Robotics & Automation

Lecture 05

Rigid Body Description, Euclidean Frame, Homogeneous Transformation

John T. Wen

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Description of a Rigid Body

A rigid body is given by a point O and a frame $\mathcal{E}$.

Given a rigid body $(O_b, \mathcal{E}_b)$, it may be represented as a displacement and a rotation with respect to another rigid body $(O_o, \mathcal{E}_o)$: $(\vec{p}_{ob}, \mathrm{rot}(\vec{k}, \theta))$ where $\vec{p}_{ob}$ is the vector from O_o to O_b and $\mathcal{E}_b = \mathrm{rot}(\vec{k}, \theta)\mathcal{E}_o$.

Represented in $\mathcal{E}_o$, we obtain an $\mathbb{R}^3$ vector and an $SO(3)$ matrix: $(p_{ob}, R_{ob}) \in SE(3)$, Special Euclidean group of order 3.

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Homogeneous Transformation

$SE(3)$ is frequently represented as a 4×4 homogeneous matrix:

$$H_{ob} = \begin{bmatrix} R_{ob} & p_{ob} \\ 0 & 1 \end{bmatrix}$$

Note that we have the group structure as in $SO(3)$:

$$SO(3) : R_{ob}^{-1} = R_{ob}^{T} = R_{bo},\ R_{oc} = R_{ob}R_{bc}$$
$$SE(3) : H_{ob}^{-1} = H_{bo},\ H_{oc} = H_{ob}H_{bc}.$$

Note that p_{bc} in H_{bc} is $(\vec{p}_{bc})_b$.

Exponential representation: $SO(3) : R = e^{\hat{k}\theta}$

$$SE(3) : H = e^{\hat{\xi}\theta}, \xi = [k^T, v^T]^T, \hat{x}i := \begin{bmatrix} \hat{k} & v \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} e^{\hat{k}\theta} & (I - e^{\hat{k}\theta})\hat{k}v + kk^T v\theta \\ 0 & 1 \end{bmatrix}.$$

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Kinematic Propagation in Rigid Bodies

Translational kinematics:

$$\vec{p}_{ob} = \vec{p}_{oa} + \vec{p}_{ab}$$

$$\text{in } \mathcal{E}_o : \qquad p_{ob} = p_{oa} + R_{oa} p_{ab}$$

Rotational kinematics:

$$R_o b = R_{oa} R_{ab}$$

Homogeneous transformation:

$$H_{ob} = H_{oa} H_{ab}$$

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Spatial Velocity propagation

Coordinate-free form:

$$\underbrace{\begin{bmatrix} \vec{\omega}_b \\ \vec{v}_b \end{bmatrix}}_{\vec{V}_b} = \underbrace{\begin{bmatrix} \bar{I} & 0 \\ -\vec{p}_{ab}\times & \bar{I} \end{bmatrix}}_{\bar{\Phi}_{ba}} \underbrace{\begin{bmatrix} \vec{\omega}_a \\ \vec{v}_a \end{bmatrix}}_{\vec{V}_a} + \begin{bmatrix} \vec{\omega}_{b/a} \\ \frac{d\vec{p}_{ab}}{dt^a} \end{bmatrix}$$

Representing $\vec{V}_b$ in $\mathcal{E}_b$ and $\vec{V}_a$ in $\mathcal{E}_a$:

$$\underbrace{\begin{bmatrix} \omega_b \\ v_b \end{bmatrix}}_{V_b} = \underbrace{\begin{bmatrix} R_{ba} & 0 \\ -R_{ba}\hat{p}_{ab} & R_{ba} \end{bmatrix}}_{\Phi_{ba}} \underbrace{\begin{bmatrix} \omega_a \\ v_a \end{bmatrix}}_{V_a} + \begin{bmatrix} R_{ba}(\omega_{b/a})_a \\ R_{ba}\frac{dp_{ab}}{dt} \end{bmatrix}$$

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Spatial Acceleration propagation

Coordinate-free form:

$$\underbrace{\begin{bmatrix} \frac{d\vec{\omega}_b}{dt^0} \\ \frac{d\vec{v}_b}{dt^0} \end{bmatrix}}_{\vec{\alpha}_b} = \begin{bmatrix} \bar{I} & 0 \\ -\vec{p}_{ab}\times & \bar{I} \end{bmatrix} \underbrace{\begin{bmatrix} \frac{d\vec{\omega}_a}{dt^0} \\ \frac{d\vec{v}_a}{dt^0} \end{bmatrix}}_{\vec{\alpha}_a} + \begin{bmatrix} \frac{d\vec{\omega}_{b/a}}{dt^a} \\ \frac{d^2\vec{p}_{ab}}{dt^{a2}} \end{bmatrix} + \begin{bmatrix} \omega_a \times \omega_b \\ \omega_a \times (\vec{\omega}_a \times \vec{p}_{ab}) + 2\vec{\omega}_a \times \frac{d\vec{p}_{ab}}{dt^a} \end{bmatrix}$$

Representing $\vec{\alpha}_b$ in $\mathcal{E}_b$ and $\vec{\alpha}_a$ in $\mathcal{E}_a$:

$$\begin{bmatrix} \dot{\omega}_b \\ \dot{v}_b \end{bmatrix} = \begin{bmatrix} R_{ba} & 0 \\ -R_{ba}\hat{p}_{ab} & R_{ba} \end{bmatrix} \begin{bmatrix} \dot{\omega}_a \\ \dot{v}_a \end{bmatrix} + \begin{bmatrix} R_{ba}(\dot{\omega}_{b/a})_a \\ R_{ba}\ddot{p}_{ab} \end{bmatrix} + \begin{bmatrix} \widehat{R_{ba}\omega_a}\omega_b \\ R_{ba}\hat{\omega}_a\hat{\omega}_a p_{ab} + 2R_{ba}\hat{\omega}_a\dot{p}_{ab} \end{bmatrix}$$

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