Robotics I

Lecture 17

Manipulability Ellipsoid and Serial Robot Singularity

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Jacobian Singularity

$$
\begin{aligned}
\textbf{rank}(J) &= \textbf{dimension of manipulability ellipsoid} \\
&= \textbf{\# of independent columns in } J \\
&= \textbf{\# of independent directions in } SE(3) \textbf{ end effector can move} .
\end{aligned}
$$

$$
\begin{aligned}
& J \textbf{ is singular} \\
\iff & J \textbf{ loses rank } (R(J) \textbf{ reduces in dimension }) \\
\iff & \textbf{ manipulability ellipsoid becomes degenerate} \\
\iff & J \textbf{ gain addition DOF in self motion } (N(J) \textbf{ gains dimension}).
\end{aligned}
$$

Manipulability Ellipsoid

Jacobian may be interpreted as mapping of a unit ball in the joint velocity space to an ellipsoid in the task space:

$$J = U\Sigma V^T = \begin{bmatrix} u_1 & \ldots & u_N \end{bmatrix} \begin{bmatrix} \sigma_1 & & 0 \\ & \ddots & \\ 0 & & \sigma_n \end{bmatrix} \begin{bmatrix} v_1^T \\ \vdots \\ v_n^T \end{bmatrix}$$

where U and V are othorgonal matrices: $U^TU = I$ and $V^TV = I$.

The principal axes of the ellipsoid are given by the left singular vectors of the Jacobian, $\{u_1, \ldots, u_n\}$, and the lengths of the principal axes are given by the singular values of the Jacobian, $\{\sigma_1, \ldots, \sigma_n\}$. Large singular value indicates the direction that the arm is easy to move, and, conversely, small singular value indicates the direction that the arm is difficult to move (near singularities). For this reason, this ellipsoid is called the *manipulability ellipsoid*.

Dual Manipulability Ellipsoid

Dual perspective: the torque induced by the spatial force (force/torque) applied at the end effector is given by

$$\tau = J^T f \text{ or } f = J^{-T}\tau.$$

Again consider a unit ball in τ and the corresponding ellipsoid in f:

$$\begin{aligned} J^T &= V\Sigma^T U^T = V\Sigma U^T \\ J^{-T} &= U\Sigma^{-1}V^T \end{aligned}$$

Then the ellipsoid has the same principal axes but the lengths along these axes are the reciprocal of the manipulability ellipsoid. This is sometimes called the force ellipsoid, or the dual manipulability ellipsoid.

In general, a direction that is easy (hard) to manipulate would require large (small) torque to resist spatial force.

Example: 3-DOF wrist

$(h_1, h_2, h_3) = (x, y, z)$**, therefore, the Jacobian is**

$$(J)_1 = [h_1, \mathbf{rot}(h_1, \theta_1)h_2, \mathbf{rot}(h_1, \theta_1)\mathbf{rot}(h_2, \theta_2)h_3].$$

Or more conveniently expressed in frame 2:

$$(J)_2 = [h_1, h_2, \mathbf{rot}(h_2, \theta_2)h_3] = \begin{bmatrix} 1 & 0 & s_2 \\ 0 & 1 & 0 \\ 0 & 0 & c_2 \end{bmatrix}.$$

Singularity occurs at $det(J) = 0$ **or** $\cos(\theta_2) = 0$ **or** $\theta_2 = \pm\frac{\pi}{2}$**.**

In coordinate-free perspective, $J = [\vec{h}_1, \vec{h}_2, \vec{h}_3]$**, where** $\vec{h}_1 \perp \vec{h}_2$ **and** $\vec{h}_2 \perp \vec{h}_3$**. Then** J **will lose rank if and only if** $\vec{h}_1$ **and** $\vec{h}_3$ **are linearly dependent.**

Example: 3-DOF planar arm

$h_1 = h_2 = h_3 = z$, **and**

$$J = \begin{bmatrix} \vec{z} & \vec{z} & \vec{z} \\ \vec{z} \times \vec{p}_{13} & \vec{z} \times \vec{p}_{23} & 0 \end{bmatrix}.$$

J **loses rank (rank less than 3) if and only if** $\vec{p}_{13}$ **and** $\vec{p}_{23}$ **are collinear, i.e.,** $\theta_2 = 0$ **or** π**. In coordinate frame,**

$$J_2 = \begin{bmatrix} z & z & z \\ \hat{z}(\mathbf{rot}(-z,\theta_2)p_{12} + p_{23}) & \hat{z}p_{23} & 0 \end{bmatrix} \sim \begin{bmatrix} 0 & 0 & z \\ \hat{z}(\mathbf{rot}(-z,\theta_2)p_{12}) & \hat{z}p_{23} & 0 \end{bmatrix}.$$

Since $p_{12} = \ell_1 x$, $p_{23} = \ell_2 x$, **we have the determinant of the (2×2 portion of) Jacobian** $= \ell_1 \ell_2 \sin\theta_2$.

Example: PUMA 560

Choose A at the origin of the spherical wrist joints, then

$$J_A = \begin{bmatrix} \vec{h}_1 & \vec{h}_2 & \vec{h}_3 & \vec{h}_4 & \vec{h}_5 & \vec{h}_6 \\ \vec{h}_1 \times \vec{p}_{14} & \vec{h}_2 \times \vec{p}_{24} & \vec{h}_3 \times \vec{p}_{34} & 0 & 0 & 0 \end{bmatrix}.$$

Singularities:

- $\begin{bmatrix} \vec{h}_4 & \vec{h}_5 & \vec{h}_6 \end{bmatrix}$ **loses rank.**
- $\begin{bmatrix} \vec{h}_1 \times \vec{p}_{14} & \vec{h}_2 \times \vec{p}_{24} & \vec{h}_3 \times \vec{p}_{34} \end{bmatrix}$ **loses rank**

First consider $\begin{bmatrix} \vec{h}_4 & \vec{h}_5 & \vec{h}_6 \end{bmatrix}$ losing rank. This is the same as the first example, and singularity occurs at $\sin\theta_5 = 0$.

PUMA 560 Singularities

Now consider $\begin{bmatrix} \vec{h}_1 \times \vec{p}_{14} & \vec{h}_2 \times \vec{p}_{24} & \vec{h}_3 \times \vec{p}_{34} \end{bmatrix}$ **losing rank. Noting that** $\vec{p}_{12}$ **is collinear with** $\vec{h}_1$**,** $\vec{h}_2 = \vec{h}_3$**, and applying elementary column reduction, we obtain**

$$\begin{bmatrix} \vec{h}_1 \times \vec{p}_{24} & \vec{h}_2 \times \vec{p}_{23} & \vec{h}_2 \times \vec{p}_{34} \end{bmatrix}.$$

From before,

$$h_1 = z, h_2 = -y, h_3 = -y, h_4 = z, h_5 = -y, h_6 = z$$

$$p_{01} = 0, p_{12} = d_1 z, p_{23} = \ell_2 x - d_3 y, p_{34} = -\ell_3 x + d_4 z, p_{45} = 0, p_{56} = 0.$$

Now represent the submatrix in 2**-frame, we obtain:**

$$\begin{bmatrix} d_3 c_2 & 0 & s_3\ell_3 - c_3 d_4 \\ \ell_2 c_2 - \ell_3 c_{23} - d_4 s_{23} & 0 & 0 \\ -d_3 s_2 & \ell_2 & -c_3\ell_3 - s_3 d_4 \end{bmatrix}.$$

The determinant is $\ell_2(\ell_2 c_2 - \ell_3 c_{23} - d_4 s_{23})(s_3\ell_3 - c_3 d_4)$. **Therefore, the singularities are at:**

- $s_3\ell_3 - c_3 d_4 = 0$**, or** $\tan\theta_3 = d_4/\ell_3$ **(boundary singularity)**
- $\ell_2 c_2 - \ell_3 c_{23} - d_4 s_{23} = 0$**. This is an interior singularity.**

Geometric conditions for singularities

- **A 6-DOF arm is singular if for any two revolute joints, the joint axes are collinear (e.g., spherical wrist).**
- **A 6-DOF arm is singular if any three parallel rotational axes lie in a plane (e.g., boundary singularity of an elbow arm).**
- **A 6-DOF arm is singular if any four rotational axes intersect at a point or three coplanar revolute axes intersect at a point.**

Two Collinear revolute axes

Suppose that $\vec{h}_i = \vec{h}_j$ **and** i, j **are both revolute axes. Then the corresponding columns in the Jacobian is**

$$\begin{bmatrix} \vec{h}_i & \vec{h}_j \\ \vec{h}_i \times \vec{p}_{iE} & \vec{h}_j \times \vec{p}_{jE} \end{bmatrix} \sim \begin{bmatrix} 0 & \vec{h}_j \\ \vec{h}_i \times \vec{p}_{ij} & \vec{h}_j \times \vec{p}_{jE} \end{bmatrix}.$$

Since h_i **and** $\vec{p}_{ij}$ **are collinear, the Jacobian loses rank.**

Three Coplanar revolute axes

Let i, j, k **be three revolute joints and**

$$\vec{h}_i = \vec{h}_j = \vec{h}_k = \vec{h}.$$

The corresponding columns in the Jacobian are:

$$\begin{bmatrix} \vec{h}_i & \vec{h}_j & \vec{h}_k \\ \vec{h}_i \times \vec{p}_{iE} & \vec{h}_j \times \vec{p}_{jE} & \vec{h}_k \times \vec{p}_{kE} \end{bmatrix} \sim \begin{bmatrix} 0 & 0 & \vec{h} \\ \vec{h} \times \vec{p}_{ij} & \vec{h} \times \vec{p}_{jk} & \vec{h} \times \vec{p}_{kE} \end{bmatrix}.$$

Since $(\vec{h}, \vec{p}_{ij}, \vec{p}_{jk})$ **are coplanar, the first two columns are linearly dependent.**

Four intersecting revolute axes

Let (i,j,k,m) **be four revolute joints, and the corresponding rotational axes,** $(\vec{h}_i, \vec{h}_j, \vec{h}_k, \vec{h}_m)$ **intersect at a point** s**. The corresponding columns in the Jacobian are:**

$$\begin{bmatrix} \vec{h}_i & \vec{h}_j & \vec{h}_k & \vec{h}_m \\ \vec{h}_i \times \vec{p}_{iE} & \vec{h}_j \times \vec{p}_{jE} & \vec{h}_k \times \vec{p}_{kE} & \vec{h}_m \times \vec{p}_{mE}. \end{bmatrix}$$

$$\sim \quad \begin{bmatrix} \vec{h}_i & \vec{h}_j & \vec{h}_k & \vec{h}_m \\ \vec{h}_i \times \vec{p}_{sE} & \vec{h}_j \times \vec{p}_{sE} & \vec{h}_k \times \vec{p}_{sE} & \vec{h}_m \times \vec{p}_{sE}. \end{bmatrix}.$$

Premultiply by $\begin{bmatrix} I & 0 \\ \vec{p}_{sE} & I \end{bmatrix}$**, we obtain** $\begin{bmatrix} \vec{h}_i & \vec{h}_j & \vec{h}_k & \vec{h}_m \\ 0 & 0 & 0 & 0 \end{bmatrix}$, **which has maximum rank 3.**

Three coplanar intersecting revolute axes

Repeat the same argument as before for the three coplanar axes, $\{\vec{h}_i, \vec{h}_j, \vec{h}_k\}$, we obtain the (sub-)Jacobian similar to

$$\begin{bmatrix} \vec{h}_i & \vec{h}_j & \vec{h}_k \\ 0 & 0 & 0 \end{bmatrix}.$$

Since $\{\vec{h}_i, \vec{h}_j, \vec{h}_k\}$ are coplanar, $[\vec{h}_i, \vec{h}_j, \vec{h}_k]$ is of rank 2.

