Practical I
Contravariant and Covariant Indices in Tensors (Symbolic)

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Aim

To understand and demonstrate the manipulation of contravariant and covariant indices in tensors using symbolic computation, focusing on the Schwarzschild metric, Christoffel symbols, and Riemann curvature tensors.

Theory

I. Tensors

The surface brightness of a galaxy follows the Sersic profile, and is characterized by the Sersic index. It indicates how the surface brightness of a galaxy falls off as a function of distance from the centre. Below are some sample profiles.

II. Indices

1. Covariant indices (subscripts): Indices that transform with the basis vectors. They represent the components of the gradient of a scalar field.
2. Contravariant indices (superscripts): Indices that transform inversely with the basis vectors. They represent the components of a vector field.

III. Schwarzschild Metric

A solution to Einstein's field equations that describes the gravitational field outside a spherical mass.

IV. Christoffel Symbols

Mathematical objects representing the connection coefficients in a curved space. They help define how vectors change as they are parallel transported.

V. Riemann Curvature Tensor

Describes the curvature of a manifold and how much it deviates from being flat. It can be derived from the Christoffel symbols.

The Schwarzschild metric describes the spacetime around a non-rotating, uncharged massive object. In spherical coordinates (t, r, θ, ϕ), it is given by:

$$ds^2 = -\left(1 - \frac{r_s}{r}\right) c^2 dt^2 + \left(1 - \frac{r_s}{r}\right)^{-1} dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2 \tag{1}$$

Procedure

Initialize SymPy and Load Schwarzschild Metric

1. Initialize symbolic printing with `sympy.init_printing()`.
2. Load the Schwarzschild metric using `Schwarzschild()` and display its tensor representation.
3. Calculate the inverse of the Schwarzschild metric and display it.

Compute Christoffel Symbols

1. Calculate the Christoffel symbols from the inverse metric using `ChristoffelSymbols.from_metric(sch_inv)` and display them.
2. Change the configuration of the Christoffel symbols to different combinations of covariant and contravariant indices and display the results.

Compute Riemann Curvature Tensor

1. Obtain the Riemann curvature tensor from the Christoffel symbols.
2. Display specific components of the Riemann tensor and change its configuration to various index types.
3. Simplify the Riemann tensor and verify if it remains consistent across different configurations.

Python Code

```python
import sympy
from einsteinpy.symbolic import ChristoffelSymbols, RiemannCurvatureTensor
from einsteinpy.symbolic.predefined import Schwarzschild

sympy.init_printing()

sch = Schwarzschild()
sch.tensor()

sch_inv = sch.inv()
sch_inv.tensor()

sch.order
sch.config
chr.config

new_chr = chr.change_config('lll')  # changing the configuration to (covariant, covariant,
    covariant)
new_chr.tensor()
new_chr.config

rm = RiemannCurvatureTensor.from_christoffels(new_chr)
rm[0,0,:,:]
rm.config

rm2 = rm.change_config("uuuu")
rm2[0,0,:,:]

rm3 = rm2.change_config("lulu")
rm3[0,0,:,:]

rm4 = rm3.change_config("ulll")
rm4.simplify()
rm4[0,0,:,:]
```

Outputs

```
Schwarzschild Metric Tensor:
```

$$\begin{bmatrix} 1-\frac{r_s}{r} & 0 & 0 & 0 \\ 0 & -\frac{1}{c^2\cdot\left(1-\frac{r_s}{r}\right)} & 0 & 0 \\ 0 & 0 & -\frac{r^2}{c^2} & 0 \\ 0 & 0 & 0 & -\frac{r^2\sin^2(\theta)}{c^2} \end{bmatrix}$$

Figure 1: 'sch.tensor()' outputs the Schwarzschild metric

```
Inverse of Schwarzschild Metric Tensor:
```

$$\begin{bmatrix} \frac{r}{r-r_s} & 0 & 0 & 0 \\ 0 & \frac{c^2(-r+r_s)}{r} & 0 & 0 \\ 0 & 0 & -\frac{c^2}{r^2} & 0 \\ 0 & 0 & 0 & -\frac{c^2}{r^2\sin^2(\theta)} \end{bmatrix}$$

Figure 2: The inverse of the Schwarzschild metric tensor

$$\left[\begin{bmatrix} 0 & \frac{r_s}{2r^2} & 0 & 0 \\ \frac{r_s}{2r^2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} -\frac{r_s}{2r^2} & 0 & 0 & 0 \\ 0 & \frac{r_s}{2c^2(r-r_s)^2} & 0 & 0 \\ 0 & 0 & \frac{r}{c^2} & 0 \\ 0 & 0 & 0 & \frac{r\sin^2(\theta)}{c^2} \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{r}{c^2} & 0 \\ 0 & -\frac{r}{c^2} & 0 & 0 \\ 0 & 0 & 0 & \frac{r^2\sin(2\theta)}{2c^2} \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{r\sin^2(\theta)}{c^2} \\ 0 & 0 & 0 & -\frac{r^2\sin(2\theta)}{2c^2} \\ 0 & -\frac{r\sin^2(\theta)}{c^2} & -\frac{r^2\sin(2\theta)}{2c^2} & 0 \end{bmatrix}\right]$$

Figure 3: Tensor Representation of Christoffel Symbol

$$\left[\begin{bmatrix} 0 & \frac{r_s}{2r^2\cdot\left(1-\frac{r_s}{r}\right)} & 0 & 0 \\ \frac{r_s}{2r^2\cdot\left(1-\frac{r_s}{r}\right)} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} -\frac{r_s\left(-\frac{c^2}{2}+\frac{r_sc^2}{2r}\right)}{r^2} & 0 & 0 & 0 \\ 0 & \frac{r_s\left(-\frac{c^2}{2}+\frac{r_sc^2}{2r}\right)}{r^2c^2\left(1-\frac{r_s}{r}\right)^2} & 0 & 0 \\ 0 & 0 & \frac{2r\left(-\frac{c^2}{2}+\frac{r_sc^2}{2r}\right)}{c^2} & 0 \\ 0 & 0 & 0 & \frac{2r\left(-\frac{c^2}{2}+\frac{r_sc^2}{2r}\right)\sin^2(\theta)}{c^2} \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{r} & 0 \\ 0 & \frac{1}{r} & 0 & 0 \\ 0 & 0 & 0 & -\sin(\theta)\cos(\theta) \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{r} \\ 0 & 0 & 0 & \frac{\cos(\theta)}{\sin(\theta)} \\ 0 & \frac{1}{r} & \frac{\cos(\theta)}{\sin(\theta)} & 0 \end{bmatrix}\right]$$

Figure 4: Output of Christoffel symbols with all covariant indices

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Figure 5: Riemann Tensor

1. First zero output represents components of the Riemann curvature tensor with all contravariant indices.
2. Seconds zero output represents components of the Riemann curvature tensor with the configuration ('lulu').
3. Third zero output represents components of the Riemann curvature tensor with the configuration ('ulll') after simplification.

Conclusion

This experiment demonstrates how to manipulate and transform the indices of tensors using symbolic computation. By analyzing the Schwarzschild metric, computing Christoffel symbols, and Riemann curvature tensors, we observed how changing the index configuration affects their representation. The symbolic approach provides a clear and precise method for handling complex tensor operations, essential in the study of general relativity and other fields involving curved spacetime.