Robotics & Automation

Lecture 06

Inverse Kinematics

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Last Time

Forward Kinematics: product of exponential approach.

Denavit-Hartenberg parameterization: 4-parameters per link instead of 6. Standard vs. modified.

Inverse Kinematics

Last time we considered the forward kinematics of an open chain: given joint displacement, q, find the end effector configuration with respect to the base configuration: $(R_{0T}(q),(p_{0T})(q)) \in SE(3)$.

Today, we'll consider the converse problem: given $(R_{0T}(q),(p_{0T})(q)) \in SE(3)$, find all corresponding q's.

This is called the inverse kinematics problem.

If the arm is non-redundant (6-DOF for spatial manipulator, 3-DOF for spatial translational only 3-DOF for planar translation and rotation, 2-DOF for planar translational only), there are typically finite number of solutions.

If the arm is redundant (more internal DOF than task DOF), then there are infinitely many solutions, and we'll need to impose additional criteria to resolve the redundancy (called redundancy resolution problem).

Inverse Kinematics Solution: Direct Approach

Write R_{0T} and p_{0T} in terms of $(q_1, \ldots, q_6)$. Then for a given (R_{0T}, p_{0T}), use "inspection" to solve for $(q_1, \ldots, q_6)$.

Inverse Kinematics: Iterative Approach

Suppose the forward kinematics is given by $(f_p(q), f_R(q) \in SE(3)$**. The inverse kinematics problem may be posed as a nonlinear minimization problem (with some chosen norm, recall discussion on metrics on** $SO(3)$**):**

$$\min_q \frac{1}{2}\left[\|p_{0T} - f_p(q)\|^2 + \alpha\|R_{0T} - f_R(q)\|^2\right]$$

Then the problem can be solved iteratively by using gradient type of algorithm, e.g., steepest descent, Newton's algorithm, conjugate gradient, etc. (provided a reasonably close initial guess is used). Watch out for singularities (in both representation and arm singularity).

Iterative Approach (Cont.)

As an example, let β be some three-parameter representation of R_{0T}. Consider the minimization problem for inverse kinematics as:

$$\min_q \left\| \begin{bmatrix} \beta(R_{0T}) \\ p_{0T} \end{bmatrix} - \begin{bmatrix} \beta(f_R(q)) \\ f_p(q) \end{bmatrix} \right\|.$$

Write the error norm as $\|f(q) - x\|$.

The Newton descent algorithm is given by $q_{k+1} = q_k - \alpha_k (\nabla_q f(q))^{-1} (f(q) - x)$.

Note that $\nabla_q f(q)$ can be written as $\nabla_q f(q) = \begin{bmatrix} I & 0 \\ 0 & J_\beta \end{bmatrix} (J_T)_0$ where J_β is the representation Jacobian (i.e., $\dot{\beta} = J_\beta \omega$) and $(J_T)_0$ is the arm Jacobian represented in the base frame. So the iterative approach would work well as long as the arm stays away from singularities (i.e., configurations such that J_T loses rank, and representation singularities).

Inverse Kinematics: Decomposition to Subproblems

For certain arms, inverse kinematics solution may be found through the solutions of a series of subproblems. Typically, arms containing a spherical joint (i.e., three consecutive rotational joints with rotational axes intersecting at a point) can be solved this way.

Subproblem 0: Given $\vec{p}$ and $\vec{q}$, $\|\vec{p}\| = \|\vec{q}\|$, and a unit vector $\vec{k}$ perpendicular to both $\vec{p}$ and $\vec{q}$. Find θ such that $exp(\theta\vec{k}\times)\vec{q} = \vec{p}$. (A unique solution always exists.)

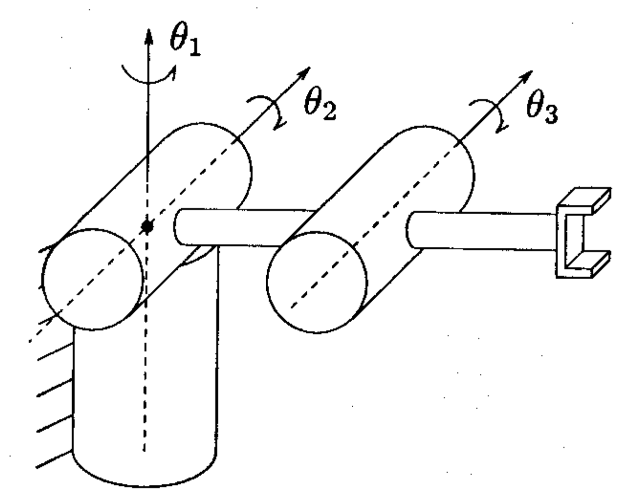
Subproblem 1: Given $\vec{p}$, $\vec{q}$, $\|\vec{p}\| = \|\vec{q}\|$, and a unit vector $\vec{k}$, $\vec{k}\cdot\vec{p} = \vec{k}\cdot\vec{q}$. Find θ such that $exp(\theta\vec{k}\times)\vec{q} = \vec{p}$. (A unique solution always exists.)

Subproblem 2: Given $\vec{p}$, $\vec{q}$, $\|\vec{p}\| = \|\vec{q}\|$, and unit vectors $\vec{k}_1$ and $\vec{k}_2$. Find θ_1, θ_2 such that $exp(\theta_1 k_1\times)exp(\theta_2 k_2\times)\vec{p} = \vec{q}$. (There may be 0, 1, or 2 solutions.)

Subproblem 3: Given $\vec{p}$, $\vec{q}$, a unit vector $\vec{k}$, and a positive constant δ. Find θ such that $\|\vec{q} - exp(\theta_k\times)\vec{p}\| = \delta$. (There may be 0, 1, or 2 solutions.)

Motivation: Elbow Manipulator

Consider a 3-DOF elbow manipulator. Since it's 3-DOF, we only consider the position kinematics. Suppose that p_{04} is given, we want to find all the corresponding $(\theta_1, \theta_2, \theta_3)$.



3-D Elbow Inverse Kinematics

First note that p_{02} is a fixed vector in the base frame, so

$$p_{24} = p_{04} - p_{02} = R_{01}R_{12}(p_{23} + R_{23}p_{34}).$$

Take the norm of both sides, we get

$$\|p_{04} - p_{02}\| = \|p_{23} + R_{23}p_{34}\|,$$

which can be used to solve θ_3 by Subproblem 3 (up to 2 solutions: elbow-up and elbow-down).

Once θ_3 is found, we can go back to

$$p_{24} = p_{04} - p_{02} = R_{01}R_{12}(p_{23} + R_{23}p_{34}).$$

and solve for (θ_1, θ_2) with Subproblem 2 (up to 2 solutions: shoulder-right, shoulder-left).

Subproblem 0

we know $\vec{p}\cdot\vec{q} = \|\vec{p}\|\,\|\vec{q}\|\cos\theta$. So we can find θ from $\cos^{-1}(\vec{p}\cdot\vec{q}/\,\|\vec{p}\|\,\|\vec{q}\|)$. This is not numerically attractive since $\cos\theta$ has near zero slope for small θ.

Alternative, we can use

$$\tan\left(\frac{\theta}{2}\right) = \frac{\|\vec{p}-\vec{q}\|}{\|\vec{p}+\vec{q}\|}.$$

If $\vec{k}$ is in the same direction as $\vec{p}\times\vec{q}$, then θ is positive, otherwise it's negative.

Recall that we also saw subproblem 0 in DH parameterization.

Subproblem 1

Spinning $\vec{p}$ (or $\vec{q}$) about $\vec{k}$ generates a cone. Let the projection of $\vec{p}$ and $\vec{q}$ to the top of the cone (corresponding to the tip of $\vec{p}$) as $\vec{p}_1$ and $\vec{q}_1$:

$$\vec{p}_1 = \vec{p} - \vec{k} \cdot \vec{p}\vec{k}, \quad \vec{q}_1 = \vec{q} - \vec{k} \cdot \vec{q}\vec{k}.$$

Then θ is the angle of rotation from $\vec{p}_1$ to $\vec{q}_1$ about $\vec{k}$. This is just subproblem 0.

Subproblem 2

If $\vec{k}_1$ and $\vec{k}_2$ are collinear, we just have subproblem 1. So assume that they are not collinear.

Now consider two cones: spinning $\vec{p}$ about $\vec{k}_2$ and spinning $\vec{q}$ about $\vec{k}_1$. There may be 0, 1, or 2 intersections between the cones, which are the solutions.

Let $\vec{z}$ be the vector of intersection. Then

$$\vec{z} = exp(-\theta_1 k_1 \times)\vec{q} = exp(\theta_2 k_2 \times)\vec{p}.$$

Represent $\vec{z}$ as

$$\vec{z} = \alpha\vec{k}_1 + \beta\vec{k}_2 + \gamma\vec{k}_1 \times \vec{k}_2.$$

Since $\vec{k} \cdot exp(-\theta k\times) = \vec{k}\cdot$ (use Euler-Rodriguez formula), we have

$$\begin{aligned} \vec{k}_1 \cdot \vec{z} &= \alpha + \beta\vec{k}_1 \cdot \vec{k}_2 = \vec{k}_1 \cdot \vec{q} \\ \vec{k}_2 \cdot \vec{z} &= \alpha\vec{k}_1 \cdot \vec{k}_2 + \beta = \vec{k}_2 \cdot \vec{p}. \end{aligned}$$

Subproblem 2 (Cont.)

This can be written as

$$\begin{bmatrix} 1 & \vec{k}_1 \cdot \vec{k}_2 \\ \vec{k}_1 \cdot \vec{k}_2 & 1 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} \vec{k}_1 \cdot \vec{q} \\ \vec{k}_2 \cdot \vec{p} \end{bmatrix}.$$

We can now solve for (α, β) **(since** $\vec{k}_1$ **and** $\vec{k}_2$ **are not collinear):**

$$\begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \frac{\begin{bmatrix} 1 & -\vec{k}_1 \cdot \vec{k}_2 \\ -\vec{k}_1 \cdot \vec{k}_2 & 1 \end{bmatrix}}{1 - (\vec{k}_1 \cdot \vec{k}_2)^2} \begin{bmatrix} \vec{q} \cdot \vec{k}_1 \\ \vec{p} \cdot \vec{k}_2 \end{bmatrix}.$$

Subproblem 2 (Cont.)

It remains to solve γ. Note that

$$\|\vec{z}\|^2 = \alpha^2 + \beta^2 + 2\alpha\beta\vec{k}_1\cdot\vec{k}_2 + \gamma^2\left\|\vec{k}_1\times\vec{k}_2\right\|^2 = \|\vec{p}\|^2 .$$

Since (α,β) have been found, γ can be solved:

$$\gamma = \pm\left[\left(\|\vec{p}\|^2 - \alpha^2 - \beta^2 - 2\alpha\beta\vec{k}_1\cdot\vec{k}_2\right)/\left\|\vec{k}_1\times\vec{k}_2\right\|^2\right]^{\frac{1}{2}} .$$

If γ's are imaginary, then there is no solution (two cones do not intersection). If $\gamma = 0$, then there is 1 solution (two cones intersection at the tangent of the cones). If γ have two positive solutions, then there are 2 solutions (two cones intersecting at two points).

Once $\vec{z}$ is found, (θ_1,θ_2) are found by solving Subproblem 1 twice.

Subproblem 3

The solutions correspond to the intersections between the cone generated by spinning $\vec{p}$ about $\vec{k}$ and the sphere centered around the tip of $\vec{q}$ with radius δ.

First project $\vec{p}$ to the top of the cone:

$$\vec{p}_1 = \vec{p} - \vec{k} \cdot \vec{p}\vec{k}.$$

Then project $\vec{q}$ to the plane parallel to the top of the cone:

$$\vec{q}_1 = \vec{q} - \vec{k} \cdot \vec{q}\vec{k}.$$

Let the distance of the projection of $\vec{q} - exp(\theta\vec{k}\times)\vec{p}$ to the top of the cone be δ_1. Then

$$\delta_1^2 = \delta^2 - (\vec{k} \cdot (\vec{p} - \vec{q}))^2.$$

Subproblem 3 (Cont.)

Now we can just focus on the action at the top of the cone: $\vec{p}_1$ rotates θ to the first intersection, then over an angle, say ϕ, to be aligned with $\vec{q}_1$, then over ϕ again to the second intersection. By the law of cosine,

$$\|\vec{p}_1\|^2 + \|\vec{q}_1\|^2 - 2\,\|\vec{p}_1\|\,\|\vec{q}_1\| \cos\phi = \delta'.$$

Solving $\cos\phi$, we get

$$\cos\phi = \frac{\|\vec{p}_1\|^2 + \|\vec{q}_1\|^2 - \delta_1}{2\,\|\vec{p}_1\|\,\|\vec{q}_1\|}.$$

If the magnitude of the right hand side is greater than 1, there is no solution. If the magnitude is 1, there is one solution. If the magnitude is less than 1, there are two solutions.

MATLAB code

Subproblem solutions:

`subproblem0.m`, `subproblem1.m`, `subproblem2.m`, `subproblem3.m`

Test code: `testa.m`, `testb.m`, `testc.m`, `testd.m`.

Examples of Inverse Kinematics

- **SCARA**
- **Elbow**
- **PUMA 560**