Robotics I

Lecture 11

Inverse Kinematics: Decomposition to Subproblems

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Motivation: Elbow Manipulator

Consider a 3-DOF elbow manipulator. Since it's 3-DOF, we only consider the position kinematics. Suppose that p_{04} is given, we want to find all the corresponding $(\theta_1, \theta_2, \theta_3)$.

3-D Elbow Inverse Kinematics

First note that p_{02} is a fixed vector in the base frame, so

$$p_{24} = p_{04} - p_{02} = R_{01}R_{12}(p_{23} + R_{23}p_{34}).$$

Take the norm of both sides, we get

$$\|p_{04} - p_{02}\| = \|p_{23} + R_{23}p_{34}\|,$$

which can be used to solve θ_3 by Subproblem 3 (up to 2 solutions: elbow-up and elbow-down).

Once θ_3 is found, we can go back to

$$p_{24} = p_{04} - p_{02} = R_{01}R_{12}(p_{23} + R_{23}p_{34}).$$

and solve for (θ_1, θ_2) with Subproblem 2 (up to 2 solutions: shoulder-right, shoulder-left).

Subproblem 0

we know $\vec{p} \cdot \vec{q} = \|\vec{p}\| \, \|\vec{q}\| \cos\theta$**. So we can find** θ **from** $\cos^{-1}(\vec{p} \cdot \vec{q} / \|\vec{p}\| \, \|\vec{q}\|)$**. This is not numerically attractive since** $\cos\theta$ **has near zero slope for small** θ**.**

Alternative, we can use

$$\tan\left(\frac{\theta}{2}\right) = \frac{\|\vec{p} - \vec{q}\|}{\|\vec{p} + \vec{q}\|}.$$

If $\vec{k}$ **is in the same direction as** $\vec{p} \times \vec{q}$**, then** θ **is positive, otherwise it's negative.**

Recall that we also saw subproblem 0 in DH parameterization.

Subproblem 1

Spinning $\vec{p}$ (or $\vec{q}$) about $\vec{k}$ generates a cone. Let the projection of $\vec{p}$ and $\vec{q}$ to the top of the cone (corresponding to the tip of $\vec{p}$) as $\vec{p}_1$ and $\vec{q}_1$:

$$\vec{p}_1 = \vec{p} - \vec{k} \cdot \vec{p}\vec{k}, \quad \vec{q}_1 = \vec{q} - \vec{k} \cdot \vec{q}\vec{k}.$$

Then θ is the angle of rotation from $\vec{p}_1$ to $\vec{q}_1$ about $\vec{k}$. This is just subproblem 0.

Subproblem 2

If $\vec{k}_1$ and $\vec{k}_2$ are collinear, we just have subproblem 1. So assume that they are not collinear.

Now consider two cones: spinning $\vec{p}$ about $\vec{k}_2$ and spinning $\vec{q}$ about $\vec{k}_1$. There may be 0, 1, or 2 intersections between the cones, which are the solutions.

Let $\vec{z}$ be the vector of intersection. Then

$$\vec{z} = \mathbf{rot}(\vec{k}_1, -\theta_1)\vec{q} = \mathbf{rot}(\vec{k}_2, \theta_2)\vec{p}.$$

Represent $\vec{z}$ as

$$\vec{z} = \alpha\vec{k}_1 + \beta\vec{k}_2 + \gamma\vec{k}_1 \times \vec{k}_2.$$

Since $\vec{k} \cdot \mathbf{rot}(\vec{k}, -\theta) = \vec{k}\cdot$ (use Euler-Rodriguez formula), we have

$$\begin{aligned} \vec{k}_1 \cdot \vec{z} &= \alpha + \beta\vec{k}_1 \cdot \vec{k}_2 = \vec{k}_1 \cdot \vec{q} \\ \vec{k}_2 \cdot \vec{z} &= \alpha\vec{k}_1 \cdot \vec{k}_2 + \beta = \vec{k}_2 \cdot \vec{p}. \end{aligned}$$

Subproblem 2 (Cont.)

This can be written as

$$\begin{bmatrix} 1 & \vec{k}_1 \cdot \vec{k}_2 \\ \vec{k}_1 \cdot \vec{k}_2 & 1 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} \vec{k}_1 \cdot \vec{q} \\ \vec{k}_2 \cdot \vec{p} \end{bmatrix}.$$

We can now solve for (α, β) **(since** $\vec{k}_1$ **and** $\vec{k}_2$ **are not collinear):**

$$\begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \frac{\begin{bmatrix} 1 & -\vec{k}_1 \cdot \vec{k}_2 \\ -\vec{k}_1 \cdot \vec{k}_2 & 1 \end{bmatrix}}{1 - (\vec{k}_1 \cdot \vec{k}_2)^2} \begin{bmatrix} \vec{q} \cdot \vec{k}_1 \\ \vec{p} \cdot \vec{k}_2 \end{bmatrix}.$$

Subproblem 2 (Cont.)

It remains to solve γ. Note that

$$\|\vec{z}\|^2 = \alpha^2 + \beta^2 + 2\alpha\beta\vec{k}_1\cdot\vec{k}_2 + \gamma^2 \left\|\vec{k}_1 \times \vec{k}_2\right\|^2 = \|\vec{p}\|^2 .$$

Since (α,β) have been found, γ can be solved:

$$\gamma = \pm\left[\left(\|\vec{p}\|^2 - \alpha^2 - \beta^2 - 2\alpha\beta\vec{k}_1\cdot\vec{k}_2\right) / \left\|\vec{k}_1 \times \vec{k}_2\right\|^2\right]^{\frac{1}{2}} .$$

If γ's are imaginary, then there is no solution (two cones do not intersection). If $\gamma = 0$, then there is 1 solution (two cones intersection at the tangent of the cones). If γ have two positive solutions, then there are 2 solutions (two cones intersecting at two points).

Once $\vec{z}$ is found, (θ_1,θ_2) are found by solving Subproblem 1 twice.

Subproblem 3

The solutions correspond to the intersections between the cone generated by spinning $\vec{p}$ about $\vec{k}$ and the sphere centered around the tip of $\vec{q}$ with radius δ.

First project $\vec{p}$ to the top of the cone:

$$\vec{p}_1 = \vec{p} - \vec{k} \cdot \vec{p}\vec{k}.$$

Then project $\vec{q}$ to the plane parallel to the top of the cone:

$$\vec{q}_1 = \vec{q} - \vec{k} \cdot \vec{q}\vec{k}.$$

Let the distance of the projection of $\vec{q} - \mathbf{rot}(\vec{k}, \theta)\vec{p}$ to the top of the cone be δ_1. Then

$$\delta_1^2 = \delta^2 - (\vec{k} \cdot (\vec{p} - \vec{q}))^2.$$

Subproblem 3 (Cont.)

Now we can just focus on the action at the top of the cone: $\vec{p}_1$ rotates θ to the first intersection, then over an angle, say ϕ, to be aligned with $\vec{q}_1$, then over ϕ again to the second intersection. By the law of cosine,

$$\|\vec{p}_1\|^2 + \|\vec{q}_1\|^2 - 2\,\|\vec{p}_1\|\,\|\vec{q}_1\|\cos\phi = \delta'.$$

Solving $\cos\phi$, we get

$$\cos\phi = \frac{\|\vec{p}_1\|^2 + \|\vec{q}_1\|^2 - \delta_1}{2\,\|\vec{p}_1\|\,\|\vec{q}_1\|}.$$

If the magnitude of the right hand side is greater than 1, there is no solution. If the magnitude is 1, there is one solution. If the magnitude is less than 1, there are two solutions.

MATLAB code

Subproblem solutions:

`subproblem0.m`, `subproblem1.m`, `subproblem2.m`, `subproblem3.m`

Examples of Inverse Kinematics

- **Three-link planar arm from Homework #1**
- **Three-link robot arm (essentially a spherical joint) from Homework #3**
- **PUMA 560 (see `puma560invkin.m`)**
- **SCARA**