Homework 3

01246 Partial differential equations – 24-10-2011– Anders Hørsted (s082382)

Exercise 1

The annulus A is given in polar coordinates by $r \in (1,2), \theta \in (0, 2\pi)$. A PDE problem is defined by

$$\Delta u = 0 \text{ in } A \tag{1}$$

$$u(1,\theta) = 0, \quad \frac{\partial u}{\partial r}(2,\theta) = 1 - 2\cos(\theta), \quad \theta \in (0, 2\pi) \tag{2}$$

The solution to this PDE problem is now found. The solution formula 6.4.7 in the textbook can be used for this problem. Using the first boundary condition we get that

$$\begin{aligned} u(1,\theta) &= \frac{1}{2}(C_0 + D_0 \log(1)) + \sum_{n=1}^{\infty} (C_n + D_n)\cos(n\theta) + (A_n + B_n)\sin(n\theta) \\ &\overset{\text{set}}{=} 0 \end{aligned}$$

from which we conclude that $C_0 = 0, C_n = -D_n, A_n = -B_n$. Since

$$u_r(r,\theta) = \frac{1}{2}D_0 r^{-1} + \sum_{n=1}^{\infty}(C_n n r^{n-1} - D_n n r^{-n-1})\cos(n\theta) + (A_n n r^{n-1} - B_n n r^{-n-1})\sin(n\theta)$$

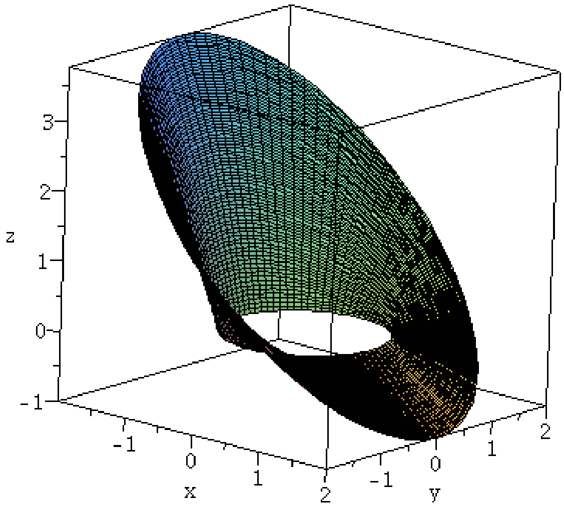
we get from the second boundary condition that

$$\begin{aligned} u_r(2,\theta) &= \frac{1}{4}D_0 + \sum_{n=1}^{\infty}(C_n n 2^{n-1} - D_n n 2^{-n-1})\cos(n\theta) + (A_n n 2^{n-1} - B_n n 2^{-n-1})\sin(n\theta) \\ &= \frac{1}{4}D_0 + \sum_{n=1}^{\infty}(2^{n-1} + 2^{-n-1})C_n n\cos(n\theta) + (2^{n-1} + 2^{-n-1})A_n n\sin(n\theta) \\ &\overset{\text{set}}{=} 1 - 2\cos(\theta) \end{aligned}$$

from which we get that $D_0 = 4, C_1 = -\frac{8}{5}, D_1 = \frac{8}{5}$ and all other coefficients should be 0. The solution is therefore given by

$$u(r,\theta) = 2\log(r) + (\frac{8}{5}r^{-1} - \frac{8}{5}r)\cos(\theta)$$

The solution is plotted and is shown in figure 1. COMMENT!!



Figur 1: Plot of solution for exercise 1

Exercise 2

Using the reflection method the Green's function G for the Laplace Equation in the half plane $H = \{(x, y) \mid x \in \mathbb{R},\ y > 0\}$ is found. Based on the derivation of the Green's function in the half space in the textbook combined with the representation formula in two dimensions (eq. 7.2.5) a good guess for G is

$$G(\mathbf{x}, \mathbf{x}_0) = \frac{1}{2\pi}(\log|\mathbf{x} - \mathbf{x}_0| - \log|\mathbf{x} - \mathbf{x}_0^*|)$$

that in coordinates becomes

$$G(\mathbf{x}, \mathbf{x}_0) = \frac{1}{2\pi}(\log((x - x_0)^2 + (y - y_0)^2)^{1/2} - \log((x - x_0)^2 + (y + y_0)^2)^{1/2}) \tag{3}$$