Sistemas Lineares (Watkins e Golub)

- Vetores - $\|\cdot\|$, $\langle \cdot,\cdot \rangle$, $\|x+y\| \leq \|x\|+\|y\|$, $|\langle x,y\rangle| \leq \|x\|\cdot\|y\|$
- Matrizes - Ax, AB, $\|\cdot\|$, det, $Ax=\lambda x$, tr, singularidade, cond, A^{-1}, A^T,
- Matrizes especiais - diagonal, triang., esparsa
- Elim. Gauss Matricialmente! Teo. diag dom.
- LU ; - Permutação ; Teo. de $\exists!$
- Análise de Erro
- Matrizes definidas positivas
- Cholesky e LDL^T

06/05

Vetor, $x \in \mathbb{R}^n$, $x = (x_1, \dots, x_n)^T$

Norma: $\|x\|_1 = \sum_{i=1}^{n} |x_i|$ Manhattan

$$\|x\|_2 = \left[\sum_{i=1}^{n} x_i^2\right]^{1/2} \quad \text{Euclidiana}$$

$$\|x\|_\infty = \max_{i=1,\dots,n} |x_i|$$

$$\|x\|_p = \left[\sum_{i=1}^{n} |x_i|^p\right]^{1/p}, \quad p \geq 1.$$

Ex.: $x = (1,2)^T$, $\|x\|_1 = |1|+|2| = 3$

$\|x\|_2 = \sqrt{1+4} = \sqrt{5}$

$\|x\|_\infty = \max\{|1|, |2|\} = 2$.

Teo.: Se $\|\cdot\|_\alpha$ e $\|\cdot\|_\beta$ são normas do $\mathbb{R}^n$, então $\exists\ c_1, c_2 > 0$ t.q.

$$c_1 \|x\|_\alpha \leq \|x\|_\beta \leq c_2 \|x\|_\alpha,$$

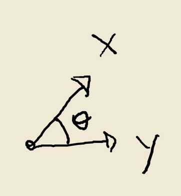
$\forall x \in \mathbb{R}^n$.

Ex.: (Vide Golub) provar valores de c_1 e c_2 nas normas acima.

Ex.: $\lim_{p \to \infty} \|x\|_p$.

Prod. Interno: $\langle x,y \rangle = \sum_{i=1}^{n} x_i y_i = x^T y$

$$\langle x,x \rangle = \|x\|_2^2$$

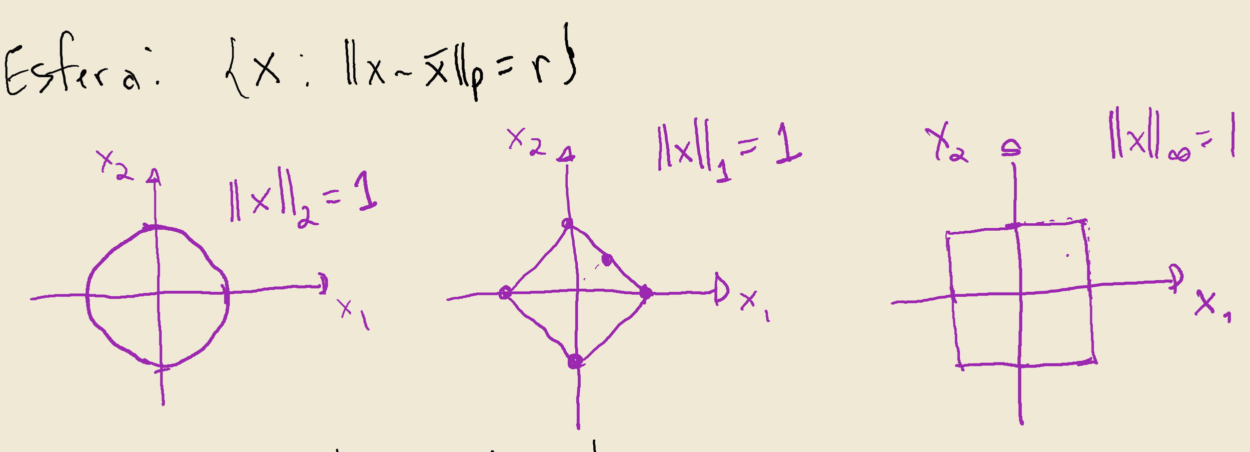


$$\cos\theta = \frac{x^T y}{\|x\| \cdot \|y\|}$$

Desig. triang.: $\|x+y\| \leq \|x\| + \|y\|$

Bolas: $B_p(\bar{x}, r) = \{x : \|x-\bar{x}\|_p < r\}$ Bola aberta

Esfera: $\{x : \|x-\bar{x}\|_p = r\}$



Des. de Cauchy-Schwartz

$$|x^T y| \leq \|x\|_2 \|y\|_2$$

Matrizes $A \in \mathbb{R}^{m \times n}$

$$y = Ax, \quad y_i = (Ax)_i = \sum_{j=1}^{n} a_{ij} x_j, \quad i=1,\dots,m$$

$$C = AB, \quad B \in \mathbb{R}^{n \times p}, \quad C \in \mathbb{R}^{m \times p}$$

$$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}, \quad i=1,\dots,m; \ j=1,\dots,p$$

Norma induzida: $$\|A\|_p = \max_{\|x\|_p=1} \overbrace{\|Ax\|_p}^{\varphi(t)} = \sup_{x \neq 0} \frac{\|Ax\|_p}{\|x\|_p}$$

$$\|A\|_p \geq 0, \quad \|A\|_p = 0 \Leftrightarrow A = 0$$

$$\|A+B\|_p \leq \|A\|_p + \|B\|_p \ ; \quad \|AB\|_p \leq \|A\|_p \cdot \|B\|_p$$

Determinante

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix}$$

$A_{ij} \leftarrow A$ sem a linha i e coluna j.

$$\det(A) = |A|$$
$$= \sum_{j=1}^{n} (-1)^{i+j} |A_{ij}|\, a_{ij} = \sum_{i=1}^{n} (-1)^{i+j} a_{ij} |A_{ij}|$$

Inversa: Se $AB = BA = I$, então B é a inversa de A, denotada A^{-1}.

$Im(A) = \{Ax : x \in \mathbb{R}^n\} = \{y \in \mathbb{R}^m : \exists\, x \in \mathbb{R}^n : y = Ax\}$

$N(A) = \{x \in \mathbb{R}^n : Ax = 0\}$

$\text{posto}(A) = \dim(Im(A))$

Teo.: $Im(A) \oplus N(A^T) = \mathbb{R}^m$

$Im(A^T) \oplus N(A) = \mathbb{R}^n$

$Im(A) \perp N(A^T)$, $Im(A^T) \perp N(A)$.

Teo.: São equivalentes as seguintes afirmações

i) $\det(A) \neq 0$
ii) $\exists\ A^{-1}$
iii) $N(A) = \{0\}$
iv) $Im(A) = \mathbb{R}^n$
v) A matriz escalonada tem diagonal não-nula

(i) $\Rightarrow$ (v) $\Rightarrow$ (iii) $\Rightarrow$ (iv) $\Rightarrow$ (ii) $\Rightarrow$ (i)

Def.: A é singular se ñ existe A^{-1}.

Autovalores: Se $Ax = \lambda x$, p/ $x \neq 0$ e $\lambda \in \mathbb{R}$, x é dito autovetor e λ autovalor.

Teo.: λ é autovalor $\Leftrightarrow$ $\det(A - \lambda I) = 0$

λ a.v. $\Leftrightarrow$ $Ax = \lambda x$ $\Leftrightarrow$ $(A - \lambda I)x = 0$ $\Leftrightarrow$ $\det(A - \lambda I) = 0$

Def.: Traço: $tr(A) = a_{11} + \dots + a_{nn}$

Teo.: $\det(A) = \lambda_1 \cdots \lambda_n$ e $tr(A) = \lambda_1 + \dots + \lambda_n$, onde $\lambda_1, \dots, \lambda_n$ são os autovalores de A. $p(\lambda) = \det(A - \lambda I)$

Teo.: Se $A \in \mathbb{R}^{n \times n}$, existem n autovalores no $\mathbb{C}$, não necessariamente distintos.

SVD: Decomposição em Valores Singulares

Teo.: Se $A \in \mathbb{R}^{m \times n}$, $r = \text{posto}(A)$, então existem bases ortonormais $\{v_1, \dots, v_n\}$ para $\mathbb{R}^n$ e $\{u_1, \dots, u_m\}$ para $\mathbb{R}^m$ e $\sigma_1 \geq \sigma_2 \geq \dots \geq \sigma_r > 0$ valores tais que

$$A = \sum_{i=1}^{r} \sigma_i u_i v_i^T.$$

Além disso, se $V = [v_1 \vdots \cdots \vdots v_n]$, $U = [u_1 \vdots \cdots \vdots u_m]$ e

$$\Sigma = \left[\begin{array}{ccc|c} \sigma_1 & & & 0 \\ & \ddots & & \\ & & \sigma_r & \\ \hline & 0 & & 0 \end{array}\right] \in \mathbb{R}^{m \times n}, \text{ então}$$

$$A = U \Sigma V^T.$$

Nota que isso implica que

$$\begin{cases} A v_i = \sigma_i u_i & i = 1, \dots, r \\ A^T u_i = \sigma_i v_i & i = 1, \dots, r \\ A v_i = 0 & i = r+1, \dots, n \\ A^T u_i = 0 & i = r+1, \dots, m \end{cases}$$

$$A = U \Sigma V^T = [\bar{U}\ \tilde{U}] \begin{bmatrix} \Sigma & \\ & 0 \end{bmatrix} \begin{bmatrix} \bar{V}^T \\ \tilde{V}^T \end{bmatrix}$$

$$= \bar{U} \Sigma \bar{V}^T$$

Ex.: Identifique os subespaços $Im(A)$, $Im(A^T)$, $N(A)$, $N(A^T)$.