Applied Math III
(MATH 2051)
Class 4

By: Melaku Berhe Belay. (PhD)
AASTU

Modeling Problems Using First-Order Linear Differential Equations

Mixing Problems

Statement of the Problem: Consider the situation depicted in Figure 1.7.1. A tank initially contains V_0 liters of a solution in which is dissolved A_0 grams of a certain chemical. A solution containing c_1 grams/liter of the same chemical flows into the tank at a constant rate of r_1 liters/minute, and the mixture flows out at a constant rate of r_2 liters/minute. We assume that the mixture is kept uniform by stirring. Then at any time t the concentration of chemical in the tank, $c_2(t)$, is the same throughout the tank and is given by

$$c_2 = \frac{A(t)}{V(t)}, \tag{1.7.1}$$

where $V(t)$ denotes the volume of solution in the tank at time t and $A(t)$ denotes the amount of chemical in the tank at time t.

Modeling Problems (Conti…)

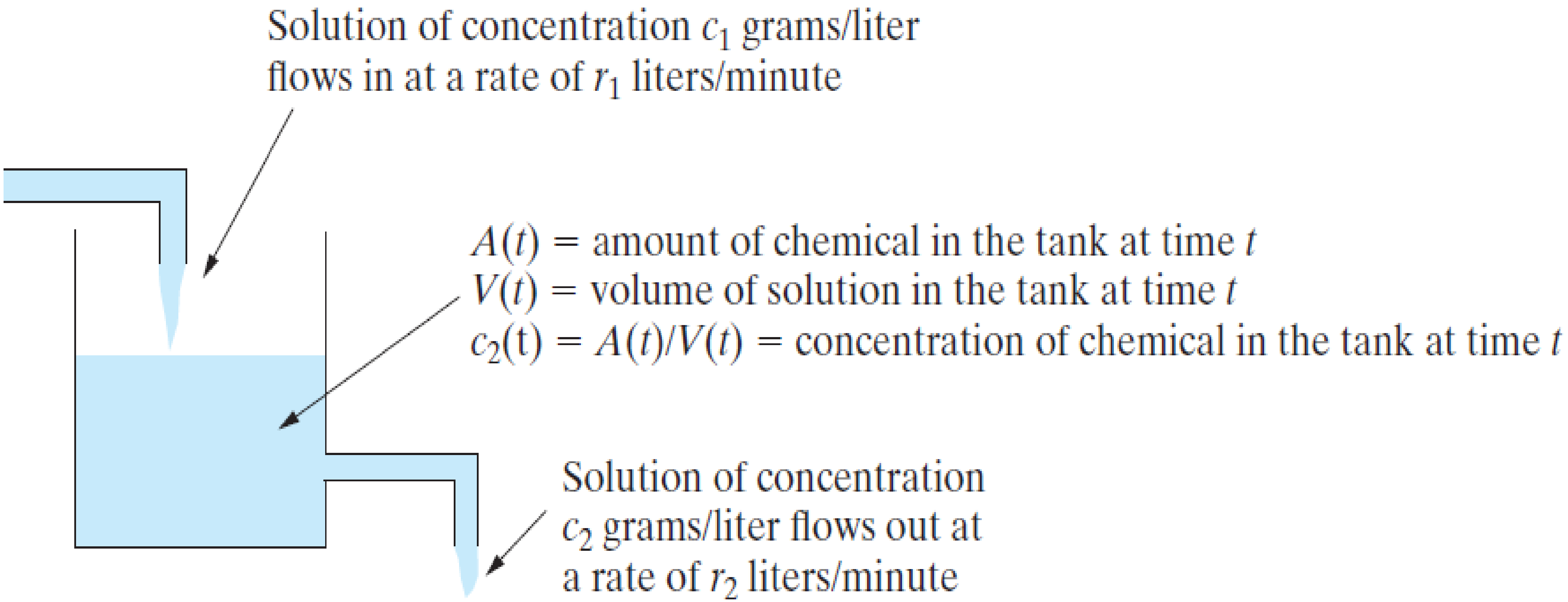

Figure 1.7.1: A mixing problem.

Modeling Problems (Conti…)

Mathematical Formulation: The two functions in the problem are $V(t)$ and $A(t)$. In order to determine how they change with time, we first consider their change during a short time interval, Δt minutes. In time Δt, $r_1\,\Delta t$ liters of solution flow into the tank, whereas $r_2\,\Delta t$ liters flow out. Thus during the time interval Δt, the *change* in the volume of solution in the tank is

$$\Delta V = r_1\,\Delta t - r_2\,\Delta t = (r_1 - r_2)\,\Delta t. \qquad (1.7.2)$$

Since the concentration of chemical in the inflow is c_1 grams/liter (assumed constant), it follows that in the time interval Δt the amount of chemical that flows into the tank is $c_1 r_1\,\Delta t$. Similarly, the amount of chemical that flows out in this same time interval is approximately[10] $c_2 r_2\,\Delta t$. Thus, the total change in the amount of chemical in the tank during the time interval Δt, denoted by ΔA, is approximately

$$\Delta A \approx c_1 r_1\,\Delta t - c_2 r_2\,\Delta t = (c_1 r_1 - c_2 r_2)\,\Delta t. \qquad (1.7.3)$$

Dividing Equations (1.7.2) and (1.7.3) by Δt yields

$$\frac{\Delta V}{\Delta t} = r_1 - r_2 \quad \text{and} \quad \frac{\Delta A}{\Delta t} \approx c_1 r_1 - c_2 r_2,$$

Modeling Problems (Conti…)

respectively. These equations describe the rates of change of V and A over the short, but finite, time interval Δt. In order to determine the instantaneous rates of change of V and A, we take the limit as $\Delta t \to 0$ to obtain

$$\frac{dV}{dt} = r_1 - r_2 \tag{1.7.4}$$

and

$$\frac{dA}{dt} = c_1 r_1 - \frac{A}{V} r_2, \tag{1.7.5}$$

where we have substituted for c_2 from Equation (1.7.1). Since r_1 and r_2 are constants, we can integrate Equation (1.7.4) directly, to obtain

$$V(t) = (r_1 - r_2)t + V_0,$$

where V_0 is an integration constant. Substituting for V into Equation (1.7.5) and rearranging terms yields the linear equation for $A(t)$

$$\frac{dA}{dt} + \frac{r_2}{(r_1 - r_2)t + V_0} A = c_1 r_1. \tag{1.7.6}$$

This differential equation can be solved, subject to the initial condition $A(0) = A_0$, to determine the behavior of $A(t)$.

Modeling Problems (Conti…)

Example

A tank contains 8 L (liters) of water in which is dissolved 32 g (grams) of chemical. A solution containing 2 g/L of the chemical flows into the tank at a rate of 4 L/min, and the well-stirred mixture flows out at a rate of 2 L/min.

1. Determine the amount of chemical in the tank after 20 minutes.
2. What is the concentration of chemical in the tank at that time?

Solution: We are given

$$r_1 = 4 \text{ L/min}, r_2 = 2 \text{ L/min}, c_1 = 2 \text{ g/L}, V(0) = 8 \text{ L}, \text{ and } A(0) = 32 \text{ g}.$$

For parts (1) and (2), we must find $A(20)$ and $A(20)/V(20)$, respectively. Now,

$$\Delta V = r_1 \, \Delta t - r_2 \, \Delta t$$

implies that

$$\frac{dV}{dt} = 2.$$

Modeling Problems (Conti…)

Integrating this equation and imposing the initial condition that $V(0) = 8$ yields

$$V(t) = 2(t + 4). \tag{1.7.7}$$

Further,

$$\Delta A \approx c_1 r_1 \, \Delta t - c_2 r_2 \, \Delta t$$

implies that

$$\frac{dA}{dt} = 8 - 2c_2.$$

That is, since $c_2 = A/V$,

$$\frac{dA}{dt} = 8 - 2\frac{A}{V}.$$

Substituting for V from (1.7.7), we must solve

$$\frac{dA}{dt} + \frac{1}{t+4}A = 8. \tag{1.7.8}$$

Modeling Problems (Conti…)

This first-order linear equation has integrating factor

$$I = e^{\int 1/(t+4)\,dt} = t + 4.$$

Consequently (1.7.8) can be written in the equivalent form

$$\frac{d}{dt}[(t+4)A] = 8(t+4),$$

which can be integrated directly to obtain

$$(t+4)A = 4(t+4)^2 + c.$$

Hence

$$A(t) = \frac{1}{t+4}[4(t+4)^2 + c].$$

Imposing the given initial condition $A(0) = 32$ g implies that $c = 64$. Consequently

$$A(t) = \frac{4}{t+4}[(t+4)^2 + 16].$$

Modeling Problems (Conti…)

Setting $t = 20$ gives us the answer for (1) and (2):

1. We have
$$A(20) = \frac{1}{6}[(24)^2 + 16] = \frac{296}{3} \text{ g}.$$

2. Furthermore, using (1.7.7),
$$\frac{A(20)}{V(20)} = \frac{1}{48} \cdot \frac{296}{3} = \frac{37}{18} \text{ g/L}. \qquad \square$$

Change of Variables

DEFINITION 1.8.1

A function $f(x, y)$ is said to be **homogeneous of degree zero**[11] if

$$f(tx, ty) = f(x, y)$$

for all positive values of t for which (tx, ty) is in the domain of f.

Remark Equivalently, we can say that f is homogeneous of degree zero if it is invariant under a re-scaling of the variables x and y.

Example

If $f(x, y) = \dfrac{x^2 + 3xy - y^2}{x^2 + 4y^2}$, then

$$f(tx, ty) = \frac{(tx)^2 + 3(tx)(ty) - (ty)^2}{(tx)^2 + 4(ty)^2} = \frac{t^2(x^2 + 3xy - y^2)}{t^2(x^2 + 4y^2)} = f(x, y),$$

so that f is homogeneous of degree zero.

Change of Variables

In the previous example, if we factor an x^2 term from the numerator and denominator, then the function f can be written in the form

$$f(x, y) = \frac{x^2\left[1 + 3(y/x) - (y/x)^2\right]}{x^2\left[1 + 4(y/x)^2\right]}.$$

That is,

$$f(x, y) = \frac{1 + 3(y/x) - (y/x)^2}{1 + 4(y/x)^2}.$$

Thus f can be considered to depend on the single variable $V = y/x$.

Theorem

A function $f(x, y)$ is homogeneous of degree zero if and only if it depends on y/x only.

Change of Variables

DEFINITION 1.8.4

If $f(x, y)$ is homogeneous of degree zero, then the differential equation

$$\frac{dy}{dx} = f(x, y)$$

is called a **homogeneous first-order differential equation**.

Theorem

The change of variables $y = xV(x)$ reduces a homogeneous first-order differential equation $dy/dx = f(x, y)$ to the separable equation

$$\frac{1}{F(V) - V}dV = \frac{1}{x}dx.$$

Change of Variables

Example

Find the general solution to

$$\frac{dy}{dx} = \frac{4x + y}{x - 4y}. \tag{1.8.3}$$

Solution: The function on the right-hand side of Equation (1.8.3) is homogeneous of degree zero, so that we have a first-order homogeneous differential equation. Substituting $y = xV$ into the equation yields

$$\frac{d}{dx}(xV) = \frac{4 + V}{1 - 4V}.$$

That is,

$$x\frac{dV}{dx} + V = \frac{4 + V}{1 - 4V},$$

or equivalently,

$$x\frac{dV}{dx} = \frac{4(1 + V^2)}{1 - 4V}.$$

Change of Variables

Separating the variables gives

$$\frac{1-4V}{4(1+V^2)}\,dV = \frac{1}{x}\,dx.$$

We write this as

$$\left[\frac{1}{4(1+V^2)} - \frac{V}{1+V^2}\right] dV = \frac{1}{x}\,dx,$$

which can be integrated directly to obtain

$$\frac{1}{4}\arctan V - \frac{1}{2}\ln(1+V^2) = \ln|x| + c.$$

Substituting $V = y/x$ and multiplying through by 2 yields

$$\frac{1}{2}\arctan(y/x) - \ln\left[(x^2+y^2)/x^2\right] = \ln(x^2) + c_1,$$

which simplifies to

$$\frac{1}{2}\arctan(y/x) - \ln(x^2+y^2) = c_1. \tag{1.8.4}$$

Change of Variables

Although this technically gives the answer, the solution is more easily expressed in terms of polar coordinates:

$$x = r\cos\theta \quad \text{and} \quad y = r\sin\theta \quad \Longleftrightarrow \quad r = \sqrt{x^2 + y^2} \quad \text{and} \quad \theta = \arctan(y/x).$$

Substituting into Equation (1.8.4) yields

$$\frac{1}{2}\theta - \ln(r^2) = c_1$$

or equivalently,

$$\ln r = \frac{1}{4}\theta + c_2.$$

Exponentiating both sides of this equation gives

$$r = c_3 e^{\theta/4}.$$

For each value of c_3, this is the equation of a logarithmic spiral. The particular spiral with equation $r = \frac{1}{2}e^{\theta/4}$ is shown in Figure 1.8.1. □

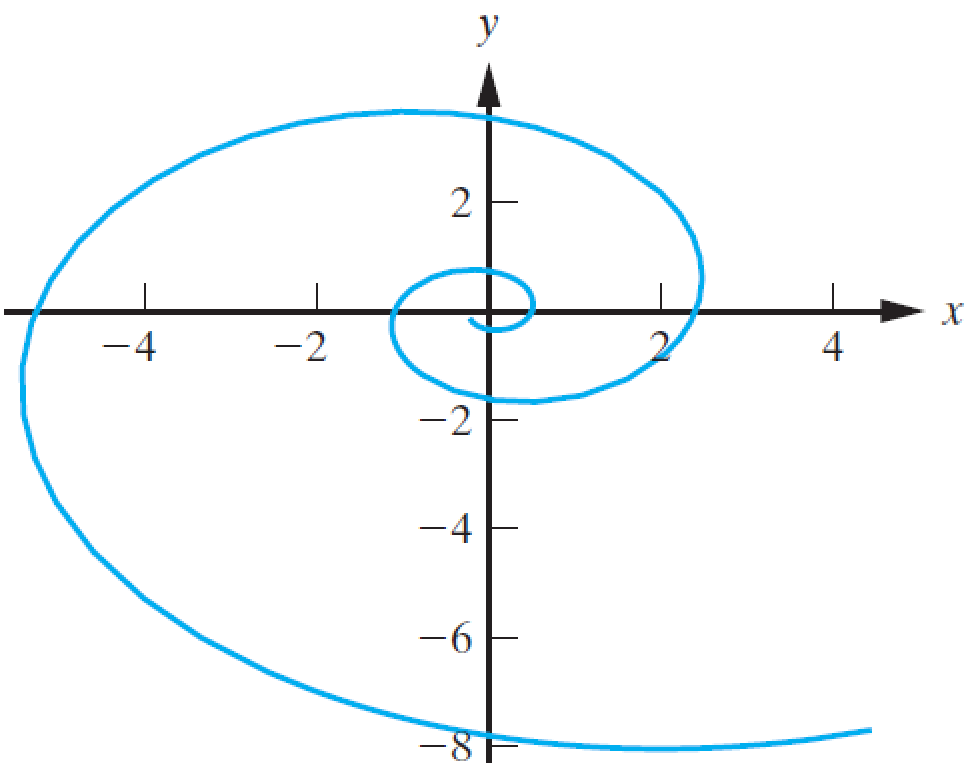

Figure 1.8.1: Graph of the logarithmic spiral with polar equation $r = \frac{1}{2}e^{\theta/4}$, $-\frac{5\pi}{6} \leq \theta \leq \frac{22\pi}{6}$.

Bernoulli Equations (Reading Assignment)

We now consider a special type of nonlinear differential equation that can be reduced to a linear equation by a change of variables.

DEFINITION 1.8.8

A differential equation that can be written in the form

$$\frac{dy}{dx} + p(x)y = q(x)y^n, \tag{1.8.9}$$

where n is a real constant, is called a **Bernoulli equation**.

If $n = 0$ or $n = 1$, Equation (1.8.9) is linear, but otherwise it is nonlinear. We can reduce it to a linear equation as follows. We first divide Equation (1.8.9) by y^n to obtain

$$y^{-n}\frac{dy}{dx} + y^{1-n}p(x) = q(x). \tag{1.8.10}$$

We now make the change of variables

Bernoulli Equations

$$u(x) = y^{1-n}, \tag{1.8.11}$$

which implies that

$$\frac{du}{dx} = (1-n)y^{-n}\frac{dy}{dx}.$$

That is,

$$y^{-n}\frac{dy}{dx} = \frac{1}{1-n}\frac{du}{dx}.$$

Substituting into Equation (1.8.10) for y^{1-n} and $y^{-n}\dfrac{dy}{dx}$ yields the linear differential equation

$$\frac{1}{1-n}\frac{du}{dx} + p(x)u = q(x),$$

or in standard form,

$$\frac{du}{dx} + (1-n)p(x)u = (1-n)q(x). \tag{1.8.12}$$

The linear equation (1.8.12) can now be solved for u as a function of x. The solution to the original equation is then obtained from (1.8.11).

Bernoulli Equations

Example

Solve

$$\frac{dy}{dx} + \frac{3}{x}y = 27y^{1/3}\ln x, \qquad x > 0.$$

Solution: The differential equation is a Bernoulli equation, with $n = 1/3$. Dividing both sides of the differential equation by $y^{1/3}$ yields

$$y^{-1/3}\frac{dy}{dx} + \frac{3}{x}y^{2/3} = 27\ln x. \tag{1.8.13}$$

We make the change of variables

$$u = y^{2/3}, \tag{1.8.14}$$

which implies that

$$\frac{du}{dx} = \frac{2}{3}y^{-1/3}\frac{dy}{dx}.$$

Substituting into Equation (1.8.13) yields

$$\frac{3}{2}\frac{du}{dx} + \frac{3}{x}u = 27\ln x$$

Bernoulli Equations

or, in standard form,

$$\frac{du}{dx} + \frac{2}{x}u = 18 \ln x. \tag{1.8.15}$$

An integrating factor for this linear equation is

$$I(x) = e^{\int (2/x)\,dx} = e^{2 \ln x} = x^2,$$

so that Equation (1.8.15) can be written as

$$\frac{d}{dx}(x^2 u) = 18x^2 \ln x.$$

Integrating, we obtain

$$x^2 u = 18\left(\frac{1}{3}x^3 \ln x - \frac{1}{9}x^3\right) + c,$$

so that

$$u(x) = 2x\,(3 \ln x - 1) + cx^{-2},$$

and so, from (1.8.14), the solution to the original differential equation is

$$y^{2/3} = 2x\,(3 \ln x - 1) + cx^{-2}. \qquad \square$$

Thank You!