3.5.5

a)

it has to be shown:

i) $X_n \geq \ 0 \ \forall \ n$

ii) $E[(X_n)] < 0 \ \forall \ n$

iii) $E[X_{n+1}|X_0, \cdots, X_n] = X_n \ \forall \ n$

i) is clear

iii) using the Markov property we get $E[X_{n+1}|X_0, \cdots, X_n] = E[X_{n+1}|X_n]$.

In case $X_n = 0$ the result is clear.

Otherwise we can calculate

$$\begin{aligned} E[X_{n+1}|X_n] &= \frac{1}{2}(X_n + 1) + \frac{1}{2}(X_n - 1) \\ &= X_n \end{aligned}$$

ii) In case $X_0 = 0$ this is clear.

Otherwise it can be shown that $E[X_n] = X_0 \ \forall \ n$. Be $X_0 = i$

$$n = 1$$

$$E[X_1] = \frac{1}{2}(E[X_0] + 1) + \frac{1}{2}(E[X_0] - 1) = E[X_0] = X_0 = i$$

Assume the assumption holds for n-1

$$E[X_n] = \frac{1}{2}(E[X_{n-1}] - 1) + \frac{1}{2}(E[X_{n-1}] + 1) = E[X_{n-1}] = X_0 = i$$

Therefore X_n is a nonnegative martingal.

b) Using the inequality (2.53) we have

$$P(\max_n X_n \geq N) \leq \frac{E[X_0]}{N}$$

since $E[X_0]$ is finite we have $\lim_{N \to \infty} \frac{E[X_0]}{N} = 0$