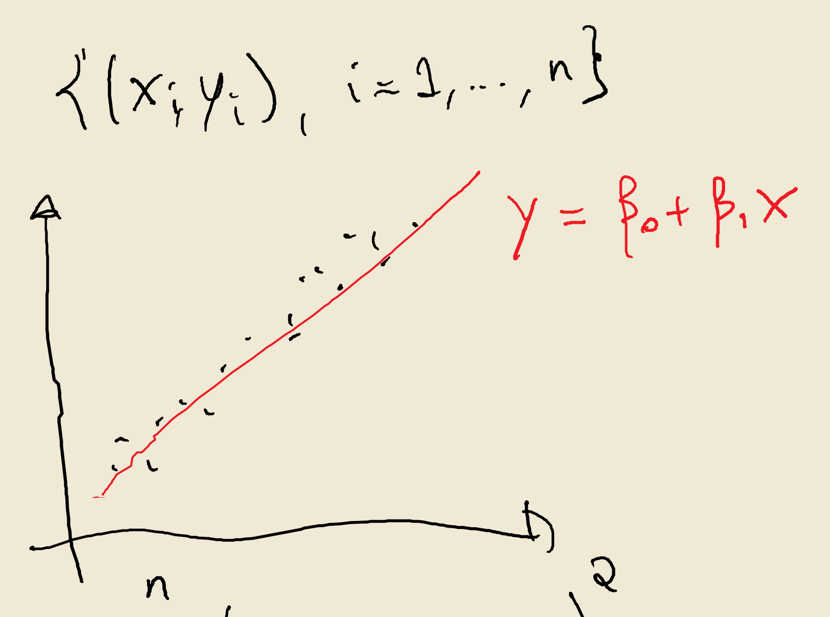
03/06

$\{(x_i, y_i),\ i = 1, \dots, n\}$



$$\min_{\beta_0, \beta_1} \frac{1}{2} \sum_{i=1}^{n} (y_i - \beta_0 - \beta_1 x_i)^2 = \min_{\beta} \frac{1}{2} \|y - X\beta\|^2$$

$$\Rightarrow \quad X^T X \beta = X^T y$$

$$\begin{bmatrix} n & \sum x_i \\ \sum x_i & \sum x_i^2 \end{bmatrix} \begin{bmatrix} \beta_0 \\ \beta_1 \end{bmatrix} = \begin{bmatrix} \sum y_i \\ \sum x_i y_i \end{bmatrix}$$

Modelo linear geral (1-dim)
em β

$$y \approx \beta_1 \varphi_1(x) + \beta_2 \varphi_2(x) + \dots + \beta_p \varphi_p(x)$$

$\varphi_k : \mathbb{R} \to \mathbb{R}$

Ex.: usual: $\varphi_1(x) = 1, \quad \varphi_2(x) = x$

$$\Rightarrow \quad y \approx \beta_1 + \beta_2 x$$

Ex.: polinomial: $\varphi_1(x) = 1, \quad \varphi_k(x) = x^{k-1}$

$$y \approx \beta_0 + \beta_1 x + \beta_2 x^2 + \dots + \beta_{p-1} x^{p-1} + \beta_p x^p$$

Ex.: Sasonal: ajustar p/ [Jan, Dez]

$$y \approx \beta_0 + \beta_1 \operatorname{sen}(Lx) + \beta_2 \cos(Lx)$$

Ex.: $y \approx \beta_0 e^{-x} + \beta_1 x + \beta_2 \ln x + \beta_3 \sin(\cos(\ln(\operatorname{tg} x)))$

Podemos escrever

$$y \approx \beta^T \varphi(x) = \langle \beta, \varphi(x) \rangle$$

$$\varphi(x) = (\varphi_1(x), \dots, \varphi_p(x))^T$$

Como encontrar os parâmetros β tais que o ajuste $y_i \approx \sum_{j=1}^{p} \beta_j \varphi_j(x)$ é o melhor?

$$\Rightarrow E(\beta) = \frac{1}{2} \sum_{i=1}^{n} \Big[\underset{i}{y_i} - \sum_{j=1}^{p} \underset{j}{\beta_j} \underset{j}{\varphi_j} (\underset{i}{x_i}) \Big]^2$$

$$\frac{\partial E}{\partial \beta_k} = \frac{1}{2} \cdot \sum_{i=1}^{n} 2 \Big[y_i - \sum_{j=1}^{p} \beta_j \varphi_j(x_i) \Big] \cdot \big[-\varphi_k(x_i) \big]$$

$$= -\sum_{i=1}^{n} y_i \varphi_k(x_i) + \sum_{i=1}^{n} \sum_{j=1}^{p} \beta_j \varphi_j(x_i) \varphi_k(x_i)$$

$$= 0 \Rightarrow \sum_{j=1}^{p} \beta_j \underbrace{\Big[\sum_{i=1}^{n} \varphi_j(x_i) \varphi_k(x_i) \Big]}_{M_{kj}} = \underbrace{\sum_{i=1}^{n} y_i \varphi_k(x_i)}_{c_k}$$

$$\therefore \quad M\beta = c$$

Se $y_i \approx \beta_0 + \beta_1 x + \dots + \beta_p x^p$

$$M = \begin{bmatrix} \sum 1 & \sum x_i & \sum x_i^2 & \cdots & \sum x_i^p \\ \sum x_i & \sum x_i^2 & & & \vdots \\ \sum x_i^2 & & \sum x_i^p & & \\ \vdots & & & \ddots & \\ \sum x_i^p & \cdots & & & \sum x_i^{2p} \end{bmatrix}$$

$$c = \begin{bmatrix} \sum_{i=1}^{n} y_i \\ \vdots \\ \sum_{i=1}^{n} y_i x_i^p \end{bmatrix}$$

Ex.: Ajuste

x	-2	-1	0	1	2
y	4	1	-3	2	3

c/ um pol. de grau 2.

$$y_i \approx \beta_0 + \beta_1 x_i + \beta_2 x_i^2$$

$M_{11} = \sum 1; \quad M_{21} = M_{12} = \sum x_i;$

$M_{31} = M_{22} = M_{13} = \sum x_i^2$

$$M = \begin{bmatrix} n & \sum x_i & \sum x_i^2 \\ \sum x_i & \sum x_i^2 & \sum x_i^3 \\ \sum x_i^2 & \sum x_i^3 & \sum x_i^4 \end{bmatrix} = \begin{bmatrix} 5 & 0 & 10 \\ 0 & 10 & 0 \\ 10 & 0 & 34 \end{bmatrix}$$

$$c = \begin{bmatrix} \sum y_i \\ \sum x_i y_i \\ \sum x_i^2 y_i \end{bmatrix} = \begin{bmatrix} 7 \\ -1 \\ 30 \end{bmatrix}$$

$$\begin{bmatrix} 5 & 0 & 10 \\ 0 & 10 & 0 \\ 10 & 0 & 34 \end{bmatrix} \begin{bmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \end{bmatrix} = \begin{bmatrix} 7 \\ -1 \\ 30 \end{bmatrix}$$

$$10\beta_1 = -1 \Rightarrow \beta_1 = -0.1$$

$$\begin{bmatrix} 5 & 10 \\ 10 & 34 \end{bmatrix} \begin{bmatrix} \beta_0 \\ \beta_2 \end{bmatrix} = \begin{bmatrix} 7 \\ 30 \end{bmatrix}$$

$$\begin{pmatrix} \beta_0 \\ \beta_2 \end{pmatrix} = \frac{1}{170 - 100} \begin{bmatrix} 34 & -10 \\ -10 & 5 \end{bmatrix} \begin{bmatrix} 7 \\ 30 \end{bmatrix}$$

$$= \frac{1}{70} \begin{bmatrix} 238 - 300 \\ -70 + 150 \end{bmatrix} = \begin{bmatrix} -62/70 \\ 80/70 \end{bmatrix}$$

$$\beta_0 = -\frac{62}{70}, \quad \beta_1 = \frac{-1}{10}, \quad \beta_2 = \frac{80}{70}$$

$$y \approx \underbrace{\beta_0 + \beta_1 x + \beta_2 x^2}_{h(x)}$$

Voltando, se $y \approx \beta_1 \varphi_1(x) + \dots + \beta_p \varphi_p(x)$ e tenho dados $\{(x_i, y_i), i = 1, \dots, n\}$,

Defino

$$X = \begin{bmatrix} \varphi_1(x_1) & \varphi_2(x_1) & \cdots & \varphi_p(x_1) \\ \varphi_1(x_2) & \varphi_2(x_2) & \cdots & \varphi_p(x_2) \\ \vdots & \vdots & & \vdots \\ \varphi_1(x_n) & \varphi_2(x_n) & \cdots & \varphi_p(x_n) \end{bmatrix}$$

Perceba que $y = (y_1, \dots, y_n)^T$ satisfaz

$$y \approx X\beta$$

Problema de Quadrados Mínimos:

$$\min_{\beta} \frac{1}{2} \|y - X\beta\|^2$$

$$\Rightarrow \underbrace{X^T X}_{M} \beta = \underbrace{X^T y}_{c} \quad \text{Equações Normais}$$

$$T = T_a + Ae^{-Bt}$$ → n̄ é linear em A e B

$$T - T_a = Ae^{-Bt}$$

$$\Rightarrow \underbrace{\ln(T - T_a)}_{Y} = \underbrace{\ln A}_{\beta_0} - \underbrace{B}_{\beta_1} t$$

$$\boxed{Y = \beta_0 - \beta_1 t}$$ → linear em β