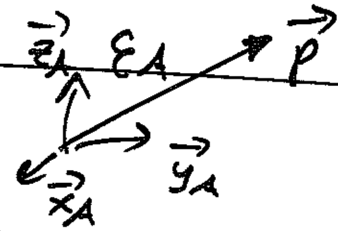
- Send team information (names + emails)
  + computer info (32-bit/64-bit, OS, express card slot, etc.)
  MATLAB version
  to instructor (wenj@rpi.edu)

HW's from last year



Rotate $\vec{p}$ about $\vec{y}_A$ by $30°$ then rotate about $\vec{x}_A$ by $45°$. Find rotation matrix represented in $\mathcal{E}_A$.

$$\mathcal{E}_B = \mathrm{rot}(\vec{x}_A, 45°)\,\mathrm{rot}(\vec{y}_A, 30°)\,\mathcal{E}_A$$

$$R_{AB} = \mathcal{E}_A^* \mathcal{E}_B = \mathcal{E}_A^* \mathrm{rot}(\vec{x}_A, 45°)\,\mathrm{rot}(\vec{y}_A, 30°)\,\mathcal{E}_A$$

$$= \underbrace{\mathcal{E}_A^* \mathrm{rot}(\vec{x}_A, 45°)\mathcal{E}_A}\underbrace{\mathcal{E}_A \mathcal{E}_A^*}_{= I} \mathrm{rot}(\vec{y}_A, 30°)\,\mathcal{E}_A$$

$$= \mathrm{rot}\Big(\underbrace{(\vec{x}_A)_A}_{= \begin{bmatrix}1\\0\\0\end{bmatrix}}, \underbrace{45°}_{\pi/4}\Big)\,\mathrm{rot}\Big(\underbrace{(\vec{y}_A)_A}_{= \begin{bmatrix}0\\1\\0\end{bmatrix}}, \underbrace{30°}_{\pi/6}\Big)$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\frac{\pi}{4} & -\sin\frac{\pi}{4} \\ 0 & \sin\frac{\pi}{4} & \cos\frac{\pi}{4} \end{bmatrix} \begin{bmatrix} \cos\frac{\pi}{6} & 0 & \sin\frac{\pi}{6} \\ 0 & 1 & 0 \\ -\sin\frac{\pi}{6} & 0 & \cos\frac{\pi}{6} \end{bmatrix}$$

Rotate $\mathcal{E}_A$ about $\vec{z}_A$ by θ, then rotate about body $\vec{x}$ by ϕ, find R_{AB}.

$$\mathcal{E}_B = \mathrm{rot}(\vec{x}_{A'}, \phi)\, \underbrace{\mathrm{rot}(\vec{z}_A, \theta)\, \mathcal{E}_A}_{\mathcal{E}_{A'}}$$

$$\mathcal{E}_A^* \mathcal{E}_B = \mathcal{E}_A^* \mathrm{rot}(\vec{x}_{A'}, \phi)\ \mathcal{E}_{A'}$$

$$= \mathcal{E}_A^* \underbrace{\mathcal{E}_{A'} \mathcal{E}_{A'}^*}_{\mathcal{I}} \mathrm{rot}(\vec{x}_{A'}, \phi) \mathcal{E}_{A'}$$

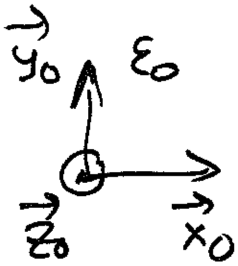
$$= \mathcal{E}_A^* \mathcal{E}_{A'}\ \mathrm{rot}(\underbrace{(\vec{x}_{A'})_{A'}}_{\begin{bmatrix}1\\0\\0\end{bmatrix}}, \phi)$$

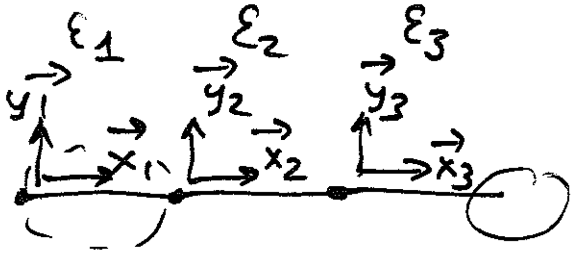
$$= \mathcal{E}_A^* \mathrm{rot}(\vec{z}_A, \theta) \mathcal{E}_A\ \mathrm{rot}\left(\begin{bmatrix}1\\0\\0\end{bmatrix}, \phi\right)$$

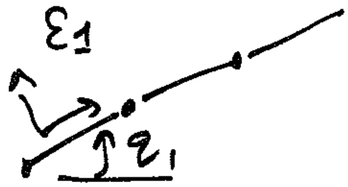
$$= \mathrm{rot}((\vec{z}_A)_A, \theta)\ \mathrm{rot}\left(\begin{bmatrix}1\\0\\0\end{bmatrix}, \phi\right)$$

$$= \mathrm{rot}\left(\begin{bmatrix}0\\0\\1\end{bmatrix}, \theta\right)\ \mathrm{rot}\left(\begin{bmatrix}1\\0\\0\end{bmatrix}, \phi\right)$$

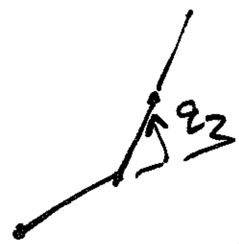
Planar Arm







$$\mathcal{E}_1 = rot(\vec{z}_0, q_1)\, \mathcal{E}_0$$



$$\mathcal{E}_2 = rot(\vec{z}_0, q_2)\, \mathcal{E}_1$$

$$\mathcal{E}_3 = rot(\vec{z}_0, q_3)\, \mathcal{E}_2$$

$$R_{01} = \mathcal{E}_0^* \mathcal{E}_1 = \mathcal{E}_0^* rot(\vec{z}_0, q_1)\, \mathcal{E}_0$$

$$= rot((\vec{z}_0)_0, q_1)$$

$$= rot\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, q_1\right)$$

$$R_{12} = \mathcal{E}_1^* \mathcal{E}_2 = \mathcal{E}_1^* rot(\vec{z}_0, q_2)\, \mathcal{E}_1$$

$$= rot((\vec{z}_0)_1, q_2)$$

$$= rot\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, q_2\right)$$

$$R_{23} = \mathcal{E}_2^* \mathcal{E}_3 = rot\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, q_3\right)$$

Orientation of tool frame (same as $\mathcal{E}_3$)

$$R_{03} = \mathcal{E}_0^* \mathcal{E}_3 = \underline{R_{01}}\,\underline{R_{12}}\,\underline{R_{23}} = \mathrm{rot}\left(\begin{bmatrix}0\\0\\1\end{bmatrix}, q_1\right)\mathrm{rot}\left(\begin{bmatrix}0\\0\\1\end{bmatrix}, q_2\right)\mathrm{rot}\left(\begin{bmatrix}0\\0\\1\end{bmatrix}, q_3\right)$$

$$= \mathrm{rot}\left(\begin{bmatrix}0\\0\\1\end{bmatrix}, \underbrace{q_1+q_2+q_3}\right)$$

Derivative of $R \in SO(3)$

Recall $SO(2)$: ← planar case

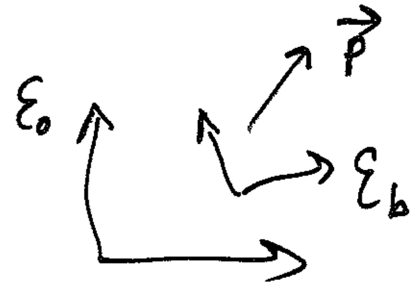
$$R = \begin{bmatrix}\cos\theta & -\sin\theta \\ \sin\theta & \cos\theta\end{bmatrix}$$

$$\dot{R} = \begin{bmatrix}-\sin\theta & -\cos\theta \\ \cos\theta & -\sin\theta\end{bmatrix}\dot{\theta}$$

$$= \dot{\theta}\underbrace{\begin{bmatrix}0 & -1 \\ 1 & 0\end{bmatrix}}_{\hat{k}} R$$

$\dot{\theta}$ ↙ scalar

$$k = \begin{bmatrix}0\\0\\1\end{bmatrix} \quad \hat{k} = \begin{bmatrix}0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0\end{bmatrix}$$



Q: How do we describe the rate of change of $\vec{p}$?

A: Use rate of change of $\vec{p}$ in a frame.

$\in \mathbb{V}^3$ $\quad \mathbb{R}^3$

$$\mathcal{E}_o^* \vec{p} = p_o \qquad \qquad \mathcal{E}_b^* \vec{p} = p_b$$

Define $\frac{d\vec{p}}{dt^o}$

$$\mathcal{E}_o^* \frac{d\vec{p}}{dt^o} = \boxed{\frac{dp_o}{dt}} \leftarrow \dot{p}_o \qquad \qquad \mathcal{E}_b^* \frac{d\vec{p}}{dt^b} = \boxed{\frac{dp_b}{dt}} \leftarrow \dot{p}_b$$

$$\frac{d\vec{p}}{dt^o} = \mathcal{E}_o \frac{dp_o}{dt}$$

$\widehat{\dot{\theta} k} \leftarrow$ call it $\hat{\omega}$

$$p_o = R_{ob}\, p_b \qquad \qquad \dot{R}_{ob} = \dot{\theta}\hat{k} R_{ob}$$

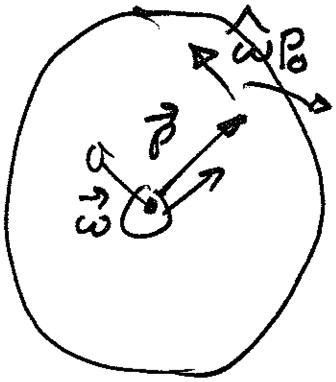
$$\dot{p}_o = \dot{R}_{ob}\, p_b + R_{ob}\, \dot{p}_b$$

$$= \hat{\omega} R_{ob}\, p_b + R_{ob}\, \dot{p}_b$$

$$= \hat{\omega} p_o + R_{ob}\, \dot{p}_b$$

if $\vec{p}$ is a position vector, $\dot{p}_o$ is the velocity seen in $\mathcal{E}_o$, $\dot{p}_b$ is velocity seen in $\mathcal{E}_b$.

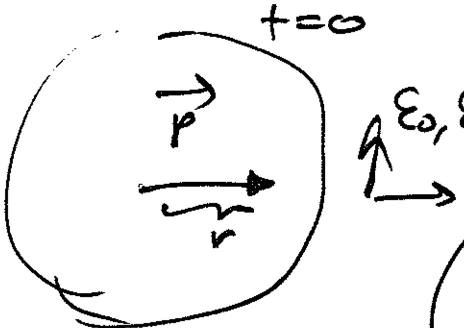
coordinate-free (or vectorial) form: 9/15/11-6



$$\underbrace{\mathcal{E}_0 \dot{P}_0}_{} = \mathcal{E}_0 \hat{\omega} P_0 + \underbrace{\mathcal{E}_0 R_{ob} \dot{P}_b}_{}$$

$$\frac{d\vec{p}}{dt^o} = \underbrace{\mathcal{E}_0 \hat{\omega} \mathcal{E}_0^T}_{\vec{\omega} \times} \underbrace{\mathcal{E}_0 P_0}_{\vec{p}} + \underbrace{\mathcal{E}_b \dot{P}_b}_{\frac{d\vec{p}}{dt^b}}$$

t=0



$$\frac{d\vec{p}}{dt^o} = \underbrace{\vec{\omega} \times \vec{p}}_{\text{linear velocity due to rotation}} + \underbrace{\frac{d\vec{p}}{dt^b}}_{} \rightarrow \text{linear velocity in the body}$$

For $\vec{p}$ fixed in $\mathcal{E}_b$, $\frac{d\vec{p}}{dt^b} = 0$ $\quad (\vec{p})_b = P_b = \begin{bmatrix} r \\ 0 \\ 0 \end{bmatrix}$

$$\therefore \frac{dP_b}{dt} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Acceleration:

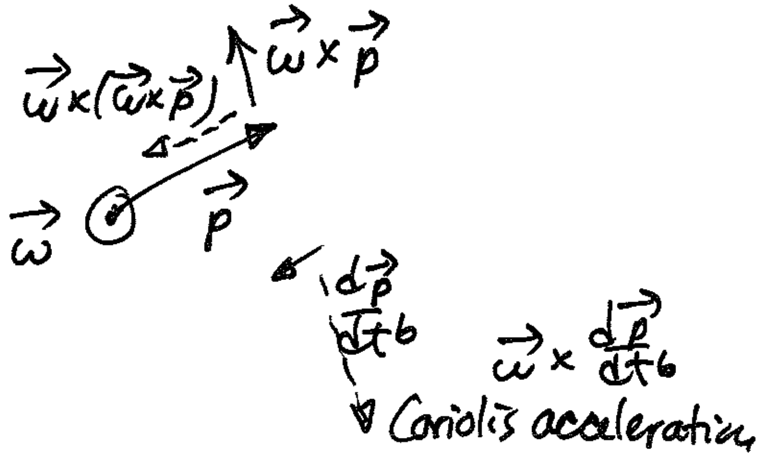
$$P_0 = R_{ob} P_b$$

$$\dot{P}_0 = \hat{\omega} P_0 + R_{ob} \dot{P}_b$$

$$\ddot{P}_0 = \dot{\hat{\omega}} P_0 + \hat{\omega} \underbrace{\dot{P}_0}_{\hat{\omega} P_0 + R_{ob} \dot{P}_b} + \dot{R}_{ob} \dot{P}_b + R_{ob} \ddot{P}_b$$

$$= \dot{\hat{\omega}} P_0 + \hat{\omega} \hat{\omega} P_0 + \hat{\omega} R_{ob} \dot{P}_b + \hat{\omega} R_{ob} \dot{P}_b + R_{ob} \ddot{P}_b$$

$$\underbrace{\ddot{P}_0}_{\text{linear acceleration in } \mathcal{E}_0} = \underbrace{R_{0b}\ddot{P}_b}_{\text{linear acceleration in } \mathcal{E}_b \text{ as seen in } \mathcal{E}_0} + \underbrace{\hat{\dot{\omega}} P_0}_{\text{linear acceleration due to angular acceleration}} + \underbrace{\hat{\omega}\hat{\omega} P_0}_{\text{Centrifugal acceleration}} + \underbrace{2\hat{\omega} R_{0b}\dot{P}_b}_{\text{Coriolis acceleration}}$$



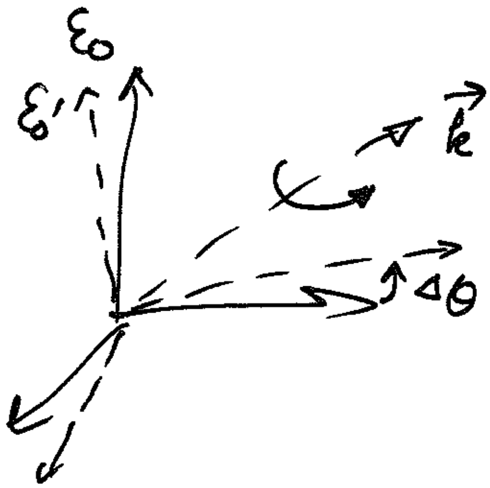
vectorial form

$\mathcal{E}_0$ (- —)

$\Downarrow$

$$\frac{d^2\vec{p}}{dt^{0\,2}} = \frac{d^2\vec{p}}{dt^{b\,2}} + \frac{d\vec{\omega}}{dt^0}\times\vec{p} + \vec{\omega}\times(\vec{\omega}\times\vec{p}) + 2\vec{\omega}\times\frac{d\vec{p}}{dt}$$

Back to Spatial ($\mathbb{V}^3$) case



Consider a small rotation about $\vec{k}$ (over $\Delta\theta$) of $\mathcal{E}_0$:

$$\mathcal{E}_0' = \mathrm{rot}(\vec{k}, \Delta\theta)\,\mathcal{E}_0$$

$$\mathcal{E}_0' = \Big(\mathcal{I} + \underbrace{\sin\Delta\theta}_{\Delta\theta}\,\vec{k}\times + \underbrace{(1-\cos\Delta\theta)}_{\approx 0}\,\vec{k}\times(\vec{k}\times\ \ )\Big)\mathcal{E}_0$$

$$\approx \left(\mathcal{I} + \Delta\theta\,\vec{k}\times\right)\mathcal{E}_0$$

Scalar

$$\frac{\mathcal{E}_0' - \mathcal{E}_0}{\Delta t} \approx \frac{\Delta\theta}{\Delta t}\vec{k}\times\mathcal{E}_0$$

$$\downarrow$$

Angular velocity of $\mathcal{E}_0$

$$\frac{d\mathcal{E}_0}{dt} = \vec{\omega}_0 \times \mathcal{E}_0$$

If $\vec{k}$ is fixed, $\underline{\underline{\vec{\omega}_0 = \vec{k}\,\dot{\theta}}}$ (like the planar case)

Now consider another frame $\mathcal{E}_b$: 9/07/11-8

$$\frac{d\mathcal{E}_b}{dt} = \vec{\omega}_b \times \mathcal{E}_b$$

$$\frac{dR_{ob}}{dt} = \frac{d}{dt}\left(\mathcal{E}_0^* \mathcal{E}_b\right) = \frac{d\mathcal{E}_0^*}{dt}\mathcal{E}_b + \mathcal{E}_0^* \frac{d\mathcal{E}_b}{dt}$$

$$= -\mathcal{E}_0^* \vec{\omega}_0 \times \mathcal{E}_b + \mathcal{E}_0^* \vec{\omega}_b \times \mathcal{E}_b$$

$$= \mathcal{E}_0^* (\vec{\omega}_b - \vec{\omega}_0) \times \mathcal{E}_b$$

$$= \underbrace{\mathcal{E}_0^* (\vec{\omega}_b - \vec{\omega}_0) \times \mathcal{E}_0}_{\widehat{\omega_b - \omega_0}} \underbrace{\mathcal{E}_0^* \mathcal{E}_b}_{R_{ob}}$$

$\omega_b - \omega_0 \rightarrow$ represented in $\mathcal{E}_0$

$$= \widehat{(\omega_{b/o})_0} R_{ob}$$

In robotics literature:

$$\boxed{\dot{R} = \hat{\omega} R}$$

$$\frac{dR_{ob}}{dt} = R_{ob} \widehat{(\omega_{b/o})_b} \qquad (\dot{R} = R\hat{\omega})$$

show it

In spacecraft dynamics literature

$$\frac{dR_{bo}}{dt} = \widehat{(\omega_{o/b})_b} R_{bo} \qquad (\dot{R} = -\hat{\omega} R)$$

$$= -\widehat{(\omega_{b/o})_b} R_{bo}$$

$$\frac{dR_{bo}}{dt} = -R_{bo} \widehat{(\omega_{b/o})_o}$$