Homework 3

01246 Partial differential equations – 25-10-2011– Anders Hørsted (s082382)

Exercise 1

The annulus A is given in polar coordinates by $r \in (1,2), \theta \in (0, 2\pi)$. A PDE problem is defined by

$$\Delta u = 0 \text{ in } A \tag{1}$$

$$u(1,\theta) = 0, \quad \frac{\partial u}{\partial r}(2,\theta) = 1 - 2\cos(\theta), \quad \theta \in (0, 2\pi) \tag{2}$$

The solution to this PDE problem is now found. The solution formula 6.4.7 in the textbook can be used for this problem. Using the first boundary condition we get that

$$\begin{aligned} u(1,\theta) &= \frac{1}{2}(C_0 + D_0 \log(1)) + \sum_{n=1}^{\infty}(C_n + D_n)\cos(n\theta) + (A_n + B_n)\sin(n\theta) \\ &= 0 \end{aligned}$$

from which we conclude that $C_0 = 0, C_n = -D_n, A_n = -B_n$. Since

$$u_r(r,\theta) = \frac{1}{2}D_0 r^{-1} + \sum_{n=1}^{\infty}(C_n n r^{n-1} - D_n n r^{-n-1})\cos(n\theta) + (A_n n r^{n-1} - B_n n r^{-n-1})\sin(n\theta)$$

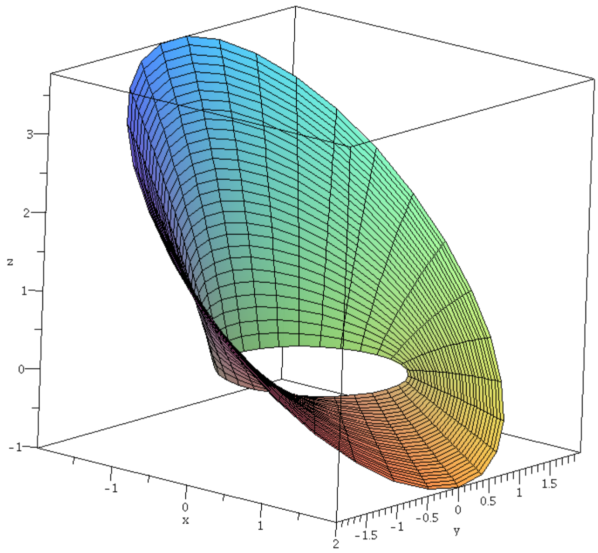
we get from the second boundary condition that

$$\begin{aligned} u_r(2,\theta) &= \frac{1}{4}D_0 + \sum_{n=1}^{\infty}(C_n n 2^{n-1} - D_n n 2^{-n-1})\cos(n\theta) + (A_n n 2^{n-1} - B_n n 2^{-n-1})\sin(n\theta) \\ &= \frac{1}{4}D_0 + \sum_{n=1}^{\infty}(2^{n-1} + 2^{-n-1})C_n n\cos(n\theta) + (2^{n-1} + 2^{-n-1})A_n n\sin(n\theta) \\ &= 1 - 2\cos(\theta) \end{aligned}$$

from which we get that $D_0 = 4, C_1 = -\frac{8}{5}, D_1 = \frac{8}{5}$ and all other coefficients should be 0. The solution is therefore given by

$$u(r,\theta) = 2\log{(r)} + (\frac{8}{5}r^{-1} - \frac{8}{5}r)\cos(\theta)$$

The solution is plotted and is shown in figure 1. From the figure it is seen that the solution takes on its minima at $(2,0)$ and maxima at $(2,\pi)$. Both point are on the boundary as expected from the maximum principle.



Figur 1: Plot of solution for exercise 1

Exercise 2

Using the reflection method the Green's function G for the Laplace Equation in the half plane $H = \{(x, y) \,|\, x \in \mathbb{R},\ y > 0\}$ is found. Based on the derivation of the Green's function in the half space in the textbook combined with the representation formula in two dimensions (eq. 7.2.5) a good guess for G is

$$G(\mathbf{x}, \mathbf{x}_0) = \frac{1}{2\pi}(\log|\mathbf{x} - \mathbf{x}_0| - \log|\mathbf{x} - \mathbf{x}_0^*|)$$

that in coordinates becomes

$$G(\mathbf{x}, \mathbf{x}_0) = \frac{1}{2\pi}(\log((x - x_0)^2 + (y - y_0)^2)^{1/2} - \log((x - x_0)^2 + (y + y_0)^2)^{1/2}) \tag{3}$$

By exactly the same arguments as in the textbook p. 191-192 it can be "proved" that G is the Green's function for D at $\mathbf{x}_0$.

We now use G to solve the Dirichlet problem

$$\Delta u = 0 \text{ in } H,$$
$$h(x) = u(x, 0) = \begin{cases} 1, & x \in (-1, 1) \\ 0, & \text{elsewhere} \end{cases}$$

To solve the problem, $\frac{\partial G}{\partial n} = -\frac{\partial G}{\partial y}|_{y=0}$ must be determined.

$$-\frac{\partial G}{\partial y} = \frac{1}{2\pi}\left(\frac{y+y_0}{|\mathbf{x}-\mathbf{x}_0^*|^2} - \frac{y-y_0}{|\mathbf{x}-\mathbf{x}_0|^2}\right)$$

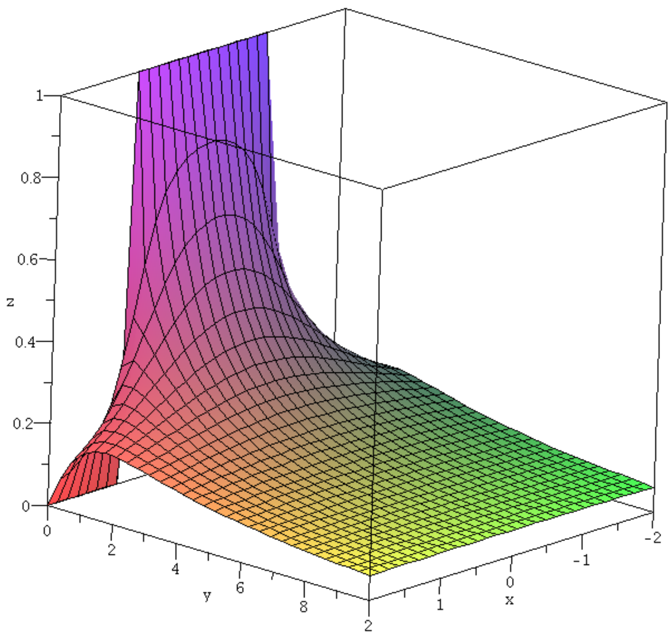
that at the boundary ($y = 0$) becomes

$$= \frac{y_0}{\pi((x-x_0)^2 + y_0^2)}$$

Using theorem 7.3.1 the solution of the Dirichlet problem is then given as

$$\begin{aligned} u(x_0, y_0) &= \int_{\infty}^{\infty} h(x) \frac{y_0}{\pi((x-x_0)^2 + y_0^2)}\, dx \\ &= \frac{1}{\pi}\int_{-1}^{1} \frac{y_0}{((x-x_0)^2 + y_0^2)}\, dx \\ &= \frac{1}{\pi}\left(\arctan\left(\frac{x_0+1}{y_0}\right) + \arctan\left(\frac{1-x_0}{y_0}\right)\right) \end{aligned}$$

The solution is plotted and shown in figure 2. It is again found that both the minimum and maximum is obtained at the boundary.



Figur 2: Plot of solution for exercise 2

Exercise 3

A PDE problem is given by

$$\begin{gathered}\Delta u = 0, \quad (x,y) \in (0,\pi)^2, \\ u(0,y) = 0, \;\; u_x(\pi,y) + 2u(\pi,y) = 0, \\ u(x,0) = 0, \;\; u(x,\pi) = x(1-x)\end{gathered}$$

The problem is solved using separation of variables. Therefore the solutions is given by

$$u_n(x,y) = X_n(x)Y_n(y)$$

From $\Delta u = 0$ we get

$$\begin{aligned}X_n''Y_n + X_nY_n'' = 0 \quad &\Leftrightarrow \\ \frac{X_n''}{X_n} = -\frac{Y_n''}{Y_n}&\end{aligned}$$

and since the left side is independent of y and the right side independent of x we get

$$\frac{X_n''}{X_n} = -\frac{Y_n''}{Y_n} = -\lambda_n \tag{4}$$

Focusing on X_n we get the ODE

$$\begin{gathered}X_n'' + \lambda_n X_n = 0, \\ X_n(0) = 0, \quad X_n'(\pi) + 2X_n(\pi) = 0\end{gathered}$$

Now we are given that all eigenvalues $\lambda_n = \beta_n^2$ are positive. Then the solutions for the ODE of X are

$$X_n(x) = A_n \cos(\beta_n x) + B_n \sin(\beta_n x)$$

Using the first boundary condition gives

$$X_n(0) = A_n = 0$$

The second boundary condition then gives

$$\begin{aligned}X_n'(\pi) + 2X_n(\pi) &= B_n\beta_n \cos(\beta_n\pi) + 2B_n \sin(\beta_n\pi) \\ &= 0 \quad \Leftrightarrow \\ \beta_n \cos(\beta_n\pi) &= -2\sin(\beta_n\pi) \\ -\frac{\beta_n}{2} &= \tan(\beta_n\pi)\end{aligned}$$

The eigenvalues are given as solutions to this equation and the corresponding eigenfunctions are $\sin(\beta_n x)$.

From (4) and the boundary conditions of the original PDE the ODE for Y_n is given by

$$Y_n'' - \lambda Y_n = 0, \quad Y_n(0) = 0$$

This ODE has exponential solutions that for convenience are written

$$Y_n(y) = C_n \cosh(\beta_n y) + D_n \sinh(\beta_n y)$$

From the homogeneous boundary condition we get

$$Y_n(0) = C_n = 0$$

and we can then write the solution that satifies all the homogeneous boundary conditions for the PDE as

$$u(x,y) = \sum_{n=0}^{\infty} D_n \sinh(\beta_n y) \sin(\beta_n x)$$

To determine the coefficients D_n we use the inhomogeneous boundary condition

$$u(x,\pi) = \sum_{n=0}^{\infty} D_n \sinh(\beta_n \pi) \sin(\beta_n x) = x(1-x)$$

This is the Fourier sine series for the function $\phi(x) = x(1-x)$ and from equation 5.1.4 we find the coefficients D_n by

$$D_n = \frac{2}{\pi \sinh(\beta_n \pi)} \int_0^{\pi} x(1-x) \sin(\beta_n x)\, dx$$