NE 250, F15

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In solving the neutron transport equation, we're interested in solving for $\phi(\vec{r}, E, \hat{\Omega}, t)$.

One quantity in the neutron transport equation to be considered is the macroscopic cross section. We'll start off by noting that the microscopic cross section is a function of only energy [$\sigma(E))$] while the macroscopic cross section depends on both space and energy [$\Sigma(\vec{r}, E) = N(\vec{r})\sigma(E)$]. This is because the macroscopic cross section is for an entire material and must take the number density of that material into account, which introduces a spatial dependence.

Absorption reactions (radiative capture, fission, etc.) are assumed to be isotropic (that is, independent of $\hat{\Omega}$), but this assumption doesn't hold for scattering: $\Sigma_s(\vec{r}, E' \to E, \hat{\Omega}' \to \hat{\Omega})$.

Additionally, leakage strongly depends on $\hat{\Omega}$. Since we're solving for $\phi(\vec{r}, E, \hat{\Omega}, t)$ and that solution is made difficult by other angularly-dependent variables, let's try to eliminate the dependence of ϕ on $\hat{\Omega}$. One approach to this simplification is the S_N (discrete ordinates) method, which changes $\hat{\Omega}$ from continuous to discrete ($\hat{\Omega} \to \hat{\Omega}_n; n = 1, \ldots, N$). This also discretizes ϕ: $\phi_n = \phi(\hat{\Omega}_n)$.

This discretization changes the integral of ϕ over $\hat{\Omega}$ to a sum:

$$\int d\hat{\Omega}\phi(\hat{\Omega}) \to \sum_{n=1}^{N} w_n \phi_n,$$

where w_n is the n^{th} weighting factor. The neutron transport equation then becomes

$$\frac{1}{v}\frac{\partial \phi_n}{\partial t} = S_n + \int_0^\infty dE' \sum_{n'=1}^{N} \Sigma_s(\vec{r}, E' \to E, \hat{\Omega}' \to \hat{\Omega}) w_n \phi_n - \Sigma_t \phi_n - \hat{\Omega}_n \cdot \nabla \phi_n.$$

A second type of discretization is the P_N method, which does a series expansion of ϕ:

$$\phi(\vec{r}, E, \hat{\Omega}, t) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} Y_{\ell m}(\hat{\Omega}) \phi(\ell, m)(\vec{r}, E, t),$$

where $Y_{\ell m}(\hat{\Omega})$ is a spherical harmonic. Notes on spherical harmonics can be found in Appendix D of "Computational Methods of Neutron Transport" by Lewis and Miller.

From here, let's assume 1D plane symmetry for the sake of simplicity. Using the P_N method, we have

$$\phi(z, E, \mu, t) = \sum_{\ell=0}^{\infty} \frac{2\ell+1}{2} P_\ell(\mu)\phi_\ell(z, E, t),$$

where $P_\ell(\mu)$ is a Legendre polynomial. Important notes on Legendre polynomials:

$$P_0(\mu) = 1$$

$$P_1(\mu) = \mu$$

$$\int_{-1}^{1} P_n(\mu)P_m(\mu)d\mu = \frac{2}{2n+1}\delta_{n,m} = \frac{2}{2n+1}\text{if n = m, 0 if n} \neq \text{m.}$$

Using $N = 1$ with Legendre polynomials (the "P1 method") gives

$$\phi(z, E, \mu, t) = \frac{1}{2}P_0(\mu)\phi_0(z, E, t) + \frac{3}{2}P_1(\mu)\phi_1(z, E, t).$$

In general,

$$\phi_\ell(z, E, t) = \int_{-1}^{1} d\mu\phi(z, E, \mu, t)P_\ell(\mu) = \sum_{\ell'=0}^{\infty} \frac{2\ell'+1}{2} P_{\ell'}(\mu)\phi_{\ell'}P_\ell(\mu).$$

Plugging the new approximate definition of ϕ into the transport equation gives

$$\begin{aligned}
&\frac{1}{v}\frac{\partial\phi_n}{\partial t}\left[\frac{1}{2}P_0(\mu)\phi_0(z, E, t) + \frac{3}{2}P_1(\mu)\phi_1(z, E, t)\right] = \frac{1}{2}P_0(\mu)S_0 + \frac{3}{2}P_1(\mu)S_1 + \\
&\quad\int_0^\infty dE' \int_{-1}^{1} d\mu' \Sigma_s(\vec{r}, E' \to E, \hat{\Omega}' \to \hat{\Omega}) \left[\frac{1}{2}P_0(\mu')\phi_0(z, E, t) + \frac{3}{2}P_1(\mu')\phi_1(z, E, t)\right] \\
&-\Sigma_t\left[\frac{1}{2}P_0(\mu)\phi_0(z, E, t) + \frac{3}{2}P_1(\mu)\phi_1(z, E, t)\right] - \mu\frac{\partial}{\partial z}\left[\frac{1}{2}P_0(\mu)\phi_0(z, E, t) + \frac{3}{2}P_1(\mu)\phi_1(z, E, t)\right]
\end{aligned}$$

Now, let's take the zeroth moment of this equation (that is, multiply it by $P_0(\mu)$) and integrate the equation over $\int_{-1}^{1} d\mu$. Let $\mu_s = cos(\theta' - \theta) = \hat{\Omega}' \cdot \hat{\Omega}$ such that $\Sigma_s(\mu' \to \mu) = \Sigma_s(\mu_s)$. Then,

$$\Sigma_s(\mu_s) = \frac{1}{2}P_0(\mu_s)\Sigma_{s,0} + \frac{3}{2}P_1(\mu_s)\Sigma_{s,1}$$

$$\int_{-1}^{1} d\mu \int_{-1}^{1} d\mu' \left[\frac{1}{2}P_0(\mu_s)\Sigma_{s,0} + \frac{3}{2}P_1(\mu_s)\Sigma_{s,1}\right]\left[\frac{1}{2}P_0(\mu')\phi_0(z,E,t) + \frac{3}{2}P_1(\mu')\phi_1(z,E,t)\right] P_0(\mu) =$$

$$\int_{-1}^{1} d\mu \int_{-1}^{1} d\mu' \frac{1}{2}P_0(\mu_s)\Sigma_{s,0}\frac{1}{2}P_0(\mu')\phi_0(z,E,t)P_0(\mu) = \Sigma_{s,0}\phi_0(z,E,t)$$

$$\int_{-1}^{1} d\mu\mu \left[\frac{1}{2}P_0(\mu)\phi_0(z,E,t) + \frac{3}{2}P_1(\mu)\phi_1(z,E,t)\right] P_0(\mu) = \phi_1$$

This gives us

$$\frac{1}{v}\frac{\partial \phi_0}{\partial t} = S_0 + \Sigma_{s,0}\phi_0 - \Sigma_t\phi_0 - \frac{\partial \phi_1}{\partial z}$$

Now, let's take the first moment of the transport equation with the approximate ϕ definition and integrate it over $\int_{-1}^{1} d\mu$:

$$\int_{-1}^{1} d\mu \int_{-1}^{1} d\mu' \left[\frac{1}{2}P_0(\mu_s)\Sigma_{s,0} + \frac{3}{2}P_1(\mu_s)\Sigma_{s,1}\right]\left[\frac{1}{2}P_0(\mu')\phi_0(z,E,t) + \frac{3}{2}P_1(\mu')\phi_1(z,E,t)\right] P_1(\mu) =$$

$$\int_{-1}^{1} d\mu \int_{-1}^{1} d\mu' \frac{3}{2}P_1(\mu_s)\Sigma_{s,1}\frac{3}{2}P_1(\mu')\phi_1(z,E,t)P_1(\mu) = \Sigma_{s,1}\phi_1(z,E,t)$$

$$\int_{-1}^{1} d\mu\mu \left[\frac{1}{2}P_0(\mu)\phi_0(z,E,t) + \frac{3}{2}P_1(\mu)\phi_1(z,E,t)\right] P_1(\mu) = \frac{1}{3}\phi_0$$

This gives us

$$\frac{1}{v}\frac{\partial \phi_1}{\partial t} = S_1 + \Sigma_{s,1}\phi_1 - \Sigma_t\phi_1 - \frac{1}{3}\frac{\partial \phi_0}{\partial z}$$

Scalar flux:

$$\phi(z,E,t) = \phi_0(z,E,t) = \int_{-1}^{1} d\mu\phi(z,E,\mu,t)P_0(\mu) = \int_{-1}^{1} d\mu\phi(z,E,\mu,t)$$

Scalar current:

$$J(z,E,t) = \phi_1(z,E,t) = \int_{-1}^{1} d\mu \phi(z,E,\mu,t) P_1(\mu) = \int_{-1}^{1} d\mu \mu \phi(z,E,\mu,t)$$

Neutron source(s) summed over every direction:

$$S(z,E,t) = S_0(z,E,t) = \int_{-1}^{1} d\mu S(z,E,\mu,t)$$

Scattering cross sections:

$$\Sigma_{s,0} = \Sigma_s(E')$$

$$\Sigma_{s,1} = \int_{-1}^{1} d\mu_s \Sigma_s(\mu_s) P_1(\mu_s)$$

$$\frac{d\mu_s \Sigma_s(\mu_s)}{\Sigma_s} = \text{probability that neutron will scatter in } d\mu_s \text{ about } \mu_s$$

$$\Sigma_{s,1} = \int_{-1}^{1} d\mu_s \frac{\Sigma_s \mu_s}{\Sigma_s} \mu_s \Sigma_s = \bar{\mu_0} \Sigma_s, \text{ where } \bar{\mu_0} = \text{average scattering angle}$$