Inverse of a Matrix
using Minors, Cofactors and Adjugate

You can calculate the Inverse of a Matrix by:

- Step 1: calculating the Matrix of Minors,
- Step 2: then turn that into the Matrix of Cofactors,
- Step 3: then the Adjugate, and
- Step 4: multiply that by 1/Determinant.

But it is best explained by working through an example!

Example: find the Inverse of A:

$$A = \begin{bmatrix} 3 & 0 & 2 \\ 2 & 0 & -2 \\ 0 & 1 & 1 \end{bmatrix}$$

It needs 4 steps. It is all simple arithmetic but there is a lot of it, so try not to make a mistake!

Step 1: Matrix of Minors

The first step is to create a "Matrix of Minors". This step has the most calculations:

For each element of the matrix:

- ignore the values on the current row and column
- calculate the determinant of the remaining values

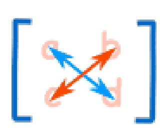
Put those determinants into a matrix (the "Matrix of Minors")

Determinant

For a 2×2 matrix (2 rows and 2 columns) the determinant is easy: **ad-bc**

Think of a cross:

- Blue means positive (+ad),
- Red means negative (-bc)



(It gets harder for a 3×3 matrix, etc)

The Calculations

Here are the first two, and last two, calculations of the "**Matrix of Minors**" (notice how I ignore the values in the current row and columns, and calculate the determinant using the remaining values):

$$\begin{bmatrix} \bullet & 0 & 2 \\ 2 & 0 & -2 \\ 0 & 1 & 1 \end{bmatrix} \quad 0{\times}1 - (-2){\times}1 = 2$$

$$\begin{bmatrix} 3 & \bullet & 2 \\ 2 & 0 & -2 \\ 0 & 1 & 1 \end{bmatrix} \quad 2{\times}1 - (-2){\times}0 = 2$$

$$\cdots$$

$$\begin{bmatrix} 3 & 0 & 2 \\ 2 & 0 & -2 \\ 0 & \bullet & 1 \end{bmatrix} \quad 3{\times}{-2} - 2{\times}2 = -10$$

$$\begin{bmatrix} 3 & 0 & 2 \\ 2 & 0 & -2 \\ 0 & 1 & \bullet \end{bmatrix} \quad 3{\times}0 - 0{\times}2 = 0$$

And here is the calculation for the whole matrix:

$$\begin{bmatrix} 0{\times}1 - (-2){\times}1 & 2{\times}1 - (-2){\times}0 & 2{\times}1 - 0{\times}0 \\ 0{\times}1 - 2{\times}1 & 3{\times}1 - 2{\times}0 & 3{\times}1 - 0{\times}0 \\ 0{\times}(-2) - 2{\times}0 & 3{\times}(-2) - 2{\times}2 & 3{\times}0 - 0{\times}0 \end{bmatrix} = \underbrace{\begin{bmatrix} 2 & 2 & 2 \\ -2 & 3 & 3 \\ 0 & -10 & 0 \end{bmatrix}}_{\text{Matrix of Minors}}$$

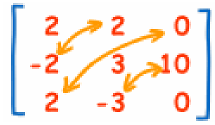
Step 2: Matrix of Cofactors

This is easy! Just apply a "checkerboard" of minuses to the "Matrix of Minors". In other words, you need to change the sign of alternate cells, like this:

$$\underbrace{\begin{bmatrix} 2 & 2 & 2 \\ -2 & 3 & 3 \\ 0 & -10 & 0 \end{bmatrix}}_{\text{Matrix of Minors}} \rightsquigarrow \begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix} \rightsquigarrow \underbrace{\begin{bmatrix} 2 & -2 & 2 \\ +2 & 3 & -3 \\ 0 & +10 & 0 \end{bmatrix}}_{\text{Matrix of CoFactors}}$$

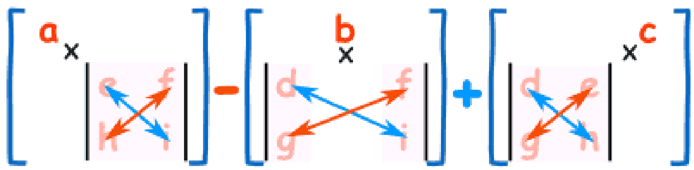
Step 3: Adjugate (also called Adjoint)

Now "Transpose" all elements of the previous matrix... in other words swap their positions over the diagonal (the diagonal stays the same):



Step 4: Multiply by 1/Determinant

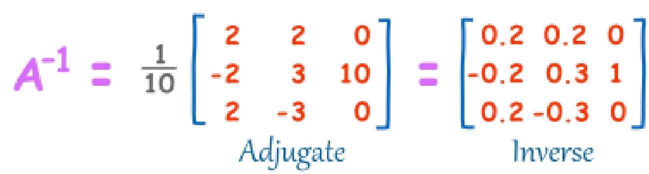
Now find the determinant of the original matrix. This isn't too hard, because we already calculated the determinants of the smaller parts when we did "Matrix of Minors".



So: multiply the top row elements by their matching "minor" determinants:

Determinant = 3×2 - 0×2 + 2×2 = **10**

And now multiply the Adjugate by 1/Determinant:



And we are done!

Compare this answer with the one we got on Inverse of a Matrix using Elementary Row Operations. Is it the same? Which method do you prefer?

Larger Matrices

It is exactly the same steps for larger matrices (such as a 4×4, 5×5, etc), but wow! there is a lot of calculation involved.

For a 4×4 Matrix you have to calculate 16 3×3 determinants. So it is often easier to use computers (such as the Matrix Calculator.)

Conclusion

- For each element, calculate the **determinant of the values not on the row or column**, to make the Matrix of Minors
- Apply a **checkerboard** of minuses to make the Matrix of Cofactors
- **Transpose** to make the Adjugate
- Multiply by **1/Determinant** to make the Inverse

Question 1 Question 2