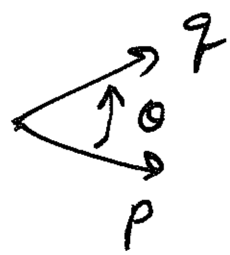
Subproblem 0



Given $p, q \in \mathbb{R}^3$, $\|p\| = \|q\|$.

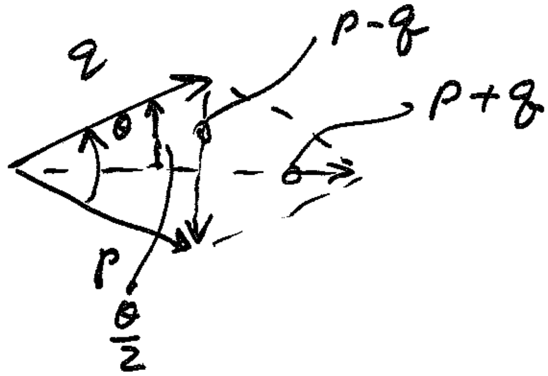
Find k, $\|k\|$, and θ, such that

$q = \mathrm{rot}(k, \theta)\, p$

$p^T q = \cos\theta$

$\theta = \cos^{-1}(p^T q)$ ← insensitive about 0

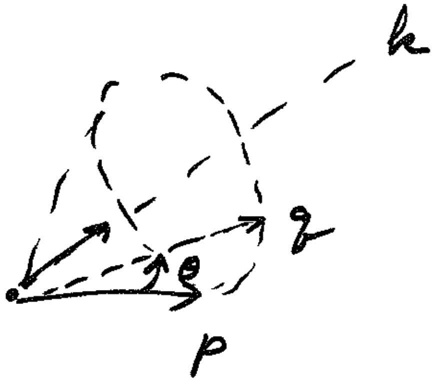
$k = \dfrac{\hat{p} q}{\|\hat{p} q\|}$ ← $\vec{p} \times \vec{q}$



$$\tan\frac{\theta}{2} = \frac{\frac{\|p-q\|}{2}}{\frac{\|p+q\|}{2}} = \frac{\|p-q\|}{\|p+q\|}$$

$$\theta = 2\tan^{-1}\left(\frac{\|p-q\|}{\|p+q\|}\right)$$

Subproblem 1

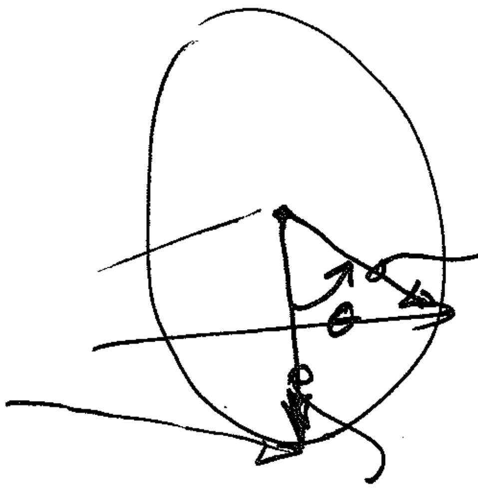


Given $p, q, k \in \mathbb{R}^3$

$\|p\| = \|q\|$ $\|k\| = 1$

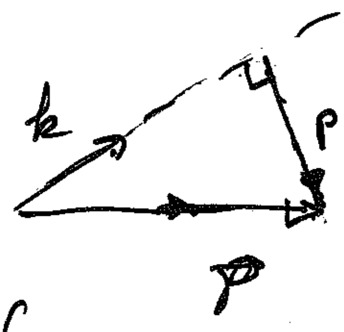
Find θ such that

$$q = \mathrm{rot}(k, \theta)\, p$$



q' = projection of q to top of cone

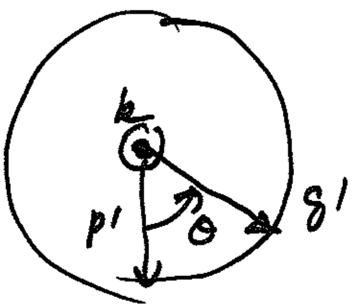
p' = projection of p to top of cone



$$p' = p - (k^T p) k$$

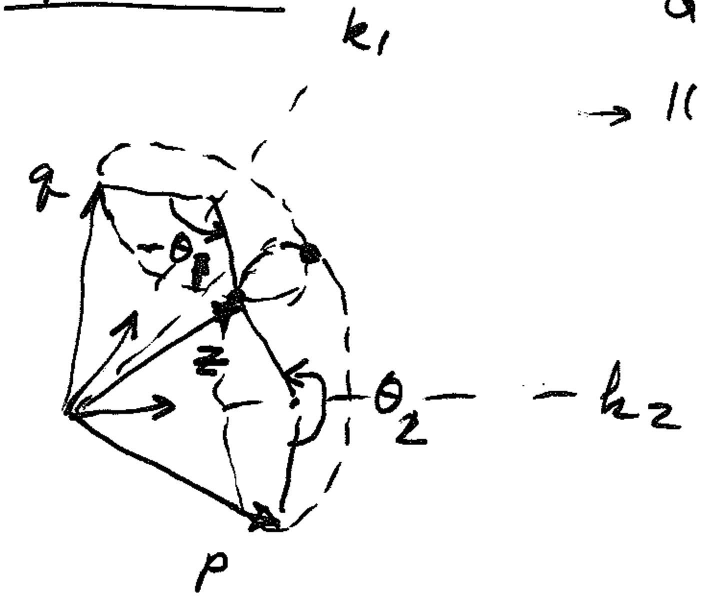
$$q' = q - (k^T q) k$$

top of cone



$\therefore$ $\theta = \mathrm{subproblem}\,0(p', q', k)$

Subproblem 2



Given $p, q, k_1, k_2 \in \mathbb{R}^3$

$\rightarrow \|p\| = \|q\|$, $\|k_1\| = \|k_2\| = 1$

Find θ_1, θ_2 such that

$$q = rot(k_1, \theta_1)\, rot(k_2, \theta_2)\, p$$

Assume $k_1 \neq k_2$

If $k_1 = k_2$, then

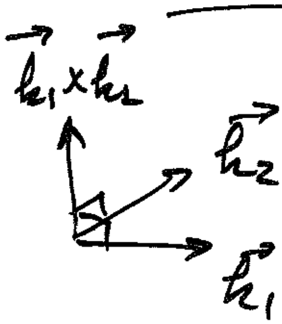
$q = rot(k_1, \theta_1 + \theta_2)\, p$

which is subproblem #1

$$rot(k_1, -\theta_1)\, q = z = rot(k_2, \theta_2)\, p$$

Represent z as:

$$z = \alpha k_1 + \beta k_2 + \gamma \underbrace{\hat{k}_1 k_2}_{\vec{k}_1 \times \vec{k}_2}$$



Solve for α, β, γ

Once z is found, (θ_1, θ_2) may be found from subproblem 1 (twice).

$$rot(k_1, -\theta_1)\, rot(k_1, \theta_1) = I$$

$$\underbrace{rot(k_1, -\theta_1)\, q}_{\text{we call this } z} = \underbrace{rot(k_1, -\theta_1)\, rot(k_1, \theta_1)}_{= I}\, rot(k_2, \theta_2)\, p$$

2

$$\mathrm{rot}(k_1,-\theta_1)q = \alpha k_1 + \beta k_2 + \gamma \hat{k}_1 k_2$$

$$\mathrm{rot}(k_2,\theta_2)p = \alpha k_1 + \beta k_2 + \gamma \hat{k}_1 k_2$$

Note: $$\underline{\underline{k^T \mathrm{rot}(k,\theta) = k^T}}$$

$$k_1^T \mathrm{rot}(k_1,-\theta_1) q = k_1^T q = \alpha \underbrace{k_1^T k_1}_{=1} + \beta k_1^T k_2 + \gamma \underbrace{k_1^T \hat{k}_1 k_2}_{=0}$$

$$= \alpha + \beta k_1^T k_2$$

$$k_2^T \mathrm{rot}(k_2,\theta_2) p = \alpha k_2^T k_1 + \beta \underbrace{k_2^T k_2}_{=1} + \gamma \underbrace{k_2^T \hat{k}_1 k_2}_{=0}$$

$$= k_2^T p$$

$$= \alpha k_1^T k_2 + \beta$$

$$\begin{bmatrix} 1 & k_1^T k_2 \\ k_1^T k_2 & 1 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} k_1^T q \\ k_2^T p \end{bmatrix}$$

$$\begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \frac{\begin{bmatrix} 1 & -k_1^T k_2 \\ -k_1^T k_2 & 1 \end{bmatrix}}{1-(k_1^T k_2)^2} \begin{bmatrix} k_1^T q \\ k_2^T p \end{bmatrix}$$

$$1-(k_1^T k_2)^2 = 0 \iff \underbrace{k_1 = \pm k_2}_{\downarrow}$$

already eliminated this case

$$\|\mathrm{rot}(k_1,-\theta_1)q\|^2 = \|q\|^2 = \|\underbrace{\alpha k_1 + \beta k_2 + \gamma \hat{k}_1 k_2}\|^2$$

$$(\alpha k_1 + \beta k_2 + \gamma \hat{k}_1 k_2)^T(\alpha k_1 + \beta k_1 + \gamma \hat{k}_1 k_2)$$

$$= \alpha^2 + 2\alpha\beta\, k_1^T k_2 + \beta^2 + \gamma^2 \, \|\hat{k}_1 k_2\|^2$$

$$\gamma^2 = \frac{\|q\|^2 - (\alpha^2 + \beta^2 + 2\alpha\beta\, k_1^T k_2)}{\|\hat{k}_1 k_2\|^2}$$

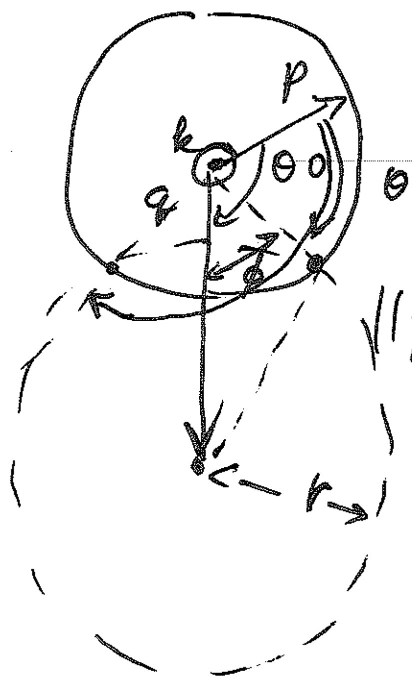
$$\gamma = \pm\sqrt{\frac{\|q\|^2 - (\alpha^2 + \beta^2 + 2\alpha\beta\, k_1^T k_2)}{\|\hat{k}_1 k_2\|^2}}$$

If $\|q\|^2 < \alpha^2 + \beta^2 + 2\alpha\beta\, k_1^T k_2$, then there is no solution (no intersection)

If $\|q\|^2 = \alpha^2 + \beta^2 + 2\alpha\beta\, k_1^T k_2$, then there is 1 solution (two cones are tangent)

If $\|q\|^2 > \alpha^2 + \beta^2 + 2\alpha\beta\, k_1^T k_2$, then there are 2 solutions

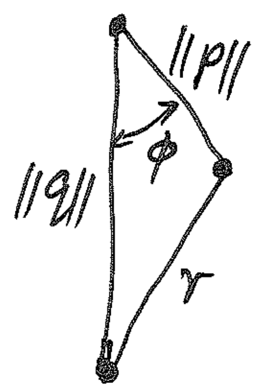
Subproblem 3'



$$\|q - \mathrm{rot}(k,\theta)p\| = r$$

Intersection between 2 circles in a plane.

Find ϕ using the cosine law:



$$\cos\phi = \frac{\|p\|^2 + \|q\|^2 - r^2}{2\|p\|\|q\|}$$

$$\phi = \pm \cos^{-1}\left(\frac{\|p\|^2 + \|q\|^2 - r^2}{2\|p\|\|q\|}\right)$$

If $\left|\frac{\|p\|^2 + \|q\|^2 - r^2}{2\|p\|\|q\|}\right| > 1$, there is no solution

If $\frac{\|p\|^2 + \|q\|^2 - r^2}{2\|p\|\|q\|} = \pm 1$, there is one solution ($\underset{+1}{\phi = 0}$ or $\underset{-1}{\pi}$)

If $\left|\frac{\|p\|^2 + \|q\|^2 - r^2}{2\|p\|\|q\|}\right| < 1$, there are 2 solutions

Find θ_0 from subproblem # 0:

$$\frac{q}{\|q\|} = \mathrm{rot}(k, \theta_0)\,\frac{p}{\|p\|}$$

Finally, $\left.\theta = \theta_0 \pm \phi \right\}$ 0, 1, or 2 solutions

<u>Special case</u>: If $q = 0$ and $\|p\| = r$, there are infinitely many solutions.

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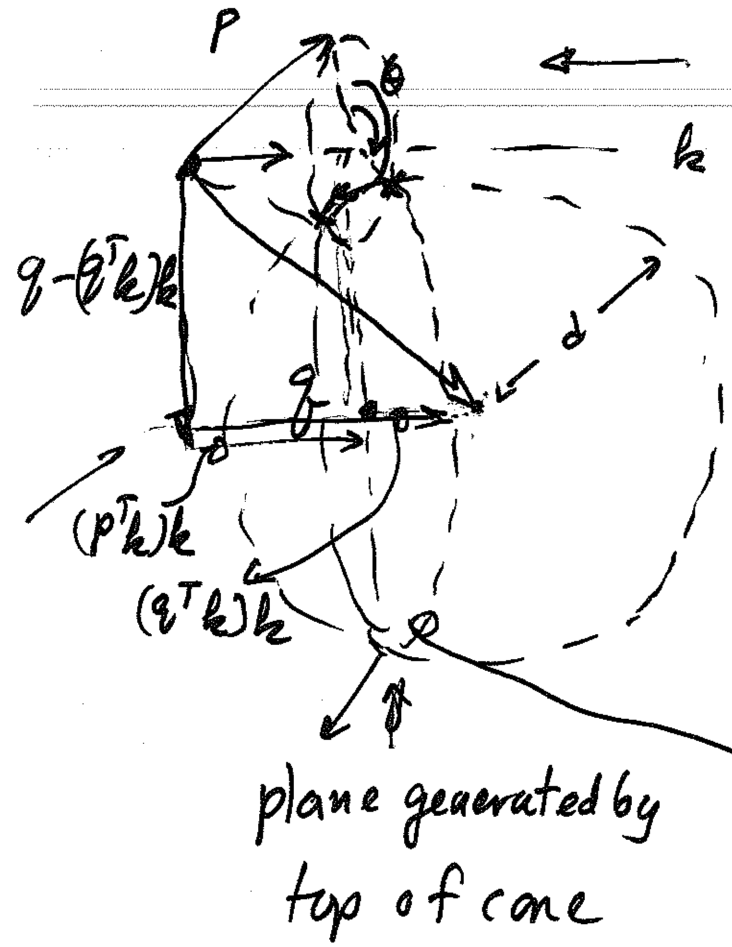
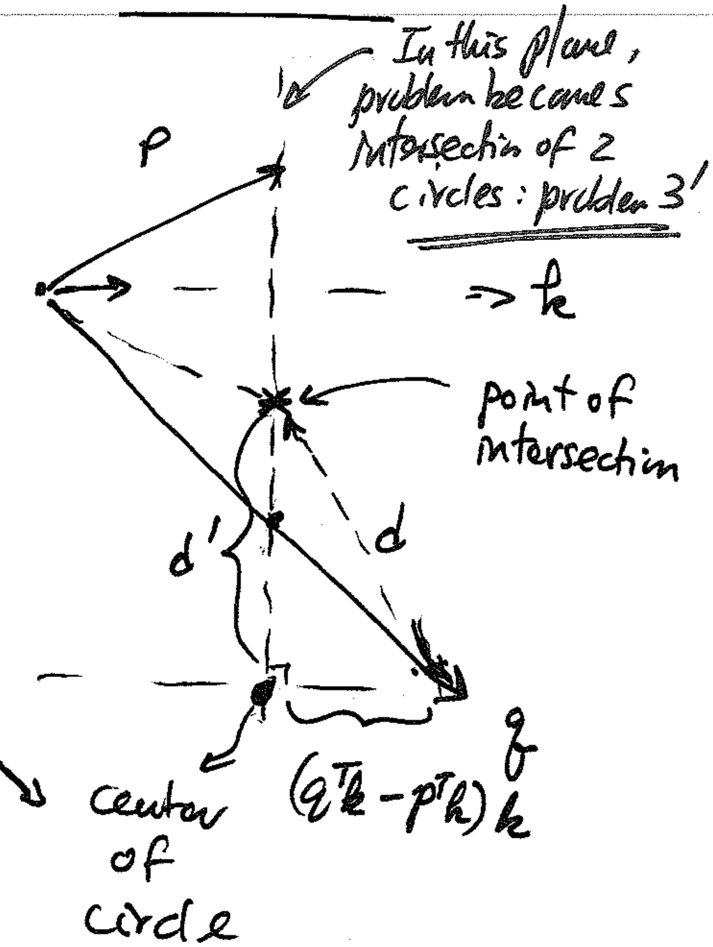
Subproblem #3

$$\| q - \mathrm{rot}(k, \theta)\, p \| = d$$

Given, $p, q \in \mathbb{R}^3$, $\|k\| = 1$, $d > 0$



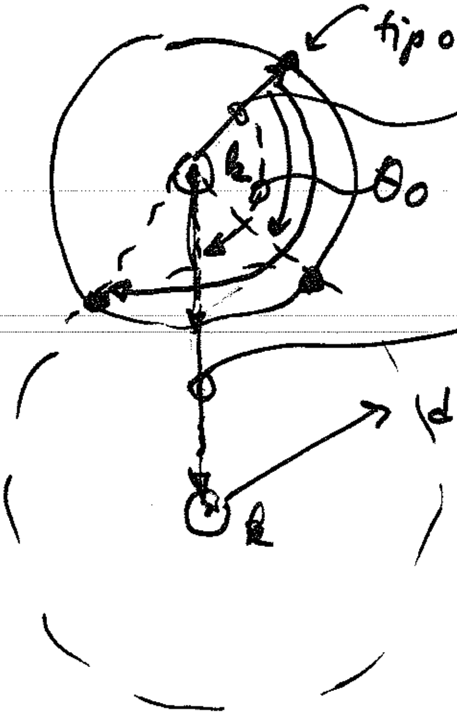
Side view



$$d' = \sqrt{d^2 - (q^Tk - p^Tk)^2}$$

radius of the circle generated by cutting the ball by top of cone.

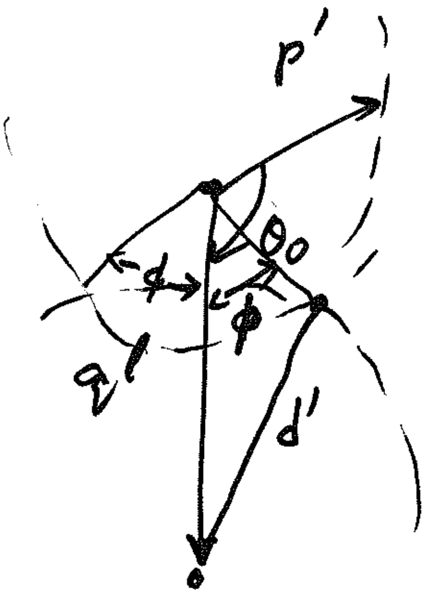
Looking down at top of cone $\qquad$ 11/20/11 - 7



$$\left.\begin{aligned} p' &= p - (p^T k)k \\ q' &= q - (q^T k)k \end{aligned}\right\}$$ Subproblem 3' with p', q', d', k

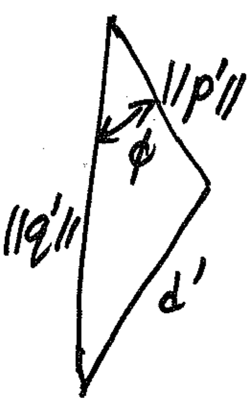
First look for the <u>rotational angle</u> from p' to q' θ_0

$$\frac{q'}{\|q'\|} = \mathrm{rot}(k, \theta_0)\,\frac{p'}{\|p'\|}$$



$$\theta = \theta_0 \pm \phi$$

Solve for ϕ from cosine rule



$$\|p'\|^2 + \|q'\|^2 - 2\|p'\|\|q'\|\cos\phi = d'^2$$

$$\cos\phi = \frac{(\|p'\|^2 + \|q'\|^2) - d'^2}{2\|p'\|\|q'\|}$$

$$\phi = \pm\cos^{-1}\left(\frac{\|p'\|^2 + \|q'\|^2 - d'^2}{2\|p'\|\|q'\|}\right)$$

If $\left|\frac{\|p'\|^2+\|q'\|^2-d'^2}{2\|p'\|\|q'\|}\right| > 1$, there is no solution

If $\frac{\|p'\|^2+\|q'\|^2-d'^2}{2\|p'\|\|q'\|} = \pm 1$, there is 1 solution

If $\left|\frac{\|p'\|^2+\|q'\|^2-d'^2}{2\|p'\|\|q'\|}\right| < 1$, there are 2 solutions

$$\underline{\theta = \theta_0 \pm \phi}$$

Inverse Kinematics for Non-redundant arm

# of joint DOF = # of task DOF

Spatial (position + orientation) : 6-DOF

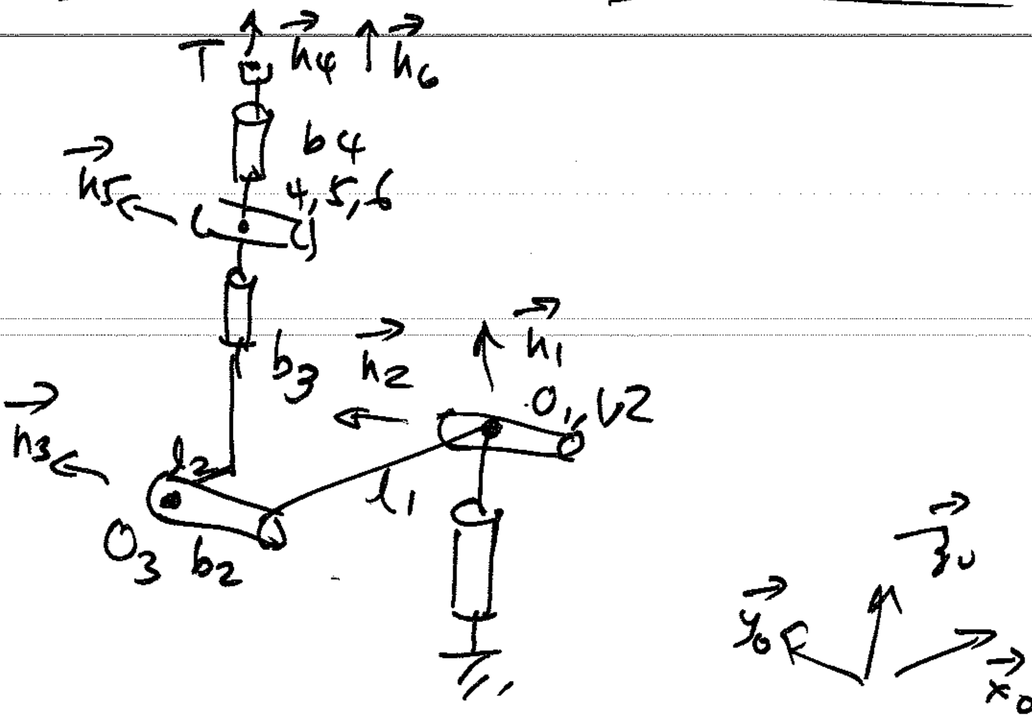
Spatial (position) : 3-DOF

planar (position + orientation): 3-DOF

planar (position) : 2-DOF

Inverse kinematics for redundant arms (more internal DOF than task DOF) needs redundancy resolution to achieve additional objectives (e.g., collision avoidance)

Inverse Kinematics of PUMA560



$$x = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad y = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad z = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$h_1 = z \quad h_2 = y \quad h_3 = y \quad h_4 = z \quad h_5 = y \quad h_6 = z$$

$$P_{01} = 0 \quad P_{12} = 0 \quad P_{23} = -l_1 x + b_2 y$$

$$P_{34} = l_2 x + b_3 z \quad P_{45} = 0 \quad P_{56} = 0 \quad P_{6T} = b_4 z$$

$$\boxed{R_{0T}} = R_{01} R_{12} R_{23} R_{34} R_{45} R_{56}$$

$$P_{0T} = \underbrace{P_{01}}_{0} + \underbrace{R_{01} P_{12}}_{0} + R_{02} P_{23} + R_{03} P_{34} + \overbrace{R_{04} P_{45}}^{0} + \overbrace{R_{05} P_{56}}^{0} + \boxed{R_{06}} \boxed{P_{6T}}$$

3 DOF position inverse kinematics