An MRA Approach to Surface Completion and Image Inpainting*

by

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Abstract

The objective of this paper is to introduce a multi-resolution approximation (MRA) approach to the study of continuous function extensions with emphasis on surface completion and image inpainting. Along the line of the notion of diffusion maps introduced by Coifman and Lafon with some "heat kernels" as integral kernels of these operators in formulating the diffusion maps, we apply the directional derivatives of the heat kernels with respect to the inner normal vectors (on the boundary of the hole to be filled in) as integral kernels of the "propagation" operators. The extension operators defined by propagations followed by the corresponding sequent diffusion processes provide the MRA continuous function extensions to be discussed in this paper. As a case study, Green's functions of some "anisotropic" differential operators are used as heat kernels, and the corresponding extension operators provide a vehicle to transport the surface or image data, along with some mixed derivatives, from the exterior of the hole to recover the missing data in the hole in an MRA fashion, with the propagated mixed derivative data to provide the surface or image "details" in the hole. An error formula in terms of the heat kernels is formulated, and this formula is applied to give the exact order of approximation for the isotropic setting.

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1. Introduction

Extension of continuous functions is a classical problem with a very long history, dating back at least to Urysohns Lemma, the Tietze Extension, and Whitneys Extension Theorem (see, for example, [2, 24, 29], and references therein). However, it was perhaps the recent paper [7] by Yu Brudnyi and P. Shartsman, on the extension of smooth functions, that facilitates in stimulating the more current and exiting work of Charles Fefferman and his colleagues, in a series of papers [6, 17, 18, 19, 20, 21, 23], on the study of linear operators for smooth function extensions and data fitting.

On the other hand, the subject of image inpainting is relatively new. In fact, the terminology of "digital image inpainting" was coined only as recently as the turn of the century, by Sapiro, Ballester, Bertalmio, and Caselles in their paper [5] presented at the 2000 SIGGRAPH Conference. In a later SIGGRAPH Conference Proceeding paper [16] published in 2003, Drori, Cohen-Or, and Yeshurun studied image inpainting by considering larger holes, and/or with missing large-scale structures and smooth areas to recover, and coined this problem area as "image completion" (see also [26]).

While the spirit of image inpainting/completion is no different from that of smooth function extension, the general approaches of these two problem areas are quite different. Indeed, for image inpainting, with the exception of a few attempts, such as using non-decimated tight wavelet frames to better achieve redundancy [8] or variational formulation in terms of wavelet coefficients [12], the universal approach has been based on numerical solution of certain partial differential equations, which are either formulated in terms of steepest decent applied to the Euler-Lagrange equations of minimum-energy models (see, for example, [4, 11, 28]), or modeled as Navier-Stokes equations for incompressible fluid [3]. Here, total variation (or TV) is the commonly used density function for the internal energy functional.

In this paper, we introduce an MRA approach to filling in holes for continuous function extensions with application to surface completion and image inpainting. Along the line of study of diffusion maps with certain "heat kernels" as kernels of the corresponding integral operators, introduced by Coifman and Lafon and thoroughly studied in [15] and by others, we apply the directional derivatives of these heat kernels with respect to the inner normal vectors (on the boundary of the hole to be filled in) as integral kernels of the data "propagation" operators. In other words, while each integral transform

$$\big(T_j f\big)(x) = \int_D f(y) G_j(x,y)dy, \tag{1}$$

with heat kernel $G_j(x,y)$ can be used to introduce diffusion maps, the propagation operators P_j, defined by

$$\big(P_j v\big)(x) = \int_{\partial D} v(y(s)) g_j(x, y(s))ds, \tag{2}$$

where

$$g_j(x,y) := -c_{j-1}(y)\Big(\frac{\partial}{\partial \mathbf{N}} G_j(x,\cdot)\Big)(y) \tag{3}$$

(with the directional derivative taken with respect to the unit inner normal vectors $\mathbf{N}$ on the boundary ∂D of D), provide a vehicle for transporting certain data $d_j(F)$ from the exterior of the hole D to recover the missing data in the hole D. We will consider $d_0(F) = F\big|_{\partial D}$, where $F(x)$ denotes the surface or image data function in the exterior of D; and for increasing values of $j = 1, 2, \ldots, n$, $d_j(F)$ will denote certain mixed derivatives of $F(x)$ of increasing orders.

We now introduce the MRA "detail-extension" operators

$$E_j = T_0 \ \cdots \ T_{j-1} P_j, \quad j = 1, \cdots, n. \tag{4}$$

While the "ground-level" propagation operator P_0 is used to provide a continuous function extension by using the data $d_0(F) = F\big|_{\partial D}$ of function values, this extension is usually unsatisfactory, particularly for application to surface completion and image inpainting, since surface geometry and image features are not well represented by the data $d_0(F)$. To import geometry and/or feature details to fill in the hole D, the extension operators $E_1, \cdots, E_n$ can be applied to certain mixed derivative data $d_1(F), \cdots, d_n(F)$, respectively, and the process can be carried out consecutively to add MRA details to the ground-level continuous function extension.

As an illustrative case study, we apply the lagged-diffusivity formulation in [28] to linearize the Perona-Malik diffusion operators (see [25]) in formulating the heat kernels, and hence the detail-extension operators. For this particular consideration, the integral operators in (1) can be used to derive a compact error formula to give the sharp approximation order.

2. Anisotropic data propagation

A popular and effective mathematical model for image noise reduction (commonly called "image de-noising") is the following anisotropic diffusion partial defferential equation (PDE), with image domain $D \subset \mathbb{R}^2$ and non-constant diffusion conductivity function $c(p), 0 \leq p < \infty$, introduced and studied by Perona and Malik [25]:

$$\begin{cases} \dfrac{\partial}{\partial t}\, u & = \nabla \cdot (c(|\nabla u|)\nabla u) \quad \text{in } D, \quad t \geq 0, \\ \dfrac{\partial}{\partial \mathbf{N}}\, u\Big|_{\partial D} & = 0, \\ u(x,0) & = u_0(x), \qquad x \text{ in } D \end{cases} \tag{5}$$

Hence, $u_0(x)$ denotes the (noisy) image to be processed, and the solution $u = u(x,t),\ x \in D$ and $t > 0$, of the initial-valued Neumann PDE (5) is the one-parameter family of the de-noised images, with the time variable t as the parameter. (See, for example, [14, 25] for typical conductivity functions.)

A rigorous mathematical treatment of the diffusion PDE (5), particularly in regularization of the diffusion conductivity $c(p) = c(|\nabla u|)$ by convoluting u with some lowpass function $s(x)$ such as the Gaussian (i.e. replacing with $p = |\nabla u|$ by $|\nabla s * u|$), was thoroughly studied by Lions et al. [1] [9]. In another development, Vogel and Oman [28] considered the “lagged diffusivity iteration” approach:

$$\begin{cases} \dfrac{\partial}{\partial t}\, u^j & = \nabla \cdot (c\ (\ |\nabla u^{j-1}|\ )\ \nabla u^j) \quad \text{in} \quad D, \quad t \geq 0, \\ \dfrac{\partial}{\partial \mathbf{N}}\, u^j \Big|_{\partial D} & = 0, \\ u^j(x,0) & = u_0(x), \qquad x \in\ D \end{cases} \tag{6}$$

where $j = 1, 2, \ldots,$ and $u^0 = u_0$ thus decoupling the nonlinear PDE (5) to an iterative family of linear PDE's (6). Again by regularization of the diffusion conductivity as mentioned above, it can be proved that for any $T > 0$, there exists some $r, 0 < r < 1$, such that $u^j(x,t)$ converges uniformly for $(x,t) \in D \times [0,T]$ to the solution $u(x,t)$ of the regularized nonlinear PDE, with $o(r^j)$ rate of convergence.

We now turn to the problem of continuous function extensions. Let Ω be a simply connected domain in $\mathbb{R}^d, d \geq 1$, and let $D_1, \cdots, D_m$ be bounded and simply connected subdomains of Ω that are pairwise disjoint, so that $\tilde{\Omega} := \Omega \backslash (D_1 \cup \cdots \cup D_m)$ is a d-dimensional “Swiss cheese” with m holes. The problem is to extend a given continuous function $F(x)$ defined on $\tilde{\Omega}$ to some continuous function $F_e(x)$ on Ω, so that $F_e(x)$ possesses certain distinct features of $F(x)$ in the “holes” $D_1, \cdots, D_m$. The goal of this paper is to introduce an MRA approach to transport these features from $\tilde{\Omega}$ to the holes $D_1, \cdots, D_m$. In application to surface completion, these could be geospatial features in terrain modeling, or surface geometrical characteristics in surface design for manufacturing. On the other hand, in application to image inpainting, features of multiresolution image edges and textures are of utmost importance. Of course the problem of continuous function extension is no different from that of function data recovery, with the missing data considered as function values in the holes $D_1, \cdots, D_m$ to be recovered.

Similar to just about all methods for surface completion (see [13] and references therein) and image inpainting (see [11, 27]), our method is also local, in the sense that only data information $F(x)$ in the neighborhoods of the boundaries $\partial D_1, \cdots, \partial D_m$ will be used to extend $F(x)$ from $\tilde{\Omega}$ to $F_e(x)$ in filling in the holes $D_1, \cdots, D_m$, respectively. Hence, we may consider only one hole $D \subset \Omega$ to be filled in, and $\tilde{\Omega} := \Omega \backslash D$. Throughout our discussions, it is necessary to assume that the boundary ∂D of D satisfies certain (piecewise) smooth condition. For convenience, we will simply state that ∂D is "sufficiently smooth." Similarly we will require $F(x)$ to have certain order of partial derivatives, by simply stating that F satisfies "sufficiently high order of smoothness." More precise smoothness conditions for the validity of our study should be clear to the interested reader.

To apply the idea of anisotropic diffusion to continuous function extensions, the first obstacle is the lack of initial conditions (such as $u_0(x)$ for the Perona-Malik PDE (5)). For this reason, we eliminate the time variable by considering the steady-sate equation (such as $\frac{\partial}{\partial t} u(x,t) = 0$ in (5) and (6)). In addition, due to the necessity of "linearization" of the elliptic spatial differential operator in the anisotropic PDE model, we follow the "lagged diffusivity iteration" in [28] with or without regularization of the diffusion conductivity $c(|\nabla u|)$ as in [1,9]. Hence, in the following, we will use the notation

$$c_j(x) := c(|\nabla u^j(x)|) \quad \text{or} \quad c(|\nabla s * u^j(x)|),\ x \in D \subset \mathbb{R}^d, \tag{7}$$

for $j = 0, \ldots, n-1$, where $s(x)$ is any suitable lowpass function such as the Gaussian (with any desirable choice of variance) and u^j is the solution of the linear of PDE (6) assuming knowledge of u^{j-1}. We therefore need some suitable function $c_{-1} \in C^1(D) \cap C(\bar{D})$ to get started. If an initial continuous function extension $F_e(x)$ of $F(x)$ from $\tilde{\Omega}$ to Ω is available, we may choose $c_{-1}(x) = C(|\nabla u^{-1}(x)|)$ or $c(|\nabla s * u^{-1}(x)|)$, where $u^{-1} = F_e\big|_D$. Otherwise, we simply set $c_{-1}(x) = 1$, independent of the choice of the conductivity function $c(p)$. For this particular choice, we have

$$\nabla \cdot \big(c_{-1}(x) \nabla f(x)\big) = (\triangle f)(x), \tag{8}$$

where $\triangle = \nabla \cdot \nabla$ denotes, as usual, the Laplacian operator. For $j = 0, \ldots, n$, we will therefore consider the "lagged diffusivity" operators

$$(L_j f)(x) = \nabla \cdot \big(c_{j-1}(x) \nabla f(x)\big), \quad x \in D, \tag{9}$$

where $L_0 = \triangle$ according to (8) if no initial continuous extension of $F(x)$ is available. The heat kernels to be considered in this paper are Green's functions $G_j(x, y)$ of the differential operators L_j; that is,

$$\begin{cases} (L_j)G_j(x,y) & = \delta(x,y), \quad x, y \in D \\ G_j(x,y)\Big|_{y \in \partial D} & = 0, \quad x \in D. \end{cases}$$

As already discussed in the introduction, the "diffusion operators" T_j are the integral operators with kernels $G_j(x, y)$ as defined in (1). Hence, assuming that ∂D is sufficiently smooth, we may use the (heat) "propagation kernels"

$$g_j(x,y) := -c_{j-1}(y)\Big(\frac{\partial}{\partial \mathbf{N}} G_j(x,\cdot)\Big)(y)$$

where $\mathbf{N}$ denotes the unit inner normal vector of ∂D at $y \in \partial D$, to introduce the "propagation operators" P_j defined in (2), where the integral is taken over the boundary ∂D, with surface area differential denoted by ds. The detail-extension operators E_j defined in (4) will be used to provide details of the extension $F_e(x)$ of $F_(x)$ from $\tilde{\Omega}$ to Ω. The following extension of Green's formula will be an important tool for our discussion in the next section.

Lemma 1. *For sufficiently smooth functions $f, h, c_{j-1} \in C(D)$ with $h|_{\partial D} = 0$,*

$$\int_D f(y)(L_j h)(y)dy = \int_D h(y)(L_j f)(y)dy - \int_{\partial D} c_{j-1}(y) f(y) \frac{\partial}{\partial \mathbf{N}}\ h(y) ds, \tag{10}$$

where y is parametrized in terms of s.

Proof: By taking the difference of the two identities:

$$\nabla \cdot \big(f c_{j-1} \nabla h\big) = c_{j-1}\big(\nabla f\big) \cdot \big(\nabla h\big) + f L_j h;$$

$$\nabla \cdot \big(h c_{j-1} \nabla f\big) = c_{j-1}\big(\nabla h\big) \cdot \big(\nabla f\big) + h L_j f,$$

we have

$$f L_j h = h L_j f + \nabla \cdot \big(c_{j-1}\big(f \nabla h - h \nabla f\big)\big).$$

Hence, formula (10) follows immediately by applying the Divergence Theorem and noting that

$$\frac{\partial}{\partial \mathbf{N}} h(y) = \big(\nabla h(y)\big) \cdot \mathbf{N},$$

since $h|_{\partial D} = 0$. Observe that the negative sign in (10) results from using the inner normal. ■

3. Continuous function extension

Our MRA approach to continuous function extension to be discussed in this section was motivated by the interesting work of Chan and Shen [10,11], where the second order of approximation by the harmonic continuous function extension u^h is improved to the fourth order by incorporating some bi-harmonic function u^a that vanishes on the boundary (see [10, pp. 258–263]). More precisely, let $F(x)$ be a sufficiently smooth function defined in $\Omega \subset \mathbb{R}^d$ but with missing portion $F_D := F\big|_D$, $D \subset \Omega$, to be recovered, and let the diameter of D be diam $(D) = 2\epsilon > 0$. Also, denote the $L^\infty(D)$ norm by $\|\ \|_D$. Then it is shown in [10, 11] that the solution u^h of the Dirichlet problem

$$\begin{cases} \triangle u^h = 0 \quad \text{in} \quad D, \\ u^h\big|_{\partial D} = F\big|_{\partial D} \quad \text{on} \quad \partial D \end{cases} \tag{11}$$

provides a "linear inpainting" scheme for the recovery of F_D, i.e. with second order uniform approximation error bound, namely:

$$\big\|F_D - u^h\big\|_D = O(\epsilon^2), \tag{12}$$

and that by adding u^a to u^h, the bi-harmonic extension

$$u^{bh} := u^h + u^a \tag{13}$$

of F provides a "cubic inpainting" scheme for the recovery of F_D, namely:

$$\big\|F_D - u^{bh}\big\|_D = O(\epsilon^4), \tag{14}$$

(see [10, Theorem 6.5]), where u^a in (13) is the solution of the PDE

$$\begin{cases} \triangle u^a = \tilde{u}^h \quad \text{in} \quad D, \\ u^a\big|_{\partial D} = 0, \end{cases} \tag{15}$$

and the function $\tilde{u}^h$ in (15) is generated by linear inpainting by using the boundary data $\triangle F|_{\partial D}$; that is, $\tilde{u}^h$ solves the Dirichlet problem (11) but with boundary data $\triangle F|_{\partial D}$ instead of $F|_{\partial D}$.

Let $G^{\triangle}(x,y)$ denote Green's function of the Laplacian operator $\triangle$ and

$$g^{\triangle}(x,y) = -\Big(\frac{\partial}{\partial \mathbf{N}} G^{\triangle}(x,\cdot)\Big)(y).$$

Then in terms of the corresponding propagation operator $P^{\triangle}$ and detail-extension operator $E^{\triangle} := T^{\triangle} P^{\triangle}$, where $T^{\triangle}$ and $P^{\triangle}$ are defined in (1) and (2) with G_j and g_j replaced by $G^{\triangle}$ and $g^{\triangle}$, respectively (i.e. by considering the constant conductivity function $c(p) = 1$), the solutions u^h and u^a of (11) and (15) are given by

$$\begin{cases} u^h = P^{\triangle}(F\big|_{\partial D}) \\ u^a = E^{\triangle}(\triangle F\big|_{\partial D}), \end{cases} \tag{16}$$

by applying Lemma 1 with $c_{j-1}(y) = 1$ and $h(y) = G^{\triangle}(x,y),\ x \in D$. In other words, the "detail" u^a is used by Chan and Shen [10, 11] to improve the order of approximation of u^h as in (13), from (12) to (14).

In the present paper, this point of view is extended from a single level of detail extension to an MRA extension scheme, and from the Laplacian operator $\triangle$ to the general lagged diffusivity anisotropic operators L_j defined in (9), with $c_{j-1}(x)$ as in (7). Precisely, while the "ground level" extension

$$u_0 = P_0(F\big|_{\partial D}), \tag{17}$$

with $j = 0$, in (2) – (3), provides a continuous function extension of F from $\tilde{\Omega} = \Omega \backslash D$ to D, the MRA detail extensions are defined by

$$w_j = E_j(d_j(F)), \quad j = 1, \ldots, n, \tag{18}$$

with E_j defined in (4), where the $d_j(F)$'s denote certain desirable mixed derivatives of F of increasing orders for $j = 1, \ldots, n$, computed in $\tilde{\Omega}$. By adopting the standard convention that an empty product is the identity; namely: $T_1 \cdots T_{j-1} =$ Identity operator for $j = 1$, we observe that for each $j = 1, \ldots, n, w = w_j$ is the solution of the PDE

$$\begin{cases} L_0 w = (T_1 \cdots T_{j-1} P_j)(d_j(F)) \quad \text{in } D, \\ w\big|_{\partial D} = 0 \end{cases}$$

and hence, the extension

$$u_n := u_0 + w_1 + \cdots + w_n \quad \text{in} \;\; D \tag{19}$$

preserves the property of continuous function extension, in that

$$u_n\big|_{\partial D} = u_0\big|_{\partial D} = F\big|_{\partial D}, \quad n = 1, 2, \ldots.$$

Since each w_j in (19) is obtained without any alteration of u_{j-1}, for $j = 1, \ldots, n$, we have an MRA structure, with u_j as the j^{th} level continuous function extension of F from $\tilde{\Omega} = \Omega \backslash D$ to D by adding the j^{th}-level details w_j, for $j = 1, \ldots, n$.

By choosing the mixed derivatives

$$d_j(F) := L_{j-1} \cdots L_0 F \quad \text{on} \;\; \partial D, \tag{20}$$

we have the following error formula for the continuous function extension u_n of F from $\tilde{\Omega} = \Omega \backslash D$ to D.

Theorem 1. *Let F be a sufficiently smooth function in Ω with missing portion $F_D := F\big|_D$, and let w_j be the j^{th} level details with boundary data $d_j(F)$ as in (20). Then the error of recovery of F_D by u_n is given by*

$$F_D(x) - u_n(x) = (T_0 \cdots T_n)(L_n \cdots L_0 F_D)(x), \quad x \in D. \tag{21}$$

In particular,

$$\big\|F_D - u_n\big\|_D \leq C_n \prod_{j=0}^{n} \big\|T_j\big\|, \tag{22}$$

where $C_n := \big\|(L_n \cdots L_0)F_D\big\|_D$ *and*

$$\big\|T_j\big\| := \sup_{x\in D}\int_D |G_j(x,y)|dy. \tag{23}$$

Proof: By applying Lemma 1 with $h(y) = G_0(x,y), x \in D$, we have

$$\begin{aligned}
F_D(x) &= \int_D F_D(y)L_0G_0(x,y)dy \\
&= \int_D (L_0F_0)(y)G_0(x,y)dy - \int_{\partial D} F(y)c_{-1}(y)\frac{\partial}{\partial \mathbf{N}}G_0(x,y)ds \\
&= (T_0(L_0F_D))(x) + u_0(x)
\end{aligned}$$

or

$$F_D(x) - u_0(x) = (T_0(L_0F_D))(x), \tag{24}$$

which verifies (21) for $j = 0$. For $1 \le j \le n$, we may again apply Lemma 1 with $h(y) = G_j(x,y), x \in D$, to obtain

$$\begin{aligned}
f_j(x) :&= (L_{j-1}\cdots L_0F_D)(x) = \int_D f_j(y)L_jG_j(x,y)dy \\
&= \int_D (L_jf_j)(y)G_j(x,y)dy - \int_{\partial D} f_j(y)g_j(x,y)ds \\
&= (T_j(L_j\cdots L_0F_D))(x) - \int_{\partial D}(d_jF)(y)g_j(x,y)ds \\
&= (T_j(L_j\cdots L_0F_D))(x) + P_j(d_jF)(x).
\end{aligned}$$

Hence, by applying the operator $T_0\cdots T_{j-1}$ to each term and the definition (4), we have

$$(T_0\cdots T_{j-1})(L_{j-1}\cdots L_0F_D)(x) = (T_0\cdots T_j)(L_j\cdots L_0F_D)(x) + (E_j(d_jF))(x),$$

which, by telescoping, yields

$$(T_0(L_0F_D))(x) - \sum_{j=1}^{n}(E_j(d_jF))(x) = (T_0\cdots T_n)(L_n\cdots L_0F_D)(x).$$

It therefore follows from (18) and (24) that

$$F_D(x) - (u_0 + w_1 + \cdots + w_n)(x) = (T_0\cdots T_n)(L_n\cdots L_0F_D)(x),$$

which agrees with the error formula (21) by applying the definition of u_n in (19). ■

Therefore, to determine the order of approximation by MRA continuous function extension, we need to estimate the norms $\|T_j\|$ of the "diffusion" operators.

4. Smooth inpainting and surface completion

The objective of this section is to extend the results of smooth inpainting in Chan and Shen [10,11] and derive sharp error estimates. That is, we restrict our study to the isotropic setting, with constant conductivity function $c(p) = 1$, so that $L_j = \triangle$ and $G_j(x, y) = G^{\triangle}(x, y)$ for all j. To apply Theorem 1, we consider the boundary data:

$$d_0(F) = F\big|_{\partial D} \quad \text{and} \quad d_j(F) = \triangle^j F\big|_{\partial D}, \quad j = 1, \ldots, n, \tag{25}$$

so that the error bound in (22) becomes

$$\left\|F_D - u_n\right\|_D \leq \left\|\triangle^{n+1} F_D\right\|_D \, \left\|T^{\triangle}\right\|^{n+1} \tag{26}$$

where

$$\left\|T^{\triangle}\right\| := \sup_{x\in D} \int_D \left|G^{\triangle}(x, y) dy\right| = \sup_{x\in D} \Big|\int_D G^{\triangle}(x, y) dy\Big| \tag{27}$$

by the property of Green's function for the Laplacian operator $\triangle$. Hence, by following [10, pp. 260–261], we set

$$u(x) := \int_D G^{\triangle}(x, y) dy$$

and observe that $u(x)$ is the solution of the PDE

$$\begin{cases} \triangle u(x) = 1, & x \in D \\ \quad u(x) = 0, & x \in \partial D \end{cases} \tag{28}$$

by applying Lemma 1. For the ball $D = D_\epsilon := \{|x - x_0| < \epsilon\} \subset \mathbb{R}^d$ with diameter $= 2\epsilon$, it is clear, as observed in [10, Lemma 6.4], that this solution is given by

$$u(x) = \frac{|x - x_0|^2 - \epsilon^2}{2d}.$$

Hence, the norm $\|T^{\triangle}\|$ in (27) has the sharp estimate $\|T^{\triangle}\| \leq \epsilon^2/2d$, and it follows from (26) that

$$\|F_D - u_n\|_D \leq \|\triangle^{n+1} F_D\|_D \left(\frac{\epsilon^2}{2d}\right)^{n+1} \quad n = 0, 1, \ldots, \tag{29}$$

which yields (12) and (14) with $n = 0$ and 1, respectively for $D = D_\epsilon$, by noting that $u^h = u_0$ and $u^{bh} = u^h + u^a = u_0 + w_1$. For $D \subset D_\epsilon$ in general, the same argument in [10, Lemma 6.3] for $\mathbb{R}^2$ applies to $\mathbb{R}^d$ to yield the estimate (29).

Of course if D has diameter $\geq 2\sqrt{2d}$, the error bound in (29) is useless. Fortunately, for application to continuous surface completion and image inpainting, we are interested in dimension $d = 2$, for which Green's function $G^{\triangle}(x, y)$ of $\triangle$ for the simply connected domain D can be expressed in terms of the conformal mapping function $\phi_x(y), x \in D \subset \mathbb{R}^2$, that maps D one-to-one onto the unit disk with center at the origin, such that $\phi_x(x) = 0$ and $\phi'_x(x) > 0$. More precisely,

$$G^{\triangle}(x, y) = \frac{1}{2\pi}\ ln\ |\phi_x(y)|. \tag{30}$$

Here, as usual, $\mathbb{R}^2$ is identified with the complex plane, with x and y being considered as the complex variable and parameter, respectively. Therefore, instead of the estimate (29) as a consequence of considering $D \subset D_\epsilon$, we may compute the conformal map $\phi_x(y)$ of some polygonal region that contains D by using the Schwarz-Christoffel mapping (see, for example, [22]), and then estimate the operator norm in (27), with D replaced by the polygonal region, more directly.

Returning to the isotropic setting with $D \subset \mathbb{R}^d,\ d \geq 2$, we remark that the polyharmonic extension (or expansion) $u_n = u_0 + w_1 + \cdots + w_n$ of F, terms of its boundary data (25), is governed by the error formula

$$F(x) - u_n(x) = \int_D K_{n+1}(x, y)\ \triangle^{n+1}\ F(y) dy, \quad x \in D, \tag{31}$$

and satisfies the "Hermite interpolation" property:

$$\triangle^j\ (F - u_n)\Big|_{\partial D} = 0, \quad j = 0, \ldots, n, \tag{32}$$

where $K_1(x,y) := G^{\triangle}(x,y)$ and

$$K_{j+1}(x,y) := \int_{D^j} G^{\triangle}(x,y_1)\cdots G^{\triangle}(y_j,y)\; dy_1\cdots dy_j \tag{33}$$

for $j = 1,\ldots,n$.

References

1. L. Alvarez, P.-L. Lions, and J.-M. Morel, Image selective smoothing and edge detection by nonlinear diffusion II, SIAM J. Numer. Anal. **29** (1992), 845–866.
2. A. Bauer and A. Simpson, Two constructive embedding-extension theorems with applications to continuity principles and to Banach-Mazur computability, Math. Comp. **50** (2004), 351–369.
3. M. Bertalmio, A.L. Bertozzi, and G. Sapiro, Navier-Stokes, fluid dynamics and image and video inpainting, in *Proc. Conf. Comp. Vision Pattern Rec.*, 2001, pp. 355–362.
4. M. Bertalmio, L. Vese, G. Sapiro, and S. Osher, Simultaneous texture and structure image inpainting, IEEE Trans. Image Proc. **10** (2003), 882–889.
5. M. Bertalmio, G. Sapiro, C. Ballester, and V. Caselles, Image inpainting, in *Proc. ACM SIGGRAPH 2000*, pp. 417–424.
6. E. Bierstone, P. Milman, and W. Pavolucki, Higher-order tangents and Fefferman's paper on Whitney's extension problem, Ann. of Math. **164** (2006), 361–370.
7. Yu. Brudnyi and P. Shartsman, Whitney's extension problem for multivariate $C^{1,\omega}$ functions, Trans. Amer. Math. Soc. **353** (2001), 2487–2512.
8. J. Cai, R.H. Chan, and Z.W. Shen, A framelet-based image inpainting algorithm, Appl. Comp. Harm. Anal. **24** (2008), 131–149.
9. F. Catté, P.-L. Lions, J.-M. Morel, and T. Coll, Image selective smoothing and edge detection by nonlinear diffusion, SIAM J. Numer. Anal. **29** (1992), 182–193.
10. T.F. Chan and J. Shen, *Image Processing and Analysis*, SIAM Publ., Philadelphia, 2006.
11. T.F. Chan and J. Shen, Mathematical models for local nontexture inpaintings, SIAM J. Appl. Math. **62** (2002), 1019–1043.
12. T.F. Chan, J. Shen, and H. Zhou, Total variation wavelet inpainting, J. Mathematical Imaging and Vision **25** (2006), 107–125.
13. C.K. Chui and M.J. Lai, Filling polygonal holes using C^1 cubic triangular spline patches, Comp. Aided Geom. Design **17** (2000), 297–307.
14. C.K. Chui and J.Z. Wang, Wavelet-based minimal-energy approach to image restoration, Appl. Comp. Harm. Anal. **23** (2007), 114–130.

15. R. Coifman and S. Lafon, Diffusion maps, Appl. Comp. Harm. Anal. **21** (2006), 5–30.
16. I. Drori, D. Cohen-Or, and H. Yeshurun, Fragment-based image completion, in *Proc. ACM SIGGRAPH 2003*, pp. 303–312.
17. C. Fefferman, A sharp form of Whitney's extension theorem, Ann. of Math. **161** (2005), 509–577.
18. C. Fefferman, The structure of linear extension operators for C^m, Rev. Mat. Iberoamericana **23** (2007), 269–280.
19. C. Fefferman, Whitney's extension problem for C^m, Ann. of Math. **164** (2006), 313–359.
20. C. Fefferman and B. Klartog, Fitting C^m-smooth functions to data I, Ann. of Math., to appear.
21. C. Fefferman and B. Klartog, Fitting C^m-smooth functions to data II, Rev. Mat. Iberoamericana, to appear.
22. R. Greene and S. Krantz, *Function Theory of One Complex Variable*. Amer. Math. Soc. Graduate Studies in Mathematics, Vol. 40, Second Edition, Amer. Math. Society, Providence, RI, 2002.
23. B. Klartog and N. Zobin, C^1 extensions of functions and stabilization of Glaseser refinements, Rev. Mat. Iberoamericana **23** (2007), 635–669.
24. E. Michael, Some extension theorems for continuous functions, Pac. J. Math. **3** (1953), 789–806.
25. P. Perona and J. Malik, Scale-space and edge detection using anisotropic diffusion, IEEE PAML **12**(7) (1990), 629–639.
26. J. Shen, L. Yuan, J. Jia, and H.Y. Shum, Image completion with structural propagation, *SIGGRAPH 2005*, Vol. 24, pp. 861–868.
27. X.C. Tai, S. Osher, and R. Holm, Image inpainting using a TV-Stokes equation, in *Image Processing Based on Partial Differential Equations*, Tai, Lie, Chan, and Osher, Eds., Springer Heidelberg, 2007, pp. 3–27.
28. C. Vogel and M. Oman, Iterative methods for total variation denoising, SIAM J. Sci. Comput. **17** (1996), 227–238.
29. N. Zobin, Extension of smooth functions from finitely connected planar domains, J. Geom. Anal. mathnet. kaist, ac. kr. **9** (1999), 492–511.