MAT 499
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Virtual Math Models: Illustrating Fundamentals of Differential Geometry in Virtual Reality Experiences

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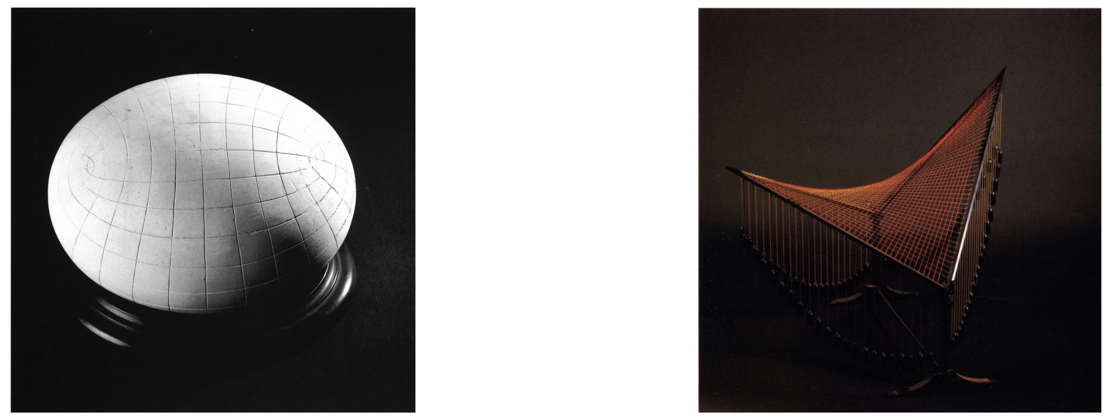
Introduction

In the late 19th century, German mathematicians started producing physical, three-dimensional models of geometric structures. They created these models in order to help them visualize what they could not using just a pen and paper [4]. One particular subset of the models made during this time were those concerned with the field of differential geometry. These models interrogated the curvature properties of surfaces; lines with special measures of curvature and points where curvature appears the same in all directions. Other models focused on the constructability of certain surfaces and the existence of minimal surfaces spanning closed boundaries [1]. Whether the insight gained was from the process of creating the models or from engaging with them as a student during lecture, these models took on an important role in how mathematicians engaged with the processes of learning and understanding mathematics.

Figure 1: Left: a plaster model three-axial ellipsoid, Fischer photo 65 [1]. Right: a string model of a hyperbolic paraboloid, Fischer photo 8 [1].



My project is concerned with investigating the role of the mathematical models produced by German mathematicians in the late 19th century by using the affordances of modern computer graphics and virtual reality to create my own mathematically engaging experiences. I developed two virtual reality experiences, each based on a set of physical models made by German mathematicians in the late 19th century. My virtual reality experiences attempt to use the affordances of computer graphics and virtual reality to create a rich process of learning and understanding that mimics why these models were created in the first place. Specifically, I use the modern visualization techniques and physical embodiment and interaction that virtual reality offers in order to create mathematically engaging experiences.

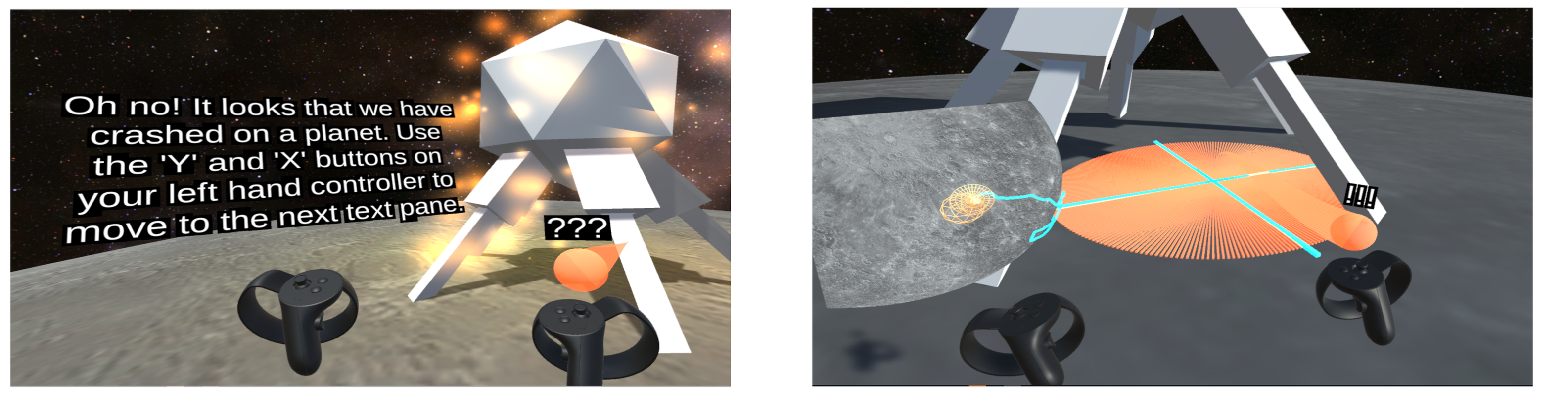
Normal Curvature

The value of curvature for a surface S at a point p is the measure of how quickly the unit tangent vector of a curve lying in S passing through p changes as we move in a particular direction. For example, the unit tangent vector of a longitude of a sphere with a small radius will change faster than that of a sphere with a larger radius. This matches up with our intuition about the smaller sphere being 'curved' more than the larger sphere.

The Curvature Experience

My first virtual reality experience is grounded in the models of quadric sufaces and the special properties of curvature that such models demonstrate. The key concept illustrated in this experience will be that of curvature, which describes how a surface behaves locally around a given point. At each point on a surface, we can ask how the surface curves as we 'look in a certain direction'. We call points that are locally spherical *umbilical points* and it turns out that different quadric surfaces have unique configurations of such points [1].

Figure 2: Left: the first scene of the curvature experience. Right: using the compass and minimap to discover an umbilical point.



In the first experience, our player has crash-landed on an unknown planet, and they are tasked with finding all of the umbilical points on the planet in order to escape. This unknown planet is a three-axial ellipsoid, and it turns out that there are exactly four umbilical points the player will have to find [1]. Looking at the engraved lines of curvature on the model of the ellipsoid in Figure 1 can give an intuition of where these umbilical points are. However, if one was not told that the umbilical points are where these lines start to converge or if one was not given these lines at all, with a definition of curvature one might not easily understand where such points lie.

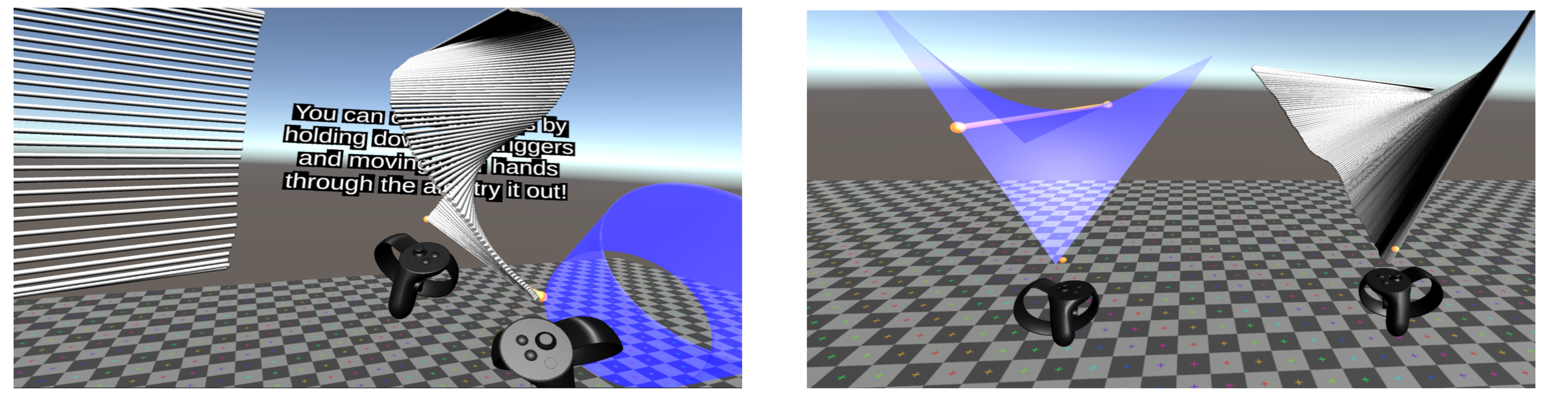
My first experience uses the affordances of virtual reality to let a player 'walk' about on an ellipsoid and attempt to gain an intuition of curvature using a 'compass' and 'minimap.' The compass shows the player how the surface that they are walking on curves at a given point by visually displaying the measure of curvature in different directions. The player uses the compass by pointing their right controller at a point on the ground in front of them and pressing down the right trigger. Then, the compass displays a 'peanut shaped' array of orange tangent directions. Each of these orange tangent directions is scaled to demonstrate the 'measure of curvature' in that direction, and two cyan tangents display the most important curvatures called the principle curvatures. The minimap allows the player to keep track of their discovered umbilical points as well as their path on the ellipsoid. Additionally, the minimap is intended to give the player another way of visualizing the curvature of the ellipsoid.

In the beginning of the experience, the player is positioned on the ellipsoid where one can visually see how the surface is curving differently in two directions. To demonstrate this to the player, the instructions urge them to look at where the 'peanut shape' of the compass stretches the most and the least. The instructions ask the player if the curvature of the horizon matches up with the shape of the curvature compass. After finding all of the umbilical points of the ellipsoid, the player can escape from the planet and has finished the experience.

The Ruled Surfaces Experience

My second virtual reality experience is grounded in the models of ruled surfaces and their construction out of strings and wire. They key concept illustrated in this experience is that of how a ruled surface can be constructed out of a continuous 'sweeping' of lines. The second experience, unlike the first, is a sandbox experience where players are challenged to figure out how to construct a ruled surface out of lines. I take full advantage of the affordance of embodiment in virtual reality by tasking the players to use the positions of their hands in space to draw out lines constructing four possible ruled surfaces. The experience interactively explains what a ruled surface is to the player by showing them how to construct a cylinder. The experience visualizes the challenge as a blue transparent blueprint that the player can reach out and 'grab.' The experience also demonstrates what a ruled surface is by drawing animated hints that show how one can move their hands to create the challenges. Each challenge comes in the form of a transparent 'blueprint' that the players have to fill in with their rulings and decide whether or not they have solved how to construct the ruled surfaces.

Figure 3: Left: drawn rulings. Right: comparing a completed hyperbolic parboloid to the blueprint.



I try to give the player as much of an embodied experience as possible by not only letting the players draw in the physical space around them but also letting them use the controllers to 'grab' their drawings and blueprint visualizations. In this fashion, the players have total control over how they want to engage with the problem. Some of the people who tried out this project were crouching and walking around the virtual objects; it is a really immersive experience. Even though this project is not as technically engaged as the first, it uses and engages with the affordances of virtual reality in a much more tangible way.

Conclusion

With some of the framework that I have produced, I can imagine extending the functionality of the curvature experience to explore surfaces with different types of curvature. I imagine a future virtual reality experience where players have to walk in 'straight paths' in differently curved surfaces in order to solve mazes or puzzles. Furthermore, such an experience might demonstrate how the curvature of a surface affects the convergence and divergence of 'straight paths.'

As we have seen, the affordances of virtual reality allow us to engage with mathematical models in similar but different ways to those of German mathematicians in the late 19th century. Virtual reality allows our mathematical models to become mathematical experiences that are a lot more engaging and facilitate a conversation between the player and mathematics. My virtual reality projects suggest that the process of creating mathematical models can still be useful to learning mathematics and that the resulting mathematical experiences can be used as valuable teaching tools.

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