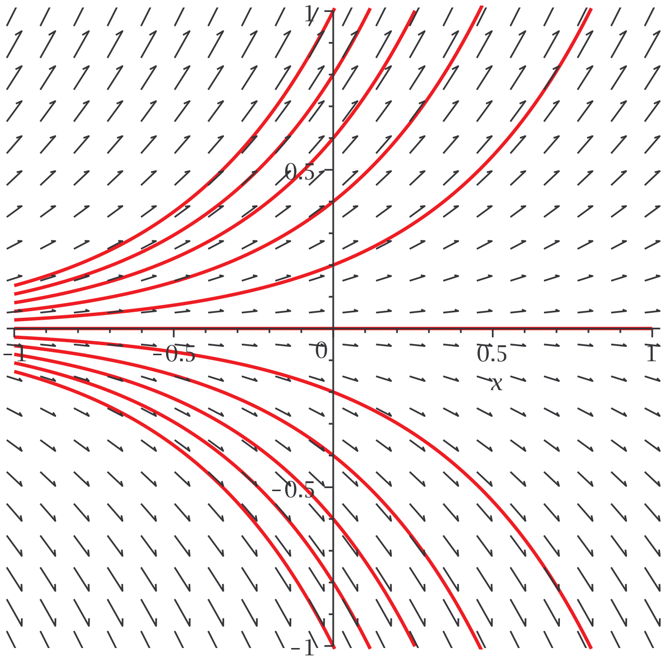


Figur 1: Characteristic curves of the PDE given in (1)

Exercise 1

A linear, first-order PDE is given by

$$u_x + 2yu_y = 0, \quad x \in \mathbb{R},\ y \in \mathbb{R} \tag{1}$$

Therefore the directional derivative in direction of the vector $v = (1, 2y)^T$ is zero for any solution u of (1). A solution u will therefore be constant along all curves where

$$\frac{dy}{dx} = 2y$$

This linear, first-order ODE can easily be solved, and gives that u will be constant along curves given by

$$y = Ce^{2x} \tag{2}$$

where $C \in \mathbb{R}$.

a)

Some of these curves are plotted in figure 1. Along any characteristic line u is constant and the specific constant value is related to the constant C in (2). From (2) we get

$C = ye^{-2x}$ and therefore

$$
\begin{aligned}
u(x,y) &= f(C) \\
&= f(ye^{-2x})
\end{aligned}
\tag{3}
$$

where f is an arbitrary function of one real variable. From this we conclude that the general solution of (1) is given by $u(x,t) = f(ye^{-2x})$.

b)

We now wish to find the solution u_p that satisfies the condition $u(0,y) = y$. By inserting in (3) we get

$$
\begin{aligned}
u(0,y) &= y \quad \Leftrightarrow \\
f(ye^{-2\cdot 0}) &= y \quad \Leftrightarrow \\
f(y) &= y
\end{aligned}
$$

so f is just the identity function and the particular solution is then given by

$$u_p(x,y) = ye^{-2x}$$

c)

We now consider the inhomogeneous equation given by

$$u_x + 2yu_y = x \tag{4}$$

The general solution to the equation should be found. As the equation is linear the complete general solution will be given as the sum of the general solution to the homogeneous equation and a particular solution to the inhomogeneous equation. The general solution to the homogeneous equation was found in (3) and therefore only the particular solution u_p to (4) needs to be determined. From (4) it is directly seen that $u_p(x,y) = \frac{1}{2}x^2$ is a particular solution. The general solution to (4) is then given as

$$u(x,y) = f(ye^{-2x}) + \frac{1}{2}x^2$$

Exercise 2

For $n \in \mathbb{N}$ a second order, inhomogeneous PDE is given by

$$
\begin{aligned}
\Delta u(x,y) = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} &= 0, \quad x > 0, y \in \mathbb{R} \\
u(0,y) = 0, \quad \frac{\partial u}{\partial x}(0,y) &= \frac{1}{n}\sin(ny), \quad y \in \mathbb{R}
\end{aligned}
\tag{5}
$$

a)

It is shown that

$$u_n(x,y) = \frac{1}{n^2}\sin(ny)\sinh(nx) \tag{6}$$

solves the problem (5). Since $\sinh(0) = 0$ it is immediately seen that $u(0,y) = 0$. Now the partial derivatives of u is calculated.

$$\begin{aligned} u_x(x,y) &= \frac{1}{n}\sin(ny)\cosh(nx) \\ u_{xx}(x,y) &= \sin(ny)\sinh(nx) \\ u_y(x,y) &= \frac{1}{n}\cos(ny)\sinh(nx) \\ u_{yy}(x,y) &= -\sin(ny)\sinh(nx) \end{aligned}$$

From this it is seen that

$$\begin{aligned} u_x(0,y) &= \frac{1}{n}\sin(ny)\cosh(0) \\ &= \frac{1}{n}\sin(ny) \end{aligned}$$

and

$$\begin{aligned} u_{xx} + u_{yy} &= \sin(ny)\sinh(nx) - \sin(ny)\sinh(nx) \\ &= 0 \end{aligned}$$

and therefore (6) solves (5).

b)

It now needs to be shown that the solution (6) is unique. First we look at the homogeneous problem related to (5). This is just the same problem, with the condition $u_x(0,y) = \frac{1}{n}\sin(ny)$, replaced by $u_x(0,y) = 0$. A solution to the homogeneous problem is denoted u_∞ and it is assumed that u_∞ is independent of y. Since u_∞ is a solution to the homogeneous problem and independent of y we know that

$$\frac{\partial^2 u_\infty}{\partial x^2} = 0 \quad , \quad \frac{\partial u_\infty}{\partial x}(0,y) = 0 \quad , \quad u_\infty(0,y) = 0$$

Since $\frac{\partial^2 u_\infty}{\partial x^2} = 0$ we have that $\frac{\partial u_\infty}{\partial x}$ must be constant, and since $\frac{\partial u_\infty}{\partial x}(0,y) = 0$, we conclude that $\frac{\partial u_\infty}{\partial x} = 0$. Since $\frac{\partial u_\infty}{\partial x} = 0$, u_∞ must be a constant and since $u_\infty(0,y) = 0$ we get that $u_\infty = 0$. With this result it can now be shown that (6) is the unique solution to (5). Assume that u_k is another solution for (5). Since the problem is linear $u_0 = u_n - u_k$ is a solution to the homogeneous problem. But from the previous result $u_0 = 0$ and therefore

$$\begin{aligned} u_n - u_k &= 0 \quad \Leftrightarrow \\ u_n &= u_k \end{aligned}$$

and so (6) is the unique solution of (5).

c)

It now needs to be determined whether (5) is a well-posed problem or not. The previous two sections have established both the existence and the uniqueness of the solution to (5) and only the stability of the problem remains to be examined.
For $x = 0$ we get

$$\begin{aligned} u_n(0, y) &= 0 \\ &\to 0 \text{ for } n \to \infty \end{aligned}$$

but for $x > 0$

$$\begin{aligned} \frac{\sinh(nx)}{n^2} &= \frac{e^{nx} - e^{-nx}}{2n^2} \\ &\to \infty \text{ for } n \to \infty \end{aligned}$$

and so u_n is unbounded as $n \to \infty$. In other words, for an arbitrary fixed y,* a very small $\varepsilon > 0$ and an arbitrarily large δ, an $N \in \mathbb{N}$ exists, such that $u_N(\varepsilon, y) > \delta$. So for some solution even the smallest perturbation can alter the solution by a large amount, and therefore the problem (5) is not well-posed.

Exercise 3

A solution for the initial value problem for the wave equation

$$\begin{gathered} u_{tt} - 4u_{xx} = 0, \quad x, t \in \mathbb{R} \\ u(x, 0) = 0, \quad u_t(x, 0) = e^{-x^2}, \quad x \in \mathbb{R} \end{gathered}$$

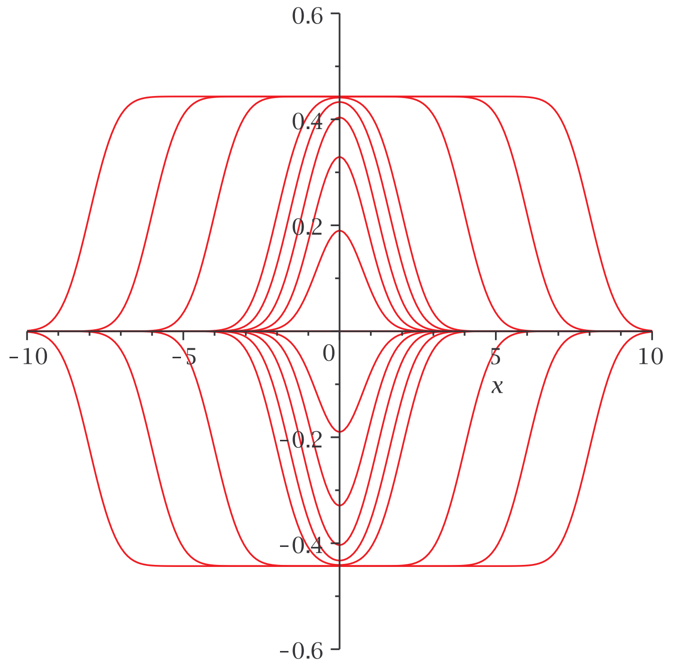
is found, and the solution is expressed in terms of the error function. Using d'Alembert formula (eq. 2.1.8 in the course textbook) with $c = 2$, $\phi(x) = 0$ and $\psi(x) = e^{-x^2}$, we get

$$\begin{aligned} u(x, t) &= \frac{1}{2}(0 + 0) + \frac{1}{4}\int_{x-2t}^{x+2t} e^{-s^2}\, ds \\ &= \frac{1}{4}\int_{x-2t}^{x+2t} e^{-s^2}\, ds \end{aligned}$$

To express the solution in terms of the error function, the sign of $x + 2t$ and $x - 2t$ as well as the sign of t should be considered. With the help of Maple it turns out that all cases can be expressed in a single expression. That is

$$\begin{aligned} u(x, t) &= \frac{1}{4}\int_{x-2t}^{x+2t} e^{-s^2}\, ds \\ &= \frac{1}{4}\left(\int_0^{-x+2t} e^{-s^2}\, ds + \int_0^{x+2t} e^{-s^2}\, ds\right) \end{aligned}$$

*To be precise we should choose y such that $y \in \mathbb{R}\backslash\{p\pi \mid p \in \mathbb{Z}\}$, since $u_n(x, y) = 0$ for $y \in \{p\pi \mid p \in \mathbb{Z}\}$



Figur 2: Solutions for exercise 3 for $t = -4, -3, \ldots, -1, -0.8, -0.6, \ldots, 0.6, 0.8, 1, \ldots, 3, 4$ and $x \in [-10, 10]$

The validity of the above can easily be checked for all possible combinations of the sign of $x + 2t$, $x - 2t$ and t. Now the solution can be expressed as

$$u(x,t) = \frac{\sqrt{\pi}}{8} \left(\operatorname{erf}(-x + 2t) + \operatorname{erf}(x + 2t) \right)$$

The solution curves for $t = -4, -3, \ldots, -1, -0.8, -0.6, \ldots, 0.6, 0.8, 1, \ldots, 3, 4$ and $x \in [-10, 10]$ are shown in figure 2. From the figure the symmetry of the solution with regard to time is seen. When t is negative we just get a reflection of the wave for positive t. Also for the solutions with a time difference of 1, the distance between the x-values, for which the solution gets "close"[†] to 0, is seen to be 2. This corresponds to the speed of the wave $c = 2$.

Exercise 4

The solution for the initial value problem for the diffusion equation

$$u_t - 4u_{xx} = 0, \quad x \in \mathbb{R}, t > 0$$
$$u(x,0) = e^{-x^2}, \quad x \in \mathbb{R}$$

[†]Close as in "can't separate the graph of the solution and the x-axis"

should be found. Using equation 2.4.8 in the course textbook, with $\phi(y) = e^{-y^2}$ and $k = 2$, we get

$$\begin{aligned} u(x,t) &= \frac{1}{\sqrt{8\pi t}} \int_{\infty}^{\infty} e^{-(x-y)^2/8t} e^{-y^2}\, dy \\ &= \frac{1}{\sqrt{8\pi t}} \int_{\infty}^{\infty} e^{-((x-y)^2+8ty^2)/8t}\, dy \end{aligned}$$

Since

$$\begin{aligned} (x-y)^2 + 8ty^2 &= x^2 + y^2 + 8ty^2 - 2xy \\ &= \left(\frac{1}{\sqrt{8t+1}}\, x - \sqrt{8t+1}\, y \right)^2 + \frac{8t}{8t+1} x^2 \end{aligned}$$

we get

$$u(x,t) = \frac{1}{\sqrt{8\pi t}} e^{-\frac{x^2}{8t+1}} \int_{\infty}^{\infty} e^{-(x/\sqrt{8t+1} - \sqrt{8t+1}\, y)^2/8t}\, dy$$

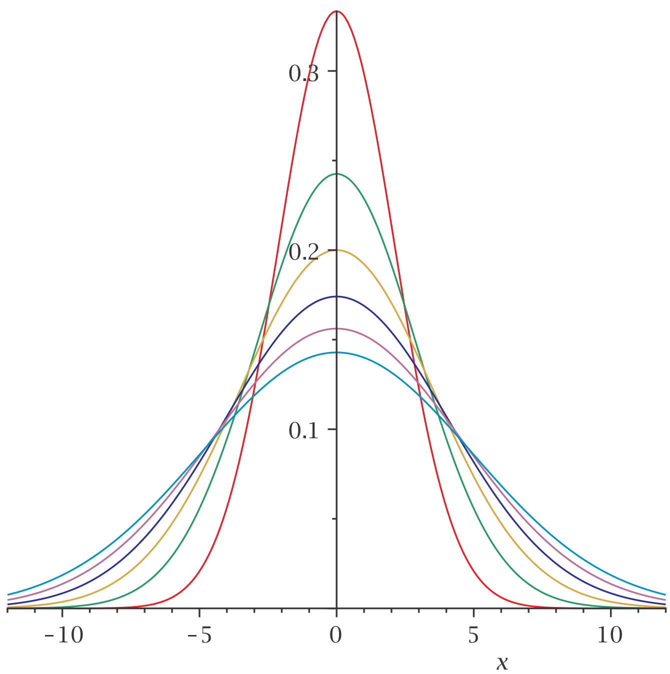
Making the substitution

$$z = \frac{(x/\sqrt{8t+1} \; - \; \sqrt{8t+1}\, y)}{\sqrt{8t}} \qquad \Rightarrow \qquad dy = \frac{\sqrt{8t}}{\sqrt{8t+1}} dz$$

finally gives

$$\begin{aligned} u(x,t) &= \frac{1}{\sqrt{\pi}\sqrt{8t+1}} e^{-\frac{x^2}{8t+1}} \int_{\infty}^{\infty} e^{-z^2}\, dz \\ &= \frac{1}{\sqrt{8t+1}} e^{-\frac{x^2}{8t+1}} \end{aligned}$$

The solution for $t = 1, 2, \ldots, 6$ is plotted in figure 3. From the figure the typical behaviour of the diffusion equation is recognized, as the solution tends to smooth out as time increases. This results in the maximum decreasing and the minimum increasing as time increases.



Figur 3: Solutions for exercise 4 for $t = 1, 2, \ldots, 6$ and $x \in [-12, 12]$