answer: 81

57. What is the number of elements in the cyclic subgroup of $\mathbb{Z}_{30}$ generated by 25? (Numerical Answer Type)

Solution: The "hard" way to do this is to actually list all the elements of $\langle 25 \rangle \le \mathbb{Z}_{30}$. This isn't actually difficult, since the subgroup is so small, $\langle 25 \rangle = \{25, 20, 15, 10, 5, 0\}$. The order of the subgroup is 6.

The clever way to find the order is to use the theorem: In $\mathbb{Z}_n$, $|\langle i \rangle| = \frac{n}{\gcd(n,i)}$. Hence, $|\langle 25 \rangle| = \frac{30}{\gcd(30,25)} = \frac{30}{5} = 6$.

Correct Answer: 6

58. Let G be a group with order 45, and H a non-abelian subgroup of G.
Assuming $H \neq G$,what is the largest order of H?
(Numerical Answer Type)
Solution: The only possible orders are (proper) divisors of 45 which are $1, 3, 5, 9, 15$ All groups of order $1, 3, 5$ are abelian.

1 is trivial, and any group of prime order must be cyclic, generated by any nontrivial element.The fact that Group of order less than 6 is Abelian.
That leaves $9, 15$.
We choose 15, because,it needs largest subgroup.
Therefore $\mid H \mid = 15, \mid G \mid = 45$, and H is non-abelian.

Theorem: All Groups with less than 6 is Abelian.

Proof: Let G be a non-abelian group.

From Non-Abelian Group has Order Greater than 4, the order of G must be at least 5.

But 5 is a prime number.

By Prime Group is Cyclic it follows that a group of order 5 is cyclic.

By Cyclic Group is Abelian this group is abelian.

Hence proved.

Correct Answer: 15

59. Suppose G is a group that has exactly 48 distinct elements of order 5.

How many distinct subgroups of order 5 does G have?

A.12

B.16

C.6

D.4

Solution: Let $'a'$ be an element in G of order 5. Then the subgroup $\langle g \rangle$ generated by $'a'$ is a cyclic group of order 5.

That is, $\langle a \rangle = \{e, a, a^2, a^3, a^4\}$, where $'e'$ is the identity element in G.

Note that the order of each non-identity element in $\langle a \rangle$ is 5.

Also, if $'b'$ is another element in $'G'$ of order 5, then we have either $\langle a \rangle = \langle b \rangle$ (or) $\langle a \rangle \cap \langle b \rangle = \{e\}$.

This follows from the fact that the intersection $\langle a \rangle \cap \langle b \rangle = \{e\}$ is a subgroup of the order 5 group $\langle a \rangle$, and thus the order of $\langle a \rangle \cap \langle b \rangle = \{e\}$ is either 5 (or) 1.

On the other hand, if H is a subgroup of G of order 5, then every non-identity element in H has order 5.

These observations imply that each subgroup of order 5 contains exactly 4 elements of order 5 and each element of order 5 appears in exactly one of such subgroups.

As there are 48 elements of order 5, there are $\frac{48}{4} = 12$ subgroups of order 5.
Correct Answer: A

60. Let $G = \langle a \rangle$. Find the smallest subgroup of G containing a^{2020} and a^{1719} is?

A. a^4

B. a^{1719}

C. a^{2020}

D. a^1

Solution:

Theorem: Let $G = \langle a \rangle$, and let H be the smallest subgroup of G that contains a^m and a^n, then $H = \left(a^{\gcd(n,m)} \right)$.

By above theorem, the smallest subgroup of G containing a^{2020} and a^{1719} is $\left(a^{\gcd(1719,2020)} \right) = (a^1)$.

Correct Answer: D

3 Graph Theory

3.1 Connectivity

61. How many edges are there in a graph with 40 vertices each of degree 4? (Numerical Answer Type)

Solution: The handshaking lemma is a consequence of the degree sum formula (also sometimes called the handshaking lemma), $\sum_{v \in V} \deg v = 2 \mid E \mid$

In graph G, the sum of the degrees of the vertices is equal to twice the number of edges. Consequently, the number of vertices with odd degrees is even.

Here, $2 \mid E \mid = 4 \times 40 \implies \mid E \mid = 80$

Correct Answer: 80

62. What is the largest possible number of vertices in a graph with 35 edges and all vertices of degree at least 3? (Numerical Answer Type)

Solution: Let there are n vertices $\mathrm{v_1, v_2, v_3, \ldots, v_n}$

Degree of $\mathrm{v_1} = 3 + \mathrm{k_1}\,(\mathrm{k_1} \geq 0)$

Degree of $\mathrm{v_2} = 3 + \mathrm{K_2}\,(\mathrm{k_2} \geq 0)$

Degree of $v_n = 3 + k_n\,(k_n \geq 0)$

By the sum of degrees theorem

$$\sum \mathrm{d}(\mathrm{v_i}) = 2^{*}|\mathrm{E}|$$

The given degree is at least 3 so

$$\sum \mathrm{d}(\mathrm{v_i}) \leq 70$$
$$3n \leq 70$$
$$n \leq 23.33$$
$$n = 23$$

Correct Answer: 23

63. Which of the following is true?

S1: The number of directed simple graphs with n-vertices is 2^{n^2-n}.

S2: The number of un-directed simple graphs with n-vertices is $2^{\binom{n}{2}}$.

A. Only S1

B. Only S2

C. Both S1 and S2

D. Neither S1 nor S2

Solution: S1: The number of such graphs is equivalent to the number of irreflexive relations on a set of size $n = 2^{n^2-n}$.

S2: The number of such graphs is equivalent to the number of irreflexive and symmetric relations on a set of size $n = 2^{\binom{n}{2}}$.

Correct Answer: C

64. A graph has 26 vertices and 58 edges. There are five vertices of degree 4, six vertices of degree 5, and seven vertices of degree 6. If the remaining vertices all have the same degree, what is this degree? (Numerical Answer Type)

Solution: There are $26 - 5 - 6 - 7 = 8$ vertices of degree x. Applying Euler's Theorem:

$$5 \cdot 4 + 6 \cdot 5 + 7 \cdot 6 + 8 \cdot x = 2 \cdot 58$$

Rearranging we obtain $x = 3$. Thus the degree of the remaining eight vertices is 3.

Correct Answer: 3

65. A graph has 24 vertices and 30 edges. It has five vertices of degree 4, seven pendant vertices, and seven vertices of degree 2. All other vertices have degree 3 or 4. How many vertices of degree 4 are there? (Numerical Answer Type)

Solution: Let x be the number of vertices of degree 3, and y the number of vertices of degree 4. The order of the graph is 24 therefore:

$$5+7+7+x+y=24$$

Applying Euler's Theorem:

$$5\cdot 4+7\cdot 1+7\cdot 2+3\cdot x+4\cdot y=3\cdot 30$$

We now have two equations with two unknowns and can solve this system of equations. Isolating for x in the first equation, $x=5-y$, and substituting it into the second equation, with some arithmetic we obtain $y=4$. Therefore there are exactly four vertices of degree 4.

Correct Answer: 4

66. What is the maximum number of vertices on a graph that has 35 edges and every vertex has degree ≥ 3? (Numerical Answer Type)

Solution: The number of vertices will be maximized by minimizing the degree of the vertices.

First, we attempt to create a 3-regular graph. Let n be the number of vertices of this graph, by Euler's Theorem we see:

$$2\cdot 35=3n$$

There is no integral solution for n, so this graph is not possible. We next try a graph where every vertex, but one, is degree 3. We see:

$$70=3(n-1)+4$$

Which rearranges to give us $n=23\in\mathbb{Z}^{+}$. Note that this means that there are 22 (an even number of) vertices of (odd) degree 3.

Correct Answer: 23

67. The following statements are given below:

S1: If all vertices in a graph, G, have odd degree, k. Then k divides $|E(G)|$.

S2: The compliment of a graph, G, of order n, denoted $\overline{G}$, has the same vertex set as G with $E(\overline{G})=E(K_n)-E(G)$. If every vertex of G has an odd degree, except for one, then there are $n-1$ vertices that have odd degrees in $\overline{G}$.

Which of the following is correct?

A. S2 Only

B. S1 Only

C. S2 and S1 both

D. Neither S2 nor S1

Solution:

S1: Suppose there are n vertices. By Euler's Theorem, $\sum_{i=1}^{n}\deg(v_i)=n\cdot k=2|E(G)|$. We know $n\in\mathbb{Z}$ and $\frac{2|E(G)|}{k}\in\mathbb{Z}$, therefore k divides $|E(G)|$.

S2: All vertices of G, except one, have odd degrees. To guarantee there are an even number of odd-degree vertices, n must be odd.

If the degree of a vertex in G is k, the degree of that same vertex in $\overline{G}$ is $n-1-k$. $n-1$ is certainly even as n is odd, therefore if k is odd, $n-1-k$ is odd as well. The vertex in G of even degree will still have an even degree in $\overline{G}$ by the same argument. Thus, there are also $n-1$ vertices of odd degree in $\overline{G}$.

Correct Answer: C

68. Which of the following is/are correct? (Mark all the appropriate choices)

A. If two graphs have the same number of vertices with the same quantity and order cycles, then they are isomorphic.

B. Two isomorphic graphs must have the same number of edges and vertices.

C. Two isomorphic graphs always look exactly the same.

D. Isomorphism is an equivalence relation on all graphs.

Solution:

A. False.

Consider the following counterexample: P_5 and the empty graph, $\overline{K_5}$. Both have five vertices and no cycles but are clearly not isomorphic.

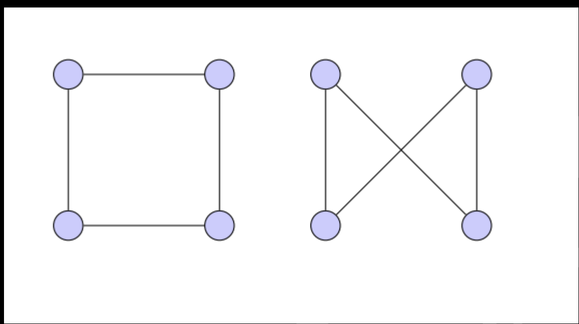
B. True.

By definition, an isomorphism is a bijection, so certainly the sizes of the vertex sets must be the same. An isomorphism also preserves adjacencies, hence the number of edges of the two graphs are necessarily the same.

C. False.

It is possible to draw a graph in multiple ways. The key to isomorphism is that the structures are identical, not how they are drawn.

For example, this is the same graph (hence isomorphic to itself) drawn in two different ways: $K_{2,2}$:



D. True.

We show that an isomorphism function is an equivalence relation by proving the three properties of an equivalence relation individually.

Reflexive: Yes.

Any graph is certainly isomorphic to itself, just let the isomorphism be the identity function.

Symmetric: Yes.

Suppose $G \cong H$, then there exists a mapping $f : V(G) \longrightarrow V(H)$. By definition, f is a bijection so there exists an inverse mapping, $f^{-1} : V(H) \longrightarrow V(G)$, that maintains adjacencies, telling us that $H \cong G$.

Transitive: Yes.

Let us consider three graphs G, H, I with $G \cong H$ and $H \cong I$. We know there exists a mapping from $V(G) \longrightarrow V(H)$ that maintains adjacencies, and similarly a mapping from $V(H) \longrightarrow V(I)$. Composing these functions we obtain the mapping: $V(G) \longrightarrow V(H) \longrightarrow V(I)$ which maintains adjacencies, hence $G \cong I$.

Correct Answer: B;D

69. Which of the following is/are incorrect? (Mark all the appropriate choices)

A. The degree sequence of two isomorphic graphs must be the same.

B. $K_{3,2}$ is isomorphic to C_5.

C. $K_{4,2}$ is isomorphic to $K_{2,4}$.

D. If G contains no cycles, all graphs isomorphic to G also have no cycles.

Solution:

A. True. If this were not true, then it would not be possible to create a bijection that maintains every adjacency.

B. False. The degree sequences of these two graphs are different! Two vertices in $K_{3,2}$ have degree 3, while all vertices in C_5 have degree 2.

C. True. These graphs are identical in structure.

D. True. Let $G \cong H$. If H contains a cycle, in order to maintain all adjacencies in G, G would contain a cycle too.

Correct Answer: B

70. Which of the following is/are incorrect? (Mark all the appropriate choices)

A. Any graph in which all vertices have even degree contains an Eulerian circuit.

B. A closed walk contains a cycle.

C. A graph with multiple components can contain an Eulerian cycle.

D. If a connected graph has $n = 2k$ vertices, for some positive integer k, all with odd degree, then there are k disjoint trails containing every edge.

Solution:

A. False.

This graph also must be connected with all vertices a non-zero, even degree.

Counterexample: Two disjoint K_3 's. The degree of every vertex is even, but there is no path that uses every edge exactly once.

B. True.

From question 5, the existence of a uv walk implies the existence of a uv path, so if there exists a closed uu walk, there exists a closed uu path, which is a cycle.

C. False.

An Eulerian circuit requires crossing every edge exactly once, but if there are no edges between the components of the graph, there is no way to reach every edge. Counterexample: Two disjoint K_3 's.

D. True.

Consider pairing off the vertices of the graph (which we can do since there are an even number of them), and add an edge between each pair of vertices. Now the degree of every vertex is even and there exists an Eulerian circuit. Removing the edges between the vertex pairs leaves us with k disjoint trails that contain all the edges.

Correct Answer: A;C

71. Which of the following is/are correct? (Mark all the appropriate choices)

A. A graph of order $n \geq 4$ that contains a triangle cannot be Hamiltonian.

B. Every Hamiltonian graph contains a Hamiltonian path.

C. If there exists a Hamiltonian path between any two vertices in a graph, then the graph is Hamiltonian.

D. A graph $K_{m,n}$ is Hamiltonian if and only if $m = n$, with $m, n \geq 2$.

Solution:

A. False.

Consider the complete graph K_4. This graph contains many triangles but, as with all complete graphs, is Hamiltonian.

B. True.

This can be seen by simply deleting one edge from a Hamiltonian cycle of the graph. This will leave a path that goes through every vertex, but is not a cycle.

C. True.

If there is a Hamiltonian path between any two vertices then graph is connected. Take two adjacent vertices u and v. Add the edge connecting them to the Hamiltonian path between them, which will create a Hamiltonian cycle.

D. True.

$K_{1,1}$ cannot contain a Hamiltonian cycle as it is acyclic. So we consider when $m, n \geq 2$.

Assume $K_{m,n}$ is Hamiltonian with $m, n \geq 2$. Assume for a contradiction that $m \neq n$. Without loss of generality suppose that $m < n$. A cycle in a bipartite graph is necessarily of even length and will alternate between the two partite sets. Let a cycle begin at some vertex in the partite set of size n. Once the cycle is length $2m$, it will return to the partite set of size n and all vertices in the partite set of size m have been visited. There will however be $n - m > 0$ vertices unreached by this cycle in the partite set of size n. This means $K_{m,n}$ cannot have a Hamiltonian cycle, which is a contradiction.

Conversely, suppose $K_{m,n}$ has $m = n$. It is easy to see that there exists a Hamiltonian cycle.

Correct Answer: B;C;D

72. For which of the following does there exist a simple graph $G = (V, E)$ satisfying the specified conditions? (Mark all the appropriate choices)

A. It has 6 vertices, 11 edges, and more than one component.

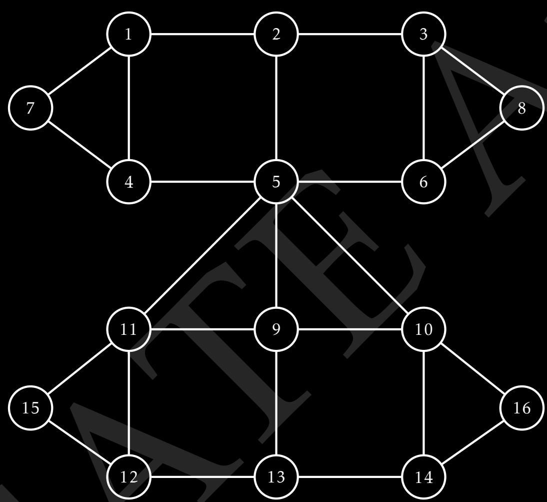
B. It is connected and has 10 edges 5 vertices and fewer than 6 cycles.

C. It has 7 vertices, 10 edges, and more than two components.

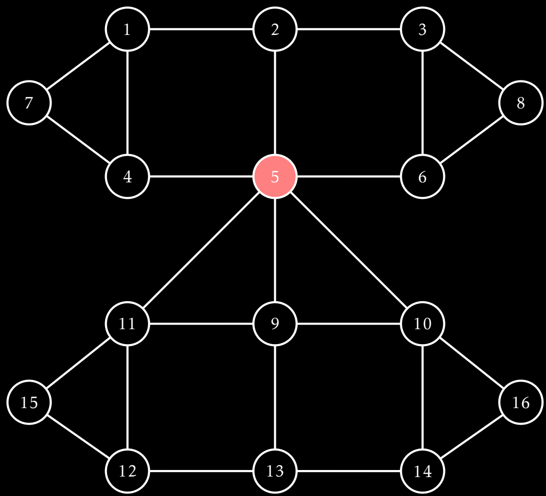
D. It has 8 vertices, 8 edges, and no cycles.

Solution: https://gateoverflow.in/244385/dsa-test-3-question-10

73. If the number of articulation points in the graph is α then the value of $2024^{\alpha!}$ =? (Numerical Answer Type)



Solution: In a graph, a vertex is called an articulation point if removing it and all the edges associated with it results in an increase of the number of connected components in the graph.
A vertex in an undirected connected graph is an articulation point
(or cut vertex) iff removing it (and edges through it) disconnects the graph.

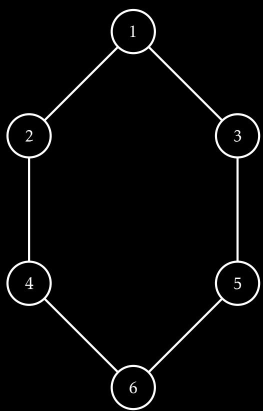


The number of articulation points is $\alpha = 1$.

$\therefore$ The value of $2024^{\alpha!} = 2024^{1!} = 2024$.

Correct Answer: 2024

74. How many Hamiltonian paths are there on the graph? (Numerical Answer Type)



Solution: A Hamiltonian path, also called a Hamilton path, is a graph path between two vertices of a graph that visits each vertex exactly once.
Hamiltonian path in clockwise direction:
$1, 3, 5, 6, 4, 2$
$3, 5, 6, 4, 2, 1$
$5, 6, 4, 2, 1, 3$
$6, 4, 2, 1, 3, 5$
$4, 2, 1, 3, 5, 6$
$2, 1, 3, 5, 6, 4$
Similarly, Hamiltonian path in anti-clockwise direction:
$1, 2, 4, 6, 5, 3$
$2, 4, 6, 5, 3, 1$
$4, 6, 5, 3, 1, 2$
$6, 5, 3, 1, 2, 4$
$5, 3, 1, 2, 4, 6$
$3, 1, 2, 4, 6, 5$
There are 12 Hamiltonian paths possible.

Correct Answer: 12

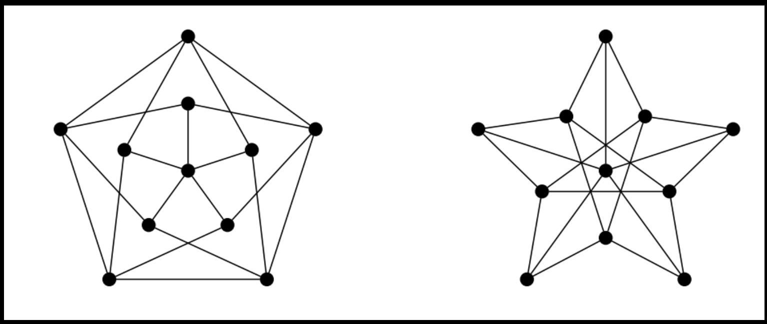
75. If a simple graph has 5 component and these components have $4, 5, 6, 7, 8$ vertices, then the maximum number of edges present in the graph? (Numerical Answer Type)

Solution: Each of the connected components can have at most $\frac{n(n-1)}{2}$ edges, where n is the number of vertices of the respective connected component; the formula is the number of edges of a complete graph with n vertices.

In total, one obtains a maximum number of edges $= \frac{4 \cdot 3}{2} + \frac{5 \cdot 4}{2} + \frac{6 \cdot 5}{2} + \frac{7 \cdot 6}{2} + \frac{8 \cdot 7}{2}$
$= 6 + 10 + 15 + 21 + 28 = 80$.

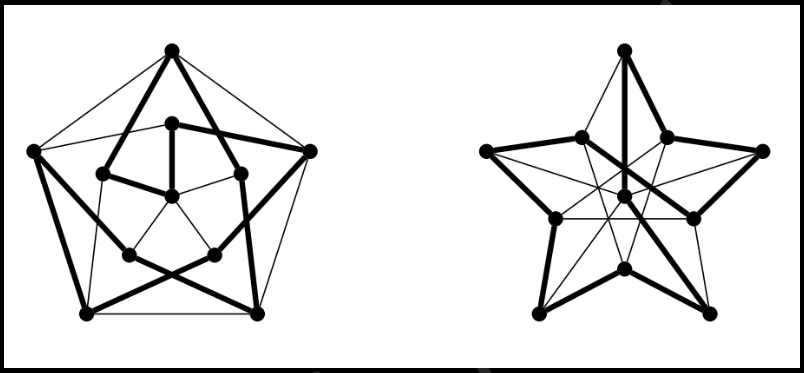
Correct Answer: 80

76. If the girth and circumference of the following graphs are α and β respectively, then the value of $\alpha + \beta =$? (Numerical Answer Type)



Solution: The graph on the left has girth 4; it's easy to find a 4-cycle and see that there is no 3-cycle. It has circumference 11, since below is an 11-cycle (a Hamilton cycle).

The graph on the right also has girth 4 . It also has circumference 11, since below is an 11-cycle. Actually, one can check that the two graphs are isomorphic.

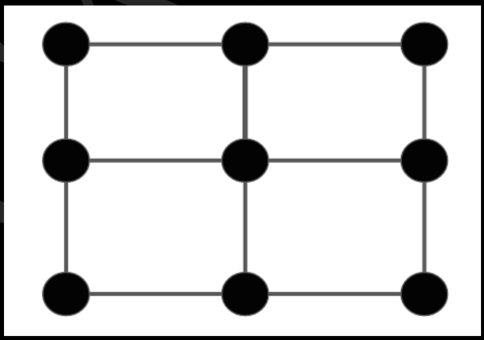


$\therefore$ The value of $\alpha + \beta = 4 + 11 = 15$.

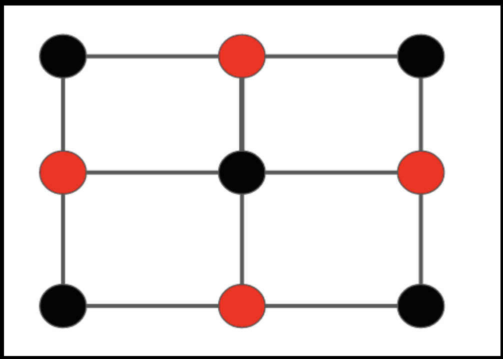
Correct Answer: 15

3.2 Matching

77. What is the least number of nodes you can have to make a minimum vertex cover of this graph? (Numerical Answer Type)



Solution:



One way to think about the problem is to color all of the nodes red, this is certainly a vertex cover since all of the nodes are in the cover and all of the edges must touch one of those nodes. Start removing nodes by coloring them black. If coloring a node black causes any of the edges to become disconnected from the cover, you have to turn one of the nodes touching the soon-to-be disconnected edge red.

Any set of red vertices that keeps all the edges touching a red vertex is a vertex cover, but the minimum vertex cover is the lowest number of nodes needed to still have a cover. This means that if we disconnect any node in a minimum vertex cover, the resulting set of vertices is no longer a cover.

Correct Answer: 4

78. The number of perfect matchings in complete graphs with 6 vertex is α. Then the value of α^2 is ______. (Numerical Answer Type)

Solution: If n is odd then perfect matching 0. because in perfect matching degree of each vertex must be 1, which is not possible if n is odd.

And if n is even then number of perfect matching in $K_{2m} = \dfrac{(2m!)}{(2^m \cdot m!)}$

where $n = 2m$ = number of vertices in the complete graph K_{2m}.

Here $n = 2m = 6 \implies m = 3$

Number of perfect matching in $K_6 = \dfrac{6!}{2^3 \cdot 3!} = \dfrac{6 \cdot 5 \cdot 4 \cdot 3!}{8 \cdot 3!} = 15.$

Short Method: vertex 1 can go with any of the 5 others, then choose one of the 4 remaining, it can go with any of three others, then there are no more choices to make. $5 \times 3 = 15$

$\therefore$ The value of $\alpha^2 = 15^2 = 225.$

Correct Answer: 225

79. The number of perfect matching in $K_{7,7}$ is β, then the value of $\dfrac{\beta}{6!}$ is ______. (Numerical Answer Type)

Solution: A perfect matching in $K_{n,n}$ simply matches the vertices from the first part to the vertices from the second in some order, and because all the edges are there, any order is permitted. So what we are looking for is just the number of different orderings of the second part, i.e. $n!$.

Equivalently we can say there are n ways to match the first vertex from the first part; once you have done that there are $(n-1)$ ways to match the second vertex from the first part (if $n > 1$), and so on.

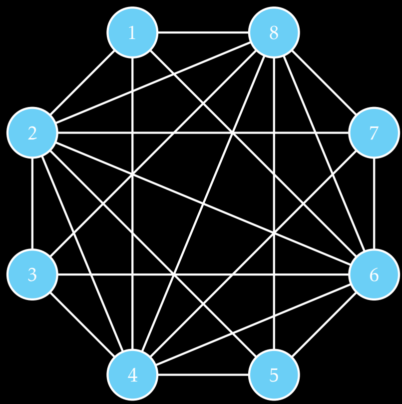
Number of perfect matching in $K_{n,n} = n \cdot (n-1) \cdot (n-2) \cdot (n-3) \cdots 1 = n!$

Number of perfect matching in $K_{7,7} = 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 7! = \beta$

$\therefore$ The value of $\dfrac{\beta}{6!} = \dfrac{7!}{6!} = \dfrac{7 \times 6!}{6!} = 7.$

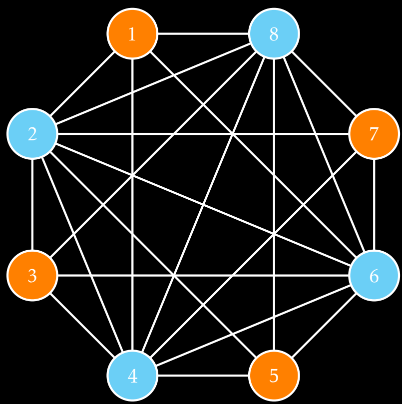
Correct Answer: 7

80. Number of the maximum independent vertex set of this graph is γ, then the value of $(\gamma!)^\gamma$ is ________. (Numerical Answer Type)



Solution: An independent vertex set of a graph G is a subset of the vertices such that no two vertices in the subset represent an edge of G.
A maximum independent vertex set is an independent vertex set containing the largest possible number of vertices for a given graph.

The independence number of a graph is the cardinality of the maximum independent set.



So, independence vertex set(orange color)= $\{1,3,5,7\}$

$\therefore$ The value of $(\gamma!)^{\gamma} = (4!)^4 = 24^4 = 331776$.

Correct Answer: 331776

81. Which of the following is/are correct? (Mark all the appropriate choices)

A. If G is a simple graph with 15 edges and $\overline{G}$ has 13 edges, then the number of vertices in G is 8.

B. If The maximum number of edges in a simple graph with 10 vertices and 4 components is 21.

C. For which value of k an acyclic graph G with 17 vertices, 8 edges, and k components exist? Let $e_i, n_i, 1 \leq i \leq k$ represents the number of edges, and number of vertices in component i, respectively.

D. Find the minimum number of vertices in a simple graph with 13 edges and having 5 vertices of degree 4 and the rest having a degree less than 3.

Solution:

A. Total number of edges possible $= \begin{pmatrix} n \\ 2 \end{pmatrix} = 15 + 13 = 28. \implies n = 8.$

B. Three components with K_1's and one component with K_7. Number of edges $= \begin{pmatrix} 7 \\ 2 \end{pmatrix} = 21.$

C. Given $\sum_{i=1}^{k} e_i = 8$

Since the graph is acyclic it follows that $e_i = n_i - 1, 1 \leq i \leq k$.

Therefore, $\sum_{i=1}^{k} (n_i - 1) = 8 \implies \sum_{i=1}^{k} n_i - k = 8$ Since $\sum_{i=1}^{k} n_i = 17, k = 17 - 8 = 9$

D.$\sum_{i=1}^{n} d_i = 13 \times 2 = 26$. Number of 4 degree vertices = 5. This contributes 20 to the degree. For the rest, we can use vertices of degree less than 3. i.e., there should exist three 2-degree vertices. Therefore, the total number of vertices is at least $5 + 3 = 8$.

Correct Answer: A;B;C;D

82. The maximum number of edges in a bipartite graph with 19 vertices is? (Numerical Answer Type)

Solution: If n is even, then complete bipartite graph $K_{\frac{n}{2},\frac{n}{2}}$ has maximum edges, which equals $\frac{n^2}{4}$.

If n is odd, then $K_{\frac{n-1}{2},\frac{n+1}{2}}$ has maximum edges which is

$\left(\frac{n-1}{2}\right)\cdot\left(\frac{n+1}{2}\right) = \frac{n^2-1}{4}$.

Here, $n = 19$ is odd, then $K_{9,10}$ has maximum edges which is $\frac{19^2-1}{4} = 90$.

So, the correct answer is 90.

3.3 Coloring

83. Which of the following is/are correct? (Mark all the appropriate choices)

A. If G is a graph with $E \leq 3V - 6$ then G is planar.

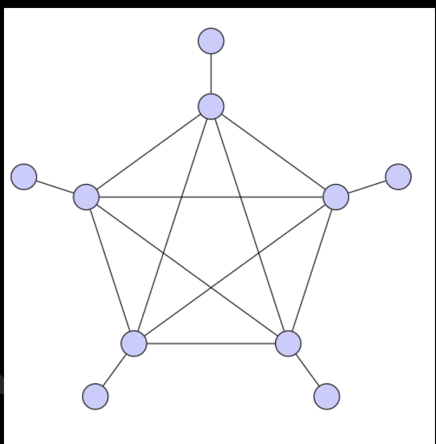
B. The subgraph of any planar graph is planar.

C. Every planar graph of order 4 or more contains at least one vertex of degree 5 or less.

D. If G has order 11, then at least one of G or $\overline{G}$ is non-planar.

Solution: A. False.

If G is planar then $E \leq 3V - 6$. The converse of this statement is false however. The graph below has 10 vertices and 15 edges, so $E \leq 3(V) - 6 = 3(10) - 6 = 24$. However, it contains a subgraph that is isomorphic to K_5 and so the graph is nonplanar.



B. True.

If a graph contains a nonplanar subgraph, then the graph must be nonplanar.

C. True.

For every planar graph of order 3 or more, $E \leq 3V - 6$. Suppose for a contradiction that there exists a planar graph of order at least 4 where every vertex is of degree 6 or more.

Then, by Euler's Theorem, $6V \leq 2E$. It then follows that $6V \leq 6V - 12$, a contradiction.

D. True.

The total number of edges in a complete graph is $\binom{11}{2} = 55$. Hence, if G has x edges then $\overline{G}$ has $55 - x$ edges. If a graph is planar then $E \leq 3V - 6$, so it follows that if a graph has more than $3V - 6$ edges then it is nonplanar. Notice that $2(11) - 6 = 27$. Thus, if G has 28 or more edges we are done. Let us suppose instead that G has no more than 27 edges. Then $\overline{G}$ at least $55 - 27 = 28$ edges, and so is nonplanar.

Correct Answer: B;C;D

84. Which of the following is/are correct? (Mark all the appropriate choices)

A. Let G be a connected, planar graph with at least 4 vertices. Prove that the number of regions is bounded above by 2 V $-$ 4.

B. If G is a connected planar graph where $E = 3V - 6$ then every region of G is a triangle.

C. Let G be a planar graph where $\delta(G) \geq 5$. Show that G has at least 12 vertices.

D. A connected, planar graph with order 22 has no more than 60 edges.

Solution:

A. As G is a connected, planar graph it follows from Euler's Planar Graph Theorem that the number of edges in G is $E = V + R - 2$.

Further, as G is planar there is an upper bound on the number of edges in $G : E \leq 3V - 6$.

If follows then that:

$$\begin{aligned} V + R - 2 &\leq 3V - 6 \\ R &\leq 2V - 4 \end{aligned}$$

So, the upper limit on the number of regions in G is $2V - 4$, as desired.

B. First, notice that G cannot be a tree as if G were a tree then $E = V - 1 = 3V - 6$. This would imply that $2V = 5$, a contradiction that the number of vertices in a graph is an integer. This means each region is bounded by at least 3 edges. Since G is planar, $V - E + R = 2$. Thus, $3V - 6 = 3E - 3R$ and so $2E = 3R$. If any region were bounded by more than 3 edges, then $2E > 3R$. As this is not true, it follows that every region is bounded by three edges. That is, each region is a triangle.

C. As the minimum degree of any vertex in G is 5, it follows from Euler's Theorem that: $5V \leq 2E$.

As G is planar, it follows from Euler's Planar Graph Theorem that $E \leq 3V - 6$ as G must have at least 3 vertices.

Thus, by transitivity of order, it follows that $\frac{5}{2}V \leq 3V - 6$. Rearranging this inequality shows that $12 \leq V$, as desired.

D. If G is a planar graph with $V = 22$, then we can apply Euler's Planar Graph Theorem. It follows that, $E \leq 3V - 6 = 3(22) - 6 = 60$, as desired.

Correct Answer: A;B;C;D

85. Which of the following is/are correct? (Mark all the appropriate choices)

A. If $\chi(G) = 3$ then G contains a triangle.

B. If a planar graph contains a triangle, then $\chi(G) = 3$.

C. Isomorphic graphs have the same chromatic number.

D. Homeomorphic graphs have the same chromatic number.

Solution: A. False. Every odd cycle with length $n \geq 3$ has chromatic number 3 and no triangles.

B. False. K_4 is planar and contains a triangle, but $\chi(K_4) = 4$.

C. True. An isomorphism preserves adjacencies between vertices and hence will also preserve a proper colouring.

D. False. C_4 and $K_3 = C_3$ are homeomorphic but $\chi(C_4) = 2$ since it is an even cycle, while $\chi(K_3) = 3$ since it is an odd cycle.

Correct Answer: C

86. Which of the following is/are correct? (Mark all the appropriate choices)

A. Any Hamiltonian graph with $\chi(G) = 2$ is planar.

B. A graph is bipartite if and only if it has chromatic number 2.

C. If $\chi(G) \leq 4$ then G is planar.

D. If $\chi(G) = n$, then G contains a subgraph isomorphic to K_n.

Solution: A. False. Consider $K_{3,3}$ which is Hamiltonian (proved in 2.5 question 9) and bipartite, hence $\chi(K_{3,3}) = 2$, but $K_{3,3}$ is certainly not planar by Kuratowski.

B. True. A graph is bipartite if and only if it contains no odd cycles. We know that if a graph contains an odd cycle, its chromatic number is at least 3 . Hence if the chromatic number is two the graph contains no odd cycles and is bipartite.

C. False. Consider $K_{3,3}$. This graph has chromatic number $2 \geq 4$, but is not planar.

D. False. Consider any odd cycle, C_n, with $n \geq 5$, such as C_5. These graphs have chromatic number 3, but do not contain K_3 subgraphs.

Correct Answer: B

87. Which of the following is/are correct? (Mark all the appropriate choices)

A. If there exists a 4 -colouring of G then $\chi(G) = 4$.

B. If G contains a subgraph isomorphic to K_n then $\chi(G) \geq n$.

C. If we can prove G has no 3-colouring then $\chi(G) = 4$.

D. If every region of a planar graph is bounded by an even number of edges, then there exists a 2-colouring of the graph.

Solution: A. False. All graphs with $\chi(G) \leq 3$ and order at least four can be coloured using four colours, this colouring just may not be a minimum colouring.

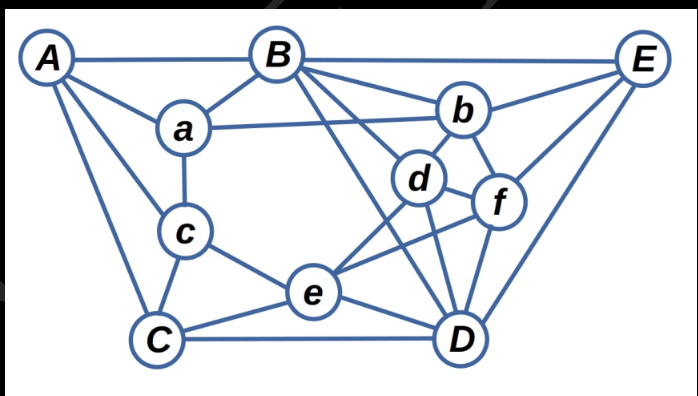
B. True. The chromatic number of a graph is at least as large as the chromatic number of all its subgraphs.

C. False. K_5 does not have a 3 -colouring, but it also does not have a 4-colouring, hence $\chi(G) \neq 4$.

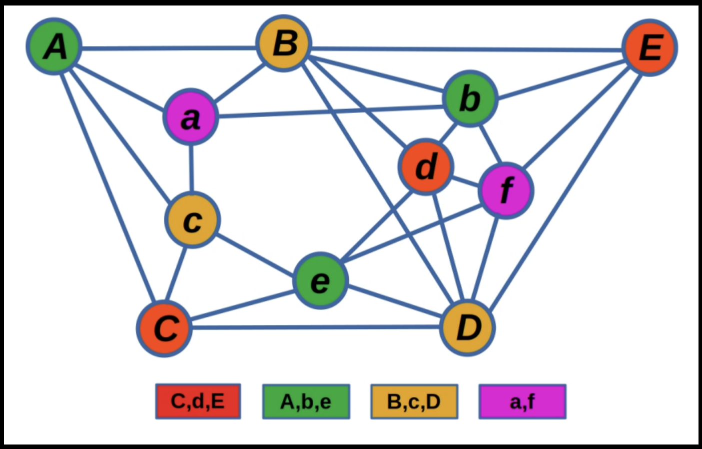
D. The statement is true. Consider a planar embedding of such a graph. Identify one region, since it is bounded by an even cycle, we can colour this cycle with two colours. We can do the same for every region, and if we have already coloured one of the vertices of the boundary cycle then colour the adjacent vertices the opposite colour.

Correct Answer: B;D

88. If the Chromatic number of below graph is α, then the value of $\alpha!$ is ______. (Numerical Answer Type)



Solution: The chromatic number of a graph is the minimum number of colors in a proper coloring of that graph.

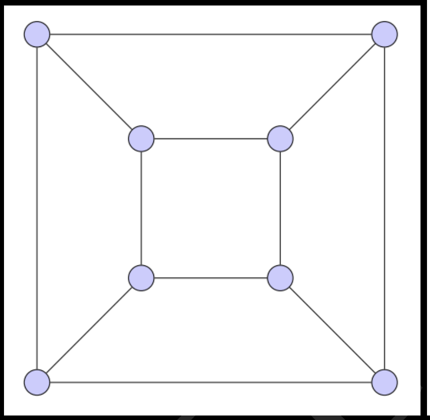


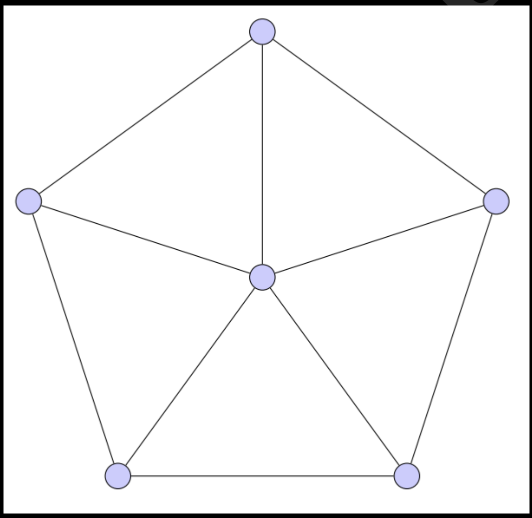
So, the value of $\alpha = 4$

$\therefore$ The value of $\alpha! = 4! = 24$

Correct Answer: 24

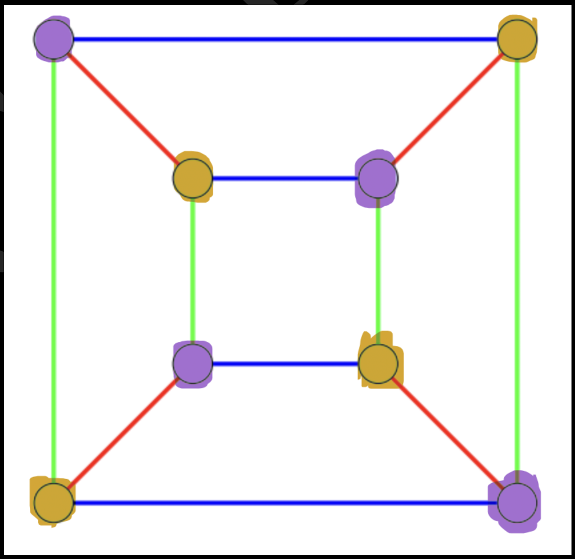
89. If the minimum edge colouring, and minimum vertex colouring for the following graphs are α, β respectively, then the value of $\alpha + \beta$ =?





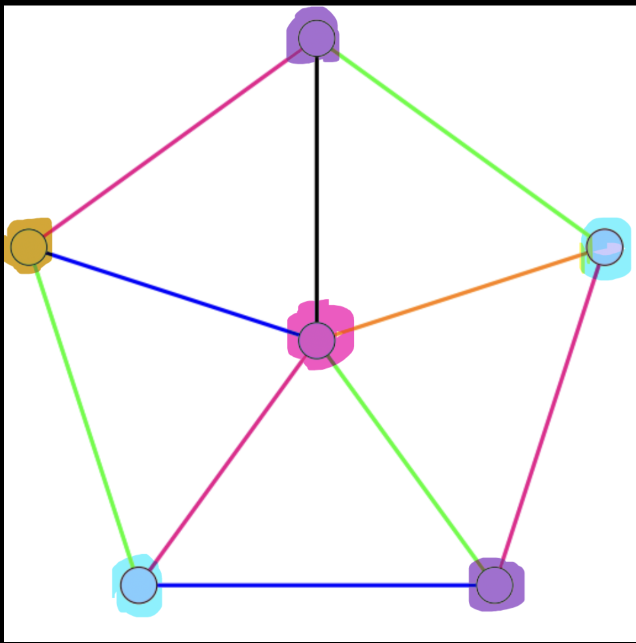
Solution:

Here is one such minimum edge colouring:



We see a minimum edge colouring requires three colours while a minimum vertex colouring required two.

Consider one such minimum edge colouring:



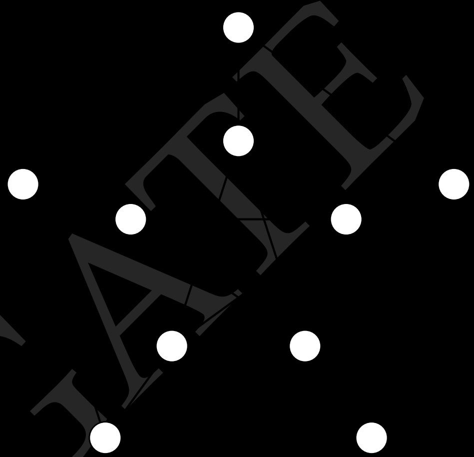
We see a minimum edge colouring requires five colours while a minimum vertex colouring required four

So, $\alpha = 3 + 5 = 8, \beta = 2 + 4 = 6$

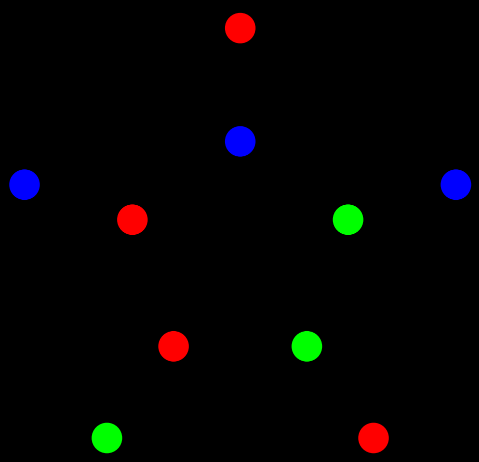
$\therefore$ The value of $\alpha + \beta = 8 + 6 = 14$

Correct Answer: 14

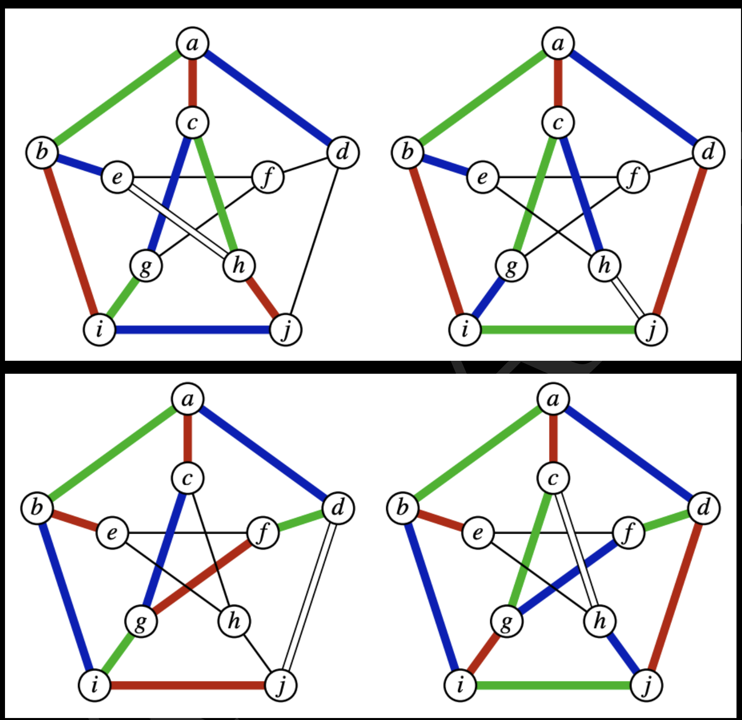
90. If the chromatic number of this graph is α, and the edge chromatic number is β, then the value of $\alpha! + \beta! =?$ (Numerical Answer Type)



Solution: The Chromatic number $\chi(G)$ is the least k, such that G is $k-$colorable.

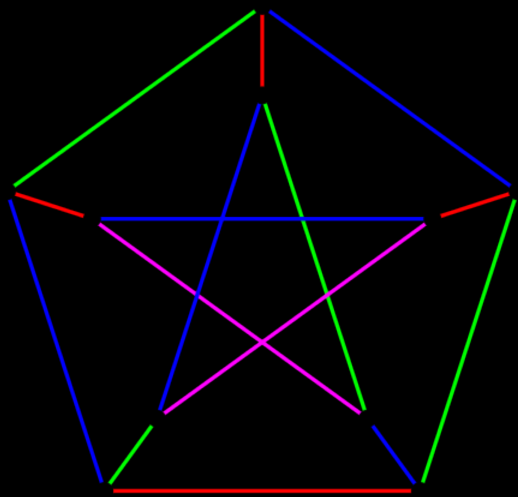


Chromatic number $\chi(G) = \alpha = 3$.
The edge chromatic number, sometimes also called the chromatic index, of a graph G is the fewest number of colors necessary to color each edge of G such that no two edges incident on the same vertex have the same color. In other words, it is the number of distinct colors in a minimum edge coloring. It is generally denoted as $\chi'(G)$.

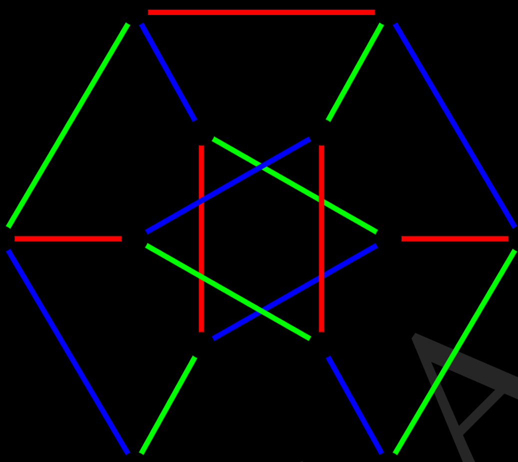


In all cases, it is not possible to edge color with 3 colors, so $\chi'(G) = \beta = 4$.

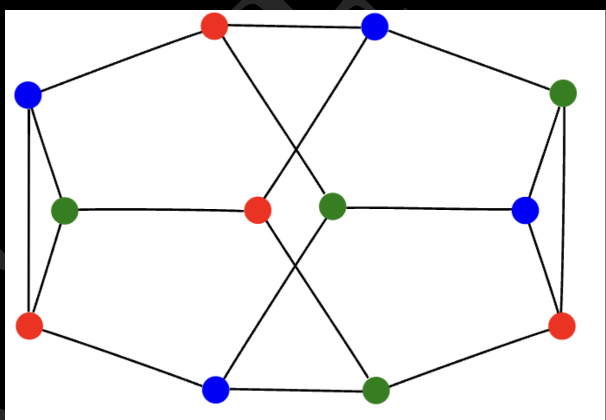
We can edge color using 4 colors.



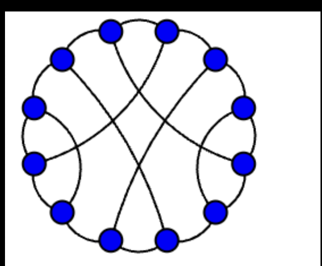
Dürer Graph:



The chromatic index of the Dürer graph is 3.



The chromatic number of the Dürer graph is 3.



The Dürer graph is Hamiltonian.

$\therefore$ The value of $\alpha! + \beta! = 3! + 4! = 6 + 24 = 30$.

Correct Answer: 30

4 Combinatorics

4.1 Counting

91. In New Hampshire, license plates consisted of two letters followed by 3 digits. How many possible license plates are there? (Numerical Answer Type)

    Solution: 26 choices for the first letter, 26 for the second, 10 choices for the first number, the second number, and the third number:

    $$26^2 \times 10^3 = 676,000$$

    Correct Answer: 676,000

92. How many legal configurations are there in Towers of Hanoi with n rings?

    A. 3^n

    B. 2^n

    C. 2^n

    D. 4^n

    Solution: The product rule again:

    - Each ring gets to "vote" for which pole it's on.
    - Once you've decided which rings are on each pole, their order is determined.
    - The total number of configurations is 3^n

    Correct Answer: A

93. Joselyn stops by a sandwich shop on her way home from class. The shop sells 4 types of potato chips, 3 types of cookies, 7 different drinks, and 10 different sandwiches. She is interested in determining how many different ways there are to order if she'd either like a drink and a cookie, or a meal which includes a sandwich, a drink, and chips. (Numerical Answer Type)

    Solution:

    Consider first the number of options if Joselyn orders a drink and a cookie. Then, consider the number of options if she orders a sandwich, drink, and chips.

    There are 3 types of cookies and 7 drinks to choose from, so there are $3 \cdot 7 = 21$ possible combinations of a cookie and a drink.

    There are 10 different sandwiches, 7 drink options, and 3 types of cookies. Therefore there are $10 \cdot 7 \cdot 3 = 210$ combinations for this meal.

    In total Joselyn can order $21 + 210 = 231$ possible combinations at the shop.

    Correct Answer: 231

94. How many nonempty sets of letters can be formed from 3 $X's$ and 5 $Y's$? (Numerical Answer Type)

    Solution: Consider an ordered pair, (X, Y) where $0 \leq X \leq 3$ and $0 \leq Y \leq 5$. Let the ordered pair represent the number of X 's and Y 's in the given non-empty set.

    There are 4 possibilities for the value of X 's and 6 possibilities for the value of Y 's in the pair. In total, there are $6 \times 4 - 1 = 23$ possible pairs. We must subtract 1 because one of our pairs represents the empty set, $(0, 0)$. Thus, there are 22 possible non-empty sets.

    Correct Answer: 22

95. If $A = \{1,2,3\}, B = \{a,b,c,d\}, C = \{\alpha,\beta,\gamma,\chi,\lambda\}$, how many distinct 3-tuples are there in the set $A \times B \times C$? (Numerical Answer Type)

Solution: Each tuple will be of the form (a,b,c) where $a \in A, b \in B$ and $c \in C$. The number of possible options in the first position is 3, the number of possible options in the second position is 4, and the number of possible options in the last position is 5. Using the rule of product it follows that there are $3 \cdot 4 \cdot 5 = 60$ distinct elements in $A \cdot B \cdot C$.

Correct Answer: 60

96. A group of eight would sit in a row at the movie theater, how many ways can arrange themselves if Andrew and Asiya refuse to sit beside each other? (Numerical Answer Type)

Solution: We begin by sitting the six individuals who are not Andrew or Asiya, there are $P(6,6)$ ways to do this. To guarantee Andrew and Asiya are not beside each other, they will either sit on an end or between two other friends. There are five seats between friends and two on the ends, which gives seven possible seats 127 for Andrew and Asiya thus $P(7,2)$ choices. By applying the Rule of Product it follows that there are $P(7,2) \cdot P(6,6) = 30240$ ways to seat the group.

Correct Answer: 30240

97. Eli wakes up every morning and makes himself a smoothie with frozen fruit. He picks 3 fruits every day to make his smoothie with out of the 10 options types of fruit in his freezer. He likes any combination of fruit in his smoothie except banana with apple. How many ways are there for Eli to make his smoothie? (Mark all the appropriate choices)

A. $\binom{10}{3} - \binom{8}{1}$

B. $\binom{9}{1} + \binom{9}{2} + \binom{8}{3}$

C. $\binom{10}{2} - \binom{8}{1}$

D. $\binom{9}{2} + \binom{9}{2} + \binom{8}{3}$

Solution:

The total number of smoothies without any restriction is $\binom{10}{3}$. The number of smoothies that include both banana and apple are $\binom{8}{1}$ so, the total possible smoothies are: $\binom{10}{3} - \binom{8}{1}$.

Alternatively, we can find the total number of smoothies using 3 cases:

- Case 1: Banana is chosen for the smoothie. The total number of smoothies, in this case, is: $\binom{8}{2}$, since apple cannot be chosen.
- Case 2: Apple is chosen for the smoothie. The total number of smoothies, in this case, is: $\binom{8}{2}$, since banana cannot be chosen.
- Case 3: Neither banana nor apple is chosen for the smoothie. The total number of smoothies in this case is: $\binom{8}{3}$.

Thus, the total smoothies is the sum of all these cases which is: $\binom{9}{2} + \binom{9}{2} + \binom{8}{3}$.

Correct Answer: A;D

98. Parents are distributing the last of the Halloween candies between their four children. There are seven packs of Skittles and six chocolate bars, in how many ways can these parents distribute the candy such that each child gets at least one pack of Skittles? (Numerical Answer Type)

Solution:

After giving each child one Skittle, we count how to distribute the 3 remaining Skittles and 6 chocolate bars. First, we count the ways to distribute the remaining Skittles, then the chocolate, and then we'll apply the Rule of Product.

Skittles: We are arranging three identical objects, potentially with repetition, between four distinct individuals, so there are $\binom{4+3-1}{3} = \binom{6}{3}$ ways to distribute the Skittles.

Chocolate: Arranging six identical chocolate bars between four distinct children gives $\binom{4+6-1}{6} = \binom{9}{6}$ ways to distribute the chocolate.

In total there are $\binom{6}{3} \cdot \binom{9}{6} = 1680$ ways to distribute this candy.

Correct Answer: 1680

99. How many integer solutions are there to the inequality,

$$x_1 + x_2 + x_3 + x_4 + x_5 < 20,$$

where $x_i \geq 0$ for $1 \leq i \leq 5$? (Numerical Answer Type)

Solution: We can re-express this problem as how many solutions are there to the equation

$$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 19$$

where $x_i \geq 0$ for $i = 1, \ldots, 5$ and $x_6 \geq 0$. x_6 is a placeholder variable which accounts for when $\sum_{i=1}^{5} x_i < 20$, since every positive value of x_6 corresponds to,

$$x_1 + x_2 + x_3 + x_4 + x_5 = 19 - x_6 < 20$$

Therefore we are distributing 19 identical objects between 6 distinct containers, hence there are $\binom{6+19-1}{19} = \binom{24}{19} = 42504$ solutions to the original inequality.

Correct Answer: 42504

100. Hannah is buying two dozen loaves of bread at a bakery out of n options. The bakery has more than three dozen of each type and Hannah is okay with repetition. If she can select the bread in 593 775 different ways, how many different types of bread does the bakery have? (Numerical Answer Type) Note: one dozen = 12.

Solution: If we know Hannah has 593775 different ways of choosing two dozen loaves with repetition from n distinct bread varieties, then we know that,

$$\binom{n+24-1}{24} = \binom{n+23}{24} = 593775$$

With trial and error, we can see that,

$$\binom{30}{24} = 593775$$

Which means that $n + 24 - 1 = 30$, and so $n = 7$.

Correct Answer: 7

101. How many people must attend a conference to ensure that at least two attendees share the same first and last initial? (Numerical Answer Type)

Solution:

First determine the number of possible, distinct initials. There are 26 options for each one's first and last initial. Therefore, there are $26^2 = 676$ different possible initials.

Let the attendees represent the 'pigeons' and the possible initials represent the 'pigeonholes'. To guarantee there are at least two attendees with the same initials, it follows from the Pigeonhole Principle that we need more than 167 attendees. So, we require at least 677 attendees.

Correct Answer: 676

102. How many integers in $X = \{0, \dots, 60\}$ must be chosen to ensure that an odd integer is selected? (Numerical Answer Type)

Solution: There are exactly 30 odd numbers bin X, and 61 numbers to choose from. Therefore at least $61 - 30 + 1 = 32$ numbers must be selected to guarantee that at least one is odd.

Correct Answer: 32

103. There are 500 families that live in the neighbourhood of South Brambleton. 100 of these families have no children and no pets. 300 families have pets, and 400 have children. How many homes in South Brambleton have both of children and pets living in them? (Numerical Answer Type)

Solution: Let P be the set of families with pets and C the set of families with children. The question asks for $|P \cap C|$. In the question, they tell us that $|P \cup C| = 500 - 100 = 400$. By Inclusion-Exclusion,

$$|P \cup C = |P| + |C| - |P \cap C|$$

Rearranging, we get

$$|P \cap C| = |P| + |C| - |P \cup C| = 300 + 400 - 400 = 300.$$

Thus, 300 homes have both pets and children in them.

Correct Answer: 300

104. A standard 12-hour clock has hour, minute, and second hands. How many times do two hands cross between 1:00 and 2:00 (not including 1:00 and 2:00 themselves)? (Numerical Answer Type)

Solution: We know that the hour and minute hands cross exactly once. Let m be the number of minutes past one o'clock that this happens. The angle between the minute hand and the 12 must be equal to the angle between the hour hand and the 12. Since 1 minute is $\frac{360^\circ}{60} = 6^\circ$ on the clock and 1 hour is $\frac{360^\circ}{12} = 30^\circ$, we have $6m = 30\left(1 + \frac{m}{60}\right)$, so $m = \frac{60}{11} = 5\frac{5}{11}$. Note that the second hand is not at the same position at this time, so we do not have to worry about a triple crossing.

On the other hand, the second-hand crosses the hour hand once every minute, for a total of 60 crossings. Also, the second-hand crosses the minute hand once every minute except the first and last, since those crossings take place at 1:00 and 2:00, for a total of 58 crossings. There is a grand total of $1 + 60 + 58 = 119$ crossings.

Correct Answer: 119

105. A permutation of the first n positive integers is quadratic if, for some positive integers a and b such that $a + b = n, a \neq 1$, and $b \neq 1$, the first a integers of the permutation form an increasing sequence and the last b integers of the permutation form a decreasing sequence, or if the first a integers of the permutation form a decreasing sequence and the last b integers of the permutation form an increasing sequence. How many permutations of the first 10 positive integers are quadratic? (Numerical Answer Type)

Solution: Clearly, either 1 or 10 must be in the middle of the permutation. Assume without loss of generality that 10 is; we can construct an equivalent permutation with 1 in the middle by replacing each number i with $11 - i$. We can pick any nonempty strict subset of the first 9 positive integers, sort it, place it at the beginning of the permutation, then place 10, then place the unchosen numbers in decreasing order. There are $2^9 - 2 = 510$ ways to do this. Therefore, there are $2 \times 510 = 1020$ quadratic permutations of the first 10 positive integers.

Correct Answer: 1020

106. The population of the US is 300 million. Every person has written somewhere between 0 and 10 million lines of code. What's the maximum number of people that we can say must have written the same number of lines of code? (Numerical Answer Type)

Solution: There are $10 \cdot 10^6 + 1$ different number of lines of code you can write. So, there exists a number of line of codes with at least $\left\lceil 300 \cdot 10^6 / \left(10^6 + 1\right)\right\rceil = 30$ people.

Correct Answer: 30

107. There are 38 different time periods during which classes at a university can be scheduled. If there are 677 different classes, what is the minimum number of different rooms that will be needed? (Numerical Answer Type)

Solution: There exists a time period will have at least $\lceil 677/38 \rceil = 18$ classes during it. So 18 different rooms will be needed.

Correct Answer: 18

108. Kalil is interviewing for jobs at 5 different companies where each job has a two-part interview. He has 5 interview-appropriate outfits. Kalil wants to wear each outfit once for each round of interviews, but does not want to wear the same outfit to the second part of an interview as he wore to the first. How many ways can he do this? (Numerical Answer Type)

Solution:

For the first round of interviews, Kalil can simply assign one outfit to each interview, giving 5 ! possible distributions of these outfits.

For the second round of interviews, the outfit must be different at that job than worn in the first round.

Hence, we would like to derange the five outfits, $d(5)$, from the first round.

By the Rule of Product, there are $5! \cdot d(5) = 120 \times 44 = 5280$ ways for Kalil dress for these interviews.

Correct Answer: 5280

109. How many possible six-figure salaries (in whole dollar amounts) are there that contain at least three distinct digits? (Numerical Answer Type)

Solution: First note that there are $999{,}999 - 100{,}000 + 1 = 900{,}000$ total possible salaries. Let us now count the complement case - number of six-figure salaries that contain at most two distinct digits. There are 9 salaries with 1 distinct digit $(111111, 222222, \ldots, 999{,}999)$. Now let us count the salaries that have exactly two distinct digits. Since the first digit cannot be 0 , we need to consider the case when one of the two distinct digits is 0 differently. So suppose one of the two distinct digits is 0 , then we have 9 choices for the other digit. The first position needs to contain this non-zero digit and we get to place either of the two in the last 5 positions however we want. There are $2^5 - 1$ different ways to do this, since we can consider a bijection between 5 digit binary numbers where 0 means that 0 is in the position, 1 means that the other digit is in the position. So there are in total $9 \times \left(2^5 - 1\right)$ different salaries with two distinct digits, one of which is 0 (since a binary value of 11111 gives all 1 digit, we take it out). For the case of two distinct digits when neither digit is 0, we get $\binom{9}{2} \times \left(2^6 - 2\right)$. First we choose two digits, and then similar to the 0 case, we have $2^6 - 2$ different ways of arranging the two numbers for two distinct digits. Notice that we have have to disclude the values 000000 and 111111, which would only include 1 distinct digit. Therefore, in total, we have $900{,}000 - 9 - 9 \times \left(2^5 - 1\right) - \binom{9}{2} \times \left(2^6 - 2\right) = 897{,}480$

Correct Answer: 897,480

110. Mickey the mailman is very lazy. He has received 10 parcels to 10 different people. However, because he is lazy, he doesn't bother reading the address and delivers them off randomly. In how many ways can Mickey deliver the parcels such that no one gets the right parcel? (Numerical Answer Type)

Solution:

By the derangements formula, the number of possible derangements is

$$10!\left(\frac{1}{0!}-\frac{1}{1!}+\cdots+\frac{1}{10!}\right)=10!-10!+\frac{10!}{2!}-\cdots+1$$
$$=1334961$$

Therefore, there are a total of 1334961 ways to deliver the parcels such that no one gets the right parcel.

Correct Answer: 1334961

4.2 Recurrence relations

111. Suppose $a_0 = 2, a_1 = 7$ and $a_{n+1} = -a_n + 5a_{n-1}$ for $n \geq 1$. The value of a_6 (without solving the recurrence relation) is ______. (Numerical Answer Type)

Solution: As we are not solving the recurrence relation, we will use the provided recurrence formula to find the terms for $n = 2, 3, 4, 5$ to then find the value for $n = 6$.

$$\begin{aligned}
a_0 &= 2\\
a_1 &= 7\\
a_2 &= -7+5(2)=3\\
a_3 &= -3+5(7)=32\\
a_4 &= -32+5(3)=-17\\
a_5 &= -(-17)+5(32)=177\\
a_6 &= -(177)+5(-17)=-262
\end{aligned}$$

Solving a recurrence relation is better as you do not need to first find all the preceding terms in order to find the n^{th} term. As n gets large, computing each term by hand will become extremely difficult and tedious.

Correct Answer: −262

112. Solve the recurrence relation $a_n = -2a_{n-1}$, where $a_0 = 5$. (Mark all the appropriate choices)

A. $a_n = 5 \cdot (-3)^n$ for $n \geq 0$.

B. $a_n = 5 \cdot (-1)^n$ for $n \geq 0$.

C. $a_n = 5 \cdot (-2)^n$ for $n \geq 0$.

D. $a_n = 3 \cdot (-2)^n$ for $n \geq 0$.

Solution:

Determining the first few terms we see that:

$$\begin{aligned}
a_0 &= 5\\
a_1 &= -2(5)\\
a_2 &= -2[-2(5)]=(-2)^2\cdot 5\\
a_3 &= -2\left[(-2)^2\cdot 5\right]=(-2)^3\cdot 5.
\end{aligned}$$

We can identify the pattern and see that $a_n = 5 \cdot (-2)^n$ for $n \geq 0$.

Correct Answer: C

113. Leora puts money in a high-interest savings account to help save for university. The interest is 8% annually and compounds monthly. If she deposits \$1500.00 on the day she opens the account, how much money will she have after 16 months? Use recurrence relations to solve this problem.

A. \$1668.25

B. \$1667.25

C. \$1666.25

D. \$1665.25

Solution:

If the annual interest rate is 8% then the monthly rate will be $\frac{8\%}{12} = 0.\overline{6}\% = 0.00\overline{6}$. For $0 \le n \le 16$, let a_n denote the money in the savings account at the end of the n^{th} month. Certainly $a_0 = \$1500$ and we can express $a_{n+1} = a_n + 0.00\overline{6}a_n = a_n(1.00\overline{6})$ for $n \ge 1$.

Solving this recurrence relation we see that $a_n = (\$1500)(1.00\overline{6})^n$ for $n \ge 0$. We can now simply solve for the 16^{th} month by plugging in $n = 16$, and we obtain $a_{16} = (\$1500)(1.00\overline{6})^{16} =$ \$1668.25.

Correct Answer: A

114. Suppose the amount of bacteria in a container triples every hour. If initially there are only 5 bacteria, how many bacteria are in the container after a day and a half? (Mark all the appropriate choices)

A. $5 \cdot \left(3^{36}\right)$

B. $5 \cdot \left(3^{35}\right)$

C. $5 \cdot \left(3^{34}\right)$

D. $5 \cdot \left(3^{32}\right)$

Solution:

Let n represent the number of hours the bacteria has been in the container, so $a_0 = 5$. Thus, we can use the recurrence relation $a_{n+1} = 3a_n$ where $n \ge 1$ to represent the bacteria growth. The unique solution to this relation is $a_n = 5 \cdot 3^n$ where $n \ge 0$.

One and a half days is equal to 36 hours, so we compute a_n for $n = 36$. Hence, there are $a_{36} = 5 \cdot \left(3^{36}\right)$ bacteria after a day and a half.

Correct Answer: A

115. Consider the recurrence relation $a_1 = 8, a_n = 6n^2 + 2n + a_{n-1}$. The value of a_{99} is ______. (Numerical Answer Type)

Solution: $a_n = 6n^2 + 2n + a_{n-1} \quad \longrightarrow (1)$

$a_1 = 8, a_0 = 0$

We can rewrite the equation (1), and get

$\implies a_n - a_{n-1} = 6n^2 + 2n$

$a_1 - a_0 = 6(1)^2 + 2(1)$

$a_2 - a_1 = 6(2)^2 + 2(2)$

$a_3 - a_2 = 6(3)^2 + 2(3)$

$a_4 - a_3 = 6(4)^2 + 2(4)$

$\vdots$

$a_{98} - a_{97} = 6(98)^2 + 2(98)$

$a_{99} - a_{98} = 6(99)^2 + 2(99)$

$a_{99} - a_0 = 6\left[1^2 + 2^2 + 3^2 + \cdots + 99^2\right] + 2[1 + 2 + 3 + \cdots + 99]$

$\implies a_{99} - 0 = 6\left[\frac{99 \times 100 \times 199}{6}\right] + 2\left[\frac{99 \times 100}{2}\right]$

$\implies a_{99} = 9900 \times 199 + 9900$

$\implies a_{99} = 9900(199 + 1)$

$\implies a_{99} = 9900 \times 200$

$\implies a_{99} = 1980000$

Second Method:

Here's the easiest way to solve this:

Taking a_{n-1} to LHS,

$a_n - a_{n-1} = 6n^2 + 2n$

Taking summation,

$\sum a_n - \sum a_{n-1} = 6\sum n^2 + 2\sum n = (n)(n+1)(2n+1) + n(n+1)$

LHS becomes:

$\sum a_n - \sum a_{n-1} = (a_n + \sum a_{n-1}) - \sum a_{n-1} = a_n$

so our equation reduces to:

$a_n = (n)(n+1)(2n+1) + n(n+1)$

Putting $n = 99$ as asked in the question,

$a_{99} = 99 * 100 * 199 + 99 * 100 = 1980000$

PS:

- $1 + 2 + 3 + \ldots + n = \sum n = \frac{n(n+1)}{2}$
- $1^2 + 2^2 + 3^2 + \ldots + n^2 = \sum n^2 = \frac{n(n+1)(2n+1)}{6}$
- $1^3 + 2^3 + 3^3 + \ldots + n^3 = \sum n^3 = \left[\frac{n(n+1)}{2}\right]^2 = \frac{n^2(n+1)^2}{4}$

Correct Answer: 1980000

116. A bus driver pays all tolls, using only nickels and dimes, by throwing one coin at a time into the mechanical toll collector.

Find a recurrence relation for the number of different ways the bus driver can pay a toll of n cents (where the order in which the coins are used matters). (Mark all the appropriate choices)

A. $a_{\mathbf{k}} = a_{\mathbf{k}+1} + a_{\mathbf{k}-2}$, where $\mathrm{a}_{\mathbf{0}} = 1, \mathrm{a}_{\mathbf{1}} = 1$

B. $a_{\mathbf{k}} = a_{\mathbf{k}-1} + a_{\mathbf{k}+2}$, where $\mathrm{a}_{\mathbf{0}} = 1, \mathrm{a}_{\mathbf{1}} = 1$

C. $a_{\mathbf{k}} = a_{\mathbf{k}+1} + a_{\mathbf{k}+2}$, where $\mathrm{a}_{\mathbf{0}} = 1, \mathrm{a}_{\mathbf{1}} = 1$

D. $a_{\mathbf{k}} = a_{\mathbf{k}-1} + a_{\mathbf{k}-2}$, where $\mathrm{a}_{\mathbf{0}} = 1, \mathrm{a}_{\mathbf{1}} = 1$

Solution: Let a_n be the number of ways the bus driver can pay a toll of n cents by using nickels and dimes (where the order in which the coins are used matters).

So in convenience we denote $a_{\mathbf{k}}$ be the number of ways the bus driver can pay a toll of $n = 5k$ cents by using nickels and dimes (where the order in which the coins are used matters).

If the last coin is a nickel, then we need to find all ways of paying $5(\mathrm{k}-1)$, which is $\mathrm{a}_{\mathbf{k-1}}$.

If the last coin is a dime, it is preceeded by all the possible ways of paying $5(\mathrm{k}-2)$ which is $\mathrm{a}_{\mathbf{k}-2}$.

So we got the recurrence relation: $a_{\mathbf{k}} = a_{\mathbf{k}-1} + a_{\mathbf{k}-2}$, where $\mathrm{a}_{\mathbf{0}} = 1, \mathrm{a}_{\mathbf{1}} = 1$

Correct Answer: D

117. A bus driver pays all tolls, using only nickels and dimes, by throwing one coin at a time into the mechanical toll collector.

In how many different ways can the driver pay a toll of 45 cents? (where the order in which the coins are used matters). (Numerical Answer Type)

Solution:

$$\begin{aligned}
&a_k = a_{k-1} + a_{k-2}, \text{ where } a_0 = 1, a_1 = 1\\
&\mathrm{a}_0 = 1,\\
&\mathrm{a}_1 = 1,\\
&\mathrm{a}_2 = \mathrm{a}_0 + \mathrm{a}_1 = 1 + 1 = 2\\
&\mathrm{a}_3 = \mathrm{a}_1 + \mathrm{a}_2 = 1 + 2 = 3\\
&\mathrm{a}_4 = \mathrm{a}_2 + \mathrm{a}_3 = 2 + 3 = 5\\
&\mathrm{a}_5 = \mathrm{a}_3 + \mathrm{a}_4 = 3 + 5 = 8\\
&\mathrm{a}_6 = \mathrm{a}_4 + \mathrm{a}_5 = 5 + 8 = 13\\
&a_7 = a_5 + a_6 = 8 + 13 = 21\\
&\mathrm{a}_8 = \mathrm{a}_6 + \mathrm{a}_7 = 13 + 21 = 34\\
&\mathrm{a}_9 = \mathrm{a}_7 + \mathrm{a}_8 = 21 + 34 = 55 \text{ ways}
\end{aligned}$$

Correct Answer: 55

118. What is the characteristic equation of

$$C_0 a_n = -C_1 a_{n-1} + C_2 a_{n-2}$$

for $n \geq 2$?

A. $C_0 r^2 - C_1 r + C_2 = 0$

B. $C_0 r^2 + C_1 r + C_2 = 0$

C. $C_0 r^2 + C_1 r - C_2 = 0$

D. $C_0 r^2 - C_1 r - C_2 = 0$

Solution: To determine the characteristic equation of this second-order homogeneous recurrence relation, first we isolate for 0 on one side:

$$C_0 a_n + C_1 a_{n-1} - C_2 a_{n-2} = 0$$

Next, we substitute $a_n = cr^n$ to obtain:

$$C_0 \cdot cr^n + C_1 \cdot cr^{n-1} - C_2 \cdot cr^{n-2} = 0$$

And so, by dividing every term by $c \cdot r^{n-2}$, we arrive at our characteristic equation:

$$C_0 r^2 + C_1 r - C_2 = 0$$

Correct Answer: C

119. Find a recurrence relation for the number of binary sequences of length n that have no consecutive 0's. Note: A binary sequence is sequence made up of only the digits " 0 " and " 1 ".

A. $a_n = a_{n-1} - a_{n-2}$ for $n \geq 3$ with $a_1 = 2$ and $a_2 = 3$

B. $a_n = -a_{n-1} + a_{n-2}$ for $n \geq 3$ with $a_1 = 2$ and $a_2 = 3$

B. $a_n = -a_{n-1} - a_{n-2}$ for $n \geq 3$ with $a_1 = 2$ and $a_2 = 3$

C. $a_n = a_{n-1} + a_{n-2}$ for $n \geq 3$ with $a_1 = 2$ and $a_2 = 3$

Solution:

Let a_n be the number of binary sequences of length n that have no consecutive 0 's. We will split up a_n even further into the sequences that end in $0, a_n^0$, and those that end in $1, a_n^1$. Then certainly $a_n = a_n^0 + a_n^1$, for $n \geq 1$. First, we notice that $a_1 = 2$, as the only possible sequences are " 0 " and " 1 ". We can build each sequence of length $n+1$ from sequences of length n by adding

one addition term at the end as since the sequences of length n already satisfy that there are no consecutive 0 's. Thus, we can see that

$$a_n = 2 \cdot a_{n-1}^1 + 1 \cdot a_{n-1}^0$$

This is because for any sequence of length $n-1$ that ends in " 1 ", either " 1 " or " 0 " can be added to the end to form a sequence of length n while a sequence of length $n-1$ that ends in " 0 " can only have " 1 " added to the end to form a sequence of length n, else there would be consecutive " 0 "'s.

Now consider some sequences that belong to a_{n-2}. If the sequence $y \in a_{n-2}$ then $y1 \in a^1_{n-1}$, and vice versa. Thus, there is a one-to-one correspondence between the sets and so it follows that $a_{n-2} = a^1_{n-1}$.

We can now find our recurrence relation by putting all of this together:

$$\begin{aligned} a_n &= 2 \cdot a^1_{n-1} + a^0_{n-1} \\ &= a^1_{n-1} + a^1_{n-1} + a^0_{n-1} \\ &= a^1_{n-1} + a_{n-1} \\ &= a_{n-1} + a_{n-2} \end{aligned}$$

Thus, our recurrence relation is:

$$a_n = a_{n-1} + a_{n-2} \text{ for } n \geq 3 \text{ with } a_1 = 2 \text{ and } a_2 = 3$$

Correct Answer: D

120. Suppose a recurrence relation of the form $a_n = c_1 a_{n-1} + c_2 a_{n-2}$ has a general solution $a_n = A_1 3^n + A_2 6^n$. The value of $c_1 - c_2 =$? (Numerical Answer Type)

Solution:

We know from the general solution that the characteristic roots of the characteristic equation must be $r_1 = 3, r_2 = 6$. The characteristic equation, in terms of c_1, c_2 is $r^2 - c_1 r - c_2 = 0$. Since we know that r_1, r_2 must solve this equation, we can solve for our two unknowns:

$$\begin{aligned} r_1 &: 3^2 - 3c_1 - c_2 = 0, \\ r_2 &: 6^2 - 6c_1 - c_2 = 0. \end{aligned}$$

The first equation implies that $c_2 = 9 - 3c_1$, so we can substitute this into the second equation: $36 - 6c_1 - 9 + 3c_1 = 27 - 3c_1 = 0$.

So, $c_1 = 9$ and $c_2 = -18$.

$\therefore$ The value of $c_1 - c_2 = 9 - (-18) = 9 + 18 = 27$.

Correct Answer: 27

4.3 Generating functions

121. In how many ways can 1000 identical pamphlets be distributed to five different counselling centers, where pamphlets are put in stacks of 50, such that each center receives at least 50 but no more than 500 pamphlets? (Numerical Answer Type)

Solution:

We first notice that thinking of this problem in terms of stacks of pamphlets rather than the pamphlets themselves reduces it to: "In how many ways can $\frac{1000}{50} = 20$ stacks be distributed to five different counselling centers such that each center receives at least $\frac{50}{50} = 1$ but no more than $\frac{500}{50} = 10$ stacks?" The generating function that represents this set up is,

$$g(x) = \left(x^1 + x^2 + \ldots + x^{10}\right)^5$$

and we are interested in determining the coefficient of x^{20}. Alternatively, we can identify the coefficient of x^{15} in,

$$g'(x) = \left(1 + x + \ldots + x^9\right)^5$$

which was obtained by factoring out an x.

We now rewrite this using what we know about series,

$$\begin{aligned} g'(x) &= \left(\frac{1-x^{10}}{1-x}\right)^5 \\ &= \left(-x^{50}+5x^{40}-10x^{30}+10x^{20}-5x^{10}+1\right)\frac{1}{(1-x)^5} \\ &= \left(-x^{50}+5x^{40}-10x^{30}+10x^{20}-5x^{10}+1\right)\cdot\sum_{r=0}^{\infty}\binom{r+5-1}{5-1}x^r. \end{aligned}$$

When this expression is expanded, we are interested in the coefficients of x when $r = 15, 5$, which correspond to the coefficients of x^{15}.

When $r = 15$, $\binom{15+5-1}{5-1} = 3\,876$.

When $r = 5$, $\binom{5+5-1}{5-1} = 126$.

Therefore the coefficient of x^{15} in $g'(x)$ is $(-5)(126)+(1)3876 = 3246$, which is the number of ways these stacks of pamphlets can be distributed.

Correct Answer: 3246

122. Determine the number of ways that \$12 in loonies can be distributed between a father's three children so that the eldest gets at least four dollars, the middle and youngest child are both guaranteed at least two dollars, but the youngest cannot receive any more than \$5 since he will spend it all on candy and rot his teeth. (Numerical Answer Type)

Solution:

We represent the eldest child's potential share of the money by $x^4+x^5+x^6+\ldots$ The middle child's, $x^2+x^3+x^4+\ldots$ The youngest child's, $x^2+x^3+x^4+x^5$. To determine the number of ways the loonies can be distributed, we are looking for the coefficient of x^{12} in the product,

$$g(x) = \left(x^4+x^5+x^5+\ldots\right)\left(x^2+x^3+x^4+x^5+\ldots\right)\left(x^2+x^3+x^4+x^5\right)$$

We can simplify $g(x)$,

$$\begin{aligned} g(x) &= \left(x^4+x^5+x^5+\ldots\right)\left(x^2+x^3+x^4+x^5+\ldots\right)\left(x^2+x^3+x^4+x^5\right) \\ &= x^8\left(1+x+x^2+\ldots\right)\left(1+x+x^2+x^3+\ldots\right)\left(1+x+x^2+x^3\right) \\ &= x^8\left(1+x+x^2+\ldots\right)^2\left(1+x+x^2+x^3\right). \end{aligned}$$

Alternatively, we can reduce this problem to identifying the coefficient of x^4 in,

$$g'(x) = \left(1+x+x^2+\ldots\right)^2\left(1+x+x^2+x^3\right)$$

Using identities and some substitutions we rewrite $g'(x)$ as,

$$\begin{aligned} g'(x) &= \left(1+x+x^2+\ldots\right)^2\left(1+x+x^2+x^3\right) \\ &= \frac{1}{(1-x)^2}\cdot\frac{1-x^4}{1-x} \\ &= \frac{\left(1-x^4\right)}{(1-x)^3} \\ &= \left(1-x^4\right)\sum_{r=0}^{\infty}\binom{r+3-1}{3-1}x^r. \end{aligned}$$

The coefficient of x^4 will occur when $r = 0, 4$. When $r = 0, \binom{0+3-1}{3-1} = 1$, when $r = 4, \binom{4+3-1}{3-1} = 15$.

Putting this together, there are, $15(1) - 1 = 14$ ways to distribute the loonies.

Correct Answer: 14

123. The coefficient of x^4 in the expansion of $\left(1+x+x^2+x^3\right)^{11}$ is _______.

A. 900

B. 909

C. 990

D. 999

Solution:

$$\begin{aligned}&\left(1+x+x^2+x^3\right)^{11}\\&=\left[(1+x)\left(1+x^2\right)\right]^{11}\\&=(1+x)^{11}\cdot\left(1+x^2\right)^{11}\end{aligned}$$

Applying the Binomial theorem for the positive index, we get

$$\left({}^{11}C_0 + {}^{11}C_1 x + {}^{11}C_2 x^2 + {}^{11}C_3 x^3 + {}^{11}C_4 x^4 + \ldots\right) \times \left({}^{11}C_0 + {}^{11}C_1 x^2 + {}^{11}C_2 x^4 + \ldots\right)$$

$\therefore$ Coefficient of x^4 in $\left(1+x+x^2+x^3\right)^{11}$ (obtained by multiplying only those terms whose product will involve x^4)

$$= {}^{11}C_0 \cdot {}^{11}C_2 + {}^{11}C_2 \cdot {}^{11}C_1 + {}^{11}C_4 \cdot {}^{11}C_0 = 990$$

Correct Answer: C

124. Let the coefficient of the middle term of the binomial expansion of $(1+x)^{2n}$ be α and those of two middle terms of the binomial expansion of $(1+x)^{2n-1}$ be β and γ. Which one of the following relations is correct?

A. $\alpha > \beta + \gamma$

B. $\alpha < \beta + \gamma$

C. $\alpha = \beta + \gamma$

D. $\alpha = \beta\gamma$

Solution:

We have $(1+x)^{2n}$ Middle term $= \left(\frac{2n}{2}+1\right)^{\text{th}}$ term $= (n+1)^{\text{th}}$ term Coefficient of $(n+1)^{\text{th}}$ term $= {}^{2n}C_n$ $\alpha = {}^{2n}C_n$ Again, we have binomial expansion of $(1+x)^{2n-1}$ coefficient of middle terms are $\left(\frac{2n-1+1}{2}\right)^{\text{th}}$ term and $\left(\frac{2n-1+1}{2}+1\right)^{\text{th}}$ term,

$$\begin{aligned}&\therefore \beta = (n+1)^{\text{th}} \text{ term } = {}^{2n-1}C_n\\&\text{and } \gamma = n^{\text{th}} \text{ term } = {}^{2n-1}C_{n-1}\\&\text{Now, } \beta+\gamma = {}^{2n-1}C_n + {}^{2n-1}C_{n-1}\left[\because {}^{n}C_r + {}^{n}C_{r-1} = {}^{n+1}C_r\right]\\&= {}^{2n}C_n = \alpha\\&\therefore \alpha = \beta + \gamma\end{aligned}$$

Correct Answer: C

125. If $a_i (i = 0, 1, 2, \ldots, 16)$ be real constants such that for every real value of x, $\left(1+x+x^2\right)^8 = a_0 + a_1 x + a_2 x^2 + \ldots + a_{16} x^{16}$, then a_5 is equal to _____.

A. 502

B. 504

C. 506

D. 508

Solution: We have,

$$\left(1+x+x^2\right)^8=\left(\frac{1-x^3}{1-x}\right)^8=\left(1-x^3\right)^8(1-x)^{-8}$$

On expanding by binomial theorem, we get

$$\begin{aligned}&=\left(1-{}^8C_1x^3+{}^8C_2x^6-\ldots\right)\left(1+8x+\frac{8\times 9}{2}x^2+\frac{8\times 9\times 10}{3\times 2\times 1}x^3+\ldots\right)\\&=\left(1-{}^8C_1x^3+{}^8C_2x^6-\ldots\right)\left(1+8x+36x^2+120x^3+330x^4+792x^5+\ldots\right)\end{aligned}$$

Now, a_5 = Coefficient of x^5

$$\begin{aligned}&=792-{}^8C_1\times 36\\&=792-288=504\end{aligned}$$

Correct Answer: B

126. What is the generating function corresponding to the Fibonacci series?

$$F_n=F_{n-1}+F_{n-2}.$$

Note that $F_0=F_1=1$.

A. $G(x)=\frac{1}{1+x-x^2}$

B. $G(x)=\frac{1}{1-x-x^2}$

C. $G(x)=\frac{1}{1-x+x^2}$

D. $G(x)=\frac{1}{1+x+x^2}$

Solution: The Fibonacci sequence can be written as per the definition given :

$a_0,a_1,a_2,a_3,a_4,a_5,a_6,a_7,\ldots=1,1,2,3,5,8,13,21,\ldots$

Therefore the ordinary generating function corresponding to the given sequence:

$G(x)=1.x^0+1.x^1+2.x^2+3.x^3+5.x^4+8.x^5+\ldots\ldots \quad\longrightarrow(1)$

$G(x)=1.+1.x^1+2.x^2+3.x^3+5.x^4+8.x^5+\ldots$

$xG(x)=1.x^1+1.x^2+2.x^3+3.x^4+5.x^5+\ldots \quad\longrightarrow(2)$

Subtracting **(2)** from **(1)**

$G(x)-xG(x)=1+0x^1+1x^2+1x^3+2x^4+3x^5+5x^6+\ldots..$

$=1+0x^1+x^2(1x^0+1x^1+2x^2+3x^3+\ldots..)$

$=1+0x^1+x^2G(x)$

$\Rightarrow G(x)=\dfrac{1}{1-x-x^2}$

Correct Answer: B

127. The coefficient of $\left[x^{50}\right]\left(x^6+x^7+x^8+\ldots\right)^6$ is ________. (Mark all the appropriate choices)

A. $\binom{19}{5}$

B. $\binom{18}{6}$

C. $\binom{19}{14}$

D. $\binom{18}{12}$

Solution: $[x^{50}](x^6+x^7+x^8+\ldots)^6$

$[x^{50}](x^6(1+x^2+x^3+\ldots))^6$

$[x^{50}](x^{36})(1+x^2+x^3+\ldots)^6$

$[x^{14}](1+x^2+x^3+\ldots)^6$

$[x^{14}](\frac{1}{1-x})^6$

$[x^{14}](1-x)^{-6}$

$[x^{14}]\binom{-6}{14}(-1)^{14}$

$[x^{14}]\binom{14+6-1}{14}(-1)^{14}(-1)^{14}$

$[x^{14}]\binom{19}{14}$

$[x^{14}]\binom{19}{5}$

$\binom{19}{5}=\binom{19}{14}$

Correct Answer: A;C

128. What will be the coefficient of x^{100}?

$$\frac{1}{(1-x^{10})(1-x^{20})(1-x^{50})}$$

(Numerical Answer Type)

Solution:

$[x^{100}]\frac{1}{(1-x^{10})(1-x^{20})(1-x^{50})}$

$[x^{100}](1-x^{10})^{-1}(1-x^{20})^{-1}(1-x^{50})^{-1}$

$=\binom{-1}{10}\binom{-1}{0}\binom{-1}{0}+\binom{-1}{0}\binom{-1}{5}\binom{-1}{0}+\binom{-1}{0}\binom{-1}{0}\binom{-1}{2}+$

$\binom{-1}{8}\binom{-1}{1}\binom{-1}{0}+\binom{-1}{6}\binom{-1}{2}\binom{-1}{0}+\binom{-1}{4}\binom{-1}{3}\binom{-1}{0}+\binom{-1}{2}\binom{-1}{4}\binom{-1}{0}+$

$\binom{-1}{5}\binom{-1}{0}\binom{-1}{1}+\binom{-1}{3}\binom{-1}{1}\binom{-1}{1}+\binom{-1}{1}\binom{-1}{2}\binom{-1}{1}$

$\binom{-n}{k}=\binom{n+k-1}{k}$

So after calculating all the terms for each term we get 1 and there are 10 such terms above.

So, coefficient of x^{100} is 10.

There are many terms so to differentiate among them distinct colors are used:

- Red - Selecting x from any one of $(1-x^{10})^{-1}(1-x^{20})^{-1}(1-x^{50})^{-1}$
- Blue - Selecting from both $(1-x^{10})^{-1}(1-x^{20})^{-1}$
- Orange - Selecting from both $(1-x^{10})^{-1}(1-x^{50})^{-1}$
- Teal - Selecting from all $(1-x^{10})^{-1}(1-x^{20})^{-1}(1-x^{50})^{-1}$

Correct Answer: 10

129. The coefficient of x^{83} in $(x^5+x^8+x^{11}+x^{14}+x^{17})^{10}$ is ________. (Numerical Answer Type)

Solution:

$$\left[x^{83}\right]:\left[\left(x^5+x^8+x^{11}+x^{14}+x^{17}\right)^{10}\right]$$

$$\left[x^{83}\right]:\left[x^{50}\left(1+x^3+x^6+x^9+x^{12}\right)^{10}\right]$$

$$\left[x^{33}\right]:\left[\left(1+x^3+x^6+x^9+x^{12}\right)^{10}\right]$$

$$\left[x^{33}\right]:\left[\left\{1+(x^3)^1+(x^3)^2+(x^3)^3+(x^3)^4\right\}^{10}\right]$$

$$\left[x^{33}\right]:\left[\left\{\frac{1-(x^3)^{4+1}}{1-(x^3)}\right\}^{10}\right]$$

$$\left[x^{33}\right]:\left[\left(1-x^{15}\right)^{10}.\frac{1}{(1-x^3)^{10}}\right]$$

$$\left[x^{33}\right]:\left[\sum_{r=0}^{10}\binom{10}{r}\left(-x^{15}\right)^r.\sum_{k=0}^{\infty}\binom{10+k-1}{k}.(x^3)^k\right]$$

Now,

$$\left[x^{33}\right]:\left[\sum_{r=0}^{10}\binom{10}{r}\left(-x^{15}\right)^r.\sum_{k=0}^{\infty}\binom{10+k-1}{k}.(x^3)^k\right]$$

$$\left[x^{33}\right]:\begin{cases}(-1)^0.\binom{10}{0}.\binom{10+11-1}{11} & r=0,k=11\\ (-1)^1.\binom{10}{1}.\binom{10+6-1}{6} & r=1,k=6\\ (-1)^2.\binom{10}{2}.\binom{10+1-1}{1} & r=2,k=1\end{cases}$$

$$\left[x^{33}\right]:\begin{cases}+\binom{10}{0}.\binom{20}{11} & r=0,k=11\\ -\binom{10}{1}.\binom{15}{6} & r=1,k=6\\ +\binom{10}{2}.\binom{10}{1} & r=2,k=1\end{cases}$$

NOTE:

- $1+x+x^2+x^3+\ldots+x^n=\frac{1-x^{n+1}}{1-x}$
- $\dfrac{1}{(1-x)^n}=\displaystyle\sum_{r=0}^{\infty}\binom{n+r-1}{r}.x^r$
- $\left[x^{83}\right]$ means coefficient of x^{83} of the whole expression.

Correct Answer: 1294080

130. What is the coefficient of x^6 in the following series expansion? (Numerical Answer Type)

$$\frac{1}{1-x}\cdot\frac{1}{1-x^2}\cdot\frac{1}{1-x^3}\cdots$$

Solution: $\dfrac{1}{1-x}=1+x+x^2+x^3+x^4+\cdots+\infty$

Simplifying each term, we get an equivalent expression for finding the coefficient of x^6.

Finding coefficient of x^6 in $\dfrac{1}{1-x}\dfrac{1}{1-x^2}\dfrac{1}{1-x^3}\cdots$ is equivalent to finding coefficient of x^6 in :

$(1+x+x^2+\cdots+x^6)(1+x^2+x^4+x^6)(1+x^3+x^6)(1+x^4)(1+x^5)(1+x^6)$

We are interested in the coefficient of x^6. So, neglect higher powers.

$\Rightarrow (1+x+x^2+\cdots+x^6)(1+x^2+x^4+x^6)(1+x^3+x^6)(1+x^4+x^5+x^6)$

$\Rightarrow (1+x+x^2+\cdots+x^6)(1+x^2+x^4+x^6)(1+x^3+x^4+x^5+2x^6)$

$\Rightarrow (1+x+x^2+x^3+x^4+x^5+x^6)(1+x^2+x^3+2x^4+2x^5+4x^6)$

Coefficient of $x^6 = \ 4+2+2+1+1+1 = 11$

Second Method:

$$G(x) = \frac{1}{1-x}\,\frac{1}{1-x^2}\,\frac{1}{1-x^3}\cdots\infty$$

$G(x)$ is a generating function. The coefficient of x^n in $G(x)$ is equal to the number of unordered partitions of number n.

if $n = 6$, then $\longrightarrow$

$$\begin{array}{llllllll}
6 & \longrightarrow 6 & & & & & \\
 & \longrightarrow 5 & 1 & & & & \\
 & \longrightarrow 4 & 2 & & & & \\
 & \longrightarrow 4 & 1 & 1 & & & \\
 & \longrightarrow 3 & 3 & & & & \\
 & \longrightarrow 3 & 2 & 1 & & & \\
 & \longrightarrow 3 & 1 & 1 & 1 & & \\
 & \longrightarrow 2 & 2 & 2 & & & \\
 & \longrightarrow 2 & 2 & 1 & 1 & & \\
 & \longrightarrow 2 & 1 & 1 & 1 & 1 & \\
 & \longrightarrow 1 & 1 & 1 & 1 & 1 & 1
\end{array}$$

Total 11 partitions. So, coefficient of x^6 in $G(x)$ is equal to 11.