Engineering Mathematics Weekly Quiz Questions For GATE 2025

GATE And Tech
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1 Linear Algebra

1.1 Matrices

1. If A $= \begin{bmatrix} 2 & -1 & 3 \\ -4 & 5 & 1 \end{bmatrix}$ and B $= \begin{bmatrix} 2 & 3 \\ 4 & -2 \\ 1 & 5 \end{bmatrix}$, then ______.

   A. only AB is defined

   B. only BA is defined

   C. AB and BA both are defined

   D. AB and BA both are not defined.

   Solution: To compute AB (matrix multiplication), the number of columns in matrix A must be equal to the number of rows in matrix B. In this case, matrix A has 3 columns, and matrix B has 3 rows, making the product $(\text{AB})_{2\times 2}$ defined.

   To compute BA (matrix multiplication), the number of columns in matrix B must be equal to the number of rows in matrix A. In this case, matrix B has 2 columns, and matrix A has 2 rows, making the product $(\text{BA})_{3\times 3}$ defined.

   Correct Answer: C

2. If matrix A is $m \times n$ and B is $n \times p$, the number of multiplication operations and addition operations needed to calculate the matrix AB, respectively, are _________.

   A. mn^2p, mpm

   B. $mpn, mp(n-1)$

   C. mpn, mpn

   D. $mn^2p, (m+p)n$

   Solution: Given that, the Order of matrix A is $(m\times n)$. The order of matrix B is $(n\times p)$. For finding the number of addition operations and number of multiplication operations. Let $m = 2, n = 3, p = 2$, Therefore, order of matrix A is (2×3).

Order of matrix B is (3×2). For matrix multiplication of AB,

$$AB = \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}_{2\times 3} \begin{bmatrix} x & y \\ q & r \\ s & t \end{bmatrix}_{3\times 2}$$

$$AB = \begin{bmatrix} a\times x + b\times q + c\times s & a\times y + b\times r + c\times t \\ d\times x + e\times q + f\times s & d\times y + e\times r + f\times t \end{bmatrix}$$

From the above example,

The number of multiplications operation is 12 i.e. mpn.

The number of additions operation is 8 i.e. $mp(n-1)$.

Correct Answer: B

3. Total number of possible matrices of order 3×3 with each entry 2 or 0 is ______.

A. 9

B. 27

C. 81

D. 512

Solution: To form a 3×3 matrix where each entry can be either 2 or 0, there are two choices (2 or 0) for each of the 9 entries.

Using the fundamental counting principle (the number of choices for each entry multiplied together), the total number of possible matrices is: $2^9 = 512$.

Correct Answer: D

4. A is a matrix of of order $m * n$, then number of minors of order r is ______.

A. ${}^m\mathrm{P}_r$

B. ${}^m\mathrm{C}_r * {}^n\mathrm{C}_r$

C. $m * n$

D. None

Solution: In a matrix of order $m\times n$, a minor of order r is obtained by selecting any r rows and r columns from the matrix. These selected rows and columns form a square submatrix of size $r\times r$.

Now, let's calculate the number of ways to choose r rows out of m rows and r columns out of n columns. The number of ways to choose r items out of a set of n distinct items is denoted as ${}^n\mathrm{C}_r$, which represents combinations.

So, to form a minor of order r in the matrix:

1. Choose r rows out of the m rows in ${}^m\mathrm{C}_r$ ways.

2. Choose r columns out of the n columns in ${}^n\mathrm{C}_r$ ways.

Since these choices are independent, you multiply the number of ways to choose rows by the number of ways to choose columns to find the total number of minors of order r:

Total number of minors of order $r = {}^m\mathrm{C}_r \cdot {}^n\mathrm{C}_r$

So, option B, ${}^m\mathrm{C}_r \cdot {}^n\mathrm{C}_r$, correctly represents the number of minors of order r in a matrix of order $m\times n$.

Correct Answer: B

5. Let $\mathrm{A} = \begin{bmatrix} 1 & 1/3 \\ c & d \end{bmatrix}$.

If $\mathrm{A}^2 = -I$, then the value of $c-d$ is ________. (Numerical Answer Type)

Solution: We have the matrix A :

$$A = \begin{bmatrix} 1 & 1/3 \\ c & d \end{bmatrix}$$

And we're given that $A^2 = -I$, where I is the 2×2 identity matrix:

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Now, let's calculate A^2 :

$$\mathrm{A}^2 = \begin{bmatrix} 1 & 1/3 \\ c & d \end{bmatrix} * \begin{bmatrix} 1 & 1/3 \\ c & d \end{bmatrix}$$

$$\mathrm{A}^2 = \begin{bmatrix} (1*1+(1/3)*(c)) & (1*(1/3)+(1/3)*(d)) \\ (c*1+d*c) & (c*(1/3)+d*d) \end{bmatrix}$$

Simplifying:

$$A^2 = \begin{bmatrix} 1+(c/3) & 1/3+(d/3) \\ c+c*d & (c/3)+\left(d^2\right) \end{bmatrix}$$

Now, we're given that $A^2 = -I$:

$$\begin{bmatrix} 1+(c/3) & 1/3+(d/3) \\ c+c*d & (c/3)+\left(d^2\right) \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

This leads to the following equations:

$1+(c/3) = -1 \quad \rightarrow (1)$

$1/3+(d/3) = 0 \quad \rightarrow (2)$

$c+c*d = 0 \quad \rightarrow (3)$

$(c/3)+\left(d^2\right) = -1 \quad \rightarrow (4)$

Let's solve these equations step by step:

From Equation (1), we have:

$c/3 = -1-1$

$c/3 = -2$

$c = -6$

From Equation (2), we have: $(d/3) = -1/3$

$d = -1$

Therefore, the value of $c-d$ is -5.

Correct Answer: -5

6. The total number of matrices $A = \begin{bmatrix} 0 & 2y & 1 \\ 2x & y & -1 \\ 2x & -y & 1 \end{bmatrix}$, $(x, y \in R, x \neq y)$ for which $A^T A = 3I_3$ is _____.

A. 2

B. 4

C. 3

D. 6

Solution: Given matrix

$$A = \begin{bmatrix} 0 & 2y & 1 \\ 2x & y & -1 \\ 2x & -y & 1 \end{bmatrix}, (x, y \in R, x \neq y)$$

for which

$$A^T A = 3I_3$$

$$\Rightarrow \begin{bmatrix} 0 & 2x & 2x \\ 2y & y & -y \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 2y & 1 \\ 2x & y & -1 \\ 2x & -y & 1 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 8x^2 & 0 & 0 \\ 0 & 6y^2 & 0 \\ 0 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

Here, two matrices are equal, therefore equating the corresponding elements, we get

$$8x^2 = 3 \text{ and } 6y^2 = 3$$
$$x = \pm\sqrt{\frac{3}{8}}$$
$$y = \pm\frac{1}{\sqrt{2}}$$

$\because$ There are 2 different values of x and y each.

So, 4 matrices are possible such that $A^T A = 3I_3$.

Correct Answer: B

7. Let $A = \begin{bmatrix} 0 & 2q & r \\ p & q & -r \\ p & -q & r \end{bmatrix}$. If $AA^T = I_3$, then $|p|$ is ______.

A. $\frac{1}{\sqrt{5}}$

B. $\frac{1}{\sqrt{2}}$

C. $\frac{1}{\sqrt{3}}$

D. $\frac{1}{\sqrt{6}}$

Solution: Given, $AA^T = I$

$$\Rightarrow \begin{bmatrix} 0 & 2q & r \\ p & q & -r \\ p & -q & r \end{bmatrix}\begin{bmatrix} 0 & p & p \\ 2q & q & -q \\ r & -r & r \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$
$$\Rightarrow \begin{bmatrix} 0+4q^2+r^2 & 0+2q^2-r^2 & 0-2q^2+r^2 \\ 0+2q^2-r^2 & p^2+q^2+r^2 & p^2-q^2-r^2 \\ 0-2q^2+r^2 & p^2-q^2-r^2 & p^2+q^2+r^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

We know that, if two matrices are equal, then corresponding elements are also equal, so

$$q^2 + r^2 = 1 = p^2 + q^2 + r^2, \quad \to (1)$$
$$2q^2 - r^2 = 0 \Rightarrow r^2 = 2q^2 \quad \to (2)$$
$$\text{and} \quad p^2 - q^2 - r^2 = 0 \quad \to (3)$$

Using equations (2) and (3), we get

$$p^2 = 3q^2 \quad \to (4)$$

Using equations (2) and (4) in equation (1), we get

$$\begin{aligned} & 4q^2 + 2q^2 &= 1 \\ \Rightarrow \quad & 6q^2 &= 1 \\ \Rightarrow \quad & 2p^2 &= 1 \quad \text{[using equation (4)]} \\ & p^2 = \frac{1}{2} \Rightarrow |p| &= \frac{1}{\sqrt{2}} \end{aligned}$$

Correct Answer: B

8. For 3×3 matrices M and N, which of the following statement(s) is/are not correct? (Mark all the correct options)

A. $N^T MN$ is symmetric or skew-symmetric, according to M is symmetric or skew-symmetric.

B. $MN - NM$ is symmetric for all symmetric matrices M and N.

C. MN is symmetric for all symmetric matrices M and N.

D. $(\text{adj}\, M)(\text{adj}\, N) = \text{adj}(MN)$ for all invertible matrices M and N.

Solution:

A. $\left(N^T M N\right)^T = N^T M^T \left(N^T\right)^T = N^T M^T N$, is symmetric if M is symmetric and skew-symmetric, if M is skew-symmetric.

B.

$$\begin{aligned}(MN - NM)^T &= (MN)^T - (NM)^T \\ &= NM - MN = -(MN - NM)\end{aligned}$$

$\therefore$ Skew-symmetric, when M and N are symmetric.

C. $(MN)^T = N^T M^T = NM \neq MN$ $\therefore$ Not correct.

D. $(\operatorname{adj} MN) = (\operatorname{adj} N) \cdot (\operatorname{adj} M)$ $\therefore$ Not correct.

Correct Answer: C; D

9. If $A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ a & 2 & b \end{bmatrix}$ is a matrix satisfying the equation $AA^T = 9I$, where, I is 3×3 identity matrix, then the ordered pair (a, b) is equal to _______.

A. $(2, -1)$

B. $(-2, 1)$

C. $(2, 1)$

D. $(-2, -1)$

Solution: Given, $A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ a & 2 & b \end{bmatrix}, A^T = \begin{bmatrix} 1 & 2 & a \\ 2 & 1 & 2 \\ 2 & -2 & b \end{bmatrix}$ and

$$\begin{aligned}AA^T &= \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ a & 2 & b \end{bmatrix}\begin{bmatrix} 1 & 2 & a \\ 2 & 1 & 2 \\ 2 & -2 & b \end{bmatrix} \\ &= \begin{bmatrix} 9 & 0 & a+4+2b \\ 0 & 9 & 2a+2-2b \\ a+4+2b & 2a+2-2b & a^2+4+b^2 \end{bmatrix}\end{aligned}$$

It is given that, $AA^T = 9I$

$$\Rightarrow \begin{bmatrix} 9 & 0 & a+4+2b \\ 0 & 9 & 2a+2-2b \\ a+4+2b & 2a+2-2b & a^2+4+b^2 \end{bmatrix} = 9\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 9 & 0 & a+4+2b \\ 0 & 9 & 2a+2-2b \\ a+4+2b & 2a+2-2b & a^2+4+b^2 \end{bmatrix} = \begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix}$$

On comparing, we get

$$a + 4 + 2b = 0 \Rightarrow a + 2b = -4 \quad \rightarrow (1)$$

$$2a + 2 - 2b = 0 \Rightarrow a - b = -1 \quad \rightarrow (2)$$

$$a^2 + 4 + b^2 = 9 \quad \rightarrow (3)$$

On solving equations. (1) and (2), we get

$$a = -2, b = -1$$

This satisfies equation (3)

Hence, $(a, b) = (-2, -1)$

Correct Answer: D

10. If $A = \begin{bmatrix} \alpha & 0 \\ 1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 5 & 1 \end{bmatrix}$, then value of α for which $A^2 = B$, is ______.

A. 1

B. -1

C. 4

D. no real values

Solution: Given, $A = \begin{bmatrix} \alpha & 0 \\ 1 & 1 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 \\ 5 & 1 \end{bmatrix}$

$$\Rightarrow \quad A^2 = \begin{bmatrix} \alpha & 0 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} \alpha & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} \alpha^2 & 0 \\ \alpha+1 & 1 \end{bmatrix}$$

Also, given, $A^2 = B$

$$\Rightarrow \quad \begin{bmatrix} \alpha^2 & 0 \\ \alpha+1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 5 & 1 \end{bmatrix}$$

$$\Rightarrow \alpha^2 = 1 \text{ and } \alpha + 1 = 5$$

Which is not possible at the same time.

$\therefore$ No real values of α exists.

Correct Answer: D

1.2 Determinants

11. Let

$$B = \begin{bmatrix} 8 & x & 0 \\ 4 & 0 & 2 \\ 12 & 6 & 0 \end{bmatrix}$$

For what value of x, B will be a singular matrix?

A. 4

B. 6

C. 8

D. 12

Solution: A matrix is singular if its determinant is equal to zero. Therefore, to find the value of x for which matrix B is singular, we need to calculate the determinant of B and set it equal to zero:

$$\det(B) = \begin{vmatrix} 8 & x & 0 \\ 4 & 0 & 2 \\ 12 & 6 & 0 \end{vmatrix}$$

We can calculate the determinant using the cofactor expansion method along the first row:

$$\det(B) = 8 \cdot \begin{vmatrix} 0 & 2 \\ 6 & 0 \end{vmatrix} - x \cdot \begin{vmatrix} 4 & 2 \\ 12 & 0 \end{vmatrix} + 0 \cdot \begin{vmatrix} 4 & 0 \\ 12 & 6 \end{vmatrix}$$

Now, calculate the determinants of the 2×2 matrices: For the first term:

$$\begin{vmatrix} 0 & 2 \\ 6 & 0 \end{vmatrix} = (0 * 0) - (2 * 6) = -12$$

For the second term:

$$\begin{vmatrix} 4 & 2 \\ 12 & 0 \end{vmatrix} = (4 * 0) - (2 * 12) = -24$$

Now, plug these values back into the determinant of B :

$$\det(B) = 8 * (-12) - x * (-24) + 0 = -96 + 24x$$

To find when B is singular, set the determinant equal to zero:

$$-96+24x=0$$

Now, solve for x :

$$24x=96$$

Divide both sides by 24 :

$$x=\frac{96}{24}=4$$

So, for $x=4$, matrix B will be a singular matrix.

Correct Answer: A

12. The sum of the real roots of the equation $\begin{vmatrix} x & -6 & -1 \\ 2 & -3x & x-3 \\ -3 & 2x & x+2 \end{vmatrix}=0$, is equal to ______.

A. 0

B. 6

C. -4

D. 1

Solution: Given equation

$$\begin{vmatrix} x & -6 & -1 \\ 2 & -3x & x-3 \\ -3 & 2x & x+2 \end{vmatrix}=0$$

On the expansion of the determinant along R_1, we get

$$\begin{aligned}
& x[(-3x)(x+2)-2x(x-3)]+6[2(x+2)+3(x-3)] \\
& -1[2(2x)-(-3x)(-3)]=0 \\
\Rightarrow & x\left[-3x^2-6x-2x^2+6x\right]+6[2x+4+3x-9] \\
\Rightarrow & x\left(-5x^2\right)+6(5x-5)-1(-5x)=0 \\
\Rightarrow & -5x^3+30x-30+5x=0 \\
\Rightarrow & 5x^3-35x+30=0 \Rightarrow x^3-7x+6=0.
\end{aligned}$$

Since all roots are real

$$\therefore \text{ Sum of roots } =-\frac{\text{coefficient of } x^2}{\text{coefficient of } x^3}=0$$

Correct Answer: A

Relation between coefficients and roots:

For a cubic equation $ax^3+bx^2+cx+d=0$, let p, q, and r be its roots, then the following holds:

Root expression	Equals to
$p+q+r$	$-\frac{b}{a}$
$pq+qr+rp$	$\frac{c}{a}$
pqr	$-\frac{d}{a}$

13. Let the numbers $2, b, c$ be in an AP and $A=\begin{bmatrix} 1 & 1 & 1 \\ 2 & b & c \\ 4 & b^2 & c^2 \end{bmatrix}$.

If $\det(A) \in [2,16]$, then c lies in the interval _______.

A. $\left[3, 2+2^{3/4}\right]$

B. $\left(2+2^{3/4}\right.$

C. $[4,6]$

D. $[2,3)$

Solution: Given, matrix $A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & b & c \\ 4 & b^2 & c^2 \end{bmatrix}$, so

$$\det(A) = \begin{vmatrix} 1 & 1 & 1 \\ 2 & b & c \\ 4 & b^2 & c^2 \end{vmatrix}$$

On applying, $C_2 \to C_2 - C_1$ and $C_3 \to C_3 - C_1$,

$$\text{we get } \det(A) = \begin{vmatrix} 1 & 0 & 0 \\ 2 & b-2 & c-2 \\ 4 & b^2-4 & c^2-4 \end{vmatrix}$$
$$= \begin{vmatrix} b-2 & c-2 \\ b^2-4 & c^2-4 \end{vmatrix}$$
$$= \begin{vmatrix} b-2 & c-2 \\ (b-2)(b+2) & (c-2)(c+2) \end{vmatrix}$$
$$= (b-2)(c-2)\begin{vmatrix} 1 & 1 \\ b+2 & c+2 \end{vmatrix}$$

[taking common $(b-2)$ from C_1 and

$$= (b-2)(c-2)(c-b)$$

$(c-2)$ from C_2]

Since, $2, b$ and c are in AP, if assume common difference of AP is d, then

$$b = 2 + d \text{ and } c = 2 + 2d$$

So, $\quad |A| = d(2d)d = 2d^3 \in [2,16]$ [given]

$$\Rightarrow d^3 \in [1,8] \Rightarrow d \in [1,2]$$
$$\therefore \quad 2 + 2d \in [2+2, 2+4]$$
$$= [4,6] \Rightarrow c \in [4,6]$$

Correct Answer: C

14. Let $A = \begin{bmatrix} 2 & b & 1 \\ b & b^2+1 & b \\ 1 & b & 2 \end{bmatrix}$, where $b > 0$. Then, the minimum value of $\frac{\det(A)}{b}$ is ______.

A. $-\sqrt{3}$
B. $-2\sqrt{3}$
C. $2\sqrt{3}$
D. $\sqrt{3}$

Solution: Given matrix, $A = \begin{bmatrix} 2 & b & 1 \\ b & b^2+1 & b \\ 1 & b & 2 \end{bmatrix}$, $b > 0$ So, $\det(A) = |A| = \begin{vmatrix} 2 & b & 1 \\ b & b^2+1 & b \\ 1 & b & 2 \end{vmatrix}$

$$= 2\left[2\left(b^2+1\right) - b^2\right] - b(2b - b)$$
$$= 2\left[2b^2 + 2 - b^2\right] - b^2 - 1\left(b^2 - b^2 - 1\right)$$
$$= 2b^2 + 4 - b^2 - 1 = b^2 + 3$$

$$\Rightarrow \quad \frac{\det(A)}{b} = \frac{b^2+3}{b} = b + \frac{3}{b}$$

Now, by $AM \geq GM$, we get

$$\frac{b+\frac{3}{b}}{2} \geq \left(b \times \frac{3}{b}\right)^{1/2} \quad \{\because b > 0\}$$

$$\Rightarrow b + \frac{3}{b} \geq 2\sqrt{3}$$

So, minimum value of $\frac{\det(A)}{b} = 2\sqrt{3}$.

Correct Answer: C

15. If $\alpha, \beta \neq 0$ and $f(n) = \alpha^n + \beta^n$ and $\begin{vmatrix} 3 & 1+f(1) & 1+f(2) \\ 1+f(1) & 1+f(2) & 1+f(3) \\ 1+f(2) & 1+f(3) & 1+f(4) \end{vmatrix} = K(1-\alpha)^2(1-\beta)^2(\alpha-\beta)^2$, then K is equal to ____.

A. $\alpha\beta$

B. $\frac{1}{\alpha\beta}$

C. 1

D. -1

Solution: Use the property that, two determinants can be multiplied column-to-row or row-to-column, to write the given determinant as the product of two determinants and then expand.

Given, $f(n) = \alpha^n + \beta^n, f(1) = \alpha + \beta, f(2) = \alpha^2 + \beta^2,$

$$f(3) = \alpha^3 + \beta^3, f(4) = \alpha^4 + \beta^4$$

$$\Delta = \begin{vmatrix} 3 & 1+f(1) & 1+f(2) \\ 1+f(1) & 1+f(2) & 1+f(3) \\ 1+f(2) & 1+f(3) & 1+f(4) \end{vmatrix}$$

$$\text{Let} \Rightarrow \Delta = \begin{vmatrix} 3 & 1+\alpha+\beta & 1+\alpha^2+\beta^2 \\ 1+\alpha+\beta & 1+\alpha^2+\beta^2 & 1+\alpha^3+\beta^3 \\ 1+\alpha^2+\beta^2 & 1+\alpha^3+\beta^3 & 1+\alpha^4+\beta^4 \end{vmatrix}$$

$$= \begin{vmatrix} 1\cdot 1+1\cdot 1+1\cdot 1 & 1\cdot 1+1\cdot\alpha+1\cdot\beta & 1\cdot 1+1\cdot\alpha^2+1\cdot\beta^2 \\ 1\cdot 1+1\cdot\alpha+1\cdot\beta & 1\cdot 1+\alpha\cdot\alpha+\beta\cdot\beta & 1\cdot 1+\alpha\cdot\alpha^2+\beta\cdot\beta^2 \\ 1\cdot 1+1\cdot\alpha^2+1\cdot\beta^2 & 1\cdot 1+\alpha\cdot\alpha+\beta^2\cdot\beta & 1\cdot 1+\alpha^2\cdot\alpha^2+\beta^2\cdot\beta^2 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 1 & 1 \\ 1 & \alpha & \beta \\ 1 & \alpha^2 & \beta^2 \end{vmatrix} \begin{vmatrix} 1 & 1 & 1 \\ 1 & \alpha & \beta \\ 1 & \alpha^2 & \beta^2 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ 1 & \alpha & \beta \\ 1 & \alpha^2 & \beta^2 \end{vmatrix}^2$$

On expanding, we get $\Delta = (1-\alpha)^2(1-\beta)^2(\alpha-\beta)^2$

But given, $\Delta = K(1-\alpha)^2(1-\beta)^2(\alpha-\beta)^2$

Hence, $K(1-\alpha)^2(1-\beta)^2(\alpha-\beta)^2 = (1-\alpha)^2(1-\beta)^2(\alpha-\beta)^2 \therefore \quad K = 1$ Correct Answer: C

16. Let $P = [a_{ij}]$ be a 3×3 matrix and let $Q = [b_{ij}]$, where $b_{ij} = 2^{i+j}a_{ij}$ for $1 \leq i, j \leq 3$. If the determinant of P is 2, then the determinant of the matrix Q is _____.

A. 2^{10}

B. 2^{11}

C. 2^{12}

D. 2^{13}

Solution: Here,

$$\mathbf{P} = [a_{ij}]_{3\times 3} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$\mathbf{Q} = [b_{ij}]_{3\times 3} = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}$$

where, $b_{ij} = 2^{i+j} a_{ij}$

$$\therefore \quad |Q| = \begin{vmatrix} 4a_{11} & 8a_{12} & 16a_{13} \\ 8a_{21} & 16a_{22} & 32a_{23} \\ 16a_{31} & 32a_{32} & 64a_{33} \end{vmatrix}$$

$$= 4 \times 8 \times 16 \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ 2a_{21} & 2a_{22} & 2a_{23} \\ 4a_{31} & 4a_{32} & 4a_{33} \end{vmatrix}$$

$$= 2^9 \times 2 \times 4 \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$= 2^{12} \cdot |P| = 2^{12} \cdot 2 = 2^{13}$$

Correct Answer: D

17. The determinant $\begin{vmatrix} xp+y & x & y \\ yp+z & y & z \\ 0 & xp+y & yp+z \end{vmatrix} = 0$, if

A. x, y, z are in AP

B. x, y, z are in GP

C. x, y, z are in HP

D. xy, yz, zx are in AP

Solution: Given, $\begin{vmatrix} xp+y & x & y \\ yp+z & y & z \\ 0 & xp+y & yp+z \end{vmatrix} = 0$

Applying $C_1 \to C_1 - (pC_2 + C_3)$

$$\Rightarrow \begin{vmatrix} 0 & x & y \\ 0 & y & z \\ -\left(xp^2 + yp + yp + z\right) & xp+y & yp+z \end{vmatrix} = 0$$

$$\Rightarrow \quad -\left(xp^2 + 2yp + z\right)\left(xz - y^2\right) = 0$$

$$\therefore \quad \text{Either } xp^2 + 2yp + z = 0 \text{ or } y^2 = xz$$

$$\Rightarrow \quad x, y, z \text{ are in GP.}$$

Correct Answer: B

18. Given "Vandermonde matrix":

$$A = \begin{bmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{bmatrix}$$

What is the det(A)? (Mark all the appropriate options)

A. $(b-a)(c-a)(c-b)$

B. $(b-a)(a-c)(c-b)$

C. $(b-a)(a-c)(b-c)$

D. $(b-a)(c-a)(b-c)$

Solution: Using row operations and properties of the determinant, we have:

$$
\begin{aligned}
\det\begin{bmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{bmatrix} &= \det\begin{bmatrix} 1 & a & a^2 \\ 0 & b-a & b^2-a^2 \\ 1 & c & c^2 \end{bmatrix} \\
&= \det\begin{bmatrix} 1 & a & a^2 \\ 0 & b-a & b^2-a^2 \\ 0 & c-a & c^2-a^2 \end{bmatrix} \\
&= (b-a)\det\begin{bmatrix} 1 & a & a^2 \\ 0 & 1 & b+a \\ 1 & c-a & c^2-a^2 \end{bmatrix} \\
&= (b-a)\det\begin{bmatrix} 1 & a & a^2 \\ 0 & 1 & b+a \\ 0 & 0 & (c-a)(c-b) \end{bmatrix} \\
&= (b-a)(c-a)(c-b)\det\begin{bmatrix} 1 & a & a^2 \\ 0 & 1 & b+a \\ 0 & 0 & 1 \end{bmatrix} \\
&= (b-a)(c-a)(c-b)\det\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\
&= (b-a)(c-a)(c-b) \\
&= (b-a)(a-c)(b-c)
\end{aligned}
$$

Correct Answer: A;C

19. Suppose a 4 by 4 matrix has the same entry $\times$ throughout its first row and column. The other 9 numbers could be anything like $1, 5, 7, 2, 3, 99, \pi, e, 4$.

$$
A = \begin{bmatrix} \times & \times & \times & \times \\ \times & \text{any numbers} & & \\ \times & \text{any numbers} & & \\ \times & \text{any numbers} & & \end{bmatrix}
$$

The determinant of A is a polynomial in $\times$. What is the largest possible degree of that polynomial?

A. 2

B. 3

C. 1

D. 4

Solution: Every term in the big formula for $\det(A)$ takes one entry from each row and column, so we can choose at most two $\times$'s and the determinant has degree 2.

Correct Answer: A

20. Suppose a 4 by 4 matrix has the same entry $\times$ throughout its first row and column. The other 9 numbers could be anything like $1, 5, 7, 2, 3, 99, \pi, e, 4$.

$$A = \begin{bmatrix} \times & \times & \times & \times \\ \times & \text{any numbers} & & \\ \times & \text{any numbers} & & \\ \times & \text{any numbers} & & \end{bmatrix}$$

If those 9 numbers give the identity matrix I, then which values of $\times$ give $\det A = 0$? (Mark all the appropriate options)

$$A = \begin{bmatrix} \times & \times & \times & \times \\ \times & 1 & 0 & 0 \\ \times & 0 & 1 & 0 \\ \times & 0 & 0 & 1 \end{bmatrix}$$

A. $\times = 0$ and $\times = \frac{1}{3}$

B. $\times = -1$ and $\times = \frac{1}{3}$

C. $\times = 0$ or $\times = \frac{1}{3}$

D. $\times = 0$ or $\times = \frac{-1}{3}$

Solution:

You can find this by cofactor expansion; here's another way:

$$\det(A) = \times \det \begin{bmatrix} 1 & \times & \times & \times \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix} = \times \det \begin{bmatrix} 1-3\times & \times & \times & \times \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \times(1-3\times)\det \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$
$$= \times(1-3\times).$$

This is zero when $x = 0$ or $x = \frac{1}{3}$.

Correct Answer: C

1.3 Adjoint and Inverse of a Matrix

21. If $B = \begin{bmatrix} 5 & 2\alpha & 1 \\ 0 & 2 & 1 \\ \alpha & 3 & -1 \end{bmatrix}$ is the inverse of a 3×3 matrix A, then the sum of all values of α for which $\det(A) + 1 = 0$, is ______.

A. 0

B. -1

C. 1

D. 2

Solution: Given matrix B is the inverse matrix of 3×3 matrix A, where $B = \begin{bmatrix} 5 & 2\alpha & 1 \\ 0 & 2 & 1 \\ \alpha & 3 & -1 \end{bmatrix}$

We know that,

$$\det(A) = \frac{1}{\det(B)} \quad \left[\because \det\left(A^{-1}\right) = \frac{1}{\det(A)}\right]$$

Since, $\det(A) + 1 = 0$ (given)

$\Rightarrow \dfrac{1}{\det(B)} + 1 = 0$
$\Rightarrow \det(B) = -1$
$\Rightarrow 5(-2-3) - 2\alpha(0-\alpha) + 1(0-2\alpha) = -1$
$\Rightarrow \alpha^2 - 2\alpha - 24 = 0$
$\Rightarrow (\alpha - 4)(\alpha + 3) = 0$
$\Rightarrow \alpha = -3, 4$

So, the required sum of all values of α is $4-3=1$.

Correct Answer: C

22. Let A and B be two invertible matrices of order 3×3. If $\det\left(ABA^T\right)=8$ and $\det\left(AB^{-1}\right)=8$, then $\det\left(BA^{-1}B^T\right)$ is equal to ______.

A. 1

B. $\frac{1}{4}$

C. $\frac{1}{16}$

D. 16

Solution: Given, $\left|ABA^T\right|=8$

$$\Rightarrow \quad |A||B|\left|A^T\right|=8 \quad [\because |XY|=|X||Y|]$$
$$\therefore \quad |A|^2|B|=8 \quad ...(i) \left[\because \left|A^T\right|=|A|\right]$$

Also, we have $\left|AB^{-1}\right|=8 \Rightarrow |A|\left|B^{-1}\right|=8$

$$\Rightarrow \quad \frac{|A|}{|B|}=8 \quad \ldots \text{(ii)} \left[\because \left|A^{-1}\right|=|A|^{-1}=\frac{1}{|A|}\right]$$

On multiplying Eqs. (i) and (ii), we get

$$\begin{aligned} & |A|^3=8\cdot 8=4^3 \\ \Rightarrow \quad & |A|=4 \\ \Rightarrow \quad & |B|=\tfrac{|A|}{8}=\tfrac{4}{8}=\tfrac{1}{2} \end{aligned}$$

Now, $\left|BA^{-1}B^T\right|=|B|\frac{1}{|A|}|B|=\left(\frac{1}{2}\right)\frac{1}{4}\left(\frac{1}{2}\right)=\frac{1}{16}$

Correct Answer: C

23. If $A=\begin{bmatrix} 2 & -3 \\ -4 & 1 \end{bmatrix}$, then $\operatorname{adj}\left(3A^2+12A\right)$ is equal to _______.

A. $\begin{bmatrix} 72 & -84 \\ -63 & 51 \end{bmatrix}$

B. $\begin{bmatrix} 51 & 84 \\ 63 & 72 \end{bmatrix}$

C. $\begin{bmatrix} 51 & 63 \\ 84 & 72 \end{bmatrix}$

D. $\begin{bmatrix} 72 & -63 \\ -84 & 51 \end{bmatrix}$

Solution: We have,

$$A=\begin{bmatrix} 2 & -3 \\ -4 & 1 \end{bmatrix}$$

$$\begin{aligned} A^2=A\cdot A &= \begin{bmatrix} 2 & -3 \\ -4 & 1 \end{bmatrix}\begin{bmatrix} 2 & -3 \\ -4 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 4+12 & -6-3 \\ -8-4 & 12+1 \end{bmatrix} \\ &= \begin{bmatrix} 16 & -9 \\ -12 & 13 \end{bmatrix} \end{aligned}$$

$$\text{Now,} \quad 3A^2 + 12A = 3\begin{bmatrix} 16 & -9 \\ -12 & 13 \end{bmatrix} + 12\begin{bmatrix} 2 & -3 \\ -4 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} 48 & -27 \\ -36 & 39 \end{bmatrix} + \begin{bmatrix} 24 & -36 \\ -48 & 12 \end{bmatrix}$$
$$= \begin{bmatrix} 72 & -63 \\ -84 & 51 \end{bmatrix}$$
$$\therefore \quad \operatorname{adj}\left(3A^2 + 12A\right) = \begin{bmatrix} 51 & 63 \\ 84 & 72 \end{bmatrix}$$

Correct Answer: C

24. Let M be a 2×2 symmetric matrix with integer entries. Then, M is invertible, if ______. (Mark all the appropriate choices)

A. the first column of M is the transpose of the second row of M

B. the second row of M is the transpose of the first column of M

C. M is a diagonal matrix with non-zero entries in the main diagonal

D. the product of entries in the main diagonal of M is not the square of an integer

Solution: A square matrix M is invertible, iff dem (M) or $|M| \neq 0$. Let

$$M = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$$

A. Given, $\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} b \\ c \end{bmatrix} \Rightarrow a = b = c = \alpha$ [let] $\Rightarrow M = \begin{bmatrix} \alpha & \alpha \\ \alpha & \alpha \end{bmatrix} \Rightarrow |M| = 0 \Rightarrow M$ is non-invertible.

B. Given, $[bc] = [ab] \Rightarrow \quad a = b = c = \alpha$ [let] Again, $|M| = 0 \Rightarrow M$ is non-invertible.

C. As given $M = \begin{bmatrix} a & 0 \\ 0 & c \end{bmatrix} \Rightarrow |M| = ac \neq 0$ [$\because$ a and c are non-zero] $\Rightarrow M$ is invertible.

D. $M = \begin{bmatrix} a & b \\ b & c \end{bmatrix} \Rightarrow |M| = ac - b^2 \neq 0 \because \quad ac$ is not equal to the square of an integer. M is invertible.

Correct Answer: C;D

25. If the adjoint of a 3×3 matrix P is $\begin{bmatrix} 1 & 4 & 4 \\ 2 & 1 & 7 \\ 1 & 1 & 3 \end{bmatrix}$, then the possible value(s) of the determinant of P is/are ______. (Mark all the appropriate choices)

A. -2

B. -1

C. 1

D. 2

Solution: If $|A_{n\times n}| = \Delta$, then $|\operatorname{adj} A| = \Delta^{A-1}$

Here, adj $P_{3\times 3} = \begin{bmatrix} 1 & 4 & 4 \\ 2 & 1 & 7 \\ 1 & 1 & 3 \end{bmatrix}$

$$\Rightarrow |\operatorname{adj} P| = |P|^2$$
$$\therefore \operatorname{adj} P| = \begin{vmatrix} 1 & 4 & 4 \\ 2 & 1 & 7 \\ 1 & 1 & 3 \end{vmatrix} = 1(3-7) - 4(6-7) + 4(2-1)$$
$$= -4 + 4 + 4 = 4 \Rightarrow |P| = \pm 2$$

Correct Answer: A;D

1.4 Rank of Matrix

26. **(Numerical Answer Type)** In the given matrix A, if the number of minors of order 3 is α, the number of minors of order 2 is β, and the number of minors of order 1 is γ, then the value of $\alpha+\beta+\gamma$ is ______.

$$A = \begin{pmatrix} 1 & 0 & 2 & 1 \\ 0 & 2 & 4 & 2 \\ 0 & 2 & 2 & 1 \end{pmatrix}$$

Solution: We obtain the determinants of order 3 by keeping all the rows and deleting one column from A. So there are four different minors of order 3. We compute one of them to illustrate:

$$\begin{vmatrix} 1 & 0 & 2 \\ 0 & 2 & 4 \\ 0 & 2 & 2 \end{vmatrix} = 1 \cdot (-4) + 2 \cdot 0 = -4$$

The minors of order 3 are called the maximal minors of A, since there are no 4×4 sub-matrices of A. There are $3 \cdot 6 = 18$ minors of order 2 and $3 \cdot 4 = 12$ minors of order 1.

$\therefore$ The value of $\alpha+\beta+\gamma = 4 + 18 + 12 = 34$.

Correct Answer: 34

27. **(Numerical Answer Type)** If the rank of the matrix

$$A = \begin{pmatrix} 1 & 0 & 2 & 1 \\ 0 & 2 & 4 & 2 \\ 0 & 2 & 2 & 1 \end{pmatrix}$$

is α, then the value of $\alpha!$ is ______.

Solution:

We use elementary row operations:

$$A = \begin{pmatrix} 1 & 0 & 2 & 1 \\ 0 & 2 & 4 & 2 \\ 0 & 2 & 2 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 2 & 1 \\ 0 & 2 & 4 & 2 \\ 0 & 0 & -2 & -1 \end{pmatrix}$$

Since the echelon form has pivots in the first three columns, A has rank $\mathrm{rk}(A) = 3$. The first three columns of A are linearly independent.

(Or)

The maximal minors have order 3, and we found that the one obtained by deleting the last column is $-4 \neq 0$. Hence $\mathrm{rk}(A) = 3$.

$\therefore$ The value of $\alpha! = 3! = 3 \times 2 \times 1 = 6$.

Correct Answer: 6

28. **(Numerical Answer Type)** If the rank of the following matrix:

$$\begin{bmatrix} 0 & 16 & 8 & 4 \\ 2 & 4 & 8 & 16 \\ 16 & 8 & 4 & 2 \\ 4 & 8 & 16 & 2 \end{bmatrix}$$

is β, then the value of $\beta^{0!+1!+2!}$ is ______.

Solution: To compute the rank of a matrix, remember two key points: (i) the rank does not change under elementary row operations; (ii) the rank of a row-echelon matrix is easy to acquire. Motivated by this, we convert the given matrix into row echelon form using elementary row operations:

$$\begin{bmatrix} 0 & 16 & 8 & 4 \\ 2 & 4 & 8 & 16 \\ 16 & 8 & 4 & 2 \\ 4 & 8 & 16 & 2 \end{bmatrix} \Rightarrow \begin{bmatrix} 2 & 4 & 8 & 16 \\ 16 & 8 & 4 & 2 \\ 4 & 8 & 16 & 2 \\ 0 & 16 & 8 & 4 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 2 & 4 & 8 \\ 0 & -24 & -60 & -126 \\ 0 & 0 & 0 & -30 \\ 0 & 4 & 2 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 2 & 4 & 8 \\ 0 & 4 & 2 & 1 \\ 0 & -24 & -60 & -126 \\ 0 & 0 & 0 & -30 \\ 1 & 2 & 4 & 8 \\ 0 & 4 & 2 & 1 \\ 0 & 0 & -48 & -120 \\ 0 & 0 & 0 & -30 \end{bmatrix}$$

As this matrix has 4 non-zero rows, we conclude that the original matrix has rank 4.

$\therefore$ The value of $\beta^{0!+1!+2!} = 4^{1+1+2} = 4^4 = 256$.

Correct Answer: 256

29. **(Numerical Answer Type)** If the rank of the following matrix:

$$\begin{bmatrix} 4 & -6 & 0 \\ -6 & 0 & 1 \\ 0 & 9 & -1 \\ 0 & 1 & 4 \end{bmatrix}$$

is γ, then the value of $\underbrace{\gamma! + (\gamma+1)! + \ldots}_{\gamma \text{ times}}$ is ________.

Solution:

$$\begin{bmatrix} 4 & -6 & 0 \\ -6 & 0 & 1 \\ 0 & 9 & -1 \\ 0 & 1 & 4 \end{bmatrix} \Rightarrow \begin{bmatrix} 2 & -3 & 0 \\ -6 & 0 & 1 \\ 0 & 9 & -1 \\ 0 & 1 & 4 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 2 & -3 & 0 \\ 0 & -9 & 1 \\ 0 & 9 & -1 \\ 0 & 1 & 4 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 2 & -3 & 0 \\ 0 & -9 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 37/9 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 2 & -3 & 0 \\ 0 & -9 & 1 \\ 0 & 0 & 37/9 \\ 0 & 0 & 0 \end{bmatrix}$$

Hence, the rank of the original matrix is 3.

$\therefore$ The value of $\underbrace{\gamma! + (\gamma+1)! + \ldots}_{\gamma \text{ times}} = \underbrace{3! + 4! + 5!}_{3 \text{ times}} = 6 + 24 + 120 = 150$.

Correct Answer: 150

30. The value of q gives A a different rank compared to all other values of q.

$$A = \begin{bmatrix} 1 & 2 & 3 & 1 \\ 2 & -1 & 3 & 4 \\ 4 & 3 & 9 & q \end{bmatrix}$$

Which of the following is/are correct? (Mark all the appropriate choices)

A. If $q = 6$, then rank(A) = 2

B. If $q \neq 6$, then rank(A) = 3

C. If $q = 6$, then rank(A) = 3

D. If $q \neq 6$, then rank(A) = 2

Solution: Start the elimination: replace row $2(r_2)$ with $r_1 - 2r_2$, then replace r_3 with $r_3 - 4r_1$ to get:

$$\begin{bmatrix} 1 & 2 & 3 & 1 \\ 0 & -5 & -3 & 2 \\ 0 & -5 & -3 & q-4 \end{bmatrix}$$

Continue by replacing the third row with the difference of the third minus the second row to get the triangular matrix U :

$$U = \begin{bmatrix} 1 & 2 & 3 & 1 \\ 0 & -5 & -3 & 2 \\ 0 & 0 & 0 & q-6 \end{bmatrix}$$

When $q = 6, U$ has two pivots, and the rank of A is 2. Otherwise, the rank of A is 3.

Correct Answer: A; B

31. Which of the following is correct? (Mark all the appropriate choices)

A. Elementary row and column operations do not change the rank of a matrix.

B. The rank of a matrix A, denoted as "rank(A)," is always a non-negative integer. In other words, rank(A) ≥ 0 for any matrix A.

C. The rank of a matrix A is equal to the rank of its transpose, i.e., $\operatorname{rank}(A) = \operatorname{rank}\left(A^T\right)$.

D. For two matrices $A(m \times n)$ and $B(n \times p)$, the rank of their product AB is at most the minimum of the ranks of A and B, i.e., $\operatorname{rank}(AB) \leq \min(\operatorname{rank}(A), \operatorname{rank}(B))$.

Solution:

A. **Proof:** Elementary row (resp. column) operations can be simulated as left (resp. right) multiplication by the elementary matrices. Since the elementary matrices are invertible, such multiplication does not change the rank of a matrix.

B. **Proof:** The rank of a matrix represents the maximum number of linearly independent rows or columns in the matrix. Since linearly independent vectors cannot be scaled to become the zero vector, the rank cannot be negative.

C. **Proof:** Let A be an $m \times n$ matrix, and let $r1, r2, \ldots, rm$ be the rows of A. The rank of A is the maximum number of linearly independent rows. Now, consider the transpose of A, denoted as A^T. The rows of A^T are the columns of A, denoted as $c1, c2, \ldots, cn$. The rank of $A^\wedge T$ is the maximum number of linearly independent columns. Since the number of linearly independent rows of A is the same as the number of linearly independent columns of A^T, $\operatorname{rank}(A) = \operatorname{rank}\left(A^T\right)$

D. **Proof:** Suppose the rank of A is r, and the rank of B is s. This means that A has r linearly independent rows and B has s linearly independent columns. When you multiply A and B to obtain AB, the resulting matrix will have at most r linearly independent rows (since A has r rows) and at most s linearly independent columns (since B has s columns). Therefore, the rank of AB is at most $\min(r, s)$, which is equivalent to $\min(\operatorname{rank}(A), \operatorname{rank}(B))$.

Correct Answer: A:B;C;D

32. **(Numerical Answer Type)** Consider the 10×10 matrix M for which the entry in the i^{th} row and j^{th} column is $i + j$.

If the rank of the matrix M is ρ, then $2024 \times \rho!$ is equal to ________.

Solution:

The first two (independent) columns C_1 and C_2 span the column space since $C_j = (2-j)C_1 + (j-1)C_2$ for $j > 2$. Thus the rank (which is the dimension of the column space) is 2.

$\therefore$ The value of $2024 \times \rho! = 2024 \times 2! = 4048$.

33. **(Numerical Answer Type)** Let $v_1, \ldots, v_n \in \mathbb{R}^m$ be a collection of vectors. The Gram matrix of this collection is defined to be the n-by- n matrix whose entry in the i^{th} row and j^{th} column is $a_{ij} = v_i \cdot v_j$, where $\cdot$ denotes the dot product.

Consider the **Gram matrix G** of the collection:

$$\begin{aligned} v_1 &= (1, 2, 1) \\ v_2 &= (-3, 5, 1) \\ v_3 &= (0, -3, 6) \\ v_4 &= (4, -2, 0). \end{aligned}$$

If the rank of G is κ, then $\kappa^{(\kappa!)}$ is equal to ________.

Solution: Let A be the 3×4 matrix with $v_1^T, .., v_4^T$ as its columns. We see by inspection that v_1, v_2, v_3 are linearly independent, so that $\text{rank}(A) = 3$. Now the Gram Matrix is $A^T A$ and $\text{rank}\left(A^T A\right) = \text{rank}(A)$.

$\therefore$ The value of $\kappa^{(\kappa!)} = 3^{(3!)} = 3^6 = 729$.

Correct Answer: 729

1.5 Eigenvalues and Eigenvectors

34. **(Numerical Answer Type)** Given the matrix $A = \begin{bmatrix} 2 & 1 & 2 \\ 1 & 3 & 1 \\ 2 & 1 & -6 \end{bmatrix}$ If the sum of the eigenvalues is α and product of the eigen value is β, then the value of $\alpha - \beta =$ ________.

Solution:

The sum of the Eigenvalues = sum of the main diagonal elements = trace of the matrix = $2 + 3 - 6 \Rightarrow \alpha = -1$

$$\begin{aligned} \text{Product of the Eigen value } &= |\mathrm{A}| \\ &= 2\begin{vmatrix} 3 & 1 \\ 1 & -6 \end{vmatrix} - 1\begin{vmatrix} 1 & 1 \\ 2 & -6 \end{vmatrix} + 2\begin{vmatrix} 1 & 3 \\ 2 & 1 \end{vmatrix} \\ &= 2(-18-1) - 1(-6-2) + 2(1-6) \Rightarrow \beta = -40 \end{aligned}$$

$\therefore$ The value of $\alpha - \beta = -1 + 40 = 39$.

Correct Answer: 39

35. The eigenvalues of

$$\begin{bmatrix} 5 & 6 & 17 \\ 0 & -19 & 23 \\ 0 & 0 & 37 \end{bmatrix}$$

are

A. $-19, 5, 37$

B. $19, -5, -37$

C. $2, -3, 7$

D. $3, -5, 37$

Solution: The eigenvalues of an upper triangular matrix are simply the diagonal entries of the matrix. Hence 5,-19, and 37 are the eigenvalues of the matrix.

Alternately, look at $\det([A]-\lambda[I])=0$

$$\det\left(\begin{bmatrix} 5 & 6 & 17 \\ 0 & -19 & 23 \\ 0 & 0 & 37 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) = 0$$

$$\det\left(\begin{bmatrix} 5-\lambda & 6 & 17 \\ 0 & -19-\lambda & 23 \\ 0 & 0 & 37-\lambda \end{bmatrix}\right) = 0$$

$$(5-\lambda)(-19-\lambda)(37-\lambda) = 0$$

Then

$$\lambda = 5, -19, 37$$

are the roots of the equation; and hence, the eigenvalues of $[A]$.

Correct Answer: A

36. The eigenvalues of the following matrix

$$\begin{bmatrix} 3 & 2 & 9 \\ 7 & 5 & 13 \\ 6 & 17 & 19 \end{bmatrix}$$

are given by solving the cubic equation?

A. $\lambda^3 - 27\lambda^2 + 167\lambda - 285 = 0$

B. $\lambda^3 - 27\lambda^2 - 122\lambda - 313 = 0$

C. $\lambda^3 + 27\lambda^2 + 167\lambda + 285 = 0$

D. $\lambda^3 + 23.23\lambda^2 - 158.3\lambda + 313 = 0$

Solution: To find the equations of

$$[A] = \begin{bmatrix} 3 & 2 & 9 \\ 7 & 5 & 13 \\ 6 & 17 & 19 \end{bmatrix}$$

we solve $\det([A]-\lambda[I])=0$

$$\det\left(\begin{bmatrix} 3 & 2 & 9 \\ 7 & 5 & 13 \\ 6 & 17 & 19 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) = 0$$

$$\det\left(\begin{bmatrix} 3-\lambda & 2 & 9 \\ 7 & 5-\lambda & 13 \\ 6 & 17 & 19-\lambda \end{bmatrix}\right) = 0$$

Using the cofactor method with Row1

$$(3-\lambda)\begin{vmatrix} 5-\lambda & 13 \\ 17 & 19-\lambda \end{vmatrix} - 2\begin{vmatrix} 7 & 13 \\ 6 & 19-\lambda \end{vmatrix} + 9\begin{vmatrix} 7 & 5-\lambda \\ 6 & 17 \end{vmatrix} = 0$$

$$(3-\lambda)((5-\lambda)(19-\lambda) - 13\times 17) - 2(7(19-\lambda) - 13\times 6) + 9(7\times 17 - 6(5-\lambda)) = 0$$

$$\lambda^3 - 27\lambda^2 - 122\lambda - 313 = 0$$

Correct Answer: B

37. The eigenvalues of a 4×4 matrix $[A]$ are given as $2, -3, 13$, and 7. The $|\det(A)|$ then is ______.

A. 546

B. 19

C. 25

D. cannot be determined

Solution: If $\lambda_1, \lambda_2, \ldots \lambda_{n-1}, \lambda_n$ are the eigenvalues of a $n \times n$ matrix $[A]$, then

$$
\begin{aligned}
|\det(A)| &= |\lambda_1 \times \lambda_2 \times \lambda_3 \times \ldots \times \lambda_n| \\
&|\lambda_1 \times \lambda_2 \times \lambda_3 \times \lambda_4| \\
&= |2 \times (-3) \times 13 \times 7| \\
&= 546
\end{aligned}
$$

Correct Answer: A

38. If one of the eigenvalues of $[A]_{n \times n}$ is zero, it implies

A. The solution to $[A][X] = [C]$ system of equations is unique

B. The determinant of $[A]$ is zero

C. The solution to $[A][X] = [0]$ system of equations is trivial

D. The determinant of $[A]$ is nonzero

Solution: For a $n \times n$ matrix $[A]$ with $\lambda_1, \lambda_2, \ldots \lambda_{n-1}, \lambda_n$ as the eigenvectors

$$|\det(A)| = |\lambda_1 \times \lambda_2 \times \lambda_3 \times \ldots \times \lambda_n|$$

Since one of the eigenvalues is zero,

$$
\begin{aligned}
|\det(A)| &= 0 \\
\det(A) &= 0
\end{aligned}
$$

Correct Answer: B

39. If $\begin{bmatrix} -4.5 \\ -4 \\ 1 \end{bmatrix}$ is an eigenvector of $\begin{bmatrix} 8 & -4 & 2 \\ 4 & 0 & 2 \\ 0 & -2 & -4 \end{bmatrix}$, the eigenvalue corresponding to the eigenvector is ______.

A. 1

B. 4

C. -4.5

D. 6

Solution: If $[A]$ is a $n \times n$ matrix and λ is one of the eigenvalues and $[X]$ is a $n \times 1$ corresponding eigenvector, then

$$
\begin{aligned}
&[A][X] = \lambda[X] \\
&\begin{bmatrix} 8 & -4 & 2 \\ 4 & 0 & 2 \\ 0 & -2 & -4 \end{bmatrix} \begin{bmatrix} -4.5 \\ -4 \\ 1 \end{bmatrix} = \lambda \begin{bmatrix} -4.5 \\ -4 \\ 1 \end{bmatrix} \\
&\begin{bmatrix} -18 \\ -16 \\ 4 \end{bmatrix} = \lambda \begin{bmatrix} -4.5 \\ -4 \\ 1 \end{bmatrix} \\
&4 \begin{bmatrix} -4.5 \\ -4 \\ 1 \end{bmatrix} = \lambda \begin{bmatrix} -4.5 \\ -4 \\ 1 \end{bmatrix} \\
&\lambda = 4
\end{aligned}
$$

Correct Answer: B

40. Given that matrix $[A] = \begin{bmatrix} 8 & -4 & 2 \\ 4 & 0 & 2 \\ 0 & -2 & -3 \end{bmatrix}$ has an eigenvalue value of 4 with the corresponding eigenvectors of $[x] = \begin{bmatrix} -4.5 \\ -4 \\ 1 \end{bmatrix}$, then $[A]^5[X]$ is ____.

A. $\begin{bmatrix} -18 \\ -16 \\ 4 \end{bmatrix}$ B. $\begin{bmatrix} -4.5 \\ -4 \\ 1 \end{bmatrix}$

C. $\begin{bmatrix} -4608 \\ -4096 \\ 1024 \end{bmatrix}$ D. $\begin{bmatrix} -0.004395 \\ -0.003906 \\ 0.0009766 \end{bmatrix}$

Solution: If for a $n \times n$ matrix $[A], \lambda$ is an eigenvalue and $[X]$ is the corresponding eigenvector, then

$$[A]^m[x] = \lambda^m[X]$$
$$[A]^5[X] = \lambda^5[X]$$
$$= 4^5 \begin{bmatrix} -4.5 \\ -4 \\ 1 \end{bmatrix}$$
$$= \begin{bmatrix} -4608 \\ -4096 \\ 1024 \end{bmatrix}$$

Correct Answer: C

Important Points

For a $n \times n$ matrix [A], if λ is an eigenvalue and $[X]$ is an eigenvector prove for

$$[A]^m[x] = \lambda^m[X], m = 1, 2, 3 \ldots$$

For a $n \times n$ matrix $[A]$, if λ is an eigenvalue and $[X]$ is an eigenvector then

$$[A][X] = \lambda[X]$$
$$[A]^2[X] = [A][A][X]$$
$$= \lambda[A][X]$$
$$= \lambda \times \lambda[X]$$
$$= \lambda^2[X]$$

If

$$[A]^{m-1}[X] = \lambda^{m-1}[X], m \geq 0, m = \text{ integer}$$

Then

$$[A]^m[X] = [A][A]^{m-1}[X]$$
$$= \lambda^{m-1}[A][X]$$
$$= [A]\lambda^{m-1}[X]$$
$$= \lambda^{m-1} \times \lambda[X]$$
$$= \lambda^m[X]$$

41. Suppose the matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ has an eigenvalue 1 with associated eigenvector $x = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$. What is A^{50}?

A. $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ B. $\begin{bmatrix} a^{50} & b^{50} \\ c^{50} & d^{50} \end{bmatrix}$

C. $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$ D. $\begin{bmatrix} 2^{50} \\ 3^{50} \end{bmatrix}$

Solution: The eigenvector-eigenvalue equation is $Ax = 1x$, so $A^2x = AAx = Ax = 1x = x$. Continuing in this manner we find $A^{50}x = x$.

Correct Answer: C

42. Let A and B be the 2×2 matrices

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}$$

Which of the following statements is most accurate?

A. Neither A nor B is diagonalizable.

B. Both A and B are diagonalizable.

C. A is diagonalizable but B is not.

D. B is diagonalizable but A is not.

Solution: A is a 90-degree rotation in the plane and has characteristic polynomial λ^2+1, so there are no eigenvalues or eigenvectors. Therefore A is not diagonalizable. B is diagonalizable, for any number of reasons, one of which is that it is symmetric and all symmetric matrices are diagonalizable.

Correct Answer: D

43. **(Numerical Answer Type)** Consider the 3×3 matrix

$$A = \begin{bmatrix} a & b & c \\ 1 & d & e \\ 0 & 1 & f \end{bmatrix}$$

If

- the top left 1×1 block is a matrix with eigenvalue 2 ;
- the top left 2×2 block is a matrix with eigenvalue 3 and -3;
- the top left 3×3 block is a matrix with eigenvalue 0,1 and -2.

Then, the sum of the entries a, b, c, d, e, f is ______.

Solution: Let A_i denote the top left $i\times i$ block of A. The matrix A_1 is the matrix $[a]$. Since a is the only eigenvalue of this matrix, we conclude that $a = 2$. We now move onto determining the entries of the matrix A_2 : $A_2 = \begin{bmatrix} 2 & b \\ 1 & d \end{bmatrix}$. Since the sum of the eigenvalues of A_2 is 0 by hypothesis, and it is also equal to the trace of A_2, we obtain that $2+d = 0$ or $d = -2$. Moreover, the product of the eigenvalues of A_2 is -9 by hypothesis, and it is equal to the determinant of A_2. Thus we have

$$-9 = 2d - b = -4 - b$$

and we deduce that $b = 5$ and therefore $A_2 = \begin{bmatrix} 2 & 5 \\ 1 & -2 \end{bmatrix}$. Finally, consider $A = A_3$. Again, the sum of the eigenvalues of A is -1 and it is also equal to the trace of A. We deduce that $f = -1$. We still need to determine the entries c and e of A and we have

$$A = \begin{bmatrix} 2 & 5 & c \\ 1 & -2 & e \\ 0 & 1 & -1 \end{bmatrix}.$$

The characteristic polynomial of this matrix is

$$-\lambda^3 - \lambda^2 + (e+9)\lambda + c - 2e + 9.$$

We know that the roots of this polynomial must be 0,1 and -2. Setting $\lambda = 0$ and $\lambda = 1$, we obtain

$$c - 2e + 9 = 0$$
$$-1 - 1 + (e+9) + c - 2e + 9 = 0$$

which is equivalent to

$$c - 2e = -9$$
$$c - e = -16.$$

Thus $c = -7$ and $e = 9$ and we conclude

$$A = \begin{bmatrix} 2 & 5 & -7 \\ 1 & -2 & 9 \\ 0 & 1 & -1 \end{bmatrix}$$

$\therefore a + b + c + d + e = 2 + 5 - 7 - 2 + 9 - 1 = 6.$

Correct Answer: 6

1.6 System of Linear Equations

44. $[A]_{m\times n}x = [b]_{n\times 1}$ be a system of equations then which of the following is true? (Mark all the appropriate choices)

A. If $m = n$ then the given system of equations has always a unique solution

B. If $m < n$ then the given system of equation has no solution

C. If $m < n$ then the given system of equation has infinite or no solution

D. If $m \le n$ then the given system of equation has no solution

Solution: We know that, if the number of the equation is less than the number of unknowns then the system of linear equations has either no or infinite solutions. As, rank$[A : B]$ < number of unknowns and if rank$[A : B]$ = rank$[A]$ $\Rightarrow$ infinite solution. rank$[A : B] \neq$ rank$[A]$ $\Rightarrow$ no solution.

Correct Answer: C

45. Consider a linear system of equations $A\vec{x} = \vec{b}$ where A is a 3×3 matrix and $\vec{b} \neq 0$. Suppose the rank of the matrix of coefficients $A = \left(a_{ij}\right)$ is equal to 2 then ______. (Mark all the appropriate choices)

A. there definitely exists a solution to the system of equations

B. there exists a non-zero column vector $\vec{v}$ in $\mathbb{R}^3$ such that $A\vec{v} = \overrightarrow{0}$

C. if there exists a solution to the system of equations $A\vec{x} = \vec{b}$ then at least one equation is a linear combination of the other two equations

D. $\det(A) = 0$

Solution: A. Consider a system $A\vec{x} = \vec{b}$, where $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \vec{b} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ Clearly $\rho(A) \neq \rho([A : b])$

$\Rightarrow$ The system has no solution

$\Rightarrow$ option A is incorrect

B. If $\rho(A) < 3$ then there always exists a non-zero vector v such that $Av = 0$.

$\Rightarrow$ Option B is correct

C. Option C is correct [By the property of a system of linear equations]

D. If $\rho(A) < 3$

$\Rightarrow \det(A) = 0$

Correct Answer: B;C;D

46. The system of linear equations

$$x+\lambda y-z=0; \lambda x-y-z=0; x+y-\lambda z=0$$

has a non-trivial solution for

A. infinitely many values of λ

B. exactly one value of λ

C. exactly two values of λ

D. exactly three values of λ

Solution: Given that the system of linear equations is

$$x+\lambda y-z=0; \lambda x-y-z=0; x+y-\lambda z=0$$

Note that, the given system will have a non-trivial solution only if the determinant of the coefficient matrix is zero, i.e.

$$\begin{vmatrix} 1 & \lambda & -1 \\ \lambda & -1 & -1 \\ 1 & 1 & -\lambda \end{vmatrix} = 0$$

$$\begin{aligned} \Rightarrow \quad & 1(\lambda+1)-\lambda\left(-\lambda^2+1\right)-1(\lambda+1)=0 \\ \Rightarrow \quad & \lambda+1+\lambda^3-\lambda-\lambda-1=0 \\ \Rightarrow \quad & \lambda^3-\lambda=0 \Rightarrow \lambda\left(\lambda^2-1\right)=0 \\ \Rightarrow \quad & \lambda=0 \text{ or } \lambda=\pm 1 \end{aligned}$$

Hence, the given system of linear equations has a non-trivial solution for exactly three values of λ.

Correct Answer: D

47. Let $\alpha_1, \alpha_2, \beta_1, \beta_2$ be the roots of $ax^2+bx+c=0$ and $px^2+qx+r=0$, respectively. If the system of equations $\alpha_1 y+\alpha_2 z=0$ and $\beta_1 y+\beta_2 z=0$ has a non-trivial solution, then the value of $\frac{b^2}{q^2}=?$

A. $\frac{ar}{pc}$

B. $\frac{ap}{rc}$

C. $\frac{ac}{pr}$

D. $\frac{pr}{ac}$

Solution: Since, α_1, α_2 are the roots of $ax^2+bx+c=0$.

$\Rightarrow \quad \alpha_1+\alpha_2=-\frac{b}{a}$ and $\alpha_1\alpha_2=\frac{c}{a} \quad \longrightarrow (1)$

Also, β_1, β_2 are the roots of $px^2+qx+r=0$.

$\Rightarrow \quad \beta_1+\beta_2=-\frac{q}{p}$ and $\beta_1\beta_2=\frac{r}{p} \quad \longrightarrow (2)$

Given the system of equations

$$\alpha_1 y+\alpha_2 z=0$$

and $\beta_1 y+\beta_2 z=0$, has non-trivial solution.

$$\therefore \begin{vmatrix} \alpha_1 & \alpha_2 \\ \beta_1 & \beta_2 \end{vmatrix} = 0 \Rightarrow \frac{\alpha_1}{\alpha_2}=\frac{\beta_1}{\beta_2}$$

Applying componendo-dividendo, $\frac{\alpha_1+\alpha_2}{\alpha_1-\alpha_2}=\frac{\beta_1+\beta_2}{\beta_1-\beta_2}$

$$\begin{aligned} &\Rightarrow (\alpha_1+\alpha_2)(\beta_1-\beta_2)=(\alpha_1-\alpha_2)(\beta_1+\beta_2) \\ &\Rightarrow (\alpha_1+\alpha_2)^2\left\{(\beta_1+\beta_2)^2-4\beta_2\beta_2\right\} \\ &=(\beta_1+\beta_2)^2\left\{(\alpha_1+\alpha_2)^2-4\alpha_1\alpha_2\right\} \end{aligned}$$

From Equation (1) and (2), we get

$$\frac{b^2}{a^2}\left(\frac{q^2}{p^2}-\frac{4r}{p}\right)=\frac{q^2}{p^2}\left(\frac{b^2}{a^2}-\frac{4c}{a}\right)$$
$$\Rightarrow \frac{b^2q^2}{a^2p^2}-\frac{4b^2r}{a^2p}=\frac{b^2q^2}{a^2p^2}-\frac{4q^2c}{ap^2}$$
$$\Rightarrow \frac{b^2r}{a}=\frac{q^2c}{p}\Rightarrow \frac{b^2}{q^2}=\frac{ac}{pr}$$

Correct Answer: C

48. Let A be a 5×5 real matrix. Suppose 0 is one of the eigenvalues of A. Which of the following statements is true? (Mark all the appropriate choices)

A. System $AX=0$ has a unique solution

B. System $AX=C$ has unique solution for any C

C. $AX=0$ has a non-trivial solution

D. None of the above

Solution: If 0 is one of the eigenvalue of A

$\Rightarrow \det(A)=0 \Rightarrow AX=0$ has a non trivial solution

Correct Answer: C

1.7 LU decomposition

49. The lower triangular matrix $[L]$ in the $[L][U]$ decomposition of the matrix given below

$$\begin{bmatrix}25&5&4\\10&8&16\\8&12&22\end{bmatrix}=\begin{bmatrix}1&0&0\\\ell_{21}&1&0\\\ell_{31}&\ell_{32}&1\end{bmatrix}\begin{bmatrix}u_{11}&u_{12}&u_{13}\\0&u_{22}&u_{23}\\0&0&u_{33}\end{bmatrix}$$

is ______.

A. $\begin{bmatrix}1&0&0\\0.40000&1&0\\0.32000&1.7333&1\end{bmatrix}$ B. $\begin{bmatrix}25&5&4\\0&6&14.400\\0&0&-4.2400\end{bmatrix}$

C. $\begin{bmatrix}1&0&0\\10&1&0\\8&12&0\end{bmatrix}$ D. $\begin{bmatrix}1&0&0\\0.40000&1&0\\0.32000&1.5000&1\end{bmatrix}$

Solution: We must first complete the first step of forward elimination.

$$\begin{bmatrix}25&5&4\\10&8&16\\8&12&22\end{bmatrix}$$

First step: Multiply Row 1 by $\frac{10}{25}=0.4$, and subtract the results from Row 2

$$[\text{ Row } 2]-[\text{ Row } 1]\times(0.4)=\begin{bmatrix}25&5&4\\0&6&14.4\\8&12&22\end{bmatrix}$$

Multiply Row 1 by $\frac{8}{25}=0.32$, and subtract the results from Row 3.

$$[\text{ Row } 3]-[\text{ Row } 1]\times(0.32)=\begin{bmatrix}25&5&4\\0&6&14.4\\0&10.4&20.72\end{bmatrix}$$

To find ℓ_{21} and ℓ_{31}, what multiplier was used to make the a_{21} and a_{31} elements zero in the first step of forward elimination using the Naïve Gauss elimination method? They are

$$\ell_{21} = 0.4$$
$$\ell_{31} = 0.32$$

To find ℓ_{32}, what multiplier would be used to make the a_{32} element zero? Remember the a_{32} element is made zero in the second step of forward elimination. So

$$\ell_{32} = \frac{10.4}{6} = 1.7333$$

Hence

$$[L] = \begin{bmatrix} 1 & 0 & 0 \\ 0.40000 & 1 & 0 \\ 0.32000 & 1.7333 & 1 \end{bmatrix}$$

Correct Answer: A

50. The upper triangular matrix $[U]$ in the $[L][U]$ decomposition of the matrix given below

$$\begin{bmatrix} 25 & 5 & 4 \\ 0 & 8 & 16 \\ 0 & 12 & 22 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ \ell_{21} & 1 & 0 \\ \ell_{31} & \ell_{32} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

is ______.

A. $\begin{bmatrix} 1 & 0 & 0 \\ 0.40000 & 1 & 0 \\ 0.32000 & 1.7333 & 1 \end{bmatrix}$

B. $\begin{bmatrix} 25 & 5 & 4 \\ 0 & 6 & 14.400 \\ 0 & 0 & -4.2400 \end{bmatrix}$

C. $\begin{bmatrix} 25 & 5 & 4 \\ 0 & 8 & 16 \\ 0 & 0 & -2 \end{bmatrix}$

D. $\begin{bmatrix} 1 & 0.2000 & 0.16000 \\ 0 & 1 & 2.4000 \\ 0 & 0 & -4.240 \end{bmatrix}$

Solution: The $[U]$ matrix is the same as the coefficient matrix that is found at the end of the forward elimination steps of the Naïve Gauss elimination method, that is

$$[U] = \begin{bmatrix} 25 & 5 & 4 \\ 0 & 8 & 16 \\ 0 & 12 & 22 \end{bmatrix}$$

First step: The first step of forward elimination does not need to be conducted as a_{21} and a_{31} are already zero. Second step: Multiply Row 2 by $\frac{12}{8} = 1.5$ and subtract the results from Row 3.

$$[\text{ Row } 3] - [\text{ Row } 2] \times (1.5) = \begin{bmatrix} 25 & 5 & 4 \\ 0 & 8 & 16 \\ 0 & 0 & -2 \end{bmatrix}$$

Thus,

$$[U] = \begin{bmatrix} 25 & 5 & 4 \\ 0 & 8 & 16 \\ 0 & 0 & -2 \end{bmatrix}$$

Correct Answer: C

2 Calculus

2.1 Limits

51. If $\lim\limits_{x \to 1} \frac{x^4 - 1}{x - 1} = \lim\limits_{x \to k} \frac{x^3 - k^3}{x^2 - k^2}$, then k is ______.

A. $\frac{4}{3}$

B. $\frac{3}{8}$

C. $\frac{3}{2}$

D. $\frac{8}{3}$

Solution: Given,

$$\lim_{x \to 1} \frac{x^4 - 1}{x - 1} = \lim_{x \to k} \frac{x^3 - k^3}{x^2 - k^2}$$

$$\lim_{x \to 1} \frac{(x-1)(x+1)\left(x^2+1\right)}{x-1}$$

$$= \lim_{x \to k} \frac{(x-k)\left(x^2+k^2+xk\right)}{(x-k)(x+k)}$$

$$\Rightarrow 2 \times 2 \quad = \frac{3k^2}{2k}$$

$$\Rightarrow \quad k \quad = \frac{8}{3}$$

Correct Answer: D

52. $\lim\limits_{y \to 0} \frac{\sqrt{1 + \sqrt{1 + y^4}} - \sqrt{2}}{y^4}$

A. exists and equals $\frac{1}{4\sqrt{2}}$

B. does not exist

C. exists and equals $\frac{1}{2\sqrt{2}}$

D. exists and equals $\frac{1}{2\sqrt{2}(\sqrt{2}+1)}$

Solution: Clearly,

$$\lim_{y \to 0} \frac{\sqrt{1 + \sqrt{1 + y^4}} - \sqrt{2}}{y^4}$$

$$= \lim_{y \to 0} \frac{\sqrt{1 + \sqrt{1 + y^4}} - \sqrt{2}}{y^4} \times \frac{\sqrt{1 + \sqrt{1 + y^4}} + \sqrt{2}}{\sqrt{1 + \sqrt{1 + y^4}} + \sqrt{2}}$$

$$= \lim_{y \to 0} \frac{\left(1 + \sqrt{1 + y^4}\right)}{y^4\left(\sqrt{1 + \sqrt{1 + y^4}} + \sqrt{2}\right)} \left[\because (a+b)(a-b) = a^2 - b^2\right]$$

$$= \lim_{y \to 0} \frac{\sqrt{1 + y^4} - 1}{y^4\left(\sqrt{1 + \sqrt{1 + y^4}} + \sqrt{2}\right)} \times \frac{\sqrt{1 + y^4} + 1}{\sqrt{1 + y^4} + 1}$$

[rationalising the numerator]

$$= \lim_{y \to 0} \frac{\left(1 + \sqrt{1 + y^4}\right) - 2}{y^4\left(\sqrt{1 + \sqrt{1 + y^4}} + \sqrt{2}\right)} \left[\because (a+b)(a-b) = a^2 - b^2\right]$$

$$= \lim_{y \to 0} \frac{\sqrt{1 + y^4} - 1}{y^4\left(\sqrt{1 + \sqrt{1 + y^4}} + \sqrt{2}\right)} \times \frac{\sqrt{1 + y^4} + 1}{\sqrt{1 + y^4} + 1}$$

[again, rationalising the numerator]

$$= \lim_{y \to 0} \frac{y^4}{y^4\left(\sqrt{1+\sqrt{1+y^4}}+\sqrt{2}\right)\left(\sqrt{1+y^4}+1\right)}$$

$$= \frac{1}{2\sqrt{2} \times 2}$$

(by cancelling y^4 and then by direct substitution).

$$= \frac{1}{4\sqrt{2}}.$$

Correct Answer: A

53. $\lim_{h \to 0} \frac{f\left(2h+2+h^2\right)-f(2)}{f\left(h-h^2+1\right)-f(1)}$, given that $f'(2) = 6$ and $f'(1) = 4$,

A. does not exist

B. is equal to $-3/2$

C. is equal to $3/2$

D. is equal to 3

Solution:

Here, $\lim_{h \to 0} \frac{f\left(2h+2+h^2\right)-f(2)}{f\left(h-h^2+1\right)-f(1)}$ $[\because f'(2) = 6 \text{ and } f'(1) = 4, \text{given}]$

Applying L'Hospital's rule,

$$= \lim_{h \to 0} \frac{\left\{f'\left(2h+2+h^2\right)\right\} \cdot (2+2h) - 0}{\left\{f'\left(h-h^2+1\right)\right\} \cdot (1-2h) - 0} = \frac{f'(2) \cdot 2}{f'(1) \cdot 1}$$

$$= \frac{6 \cdot 2}{4 \cdot 1} = 3 \quad [\text{ using } f'(2) = 6 \text{ and } f'(1) = 4]$$

Correct Answer: D

54. Let $\alpha, \beta \in R$ be such that $\lim_{x \to 0} \frac{x^2 \sin(\beta x)}{\alpha x - \sin x} = 1$. Then, $6(\alpha+\beta)$ equals ______. (Numerical Answer Type)

Solution: Here, $\lim_{x \to 0} \frac{x^2 \sin(\beta x)}{\alpha x - \sin x} = 1$

$$\Rightarrow \quad \lim_{x \to 0} \frac{x^2\left(\beta x - \frac{(\beta x)^3}{3!} + \frac{(\beta x)^5}{5!} - \ldots\right)}{\alpha x - \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots\right)} = 1$$

$$\Rightarrow \quad \lim_{x \to 0} \frac{x^3\left(\beta - \frac{\beta^3 x^2}{3!} + \frac{\beta^5 x^4}{5!} - \ldots\right)}{(\alpha - 1)x + \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots} = 1$$

The limit exists only, when $\alpha - 1 = 0$

$$\Rightarrow \quad \lim_{x \to 0} \frac{x^3\left(\beta - \frac{\beta^3 x^2}{3!} + \frac{\beta^5 x^4}{5!} - \ldots\right)}{x^3\left(\frac{1}{3!} - \frac{x^2}{5!} - \ldots\right)} = 1$$

$$\Rightarrow \quad 6\beta = 1$$

Now,

$6(\alpha+\beta) = 6\alpha + 6\beta$

$\quad = 6 + 1 = 7$

Correct Answer: 7

55. Let m and n be two positive integers greater than 1.

If $\lim_{\alpha\to 0}\left(\frac{e^{\cos(\alpha^n)}-e}{\alpha^m}\right)=-\left(\frac{e}{2}\right)$, then the value of $\frac{m}{n}$ is ______. (Numerical Answer Type)

Solution: Given, $\lim_{\alpha\to 0}\left[\frac{e^{\cos(\alpha^n)}-e}{\alpha^m}\right]=-\frac{e}{2}$

$$\Rightarrow \quad \lim_{\alpha\to 0}\frac{e\left\{e^{\cos(\alpha^n)-1}-1\right\}}{\cos(\alpha^n)-1}\cdot\frac{\cos(\alpha^n)-1}{\alpha^m}=\frac{-e}{2}$$

$$\Rightarrow \quad \lim_{\alpha\to 0} e\left\{\frac{e^{\cos(\alpha^n)-1}-1}{\cos(\alpha^n)-1}\right\}\cdot\lim_{\alpha\to 0}\frac{-2\sin^2\frac{\alpha^n}{2}}{\alpha^m}=-e/2$$

$$\Rightarrow \quad e\times 1\times(-2)\lim_{\alpha\to 0}\frac{\sin^2\left(\frac{\alpha^n}{2}\right)}{\frac{\alpha^{2n}}{4}}\cdot\frac{\alpha^{2n}}{4\alpha^m}=\frac{-e}{2}$$

$$\Rightarrow \quad e\times 1\times -2\times 1\times\lim_{\alpha\to 0}\frac{\alpha^{2n-m}}{4}=\frac{-e}{2}$$

For this to be exists, $2n-m=0$

$$\Rightarrow \quad \frac{m}{n}=2$$

Correct Answer: 2

2.2 Continuity and differentiability

56. If the function f defined on $\left(\frac{\pi}{6},\frac{\pi}{3}\right)$ by $f(x)=\begin{cases}\frac{\sqrt{2}\cos x-1}{\cot x-1}, & x\neq\frac{\pi}{4}\\ k, & x=\frac{\pi}{4}\end{cases}$ is continuous, then k is equal to

A. $\frac{1}{2}$

B. 2

C. 1

D. $\frac{1}{\sqrt{2}}$

Solution: Given function is

$$f(x)=\begin{cases}\frac{\sqrt{2}\cos x-1}{\cot x-1} & ,x\neq\frac{\pi}{4}\\ k & ,x=\frac{\pi}{4}\end{cases}$$

$\because$ Function $f(x)$ is continuous, so it is continuous at $x=\frac{\pi}{4}$.

$$\therefore \quad f\left(\frac{\pi}{4}\right)=\lim_{x\to\frac{\pi}{4}}f(x)$$

$$\Rightarrow \quad k=\lim_{x\to\frac{\pi}{4}}\frac{\sqrt{2}\cos x-1}{\cot x-1}$$

Put $x=\frac{\pi}{4}+h$, when $x\to\frac{\pi}{4}$, then $h\to 0$

$$k=\lim_{h\to 0}\frac{\sqrt{2}\cos\left(\frac{\pi}{4}+h\right)-1}{\cot\left(\frac{\pi}{4}+h\right)-1}$$

$$=\lim_{h\to 0}\frac{\sqrt{2}\left[\frac{1}{\sqrt{2}}\cos h-\frac{1}{\sqrt{2}}\sin h\right]-1}{\frac{\cot h-1}{\cot h+1}-1}$$

$$= \lim_{h\to 0} \frac{\cos h - \sin h - 1}{\frac{-2}{1+\cot h}}$$

$$= \frac{\lim}{h \to 0}\left[\frac{(1-\cos h)+\sin h}{2\sin h}(\sin h + \cos h)\right]$$

$$= \lim_{h\to 0}\left[\frac{2\sin^2\frac{h}{2} + 2\sin\frac{h}{2}\cos\frac{h}{2}}{4\sin\frac{h}{2}\cos\frac{h}{2}}(\sin h + \cos h)\right]$$

$$= \lim_{h\to 0}\left[\frac{\sin\frac{h}{2}+\cos\frac{h}{2}}{2\cos\frac{h}{2}} \times (\sin h + \cos h)\right] \Rightarrow k = \frac{1}{2}$$

Correct Answer: A

57. Let $f : R \to R$ be a function defined as

$$f(x) = \begin{cases} 5, & \text{if} \quad x \le 1 \\ a+bx, & \text{if} \quad 1 < x < 3 \\ b+5x, & \text{if} \quad 3 \le x < 5 \\ 30, & \text{if} \quad x \ge 5 \end{cases}$$

Then, f is

A. continuous if $a = -5$ and $b = 10$

B. continuous if $a = 5$ and $b = 5$

C. continuous if $a = 0$ and $b = 5$

D. not continuous for any values of a and b

Solution: Key Idea A function is said to be continuous if it is continuous at each point of the domain. We have,

$$f(x) = \begin{cases} 5 & \text{if} \quad x \le 1 \\ a+bx & \text{if} \quad 1 < x < 3 \\ b+5x & \text{if} \quad 3 \le x < 5 \\ 30 & \text{if} \quad x \ge 5 \end{cases}$$

Clearly, for $f(x)$ to be continuous, it has to be continuous at $x = 1, x = 3$ and $x = 5$ [$\because$ In rest portion it is continuous everywhere]

$$\lim_{x\to 1^+}(a+bx) = a+b = 5 \quad \longrightarrow (i)$$

$$\left[\because \lim_{x\to 1^-} f(x) = \lim_{x\to 1^+} f(x) = f(1)\right]$$

$$\lim_{x\to 5^-}(b+5x) = b+25 = 30 \quad \longrightarrow (ii)$$

$$\left[\because \lim_{x\to 5^-} f(x) = \lim_{x\to 5^+} f(x) = f(5)\right]$$

On solving Eqs. (i) and (ii), we get $b = 5$ and $a = 0$

Now, let us check the continuity of $f(x)$ at $x = 3$. Here, $\lim_{x\to 3^-}(a+bx) = a+3b = 15$ and $\lim_{x\to 3^+}(b+5x) = b+15 = 20$

Hence, for $a = 0$ and $b = 5, f(x)$ is not continuous at $x = 3 \therefore f(x)$ cannot be continuous for any values of a and b.

Correct Answer: D

58. Let $f(x) = \begin{cases} -1, & -2 \le x < 0 \\ x^2 - 1, & 0 \le x \le 2 \end{cases}$ and $g(x) = |f(x)| + f(|x|)$. Then, in the interval $(-2, 2), g$ is ______.

A. not differentiable at one point

B. not differentiable at two points

C. differentiable at all points

D. not continuous

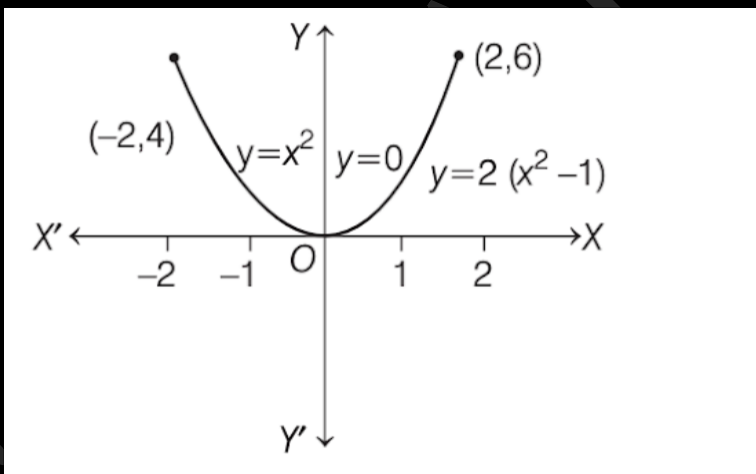
Solution: Key Idea This type of problem can be solved graphically.

We have, $f(x)=\begin{cases} -1, & -2\le x<0 \\ x^2-1, & 0\le x\le 2 \end{cases}$ and $g(x)=|f(x)|+f(|x|)$

Clearly,
$$|f(x)|=\begin{cases} 1, & -2\le x<0 \\ \left|x^2-1\right|, & 0\le x\le 2 \end{cases}$$
$$=\begin{cases} 1, & -2\le x<0 \\ -\left(x^2-1\right), & 0\le x<1 \\ x^2-1, & 1\le x\le 2 \end{cases}$$

and
$$f(|x|)=|x|^2-1, 0\le |x|\le 2$$
$$[\because f(|x|)=-1 \text{ is not possible as } |x|\not< 0]$$
$$=x^2-1, \quad |x|\le 2$$
$$=x^2-1, \quad -2\le x\le 2 \quad \left[\because |x|^2=x^2\right]$$
$$=|f(x)|+f(|x|)$$
$$\therefore g(x)=|f(x)|+f(|x|)$$
$$=\begin{cases} 1+x^2-1, & -2\le x<0 \\ -\left(x^2-1\right)+x^2-1, & 0\le x<1 \\ x^2-1+x^2-1, & 1\le x\le 2 \end{cases}$$
$$=\begin{cases} x^2, & -2\le x<0 \\ 0, & 0\le x<1 \\ 2\left(x^2-1\right), & 1\le x\le 2 \end{cases}$$

Now, let us draw the graph of $y=g(x)$, as shown in the figure.



[Here, $y=2\left(x^2-1\right)$ or $x^2=\frac{1}{2}(y+2)$ represent a parabola with vertex $(0,-2)$ and it open upward]

Note that there is a sharp edge at $x=1$ only, so $g(x)$ is not differentiable at $x=1$ only.

Correct Answer: A

59. For every twice differentiable function $f: R\to[-2,2]$ with $(f(0))^2+(f'(0))^2=85$, which of the following statement(s) is (are) TRUE? (Mark all the appropriate choices)

A. There exist $r,s\in R$, where $r<s$, such that f is one-one on the open interval (r,s)

B. There exists $x_0\in(-4,0)$ such that $|f'(x_0)|\le 1$

C. $\lim\limits_{x\to\infty} f(x)=1$

D. There exists $\alpha\in(-4,4)$ such that $f(\alpha)+f''(\alpha)=0$ and $f'(\alpha)\ne 0$

Solution: We have, and

$$(f(0))^2 + (f'(0))^2 = 85$$
$$f : R \to [-2,2]$$

A. Since, f is twice differentiable function, so f is continuous function. $\therefore$ This is true for every continuous function. Hence, we can always find $x \in (r,s)$, where $f(x)$ is one-one.

$\therefore$ This statement is true.

B. By L.M.V.T

$$f'(c) = \frac{f(b)-f(a)}{b-a} \Rightarrow \left|f'(c)\right| = \left|\frac{f(b)-f(a)}{b-a}\right|$$
$$\Rightarrow \left|f'(x_0)\right| = \left|\frac{f(0)-f(-4)}{0+4}\right| = \left|\frac{f(0)-f(-4)}{4}\right|$$

Range of f is $[-2,2]$

$$\therefore 4 \le f(0)-f(-4) \le 4 \Rightarrow 0 \le \left|\frac{f(0)-f(-4)}{4}\right| \le 1$$

Hence, $|f'(x_0)| \le 1$.

Hence, statement is true.

C. As no function is given, then we assume

$$f(x) = 2\sin\left(\frac{\sqrt{85}x}{2}\right)$$
$$\therefore \quad f'(x) = \sqrt{85}\cos\left(\frac{\sqrt{85}x}{2}\right)$$

Now, $(f(0))^2 + (f'(0))^2 = (2\sin 0)^2 + (\sqrt{85}\cos 0)^2$

$$(f(0))^2 + (f'(0))^2 = 85$$

and $\lim_{x\to\infty} f(x)$ does not exists.

Hence, statement is false.

D. From option $b, |f'(x_0)| \le 1$ and $x_0 \in (-4,0)$

$$\therefore \quad (f'(x_0))^2 \le 1$$
$$\text{Hence, } g(x_0) = (f(x_0))^2 + \left(f'(x_0)^2 \le 4+1\right.$$
$$[\because f(x_0) \in [-2,2]]$$
$$\Rightarrow \quad g(x_0) \le 5$$

Now, let $p \in (-4,0)$ for which $g(p) = 5$ Similarly, let q be smallest positive number $q \in (0,4)$ such that $g(q) = 5$

Hence, by Rolle's theorem is (p,q) $g'(c) = 0$ for $\alpha \in (-4,4)$ and since $g(x)$ is greater than 5 as we move form $x = p$ to $x = q$ and $f(x))^2 \le 4 \Rightarrow (f'(x))^2 \ge 1$ in(p,q)

Thus, $\quad g'(c) = 0 \Rightarrow \quad f'f + f'f'' = 0$ So, $f(\alpha) + f''(\alpha) = 0$ and $f'(\alpha) \ne 0$

Hence, statement is true.

Correct Answer: A; B; D

60. If $f(x)=\begin{cases}-x-\frac{\pi}{2}, & x\le -\frac{\pi}{2}\\ -\cos x, & -\frac{\pi}{2}<x\le 0\\ x-1, & 0<x\le 1\\ \ln x, & x>1\end{cases}$ Then (Mark all the appropriate choices)

A. $f(x)$ is continuous at $x=-\frac{\pi}{2}$

B. $f(x)$ is not differentiable at $x=0$

C. $f(x)$ is differentiable at $x=1$

D. $f(x)$ is differentiable at $x=-\frac{3}{2}$

Solution: $f(x)=\begin{cases}-x-\frac{\pi}{2}, & x\le -\frac{\pi}{2}\\ -\cos x, & -\frac{\pi}{2}<x\le 0\\ x-1, & 0<x\le 1\\ \log x, & x>1\end{cases}$ Continuity at $x=-\frac{\pi}{2}$,

$$\begin{aligned} f\left(-\tfrac{\pi}{2}\right) &= -\left(-\tfrac{\pi}{2}\right)-\tfrac{\pi}{2}=0\\ \text{RHL } &= \lim_{h\to 0}-\cos\left(-\tfrac{\pi}{2}+h\right)=0\end{aligned}$$

$\therefore$ Continuous at $x=\left(-\frac{\pi}{2}\right)$. Continuity at $x=0$

$$\begin{aligned} f(0) &= -1\\ \text{RHL} &= \lim_{h\to 0}(0+h)-1=-1\end{aligned}$$

$\therefore$ Continuous at $x=0$. Continuity at $x=1$,

$$f(1)=0$$

$$f'(x)=\begin{cases}-1, & x\le -\frac{\pi}{2}\\ \sin x, & -\frac{\pi}{2}<x\le 0\\ 1, & 0<x\le 1\\ \frac{1}{x}, & x>1\end{cases}$$

Differentiable at $x=0$, LHD $=0$, RHD $=1$

$\therefore$ Not differentiable at $x=0$

Differentiable at $x=1$, LHD $=1$, RHD $=1$

$\therefore$ Differentiable at $x=1$.

Also, for $x=-\frac{3}{2}$

$\Rightarrow\quad f(x)=-x-\frac{\pi}{2}$

$\therefore$ Differentiable at $x=-\frac{3}{2}$

Correct Answer: A; B; C; D

2.3 Differentiation, Maxima and Minima

61. If $f(1)=1, f'(1)=3$, then the derivative of $f(f(f(x)))+(f(x))^2$ at $x=1$ is ______.

A. 12

B. 9

C. 15

D. 33

Solution: Let $y=f(f(f(x)))+(f(x))^2$ On differentiating both sides w.r.t. x, we get

$$\frac{dy}{dx}=f'(f(f(x)))\cdot f'(f(x))\cdot f'(x)+2f(x)f'(x)$$

[by chain rule]

$$\text{So, } \left.\frac{dy}{dx}\right|_{\text{at } x=1} = f'(f(f(1))) \cdot f'(f(1)) \cdot f'(1) + 2f(1)f'(1)$$

$$\therefore \left.\frac{dy}{dx}\right|_{x=1} = f'(f(1)) \cdot f'(1) \cdot (3) + 2(1)(3)$$

$[\because f(1) = 1 \text{ and } f'(1) = 3]$

$= f'(1) \cdot (3) \cdot (3) + 6$

$= (3 \times 9) + 6 = 27 + 6 = 33$

Correct Answer: D

62. Let m_1 be the slope of the function $y = 3^x$ at the point $x = 0$ and let m_2 be the slope of the function $y = \log_3 x$ at $x = 1$ Then (Mark all the appropriate choices)

A. $m_1 m_2 = 1$

B. $m_1 = m_2$

C. $m_1 = -m_2$

D. $m_1 = 1/m_2$

Solution: The functions are inverses and the points $(0, 1)$ and $(1, 0)$ are the images of one another about the line $y = x$. Thus the slopes are reciprocals so that $m_1 = 1/m_2 \Rightarrow m_1 m_2 = 1$.

Correct Answer: A;D

63. Let $f(x) = 1 + 2x^2 + 2^2x^4 + \ldots + 2^{10}x^{20}$. Then $f(x)$ has

A. more than one minimum

B. exactly one minimum

C. at least one maximum

D. none of the above

Solution:

$$f(x) = 1 + 2x^2 + 2^2x^4 + \ldots + 2^{10}x^{20}$$
$$\Rightarrow f'(x) = 4x + 16x^3 + \ldots + 2^{10} \cdot 20 \cdot x^{19}$$
$$\Rightarrow f''(x) = 4 + 48x^2 + \ldots + 2^{10} \cdot 380 \cdot x^{18} > 0$$
$$f'(x) = 0 \text{ if } x = 0$$

Therefore, $f(x)$ is minimum at $x = 0$

Correct Answer: B

64. Let $f(x) = a - (x-3)^{8/9}$, then maxima of $f(x)$ is ______.

A. 3

B. $a - 3$

C. a

D. none of these

Solution:

$$f(x) = a - (x-3)^{8/9}$$
$$\because f'(x) = 0 - \frac{8}{9}(x-3)^{-1/9}$$

At $x = 3, f'(x)$ is not defined. Hence, $x = 3$ is the point of extremum. Therefore, the maximum value of $f(x) = a$ at $x = 3$.

Correct Answer: A

65. If $f : R \to R$ is a differentiable function such that $f'(x) > 2f(x)$ for all $x \in R$, and $f(0) = 1$ then (Mark all the appropriate choices)

A. $f(x) > e^{2x}$ in $(0, \infty)$

B. $f'(x) < e^{2x}$ in $(0, \infty)$

C. $f(x)$ is increasing in $(0, \infty)$

D. $f(x)$ is decreasing in $(0, \infty)$

Solution:

$$f'(x) > 2f(x) \Rightarrow \frac{dy}{y} > 2dx$$

$$\Rightarrow \quad \int_1^{f(x)} \frac{dy}{y} > 2\int_0^x dx$$

$$\therefore \quad \ln(f(x)) > 2x$$

$$\text{Also, as} \quad f'(x) > 2f(x)$$
$$f(x) > e^{2x}$$
$$f'(x) > 2c^{2x} > 0$$

Correct Answer: C;D

66. The least value of $\alpha \in R$ for which $4\alpha x^2 + \frac{1}{x} \geq 1$, for all $x > 0$, is ______.

A. $\frac{1}{64}$

B. $\frac{1}{32}$

C. $\frac{1}{27}$

D. $\frac{1}{25}$

Solution: Here, to find the least value of $\alpha \in R$, for which $4\alpha x^2 + \frac{1}{x} \geq 1$, for all $x > 0$. i.e. to find the minimum value of α when $y = 4\alpha x^2 + \frac{1}{x}; x > 0$ attains minimum value of α.

$\therefore$

$$\frac{dy}{dx} = 8\alpha x - \frac{1}{x^2}$$

Now,

$$\frac{d^2y}{dx^2} = 8\alpha + \frac{2}{x^3}$$

When

$$\frac{dy}{dx} = 0 \text{ then } 8x^3\alpha = 1$$

At $x = \left(\frac{1}{8\alpha}\right)^{1/3}$, $\frac{d^2y}{dx^2} = 8\alpha + 16\alpha = 24\alpha$,

Thus, y attains minimum when $x = \left(\frac{1}{8\alpha}\right)^{1/3}; \alpha > 0$.

$\therefore y$ attains minimum when $x = \left(\frac{1}{8\alpha}\right)^{1/3}$. i.e. $\quad 4\alpha\left(\frac{1}{8\alpha}\right)^{2/3} + (8\alpha)^{1/3} \geq 1$

$$\Rightarrow \quad \alpha^{1/3} + 2\alpha^{1/3} \geq 1$$

$$\Rightarrow \quad 3\alpha^{1/3} \geq 1 \Rightarrow \alpha \geq \frac{1}{27}$$

Hence, the least value of α is $\frac{1}{27}$.

Correct Answer: C

67. If $f(x)$ is a non-zero polynomial of degree four, having local extreme points at $x = -1, 0, 1$, then the set $S = \{x \in R : f(x) = f(0)\}$ contains exactly

A. four rational numbers

B. two irrational and two rational numbers

C. four irrational numbers

D. two irrational and one rational number

Solution: The non-zero four degree polynomial $f(x)$ has extremum points at $x = -1, 0, 1$, so we can assume $f'(x) = a(x+1)(x-0)(x-1) = ax\left(x^2 - 1\right)$ where, a is non-zero constant.

$$\begin{aligned} & f'(x) = ax^3 - ax \\ \Rightarrow \quad & f(x) = \tfrac{a}{4}x^4 - \tfrac{a}{2}x^2 + C \end{aligned}$$

[integrating both sides] where, C is constant of integration.

Now, since $f(x) = f(0)$

$$\begin{aligned} \Rightarrow \quad & \frac{a}{4}x^4 - \frac{a}{2}x^2 + C = C \Rightarrow \frac{x^4}{4} = \frac{x^2}{2} \\ \Rightarrow \quad & x^2\left(x^2 - 2\right) = 0 \Rightarrow x = -\sqrt{2}, 0, \sqrt{2} \end{aligned}$$

Thus, $f(x) = f(0)$ has one rational and two irrational roots.

Correct Answer: D

68. If $x = -1$ and $x = 2$ are extreme points of $f(x) = \alpha \log|x| + \beta x^2 + x$, then

A. $\alpha = -6, \beta = \frac{1}{2}$

B. $\alpha = -6, \beta = -\frac{1}{2}$

C. $\alpha = 2, \beta = -\frac{1}{2}$

D. $\alpha = 2, \beta = \frac{1}{2}$

Solution: Here, $x = -1$ and $x = 2$ are extreme points of $f(x) = \alpha \log|x| + \beta x^2 + x$, then

$$\begin{aligned} f'(x) &= \frac{\alpha}{x} + 2\beta x + 1 \\ f'(-1) &= -\alpha - 2\beta + 1 = 0 \quad \longrightarrow \text{(i)} \\ & \text{[at extreme point}, f'(x) = 0] \\ f'(2) &= \frac{\alpha}{2} + 4\beta + 1 = 0 \quad \longrightarrow \text{(ii)} \end{aligned}$$

On solving Eqs. (i) and (ii), we get

$$\alpha = 2, \beta = -\frac{1}{2}$$

Correct Answer: C

2.4 Mean Value Theorem

69. $f(x)$ is a function that is continuous and differentiable in the domain $[7, 15]$. If $f(7) = 21$ and $f'(x) \leq 14$ for all $7 \leq x \leq 15$, what is the maximum possible value of $f(15)$? (Numerical Answer Type)

Solution: Using the mean value theorem, we have

$$14 \geq f(x) = \frac{f(15) - f(7)}{15 - 7} = \frac{f(15) - 21}{8}$$

Simplifying this gives $f(15) \leq 133$.

Correct Answer: 133

70. If Rolle's theorem holds for the function $f(x) = x^3 - ax^2 + bx - 4, x \in [1,2]$ with $f'\left(\frac{4}{3}\right) = 0$, then ordered pair (a, b) is equal to :

A. $(-5, -8)$

B. $(5, -8)$

C. $(-5, 8)$

D. $(5, 8)$

Solution: **Rolle's Theorem:** For any function $f(x)$ that is continuous within the interval $[a, b]$ and differentiable within the interval (a, b), where $f(a) = f(b)$, there exists at least one point $(c, f(c))$ where $f'(c) = 0$ within the interval (a, b).

Now, $f(1) = f(2)$

$\Rightarrow 1 - a + b - 4 = 8 - 4a + 2b - 4$

$\Rightarrow 3a - b = 7 \quad \longrightarrow (1)$

$\Rightarrow f'(x) = 3x^2 - 2ax + b$

$$\Rightarrow f'\left(\frac{4}{3}\right) = 0$$

$$\Rightarrow 3 \times \frac{16}{9} - \frac{8}{3}a + b = 0$$

$\Rightarrow -8a + 3b = -16 \quad \longrightarrow (2)$

Solving equation (1) and (2) we get,

$\therefore a = 5, b = 8$

Correct Answer: D

71. If $f : R \rightarrow R$ is a twice differentiable function such that $f''(x) > 0$ for all $x \in R$, and $f\left(\frac{1}{2}\right) = \frac{1}{2}, f(1) = 1$, then

A. $f'(1) \leq 0$

B. $f'(1) > 1$

C. $0 < f'(1) \leq \frac{1}{2}$

D. $\frac{1}{2} < f'(1) \leq 1$

Solution:

Mean Value Theorem:

Suppose that a function f is

1. continuous on the closed interval $[a, b]$, and

2. differentiable on the open interval (a, b).

Then, there is a number c such that $a < c < b$ and $f'(c) = \frac{f(b)-f(a)}{b-a}$.

$f'(x)$ is increasing

For some x in $\left(\frac{1}{2}, 1\right)$

$f'(x) = 1 \quad$ [LMVT]

$\therefore \quad f'(1) > 1$

Correct Answer: B

72. Let the function, $f : [-7, 0] \rightarrow R$ be continuous on $[-7, 0]$ and differentiable on $(-7, 0)$. If $f(-7) = -3$ and $f'(x) \leq 2$, for $x \in (-7, 0)$, then for all such functions f, $f(-1) + f(0)$ lies in the interval :

A. $[-5, -7]$

B. $(-\infty, 6]$

C. $(-\infty, 20]$

D. $[-5, 3]$

Solution: Lets use LMVT for $x \in [-7, -1]$

$$\frac{f(-1) - f(-7)}{(-1+7)} \leq 2$$
$$\frac{f(-1)+3}{6} \leq 2 \Rightarrow f(-1) \leq 9$$

Also use LMVT for $x \in [-7, 0]$

$$\frac{f(0) - f(-7)}{(0+7)} \leq 2$$
$$\frac{f(0)+3}{7} \leq 2 \Rightarrow f(0) \leq 11$$
$$\therefore \quad f(0) + f(-1) \leq 20$$

Correct Answer: C

73. If $f(x)$ is twice differentiable and continuous function in $x \in [a, b]$ also $f'(x) > 0$ and $f''(x) < 0$ and $c \in (a, b)$ then $\frac{f(c)-f(a)}{f(b)-f(c)}$ is greater than

A. $\frac{b-c}{c-a}$
B. 1
C. $\frac{a+b}{b-c}$
D. $\frac{c-a}{b-c}$

Solution: Lets use LMVT for $x \in [a, c]$

$$\frac{f(c) - f(a)}{c-a} = f'(\alpha), \alpha \in (a, c)$$

also use LMVT for $x \in [c, b]$

$$\frac{f(b) - f(c)}{b-c} = f'(\beta), \beta \in (c, b)$$
$$\because f''(x) < 0 \Rightarrow f'(x) \text{ is decreasing}$$
$$f'(\alpha) > f'(\beta)$$
$$\frac{f(c) - f(a)}{c-a} > \frac{f(b) - f(c)}{b-c}$$
$$\frac{f(c) - f(a)}{f(b) - f(c)} > \frac{c-a}{b-c} \quad (\because f(x) \text{ in increasing })$$

Correct Answer: D

2.5 Integration

74. The integral $\int \sec^{23} x \operatorname{cosec}^{4/3} x dx$ is equal to (here C is a constant of integration)

A. $3 \tan^{-1/3} x + C$
B. $-3 \tan^{-1/3} x + C$
C. $-3 \cot^{-1/3} x + C$
D. $-\frac{3}{4} \tan^{-4/3} x + C$

Solution: Let $I = \int \sec^{\frac{2}{3}} x \operatorname{cosec}^{\frac{4}{3}} x dx = \int \frac{dx}{\cos^{\frac{2}{3}} x \sin^{\frac{4}{3}} x}$

$$\int \frac{dx}{\left(\frac{\sin x}{\cos x}\right)^{\frac{4}{3}} \cos^{\frac{4}{3}} x \cos^{\frac{2}{3}} x}$$

[dividing and multiplying by $\cos^{4/3} x$ in denominator]

$$= \int \frac{dx}{\tan^{\frac{4}{3}} x \cos^2 x} = \int \frac{\sec^2 x dx}{(\tan x)^{\frac{4}{3}}}$$

Now, put $\tan x = t \Rightarrow \sec^2 x dx = dt$

$$\therefore I = \int \frac{dt}{t^{4/3}} = \frac{t^{\frac{-4}{3}+1}}{\frac{-4}{3}+1} + C$$
$$= -3\frac{1}{t^{\frac{1}{3}}} + C = \frac{-3}{(\tan x)^{\frac{1}{3}}} + C = -3\tan^{-\frac{1}{3}} x + C$$

Correct Answer: B

75. The integral

$$\int \frac{\sin^2 x\cos^2 x}{\left(\sin^5 x+\cos^3 x\sin^2 x+\sin^3 x\cos^2 x\right.} dx$$

is equal to

A. $\frac{1}{3\left(1+\tan^3 x\right)}+C$

B. $\frac{-1}{3\left(1+\tan^3 x\right)}+C$

C. $\frac{1}{1+\cot^3 x}+C$

D. $\frac{-1}{1+\cot^3 x}+C$

(where C is a constant of integration)

Solution: We have,

$$I = \int \frac{\sin^2 x\cdot\cos^2 x}{\left(\sin^5 x+\cos^3 x\cdot\sin^2 x+\sin^3 x\cdot\cos^2 x+\cos^5 x\right)^2} dx$$
$$= \int \frac{\sin^2 x\cos^2 x}{\left\{\sin^3 x\left(\sin^2 x+\cos^2 x\right)+\cos^3 x\left(\sin^2 x+\cos^2 x\right)\right\}^2} dx$$
$$= \int \frac{\sin^2 x\cos^2 x}{\left(\sin^3 x+\cos^3 x\right)^2} dx = \int \frac{\sin^2 x\cos^2 x}{\cos^6 x(1+\tan^3 x)^2} dx$$
$$= \int \frac{\tan^2 x\sec^2 x}{(1+\tan^3 x)^2} dx$$

Put $\tan^3 x = t \Rightarrow 3\tan^2 x\sec^2 x dx = dt$

$$\therefore \quad I = \frac{1}{3}\int \frac{dt}{(1+t)^2}$$
$$\Rightarrow \quad I = \frac{-1}{3(1+t)} + C \Rightarrow I = \frac{-1}{3(1+\tan^3 x)} + C$$

Correct Answer: B

76. If $\int f(x)dx = \psi(x)$, then $\int x^5 f\left(x^3\right) dx$ is equal to

A. $\frac{1}{3}\left[x^3\psi\left(x^3\right)-\int x^2\psi\left(x^3\right)dx\right]+c$

B. $\frac{1}{3}x^3\psi\left(x^3\right)-3\int x^3\psi\left(x^3\right)dx+c$

C. $\frac{1}{3}x^3\psi\left(x^3\right)-\int x^2\psi\left(x^3\right)dx+c$

D. $\frac{1}{3}\left[x^3\psi\left(x^3\right)-\int x^3\psi\left(x^3\right)dx\right]+c$

Solution: Given, $\int f(x)dx = \psi(x)$

Let $\quad I = \int x^5 f\left(x^3\right) dx$

Put $\quad x^3 = t$

$$\Rightarrow \quad x^2 dx = \frac{dt}{3} \quad \longrightarrow (1)$$

$$\therefore \quad = \frac{1}{3}\int t f(t) dt$$

$$= \frac{1}{3}\left[t \cdot \int f(t)dt - \int \left\{\frac{d}{dt}(t)\int f(t)dt\right\} dt\right]$$

$$= \frac{1}{3}\left[t\psi(t) - \int \psi(t)dt\right] \quad \text{[integration by parts]}$$

$$= \frac{1}{3}\left[x^3\psi\left(x^3\right) - 3\int x^2\psi\left(x^3\right)dx\right] + c \quad \text{[from equation (1)]}$$

$$= \frac{1}{3}x^3\psi\left(x^3\right) - \int x^2\psi\left(x^3\right)dx + c$$

Correct Answer: C

77. The exact value of $\int_{0.2}^{2.2} xe^x dx$ is most nearly

A. 7.8036

B. 11.807

C. 14.034

D. 19.611

Solution: To solve this integral we must integrate by parts.

$$\int u dv = uv - \int v du$$

where

$$u = x$$
$$du = dx$$

and

$$dv = e^x dx$$
$$v = e^x$$

$$\int_{0.2}^{2.2} xe^x dx = \int_{0.2}^{2.2} xd\left(e^x\right)$$

$$= [xe^x]_{0.2}^{2.2} - \int_{0.2}^{2.2} e^x dx$$

$$= [xe^x - e^x]_{0.2}^{2.2}$$

$$= \left(2.2e^{2.2} - e^{2.2}\right) - \left(0.2e^{0.2} - e^{0.2}\right)$$

$$= 10.83 - (-0.9771)$$

$$= 11.807$$

Correct Answer: B

78. $\int_{0.2}^{2} f(x)dx$ for $f(x) = \begin{cases} x, & 0 \le x \le 1.2 \\ x^2, & 1.2 < x \le 2.4 \end{cases}$

is most nearly

A. 1.9800

B. 2.6640

C. 2.7907

D. 4.7520

Solution:

$$\int_{0.2}^{2} f(x)dx = \int_{0.2}^{1.2} x dx + \int_{1.2}^{2} x^2 dx$$
$$= \left[\frac{1}{2}x^2\right]_{0.2}^{1.2} + \left[\frac{1}{3}x^3\right]_{1.2}^{2}$$
$$= \left(\frac{1.2^2}{2} - \frac{0.2^2}{2}\right) + \left(\frac{2^3}{3} - \frac{1.2^3}{3}\right)$$
$$= 0.7 + 2.0907$$
$$= 2.7907$$

Correct Answer: C

79. Let f and g be continuous functions on $[0, a]$ such that $f(x) = f(a-x)$ and $g(x) + g(a-x) = 4$, then $\int_0^a f(x)g(x)dx$ is equal to

A. $4\int_0^a f(x)dx$

B. $\int_0^a f(x)dx$

C. $2\int_0^a f(x)dx$

D. $-3\int_0^a f(x)dx$

Solution:

$$\text{Let } I = \int_0^a f(x)g(x)dx$$
$$= \int_0^a f(a-x)g(a-x)dx$$
$$\left[\because \int_0^a f(x)dx = \int_0^a f(a-x)dx\right]$$
$$= \int_0^a f(x)[4-g(x)]dx$$
$$[\because f(x) = f(a-x) \text{ and } g(x) + g(a-x) = 4]$$
$$= \int_0^a 4f(x)dx - \int_0^a f(x)g(x)dx$$
$$\Rightarrow \quad I = 4\int_0^a f(x)dx - I$$
$$\Rightarrow \quad 2I = 4\int_0^a f(x)dx \Rightarrow \quad I = 2\int_0^a f(x)dx.$$

Correct Answer: C

80. The value of the integral $\displaystyle\int_0^{1/2} \frac{1+\sqrt{3}}{((x+1)^2(1-x)^6)^{1/4}} dx$ is ______.

(Numerical Answer Type)

Solution: Let $I = \int_0^{1/2} \frac{1+\sqrt{3}}{\left[(x+1)^2(1-x)^6\right]^{1/4}} dx$

$\Rightarrow \quad I = \int_0^{1/2} \frac{1+\sqrt{3}}{(1-x)^2\left[\left(\frac{1-x}{1+x}\right)^6\right]^{1/4}} dx$

Put $\quad \frac{1-x}{1+x} = t \Rightarrow \frac{-2dx}{(1+x)^2} = dt$ when $x = 0, t = 1, x = \frac{1}{2}, t = \frac{1}{3}$

$$I = \int_1^{1/3} \frac{(1+\sqrt{3})dt}{-2(t)^{6/4}}$$
$$\Rightarrow \quad I = \frac{-(1+\sqrt{3})}{2}\left[\frac{-2}{\sqrt{t}}\right]_1^{1/3}$$
$$\Rightarrow \quad I = (1+\sqrt{3})(\sqrt{3}-1) \Rightarrow I = 3-1 = 2$$

Correct Answer: 2

3 Probability and Statistics

3.1 Basic Probability

81. A person throws two fair dice. He wins ₹15 for throwing a doublet (same numbers on the two dice), wins ₹12 when the throw results in the sum of 9, and loses ₹6 for any other outcome on the throw. Then, the expected gain/loss (in ₹) of the person is ______.

A. $\frac{1}{2}$ gain

B. $\frac{1}{4}$ loss

C. $\frac{1}{2}$ loss

D. 2 gain

Solution: It is given that a person wins ₹15 for throwing a doublet $(1,1)(2,2),(3,3),(4,4),(5,5),(6,6)$ and win ₹12 when the throw results in a sum of 9, i.e., when $(3,6),(4,5),(5,4),(6,3)$ occurs.

Also, losses ₹6 for throwing any other outcome, i.e., when any of the rest $36-6-4=26$ outcomes occurs.

Now, the expected gain/loss $= 15\times P$ (getting a doublet) $+12\times P$ (getting sum 9) $-6\times P$ (getting any of rest 26 outcome)

$$\begin{aligned}
&= \left(15\times\frac{6}{36}\right)+\left(12\times\frac{4}{36}\right)-\left(6\times\frac{26}{36}\right)\\
&= \frac{5}{2}+\frac{4}{3}-\frac{26}{6}=\frac{15+8-26}{6}\\
&= \frac{23-26}{6}=-\frac{3}{6}=-\frac{1}{2}, \text{ means loss of ₹}\frac{1}{2}
\end{aligned}$$

Correct Answer: C

82. Two numbers are selected randomly from the set $S=\{1,2,3,4,5,6\}$ without replacement one by one. The probability that the minimum of the two numbers is less than 4, is _____.

A. 1/15

B. 14/15

C. 1/5

D. 4/5

Solution: Here, two numbers are selected from $\{1,2,3,4,5,6\}$

$\Rightarrow n(S)=6\times 5$ {as one by one without replacement}

Favourable events = the minimum of the two numbers is less than 4. $n(E)=6\times 4$ as for the minimum of the two is less than 4 we can select one from $(1,2,3,4)$ and other from $(1,2,3,4,5,6)$

$$\therefore \text{ Required probability } = \frac{n(E)}{n(S)}=\frac{24}{30}=\frac{4}{5}$$

Correct Answer: D

83. If E and F are events with $P(E)\le P(F)$ and $P(E\cap F)>0$, then which one is/are not correct? (Mark all the appropriate choices)

A. occurrence of $E\Rightarrow$ occurrence of F

B. occurrence of $F\Rightarrow$ occurrence of E

C. non-occurrence of $E\Rightarrow$ non-occurrence of F

D. None of the above

Solution: It is given that, $P(E)\le P(F)\Rightarrow E\subseteq F \quad \longrightarrow (1)$

and $P(E\cap F)>0\Rightarrow E\subset F \quad \longrightarrow (2)$

A. occurrence of $E\Rightarrow$ occurrence of F [from equation (1)]

B. occurrence of $F\Rightarrow$ occurrence of E [from equation (2)]

C. non-occurrence of $E \Rightarrow$ occurrence of F

Hence, option (C) is not correct. [from equation (1)]

Correct Answer: C

84. Three boys and two girls stand in a queue. The probability that the number of boys ahead of every girl is at least one more than the number of girls ahead of her, is ____.

A. 1/2

B. 1/3

C. 2/3

D. 3/4

Solution: The total number of ways to arrange 3 boys and 2 girls are 5!. According to the given condition, the following cases may arise.

$$\begin{matrix} B & G & G & B & B \\ G & G & B & B & B \\ G & B & G & B & B \\ G & B & B & G & B \\ B & G & B & G & B \end{matrix}$$

So, number of favourable ways $= 5 \times 3! \times 2! = 60$

$\therefore$ Required probability $= \dfrac{60}{120} = \dfrac{1}{2}$

Correct Answer: A

85. Let $S = \{1, 2, \ldots, 20\}$. A subset B of S is said to be "nice", if the sum of the elements of B is 203. Then, the probability that a randomly chosen subset of S is "nice", is ____.

A. $\frac{6}{2^{20}}$

B. $\frac{4}{2^{20}}$

C. $\frac{7}{2^{20}}$

D. $\frac{5}{2^{20}}$

Solution:

Number of subset of $S = 2^{20}$

Sum of elements in S is $1 + 2 + \ldots . + 20 = \frac{20(21)}{2} = 210$

$$\left[\because 1 + 2 + \ldots\ldots + n = \frac{n(n+1)}{2}\right]$$

Clearly, the sum of elements of a subset would be 203, if we consider it as follows

$$S - \{7\}, S - \{1, 6\} S - \{2, 5\}, S - \{3, 4\}, S - \{1, 2, 4\}$$

$\therefore$ Number of favourables cases = 5

Hence, required probability $= \dfrac{5}{2^{20}}$

Correct Answer: D

86. Three randomly chosen non-negative integers x, y and z are found to satisfy the equation $x + y + z = 10$. Then the probability that z is even, is ____.

A. $\frac{1}{2}$

B. $\frac{36}{55}$

C. $\frac{6}{11}$

D. $\frac{5}{11}$

Solution: Sample space $\rightarrow {}^{12}C_2$

The number of possibilities for z is even.

$$\begin{aligned}
z=0 &\Rightarrow {}^{11}C_1 \\
z=2 &\Rightarrow {}^{9}C_1 \\
z=4 &\Rightarrow {}^{7}C_1 \\
z=6 &\Rightarrow {}^{5}C_1 \\
z=8 &\Rightarrow {}^{3}C_1 \\
z=10 &\Rightarrow {}^{1}C_1 \\
\text{Total } &= 36 \\
\therefore \text{Required probability } &= \frac{36}{66}=\frac{6}{11}
\end{aligned}$$

Correct Answer: C

87. If 12 identical balls are to be placed in 3 different boxes, then the probability that one of the boxes contains exactly 3 balls, is ______.

A. $\frac{55}{3}\left(\frac{2}{3}\right)^{11}$

B. $55\left(\frac{2}{3}\right)^{10}$

C. $220\left(\frac{1}{3}\right)^{12}$

D. $22\left(\frac{1}{3}\right)^{11}$

Solution: We have mentioned that boxes are different and one particular box has 3 balls. Then, number of ways $=\frac{{}^{12}C_3 \times 2^9}{3^{12}}=\frac{55}{3}\left(\frac{2}{3}\right)^{11}$

Correct Answer: A

88. A group consists of 3 Norwegians, 4 Swedes, and 5 Finns, and they sit at random around a table. What is the probability that all groups end up sitting together?

A. $\frac{3!\cdot 4!\cdot 5!\cdot 2!}{11!}$

B. $\frac{3!\cdot 4!\cdot 5!}{11!}$

C. $\frac{3!\cdot 4!\cdot 5!\cdot 2!}{10!}$

D. $\frac{3!\cdot 4!\cdot 5!\cdot 2!}{12!}$

Solution: The answer is $\frac{3!\cdot 4!\cdot 5!\cdot 2!}{11!}$.

Pick, say, a Norwegian (Arne) and sit him down. Here is how you count the good events. There are 3 ! choices for ordering the group of Norwegians (and then sit them down to one of both sides of Arne, depending on the ordering). Then, there are 4 ! choices for arranging the Swedes and 5 ! choices for arranging the Finns. Finally, there are 2! choices to order the two blocks of Swedes and Finns.

Correct Answer: A

89. A bag has 6 pieces of paper, each with one of the letters, E, E, P, P, P, and R, on it. Pull 6 pieces at random out of the bag without replacement. What is the probability that these pieces, in order, spell PEPPER? (Mark all the appropriate choices)

A. $\frac{3!2!}{6!}$

B. $\frac{1}{60}$

C. $\frac{3!4!}{6!}$

D. $\frac{1}{5}$

Solution: For sampling without replacement:

1. An outcome is an ordering of the pieces of paper $E_1E_2P_1P_2P_3R$.
2. The number of outcomes thus is 6!.
3. The number of good outcomes is 3!2!.

The probability is $\frac{3!2!}{6!} = \frac{1}{60}$.

For sampling with replacement, the answer is $\frac{3^3 \cdot 2^2}{6^6} = \frac{1}{2 \cdot 6^3}$, quite a lot smaller.

Correct Answer: A;B

90. Pick an integer in $[1, 1000]$ at random. The probability that it is divisible neither by 12 nor by 15 is ______. (Upto three digits after decimal) (Numerical Answer Type)

Solution: The sample space consists of the 1000 integers between 1 and 1000 and let A_r be the subset consisting of integers divisible by r. The cardinality of A_r is $\lfloor 1000/r \rfloor$. Another simple fact is that $A_r \cap A_s = A_{\text{lcm}(r,s)}$, where lcm stands for the least common multiple. Our probability equals

$$\begin{aligned} 1 - P(A_{12} \cup A_{15}) &= 1 - P(A_{12}) - P(A_{15}) + P(A_{12} \cap A_{15}) \\ &= 1 - P(A_{12}) - P(A_{15}) + P(A_{60}) \\ &= 1 - \frac{83}{1000} - \frac{66}{1000} + \frac{16}{1000} = 0.867. \end{aligned}$$

Correct Answer: 0.867

3.2 Conditional Probability and Bayes' Theorem

91. Two integers are selected at random from the set $\{1, 2, \ldots, 11\}$. Given that the sum of selected numbers is even, the conditional probability that both the numbers are even is _____.

A. $\frac{2}{5}$

B. $\frac{1}{2}$

C. $\frac{7}{10}$

D. $\frac{3}{5}$

Solution: In $\{1, 2, 3, \ldots, 11\}$ there are 5 even numbers and 6 odd numbers. The sum even is possible only when both are odd or both are even.

Let A be the event that denotes both numbers are even and B be the event that denotes sum of numbers is even.

Then, $n(A) = {}^5C_2$ and $n(B) = {}^5C_2 + {}^6C_2$

Required probability

$$\begin{aligned} P(A/B) = \frac{P(A \cap B)}{P(B)} &= \frac{{}^5C_2/{}^{11}C_2}{\frac{({}^6C_2 + {}^5C_2)}{{}^{11}C_2}} \\ &= \frac{{}^5C_2}{{}^6C_2 + {}^5C_2} = \frac{10}{15 + 10} = \frac{2}{5} \end{aligned}$$

Correct Answer: A

92. Four persons independently solve a certain problem correctly with probabilities $\frac{1}{2}, \frac{3}{4}, \frac{1}{4}, \frac{1}{8}$. Then, the probability that the problem is solved correctly by atleast one of them, is _____.

A. $\frac{235}{256}$

B. $\frac{21}{256}$

C. $\frac{3}{256}$

D. $\frac{253}{256}$

Solution:

It is simple application of independent event, to solve a certain problem or any type of compitition each event in independent of other.

Formula used $P(A \cap B) = P(A) \cdot P(B)$, when A and B are independent events.

Probability that the problem is solved correctly by atleast one of them $=$ $1-$ (problem is not solved by all)

$\therefore \quad P($ problem is solved $) = 1 - P($ problem is not solved $)$

$$= 1 - P(\overline{A}) \cdot P(\overline{B}) \cdot P(\overline{C}) \cdot P(\overline{D})$$
$$= 1 - \left(\frac{1}{2} \cdot \frac{1}{4} \cdot \frac{3}{4} \cdot \frac{7}{8}\right) = 1 - \frac{21}{256} = \frac{235}{256}$$

Correct Answer: A

93. Let X and Y be two events such that $P(X) = \dfrac{1}{3}, P(X/Y) = \dfrac{1}{2}$ and $P(Y/X) = \dfrac{2}{5}$. Then (Mark all the appropriate choices)

A. $P(Y) = \frac{4}{15}$

B. $P(X'/Y) = \frac{1}{2}$

C. $P(X \cup Y) = \frac{7}{15}$

D. $P(X \cap Y) = \frac{2}{15}$

Solution:

$$P(X) = \frac{1}{3}$$
$$P\left(\frac{X}{Y}\right) = \frac{P(X \cap Y)}{P(Y)} = \frac{1}{2}$$
$$P\left(\frac{Y}{X}\right) = \frac{P(X \cap Y)}{P(X)} = \frac{2}{5}$$
$$P(X \cap Y) = \frac{2}{15}$$
$$P(Y) = \frac{4}{15}$$
$$P\left(\frac{X'}{Y}\right) = \frac{P(Y) - P(X \cap Y)}{P(Y)} = \frac{\frac{4}{15} - \frac{2}{15}}{\frac{4}{15}} = \frac{1}{2}$$
$$P(X \cup Y) = \frac{1}{3} + \frac{4}{15} - \frac{2}{15} = \frac{7}{15}$$

Correct Answer: A;B;C;D

94. An examination consists of multiple-choice questions, each having five possible answers. Suppose you are a student taking the exam. and that you reckon you have probability 0.75 of knowing the answer to any question that may be asked and that, if you do not know, you intend to guess an answer with probability 1/5 of being correct. What is the probability you will give the correct answer to a question? (Numerical Answer Type)

Solution:

Let A be the event that you give the correct answer. Let B be the event that you knew the answer. We want to find $\mathrm{P}(A)$. But $\mathrm{P}(A) = \mathrm{P}(A \cap B) + \mathrm{P}(A \cap B^c)$ where $\mathrm{P}(A \cap B) = \mathrm{P}(A \mid B)\mathrm{P}(B) = 1 \times 0.75 = 0.75$ and $\mathrm{P}(A \cap B^c) = \mathrm{P}(A \mid B^c)\mathrm{P}(B^c) = \frac{1}{5} \times 0.25 = 0.05$. Hence $\mathrm{P}(A) = 0.75 + 0.05 = 0.8$.

Correct Answer: 0.8

95. In a certain town, 30% of the people are Conservatives; 50% Socialists; and 20% Liberals. In this town at the last election, 65% of Conservatives voted, as did 82% of the Socialists and 50% of the Liberals. A person from the town is selected at random, and states that she voted at the last election. What is the probability that she is a Socialist? (Upto four places after decimal) (Numerical Answer Type)

Solution: We organise the problem as follows: let C,S and L be the events that a person is Conservative, Socialist, or Liberal respectively. Let V be the event that a person voted in the last election. We require to find $P(S \mid V)$, where the information we are given can be summarised as:

$$P(C) = 0.3, \quad P(S) = 0.5, \quad P(L) = 0.2$$
$$P(V \mid C) = 0.65 \quad P(V \mid S) = 0.82, \quad P(V \mid L) = 0.5$$

Now, by Bayes theorem,

$$P(S \mid V) = \frac{P(V \mid S)P(S)}{P(V)}$$

Each term is known, excepting $P(V)$ which we calculate using the idea of a partition. We can calculate $P(V)$ by associating V with the certain partition $C \cup S \cup L$:

$$\begin{aligned} P(V) &= P(V \cap (C \cup S \cup L)) \\ &= P(VC) + P(VS) + P(VL) \\ &= P(V \mid C)P(C) + P(V \mid S)P(S) + P(V \mid L)P(L) \\ &= (0.65)(0.3) + (0.82)(0.5) + (0.5)(0.2) \\ &= 0.705. \end{aligned}$$

Hence

$$\begin{aligned} P(S \mid V) &= \frac{(0.82)(0.5)}{0.705} \\ &= 0.5816 \end{aligned}$$

Correct Answer: 0.5816

96. Suppose the test for HIV is 99% accurate in both directions and 0.3% of the population is HIV positive. If someone tests positive, what is the probability they actually are HIV positive? (Upto two places after decimal) (Numerical Answer Type)

Solution: Let D is the event that a person is HIV positive, and T is the event that the person tests positive.

$$\mathbb{P}(D \mid T) = \frac{\mathbb{P}(D \cap T)}{\mathbb{P}(T)} = \frac{(0.99)(0.003)}{(0.99)(0.003) + (0.01)(0.997)} \approx 23\%.$$

A short reason why this surprising result holds is that the error in the test is much greater than the percentage of people with HIV. A little longer answer is to suppose that we have 1000 people. On average, 3 of them will be HIV positive and 10 will test positive. So the chances that someone has HIV given that the person tests positive is approximately 3/10. The reason that it is not exactly 0.3 is that there is some chance someone who is positive will test negative.

Suppose you know $\mathbb{P}(E \mid F)$ and you want to find $\mathbb{P}(F \mid E)$. Recall that

$$\mathbb{P}(E \cap F) = \mathbb{P}(E \mid F)\mathbb{P}(F)$$

and so

$$\mathbb{P}(F \mid E) = \frac{\mathbb{P}(F \cap E)}{\mathbb{P}(E)} = \frac{\mathbb{P}(E \mid F)\mathbb{P}(F)}{\mathbb{P}(E)}$$

Correct Answer: 0.23

97. Example 4.6. Suppose 30% of the women in a class received an A on the test and 25% of the men received an A. The class is 60% women. Given that a person chosen at random received an A, what is the probability this person is a women? (Upto three places after decimal) (Numerical Answer Type)

Solution:

Let A be the event of receiving an A, W be the event of being a woman, and M the event of being a man. We are given $\mathbb{P}(A \mid W) = 0.30, \mathbb{P}(A \mid M) = 0.25, \mathbb{P}(W) = 0.60$ and we want $\mathbb{P}(W \mid A)$. From the definition

$$\mathbb{P}(W \mid A) = \frac{\mathbb{P}(W \cap A)}{\mathbb{P}(A)}.$$

As in the previous example,

$$\mathbb{P}(W \cap A) = \mathbb{P}(A \mid W)\mathbb{P}(W) = (0.30)(0.60) = 0.18.$$

To find $\mathbb{P}(A)$, we write

$$\mathbb{P}(A) = \mathbb{P}(W \cap A) + \mathbb{P}(M \cap A).$$

Since the class is 40% men,

$$\mathbb{P}(M \cap A) = \mathbb{P}(A \mid M)\mathbb{P}(M) = (0.25)(0.40) = 0.10.$$

So

$$\mathbb{P}(A) = \mathbb{P}(W \cap A) + \mathbb{P}(M \cap A) = 0.18 + 0.10 = 0.28$$

Finally,

$$\mathbb{P}(W \mid A) = \frac{\mathbb{P}(W \cap A)}{\mathbb{P}(A)} = \frac{0.18}{0.28} = 0.642$$

Correct Answer: 0.642

98. A company makes optical lenses under contract to the US military. The lenes are ground to precise specifications and are shipped in lots of 100. Military inspectors check 2 different lenses on each lot of 100. Let E_1 be the event that the first lens fails the inspection and let E_2 be the event that the second lens fails the inspection. If either lens fails, the entire lot of 100 lenses is returned.

Suppose that, in a particular shipment, 3 lenses are bad. What is the probability that the shipment is rejected? (Upto four places after decimal) (Numerical Answer Type)

Solution: For the first choice, there are 100 lenses total, of which 97 are good. So the probability of choosing a good one is $\Pr\left(E_1'\right) = \frac{97}{100}$.

Given that the first lens chosen was good, there are 99 lenses left, of which 96 are good. So the probability of choosing a good one is $\Pr\left(E_2' \mid E_1'\right) = \frac{96}{99} = \frac{32}{33}$. Therefore the probability that both lenses chosen are good is

$$\Pr(E_1' \cap E_2') = \Pr(E_1')\Pr(E_2' \mid E_1') = \left(\frac{97}{100}\right)\left(\frac{32}{33}\right) = \frac{97 \cdot 8}{25 \cdot 33}.$$

So the probability that the shipment is rejected is

$$\Pr(E_1 \cup E_2) = 1 - \Pr(E_1' \cap E_2') = 1 - \frac{97 \cdot 8}{25 \cdot 33}.$$

(This is about $1 - 0.9406 = 0.0594$, or about 5.94%.)

Correct Answer: 0.0594

99. A factory production line is manufacturing bolts using three machines, A, B and C. Of the total output, machine A is responsible for 25%, machine B for 35%, and machine C for the rest. It is known from previous experience with the machines that 5% of the output from machine A is defective, 4% from machine B and 2% from machine C. A bolt is chosen at random from the production line and found to be defective. What is the probability that it came from Machine A? (Upto three places after decimal) (Numerical Answer Type)

Solution: Denote by D the event that a bolt is defective, A the event that a bolt is from machine A, and by B the event that a bolt is from machine C. Then by Bayes' theorem

$$\begin{aligned}\mathbb{P}(A \mid D) &= \frac{\mathbb{P}(D \mid A)\mathbb{P}(A)}{\mathbb{P}(D \mid A)\mathbb{P}(A) + \mathbb{P}(D \mid B)\mathbb{P}(B) + \mathbb{P}(D \mid C)\mathbb{P}(C)} \\ &= \frac{(0.05)(0.25)}{(0.05)(0.25) + (0.04)(0.35) + (0.02)(0.4)} = 0.362\end{aligned}$$

Correct Answer: 0.362

100. Suppose we have 3 cards identical in form except that both sides of the first card are colored red, both sides of the second card are colored black, and one side of the third card is colored red and the other side is colored black.

The 3 cards are mixed up in a hat, and 1 card is randomly selected and put down on the ground. If the upper side of the chosen card is colored red, what is the probability that the other side is colored black? (Upto three places after decimal) (Numerical Answer Type)

Solution:

Let RR, BB, and RB denote, respectively, the events that the chosen cars is the red-red, the black-black, or the red-black card. Letting R be the event that the upturned side of the chosen card is red, we have that the desired probability is obtained by

$$\begin{aligned} P(RB \mid R) &= \frac{P(RB \cap R)}{P(R)} \\ &= \frac{P(R \mid RB)P(RB)}{P(R \mid RR)P(RR) + P(R \mid RB)P(RB) + P(R \mid BB)P(BB)} \\ &= \frac{\left(\frac{1}{2}\right)\left(\frac{1}{3}\right)}{(1)\left(\frac{1}{3}\right) + \left(\frac{1}{2}\right)\left(\frac{1}{3}\right) + 0\left(\frac{1}{3}\right)} = \frac{1}{3} = 0.333 \end{aligned}$$

This question was actually just like the Monty Hall problem!

Correct Answer: 0.333

3.3 Random Variables

101. Varsha lives alone and dislikes cooking, so she goes out for dinner every evening. She has two favourite restaurants, Dosa Paradise and Kababs Unlimited, to which she travels by local train. The train to Dosa Paradise runs every 10 minutes, at $0, 10, 20, 30, 40$ and 50 minutes past the hour. The train to Kababs Unlimited runs every 20 minutes, at $8, 28$ and 48 minutes past the hour. She reaches the station at a random time between $7:15$ pm and $8:15$ pm and chooses between the two restaurants based on the next available train/ What is the probability that she ends up eating in Kababs Unlimited?

A. $\frac{1}{5}$

B. $\frac{1}{3}$

C. $\frac{2}{5}$

D. $\frac{1}{2}$

Solution:

If she arrives in the intervals $7:20-7:28, 7:40-7:48$, or $8:00-8:08$, the next train goes to Kababs Unlimited. This adds up to 24 minutes, out of a total of 60 minutes, so $\frac{24}{60} = \frac{2}{5}$

Correct Answer: C

102. A permutation of $1, 2, \ldots, n$ is chosen at random. Then the probability that the numbers 1 and 2 appear as neighbour equals

A. $\frac{1}{n}$

B. $\frac{2}{n}$

C. $\frac{1}{n-1}$

D. $\frac{1}{n-2}$

Solution:

The number of integers are **n**

Total number of arrangements possible = **n**!

Now, let's bundle 1 and 2 together, then number of arrangements possible = (n − 1)!2! ways.

$\because$ Total elements now are $n-1$ and one bundle is having 2 elements which can be arranged in 2! ways.

So, favorable cases = $(n-1)!2!$

So, total cases possible, i.e., sample space = $n!$

$\therefore$ Probability $= \frac{(n-1)!2!}{n!} = \frac{(n-1)!2!}{n(n-1)!} = \frac{2}{n}$

Correct Answer: B

103. A coin is such that after every toss the probability of same side coming again increases by 50% from the initial value. If initially the probability of head and tail are the same, what is the expected number of tosses until we get a head? (Numerical Answer Type)

Solution:

$E_{\text{tosses}} = 0.5 \times 1 + 0.5 \times (1 + E_{HaT})$

$\implies E_{\text{tosses}} = 1 + 0.5E_{HaT}$

$E_{HaT} = 0.25 \times 1 + 0.75 \times (1 + E_{HaT})$

$\implies E_{HaT} = 1 + 0.75E_{HaT}$

$\implies E_{HaT} = \frac{1}{0.25} = 4$

$\therefore E_{\text{tosses}} = 1 + 0.5 \times 4 = 3$

So, we must toss the coin at least 3 times to expect a Head.

Correct Answer: 3

104. Suppose X_i for $i = 1,2,3$ are independent and identically distributed random variables whose probability mass functions are $Pr[X_i = 0] = Pr[X_i = 1] = \frac{1}{2}$ for $i = 1,2,3$. Define another random variable $Y = X_1X_2 \oplus X_3$, where $\oplus$ denotes XOR. Then $Pr[Y = 0 \mid X_3 = 0] =$ ______. (Numerical Answer Type)

Solution:

As $X_3 = 0$ is given, to have Y = 0, X_1X_2 should be 0, meaning (X_1, X_2) should be one of $\{(0,0)(0,1)(1,0)\}$

So, required probability = $3 \times \frac{1}{2} \times \frac{1}{2} = 0.75$ $\because$ we can choose any of the 3 possibilities in 3 ways and then probability of each set of two combination is $\frac{1}{2} \times \frac{1}{2}$.

We can also do like follows:

There are totally 4 possibilities - $\{(0,0)(0,1)(1,0),(1,1)\}$, out of which 3 are favourable cases.

So, required probability = $\frac{3}{4} = 0.75$.

Correct Answer: 0.75

105. If the difference between the expectation of the square of a random variable $\left(E\left[X^2\right]\right)$ and the square of the expectation of the random variable $(E[X])^2$ is denoted by R, then

A. $R = 0$

B. $R < 0$

C. $R \geq 0$

D. $R > 0$

Solution:

The difference between $(E[X])$ and $(E[X])^2$ is called variance of a random variable. Variance measures how far a set of numbers is spread out. (A variance of zero indicates that all the values are identical.) A non-zero variance is always positive.

Correct Answer: C

3.4 Probability Distributions

106. A fair coin is flipped 10 times. What is the probability that it lands on heads the same number of times that it lands on tails? (Upto three decimal places) (Numerical Answer Type)

Solution:

We can imagine the different ways to obtain 5 heads and 5 tails as a permutation of 10 elements (10 throws) with two sets of identical elements (5 heads and 5 tails):

$$\frac{10!}{5! * 5!} = \binom{10}{5} = 252$$

The total number of combinations of 10 coin throws (2 possibilities: head or tail) is

$$2^{10} = 1024$$

So the probability is

$$\frac{252}{1024} = 0.246$$

Second Method:

By the binomial distribution theorem: $\dfrac{10!}{5!(10-5)!} \times 0.5^2 \times 0.5^2 = 0.246$

Correct Answer: 0.246

107. I am picking cards out of a deck. What is the probability that I pull out 2 kings out of 8 cards if I pull with replacement?

A. $\binom{8}{2}\left(\frac{1}{11}\right)^2 \cdot \left(\frac{12}{13}\right)^6$

B. $\binom{8}{2}\left(\frac{1}{13}\right)^2 \cdot \left(\frac{11}{13}\right)^6$

C. $\binom{8}{2}\left(\frac{1}{13}\right)^2 \cdot \left(\frac{12}{13}\right)^6$

D. $\binom{8}{2}\left(\frac{1}{12}\right)^2 \cdot \left(\frac{12}{13}\right)^6$

Solution: With replacement is repeated Bernoulli trials which means binomial distribution. The probability of a success or pulling out a heart is $\frac{1}{13}$. Therefore, the probability of pulling 2 kings out of 8 is

$$\binom{8}{2}\left(\frac{1}{13}\right)^2 \cdot \left(\frac{12}{13}\right)^{8-2}.$$

$$\binom{8}{2}\left(\frac{1}{13}\right)^2 \cdot \left(\frac{12}{13}\right)^{6}.$$

If do not have replacement, then this is a hyper-geometric distribution with $N = 52, n = 8, m = 4$, the the answer is

$$\frac{\binom{4}{2}\binom{48}{6}}{\binom{52}{8}}$$

Correct Answer: C

108. Consider two independent and identically distributed random variables X and Y uniformly distributed in $[0,1]$. For $\alpha \in [0,1]$, the probability that $\alpha \max(X,Y) < XY$ is ______.

A. $1/(2\alpha)$

B. $\exp(1-\alpha)$

C. $(1-\alpha)^2$
D. $1-\alpha^2$
Solution:
$P(\alpha \max(X,Y) < XY)$
$= P(\alpha X < XY \text{ AND } \alpha Y < XY)$
$= P(\alpha < Y \text{ AND } \alpha < X)$
$= P(\alpha < Y) \times P(\alpha < X)$ (Since, X and Y are independent)
$= (P(Y) - P(\alpha)) \times (P(X) - P(\alpha))$
$= (1-\alpha) \times (1-\alpha)$ (Uniform distribution in the interval 0 to 1)
$= (1-\alpha)^2$
Correct Answer: D

109. A stick of length 2024 m is broken into two pieces, at a randomly chosen break point. The expected length of the shorter piece (in meters) is ________. (Numerical Answer Type)
Solution:
Let the length of a stick be L meters.
When we break the stick into two pieces at a point x, we will be left with one piece of length x and a second piece of length $L-x$.
Since their lengths add up to L, whatever be the choice of x, one piece will always be less than $\frac{L}{2}$, and the other greater than $\frac{L}{2}$.
So, the smaller piece will have length ranging from 0 to $\frac{L}{2}$.
If the break point x is uniformly distributed throughout the length of the stick, then any point between 0 and $\frac{L}{2}$ is equally likely and their mean gives $\frac{L}{4}$.
$\therefore$ The expected (average) length of the smaller piece $= \frac{L}{4} = \frac{2024}{4} = 506$ meters.
Correct Answer: 506

110. Suppose the number of robberies in a neighbourhood can be modelled using a Poisson distribution. The probability that no robberies occur in a week is 0.135 . Given that the mean number of robberies per week is an integer, What is the probability that there are fewer than four robberies in week? (Numerical Answer Type)
Solution:
Let R be the random variable denoting the number of robberies in a week. Let the mean number of robberies be λ. Then, R follows a Poisson distribution with mean λ. We are given that $P(R = 0) = 0.135$. Then,

$$\begin{aligned}
0.135 &= P(R=0) \\
0.135 &= e^{-\lambda}\frac{\lambda^0}{0!} \\
0.135 &= e^{-\lambda} \\
\ln(0.135) &= -\lambda \\
\lambda &= -\ln(0.135) \approx 2
\end{aligned}$$

Hence, $\lambda = 2$. Thus,

$$\begin{aligned}
P(R<4) &= P(R=0) + P(R=1) + P(R=2) + P(R=3) \\
&= e^{-2} + e^{-2}\frac{(2)^1}{1!} + e^{-2}\frac{(2)^2}{2!} + e^{-2}\frac{(2)^3}{3!} \\
&= e^{-2}\left(1 + 2 + \frac{4}{2} + \frac{8}{6}\right) \\
&\approx 0.857.
\end{aligned}$$

Correct Answer: 0.857

111. Alice runs a stall at a fete in which the player is guaranteed to win \$10. Players pay a certain amount each time they throw a dice and must keep throwing the dice until a four occurs. When a four is obtained, Alice gives the player \$10. On average, Alice expects to make a profit of \$2 per game. How much does she charge for each throw? (Numerical Answer Type)

Solution:

Let \$$C$ be the charge per throw. Since the game continues until a four turns up, the number of unsuccessful throws follows a geometric distribution. The probability of success (four shows up) is $q = \frac{1}{6}$ and the probability of failure is $p = \frac{5}{6}$. The expected number of unsuccessful throws then is

$$\frac{p}{q} = \frac{5/6}{1/6} = 5$$

Including the one successful throw, on average a player therefore takes six throws. The average profit per game is given by $6C - 10$ which equals 2 ; solving for C gives

$$6C - 10 = 2 \Rightarrow C = 2.$$

Alice charges \$2 per throw.

Correct Answer: 2

112. The time taken by a food delivery firm to deliver an order is exponentially distributed. If the firm has r employees, the average delivery time is $\frac{100}{r}$. What is the least number of employees the firm should employ to ensure that more than 99% of the orders are delivered within 30 minutes?

Solution:

Let X be the random variable representing the time to deliver an order. Then X is exponentially distributed with density function $f(x) = ke^{-kx}$ where x denotes time and k satisfies the mean equation

$$\frac{1}{k} = \frac{100}{r} \Rightarrow k = \frac{r}{100}$$

Thus, the density function is $f(x) = \frac{r}{100}e^{-rx/100}$ where r, the number of employees, is yet to be determined. We require more than 99% of the orders to be delivered within 30 minutes. This means that the probability $P(X < 30)$ must be greater than 99% :

$$\begin{aligned}
P(X < 30) &> 0.99 \\
\int_0^{30} \frac{r}{100} e^{-rx/100} dx &> 0.99 \\
\frac{r}{100} \left[\frac{e^{-rx/100}}{-r/100} \right]_0^{30} &> 0.99 \\
\left[-e^{-rx/100} \right]_0^{30} &> 0.99 \\
\left[-e^{-30r/100} + 1 \right] &> 0.99 \\
e^{-3r/10} &< 1 - 0.99 \\
-\frac{3r}{10} &< \ln(0.01) \\
r &> \frac{-10 \ln(0.01)}{3} \approx 15.4.
\end{aligned}$$

Hence, the number of employees should be greater than 15 . At the very least, 16 employees should be hired to ensure that more than 99% of the orders are delivered within 30 minutes.

Correct Answer: 16

113. The number of chocolate chips in a cookie is Poisson distributed with an average of 15 chocolate chips. What is the probability that you pick up a cookie with only 10 chocolate chips in it?

A. $\frac{15^{10}e^{-10}}{10!}$

B. $\frac{15^{10}e^{-10}}{15!}$

C. $\frac{10^{15}e^{-15}}{10!}$

D. $\frac{15^{10}e^{-15}}{10!}$

Solution: This is a Poisson distribution with $\lambda = 15$.

We want to calculate $f(10) = \dfrac{\lambda^{10}e^{-\lambda}}{10!} = \dfrac{15^{10}e^{-15}}{10!}$.

Correct Answer: D

114. The number of errors on a page is Poisson distributed with approximately 1 error per 50 pages of a book. What is the probability that a novel of 300 pages contains at most 1 error?

A. $7e^{-6}$

B. $6e^{-7}$

C. $2e^{-3}$

D. $3e^{-2}$

Solution: If we average 1 error per 50 pages, then over a novel of 300 pages, we should expect $300/50 \cdot 1 = 6$ errors.

Thus, the probability of having at most one error with $\lambda = 6$ is $f(0) + f(1) = \frac{\lambda^0 e^{-\lambda}}{0!} + \frac{\lambda^1 e^{-\lambda}}{1!} = e^{-6} + 6e^{-6} = 7e^{-6}$.

Correct Answer: A

115. We roll two fair 6 sided die. What is the expected value of their product? (Numerical Answer Type)

Solution: Let X be the first value I roll and Y be the second. The rolls are independent and so $E[XY] = E[X]E[Y] = 3.5 \cdot 3.5 = 12.25$.

Correct Answer: 12.25

116. While pulling out of a box of cookies, what is the expected number of cookies I have to pull out before I pull out an oatmeal raisin if 25% of cookies are oatmeal raisin and I choose with replacement? (Numerical Answer Type)

Solution: This is a geometric distribution because I am counting the number of cookies I have to pull out before a success. The probability of success is $25\% = p = 1/4$.

So the expected number of cookies I have to pull out is $\dfrac{1-p}{p} + 1 = 3 + 1 = 4$.

The variance is $\dfrac{1-p}{p^2} = 3/(1/4) = 12$.

Correct Answer: 4

117. Cartons of milk at a supermarket are advertised as containing 1 litre, but in fact the volume is normally distributed with a mean of 1012ml and a standard deviation of 5 ml.

What is the probability that a randomly chosen carton contains more than 1010ml? (Upto four decimal places) (Numerical Answer Type)

Solution:

Let M be the random variable representing the quantity of milk in a carton (in ml). Then M is normally distributed with mean 1012 and standard deviation 5 (i.e. $M \sim N\left(1012, 5^2\right)$). The standard normal random variable is related to M by $Z = \frac{M-1012}{5}$.

Hence,

$$\begin{aligned} P(M > 1010) &= P\left(Z > \frac{1010-1012}{5}\right) \\ &= P(Z > -0.4) \\ &= 0.5 + A(0.4) \\ &= 0.5 + 0.1554 = 0.6554 \end{aligned}$$

Correct Answer: 0.6554

118. The manufacturers of a new model of car claim that it gives an average mileage of 32.4 miles per gallon with standard deviation 1.4 miles per gallon. Assuming a normal distribution, what is the probability that a randomly chosen car of that model will give a mileage of less than 30 miles per gallon? (Upto four decimal places) (Numerical Answer Type)

Solution:

Let M be the random variable representing the mileage. Then $M \sim N\left(32.4, 1.4^2\right)$ and M is related to the standard normal random variable Z by $Z = \frac{M-32.4}{1.4}$. Then,

$$\begin{aligned} P(M < 30) &= P\left(Z < \frac{30-32.4}{1.4}\right) \\ &= P(Z < -1.71) \\ &= 0.5 - A(1.71) \\ &= 0.5 - 0.4564 = 0.0436. \end{aligned}$$

Correct Answer: 0.0436

119. Cartons of milk at a supermarket are advertised as containing 1 litre, but in fact the volume is normally distributed with a mean of 1012 ml and a standard deviation of 5 ml.

What is the probability that a randomly chosen carton contains less than the advertised quantity of milk? (Upto four decimal places) (Numerical Answer Type)

Solution:

We require

$$\begin{aligned} P(M < 1000) &= P\left(Z < \frac{1000-1012}{5}\right) \\ &= P(Z < -2.4) \\ &= 0.5 - A(2.4) \\ &= 0.5 - 0.4918 = 0.0082. \end{aligned}$$

Correct Answer: 0.0082

120. Suppose X is normal with mean 6. If $\mathbb{P}(X > 16) = 0.0228$, then what is the standard deviation of X? ($\Phi(2) = 0.9772$) (Numerical Answer Type)

Solution: We know that, $\frac{X-\mu}{\sigma} = Z \sim \mathcal{N}(0,1)$ and then

$$\begin{aligned} \mathbb{P}(X > 16) = 0.0228 &\Longleftrightarrow \mathbb{P}\left(\frac{X-6}{\sigma} > \frac{16-6}{\sigma}\right) = 0.0228 \\ &\Longleftrightarrow \mathbb{P}\left(Z > \frac{10}{\sigma}\right) = 0.0228 \\ &\Longleftrightarrow 1 - \mathbb{P}\left(Z \leqslant \frac{10}{\sigma}\right) = 0.0228 \\ &\Longleftrightarrow 1 - \Phi\left(\frac{10}{\sigma}\right) = 0.0228 \\ &\Longleftrightarrow \Phi\left(\frac{10}{\sigma}\right) = 0.9772 \end{aligned}$$

Using the standard normal table we see that $\Phi(2) = 0.9772$, thus we have that

$$2 = \frac{10}{\sigma}$$

and hence $\sigma = 5$.

Correct Answer: 5

121. Website Visits: On average, visitors leave your website after 5 minutes. Assume that the length of stay is exponentially distributed. What is the probability that a user stays more than 10 minutes? (Upto four decimal places) (Numerical Answer Type)

Solution:

Let X be the number of minutes that a user stays. $X \sim \text{Exp}\left(\lambda = \frac{1}{5}\right)$.

$$\begin{aligned} P(X > 10) &= 1 - F_X(10) \\ &= 1 - \left(1 - e^{\lambda 10}\right) = e^{-2} \approx 0.1353 \end{aligned}$$

Correct Answer: 0.1353

122. Approximating Normal: Your website has 100 users and each day each user independently has a 20% chance of logging into your website. Use a normal approximation to estimate the probability that more than 21 users log in. ($\phi(0.375) = 0.6462$) (Upto four decimal places) (Numerical Answer Type)

Solution:

The number of users that log in B is binomial: $B \sim \text{Bin}(n = 100, p = 0.2)$. It can be approximated with a normal that matches the mean and variance. Let C be the normal that approximates B. We have $E[B] = np = 20$ and $\text{Var}(B) = np(1-p) = 16$, so $C \sim N\left(\mu = 20, \sigma^2 = 16\right)$. Note that because we are approximating a discrete value with a continuous random variable, we need to use the continuity correction:

$$\begin{aligned} P(B > 21) &\approx P(C > 21.5) \\ &= P\left(\frac{C - 20}{\sqrt{16}} > \frac{21.5 - 20}{\sqrt{16}}\right) \\ &= P(Z > 0.375) \\ &= 1 - P(Z < 0.375) \\ &= 1 - \phi(0.375) = 1 - 0.6462 = 0.3538 \end{aligned}$$

Correct Answer: 0.3538

123. There are **7184** undergrads at Stanford. Each student has a 13% probability of taking CS106A at some point in their four years here. Let X be the number of students who take CS106A at Stanford. What is the approximate probability that more than 900 of these students will take CS106A? (Upto three decimal places) (Numerical Answer Type)

Solution:

Since n is large, we use the normal approximation to the binomial. We know that $X \sim \text{Bin}(7184, 0.13)$.

Therefore, $E[X] = np = 933.92$ and $\text{Var}(X) = np(1-p) = 812.51$

We therefore approximate X as $Y \sim N(933.92, 812.51)$:

$$P(X > 700) \approx P(Y > 700.5) = P\left(Z > \frac{900.5 - 933.92}{\sqrt{812.51}}\right) = P(Z > -1.17) = P(Z < 1.17) = 0.879$$

Correct Answer: 0.879

124. We have an array of Boolean variables, where n of them are 0 and m of them are 1. We remove them from the bag in a randomly chosen order. What is the expected number of instances in which a 0 is immediately followed by a 1?

A. $\frac{nm}{n-m}$

B. $\frac{nm}{n+m}$

C. $\frac{n+m}{nm}$

D. $\frac{n-m}{n+m}$

Solution:

Let us define an indicator variable I_j as follows:

$$I_j = \begin{cases} 1 & \text{the } j^{th} \text{ variable is 0 and the } (j+1)^{th} \text{ variable is 1} \\ 0 & \text{otherwise} \end{cases}$$

Thus, we are trying to find:

$$X = \sum_{j=1}^{n+m-1} I_j$$

We know that the expected value of the sum is the sum of the expected value. Therefore:

$$E[X] = E\left[\sum_{j=1}^{n+m-1} I_j\right] = \sum_{j=1}^{n+m-1} E\left[I_j\right]$$

For an indicator variable, the expected value is p. Therefore:

$$E\left[I_j\right] = P\Big(\text{ the } j^{th} \text{ variable is 0 and the } (j+1)^{th} \text{ variable is 1}\Big)$$

$$= P\Big(\text{ the } (j+1)^{th} \text{ variable is 1} \mid \text{ the } j^{th} \text{ variable is 0}\Big) P\Big(\text{ the } j^{th} \text{ variable is 0}\Big)$$

$$P\Big(\text{ the } j^{th} \text{ variable is 0}\Big) = \frac{n}{n+m}$$

$$P\Big(\text{ the } (j+1)^{th} \text{ variable is 1}\Big) = \frac{m}{n+m-1}$$

Therefore:

$$E[X] = \sum_{j=1}^{n+m-1} \frac{n}{n+m} \frac{m}{n+m-1} = \frac{nm}{n+m}$$

Correct Answer: B

125. Let X be a random variable following normal distribution with mean $+1$ and variance 4. Let Y be another normal variable with mean -1 and variance unknown. If $P(X \le -1) = P(Y \ge 2)$, the standard deviation of Y is ______.

A. 3

B. 2

C. $\sqrt{2}$

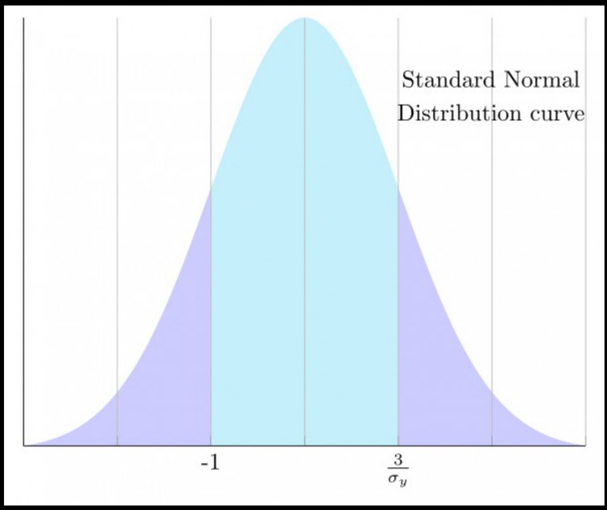
D. 1

Solution:

$P(X \le -1) = P(Y \ge 2)$ We can compare their values using standard normal distributions:

X	Y
$\mu_X = +1$	$\mu_Y = -1$
$\sigma_X{}^2 = 4$	$\sigma_Y{}^2 = ?$
$Z_X = \frac{X-1}{\sqrt{4}}$	$Z_Y = \frac{Y-(-1)}{\sigma_Y}$
$2Z_X + 1 = X$	$Y = \sigma_Y Z_Y - 1$

$$\implies P(2Z_X + 1 \le -1) = P(\sigma_Y Z_Y - 1 \ge 2) \implies P(Z_X \le -1) = P(Z_Y \ge \frac{3}{\sigma_Y})$$



$\implies -(-1) = \frac{3}{\sigma_Y}$

$\implies \sigma_Y = 3$

Correct Answer: A

3.5 Statistics

126. If the variance of the first n natural numbers is 10 and the variance of the first m even natural numbers is 16 , then $m+n$ is equal to _____. (Numerical Answer Type)

Solution:

$$\text{Var}(1,2,\ldots\ldots\ldots,\text{n}) = 10 \Rightarrow \frac{1^2+2^2+\ldots\ldots+n^2}{n} - \left(\frac{1+2+\ldots.+n}{n}\right)$$

$$\Rightarrow \frac{(n+1)(2n+1)}{6} - \left(\frac{n+1}{2}\right)^2 = 10$$

$$\Rightarrow n^2 - 1 = 120 \Rightarrow n = 11$$

$$\text{Var}(2,4,6,\ldots\ldots\ldots,2\text{ m}) = 16 \Rightarrow \text{Var}(1,2,\ldots\ldots\ldots,\text{m}) = 4$$

$$\Rightarrow \text{m}^2 - 1 = 48 \Rightarrow m = 7 \Rightarrow m+n = 18$$

Correct Answer: 18

127. 5 students of a class have an average height 150 cm and variance 18 cm^2. A new student, whose height is 156 cm, joined them. The variance (in cm^2) of the height of these six students is ______.

A. 16

B. 22

C. 20

D. 18

Solution:

Let 5 students are x_1, x_2, x_3, x_4, x_5

Given $\bar{x} = \frac{\sum x_i}{5} = 150$

$$\Rightarrow \sum_{i=1}^{5} = 750 \quad \longrightarrow (1)$$

$\frac{\sum x_i^2}{5} - (\bar{x})^2 = 18 \Rightarrow \frac{\sum x_i^2}{5} - (150)^2 = 18 \Rightarrow \sum x_i^2 = (22500+18) \times 5$

$$\Rightarrow \sum_{i=1}^{5} x_i^2 = 112590 \quad \longrightarrow (2)$$

Height of new student = 156 (Let x_6)

Then $x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 750 + 156$

$\overline{x}_{\text{new}} = \dfrac{x_1 + x_2 + x_3 + x_4 + x_5 + x_6}{6} = \frac{906}{6} = 151 \quad \longrightarrow (3)$

Variance (new) = $\frac{\sum x_i^2}{6} - (\overline{x}_{new})^2$

From equation (2) and (3)

Variance (new) = $\frac{112590+(156)^2}{6} - (151)^2 = 2281 - 22801 = 20$

Correct Answer: C

128. If the standard deviation of the numbers $-1, 0, 1, k$ is $\sqrt{5}$ where $k > 0$, then k is equal to ____.

A. $4\sqrt{\frac{5}{3}}$

B. $\sqrt{6}$

C. $2\sqrt{6}$

D. $2\sqrt{\frac{10}{3}}$

Solution:

$$S.D = \sqrt{\frac{\sum(x-\overline{x})^2}{n}}$$

$$\overline{x} = \frac{\sum x}{4} = \frac{-1+0+1+k}{4} = \frac{k}{4}$$

$$\text{Now } \sqrt{5} = \sqrt{\frac{\left(1-\frac{k}{4}\right)^2 + \left(0-\frac{k}{4}\right)^2 + \left(1-\frac{k}{4}\right)^2 + \left(k-\frac{k}{4}\right)^2}{4}}$$

$$\Rightarrow 5 \times 4 = 2\left(1 + \frac{k}{16}\right)^2 + \frac{5k^2}{8}$$

$$\Rightarrow 18 = \frac{3k^2}{4}$$

$$\Rightarrow k^2 = 24$$

$$\Rightarrow k = 2\sqrt{6}$$

Correct Answer: C

129. A bag contains 30 white balls and 10 red balls. 16 balls are drawn one by one randomly from the bag with replacement. If X be the number of white balls drawn, then $\left(\frac{\text{mean(x)}}{\text{standard deviation (x)}}\right)$ is _____.

A. 4

B. $4\sqrt{3}$

C. $3\sqrt{2}$

D. $\frac{4\sqrt{3}}{3}$

Solution:

There are 30 white balls and 10 red balls

$$\text{P(White ball)} = \frac{30}{40} = \frac{3}{4} = p$$

$$\Rightarrow q = \frac{1}{4}$$

$$\frac{\text{mean (x)}}{\text{standard deviation (x)}} = \frac{np}{\sqrt{npq}}$$

$$= \sqrt{\frac{np}{q}} = \sqrt{\frac{16 \times \left(\frac{3}{4}\right)}{\frac{1}{4}}} = 4\sqrt{3}$$

Correct Answer: B

130. The mean and the variance of five observations are 4 and 5.20 , respectively. If three of the observations are 3,4 and 4; then the absolute value of the difference of the other two observations, is ______.

A. 7

B. 5

C. 1

D. 3

Solution:

Mean $\overline{x} = 4, \sigma^2 = 5.2, n = 5, x_1 = 3x_2 = 4 = x_3$

$\sum x_i = 20$

$x_4 + x_5 = 9 \quad \longrightarrow (1)$

$\frac{\sum x_i^2}{x} - (\overline{x})^e = \sigma^2 \Rightarrow \sum x_i^2 = 106$

$x_2^4 + x_5^2 = 65 \quad \longrightarrow (2)$

Using (1) and (2) $(x_4 - x_5)^2 = 49$

$|x_4 - x_5| = 7$

Correct Answer: A

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