

NE 250, F15

Recall the one-group diffusion equation:

$$\frac{1}{v_1} \frac{\partial \phi_1(\vec{r}, t)}{\partial t} = S_1(\vec{r}, t) - \Sigma_{a,1}(\vec{r}) \phi_1(\vec{r}, t) + \nabla \cdot [D_1(\vec{r}) \nabla \phi_1(\vec{r}, t)]$$

Now, assume that space and energy dependence of the flux can be separated:

$$\phi(\vec{r}, E, t) = \phi(\vec{r}, t) \psi(E), \text{ where } \psi(E) \text{ is the neutron spectrum and } \int_0^\infty dE \psi(E) = 1.$$

Focusing on the spatial dependence of the flux, we'll assume a homogeneous steady-state, one-group system.

$$\frac{1}{v} \frac{\partial \phi(\vec{r}, t)}{\partial t} = S(\vec{r}, t) - \Sigma_s(\vec{r}) \phi(\vec{r}, t) + \nabla \cdot [D(\vec{r}) \nabla \phi(\vec{r}, t)]$$

$$\text{Steady-state: } \frac{1}{v} \frac{\partial \phi(\vec{r}, t)}{\partial t} = 0$$

Homogeneous: no material dependence on position; $\Sigma_a(\vec{r}) \rightarrow \Sigma_a$, $D(\vec{r}) \rightarrow D$

This gives us

$$0 = S(\vec{r}) - \Sigma_a \phi(\vec{r}) + D \nabla^2 \phi(\vec{r})$$

which can be rewritten as

$$\nabla^2 \phi(\vec{r}) - \frac{1}{L^2} \phi(\vec{r}) = -\frac{S(\vec{r})}{D}, \text{ where } L = \sqrt{\frac{D}{\Sigma_a}} = \text{diffusion length.}$$

$$L = \sqrt{\frac{\langle r^2 \rangle}{6}}, \langle r^2 \rangle = \text{root mean square distance from birth to absorption}$$

Now, consider a plane source of strength S_0 in an infinitely absorbing medium.

$$\phi(\vec{r}) = \phi(x)$$

$$\frac{d^2\phi(x)}{dx^2} - \frac{1}{L^2}\phi(x) = -\frac{S_0\delta(x)}{D}$$

For $x > 0$, $\frac{d^2\phi}{dx^2} - \frac{1}{L^2}\phi(x) = 0$.

Boundary conditions: $\lim_{x \rightarrow 0^+} \vec{J}(x) = \frac{S_0}{2}$, $\lim_{x \rightarrow +\infty} |\phi(x)| < \infty$, $\phi(x) \geq 0$

For $x < 0$, $\frac{d^2\phi}{dx^2} - \frac{1}{L^2}\phi(x) = 0$.

Boundary conditions: $\lim_{x \rightarrow 0^-} \vec{J}(x) = -\frac{S_0}{2}$, $\lim_{x \rightarrow -\infty} |\phi(x)| < \infty$, $\phi(x) \geq 0$

With the above equations and boundary conditions, we have a general solution form of

$$\phi(x) = c_1 e^{-x/L} + c_2 e^{x/L}.$$

From the finite flux condition, $c_2 = 0$. Then, $\phi(x) = c_1 e^{-x/L}$.

$$\lim_{x \rightarrow 0^+} \vec{J}(x) = \lim_{x \rightarrow 0^+} \left(\frac{D}{L} c_1 e^{-x/L} \right) = \frac{D}{L} c_1 = \frac{S_0}{2}$$

$$c_1 = \frac{S_0 L}{2D}$$

$$\phi(x) = \frac{S_0 L}{2D} e^{-x/L}$$

Now, consider an infinite plane centered in a slab of finite thickness a , surrounded by a vacuum.

For $x > 0$, $\frac{d^2\phi}{dx^2} - \frac{1}{L^2}\phi(x) = 0$.

Boundary conditions: $\lim_{x \rightarrow 0^+} \vec{J}(x) = \frac{S_0}{2}$, $\phi(\frac{\bar{a}}{2}) = 0$

For $x < 0$, $\frac{d^2\phi}{dx^2} - \frac{1}{L^2}\phi(x) = 0$.

Boundary conditions: $\lim_{x \rightarrow 0^-} \vec{J}(x) = -\frac{S_0}{2}$, $\phi(-\frac{\bar{a}}{2}) = 0$

$$\phi(x) = c_1 e^{-x/L} + c_2 e^{x/L}$$

or

$$\phi(x) = c_1 \cosh\left(\frac{x}{L}\right) + c_2 \sinh\left(\frac{x}{L}\right)$$

Now, consider a uniformly distributed source of strength $S_0 \frac{n}{cm^3 s}$ within a finite slab of width a with vacuum boundaries.

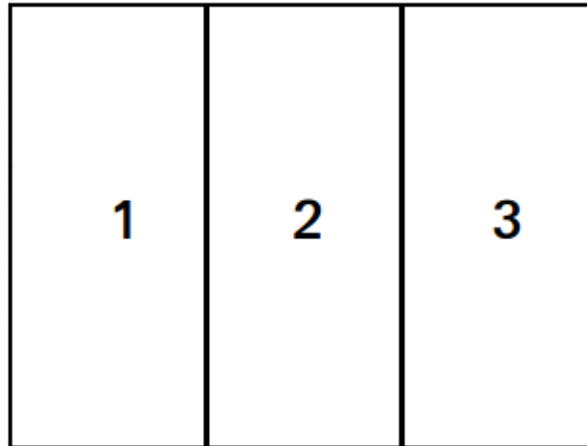
$$\frac{d^2 \phi(x)}{dx^2} - \frac{1}{L^2} \phi(x) = -\frac{S_0}{D}$$

$$\phi(x) = c_1 e^{-x/L} + c_2 e^{x/L} + \frac{S_0 L^2}{D}$$

Boundary conditions: $\phi(\pm \frac{a}{2}) = 0$

Alternative boundary condition: $\left. \frac{d\phi(x)}{dx} \right|_{x=0} = 0$ (symmetry)

Now, consider a uniform source in a reflected slab like so:



$$\frac{d^2 \phi_2(x)}{dx_2^2} - \frac{1}{L_2^2} \phi_2(x) = -\frac{S_0}{D_2}, -\frac{a}{2} < x < \frac{a}{2}$$

Boundary conditions:

$$\begin{aligned}\phi_2\left(\frac{a}{2}\right) &= \phi_3\left(\frac{a}{2}\right) \\ \vec{J}_2\left(\frac{a}{2}\right) &= \vec{J}_3\left(\frac{a}{2}\right) \\ \phi_1\left(-\frac{a}{2}\right) &= \phi_2\left(-\frac{a}{2}\right) \\ \vec{J}_1\left(-\frac{a}{2}\right) &= \vec{J}_3\left(-\frac{a}{2}\right)\end{aligned}$$

$$\frac{d^2\phi_1(x)}{dx_1^2} - \frac{1}{L_1^2}\phi_1(x) = 0, -\infty < x < -\frac{a}{2}$$

$$\frac{d^2\phi_3(x)}{dx_3^2} - \frac{1}{L_3^2}\phi_3(x) = 0, \frac{a}{2} < x < \infty$$

Boundary conditions: $\lim_{x \rightarrow -\infty} |\phi_1(x)| < \infty, \lim_{x \rightarrow \infty} |\phi_3(x)| < \infty$

What if we want to solve a generic diffusion problem?

$$D\nabla^2\phi(\vec{r}) - \Sigma_a\phi(\vec{r}) = -S(\vec{r})$$

Working in 1D:

$$M\phi(x) = f(x), \text{ where } M \text{ is an operator}$$

This is a non-homogenous order n differential equation.

$$a_0(x)\phi^{(n)}(x) + a_1(x)\phi^{(n-1)}(x) + \dots + a_n(x)\phi(x) = f(x)$$

One solution method is variation of constants.

Homogeneous problem: $M\phi(x) = 0$.

$$\phi_i(x), i = 1, \dots, n \text{ independent solutions}$$

$$\phi_{homog}(x) = \sum_{i=1}^n c_i \phi_i(x)$$

Assume that $c_i = c_i(x)$.

$$\sum_{i=1}^n c_i \phi(x) = \phi_{particular}(x)$$

If $\sum_{i=1}^n c'_i \phi_i(x) = 0$, $\sum_{i=1}^n c'_i \phi_i^{(n)}(x) = 0$, $\sum_{i=1}^n c'_i \phi_i^{(n-2)}(x) = 0$, etc.

$$\sum_{i=1}^n c'_i \phi_i^{(n-1)}(x) = \frac{f(x)}{a_0(x)}$$

$$c_i(x) = \int dx c'_i(x)$$

For $n = 2$:

$$\phi_1(x) c'_1(x) + \phi_2(x) c'_2(x) = 0$$

$$\phi_1^{(1)}(x) c'_1(x) + \phi_2^{(1)}(x) c'_2(x) = \frac{f(x)}{a_0(x)}$$

$$\phi_{part} = c_1(x) \phi_1(x) + c_2(x) \phi_2(x)$$

$$a_0(x) \phi^{(2)}(x) + a_1(x) \phi^{(1)}(x) + a_2(x) \phi(x) = 0$$

$$\phi^{(1)}(x) = c'_1(x) \phi_1(x) + c_1(x) \phi'_1(x) + c'_2(x) \phi_2(x) + c_2(x) \phi'_2(x)$$

$$c_i(x) = \int dx c'_i(x) = A_i + \psi_i(x)$$

$$\phi(x) = \sum_{i=1}^n A_i \phi_i(x) + \sum_{i=1}^n \psi_i(x) \phi_i(x)$$

For $n = 2$: $\phi(x) = A_1 \phi_1(x) + A_2 \phi_2(x) + \psi_1(x) \phi_1(x) + \psi_2(x) \phi_2(x)$

For a plane source in an infinite medium:

$$\frac{d^2\phi(x)}{dx^2} - \frac{1}{L^2}\phi(x) = -\frac{S\delta(x)}{D}$$

Homogeneous problem: $\frac{d^2\phi}{dx^2} - \frac{1}{L^2}\phi(x) = 0, \phi_1(x) = e^{-x/L}, \phi_2(x) = e^{x/L}$

$$e^{-x/L}c'_1(x) + e^{x/L}c'_2(x) = 0$$

$$-\frac{e^{-x/L}}{L}c'_1(x) + \frac{e^{x/L}}{L}c'_2(x) = -\frac{S}{D}\delta(x)$$

$$\phi_{part}(x) = c_1(x)\phi_1(x) + c_2(x)\phi_2(x)$$

$$c'_1(x) = \frac{SL}{2D}\delta(x)e^{x/L}$$

$$c'_2(x) = -\frac{SL}{2D}\delta(x)e^{-x/L}$$

$$c_1(x) = \int_{-\infty}^x dx \frac{SL}{2D}\delta(x)e^{x/L} = \frac{SL}{2D}H(x), \text{ where } H(x) \text{ is the Heaviside step function}$$

$$H(x) = \begin{cases} 1, & x > 0 \\ \frac{1}{2}, & x = 0 \\ 0, & x < 0 \end{cases}$$

$$c_2(x) = \int_{-\infty}^x dx \frac{-SL}{2D}\delta(x)e^{-x/L} = -\frac{SL}{2D}H(x),$$

$$\phi(x) = A_1\phi_1(x) + A_2\phi_2(x) + c_1(x)\phi_1(x) + c_2(x)\phi_2(x)$$

$$\phi(x) = \left(A_1 + \frac{SL}{2D}H(x)\right)e^{-x/L} + \left(A_2 - \frac{SL}{2D}H(x)\right)e^{x/L}$$

Boundary conditions:

$$\lim_{x \rightarrow \infty} |\phi(x)| < \infty \rightarrow A_2 = \frac{SL}{2D}$$

$$\lim_{x \rightarrow -\infty} |\phi(x)| < \infty \rightarrow A_1 = 0$$

$$\phi(x) = \frac{SL}{2D} H(x) e^{-x/L} + \frac{SL}{2D} [1 - H(x)] e^{x/L} = \frac{SL}{2D} e^{-|x|/L}$$

Other solution methods include the use of Green's function and the eigenfunction expansion method.

Green's function

$$\nabla \cdot [D(\vec{r}) \nabla \phi(\vec{r})] - \Sigma_a \phi(\vec{r}) = -S(\vec{r}), \forall \vec{r} \in V$$

$$\phi(\vec{r}) = 0, \forall \vec{r} \in S \equiv \partial V$$

Consider a unit source at \vec{r}' .

$G(\vec{r}, \vec{r}') = \text{flux at } \vec{r} \text{ due to a unit source } \vec{r}'$

$$\nabla \cdot [D(\vec{r}) \nabla G(\vec{r}, \vec{r}')] - \Sigma_a G(\vec{r}, \vec{r}') = -\delta(\vec{r} - \vec{r}'), \forall \vec{r} \in V$$

$$G(\vec{r}, \vec{r}') = 0, \forall \vec{r} \in S \equiv \partial V$$

$$\begin{aligned} \int_V dV \{ \nabla \cdot [D(\vec{r}) \nabla \phi(\vec{r})] G(\vec{r}, \vec{r}') - \nabla \cdot [D(\vec{r}) \nabla G(\vec{r}, \vec{r}')] \phi(\vec{r}) \} \\ = \int_S dS \hat{e}_s \cdot D(\vec{r}) [\nabla \phi(\vec{r}) \times G(\vec{r}, \vec{r}') - \phi(\vec{r}) \nabla G(\vec{r}, \vec{r}')] = 0 \end{aligned}$$

$$\int_V dV \{ S(\vec{r}) G(\vec{r}, \vec{r}') - \delta(\vec{r} - \vec{r}') \phi(\vec{r}) \} = 0$$

$$\phi(\vec{r}') = \int_V dV S(\vec{r}) G(\vec{r}, \vec{r}')$$

$$\phi(\vec{r}) = \int_V dV S(\vec{r}') G(\vec{r}', \vec{r})$$

Green's function ("kernel"): $G(\vec{r}, \vec{r}') = G(\vec{r}', \vec{r})$

The challenge is: how do we determine Green's function?

Consider a plane source in an infinite medium.

$$\frac{d^2\phi(x)}{dx^2} - \frac{1}{L^2}\phi(x) = -\frac{S}{D}\delta(x)$$

$$\lim_{x \rightarrow \pm\infty} \phi(x) = 0$$

$$\frac{d^2G(x, x')}{dx^2} - \frac{1}{L^2}G(x, x') = -\frac{\delta(x - x')}{D}$$

$$\lim_{x \rightarrow \pm\infty} G(x, x') = 0$$

Using variation of constants:

$$G(x, x') = \frac{L}{2D} H(x - x') e^{-(x-x')/L} + \frac{L}{2D} [1 - H(x - x')] e^{x-x'}$$

$$\phi(x) = \int_{-\infty}^{\infty} dx' G(x, x') \delta(x') S$$

$$\phi(x) = \frac{SL}{2D} e^{-|x|/L}$$

Eigenfunction expansion method

Term definitions:

$$Hf = \lambda f$$

f = function

H = operator

λ = scalar value

f_n = eigenfunction

λ_n = eigenvalue

Example: $x = x \rightarrow H = 1, f = x, \lambda = 1$

Consider a fairly general diffusion problem in a homogeneous volume.

$$\nabla^2 \phi(\vec{r}) - \frac{1}{L^2} \phi(\vec{r}) = -\frac{S(\vec{r})}{D}, \forall \vec{r} \in V$$

$$\phi(\vec{r}) = 0 \quad \forall \vec{r} \in S$$

Eigenproblem:

$$\nabla^2 \psi_n(\vec{r}) + B_n^2 \psi_n(\vec{r}) = 0 \quad \forall \vec{r} \in V$$

$$\nabla^2 \psi_n(\vec{r}) = -B_n^2 \psi_n(\vec{r}) \quad (H = \nabla^2, \lambda = -B_n^2)$$

$$\psi_n(\vec{r}) = 0 \quad \forall \vec{r} \in S$$

Orthonormality:

$$\int_V dV \psi_n(\vec{r}) \psi_m(\vec{r}) = 1 \text{ if } n = m, 0 \text{ if } n \neq m$$

Complete set:

$$\int_V dV \left(f - \sum_{n=1}^N f_n \psi_n \right)^2 < \varepsilon \quad (\text{for large } N \text{ and small but positive } \varepsilon)$$

$$\phi(\vec{r}) \approx \sum_{n=1}^N c_n \psi_n(\vec{r})$$

$$\nabla^2 \sum_{n=1}^N c_n \psi_n(\vec{r}) - \Sigma_a \sum_{n=1}^N c_n \psi_n(\vec{r}) = -\frac{1}{D} \sum_{n=1}^N s_n \psi_n(\vec{r})$$

$$\sum_{n=1}^N c_n [\nabla^2 \psi_n(\vec{r}) - \Sigma_a \psi_n(\vec{r})] = -\frac{1}{D} \sum_{n=1}^N s_n \psi_n(\vec{r})$$

$$\sum_{n=1}^N c_n (B_n^2 + \Sigma_a) \psi_n(\vec{r}) = \frac{1}{D} \sum_{n=1}^N s_n \psi_n(\vec{r})$$

$$\int_V dV \sum_{n=1}^N c_n (B_n^2 + \Sigma_a) \psi_n(\vec{r}) \psi_m(\vec{r}) = \int_V dV \frac{1}{D} \sum_{n=1}^N s_n \psi_n(\vec{r}) \psi_m(\vec{r})$$

Due to orthonormality,

$$c_n (B_n^2 + \Sigma_a) = \frac{1}{D} s_n$$

$$c_n = \frac{\frac{s_n}{\Sigma_a}}{1 + \frac{B_n^2}{L}}$$

$$s_n = \int_V dV S(\vec{r}) \psi_n(\vec{r})$$

$$\phi(\vec{r}) \approx \sum_{n=1}^N c_n \psi_n(\vec{r}) = \int_V dV \frac{1}{\Sigma_a} \sum_{n=1}^N \frac{\psi_n(\vec{r}')}{1 + L^2 B_n^2} \psi_n(\vec{r}) S(\vec{r}')$$

$$\frac{1}{\Sigma_a} \sum_{n=1}^N \frac{\psi_n(\vec{r}')}{1 + L^2 B_n^2} \psi_n(\vec{r}) = \text{Green's function}$$