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MRA Contextual-Recovery Extension of Smooth Functions on Manifolds

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Abstract

In a recent paper, the first author introduced an MRA (multi-resolution or multi-level approximation) approach to extend an earlier work of Chan and Shen on image inpainting, from isotropic diffusion to anisotropic diffusion and from bi-harmonic extension to multi-level lagged anisotropic diffusion extension. The objective of the present paper is to extend and generalize this work to non-stationary smooth function extension to meet the goal of inpainting missing image features, while matching the existing image content without apparent visual artifact. Our result is formulated as an MRA contextual-recovery extension for the completion of smooth functions on manifolds by deriving an error formula, from which sharp error estimates can be derived. A novel estimate for the biharmonic operator derived in this paper is a formulation of the error bound in terms the volume, as opposed to the diameter, of the image hole.

1 Introduction

Let $s \geq 2$ be a fixed integer, $\Omega \subseteq \mathbb{R}^s$ some open set, and D a domain with compact closure $\overline{D} \subset \Omega$, such that the boundary ∂D of D is sufficiently smooth for our discussion in this paper. For a smooth function $F : \Omega \to \mathbb{R}$, the problem of smooth function extension is to recover the missing portion $F|_D$ of F from information of F outside the domain D, such that the extended function is smooth in Ω and agrees with F in $\Omega \setminus D$. Although continuous function extension is a well-studied classical problem with a very long history, attention to the study of smooth function extension has not caught on till recently. In fact, it was only in the current decade that the paper [7] by Brudnyi and Shartsman played an important role in stimulating the work of Charles Fefferman and others in a series of recent papers (see [14], [15] and references therein) on the study of linear operators for smooth function extension and data fitting. On the other hand, the closely related subject of image inpainting is relatively new, with the term "digital image inpainting" coined only recently, by Bertalmio, Sapiro, Ballester, and Caselles, in their pioneering paper [5] presented at the 2000 SIGGRAPH Conference. In a later SIGGRAPH Conference Proceeding paper [13] published in 2003, Drori, Cohen-Or, and Yeshurun studied image inpainting by considering larger "image holes" D, and/or with missing large-scale structures and smooth areas to recover, and coined this problem area as "image completion" (see also [19] and references therein).

The commonly adopted approach to image inpainting is based on numerical solutions of certain PDE's (partial differential equations), with the earliest work in the form of Navier-Stokes equations for incompressible fluid (see [5] and later work by these authors and others) and the more recent research based on anisotropic diffusion PDE models formulated as steepest decent to solve the Euler-Lagrange equations of the desired minimum-energy problems. However, since no data information is available in the image hole D, the initial value corresponding to the missing image data in D is usually set to be zero. Hence, this approach puts a very heavy burden on the initial value PDE.

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In a recent work of Chan and Shen (see [9] and [10]), the initial value problem is replaced by a boundary value problem. More precisely, the inpainting data in D is first "predicted" by using the harmonic function u_0 in D with boundary value given by F on the boundary ∂D of the image hole D (where F denotes the known image data outside D), and is then "corrected" by adding the biharmonic function w_1 which vanishes on ∂D, while its Laplacian, denoted by Δw_1, matches the Laplacian of the image data ΔF on ∂D. Hence, the inpainting function introduced in [9] and [10] is the biharmonic function $u_1 := u_0 + w_1$ uniquely determined by F and ΔF on ∂D. This can be extended to multi-level inpainting, as detailed in the isotropic example in [11], namely: for any positive integer m, the inpainting function $u_m := u_0 + w_1 + \cdots + w_m$ is the $(m+1)$–fold harmonic function (i.e. $\Delta^{m+1} u_m = 0$ in D), such that $\Delta^j u_m$ matches $\Delta^j F$ for $j = 0, \cdots, m$ on ∂D, with $u_0 = F$ and $\Delta^j w_j = \Delta^j F$, for $j = 1, \cdots, m$, on ∂D.

The ultimate goal of inpainting missing image data is to recover image features in a visually seamless manner. For this purpose, it is important that the extension of $F|_{\Omega\setminus D}$ from $\Omega \setminus D$ to D, should be C^1 across the boundary ∂D in general in order to avoid visual artifact. However, the smooth functions u_m in D, obtained as above, and the image function $F|_{\Omega\setminus D}$ in $\Omega \setminus D$ do not connect in a C^1 fashion across the boundary ∂D. The main insight of the present paper is that this situation is easy to correct as follows. We may combine the two steps of the Chan–Shen algorithm, and obtain instead a biharmonic function determined uniquely by F and $\dfrac{\partial F}{\partial \nu}$ (rather than ΔF) on ∂D. Here and throughout this paper, ν denotes the unit inner normal vector defined on ∂D. As the remark after (2.1) shows, this biharmonic function on D connects with $F|_{\Omega\setminus D}$ in $\Omega \setminus D$ in a C^1 fashion across the boundary ∂D. Similarly, the solution u_m can be replaced by a $(m+1)$–fold harmonic function determined uniquely by the conditions that $\dfrac{\partial^k u_m}{\partial \nu^k} = \dfrac{\partial^k F}{\partial \nu^k}$, $k = 0, 1, \cdots, m$ on ∂D, to achieve a C^m extension. In particular, if m is odd, say $m = 2N + 1$ for some integer $N \geq 0$, then these boundary conditions are equivalent to the conditions $\Delta^k u_m = \Delta^k F$ and $\dfrac{\partial}{\partial \nu}\Delta^k u_m = \dfrac{\partial}{\partial \nu}\Delta^k F$ on ∂D for $k = 0, 1, \cdots, N$. An MRA can then be developed further as in [11] so as to achieve better and better error estimates. Finally, we note in this connection that the problem of extending F from $\Omega \setminus D$ to D in a C^m fashion using the boundary value problem paradigm is necessarily the problem of solving the problem $Lu = g$ on D, $\dfrac{\partial^k u}{\partial \nu^k} = \dfrac{\partial^k F}{\partial \nu^k}$, $k = 0, 1, \cdots, m$ on ∂D, for some function g defined in terms of $F|_{\Omega\setminus D}$, and some differential operator L. In this sense, our approach is the only possible one within the paradigm of using boundary value problems to obtain a C^m extension, apart from the choice of L and g. While the choice $g \equiv 0$ seems to be the most natural default choice, also consistent with the prevailing practice, the operator L may be chosen to achieve other goals.

In order to have the capability of recovering image features in the image hole D, the "stationary" operator Δ is generalized in [11] to some "non-stationary" operators: $L_0, \cdots, L_n$, with $L_j f := \mathsf{div}\big(c_{j-1}\nabla f\big)$, in that for each $j = 1, 2, \cdots, n+1$, the operator $L_{j-1} \cdots L_0$ replaces Δ^j, in the formulation of $u_0, w_1, \cdots, w_m$, respectively. Here, ∇ and div denote, as usual, the gradient and divergence operators respectively, and $c_{j-1} := c(|\nabla u_{j-1}|)$, for some univariate function c, called the conductivity function in anisotropic diffusion. The most popular choice of the function c is perhaps $c(y) = 1/y$, called TV (see, for example, [10, p.271]). Other effective choices of the conductivity function can be found in the pioneering paper [18] on image denoising by Perona and Malik. The motivation for introducing the non-stationary operators L_j to formulate the appropriate boundary value PDE's in [11] is due to the great success of "lagged diffusions" used in [21] and [8] to linearize the non-linear initial value PDE's for image denoising. In particular, it is shown in [8] that the solutions of the linear lagged-diffusion PDE's converge uniformly on $D \times [0, T]$, for any time interval $[0, T]$, to the solution of the (appropriately regularized) non-linear Perona-Malik PDE, at a geometrical rate.

With the rapid advancement of sensor, satellite, and camera electronic technologies, it is already feasible to capture fine details of atmospheric, terrain, and medical images in near real-time. The domains of definition of such images are not necessarily domains in $\mathbb{R}^s$, but are frequently domains in general manifolds, particularly Euclidean spheres (see, for example, [1] and [22]). Unfortunately, due to unfavorable environment, such as low-light and/or unpredictable obstacles, certain portions of the image data could be corrupted or even not acquirable. These portions could be replaced or created by image inpainting on manifolds. For instance, atmospheric effects in satellite imaging of mountainous terrains have been an ongoing research topic (see, for example, [20]) and even the most advanced sensors and camera electronics cannot avoid severe noise corruption in extreme low-light environment. This is a typical situation in exploratory medical imaging using endoscopes. Other exploratory medical imaging areas including CT (computed tomography), MRI (magnetic resonance imaging), photoacoustic imaging, and nuclear medicine, can also benefit from image inpainting on manifolds. On the other hand, we observe that in the case when the manifold has a bounded geometry, and the hole D is sufficiently small, one can assume that the manifold neighborhood Ω of D can be parametrized by a single chart; and using any coordinate system on Ω, one can always reduce a differential operator on Ω to an apparently different one on the Euclidean parameter domain.

The objective of the present paper is to extend and generalize the results in [11]. To include the applications to image inpainting and completion on manifolds together with feature recovery, the discussion in this paper will involve general linear partial differential operators L_j, $j = 0, 1, \cdots$, with smooth coefficients, rather than confining ourselves to such operators as the iterates of $\mathsf{div}(c_{j-1}\nabla)$ as discussed in [11]. In order not to introduce unnecessary notations, the functions $u_n, w_n, \cdots$, and the operators in the next two sections are different from those used in the previous discussion. Unlike the Laplacian operator, the Green's functions for these general operators are usually change sign (i.e., neither positive nor negative throughout D). Therefore, we cannot use the ideas in [11] to obtain the error bounds in the supremum norm. Instead, we will assume that the boundary value problem associated with each L_j corresponds to a strictly coercive Dirichlet form (see Section 2 for the definition and details). The error estimates will then be obtained in the L^2 sense in terms of the diameter of the hole, by first proving some lower bounds on the lowest eigenvalues of the operators L_j. An added advantage of this idea is the following. In the special case of the C^1 extension using the biharmonic operator, we may also use the Faber–Krahn (isoperimetric) inequality to obtain the bound in terms of the volume of D instead. This error bound is thus applicable to the study of inpainting images with long but narrow image holes.

The paper is organized as follows. In Section 2, we will study certain facts about the general differential operators and Green's formulas. The multi-level smooth function extension will be discussed in Section 3. We will also investigate sharp estimates on the L^2 difference between F and the inpainting function obtained by n steps of the MRA. The proof of Theorem 3.2, being somewhat technical, will be given in Section 4.

2 Preliminaries on differential operators and Green's formulas

In this section, we review certain facts about elliptic partial differential equations. The following discussion is based on [2, 16].

Let $S \subset \mathbb{R}^s$ be a domain with compact closure and sufficiently smooth boundary ∂S. For convenience of discussion, we will assume ∂S to be C^∞. Let $C_c^\infty(S)$ denote the space of all C^∞ functions f such that the closure of the set $\{\mathbf{x} : f(\mathbf{x}) \neq 0\}$ is a subset of S. For each integer $k \geq 0$, $H_k^0(S)$ will denote the space of all functions in the closure of $C_c^\infty(S)$ endowed with the norm

$$\|f\|_{k,S}^2 := \sum_{|\mathbf{m}| \leq k} \int_S |\mathbb{D}^{\mathbf{m}} f(\mathbf{x})|^2 d\mathbf{x},$$

where $\mathbb{D}^{\mathbf{m}}$ denotes the mixed partial derivative indicated by $\mathbf{m}$. The symbol $H_k(S)$ will denote the intersection of $H_k^0(\tilde{S})$ for all domains $\tilde{S} \supset \overline{S}$. It is known (cf. [16, Corollary 6.49 and remark thereafter, p. 225]) that if S has a C^∞ boundary, then $f \in H_k^0(S)$ if and only if $f \in H_k(S)$ and $\mathbb{D}^{\mathbf{m}} f = 0$ on ∂S for all $|\mathbf{m}| \leq k-1$. It is now clear from the definition that if $\tilde{S} \supset S$ is a domain in $\mathbb{R}^s$, then the space $H_k^0(S)$ consists of those functions in $H_k^0(\tilde{S})$ which are supported on S. Clearly, $\|f\|_{0,S}$ is simply the L^2 norm on S. The corresponding inner product on $L^2(S)$ will be denoted by $\langle \circ, \circ \rangle_S$.

Let $m \geq 1$ be an integer, and L be a strongly elliptic, self–adjoint, differential operator of order $2m$. We further assume that the coefficients of L are C^∞ on $\overline{S}$. We are interested in the solutions of

$$Lu = f \text{ on } S, \; \frac{\partial^k u}{\partial \nu^k}|_{\partial S} = \frac{\partial^k g}{\partial \nu^k}|_{\partial S} \; k = 0, 1, \cdots, m-1, \tag{2.1}$$

for some sufficiently smooth g defined on $\overline{S}$. We observe that the condition $u = g$ on ∂S implies that all the derivatives of u in the directions tangential to ∂S are equal to the corresponding derivatives of g. So, the condition $\dfrac{\partial u}{\partial \nu} = \dfrac{\partial g}{\partial \nu}$ on ∂S implies that all the first order derivatives of u agree with those of g on ∂S. Proceeding this way, the boundary conditions in (2.1) are equivalent to the requirement that all the partial derivatives of u of order up to $m-1$ agree with the corresponding derivatives of g on ∂S (cf. [2, p. 91]).

Clearly, the solution of (2.1) is obtained by adding g to the solution of the problem

$$Lu = f - Lg \text{ on } S,\ \frac{\partial^k u}{\partial \nu^k}|_{\partial S} = 0,\ k = 0, 1, \cdots, m-1.$$

Therefore, there is no loss of generality in restricting our attention to the problem

$$Lu = f \text{ on } S,\ \frac{\partial^k u}{\partial \nu^k}|_{\partial S} = 0,\ k = 0, 1, \cdots, m-1. \tag{2.2}$$

It can be deduced by applying integration by parts [16, p. 231] that there are C^∞ functions $a_{\mathbf{k},\mathbf{m}}$, $|\mathbf{k}|, |\mathbf{m}| \le m$, such that $a_{\mathbf{k},\mathbf{m}} = a_{\mathbf{m},\mathbf{k}}$ if $|\mathbf{k}|, |\mathbf{m}| \le m-1$, and for every $u \in H^0_m(S)$, $v \in C^\infty_c(S)$,

$$\mathcal{D}[u,v] := \int_S \sum_{|\mathbf{k}|,|\mathbf{m}|\le m} a_{\mathbf{k},\mathbf{m}}(\mathbf{y})\mathbb{D}^{\mathbf{k}}u(\mathbf{y})\mathbb{D}^{\mathbf{m}}v(\mathbf{y})d\mathbf{y} = \int_S u(\mathbf{y})(Lv)(\mathbf{y})d\mathbf{y}. \tag{2.3}$$

where $\mathcal{D}[u,v]$ is called the *Dirichlet form* for the operator L. It follows (see [2, Section 8]) that $u \in H_{2m}(S)$ is a solution of (2.2) if and only if $\mathcal{D}[u,v] = \int_S v(\mathbf{x})f(\mathbf{x})d\mathbf{x}$ for every $v \in H^0_m(S)$. We will need the following notation in the presentation of our theorems. Since the functions $a_{\mathbf{k},\mathbf{m}}$'s are in $C^\infty(\overline{S})$, the quantity $M_{\mathcal{D},S}$ defined by

$$M_{\mathcal{D},S} := \max_{|\mathbf{k}|,|\mathbf{m}|\le m} \left\{ \max_{\mathbf{x}\in\overline{S}} |a_{\mathbf{k},\mathbf{m}}(\mathbf{x})| + \sup_{\substack{\mathbf{x},\mathbf{y}\in\overline{S}\\ \mathbf{x}\ne\mathbf{y}}} \frac{|a_{\mathbf{k},\mathbf{m}}(\mathbf{x}) - a_{\mathbf{k},\mathbf{m}}(\mathbf{y})|}{\|\mathbf{x}-\mathbf{y}\|} \right\}, \tag{2.4}$$

is finite, where $\|\cdot\|$ denotes the Euclidean ℓ^2 norm on $\mathbb{R}^s$. The form $\mathcal{D}$ is called *strictly coercive over* $H^0_m(S)$ if there exists a constant $c > 0$ such that

$$\mathcal{D}[u,u] \ge c\|u\|^2_{m,S}, \qquad u \in H^0_m(S). \tag{2.5}$$

We recall the following Green's identity (cf. [2, Theorem 10.2, p. 141]), valid for every $u \in H_{2m}(\overline{S})$, $v \in H^0_m(S) \cap H_{2m}(\overline{S})$:

$$\mathcal{D}[u,v] = \int_S v(\mathbf{y})(Lu)(\mathbf{y})d\mathbf{y} = \int_S u(\mathbf{y})(Lv)(\mathbf{y})d\mathbf{y} - \sum_{k=0}^{m-1}\int_{\partial S}(N_{2m-1-k}v)(\mathbf{y})\frac{\partial^k u}{\partial \nu^k}(\mathbf{y})ds, \tag{2.6}$$

for some differential operators N_ℓ, where N_ℓ is of order ℓ, $\ell = m, \cdots, 2m-1$, satisfying certain technical conditions which we need not elaborate upon here, and ds denotes the Riemannian measure for ∂S. For example, let $Tu = \mathsf{div}(\phi(u)\nabla u)$ for a smooth function ϕ. As in [11],

$$\int_S v(\mathbf{x})(Tu)(\mathbf{x})d\mathbf{x} = \int_S u(\mathbf{x})(Tv)(\mathbf{x})d\mathbf{x} - \int_{\partial S}\phi(\mathbf{y})\left\{u(\mathbf{y})\frac{\partial v}{\partial \nu}(\mathbf{y}) - v(\mathbf{y})\frac{\partial u}{\partial \nu}(\mathbf{y})\right\}ds. \tag{2.7}$$

In particular, T is a self-adjoint operator on $H^0_2(S)$. So, one of the Dirichlet forms for $L = T^2$ is $\mathcal{D}[u,v] = \langle Tu, Tv\rangle$. Using (2.7) with Tv in place of v, we obtain

$$\mathcal{D}[u,v] = \int_S u(\mathbf{x})(Lv)(\mathbf{x})d\mathbf{x} - \int_{\partial S}\phi(\mathbf{y})\left\{u(\mathbf{y})\frac{\partial Tv}{\partial \nu}(\mathbf{y}) - (Tv)(\mathbf{y})\frac{\partial u}{\partial \nu}(\mathbf{y})\right\}ds. \tag{2.8}$$

Thus, in this example, $N_3(\mathbf{y}) = \dfrac{\partial Tv}{\partial \nu}(\mathbf{y})$, $N_2(\mathbf{y}) = -(Tv)(\mathbf{y})$.

In view of (2.6), it is not difficult to verify that u is a (weak) solution of (2.2) if and only if

$$u = \arg\min \left\{ \mathcal{D}[v,v] - 2\int_D v(\mathbf{x})f(\mathbf{x})d\mathbf{x} : v \in H_m^0(S) \right\}. \tag{2.9}$$

The following theorem is a summary of the part of the discussion in [16, pp. 248–251] relevant to the current work.

Theorem 2.1 *Let $S \subseteq \mathbb{R}^s$ be a domain with a compact closure and C^∞ boundary, and $\mathcal{D}$ be strictly coercive over $H_m^0(S)$.*
(a) *There exists an injective, continuous, linear operator $\mathbb{T}_{L,S} : L^2(S) \to H_m^0(S)$ such that for every $v \in C_c^\infty(S)$ and $f \in L^2(S)$,*

$$\mathcal{D}[v, \mathbb{T}_{L,S} f] = \int_S v(\mathbf{x})f(\mathbf{x})d\mathbf{x}. \tag{2.10}$$

If c is any constant appearing in (2.5), then the operator norm of $\mathbb{T}_{L,S}$ satisfies $\|\mathbb{T}_{L,S}\| \le c^{-1}$.
(b) *There exists a nondecreasing sequence of positive numbers $\{\lambda_{\ell,L,S}\}$ and an orthonormal basis $\{u_{\ell,L,S}\}$ of $L^2(S)$, with $u_{\ell,L,S} \in C^\infty(S) \cap H_m^0(S)$, such that $\mathcal{D}[\phi, u_{\ell,L,S}] = \lambda_{\ell,L,S}\langle \phi, u_{\ell,L,S}\rangle_S$ for each $\phi \in C_c^\infty(S)$. Furthermore, $\lim_{\ell\to\infty} \lambda_{\ell,L,S} = \infty$.*

In the sequel, we will use the notation $\|\mathbb{T}_{L,S}\|$ to denote the $L^2(S) \to L^2(S)$ operator norm of $\mathbb{T}_{L,S}$. It is clear that the bound stated in Theorem 2.1 remains valid.

We find it easier to supply the simple proof of the following proposition rather than finding a reference where it is stated explicitly.

Proposition 2.1 *The quantity $\lambda_{1,L,S}$ satisfies*

$$\lambda_{1,L,S} = \inf_{f \in C_c^\infty(S)} \frac{\mathcal{D}[f,f]}{\|f\|_{0,S}^2}. \tag{2.11}$$

Hence, the operator norm of $\mathbb{T}_{L,S}$ can be estimated by $\|\mathbb{T}_{L,S}\| \le \lambda_{1,L,S}^{-1}$.

Proof. First, let $f \in C_c^\infty(S)$, and in this proof only, we may write $f = \sum_{\ell=1}^\infty a_\ell u_{\ell,L,S}$ in the sense of $L^2(S)$. Therefore, in view of (2.3), we have for $\ell = 1, 2, \cdots$,

$$\int_S (Lf)(\mathbf{x})u_{\ell,L,S}(\mathbf{x})d\mathbf{x} = \mathcal{D}[f, u_{\ell,L,S}] = \lambda_{\ell,L,S}\langle f, u_{\ell,L,S}\rangle_S = \lambda_{\ell,L,S} a_\ell.$$

Since $Lf \in C_c^\infty(S)$ as well, it follows that $Lf = \sum_{\ell=1}^\infty \lambda_{\ell,L,S} a_\ell u_{\ell,L,S}$, again in the sense of $L^2(S)$. Thus, for every $f \in C_c^\infty(S)$, we have

$$\mathcal{D}[f,f] = \int_S f(\mathbf{x})(Lf)(\mathbf{x})d\mathbf{x} = \sum_{\ell=1}^\infty \lambda_{\ell,L,S} a_\ell^2 \ge \lambda_{1,L,S} \sum_{\ell=1}^\infty a_\ell^2 = \lambda_{1,L,S}\|f\|_{0,S}^2. \tag{2.12}$$

Let $\eta > 0$ be arbitrary. Since $u_{1,L,S} \in H_m^0$, there exists an $f \in C_c^\infty(S)$ such that $\|f\|_{0,S} = 1$, $\|f - u_{1,L,S}\|_{m,S} \le \eta/(2\lambda_{1,L,S})$, and $|\mathcal{D}[f, u_{1,L,S}] - \mathcal{D}[f,f]| \le \eta/2$. Then

$$\mathcal{D}[f,f] \le \mathcal{D}[f, u_{1,L,S}] + \eta/2 = \lambda_{1,L,S}\langle f, u_{1,L,S}\rangle_S + \eta/2 \le \lambda_{1,L,S} + \eta.$$

Together with (2.12), this leads to (2.11). The assertion about $\|\mathbb{T}_{L,S}\|$ follows from Theorem 2.1 (a). $\square$

In the sequel, we assume the existence of a *Green function* $G_{L,S} : \overline{S} \times \overline{S} \to \mathbb{R}$ such that

$$LG_{L,S}(\mathbf{x}, \circ) = \delta_{\mathbf{x}}, \text{ on } S, \ \frac{\partial^k}{\partial \nu^k} G_{L,S}(\mathbf{x}, \circ) = 0 \text{ on } \partial S, \ k = 0, 1, \cdots, m-1. \tag{2.13}$$

The operator $\mathbb{T}_{L,S}$ in Theorem 2.1 is given by

$$\mathbb{T}_{L,S}(f, \mathbf{x}) = \int_S G_{L,S}(\mathbf{x}, \mathbf{y}) f(\mathbf{y}) d\mathbf{y}. \tag{2.14}$$

In view of (2.6), the solution of the problem (2.2) is given by $\mathbb{T}_{L,S}(f)$.

Next, we introduce the operator

$$\mathbb{P}_{L,S}(f,\mathbf{x}) := \sum_{k=0}^{m-1} \int_{\partial S} (N_{m-1-k} G_{L,S}(\mathbf{x},\circ))(\mathbf{y}) \frac{\partial^k f}{\partial \nu^k}(\mathbf{y}) ds, \tag{2.15}$$

where N_ℓ's are the differential operators introduced in (2.6). We deduce from (2.6) that the solution of the problem

$$Lu = 0 \text{ on } S,\ \frac{\partial^k u}{\partial \nu^k} = \frac{\partial^k g}{\partial \nu^k} \text{ on } \partial S,\ k = 0, 1, \cdots, m-1, \tag{2.16}$$

is given by

$$u = \mathbb{P}_{L,S}(g). \tag{2.17}$$

Finally, if g is sufficiently smooth, we may take g in place of u and $G_{L,S}(\mathbf{x},\circ)$ in place of v in (2.6) to conclude that

$$g = \mathbb{T}_{L,S}(Lg) + \mathbb{P}_{L,S}(g) \text{ on } S. \tag{2.18}$$

We end this section by describing a connection between differential equations on manifolds and those on the Euclidean domain. In light of the importance of the Euclidean sphere, and not to introduce too much of background and notations for general manifolds, we illustrate this connection for the case of the sphere. For aesthetic reasons, we will denote the dimension of the ambient space by q rather than s in this example. Let $q \ge 2$ be an integer, $\mathbb{S}^q$ be the unit sphere embedded in $\mathbb{R}^{q+1}$, and Δ_q^* be the Laplace–Beltrami operator on $\mathbb{S}^q$. Any point $\mathbf{x} \in \mathbb{S}^q$ may be written in various different forms

$$\mathbf{x} = (\omega \sin\phi, \cos\phi) = (\omega\sqrt{1-t^2}, t) = \left(\frac{2r}{r^2+1}\omega, \frac{2}{r^2+1} - 1\right),$$

where $t \in [-1,1]$, $r \in [0,\infty]$, $\phi \in [0,\pi]$, and $\omega \in \mathbb{S}^{q-1}$. The first expression is the usual spherical polar coordinate representation, the second is a parametrization in terms of the last coordinate, and the last expression parametrizes $\mathbb{S}^q \setminus \{(0,\cdots,0,-1)\}$ by the steriographic projection of $\mathbb{R}^q$, itself parametrized in the polar form $r\omega$. According to Müller [17, p. 38], we have the following relations between the Laplacian Δ_q on $\mathbb{R}^q$, Δ_q^* and Δ_{q-1}^*:

$$\Delta_q f = \frac{\partial^2 f}{\partial r^2} + \frac{q-1}{r}\frac{\partial f}{\partial r} + \frac{1}{r^2}\Delta_{q-1}^* f, \tag{2.19}$$

and

$$\Delta_q^* f = (1-t^2)\frac{\partial^2 f}{\partial t^2} - qt\frac{\partial f}{\partial t} + \frac{1}{1-t^2}\Delta_{q-1}^* f. \tag{2.20}$$

(We note that the quantities denoted in [17] by $\mathbb{S}^q$ and Δ_q^* are respectively $\mathbb{S}^{q-1}$, Δ_{q-1}^* in our notations.) With the substitutions $t = \cos\phi$ and $r = \tan(\phi/2)$, a tedious computation using these identities leads one to the expression

$$\Delta_q^* f = \left(\frac{r^2+1}{2}\right)^2 \left(\Delta_q f - (q-2)\frac{2r}{r^2+1}\frac{\partial f}{\partial r}\right). \tag{2.21}$$

Thus, a desire to consider iterated Laplace–Beltrami operators on the sphere leads immediately to the consideration of general elliptic operators on the parameter space. We observe that the transit from the sphere to the Euclidian domain change the boundary conditions in the spherical analogue of (2.1). However, since all the derivatives on ∂S are included in these conditions, there is no loss of generality in assuming the boundary conditions in the stated form, as explained earlier.

3 Context-preserving multi-level smooth function extension

Let $m \ge 1$ be an integer. In order to define the MRA, we extend the ideas in [11] so as to ensure a smooth extension of F to D as a function in $C^{m-1}(\Omega)$, determined by the data information $F|_{\Omega\setminus D}$. We consider a sequence of differential operators L_j, each of order $2m$, with the properties as described in Section 2.

The operators $\mathbb{P}_{L_j,D}$, $\mathbb{T}_{L_j,D}$ and the Green's functions $G_{L_j,D}$ will be denoted by $\mathbb{P}_j$, $\mathbb{T}_j$, G_j respectively. The basis solution u_0 is defined by $u_0 = \mathbb{P}_0(F)$, and the details are defined by

$$w_j := (\mathbb{T}_0 \cdots \mathbb{T}_{j-1})(\mathbb{P}_j(L_{j-1} \cdots L_0(F)). \tag{3.1}$$

Thus, writing $v_j = (\mathbb{T}_1 \cdots \mathbb{T}_{j-1})(\mathbb{P}_j(L_{j-1} \cdots L_0(F))$, we have

$$L_0 u_0 = 0 \text{ on } D, \ \frac{\partial^k u_0}{\partial \nu^k}|_{\partial D} = \frac{\partial^k F}{\partial \nu^k}|_{\partial D}, \ k = 0, 1, \cdots, m-1, \tag{3.2}$$

$$L_0 w_j = v_j \text{ on } D, \ \frac{\partial^k w_j}{\partial \nu^k}|_{\partial D} = 0, \ k = 0, 1, \cdots, m-1, \tag{3.3}$$

We note that the operators $\mathbb{P}_j$ depend only on the values of F on $\Omega \setminus D$. Hence, only the values of $L_{j-1} \cdots L_0(F)$ on $\Omega \setminus D$ are actually used in (3.1). For integer $n \geq 1$, the reconstruction on D at level n is defined by $u_n := u_0 + \sum_{j=1}^n w_j$. Analogous to [11, Theorem 1], we have

Theorem 3.1 *Let $n \geq 0$ be an integer, $F \in C^{2m(n+1)}(\Omega)$, and*

$$F_n^*(\mathbf{x}) := \begin{cases} F(\mathbf{x}), & \text{if } \mathbf{x} \in \Omega \setminus D, \\ u_n(\mathbf{x}), & \text{if } \mathbf{x} \in \overline{D}. \end{cases}$$

Then $F_n^* \in C^{m-1}(\Omega)$, and

$$F(\mathbf{x}) - F_n^*(\mathbf{x}) = (\mathbb{T}_0 \cdots \mathbb{T}_n)(L_n \cdots L_0(F))(\mathbf{x}), \qquad \mathbf{x} \in D. \tag{3.4}$$

Proof. Since $u_0 = \mathbb{P}_0(F)$, (2.18) implies that (3.4) holds for $n = 0$. Next, let $j \geq 1$ be an integer. Using (2.18) with L_j in place of L and $L_{j-1} \cdots L_0(F)$ in place of g, we obtain that

$$\mathbb{P}_j(L_{j-1} \cdots L_0(F)) = L_{j-1} \cdots L_0(F) - \mathbb{T}_j(L_j \cdots L_0(F)).$$

Applying the operator $\mathbb{T}_0 \cdots \mathbb{T}_{j-1}$ on both sides, we deduce that

$$w_j = (\mathbb{T}_0 \cdots \mathbb{T}_{j-1})(\mathbb{P}_j(L_{j-1} \cdots L_0(F)) = (\mathbb{T}_0 \cdots \mathbb{T}_{j-1})(L_{j-1} \cdots L_0(F)) - (\mathbb{T}_0 \cdots \mathbb{T}_{j-1}\mathbb{T}_j)(L_j \cdots L_0(F)),$$

and hence, that

$$u_n = u_0 + \sum_{j=1}^n w_j = F - (\mathbb{T}_0 \cdots \mathbb{T}_n)(L_n \cdots L_0(F)).$$

This proves (3.4). □

An immediate consequence of Theorem 3.1 and Proposition 2.1 is the following error estimate in terms of the eigenvalues of the operators L_j. In the sequel, we will denote the quantities $\lambda_{\ell,L_j,S}$ and $u_{\ell,L_j,S}$ by $\lambda_{\ell,j,S}$ and $u_{\ell,j,S}$ respectively.

Proposition 3.1 *Suppose the Dirichlet forms for the operators L_j satisfy the conditions of Theorem 2.1 with D in place of S. Then with notations as in Theorem 3.1,*

$$\|F - F_n^*\|_{0,D} \leq \left\{ \prod_{j=0}^n \lambda_{1,j,D} \right\}^{-1} \|L_n \cdots L_0(F)\|_{0,D}, \tag{3.5}$$

where we recall that $\| \circ \|_{0,D}$ denotes the L^2 norm on D.

In the following sections, we will use this proposition to obtain error estimates in terms of the diameter of D, and in the special case when each $L_j = \Delta^2$, also in terms of the volume of D.

3.1 Estimates in terms of the diameter

Let the diameter of D be 2ϵ. Let $B_r := \{(x_1, \cdots, x_s) \in \mathbb{R}^s : \sum_{k=1}^{s} x_k^2 < r\}$, with $B := B_1$, denote the ball with radius r and center at the origin. In the following argument, there is no loss of generality in assuming that $\overline{\Omega} \subseteq B$ and that $D \subseteq B_\epsilon$. We will assume that each L_j is strongly elliptic on $\overline{B}$ and self-adjoint as an operator on $H_{2m}^0(B)$. We will also assume that the corresponding Dirichlet form $\mathcal{D}_j$ is strongly coercive over $H_m^0(B)$. The estimate in terms of the diameter is stated in Theorem 3.2 below.

In order to state the theorem, we need some further notation. First, we need another set of Dirichlet forms $\tilde{\mathcal{D}}_j$, obtained by replacing the coefficient functions $a_{\mathbf{k},\mathbf{m}}$'s in the definition of $\mathcal{D}_j$ by the constants $a_{\mathbf{k},\mathbf{m}}(0)$. We assume that these forms are also strictly coercive over $H_m^0(B)$, so that Theorem 2.1 is also valid for $\tilde{\mathcal{D}}_j$. Thus, for any domain $S \subset B$ with C^∞ boundary, there exists a nondecreasing sequence of positive numbers $\{\tilde{\lambda}_{\ell,j,S}\}$ and an orthonormal basis $\{\tilde{u}_{\ell,j,S}\}$ of $L^2(S)$, such that $\lim_{\ell\to\infty} \tilde{\lambda}_{\ell,j,S} = \infty$ and

$$\tilde{\mathcal{D}}_j[\phi, \tilde{u}_{\ell,j,S}] = \tilde{\lambda}_{\ell,j,S} \langle \phi, \tilde{u}_{\ell,j,S} \rangle_S, \qquad \ell = 1, 2, \cdots, \ \phi \in C_c^\infty(S).$$

We assume further that there exists $\gamma > 0$ such that

$$\min(\lambda_{1,j,B}, \tilde{\lambda}_{1,j,B}) \geq \gamma^{-1} > 0, \qquad j = 0, 1, \cdots. \tag{3.6}$$

Similarly, we assume that there exists a positive (finite) constant M such that the constants $M_{\mathcal{D}_j,B}$ defined in (2.4) satisfy

$$M_{\mathcal{D}_j,B} \leq M, \qquad j = 0, 1, \cdots. \tag{3.7}$$

Clearly, the conditions (3.6) and (3.7) are satisfied trivially in the isotropic case; i.e., when each $L_j = L$. In the anisotropic case, the operators themselves depend upon F, and hence, so do the constants γ and M. Nevertheless, we expect some uniformity for F ranging over some compact function space also in the anisotropic case.

For convenience, set

$$\gamma^* := \gamma M \binom{m+s}{s}.$$

Then we have the following result, where we recall that $\| \circ \|_{0,D}$ is just the L^2 norm on D.

Theorem 3.2 *Let $0 < \epsilon < 1$, and $D \subset B_\epsilon$ be a domain with C^∞ boundary. Then for $\epsilon \leq 1/(2\gamma^*)$,*

$$\|F - F_n^*\|_{0,D} \leq (4\epsilon^{2m})^{n+1} \left\{ \prod_{j=0}^{n} \tilde{\lambda}_{1,j,B} \right\}^{-1} \|L_n \cdots L_0(F)\|_{0,D} \leq (4\gamma\epsilon^{2m})^{n+1} \|L_n \cdots L_0(F)\|_{0,D}. \tag{3.8}$$

In light of Proposition 3.1, the proof of this theorem consists of obtaining a lower bound for the eigenvalues $\lambda_{1,j,D}$. It is convenient to present this proof in Section 4.

3.2 Estimates in terms of the volume

For the special (isotropic) setting where $L = \Delta^2$ and $m = 2$, we can derive a much more useful estimate in terms of the volume $V := \mathsf{vol}(D)$ of D, instead of the diameter 2ϵ of D as in Theorem 3.2. In the following discussion, we no longer assume that $D \subset B$. Instead, we consider

$$\rho = \left(\frac{\Gamma((s+1)/2)}{\pi^{s/2}} V \right)^{1/s}, \tag{3.9}$$

and observe that the volume of the ball B_ρ is the same as that of D, namely:

$$\mathsf{vol}(B_\rho) = \mathsf{vol}(D) = V. \tag{3.10}$$

According to [4] and references therein, we have

$$\lambda_{1,L,D} \geq \alpha(s) \lambda_{1,L,B_\rho} \tag{3.11}$$

for some constant $\alpha(s) \in (1/2, 1)$. For Euclidean dimensions $s = 2$ and 3, we may choose $\alpha(s) = 1$, and numerical evidence suggests that $\alpha(s) \geq 0.8998$ for all s, and it is known that $\alpha(s)$ tends to 1 as $s \to \infty$. It is easy to check that $\Delta^2 f(\rho\mathbf{x}) = \rho^{-4}(\Delta^2 f)(\rho\mathbf{x})$. Therefore, using the Dirichlet form $\langle \Delta u, \Delta v\rangle_B$ for Δ^2 and (2.1), we conclude that $\lambda_{1,L,B_\rho} = \rho^{-4}\lambda_{1,L,B}$. It then follows from Proposition 3.1 that

$$\|F - F_n^*\|_{0,D} \leq \left(\frac{\rho^4}{\alpha(s)\lambda_{1,L,B}}\right)^{n+1} \|L^{n+1}F\|_{0,D}, \tag{3.12}$$

where ρ is given by (3.9).

In the following we give an estimate of $\lambda_{1,L,B}$. For this purpose, recall that the Green's function G for the boundary value problem on the unit ball B is nonnegative on B (see [6]). Hence,

$$u(\mathbf{x}) := \int_B G(\mathbf{x}, \mathbf{y})d\mathbf{y} = \int_B |G(\mathbf{x}, \mathbf{y})|d\mathbf{y} \tag{3.13}$$

solves the problem

$$\Delta^2 u = 1 \text{ on } B, \quad u = \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial B, \tag{3.14}$$

and the solution is unique. Let $r \geq 0$ with $r^2 := \sum_{k=1}^s x_k^2$. Then by applying the identities

$$\Delta r^\ell = \ell(\ell + s - 2)r^{\ell-2}, \;\; \Delta^2 r^\ell = \ell(\ell - 2)(\ell + s - 2)(\ell + s - 4)r^{\ell-4},$$

it is easy to verify that

$$u(\mathbf{x}) = \frac{(r^2 - 1)^2}{8s(s + 2)} \leq \frac{1}{8s(s + 2)}.$$

Hence, together with (3.13), this shows that

$$\sup_{\mathbf{x}\in B} \int_B |G(\mathbf{x}, \mathbf{y})|d\mathbf{y} \leq \frac{1}{8s(s + 2)}.$$

The generalized Young inequality [16, p. 9] then shows that the operator norm of $\|\mathbb{T}\|$ satisfies $\|\mathbb{T}\| \leq (8s(s + 2))^{-1}$, so that

$$\lambda_{1,L,B} \geq 8s(s + 2).$$

In summary, in view of (3.12), we have proved that

$$\|F - F_n^*\|_{0,D} \leq \left(\frac{\rho^4}{8s(s + 2)\alpha(s)}\right)^{n+1} \|\Delta^{2n+2}F\|_{0,D}, \tag{3.15}$$

where ρ is a function of the volume V of D, as defined in (3.9). We remark, however, that the error bound in (3.15) is in terms of $L^2(D)-$ norm. On the other hand, in the special case when $D = B_\epsilon$, we may also follow the argument in [11] to derive the following sup-norm estimate:

$$\max_{\mathbf{x}\in B_\epsilon} |F(\mathbf{x}) - F_n^*(\mathbf{x})| \leq \left(\frac{\epsilon^4}{8s(s + 2)}\right)^{n+1} \max_{\mathbf{x}\in B_\epsilon} |\Delta^{2n+2}(F)(\mathbf{x})|. \tag{3.16}$$

Unfortunately, since the maximum principle for the harmonic operator on a general domain D does not carry over, in general, to polyharmonic operators, the sup-norm estimation in (3.16) is far too restrictive to be useful. Moreover, even if the maximum principle would hold, so that (3.16) remains valid when the maximum operation on both sides of the inequality is taken over D instead of B_ϵ, the error bound is useful only if $\epsilon < (8s(s + 2))^{1/4}$, which is still very restrictive. For instance, when $s = 2$, a domain D with very small area V, and thus small value of ρ, could have diameter greater than 3 and does not fit inside (any translate of) B_ϵ with $\epsilon < 64^{1/4} \approx 2.82$. Therefore estimation in terms of the volume of D, such as (3.15) with ρ given by the expression in terms of V in (3.9), is much more useful. We do not know, however, if the $L^2(D)$ error estimate in (3.15) can be formulated in terms of sup-norm over D in general.

The inequality (3.11) holds with 1 in place of $\alpha(s)$ also in the case when $L = \Delta$ ($m = 1$). Therefore, (3.12) is also valid in this case. It is observed in [11] in this case that $\lambda_{1,B} \le 2s$. Therefore, the analogue of (3.15) in this case is:

$$\|F - F_n^*\|_{0,D} \le \left(\frac{\rho^2}{2s}\right)^{n+1} \|\Delta^{n+1} F\|_{0,D}. \tag{3.17}$$

We conjecture that estimates similar to (3.11) are valid also in the case of $m \ge 2$, so that the theory can be generalized to arbitrary smooth function extensions.

4 Proof of Theorem 3.2.

In view of Proposition 3.1, Theorem 3.2 will be proved if we prove that $\lambda_{1,j,D} \ge (1/4)\epsilon^{-2m}\tilde{\lambda}_{1,j,B}$ for each j. Clearly, the proof is the same for every j. Hence, we drop the mention of j in this section, observing only that

$$\|f\|_{m,B}^2 \le \gamma \min(\mathcal{D}[f,f], \tilde{\mathcal{D}}[f,f]), \qquad f \in C_c^\infty(B). \tag{4.1}$$

where $\mathcal{D}$ (respectively $\tilde{\mathcal{D}}$) may be any of the forms $\mathcal{D}_j$ (respectively, $\tilde{\mathcal{D}}_j$).

First, we prove the following estimate of λ_{1,B_ϵ} in terms of ϵ.

Lemma 4.1 *For $\epsilon \le 1/(2\gamma^*)$,*

$$\lambda_{1,B_\epsilon} \ge (1/4)\epsilon^{-2m}\tilde{\lambda}_{1,B}. \tag{4.2}$$

Proof. First, we consider the case of $\tilde{\mathcal{D}}$. By replacing $\mathbf{x}$ with $\epsilon\mathbf{x}$, we see from (2.11), that

$$\tilde{\lambda}_{1,B_\epsilon} = \inf_{\phi \in C_c^\infty(B_\epsilon)} \frac{\tilde{\mathcal{D}}[\phi,\phi]}{\|\phi\|_{0,S}^2} = \inf_{f \in C_c^\infty(B),\ \|f\|_{0,B}=1} \int_B \sum_{|\mathbf{k}|,|\mathbf{m}| \le m} a_{\mathbf{k},\mathbf{m}}(0)\epsilon^{-|\mathbf{k}+\mathbf{m}|}\mathbb{D}^{\mathbf{k}} f(\mathbf{x})\mathbb{D}^{\mathbf{m}} f(\mathbf{x})d\mathbf{x}. \tag{4.3}$$

Let $f \in C_c^\infty(B)$, $\|f\|_{0,B} = 1$. In view of (4.1), we have

$$\begin{aligned}
&\left| \int_B \sum_{\substack{|\mathbf{k}+\mathbf{m}| \le 2m-1 \\ |\mathbf{k}|,|\mathbf{m}| \le m}} a_{\mathbf{k},\mathbf{m}}(0)\epsilon^{2m-|\mathbf{k}+\mathbf{m}|}\mathbb{D}^{\mathbf{k}} f(\mathbf{x})\mathbb{D}^{\mathbf{m}} f(\mathbf{x})d\mathbf{x} \right| \\
&\le M\epsilon \int_B \sum_{|\mathbf{k}|,|\mathbf{m}| \le m} \int_B |\mathbb{D}^{\mathbf{k}} f(\mathbf{x})||\mathbb{D}^{\mathbf{m}} f(\mathbf{x})|d\mathbf{x} \le M\epsilon \sum_{|\mathbf{k}|,|\mathbf{m}| \le m} \|\mathbb{D}^{\mathbf{k}} f\|_{0,B}\|\mathbb{D}^{\mathbf{m}} f\|_{0,B} \\
&= M\epsilon \left(\sum_{|\mathbf{k}| \le m} \|\mathbb{D}^{\mathbf{k}} f\|_{0,B} \right)^2 \le \binom{m+s}{s} M\epsilon \|f\|_{m,B}^2 \\
&\le \gamma M \binom{m+s}{s} \epsilon \tilde{\mathcal{D}}[f,f] = \gamma^* \epsilon \tilde{\mathcal{D}}[f,f].
\end{aligned}$$

Hence, it follows that

$$\begin{aligned}
&\left| \int_B \sum_{|\mathbf{k}|,|\mathbf{m}| \le m} a_{\mathbf{k},\mathbf{m}}(0)\epsilon^{-|\mathbf{k}+\mathbf{m}|}\mathbb{D}^{\mathbf{k}} f(\mathbf{x})\mathbb{D}^{\mathbf{m}} f(\mathbf{x})d\mathbf{x} - \epsilon^{-2m}\tilde{\mathcal{D}}[f,f] \right| \\
&= \left| \int_B \sum_{|\mathbf{k}|,|\mathbf{m}| \le m} a_{\mathbf{k},\mathbf{m}}(0)\epsilon^{-|\mathbf{k}+\mathbf{m}|}\mathbb{D}^{\mathbf{k}} f(\mathbf{x})\mathbb{D}^{\mathbf{m}} f(\mathbf{x})d\mathbf{x} - \epsilon^{-2m} \int_B \sum_{|\mathbf{k}|,|\mathbf{m}| \le m} a_{\mathbf{k},\mathbf{m}}(0)\mathbb{D}^{\mathbf{k}} f(\mathbf{x})\mathbb{D}^{\mathbf{m}} f(\mathbf{x})d\mathbf{x} \right| \\
&= \epsilon^{-2m} \left| \sum_{\substack{|\mathbf{k}+\mathbf{m}| \le 2m-1 \\ |\mathbf{k}|,|\mathbf{m}| \le m}} a_{\mathbf{k},\mathbf{m}}(0)\epsilon^{2m-|\mathbf{k}+\mathbf{m}|}\mathbb{D}^{\mathbf{k}} f(\mathbf{x})\mathbb{D}^{\mathbf{m}} f(\mathbf{x})d\mathbf{x} \right| \\
&\le \gamma^* \epsilon \frac{\tilde{\mathcal{D}}[f,f]}{\epsilon^{2m}}.
\end{aligned}$$

If $\epsilon \leq 1/(2\gamma^*)$, then we may conclude, for every $f \in C_c^\infty(B)$, that

$$\int_B \sum_{|\mathbf{k}|,|\mathbf{m}|\leq m} a_{\mathbf{k},\mathbf{m}}(0)\epsilon^{-|\mathbf{k}+\mathbf{m}|}\mathbb{D}^{\mathbf{k}} f(\mathbf{x})\mathbb{D}^{\mathbf{m}} f(\mathbf{x})d\mathbf{x} \geq (1/2)\epsilon^{-2m}\tilde{\mathcal{D}}[f,f].$$

In view of (4.3) and (2.11), we deduce that for such values of ϵ,

$$\begin{aligned}
\tilde{\lambda}_{1,B_\epsilon} &= \inf_{f\in C_c^\infty(B),\ \|f\|_{0,B}=1} \int_B \sum a_{\mathbf{k},\mathbf{m}}(0)\epsilon^{-|\mathbf{k}+\mathbf{m}|}\mathbb{D}^{\mathbf{k}} f(\mathbf{x})\mathbb{D}^{\mathbf{m}} f(\mathbf{x})d\mathbf{x} \\
&\geq (1/2)\epsilon^{-2m} \inf_{f\in C_c^\infty(B),\ \|f\|_{0,B}=1} \tilde{\mathcal{D}}[f,f] = (1/2)\epsilon^{-2m}\tilde{\lambda}_{1,B}. \qquad (4.4)
\end{aligned}$$

This proves (4.2) in the case of $\tilde{\mathcal{D}}$.

Returning to the general case, we consider $f \in C_c^\infty(B_\epsilon)$, with $\|f\|_{0,B_\epsilon} = 1$, and apply (2.4) and (4.1) to deduce that

$$\begin{aligned}
\left|\mathcal{D}[f,f] - \tilde{\mathcal{D}}[f,f]\right| &= \left|\int_{B_\epsilon} \sum_{|\mathbf{k}|,|\mathbf{m}|\leq m} (a_{\mathbf{k},\mathbf{m}}(\mathbf{x}) - a_{\mathbf{k},\mathbf{m}}(0))\mathbb{D}^{\mathbf{k}} f(\mathbf{x})\mathbb{D}^{\mathbf{m}} f(\mathbf{x})d\mathbf{x}\right| \\
&\leq M\binom{m+s}{s}\epsilon\|f\|^2_{m,B_\epsilon} = M\binom{m+s}{s}\epsilon\|f\|^2_{m,B} \\
&\leq \gamma M\binom{m+s}{s}\epsilon\tilde{\mathcal{D}}[f,f] = \gamma^*\epsilon\tilde{\mathcal{D}}[f,f]. \qquad (4.5)
\end{aligned}$$

Thus, for $\epsilon \leq 1/(2\gamma^*)$, we have $\mathcal{D}[f,f] \geq (1/2)\tilde{\mathcal{D}}[f,f]$, so that Proposition 2.1 implies that $\lambda_{1,B_\epsilon} \geq (1/2)\tilde{\lambda}_{1,B_\epsilon}$. Therefore, in view of (4.4), (4.2) follows. This completes the proof of the Lemma. □

We can now establish Theorem 3.2, as follows.

Proof of Theorem 3.2. Since $C_c^\infty(D) \subseteq C_c^\infty(B_\epsilon)$, Proposition 2.1 implies that $\lambda_{1,j,D} \geq \lambda_{1,j,B_\epsilon}$ for each integer $j \geq 0$. The theorem then follows immediately from Proposition 3.1 and Lemma 4.1. □

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