

NE 250, F15
September 14, 2015

In solving the neutron transport equation, we're interested in solving for $\phi(\vec{r}, E, \hat{\Omega}, t)$.

One quantity in the neutron transport equation to be considered is the macroscopic cross section. We'll start off by noting that the microscopic cross section is a function of only energy $[\sigma(E)]$ while the macroscopic cross section depends on both space and energy $[\Sigma(\vec{r}, E) = N(\vec{r})\sigma(E)]$. This is because the macroscopic cross section is for an entire material and must take the number density of that material into account, which introduces a spatial dependence.

Absorption reactions (radiative capture, fission, etc.) are assumed to be isotropic (that is, independent of $\hat{\Omega}$), but this assumption doesn't hold for scattering: $\Sigma_s(\vec{r}, E' \rightarrow E, \hat{\Omega}' \rightarrow \hat{\Omega})$.

Additionally, leakage strongly depends on $\hat{\Omega}$. Since we're solving for $\phi(\vec{r}, E, \hat{\Omega}, t)$ and that solution is made difficult by other angularly-dependent variables, let's try to eliminate the dependence of ϕ on $\hat{\Omega}$. One approach to this simplification is the S_N (discrete ordinates) method, which changes $\hat{\Omega}$ from continuous to discrete ($\hat{\Omega} \rightarrow \hat{\Omega}_n; n = 1, \dots, N$). This also discretizes ϕ : $\phi_n = \phi(\hat{\Omega}_n)$.

This discretization changes the integral of ϕ over $\hat{\Omega}$ to a sum:

$$\int d\hat{\Omega} \phi(\hat{\Omega}) \rightarrow \sum_{n=1}^N w_n \phi_n,$$

where w_n is the n^{th} weighting factor. The neutron transport equation then becomes

$$\frac{1}{v} \frac{\partial \phi_n}{\partial t} = S_n + \int_0^\infty dE' \sum_{n'=1}^N \Sigma_s(\vec{r}, E' \rightarrow E, \hat{\Omega}' \rightarrow \hat{\Omega}) w_n \phi_{n'} - \Sigma_t \phi_n - \hat{\Omega}_n \cdot \nabla \phi_n.$$

A second type of discretization is the P_N method, which does a series expansion of ϕ :

$$\phi(\vec{r}, E, \hat{\Omega}, t) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} Y_{\ell m}(\hat{\Omega}) \phi(\ell, m)(\vec{r}, E, t),$$

where $Y_{\ell m}(\hat{\Omega})$ is a spherical harmonic. Notes on spherical harmonics can be found in Appendix D of "Computational Methods of Neutron Transport" by Lewis and Miller.

From here, let's assume 1D plane symmetry for the sake of simplicity. Using the P_N method, we

have

$$\phi(z, E, \mu, t) = \sum_{\ell=0}^{\infty} \frac{2\ell+1}{2} P_{\ell}(\mu) \phi_{\ell}(z, E, t),$$

where $P_{\ell}(\mu)$ is a Legendre polynomial. Important notes on Legendre polynomials:

$$\begin{aligned} P_0(\mu) &= 1 \\ P_1(\mu) &= \mu \\ \int_{-1}^1 P_n(\mu) P_m(\mu) d\mu &= \frac{2}{2n+1} \delta_{n,m} = \frac{2}{2n+1} \text{ if } n = m, 0 \text{ if } n \neq m. \end{aligned}$$

Using $N = 1$ with Legendre polynomials (the ‘‘P1 method’’) gives

$$\phi(z, E, \mu, t) = \frac{1}{2} P_0(\mu) \phi_0(z, E, t) + \frac{3}{2} P_1(\mu) \phi_1(z, E, t).$$

In general,

$$\phi_{\ell}(z, E, t) = \int_{-1}^1 d\mu \phi(z, E, \mu, t) P_{\ell}(\mu) = \sum_{\ell'=0}^{\infty} \frac{2\ell'+1}{2} P_{\ell'}(\mu) \phi_{\ell'} P_{\ell}(\mu).$$

Plugging the new approximate definition of ϕ into the transport equation gives

$$\begin{aligned} \frac{1}{v} \frac{\partial \phi_n}{\partial t} \left[\frac{1}{2} P_0(\mu) \phi_0(z, E, t) + \frac{3}{2} P_1(\mu) \phi_1(z, E, t) \right] &= \frac{1}{2} P_0(\mu) S_0 + \frac{3}{2} P_1(\mu) S_1 + \\ &\int_0^{\infty} dE' \int_{-1}^1 d\mu' \Sigma_s(\vec{r}, E' \rightarrow E, \hat{\Omega}' \rightarrow \hat{\Omega}) \left[\frac{1}{2} P_0(\mu') \phi_0(z, E, t) + \frac{3}{2} P_1(\mu') \phi_1(z, E, t) \right] \\ - \Sigma_t \left[\frac{1}{2} P_0(\mu) \phi_0(z, E, t) + \frac{3}{2} P_1(\mu) \phi_1(z, E, t) \right] &- \mu \frac{\partial}{\partial z} \left[\frac{1}{2} P_0(\mu) \phi_0(z, E, t) + \frac{3}{2} P_1(\mu) \phi_1(z, E, t) \right] \end{aligned}$$

Now, let's take the zeroth moment of this equation (that is, multiply it by $P_0(\mu)$) and integrate the equation over $\int_{-1}^1 d\mu$. Let $\mu_s = \cos(\theta' - \theta) = \hat{\Omega}' \cdot \hat{\Omega}$ such that $\Sigma_s(\mu' \rightarrow \mu) = \Sigma_s(\mu_s)$. Then,

$$\Sigma_s(\mu_s) = \frac{1}{2} P_0(\mu_s) \Sigma_{s,0} + \frac{3}{2} P_1(\mu_s) \Sigma_{s,1}$$

$$\int_{-1}^1 d\mu \int_{-1}^1 d\mu' \left[\frac{1}{2} P_0(\mu_s) \Sigma_{s,0} + \frac{3}{2} P_1(\mu_s) \Sigma_{s,1} \right] \left[\frac{1}{2} P_0(\mu') \phi_0(z, E, t) + \frac{3}{2} P_1(\mu') \phi_1(z, E, t) \right] P_0(\mu) =$$

$$\int_{-1}^1 d\mu \int_{-1}^1 d\mu' \frac{1}{2} P_0(\mu_s) \Sigma_{s,0} \frac{1}{2} P_0(\mu') \phi_0(z, E, t) P_0(\mu) = \Sigma_{s,0} \phi_0(z, E, t)$$

$$\int_{-1}^1 d\mu \mu \left[\frac{1}{2} P_0(\mu) \phi_0(z, E, t) + \frac{3}{2} P_1(\mu) \phi_1(z, E, t) \right] P_0(\mu) = \phi_1$$

This gives us

$$\frac{1}{v} \frac{\partial \phi_0}{\partial t} = S_0 + \Sigma_{s,0} \phi_0 - \Sigma_t \phi_0 - \frac{\partial \phi_1}{\partial z}$$

Now, let's take the first moment of the transport equation with the approximate ϕ definition and integrate it over $\int_{-1}^1 d\mu$:

$$\int_{-1}^1 d\mu \int_{-1}^1 d\mu' \left[\frac{1}{2} P_0(\mu_s) \Sigma_{s,0} + \frac{3}{2} P_1(\mu_s) \Sigma_{s,1} \right] \left[\frac{1}{2} P_0(\mu') \phi_0(z, E, t) + \frac{3}{2} P_1(\mu') \phi_1(z, E, t) \right] P_1(\mu) =$$

$$\int_{-1}^1 d\mu \int_{-1}^1 d\mu' \frac{3}{2} P_1(\mu_s) \Sigma_{s,1} \frac{3}{2} P_1(\mu') \phi_1(z, E, t) P_1(\mu) = \Sigma_{s,1} \phi_1(z, E, t)$$

$$\int_{-1}^1 d\mu \mu \left[\frac{1}{2} P_0(\mu) \phi_0(z, E, t) + \frac{3}{2} P_1(\mu) \phi_1(z, E, t) \right] P_1(\mu) = \frac{1}{3} \phi_0$$

This gives us

$$\frac{1}{v} \frac{\partial \phi_1}{\partial t} = S_1 + \Sigma_{s,1} \phi_1 - \Sigma_t \phi_1 - \frac{1}{3} \frac{\partial \phi_0}{\partial z}$$

Scalar flux:

$$\phi(z, E, t) = \phi_0(z, E, t) = \int_{-1}^1 d\mu \phi(z, E, \mu, t) P_0(\mu) = \int_{-1}^1 d\mu \phi(z, E, \mu, t)$$

Scalar current:

$$J(z, E, t) = \phi_1(z, E, t) = \int_{-1}^1 d\mu \phi(z, E, \mu, t) P_1(\mu) = \int_{-1}^1 d\mu \mu \phi(z, E, \mu, t)$$

Neutron source(s) summed over every direction:

$$S(z, E, t) = S_0(z, E, t) = \int_{-1}^1 d\mu S(z, E, \mu, t)$$

Scattering cross sections:

$$\begin{aligned}\Sigma_{s,0} &= \Sigma_s(E') \\ \Sigma_{s,1} &= \int_{-1}^1 d\mu_s \Sigma_s(\mu_s) P_1(\mu_s) \\ \frac{d\mu_s \Sigma_s(\mu_s)}{\Sigma_s} &= \text{probability that neutron will scatter in } d\mu_s \text{ about } \mu_s \\ \Sigma_{s,1} &= \int_{-1}^1 d\mu_s \frac{\Sigma_s \mu_s}{\Sigma_s} \mu_s \Sigma_s = \bar{\mu}_0 \Sigma_s, \text{ where } \bar{\mu}_0 = \text{average scattering angle}\end{aligned}$$