Tutorial 1: Multidimensional Random Variables

Ex. 1 $P((X,Y)=(x,y))=0$ if $y>x$, $P((X,Y)=(x,y))=P(W_1=x,W_2=y)+P(W_1=y,W_2=y)=2/f$ else. Where W_1 and W_2 are the results of the dices and f the number of faces of the dices (usually $f\geq 4$). Of cource X and Y are not independent, $P(X=6,Y=1)=0\neq P(X=6)P(Y=1)$.

This is an example of *order statistic* that is often used and has lots of good properties.

Ex. 2 The density writes $\mathbf{1}_{(x,y)\in D}$, where D is the disk of center 0 and radius 1. To compute the means of x and y we just need to see that the distribution is symmetric w.r.t 0, and thus $\mathbb{E}[X]=\mathbb{E}[y]=0$.

For the covariance we need to compute:

$$\begin{aligned}\mathbb{E}[(X-\mathbb{E}[X])(Y-\mathbb{E}[Y])]=\mathbb{E}[XY]&=\int xy\mathbf{1}_{(x,y)\in D}\mathrm{d}x\mathrm{d}y\\ &=\pi\int_{x=-1}^{1}\int_{y=-\sqrt{1-x^2}}^{\sqrt{1-x^2}}xy\mathrm{d}x\mathrm{d}y\\ &=\int_{-1}^{1}0\mathrm{d}x=0.\end{aligned}$$

So the variables are uncorrelated.

Nevertheless they are not independent, as $P((X,Y)\in[0.9,0.9]^2)=0\neq P(X\geq 0.9)P(Y\geq 0.9)$.

Ex. 3 To determine the distribution of $Z=X/Y$ we need to compute $\mathbb{E}[f(Z)]$ for any f. Note that the value when $Y=0$ is not important as the densities are defined almost everywhere. We us the change of variable $(x,y)\mapsto(x/y,y)=(z,w)$ which is a C^1 diffeomorphism. This transformation has Jacobian $|w|$

$$\begin{aligned}\mathbb{E}[f(Z)]&=\frac{1}{2\pi}\int_{\mathbb{R}^2}f(x/y)\exp(-x^2/2-y^2/2)\mathrm{d}x\mathrm{d}y\\ &=\frac{1}{2\pi}\int_{\mathbb{R}^2}f(z)|w|\exp(-x^2/2-y^2/2)\mathrm{d}x\mathrm{d}y.\\ &=\frac{1}{2\pi}\int_{\mathbb{R}^2}f(z)|w|\exp(-w^2(z^2+1)/2)\mathrm{d}z\mathrm{d}w.\end{aligned}$$

We recognize that w follows conditionally on z a normal distribution with variance $1/\sqrt{z^2+1}$, we just need to add the constants so that it integrates to 1.

$$
\begin{aligned}
&= \frac{1}{2\pi}\int_{\mathbb{R}} f(z)\int_{\mathbb{R}} \frac{1}{\sqrt{2\pi(z^2+1)}}|w|\exp(-w^2(z^2+1)/2)\mathrm{d}w\mathrm{d}z \\
&= \int_{\mathbb{R}} f(z)\frac{1}{2\pi}\frac{1}{\sqrt{2\pi(z^2+1)}}\frac{2\sqrt{2}}{\sqrt{\pi(z^2+1)}}\mathrm{d}z \\
&= \int_{\mathbb{R}} f(z)\frac{1}{\pi(z^2+1)}\mathrm{d}z.
\end{aligned}
$$

Which corresponds to the standard Cauchy distribution.

Ex. 4 Similarly as the previous exercise, we use the change of variable $(x, y) = (r\cos(\theta), r\sin(\theta))$, which has jacobian r.

$$
\begin{aligned}
&\int_{\mathbb{R}^2} f(\sqrt{x^2+y^2})\exp(-(x^2+y^2)/2)2\pi)\mathrm{d}x\mathrm{d}y \\
&= \int_{\mathbb{R}^+\times[0,2\pi]} f(r)\exp(-r^2/2)r(2\pi)\mathrm{d}r\mathrm{d}\theta \\
&= \int_{\mathbb{R}^+} f(r)r\exp(-r^2/2)\mathrm{d}r.
\end{aligned}
$$

Which corresponds to the law $\chi(n)$ (that is the square root of the χ^2).

To see if X and R are independent, it suffices to say that knowing R, $X \in [-R, R]$, while marginally X has value on $\mathbb{R}$. We can compute the conditional distribution of X knowing R using once again the polar coordinates.

For that, we know that knowing R the point is uniformly distributed on the circle (see previous integral). We can proceed similarly, using the change of variable $\theta = \arccos(x/r)$ with jacobian $1/\sqrt{r^2-x^2}$.

$$
\begin{aligned}
\mathbb{E}[f(X) \mid R] &= \int_{[0,2\pi]} f(r\cos(\theta))\mathrm{d}\theta \\
&= \int_{[0,\pi]} f(r\cos(\theta))\mathrm{d}\theta + \int_{[-\pi,0]} f(r\cos(\theta))\mathrm{d}\theta \\
&= \int_{[-R,R]} f(x)\frac{1}{\sqrt{r^2-x^2}}\mathrm{d}x
\end{aligned}
$$

Ex. 5 Let's start by computing the marginals:

$$
f_X(x) = \int y = x^{2\sqrt{x}3\mathrm{d}y=3(\sqrt{x}-x^2).} \tag{1}
$$

Note that this is perfectly symmetric for Y (D is symmetric w.r.t. $x \leftrightarrow y$) and thus that:

$$
f_Y(y) = \sqrt{y} - y^2.
$$

Similarly, the conditionals will be symmetric:

$$f_{X|Y} = \mathbf{1}_D(x,y)\frac{1}{\sqrt{y-y^2}}.$$

And we can compute the mean:

$$\mathbb{E}[X \mid Y = y] = \int_R \mathbf{1}_D(x,y)\frac{x}{\sqrt{y-y^2}} = \frac{y-y^4}{\sqrt{y-y^2}}.$$

Ex. 5 This events rewrite $p = P(x \in [0,1] \cap [y-1/4, y+1/4])$. To compute this probability we need to differentiate the case $y \geq 3/4$ and $y \leq 3/4$. In the end we have:

$$\begin{aligned} p &= \int_{y=1/4}^{3/4} 1/2\mathrm{d}y + 2\int_{y=0}^{1/4} 1/4 + y\mathrm{d}y \\ &= 1/4 + 2(1/16 + 1/32) = 1/4 + 3/16 = 7/16. \end{aligned}$$

Ex. 6 We start by writing the joint distribution of $(y_1, \ldots, y_n, z)$ where $z = \sum y_i$:

$$P(Y_1 = y_1, \ldots, Y_n = y_n, Z = z) = \begin{cases} 0 & \text{if } z \neq \sum y_i \\ P(Y_1 = y_1, \ldots, Y_n = y_n) & \text{otherwise} \end{cases}$$

The marginal density of Z is Poisson (famous property) with parameter $\lambda_1 + \cdots + \lambda_n$. Then we can simply write the conditonal density:

$$P(Y_1 = y_1, \ldots, Y_n = y_n \mid \sum Y_i = n) = \frac{\lambda_1^{y_1} \ldots \lambda_n^{y_n} \exp(-\sum \lambda_i)}{y_1! \ldots y_n!} \frac{n!}{(\sum \lambda_i)^n \exp(-\sum \lambda_i)}.$$

Which is exactly what we were looking for.

Ex. 7 The sum of two normally distributed r.v. is normally distributed, which leads to: $x + y \sim \mathcal{N}(1,2)$, and then $z \sim \mathcal{N}(7/3, 2/9)$.

Ex. 8

1. We need to have $\sum_{y \in \{1,2,3\}, x \in \{1,2,3,4\}} c(x+y) = 1$. The sum equals $54c$ and thus $c = 1/54$.
2. The marginals are $P(x) = \sum_y c(x+y) = c(3x+6)$ and $P(y) = c(10+4y)$.
3. The conditional is then $P(x \mid y) = \frac{x+y}{10+4y}$, and similarly for the other conditional.
4. These variables are clearly not independent, the conditional densities show it.

Ex. 9

1. T is the minimum of two independent Exponential distributions. We have then: $P(T \geq t) = P(X \geq t)P(Y \geq t) = \exp(-2\lambda t)$, thus $T \sim \mathcal{E}(2\lambda)$.

2. We can prove that the exponential distribution is *memoryless* for that it is sufficient to prove that $P(X \geq \ell + t \mid X \geq \ell) = P(X \geq t)$, for that we write the conditional probability: $P(X \geq \ell + t \mid X \geq \ell) = P(X \geq \ell + t \cap X \geq t)/P(X \geq \ell) = P(X \geq \ell + t)/P(X \geq \ell) = \exp(-\lambda(t+\ell))/\exp(-\lambda\ell)$. Knowing that, we conclude that the time U has same distribution as Y, and that the second person will stay twice as long in mean knowing that they did not leave first.

Ex. 10 It suffices to investigate the distribution of $AD - CB$, which has an absolutely continuous distribution, as A, B, C and D have absolutely continuous distributions and are independent. That is $P(AD - CB = 0) = P(AB = CD) + P(AB = 0, CD = 0) = 0$ because everything is independent and is absolutely continuous

Ex. 11

$$Cov(Y, Z) = Cov(X - \rho Y, Y) = Cov(X, Y) - \rho Cov(Y, Y) = \rho - \rho = 0,$$

where we used the fact that if $Var(X) = Var(Y) = 1$ then $Cov(X, Y) = \rho$.