22/05 - Sist. Lin.

$$Ax = b, \quad A \in \mathbb{R}^{n\times n}, x, b \in \mathbb{R}^n$$

Existência e unicidade da solução

$$[A \vdots b] \xrightarrow{\text{escalona}^{(*)}} [U \vdots c] \longrightarrow [I \vdots A^{-1}b]$$

U: tri. sup. c/ diag. ñ-nula

(*) Escalonamento / Eliminação Gaussiana

- (i) $L_i \leftrightarrow L_j$
- (ii) $L_i \leftarrow \alpha L_i$, $\quad \alpha \neq 0$
- (iii) $L_i \leftarrow L_i - \alpha L_j$

Operação elementar do tipo $A \leftarrow EA$

E: matriz elementar

$$\text{(i)}\ P_{ij} = \begin{array}{c|ccccccc|c} & & & i & & j & & & \\ \hline & 1 & & & & & & & \\ & & \ddots & & & & & & \\ & & & 0 & & 1 & & & - i \\ & & & & \ddots & & & & \\ & & & 1 & & 0 & & & - j \\ & & & & & & \ddots & & \\ & & & & & & & 1 & \end{array}$$

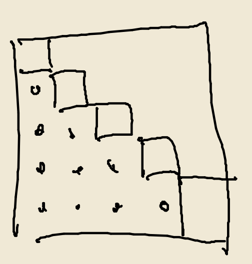
$$\text{(ii)}\ E_i^{\alpha} = \begin{bmatrix} 1 & & & & \\ & \ddots & & & \\ & & \alpha & & \\ & & & \ddots & \\ & & & & 1 \end{bmatrix} - i$$

$$\text{(iii)}\ E_{ij}^{\alpha} = \begin{bmatrix} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & -\alpha & & 1 & \\ & & & & 1 \end{bmatrix} \begin{matrix} \\ - j \\ \\ - i \\ \\ \end{matrix} \qquad \begin{bmatrix} 1 & & & \\ & 1 & -\alpha & \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} L_1 \\ L_2 \\ L_3 \\ L_4 \end{bmatrix}$$

$j \quad i \qquad \boxed{i > j}$

$L_2 \leftarrow L_2 - \alpha L_3$

Elim. Gauss. sem troca de linhas



Na col. j, linha i, quero zerar a_{ij}

$a_{ij} \leftarrow a_{ij} - m_{ij} a_{jj} = 0$

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for $j = 1:n-1$
    for $i = j+1:n$
        $m_{ij} = a_{ij} / a_{jj}$
        $L_i \leftarrow L_i - m_{ij} L_j < \begin{matrix} A \\ b \end{matrix}$
    end
end
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a_{jj}: ñ pode ser 0

$E_{ij}[A \vdots b]$

Matricialmente: (3×3)

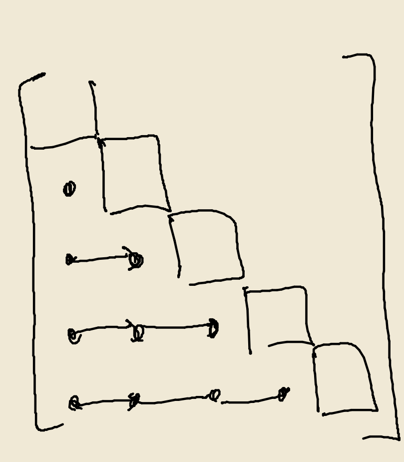
$$E_{32}E_{31}\underbrace{E_{21}A}_{a_{21}=0}x = E_{32}E_{31}E_{21}b$$

$$E_{32}E_{31}E_{21}A = U \quad \text{tri. sup.}$$

$$A = E_{21}^{-1}E_{31}^{-1}E_{32}^{-1}U \quad \leftarrow (**)$$

Veja que

$$E_{ij} = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & \vdots & \ddots & \\ & -m_{ij} & \cdots & 1 \end{bmatrix} = I - m_{ij}e_i e_j^T$$



$$E_{43}E_{42}E_{32}E_{41}E_{31}E_{21}$$

$$E_{43}E_{42}E_{41}E_{32}E_{31}E_{21}$$

$$(L_2 - m_{21}L_1) + m_{21}L_1 \rightarrow L_2$$

$$E_{n1}E_{n-1,1}\cdots E_{21} = \begin{bmatrix} 1 & & & \\ -m_{21} & 1 & & \\ \vdots & & \ddots & \\ -m_{n1} & & & 1 \end{bmatrix} = I - m_{21}e_2e_1^T - m_{31}e_3e_1^T - \cdots - m_{n1}e_ne_1^T$$

$$= I - \underbrace{(m_{21}e_2 + m_{31}e_3 + \ldots + m_{n1}e_n)}e_1^T$$

$$= I - v_1 e_1^T$$

$$v_1 = (0, m_{21}, \ldots, m_{n1})^T$$

$$E_{nj}E_{n-1,j}\cdots E_{j+1,j} = I - (m_{j+1,j}e_{j+1} + \ldots + m_{nj}e_n)e_j^T$$

$$= I - v_j e_j^T$$

$$v_j = (\underbrace{0, \ldots, 0}_{j}, m_{j+1,j}, \ldots, m_{n,j})^T$$

$$E_j = I - v_j e_j^T$$

$$U = E_{n-1}E_{n-2}\cdots E_3E_2E_1A$$

$$A = \underbrace{E_1^{-1}E_2^{-1}\cdots E_{n-2}^{-1}E_{n-1}^{-1}}_{L \text{ tri. inf. c/ diag. unitária}}U$$

Exerc.: $E_j = I - v_je_j^T$. Calcule E_j^{-1}.

Verificando apenas:

$$(I + v_je_j^T)(I - v_je_j^T) = I + \cancel{v_je_j^T} - \cancel{v_je_j^T} - v_j\underbrace{e_j^Tv_j}e_j^T$$

$$\begin{array}{l} e_j = (0, \ldots, 0, \overset{j}{1}, 0, \ldots, 0) \\ v_j = (0, \ldots, 0, 0, m_{j+1,j}, \ldots, m_{n,j}) \end{array} \Big\} \Rightarrow e_j^Tv_j = 0$$

Exerc.: Calcule $E_1^{-1}E_2^{-1}\cdots E_{n-1}^{-1}$.

$$E_j^{-1} = I + v_je_j^T$$

$$E_1^{-1}E_2^{-1} = (I + v_1e_1^T)(I + v_2e_2^T)$$

$$= I + v_1e_1^T + v_2e_2^T + v_1\cancel{e_1^Tv_2}e_2^T \quad \rightarrow 0$$

$$= I + v_1e_1^T + v_2e_2^T$$

Hipótese de indução: $E_1^{-1}E_2^{-1}\cdots E_k^{-1} = I + \sum_{j=1}^{k} v_je_j^T$

$$E_1^{-1}\cdots E_{k+1}^{-1} = \left(I + \sum_{j=1}^{k} v_je_j^T\right)\left(I + v_{k+1}e_{k+1}^T\right)$$

$$= I + \sum_{j=1}^{k+1} v_je_j^T + \sum_{j=1}^{k} v_j\cancel{e_j^Tv_{k+1}}e_{k+1}^T \quad \rightarrow 0$$

$$= I + \sum_{j=1}^{k+1} v_je_j^T$$

$$\therefore E_1^{-1}\cdots E_{n-1}^{-1} = \begin{bmatrix} 1 & & & & \\ m_{21} & 1 & & & \\ \vdots & m_{32} & 1 & & \\ \vdots & \vdots & & \ddots & \\ m_{n1} & m_{n2} & \cdots & m_{n,n-1} & 1 \end{bmatrix} \equiv L.$$

$$v_1 \quad v_2$$