CM042 - Cálculo II

24 de Outubro de 2019 - Prova 2

Gabarito

1. $\boxed{10}$ Calcule $\displaystyle\int_1^2 \int_{-1}^1 ye^{xy}\, \mathrm{d}x\mathrm{d}y$

Solution:

$$\int_1^2 \int_{-1}^1 ye^{xy}\, \mathrm{d}x\mathrm{d}y = \int_1^2 e^{xy}\Big|_{x=-1}^1 \mathrm{d}y = \int_1^2 (e^y - e^{-y})\mathrm{d}y$$
$$= (e^y + e^{-y})\Big|_1^2 = (e^2 + e^{-2}) - (e + e^{-1}) = e^2 + e^{-2} - e - e^{-1}$$

2. $\boxed{15}$ Calcule $\displaystyle\iint_D \frac{x^2}{y^3}\cos(\pi y)\mathrm{d}A$, onde D é a região trapezoidal com vértices $(0,2)$, $(0,1)$, $(2,2)$ e $(1,1)$.

Solution: A região é do tipo 2: $1 \le y \le 2$ e $0 \le x \le y$.

$$\begin{aligned}\iint_D \frac{x^2}{y^3}\cos(\pi y)\mathrm{d}A &= \int_1^2 \int_0^y \frac{x^2}{y^3}\cos(\pi y)\mathrm{d}x\mathrm{d}y \\ &= \int_1^2 \frac{x^3}{3}\Big|_0^y \frac{1}{y^3}\cos(\pi y)\mathrm{d}y \\ &= \int_1^2 \frac{y^3}{3y^3}\cos(\pi y)\mathrm{d}y \\ &= \int_1^2 \frac{1}{3}\cos(\pi y)\mathrm{d}y \\ &= \frac{1}{3}\frac{\sin(\pi y)}{\pi}\Big|_1^2 = 0.\end{aligned}$$

3. $\boxed{15}$ Calcule $\displaystyle\iint_D x\sqrt{x^2+y^2}\mathrm{d}A$, onde D é a região no primeiro quadrante entre as circunferências de raio 1 e 2, o eixo x, e a reta que passa na origem com inclinação de 30°.

Solution: Região em coordenadas polares: $1 \le r \le 2$ e $0 \le \theta \le \pi/6$.

$$\iint_D x\sqrt{x^2+y^2}\mathrm{d}A = \int_0^{\pi/6}\int_1^2 r\cos(\theta) r \cdot r\mathrm{d}r\mathrm{d}\theta = \int_0^{\pi/6}\cos(\theta)\mathrm{d}\theta \int_1^2 r^3\mathrm{d}r$$
$$= \sin(\theta)\Big|_0^{\pi/6}\frac{15}{4} = \frac{15}{8}.$$

4. $\boxed{15}$ Calcule a área da região limitada pelas curvas $2x = y^2 - 3y$ e $y = 2x$ usando integral dupla. Dica: As curvas se interceptam nos pontos $(0,0)$ e $(2,4)$.

Solution: Desenhando vemos que a região se encaixa melhor como tipo 2: $0 \le y \le 4$ e $\frac{1}{2}(y^2 - 3y) \le x \le \frac{1}{2}y$.

$$A = \iint_D \mathrm{d}A = \int_0^4 \int_{\frac{1}{2}(y^2-3y)}^{y/2} \mathrm{d}x\mathrm{d}y = \int_0^4 \Big(\tfrac{1}{2}y - \tfrac{1}{2}(y^2 - 3y)\Big)\mathrm{d}y = \frac{1}{2}\int_0^4 (4y - y^2)\mathrm{d}y$$
$$= \frac{1}{2}(2y^2 - \tfrac{1}{3}y^3)\Big|_0^4 = \frac{1}{2}\left(32 - \frac{64}{3}\right) = \frac{16}{3}.$$

5. $\boxed{15}$ Calcule $\displaystyle\iint_D \frac{x^2y}{1 + (x^2+y^2)^{5/2}}\mathrm{d}A$, onde D é a região tal que $x^2 + y^2 \le 1$ no primeiro quadrante.

Solution: Coordenadas polares: $0 \le r \le 1$ e $0 \le \theta \le \pi/2$.

$$\begin{aligned}\iint_D \frac{x^2y}{1 + (x^2+y^2)^{5/2}}\mathrm{d}A &= \int_0^{\pi/2}\int_0^1 \frac{r^2\cos^2(\theta) r\sin(\theta)}{1+r^5} r\mathrm{d}r\mathrm{d}\theta \\ &= \int_0^{\pi/2} \cos^2(\theta)\sin(\theta)\mathrm{d}\theta \int_0^1 \frac{r^4}{1+r^5}\mathrm{d}r \\ &= \int_1^0 u^2(-\mathrm{d}u)\int_1^2 \frac{1}{5v}\mathrm{d}v \\ &= \frac{1}{3}\cdot\frac{\ln 2}{5} = \frac{\ln 2}{15}.\end{aligned}$$

6. $\boxed{15}$ Calcule $\displaystyle\iiint_E x^2y\ \mathrm{d}V$, onde E é a região do primeiro octante limitada por $3y + z = 6$, $x = y$ e $x = 1$.

Solution: Região: $0 \le x \le 1$, $0 \le y \le x$ e $0 \le z \le 6 - 3y$.

$$\begin{aligned}\iiint_E x^2y\ \mathrm{d}V &= \int_0^1\int_0^x\int_0^{6-3y} x^2y\mathrm{d}z\mathrm{d}y\mathrm{d}x = \int_0^1\int_0^x x^2y(6-3y)\mathrm{d}y\mathrm{d}x \\ &= \int_0^1\int_0^x x^2(6y - 3y^2)\mathrm{d}y\mathrm{d}x = \int_0^1 x^2(3x^2 - x^3)\mathrm{d}x \\ &= \int_0^1 (3x^4 - x^5)\mathrm{d}x = \frac{3}{5} - \frac{1}{6} = \frac{13}{30}.\end{aligned}$$

7. $\boxed{15}$ Calcule $\displaystyle\iiint_E 224z^{671}\ \mathrm{d}V$, onde E é a região limitada $z = 1 - x^2 - y^2$ acima do plano xy.

Solution: Região cilíndrica: $0 \le r \le 1$, $0 \le \theta \le 2\pi$, $0 \le z \le 1 - r^2$.

$$\begin{aligned}\iiint_E 224z^{671}\ \mathrm{d}V &= \int_0^{2\pi}\int_0^1\int_0^{1-r^2} 224z^{671} r\mathrm{d}z\mathrm{d}r\mathrm{d}\theta = 448\pi\int_0^1 \frac{r(1-r^2)^{672}}{672}\mathrm{d}r\\ &= \frac{2\pi}{3}\int_1^0 u^{672}\left(\frac{-\mathrm{d}u}{2}\right) = \frac{-\pi}{3}\left(\frac{-1}{673}\right) = \frac{\pi}{2019}.\end{aligned}$$

8. $\boxed{15}$ Calcule $\iiint_E z(x^2+y^2)(x^2+y^2+z^2)\ \mathrm{d}V$, onde E é a região limitada $z^2 = x^2+y^2$ para $z \ge 0$, e pela esfera de raio 1.

Solution: Região esférica: $0 \le \rho \le 1$, $0 \le \theta \le 2\pi$ e $0 \le \varphi \le \pi/4$.

$$\begin{aligned}\iiint_E z(x^2+y^2)(x^2+y^2+z^2)\ \mathrm{d}V &= \int_0^{\pi/4}\int_0^{2\pi}\int_0^1 \rho\cos(\varphi)\cdot\rho^2\sin^2(\varphi)\cdot\rho^2\cdot\rho^2\sin(\varphi)\mathrm{d}\rho\mathrm{d}\theta\mathrm{d}\varphi\\ &= 2\pi\int_0^{\pi/4}\sin^3(\varphi)\cos(\varphi)\mathrm{d}\varphi\int_0^1\rho^7\mathrm{d}\rho\\ &= 2\pi\int_0^{\sqrt{2}/2} u^3\mathrm{d}u\mathrm{d}\varphi\int_0^1\rho^7\mathrm{d}\rho\\ &= \frac{2\pi}{8}\cdot\frac{1}{4}\left(\frac{\sqrt{2}}{2}\right)^4 = \frac{\pi}{64}.\end{aligned}$$

Por região cilíndrica: Cone é $z^2 = x^2+y^2 = r^2$ vira que $z = r$ para $z \ge 0$. A esfera é $x^2+y^2+z^2 = 1$ que vira $z = \sqrt{1-r^2}$. A intersecção dos dois é $r = \sqrt{1-r^2}$, que leva a $r = \dfrac{\sqrt{2}}{2}$. Então a região é $0 \le \theta \le 2\pi$, $0 \le r \le \sqrt{2}/2$ e $r \le z \le \sqrt{1-r^2}$.

$$\begin{aligned}\iiint_E z(x^2+y^2)(x^2+y^2+z^2)\ \mathrm{d}V &= \int_0^{2\pi}\int_0^{\sqrt{2}/2}\int_r^{\sqrt{1-r^2}} zr^2(r^2+z^2)r\mathrm{d}z\mathrm{d}r\mathrm{d}\theta\\ &= 2\pi\int_0^{\sqrt{2}/2}\int_r^{\sqrt{1-r^2}}(zr^5+z^3r^3)\mathrm{d}z\mathrm{d}r\\ &= 2\pi\int_0^{\sqrt{2}/2}\left(\tfrac{1}{2}z^2r^5+\tfrac{1}{4}z^4r^3\right)\Big|_r^{\sqrt{1-r^2}}\mathrm{d}z\mathrm{d}r\\ &= 2\pi\int_0^{\sqrt{2}/2}\left(\tfrac{1}{2}(1-2r^2)r^5+\tfrac{1}{4}(1-2r^2)r^3\right)\mathrm{d}z\mathrm{d}r\\ &= 2\pi\int_0^{\sqrt{2}/2}\left(\tfrac{1}{2}r^5-r^7+\tfrac{1}{4}r^3-\tfrac{1}{2}r^5\right)\mathrm{d}r\\ &= 2\pi\int_0^{\sqrt{2}/2}\left(\tfrac{1}{4}r^3-r^7\right)\mathrm{d}r\\ &= 2\pi\left(\frac{1}{16}\times\frac{1}{4}-\frac{1}{8}\times\frac{1}{16}\right)\\ &= \frac{\pi}{64}.\end{aligned}$$