03/06 - Cholesky

Resolução do sistema: $Ax=b$, $A = GG^T$

$$\Rightarrow \quad G\underbrace{G^T x}_{y} = b \quad \Rightarrow \quad \begin{cases} Gy = b \\ G^T x = y \end{cases}$$

Ex.: Resolva usando Cholesky:

$$\begin{bmatrix} 2 & -1 & & \\ -1 & 2 & -1 & \\ & -1 & 2 & -1 \\ & & -1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

$$\begin{bmatrix} 2 & -1 & & \\ -1 & 2 & -1 & \\ & -1 & 2 & -1 \\ & & -1 & 2 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1.41 & & & \\ -0.71 & & & \\ & & & \\ & & & \end{bmatrix}$$

$$g_{ii} = \sqrt{a_{ii} - \sum_{j=1}^{i-1} g_{ij}^2} \quad ; \quad g_{ij} = \frac{a_{ij} - \sum_{k=1}^{j-1} g_{ik} g_{jk}}{g_{jj}}$$

mesma linha

$i > j$

$g_{11} = \sqrt{2} \approx 1.41$ | $g_{31} = 0/\sqrt{2} = 0$

$g_{21} = \dfrac{-1}{\sqrt{2}} \approx -0.71$ | $g_{41} = 0/\sqrt{2} = 0$

$$g_{22} = \sqrt{2 - \left(\frac{-1}{\sqrt{2}}\right)^2} = \sqrt{2 - 1/2} = \frac{\sqrt{3}}{\sqrt{2}} \approx 1.22$$

$$g_{32} = \frac{-1 - 0 \times (-0.71)}{1.22} \approx -0.82 \quad \Big| \quad g_{42} = 0$$

$$g_{33} = \sqrt{2 - (-0.82)^2 - 0^2} \approx 1.15$$

$$g_{43} = \frac{-1 - 0 \times 0 - 0 \times (-0.82)}{1.15} \approx -0.87$$

$$g_{44} = \sqrt{2 - (-0.87)^2 - 0^2 - 0^2} \approx 1.11$$

$$G = \begin{bmatrix} 1.41 & & & \\ -0.71 & 1.22 & & \\ & -0.82 & 1.15 & \\ & & -0.87 & 1.11 \end{bmatrix} \qquad \begin{cases} Gy = b \\ G^T x = y \end{cases}$$

$$Gy = b \Rightarrow \begin{cases} 1.41\, y_1 = 1 \Rightarrow y_1 = 0.71 \\ -0.71\, y_1 + 1.22\, y_2 = 0 \Rightarrow y_2 = 0.41 \\ -0.82\, y_2 + 1.15\, y_3 = 0 \Rightarrow y_3 = 0.29 \\ -0.87\, y_3 + 1.11\, y_4 = 1 \Rightarrow y_4 = 1.13 \end{cases}$$

$$G^T x = y \Rightarrow \begin{cases} 1.41\, x_1 - 0.71\, x_2 = 0.71 \\ 1.22\, x_2 - 0.82\, x_3 = 0.41 \\ 1.15\, x_3 - 0.87\, x_4 = 0.29 \\ 1.11\, x_4 = 1.13 \end{cases}$$

$x_4 = 1.02$ $\quad x_2 = 1.02$

$x_3 = 1.02$ $\quad x_1 = 1.02$

$$\tilde{x} = \begin{bmatrix} 1.02 \\ 1.02 \\ 1.02 \\ 1.02 \end{bmatrix}$$

Resíduo: $$b - \underbrace{A\tilde{x}}_{\tilde{b}} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} - \begin{bmatrix} 2 & -1 & & \\ -1 & 2 & -1 & \\ & -1 & 2 & -1 \\ & & -1 & 2 \end{bmatrix} \begin{bmatrix} 1.02 \\ 1.02 \\ 1.02 \\ 1.02 \end{bmatrix} = \begin{bmatrix} -0.02 \\ 0 \\ 0 \\ -0.02 \end{bmatrix}$$

$\|x - \tilde{x}\|_\infty = 0.02$; $\|b - \tilde{b}\|_\infty = 0.02$

Calc. o nº de cond. de A (A^{-1}?) computacionalmente

Teo.: $K_2(A) = K_2(G)^2$ e $K(A) \leq K(G) \cdot K(G^T)$

Dem.: $K(A) = \|A\| \cdot \|A^{-1}\| = \|GG^T\| \cdot \|G^{-T} G^{-1}\| \leq \|G\| \cdot \|G^T\| \cdot \|G^{-T}\| \cdot \|G^{-1}\|$
$= K(G) \cdot K(G^T)$

P/ K_2, veja que se $G = U \Sigma V^T$ é a dec. SVD, então $A = U\Sigma V^T V \Sigma U^T = U \Sigma^2 U^T$, logo os autovalores de A são os val. sing. de G ao quadrado. Daí, basta ver que

$$K(A) = \frac{\lambda_1}{\lambda_n} \quad \text{e} \quad K(G) = \frac{\sigma_1}{\sigma_n}.$$