NE 250, F15

Recall the one-group diffusion equation:

$$\frac{1}{v_1}\frac{\partial \phi_1(\vec{r},t)}{\partial t} = S_1(\vec{r},t) - \Sigma_{a,1}(\vec{r})\phi_1(\vec{r},t) + \nabla \cdot [D_1(\vec{r})\nabla\phi_1(\vec{r},t)]$$

Now, assume that space and energy dependence of the flux can be separated:

$$\phi(\vec{r},E,t) = \phi(\vec{r},t)\psi(E), \text{ where } \psi(E) \text{ is the neutron spectrum and } \int_0^\infty dE\psi(E) = 1.$$

Focusing on the spatial dependence of the flux, we'll assume a homogeneous steady-state, one-group system.

$$\frac{1}{v}\frac{\partial \phi(\vec{r},t)}{\partial t} = S(\vec{r},t) - \Sigma_s(\vec{r})\phi(\vec{r},t) + \nabla \cdot [D(\vec{r})\nabla\phi(\vec{r},t)]$$

Steady-state: $\frac{1}{v}\frac{\partial\phi(\vec{r},t)}{\partial t} = 0$

Homogeneous: no material dependence on position; $\Sigma_a(\vec{r}) \to \Sigma_a$, $D(\vec{r}) \to D$

This gives us

$$0 = S(\vec{r}) - \Sigma_a\phi(\vec{r}) + D\nabla^2\phi(\vec{r})$$

which can be rewritten as

$$\nabla^2\phi(\vec{r}) - \frac{1}{L^2}\phi(\vec{r}) = -\frac{S(\vec{r})}{D}, \text{ where } L = \sqrt{\frac{D}{\Sigma_a}} = \text{diffusion length.}$$

$$L = \sqrt{\frac{\langle r^2\rangle}{6}}, \langle r^2\rangle = \text{ root mean square distance from birth to absorption}$$

Now, consider a plane source of strength S_0 in an infinitely absorbing medium.

$$\phi(\vec{r}) = \phi(x)$$

$$\frac{d^2\phi(x)}{dx^2} - \frac{1}{L^2}\phi(x) = -\frac{S_0\delta(x)}{D}$$

For $x > 0$, $\frac{d^2\phi}{dx^2} - \frac{1}{L^2}\phi(x) = 0$.

Boundary conditions: $\lim_{x\to 0^+} \vec{J}(x) = \frac{S_0}{2}, \lim_{x\to +\infty} |\phi(x)| < \infty, \phi(x) \geq 0$

For $x > 0$, $\frac{d^2\phi}{dx^2} - \frac{1}{L^2}\phi(x) = 0$.

Boundary conditions: $\lim_{x\to 0^-} \vec{J}(x) = -\frac{S_0}{2}, \lim_{x\to -\infty} |\phi(x)| < \infty, \phi(x) \geq 0$

With the above equations and boundary conditions, we have a general solution form of

$$\phi(x) = c_1 e^{-x/L} + c_2 e^{x/L}.$$

From the finite flux condition, $c_2 = 0$. Then, $\phi(x) = c_1 e^{-x/L}$.

$$\lim_{x\to 0^+} \vec{J}(x) = \lim_{x\to 0^+} \left(\frac{D}{L} c_1 e^{-x/L}\right) = \frac{D}{L} c_1 = \frac{S_0}{2}$$

$$c_1 = \frac{S_0 L}{2D}$$

$$\phi(x) = \frac{S_0 L}{2D} e^{-x/L}$$

Now, consider an infinite plane centered in a slab of finite thickness a, surrounded by a vacuum.

For $x > 0$, $\frac{d^2\phi}{dx^2} - \frac{1}{L^2}\phi(x) = 0$.

Boundary conditions: $\lim_{x\to 0^+} \vec{J}(x) = \frac{S_0}{2}, \phi(\frac{\tilde{a}}{2}) = 0$

For $x < 0$, $\frac{d^2\phi}{dx^2} - \frac{1}{L^2}\phi(x) = 0$.

Boundary conditions: $\lim_{x\to 0^-} \vec{J}(x) = -\frac{S_0}{2}, \phi(-\frac{\tilde{a}}{2}) = 0$

$$\phi(x) = c_1 e^{-x/L} + c_2 e^{x/L}$$

or

$$\phi(x) = c_1 cosh(\tfrac{x}{L}) + c_2 sinh(\tfrac{x}{L})$$

Now, consider a uniformly distributed source of strength $S_0 \frac{n}{cm^3 s}$ within a finite slab of width a with vacuum boundaries.

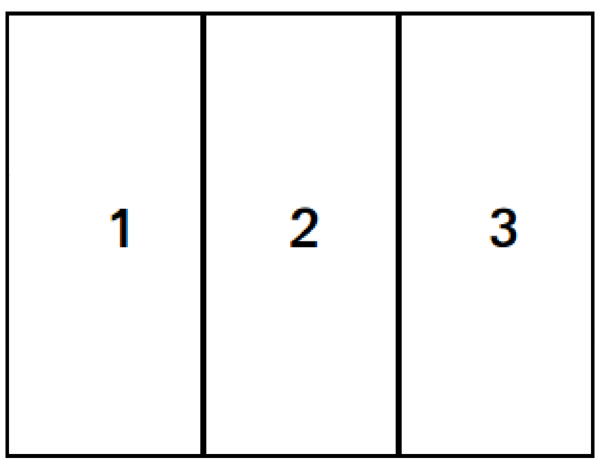
$$\frac{d^2\phi(x)}{dx^2} - \frac{1}{L^2}\phi(x) = -\frac{S_0}{D}$$

$$\phi(x) = c_1 e^{-x/L} + c_2 e^{x/L} + \frac{S_0 L^2}{D}$$

Boundary conditions: $\phi(\pm\frac{\tilde{a}}{2}) = 0$

Alternative boundary condition: $\frac{d\phi(x)}{dx}\Big|_{x=0} = 0$ (symmetry)

Now, consider a uniform source in a reflected slab like so:



$$\frac{d^2\phi_2(x)}{dx_2^2} - \frac{1}{L_2^2}\phi_2(x) = -\frac{S_0}{D_2}, -\tfrac{a}{2} < x < \tfrac{a}{2}$$

Boundary conditions:

$$\phi_2(\tfrac{a}{2}) = \phi_3(\tfrac{a}{2})$$
$$\vec{J}_2(\tfrac{a}{2}) = \vec{J}_3(\tfrac{a}{2})$$
$$\phi_1(-\tfrac{a}{2}) = \phi_2(-\tfrac{a}{2})$$
$$\vec{J}_1(-\tfrac{a}{2}) = \vec{J}_3(-\tfrac{a}{2})$$

$$\frac{d^2\phi_1(x)}{dx_1^2} - \frac{1}{L_1^2}\phi_1(x) = 0, -\infty < x < -\tfrac{a}{2}$$

$$\frac{d^2\phi_3(x)}{dx_3^2} - \frac{1}{L_3^2}\phi_3(x) = 0, \tfrac{a}{2} < x < \infty$$

Boundary conditions: $\lim\limits_{x\to-\infty} |\phi_1(x)| < \infty, \lim\limits_{x\to\infty} |\phi_3(x)| < \infty$

What if we want to solve a generic diffusion problem?

$$D\nabla^2\phi(\vec{r}) - \Sigma_a\phi(\vec{r}) = -S(\vec{r})$$

Working in 1D:

$$M\phi(x) = f(x), \text{ where } M \text{ is an operator}$$

This is a non-homogenous order *n* differential equation.

$$a_0(x)\phi^{(n)}(x) + a_1(x)\phi^{(n-1)}(x) + \ldots + a_n(x)\phi(x) = f(x)$$

One solution method is variation of constants.

Homogeneous problem: $M\phi(x) = 0$.

$$\phi_i(x), i = 1, \ldots, n \text{ independent solutions}$$

$$\phi_{homog}(x) = \sum_{i=1}^{n} c_i \phi_i(x)$$

Assume that $c_i = c_i(x)$.

$$\sum_{i=1}^{n} c_i \phi(x) = \phi_{particular}(x)$$

If $\sum_{i=1}^{n} c_i' \phi_i(x) = 0$, $\sum_{i=1}^{n} c_i' \phi_i^{(n)}(x) = 0$, $\sum_{i=1}^{n} c_i' \phi_i^{(n-2)}(x) = 0$, etc.

$$\sum_{i=1}^{n} c_i' \phi_i^{(n-1)}(x) = \frac{f(x)}{a_0(x)}$$

$$c_i(x) = \int dx c_i'(x)$$

For n = 2:

$$\phi_1(x)c_1'(x) + \phi_2(x)c_2'(x) = 0$$
$$\phi_1^{(1)}(x)c_1'(x) + \phi_2^{(1)}(x)c_2'(x) = \frac{f(x)}{a_0(x)}$$
$$\phi_{part} = c_1(x)\phi_1(x) + c_2(x)\phi_2(x)$$
$$a_0(x)\phi^{(2)}(x) + a_1(x)\phi^{(1)}(x) + a_2(x)\phi(x) = 0$$
$$\phi^{(1)}(x) = c_1'(x)\phi_1(x) + c_1(x)\phi_1'(x) + c_2'(x)\phi_2(x) + c_2(x)\phi_2'(x)$$

$$c_i(x) = \int dx c_i'(x) = A_i + \psi_i(x)$$

$$\phi(x) = \sum_{i=1}^{n} A_i \phi_i(x) + \sum_{i=1}^{n} \psi_i(x)\phi_i(x)$$

For n = 2: $\phi(x) = A_1\phi_1(x) + A_2\phi_2(x) + \psi_1(x)\phi_1(x) + \psi_2(x)\phi_2(x)$

For a plane source in an infinite medium:

$$\frac{d^2\phi(x)}{dx^2} - \frac{1}{L^2}\phi(x) = -\frac{S\delta(x)}{D}$$

Homogeneous problem: $\frac{d^2\phi}{dx^2} - \frac{1}{L^2}\phi(x) = 0, \phi_1(x) = e^{-x/L}, \phi_2(x) = e^{x/L}$

$$e^{-x/L}c_1'(x) + e^{x/L}c_2'(x) = 0$$

$$-\frac{e^{-x/L}}{L}c_1'(x) + \frac{e^{x/L}}{L}c_2'(x) = -\frac{S}{D}\delta(x)$$

$$\phi_{part}(x) = c_1(x)\phi_1(x) + c_2(x)\phi_2(x)$$

$$c_1'(x) = \frac{SL}{2D}\delta(x)e^{x/L}$$

$$c_2'(x) = -\frac{SL}{2D}\delta(x)e^{-x/L}$$

$$c_1(x) = \int_{-\infty}^{x} dx\frac{SL}{2D}\delta(x)e^{x/L} = \frac{SL}{2D}H(x), \text{ where } H(x) \text{ is the Heaviside step function}$$

$$H(x) = \begin{cases} 1, & x > 0 \\ \frac{1}{2}, & x = 0 \\ 0, & x > 0 \end{cases}$$

$$c_2(x) = \int_{-\infty}^{x} dx\frac{-SL}{2D}\delta(x)e^{-x/L} = -\frac{SL}{2D}H(x),$$

$$\phi(x) = A_1\phi_1(x) + A_2\phi_2(x) + c_1(x)\phi_1(x) + c_2(x)\phi_2(x)$$

$$\phi(x) = \left(A_1 + \frac{SL}{2D}H(x)\right)e^{-x/L} + \left(A_2 - \frac{SL}{2D}H(x)\right)e^{x/L}$$

Boundary conditions:

$$\lim_{x\to\infty}|\phi(x)\ <\infty \to A_2 = \frac{SL}{2D}$$
$$\lim_{x\to-\infty}|\phi(x)\ <\infty \to A_1 = 0$$

$$\phi(x) = \frac{SL}{2D}H(x)e^{-x/L} + \frac{SL}{2D}[1-H(x)]e^{x/L} = \frac{SL}{2D}e^{-|x|/L}$$

Other solution methods include the use of Green's function and the eigenfunction expansion method.

<u>Green's function</u>

$$\nabla\cdot[D(\vec{r})\nabla\phi(\vec{r})] - \Sigma_a\phi(\vec{r}) = -S(\vec{r}), \forall\vec{r}\in V$$

$$\phi(\vec{r}) = 0, \forall\vec{r}\in S\equiv\partial V$$

Consider a unit source at $\vec{r}'$.

$G(\vec{r},\vec{r}')=$ flux at $\vec{r}$ due to a unit source $\vec{r}'$

$$\nabla\cdot[D(\vec{r})\nabla G(\vec{r},\vec{r}')] - \Sigma_a G(\vec{r},\vec{r}') = -\delta(\vec{r}-\vec{r}'), \forall\vec{r}\in V$$

$$G(\vec{r},\vec{r}') = 0, \forall\vec{r}\in S\equiv\partial V$$

$$\int_V dV\left\{\nabla\cdot[D(\vec{r})\nabla\phi(\vec{r})]G(\vec{r},\vec{r}') - \nabla\cdot[D(\vec{r})\nabla G(\vec{r},\vec{r}')]\phi(\vec{r})\right\}$$
$$= \int_S dS\hat{e}_s\cdot D(\vec{r})[\nabla\phi(\vec{r})\times G(\vec{r},\vec{r}') - \phi(\vec{r})\nabla G(\vec{r},\vec{r}')] = 0$$

$$\int_V dV\left\{S(\vec{r})G(\vec{r},\vec{r}') - \delta(\vec{r}-\vec{r}')\phi(\vec{r})\right\} = 0$$

$$\phi(\vec{r}') = \int_V dV S(\vec{r}) G(\vec{r}, \vec{r}')$$

$$\phi(\vec{r}) = \int_V dV S(\vec{r}') G(\vec{r}', \vec{r})$$

Green's function ("kernel"): $G(\vec{r}, \vec{r}') = G(\vec{r}', \vec{r})$

The challenge is: how do we determine Green's function?

Consider a plane source in an infinite medium.

$$\frac{d^2\phi(x)}{dx^2} - \frac{1}{L^2}\phi(x) = -\frac{S}{D}\delta(x)$$

$$\lim_{x\to\pm\infty} \phi(x) = 0$$

$$\frac{d^2G(x,x')}{dx^2} - \frac{1}{L^2}G(x,x') = -\frac{\delta(x-x')}{D}$$

$$\lim_{x\to\pm\infty} G(x,x') = 0$$

Using variation of constants:

$$G(x,x') = \frac{L}{2D}H(x-x')e^{-(x-x')/L} + \frac{L}{2D}[1 - H(x-x')]e^{x-x'}$$

$$\phi(x) = \int_{-\infty}^{\infty} dx' G(x,x')\delta(x')S$$

$$\phi(x) = \frac{SL}{2D}e^{-|x|/L}$$

<u>Eigenfunction expansion method</u>

Term definitions:

$Hf = \lambda f$

$f =$ function

$H =$ operator

$\lambda =$ scalar value

$f_n =$ eigenfunction

$\lambda_n =$ eigenvalue

Example: $x = x \rightarrow H = 1, f = x, \lambda = 1$

Consider a fairly general diffusion problem in a homogeneous volume.

$$\nabla^2 \phi(\vec{r}) - \frac{1}{L^2}\phi(\vec{r}) = -\frac{S(\vec{r})}{D}, \forall \vec{r} \in V$$

$$\phi(\vec{r}) = 0 \quad \forall \vec{r} \in S$$

Eigenproblem:

$$\nabla^2 \psi_n(\vec{r}) + B_n^2 \psi_n(\vec{r}) = 0 \quad \forall \vec{r} \in V$$

$$\nabla^2 \psi_n(\vec{r}) = -B_n^2 \psi_n(\vec{r}) \quad (H = \nabla^2, \lambda = -B_n^2)$$

$$\psi_n(\vec{r}) = 0 \quad \forall \vec{r} \in S$$

Orthonormality:

$$\int_V dV \psi_n(\vec{r}) \psi_m(\vec{r}) = 1 \text{ if n = m, 0 if n} \neq \text{m}$$

Complete set:

$$\int_V dV \left(f - \sum_{n=1}^{N} f_n \psi_n \right)^2 < \varepsilon \qquad \text{(for large N and small but positive } \varepsilon\text{)}$$

$$\phi(\vec{r}) \approx \sum_{n=1}^{N} c_n \psi_n(\vec{r})$$

$$\nabla^2 \sum_{n=1}^{N} c_n \psi_n(\vec{r}) - \Sigma_a \sum_{n=1}^{N} c_n \psi_n(\vec{r}) = -\frac{1}{D} \sum_{n=1}^{N} s_n \psi_n(\vec{r})$$

$$\sum_{n=1}^{N} c_n [\nabla^2 \psi_n(\vec{r}) - \Sigma_a \psi_n(\vec{r})] = -\frac{1}{D} \sum_{n=1}^{N} s_n \psi_n(\vec{r})$$

$$\sum_{n=1}^{N} c_n (B_n^2 + \Sigma_a) \psi_n(\vec{r}) = \frac{1}{D} \sum_{n=1}^{N} s_n \psi_n(\vec{r})$$

$$\int_V dV \sum_{n=1}^{N} c_n (B_n^2 + \Sigma_a) \psi_n(\vec{r}) \psi_m(\vec{r}) = \int_V dV \frac{1}{D} \sum_{n=1}^{N} s_n \psi_n(\vec{r}) \psi_m(\vec{r})$$

Due to orthonormality,

$$c_n (B_n^2 + \Sigma_a) = \frac{1}{D} s_n$$

$$c_n = \frac{\frac{s_n}{\Sigma_a}}{1 + \frac{B_n^2}{L}}$$

$$s_n = \int_V dV S(\vec{r}) \psi_n(\vec{r})$$

$$\phi(\vec{r}) \approx \sum_{n=1}^{N} c_n \psi_n(\vec{r}) = \int_V dV \frac{1}{\Sigma_a} \sum_{n=1}^{N} \frac{\psi_n(\vec{r}')}{1 + L^2 B_n^2} \psi_n(\vec{r}) S(\vec{r}')$$

$$\frac{1}{\Sigma_a} \sum_{n=1}^{N} \frac{\psi_n(\vec{r}')}{1 + L^2 B_n^2} \psi_n(\vec{r}) = \text{Green's function}$$

$$\nabla^2\phi(\vec{r}) - \frac{1}{L^2}\phi(\vec{r}) = -\frac{S}{D}$$

Boundary condition: $\phi(\tilde{\vec{r}}) = 0$

Consider a finite slab of thickness a with a plane source at $x = 0$ and a vacuum boundary.

$$\frac{d^2\phi(x)}{dx^2} - \frac{1}{L^2}\phi(x) = -\frac{S}{D}\delta(x), \quad -\frac{a}{2} \leq x \leq \frac{a}{2}$$

$$\phi\left(\pm\tfrac{\tilde{a}}{2}\right) = 0$$

$$\frac{d^2G(x,x')}{dx^2} - \frac{1}{L^2}G(x,x') = -\frac{1}{D}\delta(x-x'), \quad -\frac{a}{2} < x < \frac{a}{2}$$

$$G\left(\pm\tfrac{\tilde{a}}{2}\right) = 0$$

$$\phi(x) = \int_{-\tilde{a}/2}^{\tilde{a}/2} dx' G(x,x')S(x')$$

$$\frac{d^2\psi(x)}{dx^2} + B^2\psi(x) = 0, \quad -\frac{a}{2} < x < \frac{a}{2}$$

$$\psi\left(\pm\tfrac{\tilde{a}}{2}\right) = 0$$

$$G(x,x') = \sum_n c_n\psi_n(x)$$

$$\psi(x) = c_1 cos(Bx) + c_2 sin(Bx)$$

$$\psi(\tfrac{\tilde{a}}{2}) = c_1 cos(B\tfrac{\tilde{a}}{2}) + c_2 sin(B\tfrac{\tilde{a}}{2}) = 0$$

$$\psi(-\tfrac{\tilde{a}}{2}) = c_1 cos(B\tfrac{\tilde{a}}{2}) - c_2 sin(B\tfrac{\tilde{a}}{2}) = 0$$

Trivial solution: $c_1 = c_2 = 0 \rightarrow \psi(x) = 0$

Non-trivial solution:

$$c_2 = 0, cos(B\tfrac{\tilde{a}}{2}) = 0 \rightarrow B\frac{\tilde{a}}{2} = \frac{n\pi}{2}, B_n = \frac{n\pi}{\tilde{a}}, \quad \text{n odd}$$
$$c_1 = 0, sin(B\tfrac{\tilde{a}}{2}) = 0 \rightarrow B\frac{\tilde{a}}{2} = \frac{n\pi}{2}, B_n = \frac{n\pi}{\tilde{a}}, \quad \text{n even}$$

$$\psi(x) = \begin{cases} cos(\frac{n\pi}{\tilde{a}}x), & \text{n odd} \\ sin(\frac{n\pi}{\tilde{a}}x), & \text{n even} \end{cases}$$

$$G(x, x') = \sum_n c_n \psi_n(x)$$

$$\frac{d^2}{dx^2} \sum_n c_n \psi_n(x) - \frac{1}{L^2} \sum_n c_n \psi_n(x) = -\frac{1}{D} \sum_n s_n \psi_n(x)$$

$$\sum_n c_n \left[\frac{d^2}{dx^2} \psi_n(x) - \frac{1}{L^2} \psi_n(x) \right] = -\frac{1}{D} \frac{2}{\tilde{a}} \sum_n \psi_n(x') \psi_n(x)$$

$$s_n = \int_{-\frac{\tilde{a}}{2}}^{\frac{\tilde{a}}{2}} dx S(x) \psi_n(x)$$

To normalize s_n, we introduce a correction factor:

$$s_n = \frac{2}{\tilde{a}} \int_{-\frac{\tilde{a}}{2}}^{\frac{\tilde{a}}{2}} dx S(x) \psi_n(x)$$

$$\int_{-\frac{\tilde{a}}{2}}^{\frac{\tilde{a}}{2}} dx \psi_n(x) \psi_m(x) = \begin{cases} \frac{\tilde{a}}{2}, n = m \\ 0, n \neq m \end{cases}$$

$$s_n = \frac{2}{\tilde{a}} \int_{-\frac{\tilde{a}}{2}}^{\frac{\tilde{a}}{2}} dx \delta(x - x')\psi_n(x) = \frac{2}{\tilde{a}} \psi_n(x')$$

Now multiply both sides of the summation equation by $\psi_n(x)$ and integrate over space:

$$\int_{-\frac{\tilde{a}}{2}}^{\frac{\tilde{a}}{2}} dx\psi_n(x) \sum_n c_n \left[\frac{d^2}{dx^2}\psi_n(x) - \frac{1}{L^2}\psi_n(x)\right] = \int_{-\frac{\tilde{a}}{2}}^{\frac{\tilde{a}}{2}} dx - \frac{1}{D}\frac{2}{\tilde{a}} \sum_n \psi_n(x')\psi_n(x)\psi_n(x)$$

$$c_n \left(B_n^2 + \frac{1}{L^2}\right) \frac{\tilde{a}}{2} = \frac{1}{D}\frac{2}{\tilde{a}}\frac{\tilde{a}}{2}\psi_n(x')$$

$$c_n = \frac{2}{\tilde{a}\Sigma_a}\frac{\psi_n(x')}{1 + B_n^2 L^2}$$

$$G(x, x') = \sum_n \frac{2}{\tilde{a}\Sigma_a}\frac{\psi_n(x')}{1 + B_n^2 L^2}\psi_n(x)$$

$$\phi(x) = \int_{-\frac{\tilde{a}}{2}}^{\frac{\tilde{a}}{2}} dxG(x, x')S(x) = \int_{-\frac{\tilde{a}}{2}}^{\frac{\tilde{a}}{2}} dx \sum_n \frac{2}{\tilde{a}\Sigma_a}\frac{\psi_n(x')}{1 + B_n^2 L^2}\psi_n(x)S_0\delta(x) = \frac{2S_0}{\tilde{a}\Sigma_a}\sum_n \frac{\psi_n(x\psi_n(0))}{1 + B_n^2 L^2}$$

$$\psi_n(0) = \begin{cases} 0, & \text{n even} \\ 1, & \text{n odd} \end{cases}$$

$$\phi(x) = \frac{2S_0}{\tilde{a}\Sigma_a}\sum_{\text{n odd}} \frac{cos(\frac{n\pi}{\tilde{a}}x)}{1 + B_n^2 L^2}$$

$$\phi(x) = \frac{S_0 L}{2D}\frac{sinh(\frac{(\tilde{a} - 2|x|)}{L})}{cosh\frac{\tilde{a}}{2L}}$$

Fission Source

$$S(\vec{r}, E, \hat{\Omega}, t) = S_{ext}(\vec{r}, E, \hat{\Omega}, t) + \int_0^\infty dE' \int_{4\pi} d\hat{\Omega}'\nu(E')\Sigma_f(\vec{r}, E')\phi(\vec{r}, E', \hat{\Omega}', t)$$

Of all the fission neutrons, we are only interested in the ones in the energy range $[E + dE]$ and in the space $d\hat{\Omega}$ about $\hat{\Omega}$. The fission term then becomes

$$\frac{\chi(E)}{4\pi}\int_0^\infty dE' \int_{4\pi} d\hat{\Omega}'\nu(E')\Sigma_f(\vec{r}, E')\phi(\vec{r}, E', \hat{\Omega}', t)$$

Integrating the entire source term equation over angle gives

$$\int_{4\pi} d\hat{\Omega}' S(\vec{r}, E, \hat{\Omega}, t) = S_{ext}(\vec{r}, E, t) + \chi(E)\int_0^\infty dE'\nu(E')\Sigma_f(\vec{r}, E')\phi(\vec{r}, E', t)$$

With the one-group approximation:

$$\int_0^\infty dES(\vec{r}, E, t) = S_{ext}(\vec{r}, t) + \nu\Sigma_f(\vec{r})\phi(\vec{r}, t)$$

Effective value:

$$\langle\nu\Sigma_f(\vec{r})\rangle = \frac{\int_0^\infty dE\nu(E)\Sigma_f(\vec{r}, E)\phi(\vec{r}, E, t)}{\int_0^\infty dE\phi(\vec{r}, E, t)}$$

We can now incorporate a fission source into the one-group diffusion equation:

$$\frac{1}{v}\frac{\partial\phi(\vec{r}, t)}{\partial t} = \nu\Sigma_f(\vec{r})\phi(\vec{r}, t) - \Sigma_a(\vec{r})\phi(\vec{r}, t) + \nabla\cdot[D(\vec{r})\nabla\phi(\vec{r}, t)]$$

Assume a steady-state homogeneous system.

$$\nu\Sigma_f(\vec{r})\phi(\vec{r}) = \Sigma_a(\vec{r})\phi(\vec{r}) - D(\vec{r})\nabla^2\phi(\vec{r})$$

$$k = \frac{\int_V dV\nu\Sigma_f(\vec{r})\phi(\vec{r})}{\int_V dV[\Sigma_a(\vec{r})\phi(\vec{r}) - D(\vec{r})\nabla^2\phi(\vec{r})]}$$

$$\nabla^2\phi(\vec{r}) + \frac{k_\infty - 1}{L^2}\phi(\vec{r}) = 0$$

$$k_\infty = \frac{\nu \Sigma_f}{\Sigma_a}$$

$$L^2 = \frac{D}{\Sigma_a}$$

For a bare reactor in a vacuum:

$$\phi(\tilde{\vec{r}}) = 0$$
$$\nabla^2 \psi(\vec{r}) + B^2 \psi(\vec{r}) = 0$$
$$\psi(\tilde{\vec{r}}) = 0$$

The system is critical iff $B^2 = \frac{k_\infty - 1}{L^2}$.

$$B_g^2 \equiv \frac{-\nabla^2 \psi(\vec{r})}{\psi(\vec{r})} = \text{ geometric buckling, proportional to the curvature of the flux}$$

$$B_m^2 \equiv \frac{k_\infty - 1}{L^2} = \text{ material buckling}$$

For a bare critical system, $B_g^2 = B_m^2$.

If $k < 1$, $\nu \Sigma_f \phi < \Sigma_a \phi - D \nabla^2 \phi$, $\nabla^2 \phi + B_m^2 \phi < 0$, $B_g^2 > B_m^2$.

If $k > 1$, $\nu \Sigma_f \phi > \Sigma_a \phi - D \nabla^2 \phi$, $\nabla^2 \phi + B_m^2 \phi > 0$, $B_g^2 < B_m^2$.

Steady-state non-critical systems can exist with non-zero external sources.

$$S_{ext} + \nu \Sigma_f(\vec{r}) \phi(\vec{r}) = \Sigma_a(\vec{r}) \phi(\vec{r}) - D(\vec{r}) \nabla^2 \phi(\vec{r})$$

A "balance factor" can be used to rewrite this equation:

$$\lambda \nu \Sigma_f(\vec{r}) \phi_\lambda(\vec{r}) = \Sigma_a(\vec{r}) \phi_\lambda(\vec{r}) - D(\vec{r}) \nabla^2 \phi_\lambda(\vec{r}),\ \lambda = \text{balance factor}$$

Here, ϕ_λ is physical $(0 \leq \phi_\lambda(\vec{r}) < \infty)$ but not realized. The multiplication factor k can be defined in terms of λ:

$$k = \frac{1}{\lambda} = \frac{\int_V dV \nu \Sigma_f(\vec{r})\phi(\vec{r})}{\int_V dV [\Sigma_a(\vec{r})\phi(\vec{r}) + D(\vec{r})\nabla^2\phi(\vec{r})]}$$

$$\Sigma_a(\vec{r})\phi(\vec{r}) - D(\vec{r})\nabla^2\phi(\vec{r}) = \frac{1}{k}\nu\Sigma_f(\vec{r})\phi(\vec{r})$$