①

CCS for $\partial q_x / \partial z$

→ At surface, $z_0 = 0$

$$* \quad \left.\frac{\partial q_x}{\partial z}\right|_0 + 3 \left.\frac{\partial q_x}{\partial z}\right|_1 = \frac{1}{\Delta z}\left[-\frac{17}{6} q_{x0} + \frac{3}{2} q_{x1} + \frac{3}{2} q_{x2} - \frac{1}{6} q_{x3}\right]$$

$$= \frac{1}{\Delta z}\left[\frac{3}{2} q_{x1} + \frac{3}{2} q_{x2} - \frac{1}{6} q_{x3}\right] - \frac{1}{\Delta x}\frac{17}{6} q_{x0}$$

→ Inner, z_i

$$* \quad \frac{1}{4}\left.\frac{\partial q_x}{\partial z}\right|_0 + \left.\frac{\partial q_x}{\partial z}\right|_1 + \frac{1}{4}\left.\frac{\partial q_x}{\partial z}\right|_2 = \frac{1}{\Delta z}\left[-\frac{3}{4} q_{x0} + \frac{3}{4} q_{x2}\right]$$

$$= \frac{1}{\Delta z}\left[\frac{3}{4} q_{x2}\right] - \frac{1}{\Delta z}\frac{3}{4} q_{x0}$$

$$* \quad \frac{1}{4}\left.\frac{\partial q_x}{\partial z}\right|_{i-1} + \left.\frac{\partial q_x}{\partial z}\right|_i + \frac{1}{4}\left.\frac{\partial q_x}{\partial z}\right|_{i+1} = \frac{1}{\Delta z}\left[-\frac{3}{4} q_{x\,i-1} + \frac{3}{4} q_{x\,i+1}\right] \quad \forall i = 2, \dots, N-3$$

$$\frac{1}{4}\left.\frac{\partial q_x}{\partial z}\right|_{N-3} + \left.\frac{\partial q_x}{\partial z}\right|_{N-2} + \frac{1}{4}\left.\frac{\partial q_x}{\partial z}\right|_{N-1} = \frac{1}{\Delta z}\left[-\frac{3}{4} q_{x\,N-3} + \frac{3}{4} q_{x\,N-1}\right]$$

$$\downarrow$$

$$* \quad \frac{1}{4}\left.\frac{\partial q_x}{\partial z}\right|_{N-3} + \left.\frac{\partial q_x}{\partial z}\right|_{N-2} = \frac{1}{\Delta z}\left[-\frac{3}{4} q_{x\,N-3} + \frac{3}{4} q_{x\,N-1}\right] - \frac{1}{4}\left.\frac{\partial q_x}{\partial z}\right|_{N-1}$$

Boundaries

$$q_{x0} = K_0 \left.\frac{\partial u}{\partial z}\right|_0 = \frac{\tau_w^x}{\rho_0}$$

$$\left.\frac{\partial q_x}{\partial z}\right|_{N-1} = -T^x_{N-1} - f V_*$$

Vectorial form

$$\tilde{M}_q^* \frac{\partial \bar{q}_x}{\partial z} = \frac{1}{\Delta z} \tilde{W}_q^* \bar{q}_x + \frac{1}{\Delta z} q_{x0} \bar{w}_q^* + (T^x_{N-1} + f V_*)\, \bar{m}_q^*$$

$$\frac{\partial \bar{q}_x}{\partial z} := \left(\left.\frac{\partial q_x}{\partial z}\right|_0, \left.\frac{\partial q_x}{\partial z}\right|_1, \dots, \left.\frac{\partial q_x}{\partial z}\right|_{N-2}\right)^T \in \mathbb{R}^{N-1}$$

$$\bar{q}_x := (q_{x1}, q_{x2}, \dots, q_{x\,N-1})^T \in \mathbb{R}^{N-1}$$

$$\tilde{M}_q^* := \begin{pmatrix} 1 & 3 & & \\ 1/4 & 1 & 1/4 & \\ & \ddots & \ddots & \ddots \\ & & 1/4 & 1 \end{pmatrix} \in \mathbb{R}^{(N-1)\times(N-1)}$$

$$\tilde{W}_q^* := \begin{pmatrix} 3/2 & 3/2 & -1/6 & & \\ 0 & 3/4 & & & \\ -3/4 & 0 & 3/4 & & \\ & \ddots & \ddots & \ddots & \\ & & -3/4 & 0 & 3/4 \end{pmatrix} \in \mathbb{R}^{(N-1)\times(N-1)}$$

$$\bar{w}_q^* := (-17/6, -3/4, 0, \dots, 0)^T \in \mathbb{R}^{N-1}$$

$$\bar{m}_q^* := (0, 0, \dots, 0, -1/4)^T \in \mathbb{R}^{N-1}$$

CCS for $\partial q_y / \partial z$

→ At surface $z_0 = 0$

$$* \quad \left.\frac{\partial q_y}{\partial z}\right|_0 + 3 \left.\frac{\partial q_y}{\partial z}\right|_1 = \frac{1}{\Delta z}\left[-\frac{17}{6} q_{y0} + \frac{3}{2} q_{y1} + \frac{3}{2} q_{y2} - \frac{1}{6} q_{y3} \right]$$

$$= \frac{1}{\Delta z}\left[\frac{3}{2} q_{y1} + \frac{3}{2} q_{y2} - \frac{1}{6} q_{y3} \right] - \frac{1}{\Delta z}\frac{17}{6} q_{y0}$$

→ Inner, z_i

$$* \quad \frac{1}{4}\left.\frac{\partial q_y}{\partial z}\right|_0 + \left.\frac{\partial q_y}{\partial z}\right|_1 + \frac{1}{4}\left.\frac{\partial q_y}{\partial z}\right|_2 = \frac{1}{\Delta z}\left[-\frac{3}{4} q_{y0} + \frac{3}{4} q_{y2} \right]$$

$$= \frac{1}{\Delta z}\left[\frac{3}{4} q_{y2} \right] - \frac{1}{\Delta z}\frac{3}{4} q_{y0}$$

$$* \quad \frac{1}{4}\left.\frac{\partial q_y}{\partial z}\right|_{i-1} + \left.\frac{\partial q_y}{\partial z}\right|_i + \frac{1}{4}\left.\frac{\partial q_y}{\partial z}\right|_{i+1} = \frac{1}{\Delta z}\left[-\frac{3}{4} q_{y\,i-1} + \frac{3}{4} q_{y\,i+1} \right] \quad \forall i = 2, \dots, N-3$$

$$\frac{1}{4}\left.\frac{\partial q_y}{\partial z}\right|_{N-3} + \left.\frac{\partial q_y}{\partial z}\right|_{N-2} + \frac{1}{4}\left.\frac{\partial q_y}{\partial z}\right|_{N-1} = \frac{1}{\Delta z}\left[-\frac{3}{4} q_{yN-3} + \frac{3}{4} q_{yN-1} \right]$$

$$\downarrow$$

$$\frac{1}{4}\left.\frac{\partial q_y}{\partial z}\right|_{N-3} + \left.\frac{\partial q_y}{\partial z}\right|_{N-2} = \frac{1}{\Delta z}\left[-\frac{3}{4} q_{yN-3} + \frac{3}{4} q_{yN-1} \right] - \frac{1}{4}\left.\frac{\partial q_y}{\partial z}\right|_{N-1}$$

Boundaries

$$q_{y0} = K_0 \left.\frac{\partial v}{\partial z}\right|_0 = \frac{\tau_w^y}{\rho_0}$$

$$\left.\frac{\partial q_y}{\partial z}\right|_{N-1} = -T_{N-1}^y + f U_*$$

Vectorial form

$$\tilde{M}_q \frac{\partial \bar{q}_y}{\partial z} = \frac{1}{\Delta z} \tilde{W}_q \bar{q}_y + \frac{1}{\Delta z} q_{y0} \bar{w}_q - (T_{N-1}^y - f U_*) \bar{m}_q$$

$$\frac{\partial \bar{q}_y}{\partial z} := \left(\left.\frac{\partial q_y}{\partial z}\right|_0, \left.\frac{\partial q_y}{\partial z}\right|_1, \dots, \left.\frac{\partial q_y}{\partial z}\right|_{N-2} \right)^T \in \mathbb{R}^{N-1}$$

$$\bar{q}_y := (q_{y1}, q_{y2}, \dots, q_{yN-1})^T \in \mathbb{R}^{N-1}$$

$$\tilde{M}_q := \begin{pmatrix} 1 & 3 & & \\ 1/4 & 1 & 1/4 & \\ & \ddots & \ddots & \ddots \\ & & 1/4 & 1 \end{pmatrix} \in \mathbb{R}^{(N-1)\times(N-1)}$$

$$\tilde{W}_q := \begin{pmatrix} 3/2 & 3/2 & -1/6 & & \\ 0 & 3/4 & & & \\ -3/4 & 0 & 3/4 & & \\ & \ddots & \ddots & \ddots & \\ & & -3/4 & 0 & 3/4 \end{pmatrix} \in \mathbb{R}^{(N-1)\times(N-1)}$$

$$\bar{w}_q := (-17/6, -3/4, 0, \dots, 0) \in \mathbb{R}^{N-1}$$

$$\bar{m}_q := (0, \dots, 0, -1/4) \in \mathbb{R}^{N-1}$$

CCS for $\partial u/\partial z$

→ Inner, z_i

$$\frac{1}{4}\frac{\partial u}{\partial z}\Big|_0 + \frac{\partial u}{\partial z}\Big|_1 + \frac{1}{4}\frac{\partial u}{\partial z}\Big|_2 = \frac{1}{\Delta z}\left[-\frac{3}{4}u_0 + \frac{3}{4}u_2\right]$$

$$* \quad \frac{\partial u}{\partial z}\Big|_1 + \frac{1}{4}\frac{\partial u}{\partial z}\Big|_2 = \frac{1}{\Delta z}\left[-\frac{3}{4}u_0 + \frac{3}{4}u_2\right] - \frac{1}{4}\frac{\partial u}{\partial z}\Big|_0$$

$$* \quad \frac{1}{4}\frac{\partial u}{\partial z}\Big|_{i-1} + \frac{\partial u}{\partial z}\Big|_i + \frac{1}{4}\frac{\partial u}{\partial z}\Big|_{i+1} = \frac{1}{\Delta z}\left[-\frac{3}{4}u_{i-1} + \frac{3}{4}u_{i+1}\right] \quad \forall i = 2, \dots, N-3$$

$$* \quad \frac{1}{4}\frac{\partial u}{\partial z}\Big|_{N-3} + \frac{\partial u}{\partial z}\Big|_{N-2} + \frac{1}{4}\frac{\partial u}{\partial z}\Big|_{N-1} = \frac{1}{\Delta z}\left[-\frac{3}{4}u_{N-3} + \frac{3}{4}u_{N-1}\right]$$

$$= \frac{1}{\Delta z}\left[-\frac{3}{4}u_{N-3}\right] + \frac{1}{\Delta z}\frac{3}{4}u_*$$

→ At bottom, $z_{N-1} = -H$

$$* \quad 3\frac{\partial u}{\partial z}\Big|_{N-2} + \frac{\partial u}{\partial z}\Big|_{N-1} = \frac{1}{\Delta z}\left[\frac{1}{6}u_{N-4} - \frac{3}{2}u_{N-3} - \frac{3}{2}u_{N-2} + \frac{17}{6}u_{N-1}\right]$$

$$= \frac{1}{\Delta z}\left[\frac{1}{6}u_{N-4} - \frac{3}{2}u_{N-3} - \frac{3}{2}u_{N-2}\right] + \frac{1}{\Delta z}\frac{17}{6}u_*$$

Boundaries

$$\frac{\partial u}{\partial z}\Big|_0 = \frac{\tau_w^x}{\rho_0 K_0}$$

Vectorial form

$$\tilde{M}_u \frac{\partial \bar{u}}{\partial z} = \frac{1}{\Delta z}\tilde{W}_u \bar{u} + \frac{1}{\Delta z}u_* \bar{w}_u + \frac{\tau_w^x}{\rho_0 K_0}\bar{m}_u$$

$$\frac{\partial \bar{u}}{\partial z} := \left(\frac{\partial u}{\partial z}\Big|_1, \frac{\partial u}{\partial z}\Big|_2, \dots, \frac{\partial u}{\partial z}\Big|_{N-1}\right)^T \in \mathbb{R}^{N-1}$$

$$\bar{u} := (u_0, u_1, \dots, u_{N-2})^T \in \mathbb{R}^{N-1}$$

$$\tilde{M}_u := \begin{pmatrix} 1 & 1/4 & & \\ 1/4 & 1 & 1/4 & \\ & \ddots & \ddots & \ddots \\ & & 3 & 1 \end{pmatrix} \in \mathbb{R}^{(N-1)\times(N-1)}$$

$$\tilde{W}_u := \begin{pmatrix} -\frac{3}{4} & 0 & \frac{3}{4} & & \\ & -\frac{3}{4} & 0 & \frac{3}{4} & \\ & & \ddots & \ddots & \ddots \\ & & & -3/4 & 0 \\ & & 1/6 & -3/2 & -3/2 \end{pmatrix} \in \mathbb{R}^{(N-1)\times(N-1)}$$

$$\bar{w}_u := (0, 0, \dots, 3/4, 17/6)^T \in \mathbb{R}^{N-1}$$

$$\bar{m}_u := (-1/4, 0, 0, \dots, 0)^T \in \mathbb{R}^{N-1}$$

CCS for $\partial v / \partial z$

→ Inner, z_i

$$\frac{1}{4}\frac{\partial v}{\partial z}\Big|_0 + \frac{\partial v}{\partial z}\Big|_1 + \frac{1}{4}\frac{\partial v}{\partial z}\Big|_2 = \frac{1}{\Delta z}\left[-\frac{3}{4}v_0 + \frac{3}{4}v_2\right]$$

$$\downarrow$$

$$* \quad \frac{\partial v}{\partial z}\Big|_1 + \frac{1}{4}\frac{\partial v}{\partial z}\Big|_2 = \frac{1}{\Delta z}\left[-\frac{3}{4}v_0 + \frac{3}{4}v_2\right] - \frac{1}{4}\frac{\partial v}{\partial z}\Big|_0$$

$$* \quad \frac{1}{4}\frac{\partial v}{\partial z}\Big|_{i-1} + \frac{\partial v}{\partial z}\Big|_i + \frac{1}{4}\frac{\partial v}{\partial z}\Big|_{i+1} = \frac{1}{\Delta z}\left[-\frac{3}{4}v_{i-1} + \frac{3}{4}v_{i+1}\right] \quad \forall i = 2, \dots, N-3$$

$$* \quad \frac{1}{4}\frac{\partial v}{\partial z}\Big|_{N-3} + \frac{\partial v}{\partial z}\Big|_{N-2} + \frac{1}{4}\frac{\partial v}{\partial z}\Big|_{N-1} = \frac{1}{\Delta z}\left[-\frac{3}{4}v_{N-3} + \frac{3}{4}v_{N-1}\right]$$

$$= \frac{1}{\Delta z}\left[-\frac{3}{4}v_{N-3}\right] + \frac{1}{\Delta z}\frac{3}{4}v_*$$

→ At bottom, $z_{N-1} = -H$

$$3\frac{\partial v}{\partial z}\Big|_{N-2} + \frac{\partial v}{\partial z}\Big|_{N-1} = \frac{1}{\Delta z}\left[\frac{1}{6}v_{N-4} - \frac{3}{2}v_{N-3} - \frac{3}{2}v_{N-2} + \frac{17}{6}v_{N-1}\right]$$

$$= \frac{1}{\Delta z}\left[\frac{1}{6}v_{N-4} - \frac{3}{2}v_{N-3} - \frac{3}{2}v_{N-2}\right] + \frac{17}{6}\frac{1}{\Delta z}v_*$$

Boundaries

$$\frac{\partial v}{\partial z}\Big|_0 = \frac{\tau_w^y}{\rho_0 K_0}$$

Vectorial form

$$\tilde{M}_v \frac{\partial \bar{v}}{\partial z} = \frac{1}{\Delta z}\tilde{W}_v \bar{v} + \frac{1}{\Delta z} v_* \bar{w}_v + \frac{\tau_w^y}{\rho_0 K_0}\bar{m}_v$$

$$\frac{\partial \bar{v}}{\partial z} := \left(\frac{\partial v}{\partial z}\Big|_1, \frac{\partial v}{\partial z}\Big|_2, \dots, \frac{\partial v}{\partial z}\Big|_{N-1}\right)^T \in \mathbb{R}^{N-1}$$

$$\bar{v} := (v_0, v_1, \dots, v_{N-2})^T \in \mathbb{R}^{N-1}$$

$$\tilde{M}_v := \begin{pmatrix} 1 & 1/4 & & \\ 1/4 & 1 & 1/4 & \\ & \ddots & \ddots & \ddots \\ & & 3 & 1 \end{pmatrix} \in \mathbb{R}^{(N-1)\times(N-1)}$$

$$\tilde{W}_v := \begin{pmatrix} -3/4 & 0 & 3/4 & & \\ & -3/4 & 0 & 3/4 & \\ & & \ddots & \ddots & \ddots \\ & & & -3/4 & 0 \\ & & 1/6 & -3/2 & -3/2 \end{pmatrix} \in \mathbb{R}^{(N-1)\times(N-1)}$$

$$\bar{w}_v := (0, \dots, 0, 3/4, 17/6)^T \in \mathbb{R}^{N-1}$$

$$\bar{m}_v := (-1/4, 0, \dots, 0) \in \mathbb{R}^{N-1}$$

$\boxed{\text{Ekman model}}$

$$\left.\begin{array}{l} \dfrac{\partial \bar{q}_x}{\partial z} + f\tilde{I}\,\bar{v} = -\bar{t}^x\,, \quad \bar{q}_x = \tilde{K}\dfrac{\partial \bar{u}}{\partial z} \\[2ex] \dfrac{\partial \bar{q}_y}{\partial z} - f\tilde{I}\,\bar{u} = -\bar{t}^y\,, \quad \bar{q}_y = \tilde{K}\dfrac{\partial \bar{v}}{\partial z} \end{array}\right\} \Rightarrow \left.\begin{array}{l} \dfrac{\partial \bar{u}}{\partial z} - \tilde{K}^{-1}\bar{q}_x = \bar{o} \\[2ex] \dfrac{\partial \bar{q}_x}{\partial z} + f\tilde{I}\,\bar{v} = -\bar{t}^x \\[2ex] \dfrac{\partial \bar{v}}{\partial z} - \tilde{K}^{-1}\bar{q}_y = \bar{o} \\[2ex] \dfrac{\partial \bar{q}_y}{\partial z} - f\tilde{I}\,\bar{u} = -\bar{t}^y \end{array}\right\} \quad \left.\begin{array}{l} \tilde{M}_u\dfrac{\partial \bar{u}}{\partial z} - \tilde{M}_u\tilde{K}^{-1}\bar{q}_x = \bar{o} \\[2ex] \tilde{M}_q\dfrac{\partial \bar{q}_x}{\partial z} + f\tilde{M}_q\bar{v} = -\tilde{M}_q\bar{t}^x \\[2ex] \tilde{M}_u\dfrac{\partial \bar{v}}{\partial z} - \tilde{M}_u\tilde{K}^{-1}\bar{q}_y = \bar{o} \\[2ex] \tilde{M}_q\dfrac{\partial \bar{q}_y}{\partial z} - f\tilde{M}_q\bar{u} = -\tilde{M}_q\bar{t}^y \end{array}\right\}$$

$$\tilde{I} := \mathrm{diag}(1, \dots, 1) \in \mathbb{R}^{(N-1)\times(N-1)}$$

$$\tilde{K} := \mathrm{diag}(K_1, K_2, \dots, K_{N-1}) \in \mathbb{R}^{(N-1)\times(N-1)}$$

$$\bar{t}^x := (T_0^x, T_1^x, \dots, T_{N-2}^x)^T \in \mathbb{R}^{N-1}$$

$$\bar{t}^y := (T_0^y, T_1^y, \dots, T_{N-2}^y)^T \in \mathbb{R}^{N-1}$$

$$\left.\begin{array}{l} \dfrac{1}{\Delta z}\tilde{W}_u\bar{u} + \dfrac{1}{\Delta z}U_*\bar{w}_u + \dfrac{\tau_w^x}{\rho_0 K_0}\bar{m}_u - \tilde{M}_u\tilde{K}^{-1}\bar{q}_x = \bar{o} \\[2ex] \dfrac{1}{\Delta z}\tilde{W}_q\bar{q}_x + \dfrac{1}{\Delta z}\dfrac{\tau_w^x}{\rho_0}\bar{w}_q - (T_{N-1}^x + fV_*)\bar{m}_q + f\tilde{M}_q\bar{v} = -\tilde{M}_q\bar{t}^x \\[2ex] \dfrac{1}{\Delta z}\tilde{W}_u\bar{v} + \dfrac{1}{\Delta z}V_*\bar{w}_u + \dfrac{\tau_w^y}{\rho_0 K_0}\bar{m}_u - \tilde{M}_u\tilde{K}^{-1}\bar{q}_y = \bar{o} \\[2ex] \dfrac{1}{\Delta z}\tilde{W}_q\bar{q}_y + \dfrac{1}{\Delta z}\dfrac{\tau_w^y}{\rho_0}\bar{w}_q - (T_{N-1}^y - fU_*)\bar{m}_q - f\tilde{M}_q\bar{u} = -\tilde{M}_q\bar{t}^y \end{array}\right\}$$

$$\begin{bmatrix} \tilde{W}_u & -\Delta z\tilde{M}_u\tilde{K}^{-1} & \tilde{o} & \tilde{o} \\ \tilde{o} & \tilde{W}_q & \Delta z f\tilde{M}_q & \tilde{o} \\ \tilde{o} & \tilde{o} & \tilde{W}_u & -\Delta z\tilde{M}_u\tilde{K}^{-1} \\ -\Delta z f\tilde{M}_q & \tilde{o} & \tilde{o} & \tilde{W}_q \end{bmatrix} \begin{bmatrix} \bar{u} \\ \bar{q}_x \\ \bar{v} \\ \bar{q}_y \end{bmatrix} = \begin{bmatrix} \bar{o} \\ -\Delta z\tilde{M}_q\bar{t}^x \\ \bar{o} \\ -\Delta z\tilde{M}_q\bar{t}^y \end{bmatrix} + \begin{bmatrix} -U_*\bar{w}_u - \Delta z\dfrac{\tau_w^x}{\rho_0 K_0}\bar{m}_u \\ -\dfrac{\tau_w^x}{\rho_0}\bar{w}_q + \Delta z(T_{N-1}^x + fV_*)\bar{m}_q \\ -V_*\bar{w}_u - \Delta z\dfrac{\tau_w^y}{\rho_0 K_0}\bar{m}_u \\ -\dfrac{\tau_w^y}{\rho_0}\bar{w}_q + \Delta z(T_{N-1}^y - fU_*)\bar{m}_q \end{bmatrix}$$