NE 250, F15

Recall the one-group diffusion equation:

$$\frac{1}{v_1}\frac{\partial \phi_1(\vec{r},t)}{\partial t} = S_1(\vec{r},t) - \Sigma_{a,1}(\vec{r})\phi_1(\vec{r},t) + \nabla \cdot [D_1(\vec{r})\nabla\phi_1(\vec{r},t)]$$

Now, assume that space and energy dependence of the flux can be separated:

$$\phi(\vec{r},E,t) = \phi(\vec{r},t)\psi(E), \text{ where } \psi(E) \text{ is the neutron spectrum and } \int_0^\infty dE\psi(E) = 1.$$

Focusing on the spatial dependence of the flux, we'll assume a homogeneous steady-state, one-group system.

$$\frac{1}{v}\frac{\partial \phi(\vec{r},t)}{\partial t} = S(\vec{r},t) - \Sigma_s(\vec{r})\phi(\vec{r},t) + \nabla \cdot [D(\vec{r})\nabla\phi(\vec{r},t)]$$

Steady-state: $\frac{1}{v}\frac{\partial \phi(\vec{r},t)}{\partial t} = 0$

Homogeneous: no material dependence on position; $\Sigma_a(\vec{r}) \to \Sigma_a, D(\vec{r}) \to D$

This gives us

$$0 = S(\vec{r}) - \Sigma_a\phi(\vec{r}) + D\nabla^2\phi(\vec{r})$$

which can be rewritten as

$$\nabla^2\phi(\vec{r}) - \frac{1}{L^2}\phi(\vec{r}) = -\frac{S(\vec{r})}{D}, \text{ where } L = \sqrt{\frac{D}{\Sigma_a}} = \text{diffusion length.}$$

$$L = \sqrt{\frac{\langle r^2\rangle}{6}}, \langle r^2\rangle = \text{ root mean square distance from birth to absorption}$$

Now, consider a plane source of strength S_0 in an infinitely absorbing medium.

$$\phi(\vec{r}) = \phi(x)$$

$$\frac{d^2\phi}{dx^2} - \frac{1}{L^2}\phi(x) = -\frac{S_0\delta(x)}{D}$$

For $x > 0$, $\frac{d^2\phi}{dx^2} - \frac{1}{L^2}\phi(x) = 0$.

Boundary conditions: $\lim_{x\to 0^+} \vec{J}(x) = \frac{S_0}{2}$, $\lim_{x\to +\infty} |\phi(x)| < \infty, \phi(x) \geq 0$

For $x > 0$, $\frac{d^2\phi}{dx^2} - \frac{1}{L^2}\phi(x) = 0$.

Boundary conditions: $\lim_{x\to 0^-} \vec{J}(x) = -\frac{S_0}{2}$, $\lim_{x\to -\infty} |\phi(x)| < \infty, \phi(x) \geq 0$

With the above equations and boundary conditions, we have a general solution form of

$$\phi(x) = c_1 e^{-x/L} + c_2 e^{x/L}.$$

From the finite flux condition, $c_2 = 0$. Then, $\phi(x) = c_1 e^{-x/L}$.

$$\lim_{x\to 0^+} \vec{J}(x) = \lim_{x\to 0^+} \left(\frac{D}{L} c_1 e^{-x/L}\right) = \frac{D}{L} c_1 = \frac{S_0}{2}$$

$$c_1 = \frac{S_0 L}{2D}$$

$$\phi(x) = \frac{S_0 L}{2D} e^{-x/L}$$

Now, consider an infinite plane centered in a slab of finite thickness a, surrounded by a vacuum.

For $x > 0$, $\frac{d^2\phi}{dx^2} - \frac{1}{L^2}\phi(x) = 0$.

$$\phi(x) = c_1 cosh(\frac{x}{L}) + c_2 sinh(\frac{x}{L})$$

Boundary conditions: $\lim_{x\to 0^+} \vec{J}(x) = \frac{S_0}{2}, \phi(\frac{\tilde{a}}{2}) = 0$