Maximal Tori I

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Throughout these notes, unless specified otherwise, we make the following conventions. G denotes a compact and connected Lie group. For a smooth map $\varphi : M \to N$ between smooth mainfolds φ_* or $D\varphi$ denotes the pushforward/differential, while φ^* denotes the pullback. The tangent space at the identity will be denoted by LG (if G is not a Lie group, LG denotes the tangent space at a distinguished point which will be clear from the context). Conjugation by an element of $g \in G$ will be denoted by $c(g)$, while the image of $g \in G$ under the adjoint representation will be denoted by $\mathrm{Ad}(g)$.

1. Tori and the Weyl Group

1.1. Abelian Lie Groups

Definition 1.1. **A torus** T **in** G is a compact, connected, abelian immersed Lie subgroup. A torus T in G is called **maximal** if, for any other torus T' we have

$$T \subseteq T' \Rightarrow T = T'. \tag{1}$$

Observation 1.2. *By (a consequence of) Cartan's theorem a torus is an embedded subgroup.*

Proof. Let $\iota : T \hookrightarrow G$ denote the inclusion of the Lie subgroup. Since T is compact, so is $i(T)$. Thus $i(T) \subseteq G$ is a closed subgroup and thus by Cartan's theorem i is an embedding of Lie groups. $\square$

Observation 1.3. *G contains a maximal torus T. However, T is in general (and usually) not unique. In fact, we will see later (Theorem 2.1) that if the maximal torus is unique, then it coincides with G.*

Proof. Let $\mathfrak{T}(G)$ denote the set of tori in G. The 0-torus $T^0 = \{e\} \subseteq G$ is a Lie subgroup which is a torus, i.e. $\mathfrak{T}(G) \neq \emptyset$. Assume not that $\mathfrak{T}(G)$ did not have a maximal element. Then for any $T \in \mathfrak{T}(G)$ there exists a $T' \neq T$ s.t. $T \subsetneq T' \subseteq G$. Since tori are compact and connected, Lemma A.1 implies that

$$\dim(T) < \dim(T') \leq \dim(G). \tag{2}$$

Since $\dim(G) < \infty$, this is a contradiction. $\square$

As the name suggests, a torus in the sense of Definition 1.1 is isomorphic to a torus in the usual sense i.e. $\mathbb{R}^d \,/\, \mathbb{Z}^d$ for some $d \geq 0$. This follows from the classification of abelian Lie groups:

Theorem 1.4 (Classification of abelian Lie groups; [BtD, I. (3.6), (3.7)]).

1. *Let G be a connected, abelian Lie group. Then $G \cong \mathbb{T}^d \times \mathbb{R}^e$ for some $d, e \geq 0$.*
2. *Let G be a compact, abelian Lie group. Then $G \cong \mathbb{T}^d \times \prod_{i=1}^k \mathbb{Z} \,/ n_i\, \mathbb{Z}$ for some $d, n_1, \ldots, n_k \geq 0$.*

Definition 1.5. Let T be a maximal torus in G, and let

$$N := \{g \in G : gTg^{-1} = T\} \tag{3}$$

be the normalizer of T in G. The the group $W := N/T$ is called the **Weyl group** of G.

Note that since T is a closed subgroup, so is

$$N = (c(\cdot)(t))^{-1}(T) \cap \bigcap_{t \in T} (c((\cdot)^{-1})(t))^{-1}(T), \tag{4}$$

where $g \mapsto c(g)(t) = gtg^{-1}$ denotes conjugation by g, applied to t.
By Definition 1.5, for a given G, the Weyl group W depends on the maximal torus T in G. However, as will be shown in Theorem 2.1 all maximal tori are conjugate to each other and as a consequence all Weyl groups are isomorphic.

The normalizer N operates on T via

$$N \times T \to T; \quad (n, t) = ntn^{-1} \tag{5}$$

and, since T is abelian and thus operates trivially on itself, the Weyl group also operates on T via

$$W \times T \to T; \quad (nT, t) = ntn^{-1}. \tag{6}$$

Theorem 1.6. *The Weyl group W of G is finite.*

Proof. Let N_0 denote the connected component of N containing the identity[1]. We will show that $N_0 \subseteq T$ and thus $N_0 = T$. It then follows that 1) since N is compact, so is $W = N/T = N/N_0$, and 2) since, as the homeomorphic image of an open set, nN_0 is open in N for every $n \in N$, and $[n] \in N/N_0$ is open precisely if $\pi^{-1}(nN_0) = nN_0 \subseteq N$ is open, that every singleton in W is open. Hence W is 1) compact and 2) discrete and thus finite.

Now, to see that $N_0 \subseteq T$ let $k := \dim(T)$. Recall that the automorphisms of a torus T are precisely those linear transformations on $\mathbb{R}^d$ that preserve the lattice $\mathbb{Z}^d$, i.e. $\mathrm{Aut}(T) = \mathrm{GL}(k, \mathbb{Z}) \subseteq \mathrm{GL}(k, \mathbb{R})$ and consider the continuous map

[1] Recall that the connected component of a topological group is necessarily a subgroup: Since $\cdot : G_0 \times G_0 \to G$ is continuous and G_0 is connected, the image is connected. Thus, since $e = e \cdot e \in \cdot(G_0 \times G_0)$, the image is contained in a connected component, which contains the identity, i.e. it is contained in G_0. The same argument works for $(\cdot)^{-1} : G_0 \to G$.

$$
\begin{aligned}
N \xrightarrow{c} \mathrm{Aut}(T) &\xrightarrow{D} \mathrm{Aut}(LT) \cong \mathrm{Aut}(\mathbb{R}^k) \cong GL(k,\mathbb{R}) && (7)\\
n \mapsto \quad c(n) &\quad \mapsto \quad \mathrm{Ad}(n). && (8)
\end{aligned}
$$

Then the image of L in (8) is precisely the subgroup $\mathrm{GL}(k,\mathbb{Z})$, which is discrete in $\mathrm{GL}(k,\mathbb{R})$. Therefore, since N_0 is connected, the image of N_0 under the above map has to be the identity in $\mathrm{GL}(k,\mathbb{R})$. In other words, N_0 acts trivially on T by conjugation.

As a consequence, for any one-parameter group $\alpha : \mathbb{R} \to N_0$ the subgroup

$$
\alpha(\mathbb{R}) \cdot T := \{\alpha(a)t : a \in \mathbb{R}, t \in T\} \subseteq G \tag{9}
$$

is (as a continuous image of connected spaces) connected and abelian: $\forall a, b \in \mathbb{R}$ and $\forall t_1, t_2 \in T$ we have

$$
\begin{aligned}
\alpha(a)t_1\alpha(b)t_2 &= \alpha(a+b)\underbrace{\alpha(b)^{-1}t_1\alpha(b)}_{=t_1}t_2 && (10)\\
&= \alpha(a+b)t_1t_2 && (11)\\
&= \alpha(b+a)t_2t_1 && (12)\\
&= \alpha(b+a)\underbrace{\alpha(a)^{-1}t_2\alpha(a)}_{=t_2}t_1 && (13)\\
&= \alpha(b)t_2\alpha(a)t_1 && (14)
\end{aligned}
$$

where we used the fact that N_0 acts trivially (by conjugation) on T. Thus by maximality of T we have $\alpha(\mathbb{R}) \cdot T = T$ and in particular $\alpha(\mathbb{R}) \subseteq T$.
Since $\exp : LG \to G$ is a local diffeomorphism around 0 (since the derivative at 0 is the identity), for any g in a neighborhood of $e \in G$ there is a one-parameter subgroup containing g. Such a subgroup is given by $t \mapsto \exp(t \log(g))$, where log denotes the local inverse of exp. Hence the one-parameter subgroups cover an open neighborhood of e in N_0, which thus is also contained in T. Thus, since N_0 is connected, by Lemma 1.7 this open neighborhood generates N_0 and hence $N_0 \subseteq T$, which concludes the proof. □

Lemma 1.7. *Let G be a topological group, let $G_0 \subseteq G$ be its connected component, and let U be an open neighborhood of $e \in G$ contained in G_0, and let $\langle U\rangle$ denote the subgroup generated by U. Then $\langle U\rangle = G_0$.*

Proof. We want to show that the subgroup $\langle U\rangle$, which is generated by U is non-empty, open, and closed. Assume without loss of generality $U^{-1} \subseteq U$ (otherwise pass to $U \cap U^{-1}$).

1. Non-empty: Since $e \in U \subseteq \langle U\rangle$ the subgroup is non-empty.
2. Open: For any $g \in \langle U\rangle$ we have $g \cdot U \subseteq \langle U\rangle$.
3. Closed: If $g \notin \langle U\rangle$, then $g \cdot U \cap \langle U\rangle = \emptyset$, as otherwise $gu = v$ for some $u \in U, v \in \langle U\rangle$ and thus $g = vu^{-1} \in \langle U\rangle$. A contradiction. Hence the complement of $\langle U\rangle$ is also open.

□

2. Conjugates of Maximal Tori in G

The main theorem of this talk will be the following:

Theorem 2.1. *Let T and T' be two maximal tori in G. Then*

(1) the conjugate of T is again a maximal torus,

(2) T and T' are conjugate; i.e. there exists a $g \in G$ s.t. $T' = gTg^{-1}$,

(3) for any $g \in G$ there exists a maximal torus T in G s.t. $g \in T$, and

(4) the Weyl group is unique up to conjugation.

Its proof relies on the mapping degree of conjugations of the torus.

Theorem 2.2 (Mapping Degree; [BtD, I. (5.19)])**.** *Let M, N be compact, connected, oriented, n-dimensional manifolds and let $f : M \to N$ be a (homotopy class of a) differentiable function. Then there is an integer* $\deg(f)$ *such that for any $\alpha \in \Omega^n(N)$ we have*

$$\int_M f^*\alpha = \deg(f) \cdot \int_N \alpha. \tag{15}$$

If $q \in N$ such that $f^{-1}(q)$ consists of $k + l$ points $p_1, \dots, p_{k+l}$ such that q is a regular value of f (i.e. that Df is bijective at every p_i), preserves orientation at $p_1, \dots, p_k$ and reverses orientation at $p_{k+1}, \dots, p_{k+1}$, then $\deg(f) = k - l$. *In particular, if f is not surjective, then there exists a $q \in N$ such that $f^{-1}(q) = \emptyset$ and thus* $\deg(f) = 0$.

In particular, it is a consequence of the following important lemma, the proof of which will be split up into several parts.

Lemma 2.3. *Let T be a maximal torus in G. Then the map*

$$q : G/T \times T \to G; \quad (g, t) \mapsto gtg^{-1} \tag{16}$$

has mapping degree $\deg(q) = |W|$, *where $|W|$ is the order of the Weyl group W associated to T. In particular, since $|W| > 0$, q is surjective.*

The proof of Lemma 2.3 is rather lengthy. Let us therefore first prove Theorem 2.1 from Lemma 2.3 and then turn to a proof of Lemma 2.3.

Proof of Thm. 2.1 from Lem. 2.3. (1) Let $g \in G$. Since $x \mapsto gxg^{-1}$ is a diffeomorphism $G \to G$ and T is compact and connected, so is gTg^{-1}. Commutativity and maximality of gTg^{-1} follow immediately from the commutativity and maximality of T.

(2) Let T and T' be two maximal tori in G. Let t' be a generator[2] of T'. By Lemma 2.3 there is a $g \in G$ such that $t' \in gTg^{-1}$ and hence, since t' generates T' we have $T' \subseteq gTg^{-1}$ and thus, by maximality of T', $T' = gTg^{-1}$.

[2] Recall that $t \in T$ is called a **generator of** T if the generated subgroup $\{t^k : k \in \mathbb{Z}\}$ is dense in T.

(3) Since q is surjective, every $g \in G$ is contained in gTg^{-1} for some $g \in G$, which, by (1) is again a maximal torus.

(4) Let N and N' be the normalizers of T and T', and let W and W' be the resulting Weyl groups, respectively. Let $g \in G$ be such that $T' = gTg^{-1}$, the existence of which is proven in (2). Then for any $n \in N, t' = gtg^{-1} \in T$ we have

$$(gng^{-1})t'(gng^{-1})^{-1} = gn\underbrace{g^{-1}t'g}_{=t}n^{-1}g^{-1} = g(ntn^{-1})g^{-1} \in gTg^{-1} = T' \tag{17}$$

Hence, $gng^{-1} \in N'$, and, by assumption $gTg^{-1} \subseteq T'$. Hence conjugation by g decends form N to $N/T = W$ and thus provides an isomorphism (with inverse being conjugation by g^{-1}). Thus the Weyl group is unique up to isomorphism (given by conjugation).

□

2.1. Some Observations

We will want to use the second part of Theorem 2.2 to compute the mapping degree of q from its fibre and conclude that q is surjective. However, for this we do not only need to understand the cardinality of the fibre of q (easy), but also the effect of q on the orientation (hard). For the latter we need orientations on $G/T \times T$ and on G that facilitate computation (e.g. are left-invariant, etc.; see Observations 2.4 and 2.6), and identifications (see. Observation 2.5) that allow the tangent map q_* to be understood as an endomorphism (not just a linear map), which then allows a computation of the determinant in the classical sense.

Before we turn to a proof of Lemma 2.3, let us first make some observations about the map

$$q : G/T \times T \to G; \quad (g, t) \mapsto gtg^{-1} \tag{18}$$

from Lemma 2.3 and the involved spaces.

Observation 2.4. *The map q is a smooth map between orientable compact manifolds of equal dimension. Note also that while T is generally not normal in G, and thus the space G/T does not carry a natural group structure, there still are the following natural left-actions:*

$$G \times T \curvearrowright G/T \times T, \quad G \curvearrowright G. \tag{19}$$

Proof. As a quotient of a compact Lie group G by a compact and connected subgroup T the orbit space G/T is a compact and orientable[3] manifold of dimension[4] $\dim(G) - \dim(T)$. The Lie groups T and G are orientable because they are parallelizable and compact and connected by assumption. □

[3] The orbit space is orientable since T is connected - see Prof. Dr. Leeb's Lecture Notes "General Facts about Lie Groups", end of page 7.

[4] See [Lee, Thm. 21.10], the "Quotient Manifold Theorem".

Observation 2.5. *Let $\langle\cdot,\cdot\rangle$ be an Ad_G-invariant inner product*[5] *on LG, let $LT \subseteq LG$ denote the Lie algebra of T, and let $LT^\perp \subseteq LG$ denote its orthogonal complement (w.r.t. $\langle\cdot,\cdot\rangle$) in LG. Then the splitting*

$$LG = LT^\perp \oplus LT \cong L(G/T) \oplus LT \tag{20}$$

is Ad_T invariant. As a consequence of the invariance, there is an induced action

$$Ad_{G/T} : T \to Aut(L(G/T)). \tag{21}$$

For the rest of the talk we will make the identification $LT^\perp \oplus LT \cong L(G/T) \oplus LT$.

Proof. 1) To see that LT is Ad_T-invariant, let $t \in T$ and $X \in LT$. Then for any $s \in \mathbb{R}$, using the fact that $c(t) \circ \exp = \exp \circ \mathrm{Ad}_t$, we have

$$\underbrace{\exp(sX)}_{\in T} = t \exp(sX) t^{-1} = c(t) \exp(sX) = \exp(s \,\mathrm{Ad}_t\, X). \tag{22}$$

Differentiating both sides of the resulting equation in s and evaluating at $s = 0$ gives $X = \mathrm{Ad}_t\, X$.

2) To see that $LT^\perp$ is invariant, let $t \in T$ and $X \in LT^\perp$; i.e. let $\langle X, Y\rangle = 0$ for any $Y \in LT$. Hence, by Ad_T invariance of LT and the inner product

$$\langle \mathrm{Ad}_t\, X, \mathrm{Ad}_t\, Y\rangle = \langle X, Y\rangle = 0. \tag{23}$$

Since $\mathrm{Ad}_t \in \mathrm{Aut}(LT)$, this implies that $\langle \mathrm{Ad}_t\, X, Z\rangle = 0$ for every $Z \in LT$ and hence $\mathrm{Ad}_t\, X \in LT^\perp$.

3) To see the second equality in (20) let $g \in G$ be arbitrary and let $\pi : G \to G/T$ denote the projection map. On the one hand, consider a smooth curve $\gamma : (-1, 1) \to gT \subseteq G$ such that $\gamma(0) = g$. Then

$$(\pi_*)_*(\gamma'(0)) = (\underbrace{\pi \circ \gamma}_{=[g]\in G/T})'(0) = 0 \tag{24}$$

Hence $T_g gT \subseteq \ker(\pi_*)_*$. On the other hand, by [Lee, Thm. 21.10], the "Quotient Manifold Theorem", π is a submersion. Thus $(\pi_*)_g : T_g G \to T_{gT}(G/T)$ is surjective and hence $\dim \ker \mathrm{d}\pi_g \le \dim T$. Thus $T_g gT = \ker \mathrm{d}\pi_g$ and hence $T_{gT}(G/T) \cong T_g G / T_g gT$. $\square$

Observation 2.6. *There are unique (up to choice of sign) invariant (under the actions in* (19)*) volume forms*

$\mathrm{d}g$	*on*	G	*invariant under action of*	G
$\mathrm{d}(gT)$	*on*	G/T	*invariant under action of*	G
$\mathrm{d}t$	*on*	T	*invariant under action of*	T

Each of them may be constructed by choosing a top-dimensional alternating form at the (image of) the identity and then defining the form at a point by pulling back through left-multiplication of the action.

[5] See Proposition A.2

In particular, $\pi^* \,\mathrm{d}(gT) \in \Omega^{n-k}(G)$ *and* $pr_2^*((\mathrm{d}t)_e) \in Alt^k(LG)$*, where* $pr_2 : LG = L(G/T) \oplus LT \to LT$*. Now, (via pullback along left-multiplication) the alternating k-form* $pr_2^*((\mathrm{d}t)_e)$ *determines a left-invariant k-form* $\mathrm{d}\tau \in \Omega^k(G)$ *such that* $\mathrm{d}\tau|_T = \mathrm{d}t$ *and such that* $\pi^* \,\mathrm{d}(gT) \wedge \mathrm{d}\tau$ *is a left-invariant volume form on* G*, since both parts of the wedge are left-invariant. Hence, we may choose the signs of the forms* $\mathrm{d}g$*,* $\mathrm{d}(gT)$*, and* $\mathrm{d}t$ *such that* $\pi^* \,\mathrm{d}(gT) \wedge \mathrm{d}\tau = c \cdot \mathrm{d}g$ *for some* $c > 0$*. One can show*[6] *that* $c = 1$*, but this is not important for our concerns.*

2.2. The determinant of the conjugation map q

Recall that for a smooth map $\varphi : M \to N$ between n-dimensional smooth manifolds, the pushforward induces a map on vector fields $\varphi_* : \Gamma(TM) \to \Gamma(TN)$, which in turn, via pullback, induces a linear map on n-forms: $\varphi^* : \Omega^n N \to \Omega^n M$. Since for each p the spaces $\mathrm{Alt}^n(T_pM)$ and $\mathrm{Alt}^n(T_{\varphi(p)}N)$ are 1-dimensional, this map can be be given, after choice of section α and β, by a real valued function $\det(\varphi) : M \to \mathbb{R}$ which then is defined by

$$\varphi^*\alpha = \det(\varphi)\beta. \tag{25}$$

If $M = N$ there is a canonical choice: $\alpha = \beta$; which then makes $\det(\varphi)$ independent of the choice of α. However, if $M \neq N$, a choice has to be made.

Observation 2.6 provides us with two reasonable volume forms on G and $G/T \times T$, respectively:

$$\mathrm{d}g = \pi^* \,\mathrm{d}(gT) \wedge \mathrm{d}\tau \in \Omega^n(G), \quad \mathrm{d}\tau|_T = \mathrm{d}t \in \Omega^k(T), \tag{26}$$

$$\alpha = \mathrm{pr}_1^* \,\mathrm{d}(gT) \wedge \mathrm{pr}_2^* \,\mathrm{d}t \in \Omega^n(G/T \times T). \tag{27}$$

which are invariant under the actions in (19). With the identification (20) we further have

$$\alpha_{(eT,e)} = \mathrm{d}g_e. \tag{28}$$

Definition 2.7. The **determinant** $\det(q) : G/T \times T \to \mathbb{R}$ of the conjugation map q is defined by the equation

$$q^* \,\mathrm{d}g = \det(q) \cdot \alpha. \tag{29}$$

Proposition 2.8. *For every* $(gT, t) \in G/T \times T$ *the determinant of the conjugation map* $q : G/T \times T \to G$ *at* (gT, t) *is given by*

$$\det(q)(gT, t) = \det(Ad_{G/T}(t^{-1}) - id_{L(G/T)}), \tag{30}$$

where $id_{L(G/T)}$ *is the identity map on* $L(G/T)$*. The determinant is to be understood as that of an endomorphism of* $L(G/T) \cong LT^{\perp}$*.*

Proof. In this proof, let us write $[g] := gT$ for equivalence classes in G/T, let $\ell.$ denote the left-action in (19) and let $([g], t) \in G/T \times T$ be fixed throughout the proof. We want to use the invariance of the involved forms to reduce the

[6] See [BtD, p. 160, 161].

computation of the determinant at $([g], t) \in G/T \times T$ to a computation at (eT, e). For this, consider the function

$$\varphi : G/T \times T \xrightarrow{\ell_{(g,t)}} G/T \times T \xrightarrow{q} G \xrightarrow{\ell_{gt^{-1}g^{-1}}} G. \tag{31}$$

Using invariance of the volume forms and the definition of the determinant of q we have

$$\begin{aligned}
\varphi^* \,\mathrm{d}g &= \ell^*_{(g,t)}(q^*(\ell^*_{gt^{-1}g^{-1}}\,\mathrm{d}g))) && (32)\\
&= \ell^*_{(g,t)}(q^*(\mathrm{d}g)) && (33)\\
&= \ell^*_{(g,t)}(\det(q)\cdot\alpha) && (34)\\
&= \det(q)\cdot(\ell^*_{(g,t)}\alpha) && (35)\\
&= \det(q)\cdot\alpha, && (36)
\end{aligned}$$

and $\varphi(eT, e) = (gt^{-1}g^{-1})gtg^{-1} = e$. Hence

$$(\varphi^* \,\mathrm{d}g)_{([e],e)} = \det(q)(g,t)\cdot\alpha_{([e],e)}. \tag{37}$$

and thus the computation of $\det(q)([g], t)$ is reduced to that of $(\varphi^* \,\mathrm{d}g)_{([e],e)}$. By (28) this amounts to computing the transformation of a degree n alternating tensor under pullback which can be done by computing the differential of φ at $([e], e)$ as an endomorphism

$$L(G/T) \oplus LT \to L(G/T) \oplus LT. \tag{38}$$

For any $([h], s) \in G/T \times T$ we may rewrite the application of φ as

$$\begin{aligned}
\varphi([h], s) &= \ell_{gt^{-1}g^{-1}}(q(\ell_{([g],t)}([h], s)) && (39)\\
&= \ell_{gt^{-1}g^{-1}}(q([gh], ts)) && (40)\\
&= \ell_{gt^{-1}g^{-1}}(([gh])ts[(gh)^{-1}]) && (41)\\
&= (gt^{-1}g^{-1})(([gh])ts[(gh)^{-1}]) && (42)\\
&= gt^{-1}[h]ts[h^{-1}]g^{-1} && (43)\\
&= c_g(c_{t^{-1}}([h])s[h^{-1}]). && (44)
\end{aligned}$$

Thus, using the chain rule and the product rule, the differential at $([e], e)$ is given by

$$(X, Y) \mapsto \mathrm{Ad}(g) \circ (\mathrm{Ad}_{G/T}(t^{-1})X + Y - X), \tag{45}$$

where $\mathrm{Ad}_{G/T}$ denotes the induced action in (21). Since the inner product on LG is Ad_G-invariant i.e. $\mathrm{Ad}(g)$ is orthogonal w.r.t. this inner product, the determinant of $\mathrm{Ad}(g)$ is ± 1. Since $\mathrm{Ad}(e) = \mathrm{id}_{LG}$ and G is connected, we have $\mathrm{Ad}(g) = 1$. Using the identification (20) and the Ad_T-invariance of the splitting in Observation 2.5, this gives an endomorphism in block form, whose determinant is thus

$$\det\begin{pmatrix} \mathrm{Ad}_{G/T}(t^{-1}) - \mathrm{id}_{L(G/T)} & 0 \\ 0 & \mathrm{id}_{LT}\end{pmatrix} = \det(\mathrm{Ad}_{G/T}(t^{-1}) - \mathrm{id}_{L(G/T)}) \tag{46}$$

This concludes the proof. ☐

Lemma 2.9. *Let $t \in T$ be a topological generator. Then*

1) $q^{-1}(t)$ consists of $|W|$ many points and

2) $\det(q)(gT, s) > 0$ for any $(gT, s) \in q^{-1}(t)$

Proof. 1) Let $N(T)$ denote the normalizer of T in G and assume that $t \in T$ is a topological generator of T. Then for a fixed $gT \in G/T$

$$\exists s \in T : q(gT, s) = gsg^{-1} = t \tag{47}$$
$$\Leftrightarrow \exists s \in T : g^{-1}tg = s \in T \tag{48}$$
$$\Leftrightarrow g^{-1}Tg \subseteq T \tag{49}$$
$$\Leftrightarrow g \in N(T). \tag{50}$$

Therefore

$$q^{-1}(t) = \{(gT, g^{-1}tg) \in G/T \times T : g \in N(T)\} \tag{51}$$

Now, note that if two elements $(gT, g^{-1}tg)$, $(hT, h^{-1}th)$ in $q^{-1}(t)$ are equal if and only if $h^{-1}g \in T$. Thus $q^{-1}(t)$ is in bijection to $W = N(T)/T$ which gives the result.
2) Recall from Proposition 2.8 that $\det(q)$ is given by the determinant of an endomorphism of $L(G/T)$. We want to show that this endomorphism has no real eigenvalues. If that is the case, as an endomorphism of a real vector space, the eigenvalues come in complex conjugated pairs and thus the determinant (as a product of eigenvalues) is non-negative. Moreover, since this implies that 0 cannot be an eigenvalue, this implies that the determinant is strictly positive.
Firstly, if $\mathrm{Ad}_{G/T}(t^{-1}) - \mathrm{id}_{L(G/T)}$ had a real eigenvalue, then so would $\mathrm{Ad}_{G/T}(t^{-1})$ (since $-\mathrm{id}_{L(G/T)}$ just shifts the spectrum of $\mathrm{Ad}_{G/T}(t^{-1})$ by -1). Since, w.r.t. the Ad_G invariant inner product, $\mathrm{Ad}_{G/T}(t^{-1})$ is an orthogonal transformation, that eigenvalue would have to be ± 1. In that case, since $\mathrm{Ad}(gh) = \mathrm{Ad}(g) \circ \mathrm{Ad}(h)$ that would imply that $\mathrm{Ad}_{G/T}(t^{-2})$ had eigenvalue 1. We show that this is a contradiction:
Assume there exists a non-zero $X \in L(G/T) \subseteq LG$ such that $\mathrm{Ad}_{G/T}(t^{-2})X = X$ and let $s \in \mathbb{R}$ be arbitrary. Then by linearity of the adjoint representation and naturality of the exponential map $\exp : G \to LG$ we have

$$c(t^{-2}) \exp(sX) = \exp(\mathrm{Ad}_{G/T}(t^{-2})sX) = \exp(sX), \tag{52}$$

and hence

$$c(t^{-2k}) \exp(sX) = \exp(sX), \quad k \in \mathbb{Z}\,. \tag{53}$$

By Kronecker's theorem A.3, t^{-2} is also a topological generator and hence

$$c(t') \exp(sX) = \exp(sX), \quad \forall t' \in T. \tag{54}$$

Thus the one parameter subgroup $H := \{\exp(sX) | s \in \mathbb{R}\}$ is left pointwise invariant by conjugation of T, i.e. every element in H commutes with every element in T. Thus $H \cdot T$ is abelian, compact and connected. Therefore $H \cdot T \subseteq T$ and hence $H \subseteq T$. Therefore $X \in LT \cap L(G/T) = \{0\}$. A contradiction. ☐

The following is a nice consequence of the proof above:

Observation 2.10. *If t topologically generates T, then $Ad_{G/T}(t)$ operates on $L(G/T)$ and has no real eigenvalues. Hence the dimension of G/T is even.*

Finally, let us complete the proof of Lemma 2.3.

Proof of Lemma 2.3. By (1) of Lemma 2.9 $q^{-1}(t)$ consists precisely of $|W|$ many points. By (2) of Lemma 2.9, q is orientation preserving at each of these points. Hence as a consequence of the second part of Theorem 2.2

$$\deg(q) = |W| > 0, \tag{55}$$

and thus by the last part of Theorem 2.2 q is surjective. □

Proposition 2.11 (Weyl's Integration Formula)**.** *Let $f : G \to \mathbb{R}$ be continuous. Then*

$$|W| \cdot \int_G f(g)\,\mathrm{d}g = \int_T \left[\det(id_{L(G/T)} - Ad_{G/T}(t^{-1})) \int_G f(gtg^{-1})\,\mathrm{d}g\right] \mathrm{d}t. \tag{56}$$

Proof. Via Lemma 2.3 and the definition of the mapping degree, the left hand side becomes

$$|W| \cdot \int_G f(g)\,\mathrm{d}g = \deg(q) \cdot \int_G f(g)\,\mathrm{d}g = \int_{G/T \times T} q^*(f\,\mathrm{d}g) = \int_{G/T \times T} (f \circ q) q^*\,\mathrm{d}g. \tag{57}$$

By (29) and (27) this gives

$$\int_{G/T \times T} (f \circ q) \underbrace{q^*\,\mathrm{d}g}_{=\det(q)\alpha} = \int_{G/T \times T} (f \circ q) \det(q)(\mathrm{pr}_1^*(\mathrm{d}gT) \wedge \mathrm{pr}_2^*\,\mathrm{d}t). \tag{58}$$

By Fubini's theorem and Proposition 2.8 we obtain

$$= \int_T \left(\int_{G/T} (f \circ q) \det(q)\,\mathrm{d}gT \right) \mathrm{d}t \tag{59}$$

$$= \int_T \left(\det(\mathrm{Ad}_{G/T}(t^{-1}) - \mathrm{id}_{L(G/T)}) \int_{G/T} (f \circ q)\,\mathrm{d}gT \right) \mathrm{d}t. \tag{60}$$

Finally, writing out q as a function on G instead of G/T this yields

$$= \int_T \left[\det(\mathrm{id}_{L(G/T)} - \mathrm{Ad}_{G/T}(t^{-1})) \int_G f(gtg^{-1})\,\mathrm{d}g\right] \mathrm{d}t. \tag{61}$$

In the last step we also used the fact that the dimension of $L(G/T)$ is even (noted in Observation 2.10), to switch sign inside the determinant. □

An interpretation of the formula: For a fixed t in the maximal torus T, define $f_t(g) = f(gtg^{-1})$. Note that f_t is constant on cosets of T and f factors into $f = f_t \circ \pi$. We may thus express the integral of f on G by first holding t fixed, integrating over the orbit gT, then weighing the result by the factor $\det(\mathrm{id}_{L(G/T)} - \mathrm{Ad}_{G/T}(t^{-1}))$ and integrating the result over T. In this sense, if we normalize $\mathrm{vol}(G) = \mathrm{vol}(G/T) = 1$, then $\det(\mathrm{id}_{L(G/T)} - \mathrm{Ad}_{G/T}(t^{-1}))$ can be interpreted as the volume of the conjugacy class of t.

A. Some Further Propositions

Lemma A.1. *Let N be a connected C^∞-manifold and let M be a compact C^∞-submanifold with inclusion $\iota : M \hookrightarrow N$. Then $dim(M) < dim(N)$, unless M and N are diffeomorphic.*

Proof. Recall that if ι is an immersion, then for any $p \in M$ the map $(D\iota)_p : T_pM \to T_{\iota(p)}N$ is injective and hence

$$\dim(M) = \dim(T_pM) \leq \dim(T_{\iota(p)N} = \dim(N). \tag{62}$$

To see that $\dim(M)$ has to be *strictly* smaller than $\dim(N)$ assume $\dim(M) = \dim(N)$ and M and N are not diffeomorphic. Then ι_* is pointwise injective and $\dim(T_pM) = \dim(T_{\iota(p)}N)$ by assumption, we conclude that ι_* is also pointwise surjective and hence a submersion. In particular, ι_* is pointwise invertible and thus as local diffeomorphism. The map ι is thus an open map and hence $\iota(M) \subseteq N$ is open. Also, since M is compact, $\iota(M) \subseteq N$ is closed and therefore closed. Thus, since N is connected $\iota(M) = N$ and ι is surjective. Hence ι is a bijective local diffeomorphism and thus a global diffeomorphism. ☐

Proposition A.2. *Let $b : LG \times LG \to \mathbb{R}$ be an inner product on LG. Then*

$$LG \times LG \to \mathbb{R} \tag{63}$$

$$(X, Y) \mapsto \langle X, Y \rangle := \int_G b\,(Ad(g)X, Ad(g)Y)\,\mathrm{d}g, \tag{64}$$

is an Ad_G-invariant inner product, where dg denotes the bi-invariant Haar measure on G.

Proof. The right hand side is finite since the integrand is a continuous function on a compact topological space and thus bounded and since the Haar measure is finite.

Bi-linearity and positivity are immediate. Assume $X \in LG$ such that

$$0 = \langle X, X \rangle = \int_G b\,(\mathrm{Ad}(g)X, \mathrm{Ad}(g)X)\,\mathrm{d}g \tag{65}$$

Then $\forall g \in G : b\,(\mathrm{Ad}(g)X, \mathrm{Ad}(g)X) =$ and thus, since b is non-degenerate, $\mathrm{Ad}(g)X = 0$. Since $\mathrm{Ad}(g) \in \mathrm{Aut}(LG)$, this implies $X = 0$.

Ad_G invariance follows from the fact that $g \mapsto \mathrm{Ad}(g)$ is a homomorhism and from the right-invariance of dg. ☐

Theorem A.3 (Kronecker)**.** *Let* $v = (v_1, \ldots, v_n) \in \mathbb{R}^n$*. Then* $\exp(v) \in T^n$ *is a topological generator if and only if* 1 *and* $v_1, \ldots, v_n$ *are linearly independent over* $\mathbb{Q}$*; i.e. for every* $q_0, q_1, \ldots, q_n \in \mathbb{Q}$

$$q_1 v_1 + \ldots + q_n v_n = q_0 \quad \Rightarrow \quad q_0 = q_1 = \ldots = q_n = 0. \tag{66}$$

Theorem A.4 ((Consequence of) Cartan's Theorem)**.** *Let* $A \subseteq G$ *be a closed subgroup of a Lie group* G*. Then* A *is an embedded Lie subgroup.*

References

[BtD] T. Bröckner, T. tom Dieck, *Representations of Compact Lie Groups*, Springer, 1985.

[Lee] J. M. Lee, *Introduction to Smooth Manifolds*, Springer, 2012.