Robotics I

Lecture 08

Forward Kinematics of Articulated Robots

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Forward Kinematics Algorithm

Given $(q_1,\ldots,q_n)$**, find** (p_{0n},R_{0n})**.**

1. **Define the base frame** $\mathcal{E}_0$ **and origin** O_0**.**
2. **Define the *zero configuration*, the configuration at which all joint displacements are defined to be zero and all frames are aligned,** $\mathcal{E}_i=\mathcal{E}_0$**,** $i=1,\ldots,n+1$**.**
3. **Choose origins of** i**th body,** O_i**, along the motion axis** $\vec{h}_i$**.**
4. **While the chain is in the zero configuration,**
   (a) **find** h_i **(**$\vec{h}_i$ **represented in** $\mathcal{E}_{i-1}$**) by expressing** $\vec{h}_i$ **in** $\mathcal{E}_0$ **(since** $\mathcal{E}_{i-1}=\mathcal{E}_0$**).**
   (b) **find** $p_{i-1,i}$ **(**$\vec{p}_{i-1,i}$ **represented in** $\mathcal{E}_{i-1}$**) by representing** $\vec{p}_{i-1,i}$ **in** $\mathcal{E}_0$ **(since** $\mathcal{E}_{i-1}=\mathcal{E}_0$**).**
5. **Apply the forward kinematic equations to obtain** (p_{0T},R_{0T})**.**
   $R_{0T}(q)=R_{01}(q_1)R_{12}(q_2)\ldots R_{n-1,n}(q_n)R_{nT}$
   $p_{0T}(q)=p_{01}+R_{01}(q_1)p_{12}+\ldots+R_{0,n-1}(q_1,\ldots,q_{n-1})p_{n-1,n}+R_{0,n}(q_1,\ldots,q_n)p_{nT}.$
   If i**th joint is revolute:** $p_{i-1,i}$ **is a constant vector,** $R_{i-1,i}=$ **rot**(h_i,q_i)**.**
   If i**th joint is prismatic:** $p_{i-1,i}=p_{i-1,i}(0)+q_ih_i$ **where** $p_{i-1,i}(0)$ **is a constant vector from the zero configuration,** $R_{i-1,i}=I$**.**

Forward Kinematics Implementation

To implement forward kinematics numerically, we need the following *constant* vectors:

- **Axes of motion:** $(h_i)_{i-1}$
- **Link vectors:** $(p_{i-1,i}(0))_{i-1}$

To find these vectors, first put the chain in the "zero configuration" (you choose!), i.e., when all joint angles are zero. In this configuration, all frames are aligned, so

$$(h_i)_{i-1} = (h_i)_0, \quad (p_{i-1,i})_{i-1} = (p_{i-1,i})_0$$

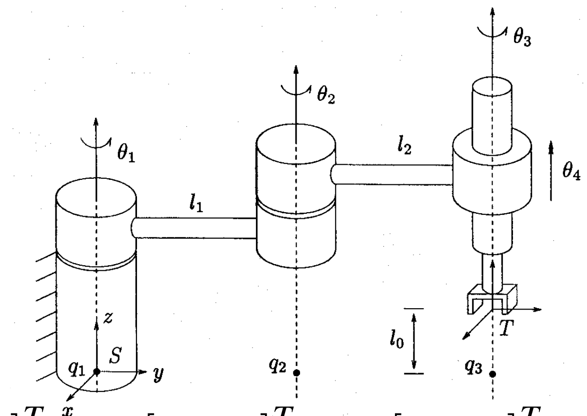
which means we just read off these vectors in the inertial frame.

Summary of procedure for forward kinematics:

- **put chain in the chosen zero configuration**
- **choose origin of ith frame along h_i**
- **Express $\vec{h}_i$ and $\vec{p}_{i,i+1}$ all in E_0.**
- **Apply forward kinematics formula.**

Now let's look at some examples!

Example: SCARA Arm (4 DOF, RRRP)



$x = [1,0,0]^T, y = [0,1,0]^T, z = [0,0,1]^T$

$h_1 = h_2 = h_3 = h_4 = z$

$p_{01} = 0, p_{12} = \ell_1 y, p_{23} = \ell_2 y, p_{34} = \ell_3 z$

$R_{01} = \textbf{rot}(z,q_1), R_{12} = \textbf{rot}(z,q_2), R_{23} = \textbf{rot}(z,q_3), R_{34} = I$

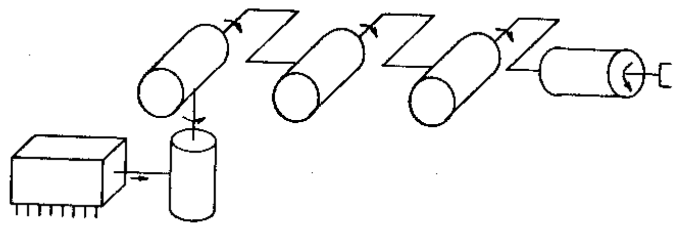
forward kinematics: rotation

$$R_{04} = R_{01}R_{12}R_{23}R_{34}$$

forward kinematics: position

$$p_{04} = R_{01}p_{12} + R_{01}R_{12}p_{23} + R_{01}R_{12}R_{23}(p_{34} + q_4 h_4).$$

Example: Rhino Arm (6 DOF, PRRRRR)



(iv) Rhino robot

$h_1 = y, h_2 = z, h_3 = h_4 = h_5 - x, h_6 = y$

$p_{01} = \ell_1 y, p_{12} = 0, p_{23} = \ell_2 z, p_{34} = \ell_3 y, p_{45} = \ell_4 y, p_{56} = \ell_6 y$

forward kinematics: rotation

$R_{06} = \mathbf{rot}(h_2, q_2)\mathbf{rot}(h_3, q_3)\mathbf{rot}(h_4, q_4)\mathbf{rot}(h_5, q_5)\mathbf{rot}(h_6, q_6)$

forward kinematics: position

$p_{06} = (p_{01} + q_1 h_1) + p_{12} + R_{12} p_{23} + R_{12} R_{23} p_{34} + R_{12} R_{23} R_{34} p_{45} + R_{12} R_{23} R_{34} R_{45} p_{56}.$

MATLAB code (on course webpage): `scara.m`**,** `rhino.m` **(also** `elbow.m` **and** `puma560.m`**) which call the general forward kinematics code** `fwdkin.m` **to compute** R_{0T} **and** p_{0T} `showarm.m` **and** `showatt.m` **for visualization.**

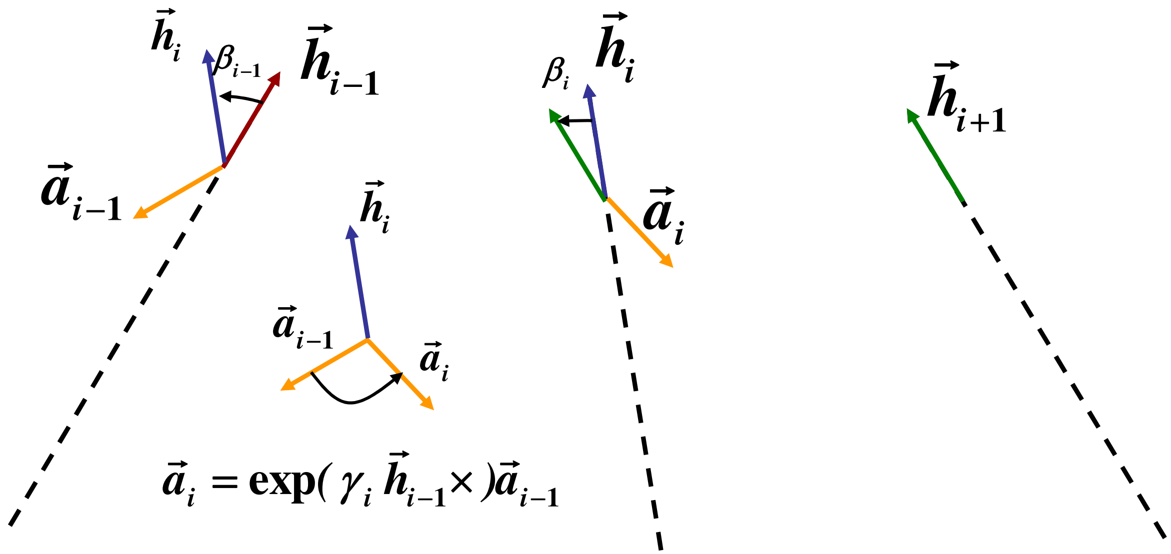
Relationship to Denavit-Hartenberg Parameters

In describing the relative rotation and translation between consecutive frames, we need $(p_{i-1,i}, R_{i-1,i}) \in SE(3)$ which can be parameterized by at least 6 parameters.

The Denavit-Hartenberg (DH) parameterization choose each frame in a particular way so that one needs only 4 parameters to describe the relative configuration between consecutive frames.

Rotation

$$\vec{h}_i = \exp(\beta_i \vec{a}_{i-1} \times) \vec{h}_{i-1}$$



$$\vec{a}_i = \exp(\gamma_i \vec{h}_{i-1} \times) \vec{a}_{i-1}$$

$$\vec{a}_{i-1} := \frac{\vec{h}_{i-1} \times \vec{h}_i}{\left\| \vec{h}_{i-1} \times \vec{h}_i \right\|}$$

DH and Modified DH: Rotation

Relative orientation:

- **If $\vec{h}_i$ is *not* parallel to $\vec{h}_{i-1}$, choose $\vec{a}_{i-1} = \frac{\vec{h}_{i-1}\times\vec{h}_i}{\|\vec{h}_{i-1}\times\vec{h}_i\|}$.**

  If $\vec{h}_i$ is parallel to $\vec{h}_{i-1}$, choose $\vec{a}_{i-1}$ as the common normal (i.e., $\vec{a}_{i-1}$ is a unit vector perpendicular to $\vec{h}_i$ and $\vec{h}_{i-1}$).

- **There exists a unique angle β_{i-1} (we shall see a numerically stable solution method for it soon) such that**

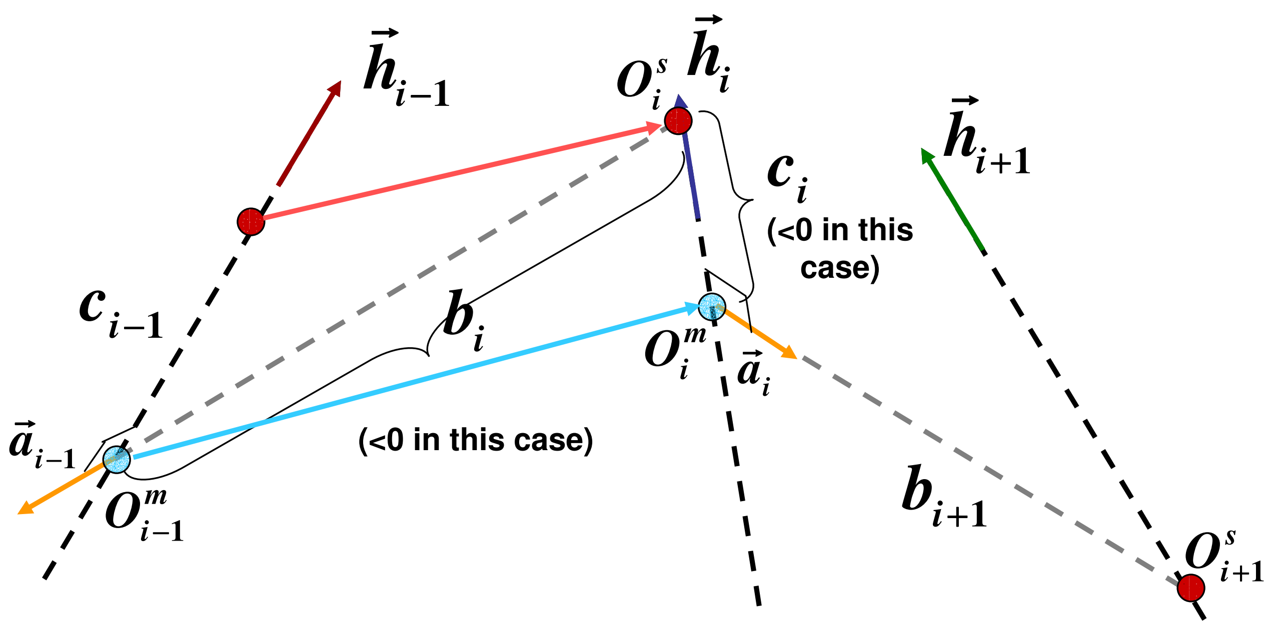
$$\vec{h}_i = \mathbf{rot}(\vec{a}_{i-1}, \beta_{i-1})\vec{h}_{i-1}.$$

- **Note that $\vec{a}_{i-1}$ and $\vec{a}_i$ are both perpendicular to $\vec{h}_i$. Therefore, there exists a unique angle γ_i such that**

$$\vec{a}_i = \mathbf{rot}(\vec{h}_i, \gamma_i)\vec{a_{i-1}}.$$

$(\vec{h}_i, \vec{a}_i)$ defines a frame (use cross product to generate the third coordinate vector). To propagate $(\vec{h_{i-1}}, \vec{a_{i-1}})$ to $(\vec{h}_i, \vec{a}_i)$, we only need (β_{i-1}, γ_i) (one less parameters than full rotation!).

Translation



$$O_{i-1}^s \rightarrow O_i^s : \vec{p}_{i-1,i} = c_{i-1}\vec{h}_{i-1} + b_i\vec{a}_{i-1}$$

$$O_{i-1}^m \rightarrow O_i^m : \vec{p}_{i-1,i} = b_i\vec{a}_{i-1} + c_i\vec{h}_i$$

DH and Modified DH: Translation

Define

$O_i^s =$ **intersection between** $\vec{a}_{i-1}$ **and** $\vec{h}_i$

$O_i^m =$ **intersection between** $\vec{a}_i$ **and** $\vec{h}_i$

If $\vec{h}_{i-1}$ **and** $\vec{h}_i$ **are parallel, choose a location of** $\vec{a}_i$ **along** $\vec{h}_i$**.**

Let

$c_i =$ **length of the vector from** O_i^s **to** O_i^m **along** $\vec{h}_i$

$b_i =$ **length of the vector from** O_{i-1}^m **to** O_i^s **along** $\vec{a}_{i-1}$

If we choose O_i^s **as the origin of frame** i**, then the vector from** O_{i-1}^s **to** O_i^s **is**

$$\vec{p}_{i-1,i} = c_{i-1}\vec{h}_{i-1} + b_i\vec{a}_{i-1} \quad \textbf{Standard D-H}\ .$$

If we choose O_i^m **as the origin of frame** i**, then the vector from** O_{i-1}^m **to** O_i^m **is**

$$\vec{p}_{i-1,i} = b_i\vec{a}_{i-1} + c_i\vec{h}_i \quad \textbf{Modified D-H}\ .$$

To describe the vector between consecutive origins, we only need (c_i, b_i) **– two parameters instead of three!**

Standard DH Parameters

In Standard DH parameterization, ith frame is given by $(\mathcal{E}_i^s, \mathcal{O}_i^s)$ with $\mathcal{E}_{i-1}^s = [\vec{a}_{i-1}, \vec{h}_i \times \vec{a}_{i-1}, \vec{h}_i]$, $i = 1, \ldots, n$ (note that $\vec{z}_{i-1} = \vec{h}_i$!). Then

$$\mathcal{E}_i^s = \mathbf{rot}(\vec{a}_i, \beta_{i+1})\mathbf{rot}(\vec{h}_i, \gamma_i)\mathcal{E}_{i-1}^s.$$

Multiply $\mathcal{E}_i^{s*}$ on both sides:

$$I = \mathbf{rot}(x, \beta_{i+1})R_{i,i-1}\mathbf{rot}(z, \gamma_i)$$

where $x = [1,0,0]^T$ and $z = [0,0,1]^T$. Then

$$R_{i-1,i} = \mathbf{rot}(z, \gamma_i)\mathbf{rot}(x, \beta_{i+1}).$$

For translation,

$$(\vec{p}_{i-1,i})_{i-1} = c_{i-1}R_{i,i-1}z + b_i x.$$

Modified DH Parameters

In Modified DH parameterization, ith frame is given by $(\mathcal{E}_i^m, O_i^m)$ with $\mathcal{E}_i^m = [\vec{a}_i, \vec{h}_i \times \vec{a}_i, \vec{h}_i]$, $i = 1, \ldots, n$. Then

$$\mathcal{E}_i^m = \mathbf{rot}(\vec{h}_i, \gamma_i)\mathbf{rot}(\vec{a}_{i-1}, \beta_i)\mathcal{E}_{i-1}^m.$$

Multiply $\mathcal{E}_i^{m*}$ on both sides:

$$I = \mathbf{rot}(z, \gamma_i) R_{i,i-1} \mathbf{rot}(x, \beta_i)$$

where $x = [1, 0, 0]^T$ and $z = [0, 0, 1]^T$. Then

$$R_{i-1,i} = \mathbf{rot}(x, \beta_i)\mathbf{rot}(z, \gamma_i).$$

For translation,

$$(\vec{p}_{i-1,i})_{i-1} = c_i R_{i-1,i} z + b_i x.$$

Summary of SDH vs. MDH

SDH

$\mathcal{E}_i = [\vec{a}_i, \vec{h}_{i+1} \times \vec{a}_i, \vec{h}_{i+1}]$**,** $i = 0, 1, \ldots, n-1$

$\alpha_i = \beta_{i+1}$ **(twist angle),** $q_i = \gamma_i$

$R_{i-1,i} = \mathbf{rot}(z, q_i)\mathbf{rot}(x, \alpha_i)$

$\ell_i = b_i$**,** $d_i = c_{i-1}$

$(p_{i-1,i})_{i-1} = d_i R_{i,i-1} z + \ell_i x.$

MDH

$\mathcal{E}_i = [\vec{a}_i, \vec{h}_i \times \vec{a}_i, \vec{h}_i]$**,** $i = 1, \ldots, n$

$\alpha_i = \beta_i$ **(twist angle),** $q_i = \gamma_i$

$R_{i-1,i} = \mathbf{rot}(x, \alpha_i)\mathbf{rot}(z, q_i)$

$\ell_{i-1} = b_i$**,** $d_i = c_i$

$(p_{i-1,i})_{i-1} = d_i R_{i-1,i} z + \ell_{i-1} x.$

$(\alpha_i, q_i, \ell_i, d_i)$ **are called the (S or M) DH parameters for joint** i**.**

Forward kinematics remains the same, except with the $(R_{i-1,i}, p_{i-1,i})$ **from above. Zero configuration corresponds to all** $\vec{a}_i$**'s are aligned.**

