Brownian Motion

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Brownian Motion

Today:

- Definition and first properties
- Reflection principle and maximum variable
- Derived processes

Next week

- Brownian motion with drift
- Ohrnstein-Uhlenbeck process

Two weeks from now

- Queueing theory

Brownian Motion: Definition

Definition

Brownian motion with diffusion coefficient σ^2 is a stochastic process $\{B(t); t \geq 0\}$ with the properties

(a) *Every increment $B(s+t) - B(s)$ is normally distributed with mean zero and variance $\sigma^2 t$; $\sigma^2 > 0$ is a fixed parameter*

(b) *For every pair of disjoint time intervals $(t_1, t_2], (t_3, t_4]$, with $0 \leq t_1 < t_2 \leq t_3 < t_4$, the increments $B(t_4) - B(t_3)$ and $B(t_2) - B(t_1)$ are independent random variables and similarly for n disjoint time intervals, where n is an arbitrary positive integer.*

(c) *$B(0) = 0$, and $B(t)$ is a continuous function of t*

Diffusion equation

$$\frac{\partial p}{\partial t} = \frac{1}{2}\sigma^2 \frac{\partial^2 p}{\partial y^2}$$

$$p(y,t|x) = \frac{1}{\sqrt{2\pi t}\sigma} e^{-\frac{(y-x)^2}{2t\sigma^2}}$$

$$\frac{\partial p}{\partial t} = \frac{1}{\sqrt{2\pi t}\sigma}\left(-\frac{1}{2t\sqrt{t}}\right) e^{-\frac{(y-x)^2}{2t\sigma^2}} + \frac{1}{\sqrt{2\pi t}\sigma} e^{-\frac{(y-x)^2}{2t\sigma^2}} \frac{(y-x)^2}{2t^2\sigma^2}$$

$$\frac{\partial p}{\partial y} = \frac{1}{\sqrt{2\pi t}\sigma} e^{-\frac{(y-x)^2}{2t\sigma^2}} \left(\frac{-(y-x)}{t\sigma^2}\right)$$

Standard Brownian motion: $\sigma^2 = 1$.

$$\phi_t(x) = \frac{1}{\sqrt{t}}\phi\left(\frac{x}{\sqrt{t}}\right), \qquad \Phi_t(x) = \Phi\left(\frac{x}{\sqrt{t}}\right)$$

Covariance Function

$$
\begin{aligned}
\mathbb{C}\mathrm{ov}[B(s), B(t)] &= \mathbb{E}[B(s)B(t)] \\
&= \mathbb{E}[B(s)\,(B(t) - B(s) + B(s))] \\
&= \mathbb{E}[B(s)^2] + \mathbb{E}[B(s)]\mathbb{E}[B(t) - B(s)] \\
&= s\sigma^2
\end{aligned}
$$

Invariance Principle

Let ξ_i be i.i.d. with $\mathbb{E}(\xi_i) = 0$ and $\mathbb{V}\text{ar}(x_i) = 1$; then

$$S_n = \xi_1 + \cdots + \xi_n$$

$$\lim_{n\to\infty} \mathbb{P}\left\{\frac{S_n}{\sqrt{n}} \leq x\right\} = \Phi(x), \qquad \text{CLT}$$

$$B_n(t) = \frac{S_{[nt]}}{\sqrt{n}}$$

$$B_n(t) = \frac{S_k}{\sqrt{n}} = \frac{S_k}{\sqrt{k}}\frac{\sqrt{k}}{\sqrt{n}}, \qquad \text{for } [nt] \leq k < [nt] + 1$$

Or

$$B_n(t) = \frac{S_k}{\sqrt{n}} = \frac{S_k}{\sqrt{k}}\frac{\sqrt{k}}{\sqrt{n}}, \qquad \text{for } \frac{k}{n} \leq t < \frac{k}{n} + 1$$

The normalized sum should show Brownian behaviour for n large

Gaussian Processes

A random vector $(X_1, \ldots, X_n)$ is said to be multivariate normal iff $Y = \alpha_1 X_1 + \ldots \alpha_n X_n$ is univariate Gaussian for all real α_i
With $\mu_i = \mathbb{E}(X_i)$ and $\Gamma_{ij} = \mathbb{C}\mathrm{ov}(X_i, X_j)$ we get

$$f(x_1, \ldots, x_n) = f(\boldsymbol{x}) = \frac{1}{(2\pi)^{\frac{n}{2}} \sqrt{\mathrm{Det}(\boldsymbol{\Gamma})}} e^{-\frac{1}{2}\boldsymbol{x}'\boldsymbol{\Gamma}^{-1}\boldsymbol{x}}$$

Gaussian Process

$$\mu(t) = \mathbb{E}[X(t)], \Gamma(s,t) = \mathbb{E}[\{X(s) - \mu(s)\}\{X(t) - \mu(t)\}]$$

$$\sum_{i=1}^{n} \sum_{j=1}^{n} \alpha_i \alpha_j \Gamma(t_i, t_j) \geq 0$$

The Reflection Principle

$$\tau = \min\{u \geq 0; B(u) = x\}$$

$$B^*(u) = \begin{cases} B(u) & \text{for } u \leq \tau, \\ x - [B(u) - x] & \text{for } \tau \leq u \end{cases}$$

$$\mathbb{P}\left\{\max_{0 \leq u \leq t} B(u) > x\right\} = 2\mathbb{P}(B(t) > x)$$

$$M(t) = \max_{0 \leq u \leq t} B(u)$$

$$\mathbb{P}(M(t) > x) = 2[1 - \Phi_t(x)]$$

Time to First Reach a Level

$$
\begin{aligned}
\tau_x &= \min\{u \geq 0; B(u) = x\} \\
\mathbb{P}(\tau_x \leq t) &= \mathbb{P}(M(t) > x) \\
2[1 - \Phi_t(x)] &= \frac{2}{\sqrt{2\pi t}} \int_x^\infty e^{-\xi^2/(2t)} \mathrm{d}\xi \\
&= \sqrt{\frac{2}{\pi}} \int_{\frac{x}{\sqrt{t}}}^\infty e^{-\eta^2/(2)} \mathrm{d}\eta \\
f_{\tau_x}(t) &= \frac{1}{\sqrt{2\pi}} \frac{x}{t\sqrt{t}} e^{-x^2/(2t)}
\end{aligned}
$$

Reflected Brownian Motion

$$
\begin{aligned}
R(t) &= \begin{cases} B(t) & \text{if } B(t) \geq 0 \\ -B(t) & \text{if } B(t) < 0 \end{cases} \\
\mathbb{E}(R(t)) &= \sqrt{2t/\pi} \\
\mathbb{V}\mathrm{ar}(R(t)) &= \left(1 - \frac{2}{\pi}\right) t \\
\mathbb{P}\{R(t) \leq y | R(0) = x\} &= \int_{-y}^{y} \phi_t(z - x) \mathrm{d}x \\
p(y, t | x) &= \phi_t(y - x) + \phi_t(y + x)
\end{aligned}
$$

Absorbed Brownian Motion

The movement ceases once the level 0 is reached.

$$\begin{aligned} G_t(x,y) &= \mathbb{P}\{A(t) > y | A(0) = x\} \\ &= \mathbb{P}\{B(t) > y, \min\{B(u) > 0; 0 \leq u \leq t | B(0) = x\} \end{aligned}$$

We first observe

$$\mathbb{P}\{B(t) > y | B(0) = x\} = G_t(x,y)$$

$$+\mathbb{P}\{B(t) > y, \min\{B(u) \leq 0; 0 \leq u \leq t\} | B(0) = x\}$$

Due to reflection the latter term is also

$$\mathbb{P}\{B(t) > y, \min\{B(u) \leq 0; 0 \leq u \leq t\} | B(0) = x\}$$

$$= \mathbb{P}\{B(t) < -y, \min\{B(u) \leq 0; 0 \leq u \leq t\} | B(0) = x\}$$

$$= \mathbb{P}\{B(t) < -y | B(0) = x\}$$

Absorbed Brownian Motion

Summarizing we get

$$
\begin{aligned}
\mathbb{P}\{A(t) > y | A(0) = x\} &= G_t(x, y) \\
&= 1 - \Phi_t(y - x) - \Phi_t(-y - x) \\
&= \Phi_t(y + x) - \Phi_t(y - x)
\end{aligned}
$$

We have already seen that

$$\mathbb{P}\{A(t) = 0 | A(0) = x\} = 2[1 - \Phi_t(x)]$$

Brownian Bridge

Distribution of $B(t); 0 \leq t \leq 1$ conditioned on $\{B(0) = 0, B(1) = 0\}$.

$$\begin{aligned}
\mathbb{E}\{B(t)|B(0) = 0, B(1) = 0\} &= 0 \\
\mathbb{V}\text{ar}\{B(t)|B(0) = 0, B(1) = 0\} &= t(1-t) \\
\mathbb{C}\text{ov}\{B(s), B(t)|B(0) = 0, B(1) = 0\} &= s(1-t)
\end{aligned}$$

The process is Gaussian

Brownian meander