Exercise 1

A PDE problem is given by

$$\begin{aligned} u_{tt} - 2u_{xx} &= x\cos(t), \qquad x,t \in \mathbb{R} \\ u(x,0) = 0, \quad u_t(x,0) &= x, \qquad x \in \mathbb{R} \end{aligned}$$

The solution u is determined. The PDE problem is the wave equation with a source where

$$c = \sqrt{2}, \quad \phi(x) = 0, \quad \psi(x) = x \quad f(x,t) = x\cos(t)$$

Using equation 3.4.3 from the course textbook the solution is given as

$$\begin{aligned} u(x,t) &= \frac{1}{2}(0+0) + \frac{1}{2\sqrt{2}}\int_{x-\sqrt{2}t}^{x+\sqrt{2}t} s\,ds + \frac{1}{2\sqrt{2}}\int_0^t \int_{x-\sqrt{2}(t-s)}^{x+\sqrt{2}(t-s)} y\cos(t)\,dy\,ds \\ &= \frac{1}{2\sqrt{2}}\left(\frac{1}{2}((x+\sqrt{2}t)^2 - (x-\sqrt{2}t)^2) + \int_0^t \cos(s)\frac{1}{2}((x+\sqrt{2}(t-s))^2 - (x-\sqrt{2}(t-s))^2)\,ds\right) \\ &= \frac{1}{2\sqrt{2}}\left(2\sqrt{2}xt + 2\sqrt{2}x\left(\int_0^t t\cos(s)\,ds - \int_0^t s\cos(s)\,ds\right)\right) \\ &= xt + x\left(t\sin(t) - \left([s\sin(s)]_0^t - \int_0^t \sin(s)\,ds\right)\right) \\ &= xt + x[-\cos(s)]_0^t \\ &= xt + x(1-\cos(t)) \end{aligned}$$

By writing the solution as $u(x,t) = x(t+1-\cos(t))$, it is seen that the solution for an arbitrary fixed time t_0 is a linear function of x. This is confirmed by plotting the solution for $t = 1,2,3,4$ and $x \in [-3,3]$. The plot is shown in figure 1

Exercise 2

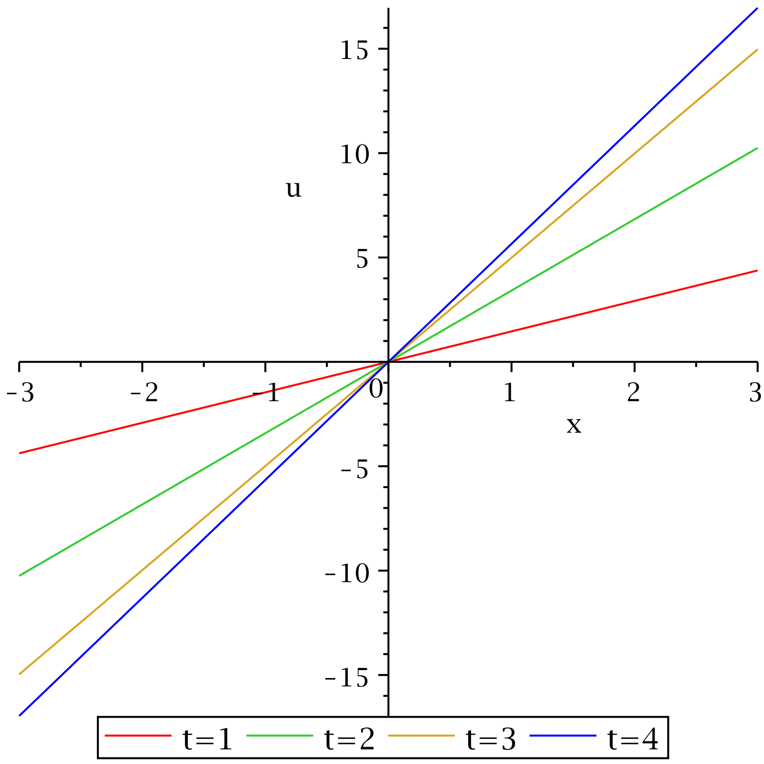
A PDE problem is given by

$$\begin{aligned} u_t - ku_{xx} = 0, \quad 0 < x < L, t > 0 \\ u(0,t) = at, \quad u(L,t) = 0, \quad t > 0 \\ u(x,0) = 0, \quad 0 < x < L \end{aligned} \tag{1}$$

where $a \in \mathbb{R}, k > 0$ and $L > 0$. The solution u for the PDE problem should be found.

The PDE is the diffusion equation with inhomogeneous boundary conditions. Therefore the expansion method can be used to find u. We look at the Fourier sine series expansion of u.

$$u(x,t) = \sum_{n=1}^{\infty} u_n(t)\sin\frac{n\pi x}{L} \tag{2}$$



Figur 1: The solution $u(x,t)$ for exercise 1 for different values of t.

The coefficients $u_n(t)$ can be determined by using equation 5.6.10 in the course textbook (with $j(t) = 0, h(t) = at$). The initial condition $u(x, 0) = 0$ implies that $u_n(0) = 0$ and since

$$\begin{aligned} u_n(0) &= Ce^{-n^2\pi^2L^{-2}k\cdot 0} - 2n\pi L^{-2}k \int_0^0 e^{-n^2\pi^2L^{-2}k(t-s)}(-as)\,ds \\ &= C \end{aligned}$$

we get that $C = 0$. Using Maple the coefficients are now found as

$$\begin{aligned} u_n(t) &= 2n\pi L^{-2}ka \int_0^t e^{-n^2\pi^2L^{-2}k(t-s)}s\,ds \\ &= \frac{2aL^2}{n^3\pi^3k}(e^{-n^2\pi^2L^{-2}kt} - 1) + \frac{2a}{n\pi}t \end{aligned}$$

Using these coefficients in (2) gives the solution to the problem in (1). To verify the solution the finite sum

$$\sum_{n=1}^{100} u_n(t) \sin \frac{n\pi x}{L}$$

is plotted for $a = 2, k = 1, L = 1$ and $t = 1, 2$. The plot is shown in figure 2. The solutions are found to “approach” the value at for x “close” to 0. The actual value is of cause $u(0, t) = 0$ but this is ok since the series isn't bound to converge at the endpoints.

Exercise 3

A PDE problem for the damped wave equation is given by

$$\begin{gathered} u_{tt} + u_t - u_{xx} = 0, \quad 0 < x < \pi, t \in \mathbb{R} \\ u(x, 0) = 0, \quad u_t(x, 0) = \sin(x), \quad 0 < x < \pi \\ u(0, t) = u(\pi, t) = 0, \quad t \in \mathbb{R} \end{gathered} \tag{3}$$

A solution for the problem is found by separation of variables. The solution is assumed to be of the form

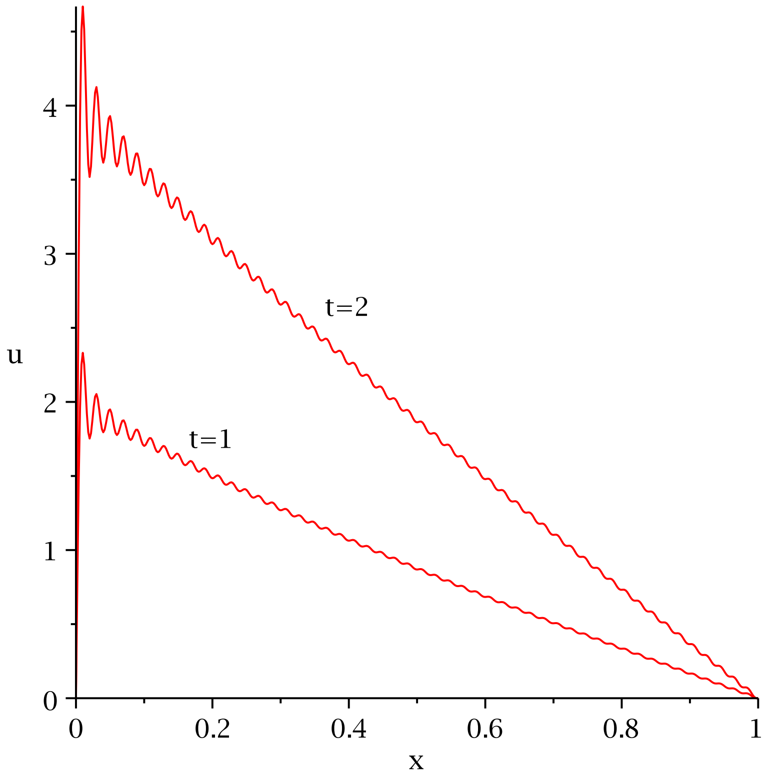
$$u(x, t) = X(x)T(t)$$

Inserted into the PDE this gives

$$\begin{gathered} X(x)T''(t) + X(x)T'(t) - X''(x)T(t) = 0 \quad \Leftrightarrow \\ \frac{T''(t) + T'(t)}{T(t)} = \frac{X''(x)}{X(x)} = -\lambda \end{gathered}$$

The ODE for $X(x)$ becomes $X''(x) + \lambda X(x) = 0$ which combined with the homogeneous Dirichlet boundary conditions have been solved on page 85 in the course textbook. The eigenvalues and eigenfunctions are therefore given by

$$\lambda_n = \left(\frac{n\pi}{l}\right)^2 = n^2, \quad X_n(x) = \sin(nx) \quad (n = 1, 2, 3, \dots)$$



Figur 2: The solution for exercise 2 for $t = 1, 2$. Only the first 100 terms in the solution sum was used and the parameters from the problem statement was chosen as $a = 2, k = 1$ and $L = 1$.

The ODE for T is then

$$T''(t) + T'(t) + n^2 T(t) = 0 \tag{4}$$

The characteristic polynomium is

$$s^2 + s + n^2 = 0 \quad \Rightarrow$$
$$s = \frac{-1 \pm \sqrt{1 - 4n^2}}{2}$$

and since $1 - 4n^2 < 0$ for all $n = 1, 2, 3, \dots$

$$s = -\frac{1}{2} \pm \sqrt{n^2 - \frac{1}{4}}\; i$$

The general solution for (4) is then given by

$$T(t) = A_n e^{-\frac{t}{2}} \cos\left(\sqrt{n^2 - \frac{1}{4}}\, t\right) + B_n e^{-\frac{t}{2}} \sin\left(\sqrt{n^2 - \frac{1}{4}}\, t\right)$$

From the original problem (3) we know $u(x,0) = X(x)T(0) = 0 \Rightarrow T(0) = A_n = 0$ and therefore

$$u_n(x,t) = B_n e^{-\frac{t}{2}} \sin\left(\sqrt{n^2 - \frac{1}{4}}\, t\right) \sin(nx)$$

The problem (3) is linear so any finite sum of solutions is also a solution, so

$$u(x,t) = \sum_n B_n e^{-\frac{t}{2}} \sin\left(\sqrt{n^2 - \frac{1}{4}}\, t\right) \sin(nx)$$

The coefficients B_n can be found from the initial condition $u_t(x,0) = \sin(x)$ and since

$$u_t(x,t) = \sum_n B_n \left(e^{-\frac{t}{2}} \cos\left(\sqrt{n^2 - \frac{1}{4}}\, t\right) \sqrt{n^2 - \frac{1}{4}} - \frac{1}{2} e^{-\frac{t}{2}} \sin\left(\sqrt{n^2 - \frac{1}{4}}\, t\right)\right) \sin(nx)$$

we get

$$u_t(x,0) = \sum_n B_n \sqrt{n^2 - \frac{1}{4}} \sin(nx) = \sin(x)$$

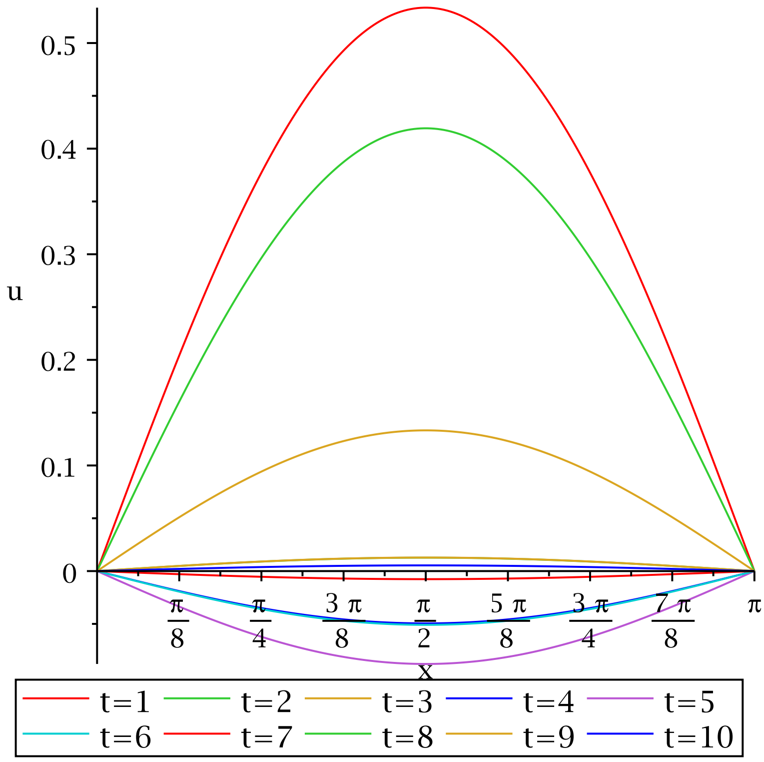
The coefficients B_n are therefore given by

$$B_n = \begin{cases} \frac{2}{\sqrt{3}} & n = 1 \\ 0 & \text{else} \end{cases}$$

which gives the final solution

$$u(x,t) = \frac{2}{\sqrt{3}} e^{-\frac{t}{2}} \sin\left(\frac{\sqrt{3}}{2} t\right) \sin(x)$$

In figure 3 the solution is plotted for different values of t and it is seen that the solution match the expected behaviour for a damped wave equation. This behaviour could also be anticipated from the exponentially decreasing function of t in the solution.



Figur 3: The solution $u(x,t)$ for exercise 3 for different values of t. The anticipated damped wave behaviour is recognized in the plot.