Project Notes

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Contents

1 Shape Functions

$$N_1 = \frac{1}{4}(1+s)(1+t),$$

$$N_2 = \frac{1}{4}(1-s)(1+t),$$

$$N_3' = \frac{1}{2}(1-t)$$

2 $x(s,t)$ and $y(s,t)$

$$\begin{aligned}
x(s,t) &= \frac{1}{4}(1+s)(1+t)x_1 + \frac{1}{4}(1-s)(1+t)x_2 + \frac{1}{2}(1-t)x_3, \\
&= \frac{1}{4}(1+s+t+st)x_1 + \frac{1}{4}(1-s+t-st)x_2 + \frac{1}{2}(1-t)x_3, \\
&= \frac{1}{4}(x_1+x_2+2x_3) + \frac{1}{4}(x_1-x_2)s + \frac{1}{4}(x_1+x_2-2x_3)t + \frac{1}{4}(x_1-x_2)st, \\
y(s,t) &= \frac{1}{4}(1+s)(1+t)y_1 + \frac{1}{4}(1-s)(1+t)y_2 + \frac{1}{2}(1-t)y_3. \\
&= \frac{1}{4}(1+s+t+st)y_1 + \frac{1}{4}(1-s+t-st)y_2 + \frac{1}{2}(1-t)y_3, \\
&= \frac{1}{4}(y_1+y_2+2y_3) + \frac{1}{4}(y_1-y_2)s + \frac{1}{4}(y_1+y_2-2y_3)t + \frac{1}{4}(y_1-y_2)st
\end{aligned}$$

3 $x(s,t)$ and $y(s,t)$ Derivatives

$$\begin{aligned}
\frac{\partial x}{\partial s} &= \frac{1}{4}(1+t)x_1 - \frac{1}{4}(1+t)x_2, \\
&= \frac{1}{4}(x_1-x_2) + \frac{1}{4}(x_1-x_2)t, \\
\frac{\partial x}{\partial t} &= \frac{1}{4}(1+s)x_1 + \frac{1}{4}(1-s)x_2 - \frac{1}{2}x_3, \\
&= \frac{1}{4}(x_1+x_2-2x_3) + \frac{1}{4}(x_1-x_2)s, \\
\frac{\partial y}{\partial s} &= \frac{1}{4}(1+t)y_1 - \frac{1}{4}(1+t)y_2, \\
&= \frac{1}{4}(y_1-y_2) + \frac{1}{4}(y_1-y_2)t, \\
\frac{\partial y}{\partial t} &= \frac{1}{4}(1+s)y_1 + \frac{1}{4}(1-s)y_2 - \frac{1}{2}y_3, \\
&= \frac{1}{4}(y_1+y_2-2y_3) + \frac{1}{4}(y_1-y_2)s
\end{aligned}$$

4 Jacobian

$$\boldsymbol{J} = \begin{bmatrix} \frac{1}{4}(x_1-x_2) + \frac{1}{4}(x_1-x_2)t & \frac{1}{4}(y_1-y_2) + \frac{1}{4}(y_1-y_2)t \\ \frac{1}{4}(x_1+x_2-2x_3) + \frac{1}{4}(x_1-x_2)s & \frac{1}{4}(y_1+y_2-2y_3) + \frac{1}{4}(y_1-y_2)s \end{bmatrix}$$

5 Jacobian Determinant

$$\frac{\partial x}{\partial s}\frac{\partial y}{\partial t} - \frac{\partial x}{\partial t}\frac{\partial y}{\partial s} = \frac{1}{8}(1+t)\left(x_3(y_1-y_2) + x_1(y_2-y_3) + x_2(y_3-y_1)\right)$$