08/05 - Condicionamento

$$\|A\|_1 = \max_{j=1,\dots,n} \sum_{i=1}^{m} |a_{ij}| \qquad \|A\|_F = \left[\sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij}^2\right]^{1/2}$$

$$\|A\|_\infty = \max_{i=1,\dots,m} \sum_{j=1}^{n} |a_{ij}|$$

Frobenius
Não é induzida

$\|A\|_2 = \sigma_1$ maior valor singular

Exercícios: Mostre que as normas, de fato, são essas, i.e., suas expressões usando o sup, viram as expressões acima.

Ex.: $\|A\|_2 = \sup_{x \neq 0} \dfrac{\|Ax\|_2}{\|x\|_2}$

ou seja $\quad \|A\|_2 \geq \dfrac{\|Ay\|_2}{\|y\|_2}, \quad \forall\, y \in \mathbb{R}^n,\ y \neq 0.$

Considere a dec. SVD de A: $A = U\Sigma V^T$.

Logo, $Av_i = \sigma_i u_i$. Daí,

$$\|A\|_2 \geq \frac{\|Av_1\|_2}{\|v_1\|_2} = \frac{\|\sigma_1 u_1\|_2}{1} = \sigma_1 \|u_1\|_2 = \sigma_1$$

Tome $x \in \mathbb{R}^n$, $x \neq 0$ qualquer. Como $\{v_1,\dots,v_n\}$ é base de $\mathbb{R}^n$, $\exists\, \lambda_1,\dots,\lambda_n \in \mathbb{R}$ t.q.

$Av_i = \sigma_i u_i$

$$Ax = \lambda_1 \overset{A}{v_1} + \dots + \lambda_n v_n$$

A*

$$Ax = \lambda_1 \overbrace{\sigma_1 u_1}^{Av_1} + \lambda_2\sigma_2 u_2 + \dots + \lambda_p \sigma_p u_p + 0$$

$$\|Ax\|_2^2 = (\lambda_1\sigma_1)^2 + (\lambda_2\underbrace{\sigma_2}_{\leq \sigma_1})^2 + \dots + (\lambda_p \underbrace{\sigma_p}_{\leq \sigma_1})^2$$

$$\leq (\lambda_1\sigma_1)^2 + (\lambda_2\sigma_1)^2 + \dots + (\lambda_p\sigma_1)^2$$

$$= \sigma_1^2(\lambda_1^2 + \dots + \lambda_p^2)$$

$\|\alpha_1 u_1 + \dots + \alpha_p u_p\|_2^2$
$= (\alpha_1 u_1 + \dots + \alpha_p u_p)^T(\alpha_1 u_1 + \dots + \alpha_p u_p)$
$= \sum_{i=1}^{p}\sum_{j=1}^{p} \alpha_i\alpha_j u_i^T u_j = \sum_{i=1}^{p} \alpha_i^2 \|u_i\|^2 = \sum_{i=1}^{p}\alpha_i^2.$

$$\|x\|_2^2 = \|\lambda_1 v_1 + \dots + \lambda_n v_n\|_2^2$$
$$= \lambda_1^2 + \dots + \lambda_p^2 + \underbrace{\lambda_{p+1}^2 + \dots + \lambda_n^2}_{\geq 0}$$
$$\geq \lambda_1^2 + \dots + \lambda_p^2 + 0$$

Logo,

$$\|A\|_2 = \sup_{x\neq 0} \frac{\|Ax\|_2}{\|x\|_2} \leq \sup_{\lambda \neq 0} \frac{\sigma_1\sqrt{\lambda_1^2+\dots+\lambda_p^2}}{\sqrt{\lambda_1^2+\dots+\lambda_p^2}}$$

$$= \sup_{\lambda\neq 0} \sigma_1 = \sigma_1$$

$\therefore \quad \sigma_1 \leq \|A\|_2 \leq \sigma_1 \Rightarrow \boxed{\|A\|_2 = \sigma_1}$

Ex.: Vide Golub. Provar com otimização.

$\det(\alpha A) = \alpha^n \det(A)$

Condicionamento ou número de condição:

$$K(A) = \|A\|\cdot\|A^{-1}\|$$

Se A não tem inversa, definimos $K(A) = \infty$.

Se for uma norma específica, põe o índice no K, e.g., $K_\infty(A) = \|A\|_\infty \|A^{-1}\|_\infty$.

Teo.: $K_2(A) = \dfrac{\sigma_1}{\sigma_n}$, se $A \in \mathbb{R}^{n\times n}$ tem inversa.

Dem.: Basta mostrar que $\|A^{-1}\|_2 = \dfrac{1}{\sigma_n}$.

Propriedade: $K(I) = \|I\|\cdot\|I^{-1}\| = \|I\|^2$

p/ norma induzida: $\|I\| = \sup_{x\neq 0} \dfrac{\|Ix\|}{\|x\|} = 1$

Logo, $K(I) = 1$

consistência

$$1 = \|I\| = \|A\cdot A^{-1}\| \leq \|A\|\cdot\|A^{-1}\| = K(A)$$

$\therefore \quad K(A) \geq 1$ p/ normas induzidas.

Exemplos: (i) $K(\alpha I) \overset{\alpha \neq 0}{=} \|\alpha I\|\cdot\|(\alpha I)^{-1}\| = \|\alpha I\|\cdot\|\frac{1}{\alpha} I\| = 1$

(ii) $K(P) = 1$, P matriz de permutação

(iii) $K(Q) = 1$, matriz ortogonal $(Q^{-1} = Q^T)$

Exemplos: $A = \begin{bmatrix} 1.01 & 2 \\ 1 & 2 \end{bmatrix}$

$\|A\|_1 = \max\{|1.01| + |1|, |2| + |2|\} = \max\{2.01, 4\} = 4$

$\|A\|_\infty = \max\{|1.01| + |2|, |1| + |2|\} = 3.01$

$$A^{-1} = \frac{1}{2.02 - 2}\begin{bmatrix} 2 & -2 \\ -1 & 1.01 \end{bmatrix} = \frac{1}{0.02}\begin{bmatrix} 2 & -2 \\ -1 & 1.01 \end{bmatrix}$$

$$\|A^{-1}\|_1 = \frac{1}{0.02}\max\{3, 3.01\} = \frac{3.01}{0.02} = 150.5$$

$$\|A^{-1}\|_\infty = \frac{1}{0.02}\max\{4, 2.01\} = \frac{4}{0.02} = 200$$

$K_1(A) = 4 \times 150.5 = 602$

$K_\infty(A) = 3.01 \times 200 = 602$

Ex.: Verifique se $K(A) \xrightarrow{\varepsilon \to 0^+} \infty$ p/

(i) $A = \begin{bmatrix} \varepsilon & \\ & \varepsilon \end{bmatrix}$, $K(A) = K(\varepsilon I) = 1$.

Exerc. K_1, K_2, K_∞ de diagonal.

(ii) $A = \begin{bmatrix} \varepsilon & \\ & 1 \end{bmatrix}$

$\|A\| = \max\{1, \varepsilon\}$, $A^{-1} = \begin{bmatrix} \varepsilon^{-1} & \\ & 1 \end{bmatrix}$, $\|A^{-1}\| = \max\{1, \varepsilon^{-1}\}$

$K(A) = \max\{1,\varepsilon\}\max\{1,\varepsilon^{-1}\} = \max\{\varepsilon, \varepsilon^{-1}\}$

se $\varepsilon \to 0^+$, eventualmente, $\varepsilon < 1$, daí $K(A) = \varepsilon^{-1} \to \infty$.

(iii) $A = \begin{bmatrix} 1+\varepsilon & 1 \\ 1 & 1 \end{bmatrix}$, $\|A\|_1 = 2+\varepsilon$, $A^{-1} = \dfrac{1}{\varepsilon}\begin{bmatrix} 1 & -1 \\ -1 & 1+\varepsilon \end{bmatrix}$, $\|A^{-1}\|_1 = \dfrac{2+\varepsilon}{\varepsilon}$

$$K_1(A) = (2+\varepsilon)\frac{2+\varepsilon}{\varepsilon} = \frac{(2+\varepsilon)^2}{\varepsilon} \xrightarrow{\varepsilon \to 0^+} +\infty$$

Resolução de sistemas $\quad (K(A) < \infty)$

$\to \quad Ax^* = b \qquad x^*$ sol. exata

$x^* + \Delta x, \quad b + \Delta b$

achou $\bar{x}$: $\quad A\bar{x} = \bar{b}$

$A(x^* - \bar{x}) = b - \bar{b} \quad \leadsto \quad x^* - \bar{x} = A^{-1}(b - \bar{b})$

$\Delta x = A^{-1}\Delta b$

$\|\Delta x\| = \|A^{-1}\Delta b\|$

$\|\Delta x\| \leq \|A^{-1}\|\cdot\|\Delta b\|$

$\|A\|\cdot\|\Delta x\| \leq \|A\|\cdot\|A^{-1}\|\cdot\|\Delta b\|$
$= K(A)\cdot\|\Delta b\|$

$\|b\| = \|Ax^*\| \leq \|A\|\cdot\|x^*\|$

$\|\Delta x\|\cdot\|b\| \leq \|A\|\cdot\|A^{-1}\|\cdot\|x^*\|\cdot\|\Delta b\|$

$$\frac{\|\Delta x\|}{\|x^*\|} \leq K(A)\cdot\frac{\|\Delta b\|}{\|b\|}$$

Exerc.: $A = \begin{bmatrix} 1+\varepsilon & 1-\varepsilon \\ 1 & 1 \end{bmatrix}$, $K(A)$

Ex.: $b = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$, $\bar{x} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$, $A\bar{x} = \begin{bmatrix} 2+2\varepsilon \\ 2 \end{bmatrix} = \bar{b}$

$\Delta b = \begin{bmatrix} -2\varepsilon \\ 0 \end{bmatrix}$, $x^* = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$