Ex.: Calcule LU 27/05

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & -1 & 4 \\ 5 & 0 & 4 \end{bmatrix}$$

$$A^{(1)} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & -9 & -8 \\ 5 & -10 & -11 \end{bmatrix} \qquad m_{21} = \frac{a_{21}}{a_{11}} = \frac{4}{1} \qquad L_2 - m_{21} L_1 \qquad m_{31} = \frac{a_{31}}{a_{11}} = \frac{5}{1}$$

$$A^{(2)} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & -9 & -8 \\ 5 & 10/9 & -19/9 \end{bmatrix} \qquad m_{32} = \frac{a_{32}^{(1)}}{a_{22}^{(1)}} = \frac{-10}{-9} = \frac{10}{9}$$

$$-11 - \frac{10}{9}(-8) = \frac{-99+80}{9} = \frac{-19}{9}$$

$$L = \begin{bmatrix} 1 & & \\ 4 & 1 & \\ 5 & 10/9 & 1 \end{bmatrix} \quad U = \begin{bmatrix} 1 & 2 & 3 \\ & -9 & -8 \\ & & -19/9 \end{bmatrix}$$

Como resolver $Ax = b$ com $A = LU$

$$Ax = b \Rightarrow L\underbrace{Ux}_{y} = b \Rightarrow \begin{cases} Ly = b \\ Ux = y \end{cases}$$

Ex.: Resolva usando LU:

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & -1 & 4 \\ 5 & 0 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 6 \end{bmatrix}$$

$$L = \begin{bmatrix} 1 & & \\ 4 & 1 & \\ 5 & 10/9 & 1 \end{bmatrix}, \quad U = \begin{bmatrix} 1 & 2 & 3 \\ & -9 & -8 \\ & & -19/9 \end{bmatrix}$$

$$Ly = b \Rightarrow \begin{cases} y_1 = 1 & \Rightarrow y_1 = 1 \\ 4y_1 + y_2 = 3 & \Rightarrow y_2 = 3 - 4 \cdot 1 = -1 \\ 5y_1 + \frac{10}{9} y_2 + y_3 = 6 \end{cases}$$

$$y_3 = 6 - 5 \cdot 1 - \frac{10}{9} \cdot (-1) = \frac{19}{9}.$$

$$Ux = y \Rightarrow \begin{cases} x_1 + 2x_2 + 3x_3 = 1 \\ -9x_2 - 8x_3 = -1 \\ -\frac{19}{9} x_3 = \frac{19}{9} \end{cases}$$

$$x_3 = -1$$

$$x_2 = \frac{-1 + 8 \cdot \overset{x_3}{(-1)}}{-9} = \frac{-9}{-9} = 1$$

$$x_1 = \frac{1 - 3 \cdot \overset{x_3}{(-1)} - 2 \cdot \overset{x_2}{1}}{1} = 2$$

LU com pivoteamento parcial

sem piv. $\quad E_{n-1} \cdots E_3 E_2 E_1 A = U$

E_2: 2ª elim. $\quad E_1$: 1ª col

piv.: P_j de permutação de linhas na j-ésima iteração; P_j troca a linha j por alguma linha $k \geq j$, onde $|a_{kj}| = \max_{i=j,\dots,n} |a_{ij}|$

Con pivoteamento:

$$E_{n-1} P_{n-1} \cdots E_3 P_3 E_2 P_2 E_1 P_1 A = U$$

Teorema: $PA = LU$.

Motivação: A 3x3

$P_2^T P_2 = I$ → Prop.: As matrizes P_j satisfazem $P_j^{-1} = P_j^T$

$$E_2 P_2 E_1 P_1 A = U$$

P_1: age $L_1 : L_3$; E_1: age $L_2 : L_3$; P_2: age $L_2 : L_3$; E_2: L_3

$$E_2 \underbrace{P_2 E_1 P_2^T} P_2 P_1 A = U$$

$$E_1 = \begin{bmatrix} 1 & & \\ m_{21} & 1 & \\ m_{31} & & 1 \end{bmatrix}, \quad P_2 E_1 = \begin{bmatrix} 1 & & \\ m_{31} & & 1 \\ m_{21} & 1 & \end{bmatrix}$$

$$P_2 E_1 P_2^T = \begin{bmatrix} 1 & & \\ m_{31} & 1 & \\ m_{21} & & 1 \end{bmatrix}$$

$$E_1 P_1 A = A^{(1)}$$

$$E_2 P_2 E_1 P_1 A = A^{(2)}$$

$$E_2 \underbrace{P_2 E_1 P_2^T} P_2 P_1 A = A^{(2)}$$

$$E_2 \tilde{E}_1 P_2 P_1 A = A^{(2)}$$

$$E_3 P_3 E_2 \tilde{E}_1 P_2 P_1 A = A^{(3)}$$

$$E_3 \underbrace{P_3 E_2 P_3^T} P_3 P_2 E_1 P_2^T P_2 P_1 A = A^{(3)}$$

$$E_3 \tilde{E}_2 \underbrace{P_3 P_2 E_1 (P_2^T P_3^T)} P_3 P_2 P_1 A = A^{(3)} \qquad P_2^T P_3^T = (P_3 P_2)^T$$

$$E_3 \tilde{E}_2 \tilde{\tilde{E}}_1 P_3 P_2 P_1 A = A^{(3)}$$

Teo.: Fazendo elim. Gauss. c/ piv., obtemos $PA = LU$, onde

$$P = P_{n-1} \cdot P_{n-2} \cdots P_3 P_2 P_1$$

$$L = (E_{n-1} \cdot \tilde{E}_{n-2} \tilde{E}_{n-3} \cdots \hat{E}_2 \tilde{E}_1)^{-1},$$

onde $\tilde{E}_{n-2} = P_{n-1} E_{n-2} P_{n-1}^T$

$\tilde{E}_{n-3} = P_{n-1} P_{n-2} E_{n-3} P_{n-2}^T P_{n-1}^T$

$\vdots$

$\tilde{E}_1 = P_{n-1} P_{n-2} \cdots P_2 E_1 P_2^T \cdots P_{n-1}^T$

e U a matriz resultante da eliminação.

Ex.: Resolva o SL c/ LU c/ pivot.

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & -1 & 4 \\ 5 & 0 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 6 \end{bmatrix}$$

No LU c/ pivoteamento, $p = [1\ 2\ 3]$

pivô: $|a_{31}| = 5 \quad L_1 \leftrightarrow L_3$

$$p = [3\ 2\ 1], \quad \tilde{A} = \begin{bmatrix} 5 & 0 & 4 \\ 4 & -1 & 4 \\ 1 & 2 & 3 \end{bmatrix}$$

$$A^{(1)} = \begin{bmatrix} 5 & 0 & 4 \\ 0.8 & -1 & 0.8 \\ 0.2 & 2 & 2.2 \end{bmatrix}$$

pivô: $|a_{32}| = 2$, $L_3 \leftrightarrow L_2$ $\quad p = [3\ 1\ 2]$

$$\tilde{A}^{(1)} = \begin{bmatrix} 5 & 0 & 4 \\ 0.2 & 2 & 2.2 \\ 0.8 & -1 & 0.8 \end{bmatrix}$$

$$A^{(2)} = \begin{bmatrix} 5 & 0 & 4 \\ 0.2 & 2 & 2.2 \\ 0.8 & -0.5 & 1.9 \end{bmatrix} \qquad 0.8 - (-0.5) \times 2.2 \quad 0.8 + 1.1$$

$$L = \begin{bmatrix} 1 & & \\ 0.2 & 1 & \\ 0.8 & -0.5 & 1 \end{bmatrix}, \quad U = \begin{bmatrix} 5 & 0 & 4 \\ & 2 & 2.2 \\ & & 1.9 \end{bmatrix}$$

$$P = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$K(A) = \|A\| \cdot \|A^{-1}\|$$

$$A = P^T L U$$

$$\boxed{\text{Prop.: } \|P\|_1 = \|P\|_2 = \|P\|_\infty = 1}$$

$$\|A\| = \|P^T L U\| \leq \|P^T\| \cdot \|L\| \cdot \|U\| = \|L\| \cdot \|U\|$$

$$\|A^{-1}\| = \|U^{-1} L^{-1} P\| \leq \|U^{-1}\| \cdot \|L^{-1}\| \cdot \|P\| = \|L^{-1}\| \cdot \|U^{-1}\|$$

$$\Rightarrow K(A) = \|A\| \cdot \|A^{-1}\| \leq \|L\| \cdot \|L^{-1}\| \cdot \|U\| \cdot \|U^{-1}\| = K(L) \cdot K(U)$$

$$K(L) \cdot K(U) \geq K(A)$$

Com pivoteamento $|l_{ij}| \leq 1$ p/ $i \geq j$.

$$\|L\|_\infty \leq n$$