Fourier Transform of a Sine–Gaussian: Detailed Derivation via Completing the Square

Convention

We use the frequency (Hz) Fourier transform

$$\mathcal{F}\{f\}(\nu) \equiv \int_{-\infty}^{\infty} f(t)\, e^{-i2\pi\nu t}\, dt, \qquad \nu \in \mathbb{R}. \tag{1}$$

Let

$$f(t) = e^{-\frac{t^2}{2\sigma^2}}\, \sin(2\pi f_0 t + \phi), \qquad \sigma > 0,\ f_0 \in \mathbb{R},\ \phi \in \mathbb{R}. \tag{2}$$

We will first compute the transform of a Gaussian and of a modulated Gaussian, then form the sine–Gaussian by linearity.

Basic result

Consider

$$I(\nu) := \int_{-\infty}^{\infty} \exp\left(-\frac{t^2}{2\sigma^2}\right) e^{-i2\pi\nu t}\, dt. \tag{3}$$

Write the exponent as a quadratic form and complete the square. Factor $-\frac{1}{2\sigma^2}$:

$$\begin{aligned}
-\frac{t^2}{2\sigma^2} - i2\pi\nu t &= -\frac{1}{2\sigma^2}\left(t^2 + i4\pi\sigma^2\nu\, t\right) && (4)\\
&= -\frac{1}{2\sigma^2}\left[\left(t + i2\pi\sigma^2\nu\right)^2 - \left(i2\pi\sigma^2\nu\right)^2\right] && (5)\\
&= -\frac{\left(t + i2\pi\sigma^2\nu\right)^2}{2\sigma^2} - \underbrace{\frac{-\left(i2\pi\sigma^2\nu\right)^2}{2\sigma^2}}_{=\ -2\pi^2\sigma^2\nu^2}. && (6)
\end{aligned}$$

Hence

$$\exp\left(-\frac{t^2}{2\sigma^2} - i2\pi\nu t\right) = e^{-2\pi^2\sigma^2\nu^2}\, \exp\left(-\frac{\left(t + i2\pi\sigma^2\nu\right)^2}{2\sigma^2}\right). \tag{7}$$

Now we change variables, obtaining:

$$\begin{aligned}
I(\nu) &= e^{-2\pi^2\sigma^2\nu^2} \int_{-\infty}^{\infty} \exp\left(-\frac{\left(t + i2\pi\sigma^2\nu\right)^2}{2\sigma^2}\right) dt && (8)\\
&= e^{-2\pi^2\sigma^2\nu^2} \int_{-\infty}^{\infty} \exp\left(-\frac{u^2}{2\sigma^2}\right) du \qquad (u = t + i2\pi\sigma^2\nu) && (9)\\
&= \sigma\sqrt{2\pi}\, e^{-2\pi^2\sigma^2\nu^2}. && (10)
\end{aligned}$$

Thus

$$\boxed{\int_{-\infty}^{\infty} e^{-\frac{t^2}{2\sigma^2}}\, e^{-i2\pi\nu t} dt = \sigma\sqrt{2\pi}\, e^{-2\pi^2\sigma^2\nu^2}.} \tag{11}$$

Frequency shift

For $f(t) = e^{-\frac{t^2}{2\sigma^2}} e^{\pm i2\pi f_0 t}$, the modulation property gives

$$\mathcal{F}\{e^{-\frac{t^2}{2\sigma^2}} e^{\pm i2\pi f_0 t}\}(\nu) = I(\nu \mp f_0) = \sigma\sqrt{2\pi}\, e^{-2\pi^2\sigma^2(\nu\mp f_0)^2}. \tag{12}$$

Final result

Using $\sin(2\pi f_0 t + \phi) = \frac{1}{2i}\left(e^{i(2\pi f_0 t+\phi)} - e^{-i(2\pi f_0 t+\phi)}\right)$ and linearity,

$$\begin{aligned}\mathcal{F}\{f\}(\nu) &= \int e^{-\frac{t^2}{2\sigma^2}} \sin(2\pi f_0 t + \phi)\, e^{-i2\pi\nu t} dt & (13)\\ &= \frac{1}{2i}\Big[e^{i\phi}\, \underbrace{\mathcal{F}\{e^{-\frac{t^2}{2\sigma^2}} e^{i2\pi f_0 t}\}(\nu)}_{\sigma\sqrt{2\pi}\, e^{-2\pi^2\sigma^2(\nu-f_0)^2}} - e^{-i\phi}\, \underbrace{\mathcal{F}\{e^{-\frac{t^2}{2\sigma^2}} e^{-i2\pi f_0 t}\}(\nu)}_{\sigma\sqrt{2\pi}\, e^{-2\pi^2\sigma^2(\nu+f_0)^2}}\Big]. & (14)\end{aligned}$$

Therefore

$$\boxed{\mathcal{F}\{e^{-\frac{t^2}{2\sigma^2}} \sin(2\pi f_0 t + \phi)\}(\nu) = \frac{\sigma\sqrt{2\pi}}{2i}\left(e^{i\phi} e^{-2\pi^2\sigma^2(\nu-f_0)^2} - e^{-i\phi} e^{-2\pi^2\sigma^2(\nu+f_0)^2}\right).} \tag{15}$$

Cosine Gaussian. For $g(t) = e^{-\frac{t^2}{2\sigma^2}} \cos(2\pi f_0 t + \phi)$, one similarly finds

$$\mathcal{F}\{g\}(\nu) = \frac{\sigma\sqrt{2\pi}}{2}\left(e^{i\phi} e^{-2\pi^2\sigma^2(\nu-f_0)^2} + e^{-i\phi} e^{-2\pi^2\sigma^2(\nu+f_0)^2}\right). \tag{16}$$

Shape parameters

The spectral standard deviation is $\sigma_\nu = \frac{1}{2\pi\sigma}$. Each lobe is a Gaussian centered at $\nu = \pm f_0$ with width σ_ν.

Converting to angular-frequency

Let $\omega = 2\pi\nu$ and define

$$\mathcal{F}_\omega\{f\}(\omega) \equiv \int_{-\infty}^{\infty} f(t)\, e^{-i\omega t}\, dt. \tag{17}$$

Then

$$\mathcal{F}_\omega\left\{e^{-\frac{t^2}{2\sigma^2}} e^{i\omega_0 t}\right\}(\omega) = \sqrt{2\pi}\,\sigma\, e^{-\frac{\sigma^2}{2}(\omega-\omega_0)^2}, \tag{18}$$

so for the sine–Gaussian

$$\mathcal{F}_\omega\{e^{-\frac{t^2}{2\sigma^2}} \sin(\omega_0 t + \phi)\}(\omega) = \frac{\sqrt{2\pi}\,\sigma}{2i}\left(e^{i\phi} e^{-\frac{\sigma^2}{2}(\omega-\omega_0)^2} - e^{-i\phi} e^{-\frac{\sigma^2}{2}(\omega+\omega_0)^2}\right). \tag{19}$$

Checks

- Symmetry: for $\phi = 0$, $\mathcal{F}\{\cdot\}$ is purely imaginary and odd, as expected from an odd time-domain signal times an even window.
- Limit $f_0 \to 0$: the sine–Gaussian tends to $\sin(\phi)$ times a base Gaussian; the transform reduces consistently.
- Energy: $\int |f(t)|^2 dt = \sigma\sqrt{\pi}\left(1 - e^{-4\pi^2 f_0^2\sigma^2}\cos 2\phi\right)$; the spectrum integrates to the same value (Parseval).