27/05 - $Ax = b$

Elem.

$$[A \mid b] \to [U \mid c] \to \text{Tri. sup.}$$

$$x = U^{-1}c$$

Teoria:

$$E_{n,n-1} \cdot \ldots \cdot E_{32} \cdot E_{n1} \cdot \ldots \cdot E_{21} A = U$$

$$E_{ij} = \begin{array}{c} \begin{bmatrix} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & -m_{ij} & & \ddots & \\ & & & & 1 \end{bmatrix} \begin{matrix} \\ \\ \\ i \\ \\ \end{matrix} \\ j \end{array}$$

$$= I - m_{ij} e_i e_j^T$$

$$E_{n,n-1} \cdot \ldots \cdot E_{32} \cdot E_{n1} \cdot \ldots \cdot E_{21} A = U$$

$$A = E_{21}^{-1} \cdot \ldots \cdot E_{n,n-1}^{-1} U$$

Teo.: $L \equiv E_{21}^{-1} \cdot \ldots \cdot E_{n,n-1}^{-1}$ é igual a

$$L = \begin{bmatrix} 1 & & & \\ m_{21} & 1 & & \\ m_{31} & m_{32} & 1 & \\ \vdots & \vdots & & \ddots \\ m_{n1} & m_{n2} & \cdots & 1 \end{bmatrix}$$

$$\therefore A = LU$$

1º: Exemplo de cálculo de LU:

$$A = \begin{bmatrix} 4 & 3 & 2 \\ 2 & 5 & -1 \\ 1 & -2 & 0 \end{bmatrix}$$

Já calcula m_{ij} e guarda em A

$$m_{21} = \frac{a_{21}}{a_{11}} = \frac{2}{4}$$

$$L_2 - \frac{1}{2} L_1 \qquad m_{31} = \frac{a_{31}}{a_{11}} = \frac{1}{4}$$

$$L_3 - \frac{1}{4} L_1$$

$$A^{(1)} = \begin{bmatrix} 4 & 3 & 2 \\ 1/2 & 7/2 & -2 \\ 1/4 & -11/4 & -1/2 \end{bmatrix}$$

$$m_{32} = \frac{a_{32}^{(1)}}{a_{22}^{(1)}} = \frac{-11/4}{7/2} = \frac{-11}{14}$$

$$L_3 - \left(\frac{-11}{14}\right) \cdot L_2$$

$$A^{(2)} = \begin{bmatrix} 4 & 3 & 2 \\ 1/2 & 7/2 & -2 \\ 1/4 & -11/14 & -\frac{29}{14} \end{bmatrix}$$

$$L = \begin{bmatrix} 1 & & \\ 1/2 & 1 & \\ 1/4 & -11/14 & 1 \end{bmatrix}, \quad U = \begin{bmatrix} 4 & 3 & 2 \\ & 7/2 & -2 \\ & & -\frac{29}{14} \end{bmatrix}$$

Tenho $A = LU$, como resolver $Ax = b$

$$Ax = b \Rightarrow L\underbrace{Ux}_{y} = b$$

$$\begin{cases} Ly = b & \leftarrow 1^{\circ} \to y \\ Ux = y & \leftarrow 2^{\circ} \to x \end{cases}$$

Ex.: Resolva o sistema abaixo usando fat. LU:

$$\begin{bmatrix} 4 & 3 & 2 \\ 2 & 5 & -1 \\ 1 & -2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 9 \\ 10 \\ 0 \end{bmatrix}$$

1º calcular LU

$$L = \begin{bmatrix} 1 & & \\ 1/2 & 1 & \\ 1/4 & -11/14 & 1 \end{bmatrix}, \quad U = \begin{bmatrix} 4 & 3 & 2 \\ & 7/2 & -2 \\ & & -\frac{29}{14} \end{bmatrix}$$

$$Ly = b: \begin{cases} y_1 = 9 \\ \frac{1}{2} y_1 + y_2 = 10 \\ \frac{1}{4} y_1 - \frac{11}{14} y_2 + y_3 = 0 \end{cases}$$

$$\to y_1 = 9$$

$$\to y_2 = 10 - \frac{1}{2} \cdot 9 = \frac{11}{2}$$

$$\to y_3 = 0 - \frac{1}{4} \cdot 9 + \frac{11}{14} \cdot \frac{11}{2} = \frac{29}{14}$$

$$Ux = y \to \begin{cases} 4x_1 + 3x_2 + 2x_3 = 9 \\ \frac{7}{2} x_2 - 2x_3 = \frac{11}{2} \\ \frac{-29}{14} x_3 = \frac{29}{14} \to x_3 = -1 \end{cases}$$

$$x_2 = \frac{11/2 + 2 \cdot \overset{x_3}{(-1)}}{7/2} = 1$$

$$x_1 = \frac{9 - 2 \cdot \overset{x_3}{(-1)} - 3 \cdot \overset{x_2}{(1)}}{4} = 2$$

$$\therefore x = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}.$$

Cobro até aqui

LU com pivoteamento (parcial / de linhas)

Temos A

Fazemos troca de linhas

$$P_{ij} = \begin{array}{c} \begin{bmatrix} 1 & & & & & \\ & 1 & & & & \\ & & 0 & \cdots & 1 & \\ & & \vdots & 1 & \vdots & \\ & & 1 & \cdots & 0 & \\ & & & & & 1 \end{bmatrix} \begin{matrix} \\ \\ i \\ \\ j \\ \\ \end{matrix} \\ i \quad j \end{array} \to I \text{ com linhas } i \text{ e } j \text{ trocadas}$$

Elim. Gauss. c/ pivot.:

$$E_{n-1} P_{n-1} \cdot \ldots \cdot E_2 P_2 E_1 P_1 A = U$$

P_1: Põe o pivo na linha 1

E_1: Elim. col. 1.

Teo.: Fazendo esse processo, você obtém

$$PA = LU,$$

onde L e U são as matrizes de antes, mas obedecendo a troca de linhas e P é a matriz identidade obedecendo a troca de linhas.

Ex.:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & -1 & 4 \\ 5 & 0 & 4 \end{bmatrix}$$

fatoração LU c/ pivot.

$p = (1, 2, 3)$

pivô: $a_{31} = 5$

$L_1 \leftrightarrow L_3$: $\quad p = (3, 2, 1)$

$$\tilde{A} = \begin{bmatrix} 5 & 0 & 4 \\ 4 & -1 & 4 \\ 1 & 2 & 3 \end{bmatrix}, \quad A^{(1)} = \begin{bmatrix} 5 & 0 & 4 \\ 0.8 & -1 & 0.8 \\ 0.2 & 2 & 2.2 \end{bmatrix}$$

pivô: $a_{32} = 2$

$L_2 \leftrightarrow L_3$: $\quad p = (3, 1, 2)$

$$\tilde{A}^{(1)} = \begin{bmatrix} 5 & 0 & 4 \\ 0.2 & 2 & 2.2 \\ 0.8 & -1 & 0.8 \end{bmatrix}$$

$$A^{(2)} = \begin{bmatrix} 5 & 0 & 4 \\ 0.2 & 2 & 2.2 \\ 0.8 & -0.5 & -0.3 \end{bmatrix}$$

$$L = \begin{bmatrix} 1 & & \\ 0.2 & 1 & \\ 0.8 & -0.5 & 1 \end{bmatrix}, \quad U = \begin{bmatrix} 5 & 0 & 4 \\ & 2 & -0.2 \\ & & -0.3 \end{bmatrix}, \quad P = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$