Option-Pricing: Theoretical Background

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1 Risk–Neutral Dynamics

Under the risk–neutral measure $\mathbb{Q}$, the FX spot S_t must earn the domestic interest rate r_d when properly discounted. Equivalently one may show (Girsanov's theorem; see *Shreve 2004*) that

$$\mathrm{d}S_t = (r_d - r_f)\, S_t \,\mathrm{d}t \;+\; \sigma\, S_t \,\mathrm{d}W_t^{\mathbb{Q}},$$

so that the log-return

$$\ln \frac{S_T}{S_0} \;\sim\; \mathcal{N}\Big((r_d - r_f - \tfrac{1}{2}\sigma^2)\, T,\; \sigma^2\, T\Big).$$

Advice: calibrate σ to market-implied vols by inverting the Black–Scholes formula on liquid vanilla options.

2 Black–Scholes Closed-Form

Setting payoff $g(S_T) = \max(S_T - K, 0)$, risk–neutral valuation gives

$$C = e^{-r_d T}\, \mathbb{E}^{\mathbb{Q}}[\, g(S_T)\,] = S_0 e^{-r_f T} N(d_1) \;-\; K e^{-r_d T} N(d_2),$$

with

$$d_{1,2} = \frac{\ln(S_0/K) + (r_d - r_f \pm \frac{1}{2}\sigma^2)\, T}{\sigma\sqrt{T}}, \quad N(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-u^2/2}\, \mathrm{d}u.$$

Further reading: see *Hull (2018)*, Chapter 21, for a detailed derivation.

3 Monte Carlo Simulation

Approximate

$$C \approx e^{-r_d T}\, \frac{1}{N} \sum_{i=1}^{N} \max\big(S_0 e^{(r_d - r_f - \frac{1}{2}\sigma^2)T + \sigma\sqrt{T} Z_i} - K,\; 0\big), \quad Z_i \sim \mathcal{N}(0,1).$$

- **Convergence:** error $O(N^{-1/2})$; to halve error, quadruple N.
- **Variance reduction:** employ antithetic variates, control variates (use the exact BS price as control), or stratified sampling for faster convergence.

4 PDE Approach (Crank–Nicolson)

The Black–Scholes PDE for $V(S,t)$,

$$V_t + (r_d - r_f)S\,V_S + \tfrac{1}{2}\sigma^2 S^2\,V_{SS} - r_d V = 0,$$

can be discretized on a grid in (S,t):

$$\underbrace{\frac{V_j^{n+1} - V_j^n}{\Delta t}}_{\text{time derivative}} + \underbrace{(r_d - r_f)S_j\,\delta_S V_j^n}_{\text{drift}} + \underbrace{\tfrac{1}{2}\sigma^2 S_j^2\,\delta_{SS} V_j^n}_{\text{diffusion}} - r_d V_j^n = 0.$$

Crank–Nicolson uses the average of explicit and implicit schemes, yielding a tridiagonal linear system at each time-step.
Tip: choose $S_{\max} \approx S_0 e^{(r_d - r_f)T + 6\sigma\sqrt{T}}$ to truncate the domain with negligible tail probability.

5 Quadrature (Simpson's Rule)

One may also compute

$$C = e^{-r_d T}\int_K^{S_{\max}} (S-K)\,f(S)\,\mathrm{d}S, \quad f(S) = \frac{1}{S\sqrt{2\pi\sigma^2 T}}\exp\Bigl(-\frac{(\ln(S/S_0) - (r_d - r_f - \tfrac{1}{2}\sigma^2)T)^2}{2\sigma^2 T}\Bigr).$$

Using Simpson's composite rule on an even partition $[K, S_{\max}]$, error is $O(h^4)$ in the step-size h. *Advice:* ensure $S_{\max}$ captures at least 99.99% of the lognormal mass.

6 Source & References

1. Hull, J. C. (2018). *Options, Futures and Other Derivatives.*
2. Shreve, S. E. (2004). *Stochastic Calculus for Finance II.*
3. Glasserman, P. (2003). *Monte Carlo Methods in Financial Engineering.*