Analytic Posteriors for a Sine–Gaussian in White Noise

1   Model

Let $y \in \mathbb{R}^n$ be uniformly sampled data with sampling times $t_1, \ldots, t_n$, corrupted by i.i.d. Gaussian noise $\varepsilon \sim \mathcal{N}(0, \sigma_n^2 I)$. The signal is a sine–Gaussian template with unit-amplitude shape

$$\phi_i(t_0) \;=\; \sin\!\big(2\pi f_0(t_i - t_0)\big)\, \exp\!\Big(-\tfrac{1}{2}\Big(\tfrac{t_i - t_0}{\sigma_{\text{env}}}\Big)^2\Big), \quad i = 1, \ldots, n,$$

so that

$$y \;=\; \mu\, \phi(t_0) + \varepsilon, \qquad \varepsilon \sim \mathcal{N}(0, \sigma_n^2 I).$$

We denote $\|\phi\|^2 := \phi^\top \phi$ and the inner product $\phi^\top y$.

2   Analytic posterior for amplitude μ (flat prior)

Assume t_0 is known and adopt a flat (improper) prior $p(\mu) \propto 1$. The likelihood is

$$p(y \mid \mu) \;\propto\; \exp\!\Big(-\frac{1}{2\sigma_n^2}\, \|y - \mu\phi\|^2\Big).$$

Expanding the quadratic and completing the square yields the Gaussian posterior

$$\boxed{\; \mu \mid y \;\sim\; \mathcal{N}\!\Big(\, \hat{\mu},\, \sigma_\mu^2 \,\Big), \qquad \hat{\mu} \;=\; \frac{\phi^\top y}{\phi^\top \phi}, \quad \sigma_\mu^2 \;=\; \frac{\sigma_n^2}{\phi^\top \phi}. \;}$$

Equivalently, the log-posterior (up to an additive constant) is

$$\log p(\mu \mid y) \;=\; -\tfrac{1}{2}\frac{\phi^\top \phi}{\sigma_n^2}\, (\mu - \hat{\mu})^2.$$

Remarks.   (i) $\hat{\mu}$ is the least-squares / matched-filter estimator. (ii) With a Gaussian prior $\mu \sim \mathcal{N}(0, \sigma_0^2)$, one obtains $\mathrm{Var}(\mu \mid y) = \big(\frac{\phi^\top \phi}{\sigma_n^2} + \frac{1}{\sigma_0^2}\big)^{-1}$ and mean $\mathbb{E}[\mu \mid y] = \mathrm{Var}(\mu \mid y)\, \frac{\phi^\top y}{\sigma_n^2}$; letting $\sigma_0^2 \to \infty$ recovers the flat-prior result above.

(Optional) Marginal posterior for t_0 with flat prior on μ

If μ is unknown with flat prior $p(\mu) \propto 1$ and t_0 has a proper uniform prior on $[t_{\min}, t_{\max}]$, one can integrate out μ:

$$p(y \mid t_0) \;\propto\; \frac{1}{\|\phi(t_0)\|}\, \exp\!\Big(\frac{(\phi(t_0)^\top y)^2}{2\sigma_n^2\, \|\phi(t_0)\|^2}\Big),$$

so the marginal posterior is

$$p(t_0 \mid y) \ \propto \ \frac{1}{\|\phi(t_0)\|} \ \exp\!\left(\frac{(\phi(t_0)^\top y)^2}{2\sigma_n^2 \, \|\phi(t_0)\|^2} \right) \quad \text{for} \ \ t_{\min} \leq t_0 \leq t_{\max}, \ \ 0 \text{ otherwise.}$$

This depends on the matched-filter SNR $\rho(t_0) := \dfrac{\phi(t_0)^\top y}{\sigma_n \|\phi(t_0)\|}$ via $\exp(\rho(t_0)^2/2)$, with a $1/\|\phi(t_0)\|$ factor from integrating over μ.

3 Analytic posterior for amplitude μ with flat prior (detailed)

We observe a time series

$$y = (y_1, \ldots, y_n)^\top, \qquad t = (t_1, \ldots, t_n)^\top,$$

with independent Gaussian noise

$$y_i = \mu \, \phi_i + \varepsilon_i, \qquad \varepsilon_i \sim \mathcal{N}(0, \sigma_n^2).$$

Here the template (unit amplitude sine–Gaussian) is

$$\phi_i \equiv \phi(t_i; t_0) = \sin\!\big(2\pi f_0(t_i - t_0)\big) \ \exp\!\Big[-\tfrac{1}{2}\big(\tfrac{t_i - t_0}{\sigma_{\mathrm{env}}}\big)^2 \Big].$$

Vector form:

$$y = \mu \, \phi + \varepsilon, \qquad \varepsilon \sim \mathcal{N}(0, \sigma_n^2 I_n).$$

Likelihood

Because the noise samples are independent Gaussians:

$$p(y \mid \mu) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi}\sigma_n} \exp\!\left[-\frac{(y_i - \mu\phi_i)^2}{2\sigma_n^2} \right].$$

Thus

$$\log p(y \mid \mu) = -\frac{1}{2\sigma_n^2} \sum_{i=1}^{n} (y_i - \mu\phi_i)^2 - \frac{n}{2} \log(2\pi\sigma_n^2).$$

We drop the additive constant independent of μ:

$$\log p(y \mid \mu) \doteq -\frac{1}{2\sigma_n^2} \sum_{i=1}^{n} (y_i - \mu\phi_i)^2.$$

Expand the square term-by-term:

$$\sum_{i=1}^{n} (y_i - \mu\phi_i)^2 = \sum_{i=1}^{n} y_i^2 - 2\mu \sum_{i=1}^{n} y_i\phi_i + \mu^2 \sum_{i=1}^{n} \phi_i^2.$$

Define the sufficient statistics

$$\phi^\top y = \sum_{i=1}^{n} y_i\phi_i, \qquad \|\phi\|^2 = \phi^\top \phi = \sum_{i=1}^{n} \phi_i^2.$$

Then

$$\log p(y \mid \mu) \doteq -\frac{1}{2\sigma_n^2} \Big(\mu^2 \|\phi\|^2 - 2\mu \, (\phi^\top y) \Big) + \text{const.}$$

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Complete the square

We want to rewrite

$$\mu^2 \|\phi\|^2 - 2\mu(\phi^\top y).$$

Factor $\|\phi\|^2$:

$$\|\phi\|^2 \left(\mu^2 - 2\mu \frac{\phi^\top y}{\|\phi\|^2} \right).$$

Write as a square minus a correction:

$$\mu^2 - 2\mu \frac{\phi^\top y}{\|\phi\|^2} = \left(\mu - \frac{\phi^\top y}{\|\phi\|^2} \right)^2 - \left(\frac{\phi^\top y}{\|\phi\|^2} \right)^2.$$

Thus

$$\mu^2 \|\phi\|^2 - 2\mu(\phi^\top y) = \|\phi\|^2 \left(\mu - \frac{\phi^\top y}{\|\phi\|^2} \right)^2 - \frac{(\phi^\top y)^2}{\|\phi\|^2}.$$

Plug back into the log likelihood:

$$\log p(y \mid \mu) \doteq -\frac{1}{2\sigma_n^2} \left[\|\phi\|^2 \left(\mu - \frac{\phi^\top y}{\|\phi\|^2} \right)^2 - \frac{(\phi^\top y)^2}{\|\phi\|^2} \right].$$

Drop the constant independent of μ:

$$\log p(y \mid \mu) \doteq -\frac{\|\phi\|^2}{2\sigma_n^2} \left(\mu - \frac{\phi^\top y}{\|\phi\|^2} \right)^2.$$

Flat prior and posterior

With flat prior $p(\mu) \propto 1$,

$$p(\mu \mid y) \propto p(y \mid \mu).$$

Thus

$$\log p(\mu \mid y) = -\frac{\|\phi\|^2}{2\sigma_n^2} \left(\mu - \frac{\phi^\top y}{\|\phi\|^2} \right)^2 + \text{const.}$$

Recognizing the kernel of a Gaussian:

$$\boxed{\mu \mid y \sim \mathcal{N}\left(\hat{\mu},\, \sigma_\mu^2 \right), \qquad \hat{\mu} = \frac{\phi^\top y}{\|\phi\|^2}, \quad \sigma_\mu^2 = \frac{\sigma_n^2}{\|\phi\|^2}.}$$

This recovers the well-known matched-filter estimator for amplitude with its variance.