Iterative Performance of Monte Carlo Linear Solver Methods

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Research supported by DOE (Office of Science)

SIAM Conference on Computational Science and Engineering 2015

March 17, 2015

Outline

- Motivations
- Standard Monte Carlo vs. Monte Carlo Synthetic Acceleration (MCSA)
- Choice of the preconditioner
- Convergence conditions
- Numerical results

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Motivations

Goal:

- development of reliable algorithms to solve highly dimensional (towards exascale) sparse linear systems in parallel

Issues:

- occurrence of **faults** (abnormal operating conditions of the computing system which cause a wrong answer)
  - hard faults
  - soft faults

[M. Hoemmen and M.A. Heroux, 2011; G. Bronevetsky and B.R. de Supinski, 2008; P. Prata and J.B. Silva, 1999]

Motivations

Resilience: ability to compute a correct output in presence of faults

Different approaches to tackle the problem:

- adaptation of Krylov subspace methods (CG, GMRES, Bi-CGStab, ...) to fault tolerance by recover-restart strategies
- abandonment of deterministic paradigm in favor of stochastic approaches

[E. Agullo, L. Giraud, A. Guernouche, J. Roman and M. Zounon, 2013]

Monte Carlo Linear Solvers

Mathematical setting

Let us consider a sparse linear system

$$A\mathbf{x} = \mathbf{b}, \tag{1}$$

where $A \in \mathbb{R}^{n \times n}$, $\mathbf{x} \in \mathbb{R}^n$, $\mathbf{b} \in \mathbb{R}^n$.
With left preconditioning (1) becomes

$$P^{-1}A\mathbf{x} = P^{-1}\mathbf{b}, \quad P \in \mathbb{R}^{n \times n}. \tag{2}$$

(2) can be reinterpreted as a fixed point scheme

$$\mathbf{x} = H\mathbf{x} + \mathbf{f}, \quad H = I - P^{-1}A, \ \mathbf{f} = P^{-1}\mathbf{b}. \tag{3}$$

Assuming $\rho(H) < 1$, (3) generates a sequence of approximate solutions $\{\mathbf{x}_k\}_{k=0}^{\infty}$ which converges to the exact solution of (1).

Monte Carlo Linear Solvers

Mathematical setting

The solution to (3) can be written in terms of a power series in H (Neumann series):

$$\mathbf{x} = \sum_{\ell=0}^{\infty} H^{\ell}\mathbf{f}.$$

By restricting the attention to a single component of $\mathbf{x}$ we have

$$x_i = \sum_{\ell=0}^{\infty} \sum_{k_1=1}^{n} \sum_{k_2=1}^{n} \cdots \sum_{k_\ell=1}^{n} H_{k_0,k_1} H_{k_1,k_2} \ldots H_{k_{\ell-1},k_\ell} f_{k_\ell}. \qquad (4)$$

(4) can be reinterpreted as the sampling of an estimator defined on a random walk.

Monte Carlo Linear Solvers

Mathematical setting: Forward (Direct) Method

Goal: evaluate a functional such as

$$J(\mathbf{x}) = \langle \mathbf{h}, \mathbf{x} \rangle = \sum_{i=1}^{n} h_i x_i, \quad \mathbf{h} \in \mathbb{R}^n.$$

Approach: define random walks to evaluate J.
Consider a random walk whose state space S is the set of indices of the forcing term $\mathbf{f}$:

$$S = \{1, 2, \ldots, n\} \subset \mathbb{N}.$$

[N. Metropolis and S. Ulam, 1949; G. E. Forsythe and A. Leibler, 1950; W. R. Wasow, 1952; J. H. Halton, 1962]

Monte Carlo Linear Solvers

Mathematical setting: Forward (Direct) Method

Initial probability: pick k_0 as initial step of the random walk:

$$\tilde{p}(k_0 = i) = \tilde{p}_{k_0} = \frac{|h_i|}{\sum_{i=1}^{n} |h_i|}.$$

Possible choices for the transition probability:
$p(:, i) : S \to [0, 1] \quad \forall i \in S$

$$p(k_i = j | k_{i-1} = i) = P_{i,j} = \frac{|H_{i,j}|}{\sum_{i=1}^{n} |H_{i,j}|} \qquad \textit{weighted}$$

$$P_{i,j} = \begin{cases} 0 & \text{if} \quad H_{i,j} = 0 \\ \frac{1}{\#(\text{non-zeros in the row})} & \text{if} \quad H_{i,j} \neq 0 \end{cases} \qquad \textit{uniform}$$

Monte Carlo Linear Solvers

Mathematical setting: Forward (Direct) Method

A related sequence of weights is defined by

$$w_{i,j} = \frac{H_{i,j}}{P_{i,j}}$$

in order to build an auxiliary sequence

$$W_0 = \frac{h_{k_0}}{\tilde{p}_{k_0}}, \quad W_i = W_{i-1} w_{k_{i-1},k_i} \quad k_{i-1}, k_i \in S.$$

This allows us to define a random variable $X(\cdot) : \Pi \to \mathbb{R}$.
Π= set of realizations of a random walk γ defined on S:

$$X(\nu) = \sum_{\ell=0}^{\infty} W_\ell f_{k_\ell}, \quad \nu = \text{permutation of } \gamma$$

Monte Carlo Linear Solvers

Mathematical setting: Forward (Direct) Method

Define the expected value of $X(\cdot)$:

$$E[X] = \sum_{\nu} P_\nu X(\nu) \quad (P_\nu = \text{probability of the permutation } \nu).$$

It can be proved that

$$E[W_\ell f_{k_\ell}] = \left\langle \mathbf{h}, H^\ell \mathbf{f} \right\rangle$$

and consequently

$$E\left[\sum_{\ell=0}^{\infty} W_\ell f_{k_\ell} \right] = \langle \mathbf{h}, \mathbf{x} \rangle .$$

Monte Carlo Linear Solvers

Mathematical setting: Forward (Direct) Method

If $\mathbf{h}$ is a vector of the standard basis: $\mathbf{h} = \mathbf{e}_j$

$$\tilde{p}(K_0 = i) = \delta_{i,j}, \quad i, j = 1, \ldots, n$$

and the expected value assumes the following form:

$$E\left[\sum_{\ell=0}^{\infty} W_\ell f_{k_\ell}\right] = x_i = \sum_{\ell=0}^{\infty} \sum_{k_1=1}^{n} \sum_{k_2=1}^{n} \cdots \sum_{k_\ell=1}^{n} P_{k_0,k_1} P_{k_1,k_2} \cdots P_{k_{\ell-1},k_\ell} w_{k_0,k_1} w_{k_1,k_2} \cdots w_{k_{\ell-1},k_\ell} f_{k_\ell}. \tag{5}$$

Drawback: the estimator is defined entry-wise.

Monte Carlo Linear Solvers

Mathematical setting: Adjoint Method

Consider the adjoint linear system

$$(P^{-1}A)^T\mathbf{y} = \mathbf{d}.$$

With a fixed point approach, we get

$$\mathbf{y} = H^T\mathbf{y} + \mathbf{d}.$$

By picking $\mathbf{d} = \mathbf{e}_j$ we have

$$J^*(\mathbf{y}) := \langle \mathbf{y}, \mathbf{f} \rangle = \langle \mathbf{x}, \mathbf{e}_j \rangle = x_j, \quad j = 1, \ldots, n.$$

Monte Carlo Linear Solvers

Mathematical setting: Adjoint Method

The initial probability and transition matrix may be defined as

$$\tilde{p}_{k0} = p(k_0 = i) = \frac{|f_i|}{\sum_{i=1}^{n} |f_i|}, \quad P_{i,j} = \frac{|H_{i,j}^T|}{\sum_{k=1}^{n} |H_{i,k}^T|} = \frac{|H_{j,i}|}{\sum_{k=1}^{n} |H_{k,i}|}.$$

Reintroduce weights as before:

$$w_{i,j} = \frac{H_{j,i}}{P_{i,j}} \Rightarrow W_0 = sign(f_i)\|\mathbf{f}\|_1, \quad W_j = W_{j-1} w_{k_{i-1},k_i}, \quad k_{i-1}, k_i \in S.$$

Expected value of the estimator for the entire solution vector $\mathbf{x}$:

$$E\left[\sum_{\ell=0}^{\infty} W_\ell d_{k_\ell}\right] = \sum_{\ell=0}^{\infty} \sum_{k_1}^{n} \sum_{k_2}^{n} \cdots \sum_{k_\ell}^{n} f_{k_0} P_{k_0,k_1} P_{k_1,k_2} \cdots P_{k_{\ell-1},k_\ell} w_{k_0,k_1} \cdots w_{k_{\ell-1},k_\ell} \delta_{k_\ell,j}. \quad (6)$$

Monte Carlo Linear Solvers

Statistical constraints for convergence

Expected value and variance of the estimator must be finite to apply the Central Limit Theorem.

Forward Method: $(H^*)_{i,j} = \frac{(H_{i,j})^2}{P_{i,j}}$

$$E\left[\sum_{\ell=0}^{\infty} W_\ell f_{k_\ell}\right] < \infty \Leftrightarrow \rho(H) < 1$$

$$Var\left[\sum_{\ell=0}^{\infty} W_\ell f_{k_\ell}\right] < \infty \Leftrightarrow \rho(H^*) < 1$$

Adjoint Method: $(H^*)_{i,j} = \frac{(H_{j,i})^2}{P_{i,j}}$

$$E\left[\sum_{\ell=0}^{\infty} W_\ell d_{k_\ell}\right] < \infty \Leftrightarrow \rho(H) < 1$$

$$Var\left[\sum_{\ell=0}^{\infty} W_\ell d_{k_\ell}\right] < \infty \Leftrightarrow \rho(H^*) < 1$$

[J. H. Halton, 1994; H. Ji, M. Mascagni and Y. Li, 2013]

Monte Carlo Linear Solvers

The choice of the transition probability

Theorem (H. Ji, M. Mascagni and Y. Li)

Let $H \in \mathbb{R}^{n \times n}$, where $\|H\|_\infty < 1$ for the Forward method and $\|H\|_1 < 1$ for the Adjoint method. Consider ν_k as the realization of a random walk γ truncated at the k-th step. Then, there always exists a transition matrix P such that $Var\left(X(\nu_k)\right) \to 0$ and $Var\left(\sum_\nu X(\nu_k)\right)$ is bounded as $k \to \infty$.

Remark If the hypotheses hold, then $P_{i,j} = \frac{|H_{i,j}|}{\sum_{k=1}^{n} |H_{i,k}|}$ (Forward) or $P_{i,j} = \frac{|H^T_{i,j}|}{\sum_{k=1}^{n} |H^T_{i,k}|}$ (Adjoint) guarantee $\|H^*\|_\infty < 1$ or $\|H^*\|_1 < 1$ respectively.

Monte Carlo Linear Solvers

First approaches to a parallelization

First attempts to apply Monte Carlo as a linear solver considered the fixed point formulation of the original linear system.

- \+ embarrassingly parallelizable (ideally we could employ as many processes as the amount of histories)
- – huge amount of time to compute a reasonably accurate solution

E.g. $\|r_0\| = \|\mathbf{b} - A\mathbf{x}_0\| \approx 1$ then $\frac{\|\mathbf{r}_k\|}{\|\mathbf{b}\|} < 10^{-m} \Rightarrow N_{histories} \approx 10^{2m}$ because of the Central Limit Theorem.

[S. Branford, C. Sahin, A. Thandavan, C. Weihrauch, V. N. Alexandrov and I. T. Dimov, 2008; P. Jakovits, I. Kromonov and S. R. Srirama, 2011]

Monte Carlo Linear Solvers

Monte Carlo Synthetic Acceleration (MCSA)

Data: A, $\mathbf{b}$, H, $\mathbf{f}$, $\mathbf{x}_0$
Result: x_{num}
$\mathbf{x}^l = \mathbf{x}_0$;
while *not reached convergence* **do**
  $\mathbf{x}^{l+\frac{1}{2}} = H\mathbf{x}^l + \mathbf{f}$;
  $\mathbf{r}^{l+\frac{1}{2}} = \mathbf{b} - A\mathbf{x}^{l+\frac{1}{2}}$;
  $\delta\mathbf{x}^{l+\frac{1}{2}} = (I - H)^{-1}\mathbf{r}^{l+\frac{1}{2}}$; "Solved" with Standard MC
  $\mathbf{x}^{l+1} = \mathbf{x}^{l+\frac{1}{2}} + \delta\mathbf{x}^{l+\frac{1}{2}}$;
end
$x_{num} = x^{l+1}$

[S. Slattery, 2013 (PhD Thesis); T. M. Evans, S. R. Slattery and P. P. H. Wilson, 2013]

Adaptivity for the number of random walks

Forward Method:

$\theta_i \in \mathbb{R}$

$\theta_i = E\left[\sum_{l=0}^{\infty} W_l b_{k_l}\right]$

$\sigma_i = \sqrt{Var[\theta_i]}$

Find $\tilde{N}_i$ s.t. $\frac{\sigma_i^{\tilde{N}_i}}{|\hat{x}_i|} < \varepsilon$

$i = 1, \ldots, n$

Adjoint Method:

$\boldsymbol{\theta} \in \mathbb{R}^n$

$\boldsymbol{\theta}_i = E\left[\sum_{l=0}^{\infty} W_l d_{k_l} \delta_{k_l,i}\right]$

$\boldsymbol{\sigma}_i = \sqrt{Var[\boldsymbol{\theta}]_i}$

Find $\tilde{N}$ s.t. $\frac{\|\boldsymbol{\sigma}^{\tilde{\boldsymbol{N}}}\|_1}{\|\hat{\mathbf{x}}\|_1} < \varepsilon$

$i = 1, \ldots, n$

Applicable preconditioning techniques

Remark: explicit knowledge of H_{ij} is needed

In fact the entry P_{ij} of the transition matrix is defined in terms of H_{ij} (Forward method) or H_{ji} (Adjoint method).

This limits the viable choices of preconditioners:

- diagonal preconditioners
- block diagonal preconditioners
- sparse approximate inverse preconditioners (AINV)

Monte Carlo Linear Solvers

Examples

Strictly diagonally dominant matrices (s.d.d.) with diagonal preconditioning.

$A \in \mathbb{R}^{n \times n}$ s.d.d. by rows: $P = diag(A)$
$H = I - P^{-1}A$, $\|H\|_\infty < 1 \quad (\Leftrightarrow \|H^*\|_\infty < 1)$

$A \in \mathbb{R}^{n \times n}$ s.d.d. by columns: $P = diag(A)$
$H = I - AP^{-1}$, $\|H\|_1 < 1 \quad (\Leftrightarrow \|H^*\|_1 < 1)$

Strict diag. dominance	Forward Method	Adjoint method
by rows	converges	not guaranteed
by columns	not guaranteed	converges

Monte Carlo Linear Solvers

Examples

Other examples:

- A M-matrix $\Rightarrow \exists D$ diagonal s.t. AD is s.d.d.
- with preconditioner $P = \text{block\_diag}(A)$:

$$\begin{cases} \displaystyle\sum_{\substack{j=1 \\ j\neq i}}^{p} \|A_{ii}^{-1}A_{ij}\|_\infty < 1. \quad \forall i = 1, \ldots, p \\ \\ \Rightarrow \text{ Forward method with block Jacobi converges} \end{cases}$$

$$\begin{cases} \displaystyle\sum_{\substack{i=1 \\ i\neq j}}^{p} \|A_{ii}^{-1}A_{ij}\|_1 < 1. \quad \forall j = 1, \ldots, p \\ \\ \Rightarrow \text{ Adjoint method with block Jacobi converges} \end{cases}$$

2D parabolic problems: test case 1

Give an open and bounded set $\Omega \subset \mathbb{R}^2$ s.t. $\Omega = (0,2)^2 \setminus (1,2)^2$

$$\begin{cases} \frac{\partial u}{\partial t} - \mu \Delta u + \boldsymbol{\beta}(\mathbf{x}) \cdot \nabla u = 0, & \mathbf{x} \in \Omega, \quad t \in (0, T] \\ u(\mathbf{x}, 0) = u_0, & \mathbf{x} \in \Omega \\ u(\mathbf{x}, t) = 0, & \mathbf{x} \in \Gamma_D, \quad t \in (0, T] \\ \frac{\partial u}{\partial n} + \chi u = 0, & \mathbf{x} \in \Gamma_R, \quad t \in (0, T] \end{cases} \tag{7}$$

where $\mu = \frac{3}{200}$, $\chi = 3$, $\boldsymbol{\beta}(\mathbf{x}) = [x, -y]^T$.
$\Gamma_R = \{\{x = 1\} \times (1,2)\} \cup \{(1,2) \times \{y = 1\}\}$, $\Gamma_D = \partial\Omega \setminus \Gamma_R$.

Discretization (`FreeFem++` employed):

- Triangular FEM
- spatial discretization step $h_{max} = 0.018$, 37,177 d.o.f.'s
- time discretization step $\Delta t = h_{max}^2$

2D parabolic problems: test case 1

Preconditioning: sparse factorized AINV with drop tolerance $\tau = 0.05$. [M. Benzi and M. Tuma, SISC, 1998]

Numerical setting:

- Adjoint MCSA employed as linear solver
- residual relative tolerance: $\varepsilon_1 = 10^{-7}$
- weighted transition probability
- maximal # steps per history: 10
- statistical error-based adaptive parameter: $\varepsilon_2 = 0.5$
- granularity of the adaptive approach: $n_{histories} = 100$
- simulations run on laptop

2D parabolic problems: test case 1

Stiffness matrix: A ($n = 37,177$).
AINV preconditioner: M.
Iteration matrix: $H = I - MA$.
Solution computed just for a generic time step.

- $\frac{nnz(H)}{nnz(A)} = 2.984$
- $\|H\|_1 = 0.597$
- # iterations employed: 9
- relative error$= 3.783 \cdot 10^{-8}$
- # total random walks employed: 10,700

2D parabolic problems: test case 2

Give an open and bounded set $\Omega = (0,1) \times (0,1)$

$$\begin{cases} \frac{\partial u}{\partial t} - \mu \Delta u + \boldsymbol{\beta}(\mathbf{x}) \cdot \nabla u = 0, & \mathbf{x} \in \Omega, \quad t \in (0, T] \\ u(\mathbf{x}, 0) = u_0, & \mathbf{x} \in \Omega \\ u(\mathbf{x}, t) = 0, & \mathbf{x} \in \partial\Omega, \quad t \in (0, T] \end{cases} \tag{8}$$

where $\mu = \frac{3}{200}$, $\boldsymbol{\beta}(\mathbf{x}) = [2,\ sin(x)]^T$.

Discretization (`FreeFem++` employed):

- Triangular FEM
- spatial discretization step $h_{max} = 0.014$, 40,501 d.o.f.'s
- time discretization step $\Delta t = h_{max}^2$

2D parabolic problems: test case 2

Stiffness matrix: A ($n = 40,501$).
AINV preconditioner: M.
Iteration matrix: $H = I - MA$.
Solution computed just for a generic time step.

- $\frac{nnz(H)}{nnz(A)} = 3.106$
- $\|H\|_1 = 0.185$
- # iterations employed: 9
- relative error$= 6.904 \cdot 10^{-8}$
- # total random walks employed: 4,500

2D parabolic problems: test case 3

Give an open and bounded set $\Omega = (0,1) \times (0,1)$

$$\begin{cases} \frac{\partial u}{\partial t} - \mu \Delta u + \boldsymbol{\beta}(\mathbf{x}) \cdot \nabla u = 0, & \mathbf{x} \in \Omega, \quad t \in (0, T] \\ u(\mathbf{x}, 0) = u_0, & \mathbf{x} \in \Omega \\ u(\mathbf{x}, t) = u_D(\mathbf{x}), & \mathbf{x} \in \partial\Omega, \quad t \in (0, T] \end{cases} \tag{9}$$

where $\mu = \frac{3}{200}$, $\boldsymbol{\beta}(\mathbf{x}) = [2y(1-x^2),\ -2x(1-y^2)]^T$,
$u_D = 0$ on $\{\{x = 0\} \times (0,1)\}, \{(0,1) \times \{y = 0\}\}, \{(0,1) \times \{y = 1\}\}$.

Discretization (`IFISS` toolbox employed):

- Quadrilateral FEM
- spatial discretization step $h = 2^{-8} \Rightarrow$ 66,049 d.o.f.'s
- time discretization step $\Delta t = h^2$

2D parabolic problems: test case 3

Stiffness matrix: A ($n = 66,049$).
AINV preconditioner: M.
Iteration matrix: $H = I - MA$.
Solution computed just for a generic time step.

- $\frac{nnz(H)}{nnz(A)} = 4.736$
- $\|H\|_1 = 0.212$
- # iterations employed: 12
- relative error$= 9.193 \cdot 10^{-8}$
- # total random walks employed: $9,300$

Conclusions and future developments

Conclusions

- MC solvers motivated by resilience
- hypotheses for convergence difficult to satisfy in general
- currently best results obtained using AINV

Future developments

- extension of the set of matrices for which MC solvers is guaranteed to converge a priori
- analysis of how $\rho(H)$ and $\rho(H^*)$ affect convergence
- refinement of the adaptive selection of histories

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Monte Carlo Linear Solvers

The choice of the transition probability

Theorem (Hao J. et al.)

Let $H \in \mathbb{R}^{n \times n}$ with spectral radius $\rho(H) < 1$. Let $H^+ \in \mathbb{R}^{n \times n}$ where $H^+_{i,j} = |H_{i,j}|$. Consider ν_k as the realization of a random walk γ truncated at the k-th step. If $\rho(H^+) > 1$, there does not exist a transition matrix P such that $Var\Big(X(\nu_k)\Big)$ converges to zero as $k \to \infty$.

Numerical experiments

Simplified P_N equations

Steady-state, multigroup, one-dimensional, eigenvalue-form of Boltzmann transport equation:

$$\mu \frac{\partial \psi^g(x,\mu)}{\partial x} + \sigma^g(x)\psi^g(x,\mu) = \sum_{g'=1}^{N_g} \int_{4\pi} \sigma_s^{gg'}(x, \hat{\Omega}\cdot\hat{\Omega}')\psi^{g'}(x,\Omega')d\Omega' + \\ + \frac{1}{k}\sum_{g'=1}^{N_g} \frac{\chi^g}{4\pi} \int_{4\pi} \nu\sigma_f^{g'}(x)\psi^{g'}(x,\Omega')d\Omega'. \tag{10}$$

$\psi^g(x,\mu)$ = angular flux for group g
σ^g = total interaction cross section
$\sigma_s^{gg'}(x, \hat{\Omega}\cdot\hat{\Omega}')$ = scattering cross section from group $g' \to g$

Numerical experiments

Simplified P_N equations

- Legendre polynomial spectral discretization (P_N equations)
- consider just odd sets of P_N equations
- removing lower order gradient terms from each equation (SP_N equations)

$$-\nabla \cdot \mathbb{D}_n \nabla \mathbb{U}_n + \sum_{m=0}^{\frac{N+1}{2}} \mathbb{A}_{nm} \mathbb{U}_m = \frac{1}{k} \sum_{m=1}^{\frac{N+1}{2}} \mathbb{F}_{nm} \mathbb{U}_m, \quad n = 1, \ldots, \frac{N+1}{2} \tag{11}$$

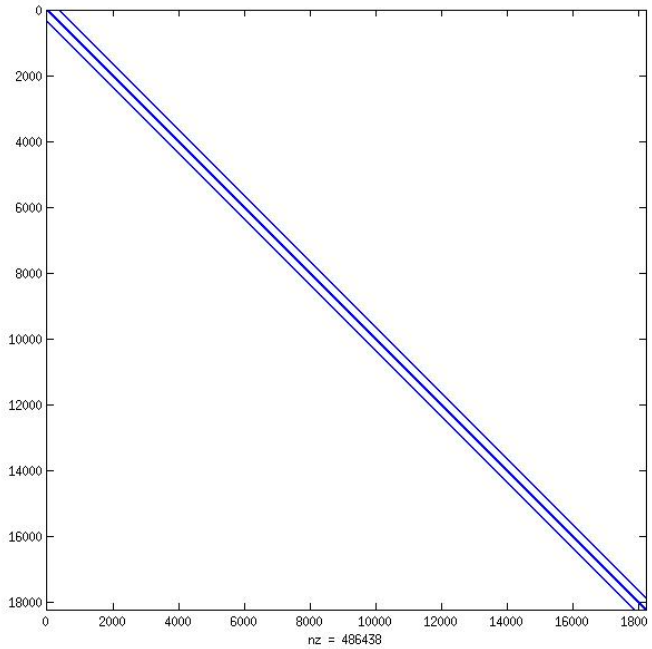
$\mathbb{U}_m$ is a linear combination of moments.

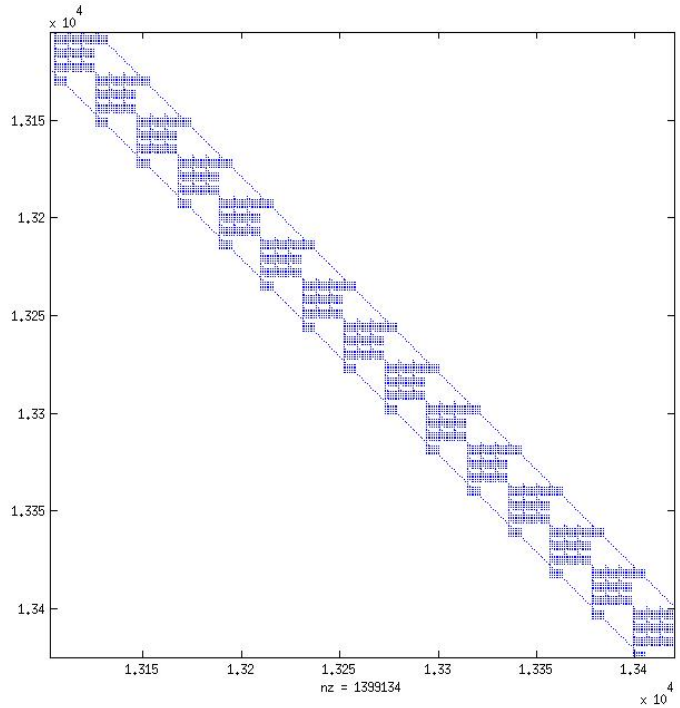
- Discretized with Finite Volumes

Numerical experiments

Simplified P_N equations

Sparsity pattern and zoom on the structure of a SP_1 matrix





Numerical experiments

Simplified P_N equations

Properties of matrices:

- discretization of SP_1 equations
- all nonzero diagonal entries

Numerical treatment:

- block Jacobi preconditioning

Kind of matrix	size	nnz	Prec. block size	nnz after prec
SP_1 (a)	18,207	486,438	63	2,644,523
SP_1 (b)	19,941	998,631	69	1,774,786

[Matrices provided by courtesy by Thomas Evans, Steven Hamilton, Stuart Slattery, ORNL]

Numerical experiments

Simplified P_N equations

MCSA parameter setting:

- Adjoint method
- residual relative tolerance: $\varepsilon_1 = 10^{-7}$
- almost optimal transition probability
- maximal # steps per history: 10
- statistical error-based adaptive parameter: $\varepsilon_2 = 0.5$
- granularity of the adaptive approach: $n_{histories} = 1,000$
- simulations run in serial mode

Numerical experiments

Simplified P_N equations

matrix	$\rho(H)$	$\rho(H^*)$	relative err.	# iterations
SP_1 (a)	0.9779	0.9758	$9.97 \cdot 10^{-6}$	340
SP_1 (b)	0.9798	0.9439	$3.89 \cdot 10^{-5}$	209

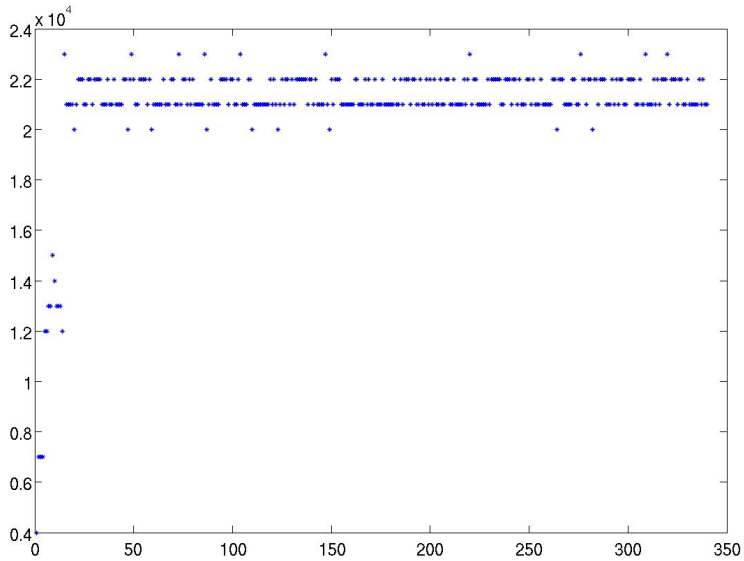


Figure 1 : # histories for (a) at each iteration.

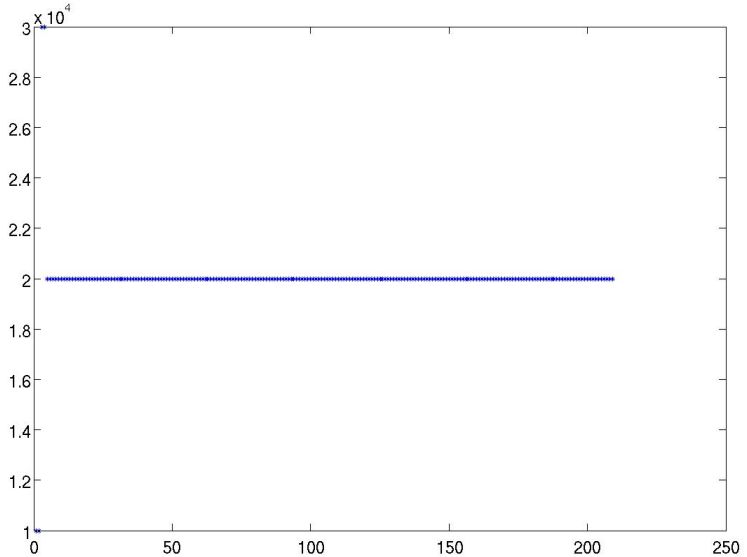


Figure 2 : # histories for (b) at each iteration.

Numerical experiments

Simplified P_N equations

Issues raised for a general SP_N matrix:

- hard to find a block Jacobi preconditioner s.t. $\rho(H^*) < 1$ for every SP_N problem
- attempt to use Approximate Inverse preconditioners was not effective
  - for sparse preconditioners $\rho(H^*) < 1$ is not respected
- attempt to use ILU preconditioners was not effective
  - for ILU(0) we got $\rho(H^*) > 1$
  - massive fill-in with ILUT
- reordering and scaling did not facilitate the convergence requirements

Numerical experiments

Diagonally dominant matrices

Set of matrices obtained by a diagonal shift of matrices associated with SP_1, SP_3 and SP_5 equations.

$$(A + sD), \quad s \in \mathbb{R}^+, \quad D = diag(A).$$

All the matrices have been turned into s.d.d. by columns.

Initial matrix	s	$\rho(H)$	$Forward - \rho(H^*)$	$Adjoint - \rho(H^*)$
SP_1 (a)	0.3	0.7597	0.7441	0.6983
SP_1 (b)	0.4	0.7046	1.1448	0.5680
SP_3	0.9	0.5869	0.4426	0.3727
SP_5	1.6	0.5477	0.3790	0.3431

Numerical experiments

Diagonally dominant matrices

$$(A + sD), \quad s \in \mathbb{R}^+, \quad D = diag(A).$$

matrix	nnz	s	relative err.	# iterations
SP_1 (a)	486,438	0.3	$6.277 \cdot 10^{-7}$	36
SP_1 (b)	998,631	0.4	$9.885 \cdot 10^{-7}$	21
SP_3	846,549	0.9	$5.919 \cdot 10^{-7}$	18
SP_5	1,399,134	1.6	$4.031 \cdot 10^{-7}$	19

Numerical experiments

Diagonal shift

Strict diagonally dominance is a sufficient condition for convergence but not necessary.

$$(A + sD), \quad s \in \mathbb{R}^+, \quad D = diag(A)$$

such that $\rho(H) < 1$ and $\rho(H^*) < 1$.

Initial matrix	s	$\rho(H)$	$Forward - \rho(H^*)$	$Adjoint - \rho(H^*)$
SP_1 (a)	0.2	0.8230	0.8733	0.8195
SP_1 (b)	0.2	0.8220	1.5582	0.7731
SP_3	0.3	0.8126	0.9459	0.7961
SP_5	0.7	0.8376	0.8865	0.8026

Numerical experiments

Diagonal shift

$$(A + sD), \quad s \in \mathbb{R}^+, \quad D = diag(A).$$

matrix	s	relative err.	# iterations
SP_1 (a)	0.2	$6.394 \cdot 10^{-7}$	48
SP_1 (b)	0.2	$2.59 \cdot 10^{-6}$	36
SP_3	0.3	$5.35 \cdot 10^{-7}$	45
SP_5	0.7	$4.21 \cdot 10^{-7}$	70