Tutorial 1: Multidimensional Random Variables

Ex. 1 We roll two dices, let X be the minimum of the dices, and Y the maximum. Write the joint law (X, Y), are X and Y independent?

Ex. 2 Let $M = (X, Y)$ be a point taken uniformly on the disk of center 0 and radius 1. Are X and Y uncorrelated? Are they independent?

Ex. 3 Let X and Y be independent RV of law $\mathcal{N}(0, 1)$, find the distribution of $Z = X/Y$ if $Y \neq 0$ and $Z = 0$ otherwise.

Ex. 4 Let X and Y be two real indepent random variables with distribution of density $f : x \mapsto (2\pi)^{-1/2} \exp(x^2/2)$. Find the law of $R = \sqrt{X^2 + Y^2}$, find the expected value of R, are X and R independent?

Ex. 5 Let (X, Y) be a random variable with value in $]0, 1[^2$ with density $f(x, y) = 3\mathbf{1}_D$, with $D = \{(x, y) \mid 0 < x^2 < y < \sqrt{x} < 1\}$. Find the conditional laws of $Y \mid X$ and $X \mid Y$, and determine their conditional means.

Ex. 5 Two points x and y are randomly chosen from the segment $[0, 1]$. Obtain the probability of the distance between x and y being greater than $1/4$.

Ex. 6 The multinomial distribution is a generalisation of the binomial distribution which models the number of outcomes falling into a range of k mutually exclusive categories out of n identical and independent trials with probabilities $p_1, \ldots, p_k$. The probability mass function is given by

$$P\left(\mathrm{x}_1 = x_1, \cdots, \mathrm{x}_k = x_k\right) = \frac{n!}{x_1! \cdots x_k!} p_1^{x_1} \cdots p_k^{x_k}$$

where $x_i \in \{0, \ldots, n\}$ and $\sum_{i=1}^k x_k = n$, with $n > 0$ and $\sum_{i=1}^k p_k = 1$.

Given a collection of independent random variates $y_1, \ldots, y_k$ such that $y_i \sim \text{Poisson}\left(\lambda_i\right), i = 1, 2, \ldots, k$, show that for any $n > 0$ the distribution of $[y_1, \ldots, y_k]$ conditional to $\sum_{i=1}^k y_i = n$ is multinomial with parameters n and $p_i = \lambda_i / \sum_{i=1}^k \lambda_i$.

Ex. 7 Given x $\sim N(0, 1)$ and y $\sim N(1, 1)$, obtain the distribution of the random variate

$$z = \frac{x + y}{3} + 2$$

assuming that x and y are independent random variates.

Ex. 8 The joint probability mass function of a certain random vector $(\mathrm{x}, \mathrm{y})'$ is given by

$$P(\mathrm{x} = x, \mathrm{y} = y) = c(x + y) \quad x = 1, 2, 3, 4; y = 1, 2, 3$$

Determine:

1. The value of c.
2. The marginal probability mass functions of x and y.
3. The conditional probability mass function of x given $\mathrm{y} = y$ and y given $\mathrm{x} = x$.
4. Investigate the independence of x and y.

Ex. 9 To people enter at the same time two phone boxes, we write X and Y the communication times with same distribution $\mathcal{E}$, assumed independent.

1. Let T be the waiting time before one of the boxes becomes available. Compute the distribution of T and its mean.
2. We write U the time the first person to finish waits the second one, compute the law of U, compare the mean waiting time to the mean length of the communication $1/\alpha$.

Ex. 10 Let $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$, be a random matrix with A, B, C and D rvs independent with absolutely continuous distribution. Show that M is almost surely invertible (this result can be generalised to larger matrices).

Ex. 11 Let X and Y be two real rvs centered with variance 1. Let ρ be their correlation coefficient. Show that $Z = X - \rho Y$ and Y are decorrelated.