Cálculo Diferencial e Integral I - Prova 3

21 de Junho de 2017

Questão 1 .... $\boxed{50}$

Calcule as integrais a seguir.

(a) (10 points) $\displaystyle\int x^2\cos(x^3)\mathrm{d}x$

Solution: $u = x^3 \to \mathrm{d}u = 3x^2\mathrm{d}x$

$$\int x^2\cos(x^3)\mathrm{d}x = \frac{1}{3}\int cos(u)\mathrm{d}u = \frac{1}{3}sin(u) + C = \frac{1}{3}sin(x^3) + C$$

(b) (10 points) $\displaystyle\int_0^{\pi/3} x^2\cos(3x)\mathrm{d}x$

Solution: Usando integração por partes duas vezes, $\sin(0) = 0$, $\sin(\pi) = 0$, $\cos(0) = 1$ e $\cos(\pi) = -1$, temos

$$\begin{aligned}\int_0^{\pi/3} x^2\cos(3x)\mathrm{d}x &= \cancelto{0}{\left.\frac{x^2\sin(3x)}{3}\right|_0^{\pi/3}} - \frac{2}{3}\int_0^{\pi/3} x\sin(3x)\mathrm{d}x \\ &= -\frac{2}{3}\left[\left.\frac{-x\cos(3x)}{3}\right|_0^{\pi/3} + \frac{1}{3}\int_0^{\pi/3}\cos(3x)\mathrm{d}x\right] \\ &= -\frac{2}{3}\left(\frac{\pi}{9}\right) - \cancelto{0}{\frac{2}{9}\sin(3x)\Big|_0^{\pi/3}} \\ &= -\frac{2\pi}{27}.\end{aligned}$$

(c) (10 points) $\displaystyle\int \frac{x+11}{x^2-2x-15}\mathrm{d}x$

Solution: Temos $x^2 - 2x - 15 = (x-5)(x+3)$, logo

$$\frac{x+11}{(x-5)(x+3)} = \frac{A}{x-5} + \frac{B}{x+3}.$$

Temos

$$x + 11 = A(x+3) + B(x-5) = (A+B)x + (3A - 5B),$$

dando o sistema

$$\begin{cases} A + B &= 1, \\ 3A - 5B &= 11. \end{cases}$$

A solução é $A = 2$ e $B = -1$. Logo

$$\begin{aligned}\int \frac{x+11}{x^2-2x-15}\mathrm{d}x &= 2\int \frac{1}{x-5}\mathrm{d}x - \int \frac{1}{x+3}\mathrm{d}x \\ &= 2\ln|x-5| - \ln|x+3| + C.\end{aligned}$$

(d) (10 points) $\int \sin^4(x)\cos^5(x)\mathrm{d}x$

Solution: Faremos a mudança $u = \sin(x)$ e $\mathrm{d}u = \cos(x)\mathrm{d}x$.

$$\begin{aligned}\int \sin^4(x)\cos^5(x)\mathrm{d}x &= \int \sin^4(x)[1-\sin^2(x)]^2\cos x\mathrm{d}x \\ &= \int u^4(1-u^2)^2\mathrm{d}u \\ &= \int (u^8 - 2u^6 + u^4)\mathrm{d}u \\ &= \frac{u^9}{9} - \frac{2u^7}{7} + \frac{u^5}{5} + C \\ &= \frac{\sin^9(x)}{9} - \frac{2\sin^7(x)}{7} + \frac{\sin^5(x)}{5} + C.\end{aligned}$$

(e) (10 points) $\int \frac{1}{x^2\sqrt{9+x^2}}\mathrm{d}x$

Solution: Faremos a mudança $x = \tan t$ e depois $u = \sin t$.

$$\begin{aligned}\int \frac{1}{x^2\sqrt{9+x^2}}\mathrm{d}x &= \frac{1}{3}\int \frac{1}{x^2\sqrt{1+(\frac{x}{3})^2}}\mathrm{d}x = \frac{1}{3}\int \frac{3\sec^2 t}{9\tan^2 t\sec t}\mathrm{d}t \\ &= \frac{1}{9}\int \frac{\cos t}{\sin^2 t}\mathrm{d}t = \frac{1}{9}\int \frac{\mathrm{d}u}{u^2} = \frac{-1}{9u} + C. \\ &= \frac{-1}{9\sin t} + C = \frac{-\cos t}{\sin t\cos t} + C \\ &= \frac{-\sec t}{\tan t} + C = \frac{-\sqrt{1+\tan^2 t}}{\tan t} + C = \frac{-1\sqrt{1+x^2}}{x} + C.\end{aligned}$$

Questão 2 . . . . $\boxed{20}$

Faça o que se pede

(a) (5 points) Calcule $\int \ln x\mathrm{d}x$.

Solution:

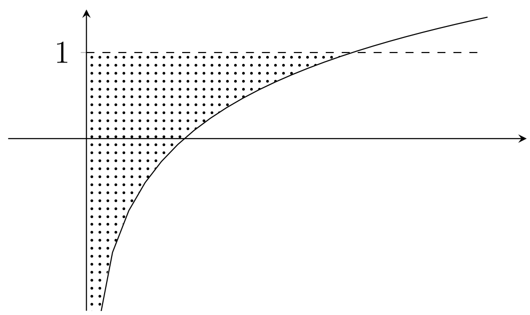
$$\int \ln x \mathrm{d}x = x \ln x - \int \mathrm{d}x = x \ln x - x + C.$$

(b) (5 points) Calcule $\lim_{x \to 0^+} x \ln x$.

Solution:

$$\lim_{x \to 0^+} x \ln x = \lim_{x \to 0^+} \frac{\ln x}{1/x} = \lim_{x \to 0^+} \frac{1/x}{-1/x^2} = \lim_{x \to 0^+} -x = 0.$$

(c) (10 points) Calcule a área da região formada pelos pontos (x, y) que satisfazem $\ln x \leq y \leq 1$. Essa área está representada na figura ao lado.



Solution:

$$\begin{aligned} A &= \int_0^e (1 - \ln x)\mathrm{d}x = \lim_{t \to 0^+} \int_t^e (1 - \ln x)\mathrm{d}x = \lim_{t \to 0^+} [2x - x \ln x]\Big|_t^e \\ &= 2e - e \ln e - \lim_{t \to 0^+} (2t - t \ln t) = e. \end{aligned}$$

Questão 3 .... $\boxed{10}$

Calcule a derivada da função

$$F(x) = \int_1^{x^2+1} e^{-t^2} \mathrm{d}t,$$

usando o Teorema Fundamental do Cálculo e a regra da cadeia.

Solution:

$$
\begin{aligned}
F'(x) &= \frac{\mathrm{d}}{\mathrm{d}x}\int_1^{x^2+1} e^{-t^2}\mathrm{d}t = \frac{\mathrm{d}u}{\mathrm{d}x}\frac{\mathrm{d}}{\mathrm{d}u}\int_1^u e^{-t^2}\mathrm{d}t \\
&= 2xe^{-u^2} = 2xe^{-(x^2+1)^2}
\end{aligned}
$$

Questão 4 .... $\boxed{30}$

Escolha **duas** das questões abaixo para resolver. Cada uma vale 15 pontos. Deixe indicado **claramente** quais duas foram escolhidas, ou esta questão não será pontuada.

(a) $\displaystyle\int \frac{1}{(5-4x-x^2)^{3/2}}\mathrm{d}x$

Solution: Temos $5-4x-x^2 = 9\left[1-\left(\frac{x+2}{3}\right)^2\right]$. Fazendo $z=(x+2)/3$, temos

$$
\begin{aligned}
\int \frac{1}{(5-4x-x^2)^{3/2}}\mathrm{d}x &= \frac{1}{9}\int \frac{1}{(1-z^2)^{3/2}}\mathrm{d}z = \frac{1}{9}\int \frac{\cos t}{\cos^3 t}\mathrm{d}t \\
&= \frac{1}{9}\int \sec^2 t\mathrm{d}t = \frac{1}{9}\tan t + C = \frac{1}{9}\frac{z}{\sqrt{1-z^2}} + C \\
&= \frac{x+2}{9\sqrt{5-4x-x^2}} + C.
\end{aligned}
$$

(b) $\displaystyle\int_{-\infty}^0 \frac{1}{1+e^{-x}}\mathrm{d}x$

Solution: Faremos a mudança $u = e^x + 1$.

$$
\int_{-\infty}^0 \frac{1}{1+e^{-x}}\mathrm{d}x = \int_{-\infty}^0 \frac{e^x}{e^x+1}\mathrm{d}x = \int_1^2 \frac{1}{u}\mathrm{d}u = \ln u\Big|_1^2 = \ln 2 - \cancelto{0}{\ln 1} = \ln 2.
$$

(c) $\displaystyle\int_{-1}^1 \frac{x^2+\sin(x^3)}{x^2+1}\mathrm{d}x$

Solution:

$$
\begin{aligned}
\int_{-1}^{1} \frac{x^2+\sin(x^3)}{x^2+1}\mathrm{d}x &= \int_{-1}^{1} \frac{x^2}{x^2+1}\mathrm{d}x + \cancelto{0}{\int_{-1}^{1} \frac{\sin(x^3)}{x^2+1}\mathrm{d}x} \\
&= 2\int_{0}^{1} \frac{x^2+1-1}{x^2+1}\mathrm{d}x = 2\int_{0}^{1} \left[1-\frac{1}{x^2+1}\right]\mathrm{d}x \\
&= 2\left[x-\arg\tan x\right]\Big|_{0}^{1} = 2-\frac{\pi}{2}.
\end{aligned}
$$