NE 155, Class 3, S15

Types of Equations in the Engineering Fields

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Introduction

In science and engineering in general, and nuclear engineering and reactor analysis in specific, we encounter a wide range of mathematical physics equations. In today's lecture we will introduce some of them.

- Ordinary differential equations (ODEs)
- Partial differential equations (PDEs)
  - Elliptic PDEs
  - Parabolic PDEs
  - Hyperbolic PDEs
- Integro-differential equations
- Integral equations

1 ODEs

The most general form of an n$^{\text{th}}$ order linear ordinary differential eqn. is

$$a_n(x)y^{(n)}(x) + a_{n-1}(x)y^{(n-1)}(x) + \cdots + a_2(x)y^{(2)}(x) + a_1(x)y(x) + a_0(x) = f(x)$$

where

- a_n are coefficients
- $y^{(n)}$ is the n$^{\text{th}}$ derivative of y.

Boundary conditions:

1. Initial Value Problem (**IVP**): if y and its derivatives are given at one end of the domain/interval (e.g. time 0 if there's time or x 0 if there's only space, etc.)

2. Boundary Value Problem (**BVP**): if y and/or its derivatives are given at each end of the interval

Linear 1st order ODE's

Reminders

- Linear means each term is either a constant or the product of a constant and the first power of a single variable (in this case y).
- the highest derivative is to the first power.

Linear 1st order ODE Example:

$$\frac{dy}{dx} + 3y(x) = sin(x) \qquad x \in [0, 1]$$

- IVP if boundary conditions are y(0) = 1; y'(0) = 2
- BVP if boundary conditions are y(0)=-1, y(1) = 3

In this case the general solution is obtained through the use of an integrating factor.

Linear 1st order ODE Example:

Point Kinetics analysis of a nuclear reactor is an IVP, linear 1st order ODE.

$$\begin{aligned}
\frac{dn(t)}{dt} &= \frac{\rho(t) - \beta}{l^*} n(t) + \sum_{i=1}^{N} \lambda_i C_i(t) \\
\frac{dC_i(t)}{dt} &= \frac{\beta_i}{l^*} n(t) - \lambda_i C_i(t) \qquad i = 1, \ldots, N
\end{aligned}$$

Where (we'll talk more about what these terms mean later)

- n = # neutrons / s
- β = fraction of delayed neutrons
- λ_i = effective decay constant of the ith precursor
- $C_i(t)$ = delayed neutron concentration of the ith precursor
- l^* = mean neutron lifetime

- $\rho = \frac{k-1}{k}$ = reactivity

BCs: $n(0) = n_0$ and $C_i(0) = C_{i,0}$ for $i = 1, \ldots, N$.

Linear 1st order ODE Example:

The number of atoms in during radioactive decay (assuming decay only here) is described by the Bateman equation, which is a linear, 1st order ODE that is in a IVP:

$$\begin{aligned}
\frac{dN_1(t)}{dt} &= -\lambda_1 N_1(t) \\
\frac{dN_i(t)}{dt} &= -\lambda_i N_i(t) + \lambda_{i-1} N_{i-1}(t) \qquad 1 < k < I \\
\frac{dN_I(t)}{dt} &= \lambda_{I-1} N_{I-1}(t) \\
\text{BC: } N_i(t = 0) &= N_{i,0}
\end{aligned}$$

note: isotope i decays into $i + 1$. This can be adapted for decay branches, and becomes more complicated if we have neutrons that transmute isotopes.

2nd order ODE Example:

$$-\frac{d}{dx} p(x) \frac{d}{dx} \phi(x) + q(x)\phi(x) = S(x)$$

$$\text{defined for } \alpha \le x \le \beta$$

$$\text{BC: } \frac{d\phi}{dx} + \gamma\phi = \sigma \qquad \text{at } x = \alpha \text{ and } x = \beta \text{ (BVP)}$$

This has

- Neumann BCs if $\gamma = 0$ (specifies the values that the *derivative* of a solution is to take on the boundary of the domain)

- Dirichet BCs: if $\frac{d\phi}{dx} = 0$ (specifies the values that a *solution* is to take on the boundary of the domain)

- Mixed BCs if [$\gamma \neq 0$ and $\frac{d\phi}{dx} = 0$ at $x = \alpha$ and $\gamma = 0$] and [$\frac{d\phi}{dx} \neq 0$ at $x = \beta$] (the solution is required to satisfy a Dirichlet or a Neumann boundary condition in a mutually exclusive way on disjoint parts of the boundary)

- If $S(x)$ is nonzero at least somewhere over the physical range, a unique solution exists.

2nd order ODE Example:

The homogeneous equation

$$-\frac{d}{dx}p(x)\frac{d}{dx}\phi(x) + q(x)\phi(x) = \lambda f(x)\phi(x)$$

where $p(x) > 0$, $f(x) \geq 0$,$\lambda =$ eigenvalue;

$$\frac{d\phi}{dx} + \gamma_L\phi = \Sigma \qquad \text{at } x = \alpha \qquad \gamma_L \geq 0$$
$$\frac{d\phi}{dx} + \gamma_R\phi = \Sigma \qquad \text{at } x = \beta \qquad \gamma_R \geq 0$$

is known as the Sturm-Liouville eigenvalue problem. If $\gamma_L = \gamma_R = 0$, the BCs become $\phi(\alpha) = \phi(\beta) = 0$.

The solution has an infinite number of eigenfunctions, $\phi_i(x)$ with corresponding **REAL** and **DISTINCT** eigenvalues, λ_i.

If we number them in sequence: $\lambda_0 < \lambda_1 < \lambda_2 < \cdots < \lambda_\infty$, then λ_0 is the lowest eigenvalue and $\phi_0(x)$ is the fundamental eigenmode.

If $q(x) \geq 0$, then all λ_i are positive.

2nd order ODE Example:

1-D, 1-group, time-independent neutron diffusion equation:

$$-\frac{d}{dx}D(x)\frac{d}{dx}\phi(x) + \Sigma_a(x)\phi(x) = S(x) \qquad \text{Fixed Source}$$
$$-\frac{d}{dx}D(x)\frac{d}{dx}\phi(x) + \Sigma_a(x)\phi(x) = \frac{1}{k}\nu\Sigma_f(x)\phi(x) \qquad \text{Fission / Eigenvalue}$$

BCs: (BVP) vacuum, $\phi(\pm a) = 0$

2 PDEs

A partial differential equation is an equation containing an unknown function of two or more variables and its derivatives with respect to those variables.

If the PDE is linear in u and all derivatives of u, then we say that the PDE is linear.

$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2} + D\frac{\partial u}{\partial x} + E\frac{\partial u}{\partial y} + Fu(x) = G$$

This equation is a 2nd order PDE in two variables. It is linear if A through G do not depend on u (they may depend on x and/or y).

Classification:

To think about classification, think about replacing ∂x by x and ∂y by y (formally this is done via Fourier transform). This converts the PDE into a polynomial of the same degree.

Just as one classifies conic sections and quadratic forms into parabolic, hyperbolic, and elliptic based on the discriminant $B^2 - 4AC$, the same can be done for a second-order PDE at a given point.

Note: these classifications only apply to second order PDEs.

- **Elliptic** if $B^2 - 4AC < 0$. Some famous elliptic PDEs:

$$\nabla^2 u = 0 \qquad \text{Laplace's eqn.}$$
$$\nabla^2 u = f(x) \qquad \text{Poisson's eqn.}$$
$$-\frac{\partial}{\partial x}D(x,y)\frac{\partial}{\partial x}\phi(x,y) - \frac{\partial}{\partial y}D(x,y)\frac{\partial}{\partial y}\phi(x,y) + \big(\Sigma_a(x,y) - \frac{1}{k}\nu\Sigma_f(x,y)\big)\phi(x,y) = 0$$

  There's no B term, so $-4AC < 0$ since D is positive.

  One property of constant coefficient elliptic equations is that their solutions can be studied using the Fourier transform.

  Recall

$$\text{gradient is } \nabla T = \vec{i}\frac{\partial T}{\partial x} + \vec{j}\frac{\partial T}{\partial y} + \vec{k}\frac{\partial T}{\partial z}$$
$$\text{divergence is } \nabla \cdot \vec{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}$$

- **Hyperbolic** if $B^2 - 4AC > 0$, e.g.

$$\frac{\partial^2 u}{\partial t^2} + c^2\frac{\partial^2 u}{\partial x^2} = 0 \qquad \text{1-D wave eqn.}$$

There's no B term, and C is negative so $-4AC > 0$

- if u and its first t derivative are arbitrarily specified with initial data on the initial line $t = 0$ (with sufficient smoothness properties), then there exists a solution for all of t.
- The solutions of hyperbolic equations are "wave-like." If a disturbance is made in the initial data of a hyperbolic differential equation, then not every point of space feels the disturbance at once.
- Relative to a fixed time coordinate, disturbances have a finite propagation speed. They travel along the characteristics of the equation.

- **Parabolic** if $B^2 - 4AC = 0$, e.g.

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} \qquad \text{1-D heat eqn.}$$
$$\frac{1}{v}\frac{\partial \phi(x,t)}{\partial t} = \frac{\partial}{\partial x}D(x,t)\frac{\partial}{\partial x}\phi(x,t) + \big(\nu\Sigma_f(x,t) \; - \Sigma_a(x,t)\big)\phi(x,t) + S(x,t)$$

There aren't B or C terms, so $-4AC = 0$

A perturbation of the initial (or boundary) data of an elliptic or parabolic equation is felt at once by essentially all points in the domain.

Equations that are parabolic at every point can be transformed into a form analogous to the heat equation by a change of independent variables. Solutions smooth out as the transformed time variable increases.

higher order PDE classification

If there are n independent variables $x_1, x_2, \ldots, x_n$, a general linear partial differential equation of second order has the form

$$Lu = \sum_{i=1}^{n}\sum_{j=1}^{n} a_{i,j}\frac{\partial^2 u}{\partial x_i x_j} + \text{ Lower Order Terms } = 0$$

The classification depends upon the signature of the eigenvalues of the coefficient matrix $a_{i,j}$.

1. Elliptic: The eigenvalues are all positive or all negative.
2. Parabolic: The eigenvalues are all positive or all negative, save one that is zero.
3. Hyperbolic: There is only one negative eigenvalue and all the rest are positive, or there is only one positive eigenvalue and all the rest are negative.

4. Ultrahyperbolic: There is more than one positive eigenvalue and more than one negative eigenvalue, and there are no zero eigenvalues. There is only limited theory for ultrahyperbolic equations (Courant and Hilbert, 1962).

The reason we care based on the TE

- In a void, the transport equation is like a hyperbolic wave equation.
- For highly-scattering regions where Σ_s is close to Σ, the equation becomes elliptic for the steady-state case.
- If the scattering is forward-peaked then the equation is parabolic.