NE 250, F15

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We have now eliminated the dependence of ϕ on $\hat{\Omega}$, so let's take a look at the two equations at which we arrived last time. In 1D,

$$\frac{1}{v}\frac{\partial \phi(z,E,t)}{\partial t} = S(z,E,t) + \int_0^\infty dE' \Sigma_s(E' \to E)\phi(z,E,t) - \Sigma_t(E)\phi(z,E,t) - \frac{\partial}{\partial z}J(z,E,t)$$

$$\frac{1}{v}\frac{\partial J(z,E,t)}{\partial t} = S_1(z,E,t) + \int_0^\infty dE' \bar{\mu}_0 \Sigma_s(E' \to E)J(z,E,t) - \Sigma_t(E)J(z,E,t) - \frac{1}{3}\frac{\partial}{\partial z}\phi(z,E,t)$$

In 3D,

$$\frac{1}{v}\frac{\partial \phi(\vec{r},E,t)}{\partial t} = S(\vec{r},E,t) + \int_0^\infty dE' \Sigma_s(E' \to E)\phi(\vec{r},E,t) - \Sigma_t(E)\phi(\vec{r},E,t) - \nabla \cdot \vec{J}(\vec{r},E,t)$$

$$\frac{1}{v}\frac{\partial \vec{J}(\vec{r},E,t)}{\partial t} = S_1(\vec{r},E,t) + \int_0^\infty dE' \bar{\mu}_0 \Sigma_s(E' \to E)\vec{J}(\vec{r},E,t) - \Sigma_t(E)\vec{J}(\vec{r},E,t) - \frac{1}{3}\nabla\phi(\vec{r},E,t)$$

Assuming that all neutrons have the same energy ("one-speed approximation"),

$$\frac{1}{v}\frac{\partial \phi(\vec{r},t)}{\partial t} = S(\vec{r},t) + \Sigma_s\phi(\vec{r},t) - \Sigma_t\phi(\vec{r},t) - \nabla \cdot \vec{J}(\vec{r},t)$$

$$\frac{1}{v}\frac{\partial \vec{J}(\vec{r},t)}{\partial t} = S_1(\vec{r},t) + \bar{\mu}_0\Sigma_s\vec{J}(\vec{r},t) - \Sigma_t\vec{J}(\vec{r},t) - \frac{1}{3}\nabla\phi(\vec{r},t)$$

Now, assume that the source is isotropic:

$$S(\vec{r},E,\hat{\Omega},t) = \frac{S(\vec{r},E,t)}{4\pi}$$

$$S_1(\vec{r},E,\hat{\Omega},t) = \int_{4\pi} d\hat{\Omega} S(\vec{r},E,\hat{\Omega},t)\hat{\Omega} = \frac{S(\vec{r},E,t)}{4\pi}\int_{4\pi} d\hat{\Omega}\hat{\Omega} = 0$$

Rearranging the current equation gives:

$$\frac{1}{|\vec{J}(\vec{r},t)|}\frac{\partial \vec{J}(\vec{r},t)}{\partial t} = \frac{\bar{\mu_0}\Sigma_s v \vec{J}(\vec{r},t)}{|\vec{J}(\vec{r},t)|} - v\Sigma_t - \frac{1}{3|\vec{J}(\vec{r},t)|}\nabla\phi(\vec{r},t)$$

In general, $\frac{1}{|\vec{J}(\vec{r},t)|}\frac{\partial \vec{J}(\vec{r},t)}{\partial t} \ll v\Sigma_t$, so we assume/approximate $\frac{1}{|\vec{J}(\vec{r},t)|}\frac{\partial \vec{J}(\vec{r},t)}{\partial t} \approx 0$. The current equation then becomes

$$\vec{J}(\vec{r},t) = \frac{-1}{3(\Sigma_t - \bar{\mu_0}\Sigma_s)}\nabla\phi(\vec{r},t),$$

which is known as Fick's law. Let us define the diffusion coefficient as

$$D(\vec{r}) = \frac{1}{3(\Sigma_t - \bar{\mu_0}\Sigma_s)} = \frac{1}{3\Sigma_{tr}},$$

where $\Sigma_{tr} = \Sigma_t - \bar{\mu_0}\Sigma_s$ is the "transport" cross section. Also note that

$$\bar{\mu_0} \approx \frac{2}{3A} > 0,$$

which means that scattering is forward-biased in the lab frame. For large A, $\bar{\mu_0} \to 0$, meaning that scattering is isotropic in the lab frame for large target nuclei. Plugging Fick's law back into the flux equation gives

$$\frac{1}{v}\frac{\partial\phi(\vec{r},t)}{\partial t} = S(\vec{r},t) - \Sigma_a\phi(\vec{r},t) + \nabla\cdot[D(\vec{r})\nabla\phi(\vec{r},t)],$$

which is the one-speed diffusion equation. The diffusion equation is of interest for several reasons. The P_1 approximation is not valid at interface boundaries, in highly absorbing media, or near sources. The diffusion equation is valid a few mean paths away from a source or a boundary but is also not valid in highly absorbing media.

Now, assume an isotropic flux $\phi(\vec{r},E,\hat{\Omega},t)$. The net current, $\vec{J}(\vec{r},\hat{\Omega},t)$i, is then zero.

Next, consider $J_x(\vec{r},\hat{\Omega},t) > 0$, $J_y(\vec{r},\hat{\Omega},t) = J_z(\vec{r},\hat{\Omega},t) = 0$.

With the P_1 approximation, we have

$$\phi(\vec{r}, E, \hat{\Omega}, t) \approx \frac{1}{4\pi}\phi(\vec{r}, E, t) + \frac{3}{4\pi}\nabla\vec{J}(\vec{r}, t)\cdot\hat{\Omega}$$

In order for both sides of this equation to be positive, it is required that

$$-\vec{J}(\vec{r}, t)\cdot\hat{\Omega} < \frac{1}{3}\phi(\vec{r}, E, t).$$

For this to be valid, we require that

$$J_x < \frac{1}{3}\phi(\vec{r}, E, t).$$

To solve the diffusion equation, we need initial and boundary conditions.

Initial condition: $\phi(\vec{r}, 0) = \phi_0(\vec{r})\forall\vec{r}$

Interface boundary conditions:

Transport

$\phi_1(\vec{r}_s, E, \hat{\Omega}, t) = \phi_2(\vec{r}_s, E, \hat{\Omega}, t)$
$\forall\vec{r}_s \in S, S \equiv \partial V, \forall E, \forall\hat{\Omega}, \forall t$

Diffusion

$\phi_1(\vec{r}_s, t) = \phi_2(\vec{r}_s, t)$
$\forall\vec{r}_s \in S, S \equiv \partial V, \forall t$
$\vec{J}_1(\vec{r}_s, t) = \vec{J}_2(\vec{r}_s, t)$

Vacuum boundary conditions:

Transport

$\phi(\vec{r}_s, E, \hat{\Omega}, t) = 0$
$\forall\vec{r}_s \in S, S \equiv \partial V, \forall E, \forall\hat{\Omega} : \hat{\Omega}\cdot\hat{e}_s < 0, \forall t$

Diffusion

$$\begin{aligned}\vec{J}_-(\vec{r}_s) &= \int_{2\pi-} d\hat{\Omega}\hat{\Omega}\cdot\hat{e}_s\left[\frac{\phi(\vec{r}, t)}{4\pi} + \frac{3}{4\pi}\hat{\Omega}\cdot\vec{J}(\vec{r}, t)\right] \\ &= \frac{1}{4}\phi(\vec{r}_s, t) + \frac{D(\vec{r})}{2}\nabla\phi(\vec{r}_s, t) \\ &= 0\end{aligned}$$

$$\vec{J}(\vec{r}, t) = \int_{4\pi} d\hat{\Omega}\hat{\Omega}\phi(\vec{r}, E, \hat{\Omega}, t) = \text{current at } \vec{r}$$

$$J = \int_S ds\vec{J}(\vec{r}, t)\cdot\hat{e}_s = \text{current through a surface } S$$

$$J_{net} = \int_{4\pi} d\hat{\Omega}\hat{\Omega} \cdot \hat{e}_s \phi(\vec{r}, E, \hat{\Omega}, t) = \text{net current at } \vec{r} \text{ with respect to } \hat{e}_s$$

Scalar currents:

$$J_{\pm} = \int_{2\pi^{\pm}} d\hat{\Omega}\hat{\Omega} \cdot \hat{e}_s \phi(\vec{r}, E, \hat{\Omega}, t)$$

$$2\pi^{+} \equiv 0 < \theta < \pi/2, 2\pi^{-} \equiv \pi/2 < \theta < \pi$$

In 1D,

$$J_{-}(z_s, t) = \frac{1}{4}\phi(z_s, t) + \frac{D(z_s)}{2}\frac{d\phi(z_s, t)}{dz}\Big|_{z_s} = 0$$

$$\frac{d\phi(z_s, t)}{dz}\Big|_{z_s} = \frac{-\phi(z_s, t)}{2D(z_s)}$$

In order to use the diffusion approximation, we must accept that the above solution at the vacuum is wrong, necessitating the introduction of an extrapolation distance:

$$\tilde{z}_s = z_s + 2D(z_s)$$

This allows us to obtain the correct solution away from the boundary.

And now, back to discussion of the P_1 approximation...

In the P_1 approximation of the neutron transport equation, the angular flux is linearly anisotropic. The approximation is valid away from boundaries, away from neutron sources and sinks, and in media that are not highly absorbing.

Other assumptions we'll introduce here are the one-speed approximation, an isotropic source term, and the approximation that $\frac{1}{|\vec{J}(\vec{r},t)|}\frac{\partial \vec{J}(\vec{r},t)}{\partial t} \ll v\Sigma_t$. This leads to the one-speed diffusion equation, which was derived above:

$$\frac{1}{v}\frac{\partial \phi(\vec{r}, t)}{\partial t} = S(\vec{r}, t) - \Sigma_a \phi(\vec{r}, t) + \nabla \cdot [D(\vec{r})\nabla\phi(\vec{r}, t)],$$

The boundary conditions necessary to solve this equation are listed above. Now, again consider the energy-dependent P_1 equations:

$$\frac{1}{v}\frac{\partial \phi(\vec{r},E,t)}{\partial t} = S(\vec{r},E,t) + \int_0^\infty dE' \Sigma_s(E' \to E)\phi(\vec{r},E,t) - \Sigma_t(E)\phi(\vec{r},E,t) - \nabla \cdot \vec{J}(\vec{r},E,t)$$

$$\frac{1}{v}\frac{\partial \vec{J}(\vec{r},E,t)}{\partial t} = S_1(\vec{r},E,t) + \int_0^\infty dE' \bar{\mu}_0 \Sigma_s(E' \to E)\vec{J}(\vec{r},E,t) - \Sigma_t(E)\vec{J}(\vec{r},E,t) - \frac{1}{3}\nabla\phi(\vec{r},E,t)$$

From the isotropic source assumption, $S_1(\vec{r},E,t) = 0$.

From the $\frac{1}{|\vec{J}(\vec{r},t)|}\frac{\partial \vec{J}(\vec{r},t)}{\partial t} \ll v\Sigma_t$ assumption, $\frac{1}{v}\frac{\partial \vec{J}(\vec{r},t)}{\partial t} = 0$.

With the one-speed approximation,

$$\vec{J}(\vec{r},t) \approx -D(\vec{r})\nabla\phi(\vec{r},t)$$

Similarly,

$$\vec{J}(\vec{r},E,t) \approx -D(\vec{r},E)\nabla\phi(\vec{r},E,t)$$

Plugging this into the first P_1 equation gives

$$\frac{1}{v}\frac{\partial \phi(\vec{r},E,t)}{\partial t} = S(\vec{r},E,t) + \int_0^\infty dE' \Sigma_s(E' \to E)\phi(\vec{r},E,t) - \Sigma_t(E)\phi(\vec{r},E,t) + \nabla \cdot [D(\vec{r},E)\nabla\phi(\vec{r},E,t)]$$

Now, let's make the *one-group* (not one-speed) assumption. This means that we will integrate the entire equation over all energy space $[\int_0^\infty dE(\cdot)]$.

"Group constants" are defined as follows:

$$\Sigma_{t,1}(\vec{r}) = \frac{\int_0^\infty dE \Sigma_t(\vec{r},E)\phi(\vec{r},E,t)}{\int_0^\infty dE \phi(\vec{r},E,t)} = \text{effective cross section}$$

$$\phi_1(\vec{r},t) = \int_0^\infty dE\phi(\vec{r},E,t) = \text{group flux}$$

Thus, $\int_0^\infty dE\Sigma_t(\vec{r},E)\phi(\vec{r},E,t) = \Sigma_{t,1}(\vec{r})\phi_1(\vec{r},t)$.

$$\int_0^\infty dE\frac{1}{v}\frac{\partial\phi(\vec{r},E,t)}{\partial t} = \frac{1}{v_1}\frac{\partial\phi_1(\vec{r},t)}{\partial t},\ \text{where } \frac{1}{v_1} = \frac{\int_0^\infty dE\frac{1}{v}\phi(\vec{r},E,t)}{\phi_1(\vec{r},t)}$$

$$\int_0^\infty dES(\vec{r},E,t) = S_1(\vec{r},t)$$

$$\int_0^\infty dE\int_0^\infty dE'\Sigma_s(E'\to E)\phi(\vec{r},E',t) = \int_0^\infty dE'\Sigma_s(E')\phi(\vec{r},E',t) = \Sigma_{s,1}\phi_1$$

$$\int_0^\infty dE\nabla\cdot[D(\vec{r},E)\nabla\phi(\vec{r},E,t)] = \nabla\cdot[D_1(\vec{r})\nabla\phi_1(\vec{r},t)]$$

Note that although the subscripts used here for the one-group approximation are the same as those used in the P_1 approximation derivation, the quantities are not the same. Combining all of the above terms, we have

$$\frac{1}{v_1}\frac{\partial\phi_1(\vec{r},t)}{\partial t} = S_1(\vec{r},t) - \Sigma_{a,1}(\vec{r})\phi_1(\vec{r},t) + \nabla\cdot[D_1(\vec{r})\nabla\phi_1(\vec{r},t)]$$

This is the *one-group* diffusion equation.