Classification of Rank 1 Lie Groups

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Throughout these notes, let G be a connected, compact Lie group, let $N(H)$ denote the normalizer of a subgroup $H \subseteq G$ and let 0 denote the trivial group.

1 Definitions and Basic Results

Definition 1.1. The **rank** k of G is the dimension of its maximal torus.

The notion of rank above is well defined since all maximal tori in G are conjugate and thus have the same dimension. The dimension of a torus is its only invariant as a Lie group. While the Weyl group of a maximal torus is a more refined invariant, it seems reasonable that, even though only a crude invariant, the classification of Lie groups of low rank is possible. This is the case for $k = 0, 1$.

Theorem 1.2. *If G has rank $k = 0$, then G is trivial.*

Proof. Assume G is non-trivial, then since it is assumed to be connected, it must have dimension at least 1. Hence there is a non-zero element $X \in LG$. Let $s \mapsto \alpha(s) := \exp(sX)$ be the one-parameter subgroup tangent to X. Since exp is a local diffeomorphism, α is a non-trivial subgroup. Hence $\overline{\alpha(\mathbb{R}))}$ is a torus in G. $\square$

Theorem 1.3. *If G has rank $k = 1$, then either $G \cong U(1)$, $G \cong SO(3)$, or $G \cong SU(2)$. The latter two have dimension 3 and are non-commutative, while the first has dimension 1 and is abelian.*

The proof will proceed in a few steps.

2 The abelian case

Proof. Assume G is abelian, then since G is also compact and connected by assumption, it has to be a torus. Thus it is its own maximal torus. Hence $G \cong U(1)$. $\square$

3 The non-abelian case

Note immediately that if G is not abelian, then $\dim G > 1$ since one-dimensional Lie algebras must be abelian.

Lemma 3.1. *Let α be a one-parameter subgroup of G which is not periodic. Then G contains a torus of dimension $k > 1$.*

Proof. The subgroup $\alpha(\mathbb{R})$ is a connected abelian subgroup. Then $\overline{\alpha(\mathbb{R})}$ is a compact, connected abelian group, and thus a torus. Assume the dimension of the torus was $k \leq 1$. If it were $k = 0$, then it were periodic, and if it were $k = 1$, then $\alpha(\mathbb{R})$ would be a connected subgroup of $\overline{\alpha(\mathbb{R})} \cong U(1)$ which is dense, of which the only example is $U(1)$ itself, which would show that α was periodic. A contradiction to the assumption. $\square$

3.1 Identifying $G/T \cong \mathbb{S}^{n-1}$

From now on set $\dim G = n$ and equip G with a bi-invariant Riemannian metric. Such a metric exists, since G is assumed to be compact.

Let $X \in LG$ and let $s \mapsto \alpha(s) := \exp(sX)$ be the one-parameter subgroup tangent to X. Then by assumption and Lemma 3.1 α is periodic i.e. is has compact image and of dimension 1 and thus is a 1-torus. On the other hand, every 1-torus can be constructed that way. In other words, any one-parameter group α must close up, and the images of the various one-parameter groups are precisely the maximal tori of G.

Proposition 3.2. *The adjoint representation of G on LG restricts to the unit sphere (w.r.t. the bi-invariant Riemannian metric) in LG and the induced action*

$$\Psi : G \to O(n) = Aut(\mathbb{S}^{n-1}); \quad \Psi(g) = Ad(g) \tag{1}$$

is transitive. Here $\mathbb{S}^{n-1} \subseteq LG$ denotes the unit sphere (w.r.t. the inner product fixed on LG).

Proof. Fix $g \in G$. Since the Riemannian metric was assumed to be bi-invariant, $\mathrm{Ad}(g)$ is an isometry on LG. Hence the adjoint representation restricts to $\mathbb{S}^{n-1} \subseteq LG$.
Since X and $-X$ generate the same one-parameter group, the action descends to the projectivization $P(LG) \cong \mathbb{S}^{n-1}/\{\pm 1\}$; i.e. the pairs of antipodes in $\mathbb{S}^{n-1}$ and since all maximal tori are conjugate to each other, the action is transitive on $P(LG)$. Thus, the action on $\mathbb{S}^{n-1}$ has at most[1] 2 orbits. However, since these orbits are compact, admitting 2 orbits would imply that $\mathbb{S}^{n-1}$ is not connected, which is not the case since $n > 1$. Hence the action can only have one orbit and thus the action (1) is transitive. $\square$

Proposition 3.3. *For a given $X \in \mathbb{S}^{n-1} \subseteq LG$ as above, the map*

$$\phi_X : G/T \to \mathbb{S}^{n-1} \subseteq LG; g \mapsto Ad(g)X \tag{2}$$

is a well-defined, G-equivariant bijection between compact homogeneous G-spaces.

Proof. *Well-defined*: Let $t \in T$. Then for any $a \in \mathbb{R}$, using the naturality of the exponential map,

$$\exp(a\,\mathrm{Ad}(t)X) = c(t)\underbrace{\exp(aX)}_{\in T} = \exp(aX). \tag{3}$$

[1]For example, one orbit could be the upper hemisphere, while the other is the lower hemisphere, where the action on the lower hemisphere is dictated by the one on the upper in order to respect the quotient.

Differentiating both sides in s and evaluating at $s = 0$ gives $\mathrm{Ad}(t)X = X$. Hence for any $t_1^{-1}t_2 \in T$ we have

$$\mathrm{Ad}(t_1^{-1})\,\mathrm{Ad}(t_2)X = \mathrm{Ad}(t_1^{-1}t_2)X = X. \tag{4}$$

That is, $\mathrm{Ad}(t_1)X = \mathrm{Ad}(t_2)X$.
Equivariant: Since G acts on G/T by left-multiplication and on $\mathbb{S}^{n-1} \subseteq LG$ via the adjoint representation, ϕ_X is equivariant.
G-spaces: By Proposition 3.2 the adjoint action on $\mathbb{S}^{n-1}$ has a single orbit and thus gives a G-space.
Injective: Assume $\mathrm{Ad}(g)X = \mathrm{Ad}(h)X$. Then $\mathrm{Ad}(h^{-1}g)X = X$. Then, for any $a \in \mathbb{R}$

$$c(h^{-1}g)\exp(aX) = \exp(a\,\mathrm{Ad}(h^{-1}g)X) = \exp(aX). \tag{5}$$

Since $T = \exp(\mathbb{R}\,X)$, this implies that $h^{-1}g$ lies in $Z(T)$, which coincides with T itself, since T is a maximal torus.
Surjective: Since the action Ψ is transitive, the map is surjective. □

Corollary 3.4. *The map ϕ_X is a diffeomorphism*

$$G/T \cong \mathbb{S}^{n-1}\,. \tag{6}$$

Proof. By the Equivariant Rank Theorem (Lee, 2012, Thm. 7.25), as a bijective G-equivariant map between G-spaces, ϕ_X is a diffeomorphism. □

3.2 Characterizing the Weyl Group

Proposition 3.5. *The image of the normalizer under the map ϕ_X is $\{X, -X\}$. Thus the degree of T in $N(T)$ is $[N(T) : T] = 2$, $W \cong \mathbb{Z}/2\,\mathbb{Z}$, and the non-trivial element of W acts on T by orientation reversal.*

Proof. Let $g \in N(T)$. Then, since $T = \exp(\mathbb{R}\,X)$ there exists a $a \in \mathbb{R}$ such that

$$\exp(\mathrm{Ad}_g X) = \underbrace{c(g)\exp(X)}_{\in T} = \exp(aX). \tag{7}$$

However, by (1) we conclude $a = \pm 1$. Thus, by Corollary 3.4 this means that T has precisely 2 cosets in N. Thus $[N(T) : T] = 2$ and $W \cong \mathbb{Z}/2\,\mathbb{Z}$. The action of the non-trivial element $n \in W$ on T is thus given by

$$c(n)\exp(aX) = \exp(a\,\mathrm{Ad}(n)X) = \exp(-aX), \quad a \in \mathbb{R}\,. \tag{8}$$

□

3.3 Deducing $\dim = 3$ (topological argument)

Since $T \cong \mathbb{S}^1$ and $G/T \cong \mathbb{S}^{n-1}$, their homotopy groups (at least in low degree) are very tractable. We will thus make a homotopy theoretic argument for the dimension of G. Recall that by (Bröcker und Tom Dieck, 2013, Thm. (4.3))

$$T \xrightarrow{i} G \xrightarrow{p} G/T \tag{9}$$

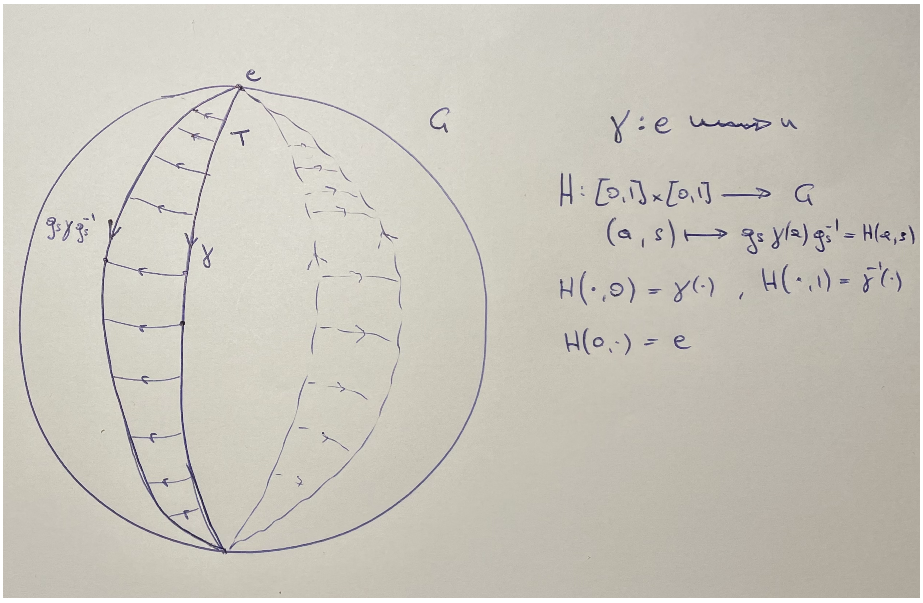

Figure 1: Illustration of $i_*([\gamma]) = i_*([\gamma])^{-1}$.

is a fibre bundle and therefore induces a long exact sequence in homotopy[2]

$$\ldots \to \pi_2(G/T) \xrightarrow{\delta} \underbrace{\pi_1(T)}_{\cong \mathbb{Z}} \xrightarrow{i_*} \pi_1(G) \xrightarrow{p_*} \pi_1(G/T) \to \ldots \tag{10}$$

where all homotopy groups are based at e and eT.

Proposition 3.6. *Let $[\gamma] \in \pi_1(T)$ be a generator of $\pi_1(T)$. Then $i_*([\gamma]) = i_*([\gamma])^{-1}$. In particular, $i_*([\gamma]) \in \pi_1(G)$ is of order 2.*

Proof. Since G is connected and thus, as a manifold, also path-connected, there is a path $g_\bullet : [0,1] \to G$ such that $g_0 = e$ and $g_1 = n$, where $n \in G$ is an element representing the non-trivial element in the Weyl group. By Proposition 3.5 we have $\operatorname{Ad}_n X = -X$. Let $[\gamma] \in \pi_1(T)$ be a generator as above. Then

$$H : [0,1] \times [0,1] \to G; \quad (a, s) \mapsto g_s \gamma(a) g_s^{-1} \tag{11}$$

is a homotopy in G from $H(\cdot, 0) = \gamma$ to $H(\cdot, 1) = \gamma^{-1}$, which fixes $H(0, \cdot) = e$. Hence γ and γ^{-1} are homotopic (relative to e) and thus represent the same element in $\pi_1(G)$. Hence $i_*([\gamma])^2 = i_*([\gamma]) i_*([\gamma])^{-1} = 1$.

□

As a result of Proposition 3.6 the image $i_*(\pi_1(T)) \subseteq \pi_1(G)$ is either isomorphic to $\mathbb{Z}/2\mathbb{Z}$ or 0. Therefore, the kernel of i_* must be infinite, which, by the exactness of the sequence, implies that $\pi_2(G/T)$ is also infinite. Since $G/T \cong \mathbb{S}^{n-1}$ and the homotopy groups of spheres in low degree and dimension are

[2]See (Hatcher, 2005, Sec. 4.2).

known, this leaves only $n = 3$. It then follows furthermore that $\pi_1(G/T) = \pi_1(\mathbb{S}^{n-1}) = 0$, and then, again by exactness, that i_* is surjective. Therefore, $\pi_1(G)$ is either isomorphic to $\mathbb{Z}/2\mathbb{Z}$ or 0.

3.4 Concluding via coverings

We now know that G must be of dimension $n = 3$ and that the $\pi_1(G)$ is either isomorphic to $\mathbb{Z}/2\mathbb{Z}$ or 0. To conclude with a full classification we appeal to the theory of covering spaces.[3]

Recall that G acts on LG via the adjoint representation as isometries w.r.t. the bi-invariant inner product fixed in the beginning of the section; that is, Ψ, as defined in (1), takes values in the $\mathrm{Isom}(\mathbb{S}^2) = \mathrm{O}(3)$. More specifically, since G is connected, its image has to lie in the connected component containing e. We summarize:

$$\Psi : G \to \mathrm{Isom}(\mathbb{S}^2)_0 = \mathrm{SO}(3); g \mapsto \mathrm{Ad}(g). \tag{12}$$

Proposition 3.7. *The kernel of Ψ equals the center $Z(G)$ of G and is therefore discrete.*

Proof. On the one hand, let $g \in Z(G)$ and $X \in LG$. Then for any $a \in \mathbb{R}$

$$\exp(a\,\mathrm{Ad}(g)X) = c(g)\underbrace{\exp(aX)}_{\in T} = \exp(aX). \tag{13}$$

Differentiating both sides in a and evaluating at $a = 0$ shows that $g \in \ker\Psi$.

On the other hand, assume $g \in \ker\Psi$. Since G is compact and connected, exp is surjective[4] and hence for any $h \in G$ there is a $X_h \in LG$ such that $h = \exp(X_h)$ and thus

$$c(g)h = c(g)\exp(X_h) = \exp(\underbrace{\mathrm{Ad}(g)}_{=\mathrm{id}_{LG}} X_h) = \exp(X_h) = h, \quad \forall h \in G. \tag{14}$$

Thus $g \in Z(G)$. Hence, since the center of a compact Lie group equals the intersection of all its maximal tori[5], which are 1-dimensional in our case, the center is discrete. $\square$

Therefore, SO(3) is the quotient of G by a finite subgroup (which acts properly). Hence $\dim(\mathrm{SO}(3)) = \dim G = 3$, Ψ is a submersion, and thus Ψ is a local diffeomorphism.

Thus G must be a covering space of SO(3). Luckily, the fundamental group of SO(3) is known to be $\mathbb{Z}/2\mathbb{Z}$, and thus, by standard covering space theory, there are (up to covering space isomorphism) precisely two covering spaces (corresponding to the subgroups of $\mathbb{Z}/2\mathbb{Z}$, which are $\mathbb{Z}/2\mathbb{Z}$ itself and 0). These are the trivial covering $\mathrm{SO}(3) \to \mathrm{SO}(3)$ and the 2:1-covering $\mathrm{SU}(2) \to \mathrm{SO}(3)$. Thus G must be isomorphic to either SU(2) or SO(3).

[3] See (Hatcher, 2005, Sec. 1.3).
[4] See (Bröcker und Tom Dieck, 2013, Thm. (2.2))
[5] See (Bröcker und Tom Dieck, 2013, Thm. (2.3) (iii)).

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