Robotics & Automation

Lecture 03

Rotation Operator

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Geometric Objects and Operations

Point: O

Vector: $\vec{p}$

Frames: $(\vec{e}_1, \vec{e}_2, \vec{e}_3)$, $\vec{e}_i \cdot \vec{e}_j = \delta_{ij}$, $\vec{e}_1 \times \vec{e}_2 = \vec{e}_3$

Bound Vector: $(O, \vec{p})$ vector attached to a point

Linear transforms: $\mathcal{L}$.

Representation of vector in a frame: $v = \mathcal{E}^* \vec{v}$, $\vec{v} = \mathcal{E} v$

Representation of linear transform in a frame: $L = \mathcal{E}^* \mathcal{L} \mathcal{E}$, $\mathcal{L} = \mathcal{E} L \mathcal{E}^*$

Examples: I, $\vec{k}\times$, $\mathrm{rot}(\vec{k}, \theta))$

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Rotation Operator

Planar: $SO(2)$ (special orthogonal group of dimension 2), Spatial: $SO(3)$ (special orthogonal group of dimension 3, Orthogonal $\mathbb{R}^{3\times 3}m$ matrix, with determinant 1)

$$\begin{aligned} R_{ob} &= \text{representation of } \mathcal{E}_b \text{ in } \mathcal{E}_0 \\ &= \text{representation of rotation operator in } \mathcal{E}_0 \\ &= \text{transformation of representations in } \mathcal{E}_0 \text{ to } \mathcal{E}_b \end{aligned}$$

Properties: $R_{ob}^T R_{ob} = I$, $R_{oo} = I$, $R_{oc} = R_{ob}R_{bc}$, $R_{ob}^{-1} = R_{ob}^T = R_{bo}$

Transformation of representations in different frames (backward rotation):

$$\begin{aligned} v_0 &= R_{0b} v_b \\ L_0 &= R_{0b} L_b R_{b0}. \end{aligned}$$

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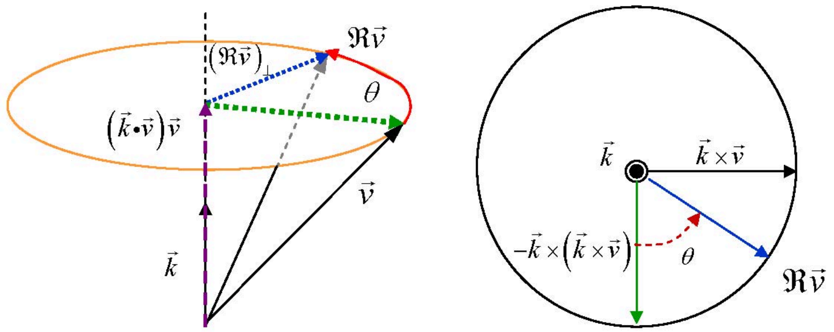
Representation of Rotation Operator

Consider a general rotation $\mathcal{R}$ about a unit vector $\vec{k}$ over an angle θ.



Consider $\mathcal{R}\vec{v}$. Decompose $\mathcal{R}\vec{v}$ into a component along $\vec{k}$ and a component perpendicular to $\vec{k}$:

$$\mathcal{R}\vec{v} = (\vec{k}\cdot\vec{v})\vec{k} + (\mathcal{R}\vec{v})_\perp.$$

Consider the disk formed by the tip of $\vec{v}$ rotating about $\vec{k}$. Then $(\mathcal{R}\vec{v})_\perp$ may be decomposed as

$$(\mathcal{R}\vec{v})_\perp = \sin\theta\vec{k}\times\vec{v} - \cos\theta\vec{k}\times(\vec{k}\times\vec{v}).$$

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Continuing...

Let r be the radius of the disk. From simple trigonometry,

$$r^2 = \|\vec{v}\|^2 - (\vec{k}\cdot\vec{v})^2 = v^T v - v^T k k^T v = v^T v - v^T(\hat{k}\hat{k}+I)v = -v^T\hat{k}\hat{k}v = (\hat{k}v)^T(\hat{k}v) = \left\|\hat{k}v\right\|^2.$$

Now decompose $(R\vec{v})_\perp$ to components along $\vec{k}\times\vec{v}$ and $\vec{k}\times(\vec{k}\times v)$:

$$(R\vec{v})_\perp = r\sin\theta\vec{k}\times\vec{v}/\left\|\vec{k}\times\vec{v}\right\| - r\cos\theta\vec{k}\times(\vec{k}\times\vec{v})/\left\|\vec{k}\times(\vec{k}\times\vec{v})\right\|.$$

Note that $\left\|\vec{k}\times(\vec{k}\times\vec{v})\right\| = \left\|\vec{k}\times\vec{v}\right\|$.

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Euler-Rodrigues Formula

Hence

$$\mathcal{R}\vec{v} = (\vec{k}\vec{k}\cdot + \sin\theta\vec{k}\times - \cos\theta\vec{k}\times(\vec{k}\times))\vec{v}.$$

Representing $\vec{v}$ and $\vec{k}$ with respect to a frame, we have $\mathcal{R}$ as a 3×3 matrix:

$$R = I_3 + \sin\theta\hat{k} + (1 - cos\theta)\hat{k}^2.$$

This is called the Euler-Rodrigues Formula. We'll denote R as the $so(3)$ matrix rot(k,θ). Verify that if $\mathcal{E}_b = \mathcal{R}\mathcal{E}_0$, then $\mathcal{R}$ represented in both $\mathcal{E}_0$ and $\mathcal{E}_b$ is rot(k,θ) (and $\vec{k}$ in both frames is k).

In spacecraft literature, you'll see

$$R^T = I_3 - \sin\theta\hat{k} + (1 - cos\theta)\hat{k}^2.$$

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Consecutive Rotations

Consecutive rotations about the same axes is additive:

$$\mathrm{rot}(\vec{k},\theta_2)\mathrm{rot}(\vec{k},\theta_1) = \mathrm{rot}(\vec{k},\theta_1+\theta_2).$$

Rotations about body axes, e.g., rotate about $\vec{x}$ for θ_1, then body $\vec{y}$ for θ_2, then new body $\vec{z}$ for θ_3:

$$\begin{aligned}
&\mathcal{E}_1 = \mathrm{rot}(\vec{x}_0,\theta_1)\mathcal{E}_0,\ \mathcal{E}_2 = \mathrm{rot}(\vec{y}_1,\theta_2)\mathcal{E}_1,\ \mathcal{E}_b = \mathrm{rot}(\vec{z}_2,\theta_2)\mathcal{E}_2 \\
&\mathcal{E}_b = \mathrm{rot}(\vec{z}_2,\theta_3)\mathrm{rot}(\vec{y}_1,\theta_2)\mathrm{rot}(\vec{x}_0,\theta_1)\mathcal{E}_0 \\
&R_{0b} = \mathcal{E}_0^*\mathcal{E}_b = \mathrm{rot}(x,\theta_1)\mathrm{rot}(y,\theta_2)\mathrm{rot}(z,\theta_3) \\
&x = [1,0,0]^T,\ y = [0,1,0]^T,\ z = [0,0,1]^T.
\end{aligned}$$

Rotations about fix axes:

$$\begin{aligned}
&\mathcal{E}_b = \mathrm{rot}(\vec{z}_0,\theta_3)\mathrm{rot}(\vec{y}_0,\theta_2)\mathrm{rot}(\vec{x}_0,\theta_1)\mathcal{E}_0 \\
&R_{0b} = \mathcal{E}_0^*\mathcal{E}_b = \mathrm{rot}(z,\theta_3)\mathrm{rot}(y,\theta_2)\mathrm{rot}(x,\theta_1)
\end{aligned}$$

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