Completely positive maps and Stinespring's theorem

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Fragen:

- Why only consider countably many states? i.e. why only $\mathbb{C}^n$ and not arbitrary Hilbert spaces?

In the following let H be a Hilbert space, $\mathbb{B}(H)$ and $\mathcal{L}(H)$ the space of bounded linear operators and linear operators $H \to H$, respectively.

Let $\mathfrak{O}$ denote a C^*-algebra of observables.

What should be the most general evolution of a quantum systems? While in closed systems, evolutions $0 \mapsto t$ are given by unitary transformations

$$\psi_S \mapsto U(t)\psi_S \quad \psi_S \in H \tag{1}$$

on states and

$$A \mapsto U(t)^* A U(t), \quad A \in \mathfrak{A} \tag{2}$$

on observables. Here we want to consider a more general notion of evolution of quantum systems that is also suitable for the study of open systems. An evolution of a quantum system should be a function $T : \mathfrak{S} \to \mathfrak{S}$ satisfying the following (physically sound) postulates:

(i) For any state the expectation of the trivial observable should be 1 i.e. for any $S \in \mathfrak{S}$: $\mathbb{E}_S[A(1)] = 1$ or terms of density operators $\mathrm{Tr}(SA(1)) = 1$

(ii) Any measurement (observable) which only takes on positive values should also take

However, these assumptions are not strong enough. Condition (iii) physically corresponds to an evolution taking a state of a closed system $\mathcal{H}$ to another state of that system. However, it does not consider entanglement.

Definition 1. A linear map $T : \mathcal{B}(H) \to \mathcal{B}(H)$ is completely positive if for every $n \in \mathbb{N}$ the map

$$\mathcal{B}(H) \otimes \mathbb{C}^n \to \mathcal{B}(H) \otimes \mathbb{C}^n, \quad A \otimes B \mapsto T(A) \otimes B \tag{3}$$

is positive. A map is called n-positive if the above is true up to a fixed $n \in \mathbb{N}$.

Example 1. • every $*$-homomorphism is completely positive

- unitary transformation
- non example: transpose

Not

Note that every $*$-homomorphism is completely positive. The converse is not true – but almost.

Theorem 1. *(Stinespring) Let $\mathcal{A}$ be a C^*-algebra, H a Hilbert space and $T : \mathcal{A} \to \mathcal{B}(H)$ a completely positive map. Then there exists*

- *a Hilbert space K,*
- *a $*$-representation π, and*
- *bounded linear operator $V : H \to K$ s.t.*

$$Tx = V^* \pi(x) V, \quad x \in \mathcal{A} \tag{4}$$

Furthermore, K is unique in the following sense:

Proof. □

References

[JL] G. James, M. Liebeck, *Representations and characters of groups. Second edition*, Cambridge University Press, 2001.