Applied Math III
(MATH 2051)
Class 5

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Exact Differential Equations

Recall from a previous calculus course that if $\phi = \phi(x, y)$ is a function of two Variables , *x* and *y*, then the differential of ϕ, denoted $d\phi$, is defined by

$$d\phi = \frac{\partial \phi}{\partial x} dx + \frac{\partial \phi}{\partial y} dy. \tag{1.9.2}$$

DEFINITION 1.9.2

The differential equation

$$M(x, y)\, dx + N(x, y)\, dy = 0$$

is said to be **exact** in a region R of the xy-plane if there exists a function $\phi(x, y)$ such that

$$\frac{\partial \phi}{\partial x} = M, \qquad \frac{\partial \phi}{\partial y} = N, \tag{1.9.4}$$

for all (x, y) in R.

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Any function ϕ satisfying (1.9.4) is called a **potential function** for the differential equation

$$M(x, y)\,dx + N(x, y)\,dy = 0.$$

We emphasize that if such a function exists, then the preceding differential equation can be written as

$$d\phi = 0.$$

Theorem

The general solution to an exact equation

$$M(x, y)\,dx + N(x, y)\,dy = 0$$

is defined implicitly by

$$\phi(x, y) = c,$$

where ϕ satisfies (1.9.4) and c is an arbitrary constant.

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Proof We rewrite the differential equation in the form

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0.$$

Since the differential equation is exact, there exists a potential function ϕ (see (1.9.4)) such that

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0.$$

But this implies that $\dfrac{d}{dx}[\phi(x, y(x))] = 0$, so that $\phi(x, y) = c$, where c is a constant.

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Exact Differential Equations

Remarks

1. The potential function ϕ is a function of two variables x and y, and we interpret the relationship $\phi(x, y) = c$ as defining y implicitly as a function of x. The preceding theorem states that this relationship defines the general solution to the differential equation for which ϕ is a potential function.

2. Geometrically, Theorem 1.9.3 says that the solution curves of an exact differential equation are the family of curves $\phi(x, y) = k$, where k is a constant. These are called the **level curves** of the function $\phi(x, y)$.

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Theorem

(Test for Exactness)

Let M, N, and their first partial derivatives M_y and N_x, be continuous in a (simply connected[13]) region R of the xy-plane. Then the differential equation

$$M(x, y)\,dx + N(x, y)\,dy = 0$$

is exact for all x, y in R if and only if

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}. \tag{1.9.5}$$

Example

Determine whether the given differential equation is exact.

1. $2xe^y\,dx + (x^2e^y + \cos y)\,dy = 0.$
2. $x^2y\,dx - (xy^2 + y^3)\,dy = 0.$

Solution:

1. In this case, $M = 2xe^y$ and $N = x^2e^y + \cos y$, so that $M_y = 2xe^y = N_x$. It follows from the previous theorem that the differential equation is exact.
2. In this case, we have $M = x^2y$ and $N = -(xy^2 + y^3)$, so that $M_y = x^2$, whereas $N_x = -y^2$. Since $M_y \neq N_x$, the differential equation is not exact. $\square$

Example

Determine the general solution to $(y/x)\,dx + [1 + \ln(xy)]\,dy = 0, \quad x > 0.$

Solution: We have

$$M(x, y) = y/x, \qquad N(x, y) = 1 + \ln(xy),$$

so that

$$M_y = 1/x = N_x.$$

Hence the given differential equation is exact, and so there exists a potential function ϕ such that (see Definition 1.9.2)

$$\frac{\partial \phi}{\partial x} = y/x, \tag{1.9.10}$$

$$\frac{\partial \phi}{\partial y} = 1 + \ln(xy). \tag{1.9.11}$$

Integrating Equation (1.9.10) with respect to x, holding y fixed, yields

$$\phi(x, y) = y \ln x + h(y), \tag{1.9.12}$$

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where h is an arbitrary function of y. We now determine $h(y)$ such that (1.9.12) also satisfies Equation (1.9.11). Taking the derivative of (1.9.12) with respect to y yields

$$\frac{\partial\phi}{\partial y} = \ln x + \frac{dh}{dy}. \tag{1.9.13}$$

Equations (1.9.11) and (1.9.13) give two expressions for $\frac{\partial\phi}{\partial y}$. This allows us to determine h. Subtracting Equation (1.9.11) from Equation (1.9.13) gives the consistency requirement

$$\ln x + \frac{dh}{dy} - 1 - \ln(xy) = 0,$$

which simplifies to

$$\frac{dh}{dy} = 1 + \ln y.$$

Integrating the preceding equation yields

$$h(y) = y \ln y,$$

where we have set the integration constant equal to zero without loss of generality since we only require one potential function. Substitution into (1.9.12) yields the potential function

$$\phi(x, y) = y \ln x + y \ln y = y \ln(xy).$$

Consequently, the given differential equation can be written as

$$d[y \ln(xy)] = 0,$$

and so, from Theorem 1.9.3, the general solution is

$$y \ln(xy) = c. \qquad \square$$

Integrating Factors:

DEFINITION 1.9.8

A nonzero function $I(x, y)$ is called an **integrating factor** for the differential equation $M(x, y)\,dx + N(x, y)\,dy = 0$ if the differential equation

$$I(x, y)M(x, y)\,dx + I(x, y)N(x, y)\,dy = 0$$

Example

Show that $I = x^2 y$ is an integrating factor for the differential equation

$$(3y^2 + 5x^2 y)\,dx + (3xy + 2x^3)\,dy = 0. \qquad (1.9.18)$$

Solution: Multiplying the given differential equation (which is not exact) by $x^2 y$ yields

$$(3x^2 y^3 + 5x^4 y^2)\,dx + (3x^3 y^2 + 2x^5 y)\,dy = 0. \qquad (1.9.19)$$

Thus,

$$M_y = 9x^2 y^2 + 10x^4 y = N_x,$$

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so that the general solution to Equation (1.9.19) (and hence the general solution to Equation (1.9.18)) is defined implicitly by

$$x^3y^3 + x^5y^2 = c.$$

That is,

$$x^3y^2(y + x^2) = c. \qquad \square$$

Theorem

The function $I(x, y)$ is an integrating factor for

$$M(x, y)\,dx + N(x, y)\,dy = 0 \tag{1.9.20}$$

if and only if it is a solution to the partial differential equation

$$N\frac{\partial I}{\partial x} - M\frac{\partial I}{\partial y} = \left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) I. \tag{1.9.21}$$

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Theorem

Consider the differential equation $M(x, y)\,dx + N(x, y)\,dy = 0$.

1. There exists an integrating factor that depends only on x if and only if $(M_y - N_x)/N = f(x)$, a function of x only. In such a case, an integrating factor is

$$I(x) = e^{\int f(x)\,dx}.$$

2. There exists an integrating factor that depends only on y if and only if $(M_y - N_x)/M = g(y)$, a function of y only. In such a case, an integrating factor is

$$I(y) = e^{-\int g(y)\,dy}.$$

Example

Solve

$$(2x - y^2)\,dx + xy\,dy = 0, \qquad x > 0. \tag{1.9.23}$$

Solution: The equation is not exact ($M_y \neq N_x$). However,

$$\frac{M_y - N_x}{N} = \frac{-2y - y}{xy} = -\frac{3}{x},$$

which is a function of x only. It follows from (1) of the preceding theorem that an integrating factor for Equation (1.9.23) is

$$I(x) = e^{-\int (3/x)\,dx} = e^{-3\ln x} = x^{-3}.$$

Multiplying Equation (1.9.23) by I yields the exact equation

$$(2x^{-2} - x^{-3}y^2)\,dx + x^{-2}y\,dy = 0. \tag{1.9.24}$$

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$$\phi(x, y) = \frac{1}{2}x^{-2}y^2 - 2x^{-1},$$

and hence the general solution to (1.9.23) is given implicitly by

$$\frac{1}{2}x^{-2}y^2 - 2x^{-1} = c,$$

or equivalently,

$$y^2 - 4x = c_1x^2. \qquad \square$$

Thank You!