Simultaneous Linear Equations

GongYi龚怡

2167570874@qq.com

Link

Slides：

https://pan.baidu.com/s/1uqWdtLXJhJ9TvKN1CsK_9w

Access code ：math

Software:

https://pan.baidu.com/s/1iXhXryPJG-YNYF-RedTZ1Q

Access code : 57fs

What does a matrix look like?

Matrices are everywhere. If you have used a spreadsheet such as Excel or Lotus or written a table, you have used a matrix. Matrices make presentation of numbers clearer and make calculations easier to program.

Look at the matrix below about the sale of tires in a Blowoutr'us store – given by quarter and make of tires.

$$
\begin{array}{l c}
 & \begin{array}{cccc} Q1 & Q2 & Q3 & Q4 \end{array} \\
\textbf{Tirestone} & \\
\textbf{Michigan} & \begin{bmatrix} 25 & 20 & 3 & 2 \\ 5 & 10 & 15 & 25 \\ 6 & 16 & 7 & 27 \end{bmatrix} \\
\textbf{Copper} &
\end{array}
$$

If one wants to know how many *Copper* tires were sold in *Quarter 4*, we go along the row *Copper* and column *Q4* and find that it is 27.

What is a matrix?

A *matrix* is a rectangular array of elements. The elements can be symbolic expressions or numbers. Matrix $[A]$ is denoted by

$$[A] = \begin{bmatrix} a_{11} & a_{12} & \cdots\cdots & a_{1n} \\ a_{21} & a_{22} & \cdots\cdots & a_{2n} \\ \vdots & & & \vdots \\ a_{m1} & a_{m2} & \cdots\cdots & a_{mn} \end{bmatrix}$$

What is a matrix?

Row i of $[A]$ has n elements and is

$$\begin{bmatrix} a_{i1} & a_{i2} \ldots . a_{in} \end{bmatrix}$$

and column j of $[A]$ has m elements and is

$$\begin{bmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{mj} \end{bmatrix}$$

What is a matrix?

Each matrix has rows and columns and this defines the size of the matrix. If a matrix $[A]$ has m rows and n columns, the size of the matrix is denoted by $m \times n$. The matrix $[A]$ may also be denoted by $[A]_{mxn}$ to show that $[A]$ is a matrix with m rows and n columns.

Each entry in the matrix is called the entry or element of the matrix and is denoted by a_{ij} where I is the row number and j is the column number of the element.

What is a matrix?

The matrix for the tire sales example could be denoted by the matrix $[A]$ as

$$[A]=\begin{bmatrix} 25 & 20 & 3 & 2 \\ 5 & 10 & 15 & 25 \\ 6 & 16 & 7 & 27 \end{bmatrix}$$

There are 3 rows and 4 columns, so the size of the matrix is 3×4. In the above $[A]$ matrix, $a_{34}=27$.

Special Types of Matrices

- Row Vector
- Column Vector
- Submatrix
- Square Matrix
- Upper Triangular Matrix
- Lower Triangular Matrix

- Diagonal Matrix
- Identity Matrix
- Zero Matrix
- Tri-diagonal Matrices
- Diagonally Dominant Matrix

What Is a Vector?

What is a vector?

A vector is a matrix that has only one row or one column. There are two types of vectors – row vectors and column vectors.

Row Vector:

If a matrix $[B]$ has one row, it is called a row vector $[B] = [b_1 \; b_2 \ldots\ldots b_n]$ and n is the dimension of the row vector.

Column vector:

If a matrix $[C]$ has one column, it is called a column vector

$$[C] = \begin{bmatrix} c_1 \\ \vdots \\ \vdots \\ c_m \end{bmatrix}$$

and m is the dimension of the vector.

Row Vector

Example 1

An example of a row vector is as follows,

$$[B]=[25\ \ 20\ \ 3\ \ 2\ \ 0]$$

$[B]$ is an example of a row vector of dimension 5.

Column Vector

Example 2

An example of a column vector is as follows,

$$[C] = \begin{bmatrix} 25 \\ 5 \\ 6 \end{bmatrix}$$

$[C]$ is an example of a row vector of dimension 5.

Submatrix

If some row(s) or/and column(s) of a matrix $[A]$ are deleted (no rows or columns may be deleted), the remaining matrix is called a submatrix of $[A]$.

Example 3

Find some of the submatrices of the matrix

$$[A]=\begin{bmatrix}4 & 6 & 2\\ 3 & -1 & 2\end{bmatrix}$$

Square Matrix

If the number of rows m a matrix is equal to the number of columns n of a matrix $[A]$, $(m=n)$, then $[A]$ is called a square matrix. The entries $a_{11}, a_{22}, \ldots, a_{nn}$ are called the *diagonal elements* of a square matrix. Sometimes the diagonal of the matrix is also called the *principal or main of the matrix*.

Example 4

Give an example of a square matrix.

$$[A] = \begin{bmatrix} 25 & 20 & 3 \\ 5 & 10 & 15 \\ 6 & 15 & 7 \end{bmatrix}$$

is a square matrix as it has the same number of rows and columns, that is, 3. The diagonal elements of $[A]$ are $a_{11} = 25, a_{22} = 10, a_{33} = 7$.

Upper Triangular Matrix

A $m \times n$ matrix for which $a_{ij} = 0, i > j$ is called an upper triangular matrix. That is, all the elements below the diagonal entries are zero.

Example 5

Give an example of an upper triangular matrix.

$$[A] = \begin{bmatrix} 10 & -7 & 0 \\ 0 & -0.001 & 6 \\ 0 & 0 & 15005 \end{bmatrix}$$

is an upper triangular matrix.

Lower Triangular Matrix

A $m \times n$ matrix for which $a_{ij} = 0, \; j > i$ is called an lower triangular matrix. That is, all the elements above the diagonal entries are zero.

Example 6

Give an example of a lower triangular matrix.

$$[A] = \begin{bmatrix} 1 & 0 & 0 \\ 0.3 & 1 & 0 \\ 0.6 & 2.5 & 1 \end{bmatrix}$$

is a lower triangular matrix.

Diagonal Matrix

A square matrix with all non-diagonal elements equal to zero is called a diagonal matrix, that is, only the diagonal entries of the square matrix can be non-zero, $(a_{ij} = 0, i \neq j)$.

Example 7

An example of a diagonal matrix.

$$[A] = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2.1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Any or all the diagonal entries of a diagonal matrix can be zero.

$$[A] = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2.1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

is also a diagonal matrix.

Identity Matrix

A diagonal matrix with all diagonal elements equal to one is called an identity matrix, ($a_{ij} = 0, i \neq j$ and $a_{ii} = 1$ for all i).

An example of an identity matrix is,

$$[A] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Zero Matrix

A matrix whose all entries are zero is called a zero matrix, ($a_{ij} = 0$ for all i and j).

Some examples of zero matrices are,

$$[A] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad [\mathrm{B}] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Tridiagonal Matrix

A tridiagonal matrix is a square matrix in which all elements not on the following are zero - the major diagonal, the diagonal above the major diagonal, and the diagonal below the major diagonal.

An example of a tridiagonal matrix is,

$$[A] = \begin{bmatrix} 2 & 4 & 0 & 0 \\ 2 & 3 & 9 & 0 \\ 0 & 0 & 5 & 2 \\ 0 & 0 & 3 & 6 \end{bmatrix}$$

Non-square Matrix

Do non-square matrices have diagonal entries?

Yes, for a $m \times n$ matrix $[A]$, the diagonal entries are $a_{11}, a_{22} \ldots, a_{k-1,k-1}, a_{kk}$ where k=min$\{m,n\}$.

Example 11

What are the diagonal entries of

$$[A] = \begin{bmatrix} 3.2 & 5 \\ 6 & 7 \\ 2.9 & 3.2 \\ 5.6 & 7.8 \end{bmatrix}$$

The diagonal elements of $[A]$ are $a_{11} = 3.2$ and $a_{22} = 7$.

Diagonally Dominant Matrix

A $n \times n$ square matrix $[A]$ is a diagonally dominant matrix if

$$|a_{ii}| \geq \sum_{\substack{j=1 \\ i \neq j}}^{n} |a_{ij}| \quad \text{for all} \quad i = 1,2,\ldots\ldots,n \qquad \text{and}$$

$$|a_{ii}| > \sum_{\substack{j=1 \\ i \neq j}}^{n} |a_{ij}| \quad \text{for at least one } i,$$

that is, for each row, the absolute value of the diagonal element is greater than or equal to the sum of the absolute values of the rest of the elements of that row, and that the inequality is strictly greater than for at least one row. Diagonally dominant matrices are important in ensuring convergence in iterative schemes of solving simultaneous linear equations.

Example 12

Give examples of diagonally dominant matrices and not diagonally dominant matrices.

$$[A]=\begin{bmatrix}15 & 6 & 7\\ 2 & -4 & -2\\ 3 & 2 & 6\end{bmatrix}$$

is a diagonally dominant matrix as

$$|a_{11}|=|15|=15\geq|a_{12}|+|a_{13}|=|6|+|7|=13$$

$$|a_{22}|=|-4|=4\geq|a_{21}|+|a_{23}|=|2|+|-2|=4$$

$$|a_{33}|=|6|=6\geq|a_{31}|+|a_{32}|=|3|+|2|=5$$

and for at least one row, that is Rows 1 and 3 in this case, the inequality is a strictly greater than inequality.

Example 12 (cont.)

$$[B] = \begin{bmatrix} -15 & 6 & 9 \\ 2 & -4 & 2 \\ 3 & -2 & 5.001 \end{bmatrix}$$

is a diagonally dominant matrix as

$$|b_{11}| = |-15| = 15 \geq |b_{12}| + |b_{13}| = |6| + |9| = 15$$

$$|b_{22}| = |-4| = 4 \geq |b_{21}| + |b_{23}| = |2| + |2| = 4$$

$$|b_{33}| = |5.001| = 5.001 \geq |b_{31}| + |b_{32}| = |3| + |-2| = 5$$

The inequalities are satisfied for all rows and it is satisfied strictly greater than for at least one row (in this case it is Row 3).

Example 12 (cont.)

$$[C]=\begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix}$$

is not diagonally dominant as

$$|c_{22}| = |8| = 8 \leq |c_{21}| + |c_{23}| = |64| + |1| = 65$$

Example 13

When are two matrices considered to be equal?

Two matrices [*A*] and [*B*] is the same (number of rows and columns are same for [*A*] and [*B*]) and $a_{ij}=b_{ij}$ for all *i* and *j*.

What would make

$$[A]=\begin{bmatrix}2 & 3\\ 6 & 7\end{bmatrix}$$

to be equal to

$$[B]=\begin{bmatrix}b_{11} & 3\\ 6 & b_{22}\end{bmatrix}$$

The two matrices [*A*] and [*B*] would be equal if $b_{11}=2$ and $b_{22}=7$.

Key Terms:

Matrix

Vector

Submatrix

Square matrix

Equal matrices

Zero matrix

Identity matrix

Diagonal matrix

Upper triangular matrix

Lower triangular matrix

Tri-diagonal matrix

Diagonally dominant matrix

Vectors

Unary Matrix Operations

What is a vector?

A vector is a collection of numbers in a definite order. If it is a collection of n numbers it is called a n-dimensional vector. So the vector $\vec{A}$ given by

$$\vec{A} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$$

Is a n-dimensional column vector with n components, $a_1, a_2, \ldots\ldots, a_n$. The above is a column vector. A row vector [B] is of the form $\vec{B} = [b_1, b_2, \ldots\ldots, b_n]$ where $\vec{B}$ is a n-dimensional row vector with n components $b_1, b_2, \ldots\ldots, b_n$

Example 1

Give an example of a 3-dimensional column vector.

Solution

Assume a point in space is given by its (*x,y,z*) coordinates. Then if the value of $x = 3, y = 2, z = 5$ the column vector corresponding to the location of the points is

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 5 \end{bmatrix}$$

When are two vectors equal?

Two vectors $\vec{A}$ and $\vec{B}$ are equal if they are of the same dimension and if their corresponding components are equal.

Given

$$\vec{A} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$$

and

$$\vec{B} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

Then $\vec{A} = \vec{B}$ if $a_i = b_i,\ i = 1,2,\ldots\ldots,n$

Example 2

What are the values of the unknown components in $\vec{B}$ if

$$\vec{A} = \begin{bmatrix} 2 \\ 3 \\ 4 \\ 1 \end{bmatrix}$$

and

$$\vec{B} = \begin{bmatrix} b_1 \\ 3 \\ 4 \\ b_4 \end{bmatrix}$$

and $\vec{A} = \vec{B}$

Example 2 (cont.)

Solution

$$b_1 = 2, b_4 = 1$$

How do you add two vectors?

Two vectors can be added only if they are of the same dimension and the addition is given by

$$[A]+[B]=\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}+\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

$$=\begin{bmatrix} a_1+b_1 \\ a_2+b_2 \\ \vdots \\ a_n+b_n \end{bmatrix}$$

Example 3

Add the two vectors

$$\vec{A} = \begin{bmatrix} 2 \\ 3 \\ 4 \\ 1 \end{bmatrix}$$

and

$$\vec{B} = \begin{bmatrix} 5 \\ -2 \\ 3 \\ 7 \end{bmatrix}$$

Example 3 (cont.)

Solution

$$\vec{A}+\vec{B}=\begin{bmatrix}2\\3\\4\\1\end{bmatrix}+\begin{bmatrix}5\\-2\\3\\7\end{bmatrix}$$

$$=\begin{bmatrix}2+5\\3-2\\4+3\\1+7\end{bmatrix}$$

$$=\begin{bmatrix}7\\1\\7\\8\end{bmatrix}$$

Example 4

A store sells three brands of tires: Tirestone, Michigan and Copper. In quarter 1, the sales are given by the column vector

$$\vec{A}_1 = \begin{bmatrix} 25 \\ 5 \\ 6 \end{bmatrix}$$

where the rows represent the three brands of tires sold – Tirestone, Michigan and Copper respectively. In quarter 2, the sales are given by

$$\vec{A}_2 = \begin{bmatrix} 20 \\ 10 \\ 6 \end{bmatrix}$$

What is the total sale of each brand of tire in the first half of the year?

Example 4 (cont.)

Solution

The total sales would be given by

$$\vec{C} = \vec{A}_1 + \vec{A}_2$$

$$= \begin{bmatrix} 25 \\ 5 \\ 6 \end{bmatrix} + \begin{bmatrix} 20 \\ 10 \\ 6 \end{bmatrix}$$

$$= \begin{bmatrix} 25+20 \\ 5+10 \\ 6+6 \end{bmatrix}$$

$$= \begin{bmatrix} 45 \\ 15 \\ 12 \end{bmatrix}$$

So the number of Tirestone tires sold is 45, Michigan is 15 and Copper is 12 in the first half of the year.

What is a null vector?

A null vector is where all the components of the vector are zero.

Example 5

Give an example of a null vector or zero vector.

Solution

The vector

$$\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Is an example of a zero or null vector

What is a unit vector?

A unit vector $\vec{U}$ is defined as

$$\vec{U} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}$$

where

$$\sqrt{u_1^2 + u_2^2 + u_3^2 + \ldots + u_n^2} = 1$$

Example 6

Give examples of 3-dimensional unit column vectors.

Solution

Examples include

$$\begin{bmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \text{etc.}$$

How do you multiply a vector by a scalar?

If k is a scalar and $\vec{A}$ is a n-dimensional vector, then

$$k\vec{A} = k\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$$

$$= \begin{bmatrix} ka_1 \\ ka_2 \\ \vdots \\ ka_n \end{bmatrix}$$

Example 7

What is $2\vec{A}$ if

$$\vec{A} = \begin{bmatrix} 25 \\ 20 \\ 5 \end{bmatrix}$$

Example 7 (cont.)

Solution

$$2\vec{A} = 2\begin{bmatrix} 25 \\ 20 \\ 5 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \times 25 \\ 2 \times 20 \\ 2 \times 5 \end{bmatrix}$$

$$= \begin{bmatrix} 50 \\ 40 \\ 10 \end{bmatrix}$$

Example 8

A store sells three brands of tires: Tirestone, Michigan and Copper. In quarter 1, the sales are given by the column vector

$$\vec{A} = \begin{bmatrix} 25 \\ 25 \\ 6 \end{bmatrix}$$

If the goal is to increase the sales of all tires by at least 25% in the next quarter, how many of each brand should be sold?

Example 8 (cont.)

Solution

Since the goal is to increase the sales by 25%, one would multiply the $\vec{A}$ vector by 1.25,

$$\vec{B} = 1.25\begin{bmatrix} 25 \\ 25 \\ 6 \end{bmatrix}$$

$$= \begin{bmatrix} 31.25 \\ 31.25 \\ 7.5 \end{bmatrix}$$

Since the number of tires must be an integer, we can say that the goal of sales is

$$\vec{B} = \begin{bmatrix} 32 \\ 32 \\ 8 \end{bmatrix}$$

What do you mean by a linear combination of vectors?

Given

$$\vec{A}_1, \vec{A}_2, \ldots\ldots, \vec{A}_m$$

as m vectors of same dimension n, and if $k_1, k_2, \ldots, k_m$ are scalars, then

$$k_1\vec{A}_1 + k_2\vec{A}_2 + \ldots\ldots + k_m\vec{A}_m$$

is a linear combination of the m vectors.

Example 9

Find the linear combinations

a) $\vec{A}-\vec{B}$ and

b) $\vec{A}+\vec{B}-3\vec{C}$

where

$$\vec{A}=\begin{bmatrix}2\\3\\6\end{bmatrix},\vec{B}=\begin{bmatrix}1\\1\\2\end{bmatrix},\vec{C}=\begin{bmatrix}10\\1\\2\end{bmatrix}$$

Example 9 (cont.)

Solution

a)

$$\vec{A}-\vec{B}=\begin{bmatrix}2\\3\\6\end{bmatrix}-\begin{bmatrix}1\\1\\2\end{bmatrix}$$

$$=\begin{bmatrix}2-1\\3-1\\6-2\end{bmatrix}$$

$$=\begin{bmatrix}1\\2\\4\end{bmatrix}$$

Example 9 (cont.)

Solution

b)

$$\vec{A}+\vec{B}-3\vec{C}=\begin{bmatrix}2\\3\\6\end{bmatrix}+\begin{bmatrix}1\\1\\2\end{bmatrix}-3\begin{bmatrix}10\\1\\2\end{bmatrix}$$

$$=\begin{bmatrix}2+1-30\\3+1-3\\6+2-6\end{bmatrix}$$

$$=\begin{bmatrix}-27\\1\\2\end{bmatrix}$$

What do you mean by vectors being linearly independent?

A set of vectors $\vec{A}_1, \vec{A}_2, \ldots, \vec{A}_m$ are considered to be linearly independent if

$$k_1\vec{A}_1 + k_2\vec{A}_2 + \ldots\ldots + k_m\vec{A}_m = \vec{0}$$

has only one solution of

$$k_1 = k_2 = \ldots\ldots = k_m = 0$$

Example 10

Are the three vectors

$$\vec{A}_1 = \begin{bmatrix} 25 \\ 64 \\ 144 \end{bmatrix}, \vec{A}_2 = \begin{bmatrix} 5 \\ 8 \\ 12 \end{bmatrix}, \vec{A}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

linearly independent?

Example 10 (cont.)

Solution

Writing the linear combination of the three vectors

$$k_1\begin{bmatrix}25\\64\\144\end{bmatrix}+k_2\begin{bmatrix}5\\8\\12\end{bmatrix}+k_3\begin{bmatrix}1\\1\\1\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

gives

$$\begin{bmatrix}25k_1+5k_2+k_3\\64k_1+8k_2+k_3\\144k_1+12k_2+k_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

Example 10 (cont.)

The previous equations have only one solution, $k_1 = k_2 = k_3 = 0$. However, how do we show that this is the only solution? This is shown below.

The previous equations are

$$25k_1 + 5k_2 + k_3 = 0 \quad (1)$$

$$64k_1 + 8k_2 + k_3 = 0 \quad (2)$$

$$144k_1 + 12k_2 + k_3 = 0 \quad (3)$$

Subtracting Eqn (1) from Eqn (2) gives

$$39k_1 + 3k_2 = 0$$

$$k_2 = -13k_1 \quad (4)$$

Multiplying Eqn (1) by 8 and subtracting it from Eqn (2) that is first multiplied by 5 gives

$$120k_1 - 3k_3 = 0$$

$$k_3 = 40k_1 \quad (5)$$

Example 10 (cont.)

Remember we found Eqn (4) and Eqn (5) just from Eqns (1) and (2).

Substitution of Eqns (4) and (5) in Eqn (3) for and gives

$$144k_1 + 12(-13k_1) + 40k_1 = 0$$

$$28k_1 = 0$$

$$k_1 = 0$$

This means that k_1 has to be zero, and coupled with (4) and (5), k_2 and k_3 are also zero. So the only solution is $k_1 = k_2 = k_3 = 0$. The three vectors hence are linearly independent

Example 11

Are the three vectors

$$\vec{A}_1 = \begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix}, \vec{A}_2 = \begin{bmatrix} 2 \\ 5 \\ 7 \end{bmatrix}, A_3 = \begin{bmatrix} 6 \\ 14 \\ 24 \end{bmatrix}$$

linearly independent?

Example 11 (cont.)

By inspection,

$$\vec{A}_3 = 2\vec{A}_1 + 2\vec{A}_2$$

Or

$$-2\vec{A}_1 - 2\vec{A}_2 + \vec{A}_3 = \vec{0}$$

So the linear combination

$$k_1\vec{A}_1 + k_2\vec{A}_2 + k_3\vec{A}_3 = \vec{0}$$

Has a non-zero solution

$$k_1 = -2,\ k_2 = -2,\ k_3 = 1$$

Example 11 (cont.)

Hence, the set of vectors is linearly dependent.

What if I cannot prove by inspection, what do I do? Put the linear combination of three vectors equal to the zero vector,

$$k_1\begin{bmatrix}1\\2\\5\end{bmatrix}+k_2\begin{bmatrix}2\\5\\7\end{bmatrix}+k_3\begin{bmatrix}6\\14\\24\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

to give

$$k_1+2k_2+6k_3=0 \qquad (1)$$

$$2k_1+5k_2+14k_3=0 \qquad (2)$$

$$5k_1+7k_2+24k_3=0 \qquad (3)$$

Example 11 (cont.)

Multiplying Eqn (1) by 2 and subtracting from Eqn (2) gives

$$k_2 + 2k_3 = 0$$

$$k_2 = -2k_3 \qquad (4)$$

Multiplying Eqn (1) by 2.5 and subtracting from Eqn (2) gives

$$-0.5k_1 - k_3 = 0$$

$$k_1 = -2k_3 \qquad (5)$$

Example 11 (cont.)

Remember we found Eqn (4) and Eqn (5) just from Eqns (1) and (2).

Substitute Eqn (4) and (5) in Eqn (3) for k_1 and k_2 gives

$$5(-2k_3) + 7(-2k_3) + 24k_3 = 0$$

$$-10k_3 - 14k_3 + 24k_3 = 0$$

$$0 = 0$$

This means any values satisfying Eqns (4) and (5) will satisfy Eqns (1), (2) and (3) simultaneously.

For example, chose

$k_3 = 6$ then

$k_2 = -12$ from Eqn(4), and

$k_1 = -12$ from Eqn(5).

Example 11 (cont.)

Hence we have a nontrivial solution of $[k_1 \quad k_2 \quad k_3] = [-12 \quad -12 \quad 6]$. This implies the three given vectors are linearly dependent. Can you find another nontrivial solution?

What about the following three vectors?

$$\begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \\ 7 \end{bmatrix}, \begin{bmatrix} 6 \\ 14 \\ 25 \end{bmatrix}$$

Are they linearly dependent or linearly independent?

Note that the only difference between this set of vectors and the previous one is the third entry in the third vector. Hence, equations (4) and (5) are still valid. What conclusion do you draw when you plug in equations (4) and (5) in the third equation: $5k_1 + 7k_2 + 25k_3 = 0$?

What has changed?

Example 12

Are the three vectors

$$\vec{A}_1 = \begin{bmatrix} 25 \\ 64 \\ 89 \end{bmatrix}, \vec{A}_2 = \begin{bmatrix} 5 \\ 8 \\ 13 \end{bmatrix}, \vec{A}_3 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$$

linearly dependent?

Example 12 (cont.)

Solution

Writing the linear combination of the three vectors and equating to zero vector

$$k_1\begin{bmatrix}25\\64\\89\end{bmatrix}+k_2\begin{bmatrix}5\\8\\13\end{bmatrix}+k_3\begin{bmatrix}1\\1\\2\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

gives

$$\begin{bmatrix}25k_1+5k_2+k_3\\64k_1+8k_2+k_3\\89k_1+13k_2+2k_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

Example 12 (cont.)

In addition to $k_1 = k_2 = k_3 = 0$, one can find other solutions for which k_1, k_2, k_3 are not equal to zero. For example $k_1 = 1, k_2 = -13, k_3 = 40$ is also a solution. This implies

$$1\begin{bmatrix} 25 \\ 64 \\ 89 \end{bmatrix} - 13\begin{bmatrix} 5 \\ 8 \\ 13 \end{bmatrix} + 40\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

So the linear combination that gives us a zero vector consists of non-zero constants. Hence $\vec{A}_1, \vec{A}_2, \vec{A}_3$ are linearly dependent

What do you mean by the rank of a set of vectors?

From a set of n-dimensional vectors, the maximum number of linearly independent vectors in the set is called the rank of the set of vectors. *Note that the rank of the vectors can never be greater than the vectors dimension.*

Example 13

What is the rank of

$$\vec{A}_1 = \begin{bmatrix} 25 \\ 64 \\ 144 \end{bmatrix}, \vec{A}_2 = \begin{bmatrix} 5 \\ 8 \\ 12 \end{bmatrix}, \vec{A}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Solution

Since we found in Example 10 that $\vec{A}_1, \vec{A}_2, \vec{A}_3$ are linearly dependent, the rank of the set of vectors $\vec{A}_1, \vec{A}_2, \vec{A}_3$ is 3

Example 14

What is the rank of

$$\vec{A}_1 = \begin{bmatrix} 25 \\ 64 \\ 89 \end{bmatrix}, \vec{A}_2 = \begin{bmatrix} 5 \\ 8 \\ 13 \end{bmatrix}, \vec{A}_3 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$$

Solution

In Example 12, we found that $\vec{A}_1, \vec{A}_2, \vec{A}_3$ are linearly dependent, the rank of $\vec{A}_1, \vec{A}_2, \vec{A}_3$ is hence not 3, and is less than 3. Is it 2? Let us choose

$$\vec{A}_1 = \begin{bmatrix} 25 \\ 64 \\ 89 \end{bmatrix}, \vec{A}_2 = \begin{bmatrix} 5 \\ 8 \\ 13 \end{bmatrix}$$

Linear combination of $\vec{A}_1$ and $\vec{A}_2$ equal to zero has only one solution. Therefore, the rank is 2.

Example 15

What is the rank of

$$\vec{A}_1 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \vec{A}_2 = \begin{bmatrix} 2 \\ 2 \\ 4 \end{bmatrix}, \vec{A}_3 = \begin{bmatrix} 3 \\ 3 \\ 5 \end{bmatrix}$$

Solution

From inspection

$$\vec{A}_2 = 2\vec{A}_1$$

that implies

$$2\vec{A}_1 - \vec{A}_2 + 0\vec{A}_3 = \vec{0}.$$

Example 15 (cont.)

Hence

$$k_1\vec{A}_1 + k_2\vec{A}_2 + k_3\vec{A}_3 = \vec{0}.$$

has a nontrivial solution

So $\vec{A}_1, \vec{A}_2, \vec{A}_3$ are linearly dependent, and hence the rank of the three vectors is not 3.

Since $\vec{A}_2 = 2\vec{A}_1$

$\vec{A}_1$ and $\vec{A}_2$ are linearly dependent, but

$$k_1\vec{A}_1 + k_3\vec{A}_3 = \vec{0}.$$

has trivial solution as the only solution. So $\vec{A}_1$ and $\vec{A}_3$ are linearly independent. The rank of the above three vectors is 2.

Linearly Dependent

,

Prove that if a set of vectors contains the null vector, the set of vectors is linearly dependent.

Let $\vec{A}_1, \vec{A}_2, \ldots\ldots\ldots, \vec{A}_m$ be a set of n-dimensional vectors, then

$$k_1\vec{A}_1 + k_2\vec{A}_2 + \ldots + k_m\vec{A}_m = \vec{0}$$

is a linear combination of the m vectors. Then assuming if $\vec{A}_1$ is the zero or null vector, any value of k_1 coupled with $k_2 = k_3 = \ldots = k_m = 0$ will satisfy the above equation. Hence, the set of vectors is linearly dependent as more than one solution exists.

Linearly Independent

,

Prove that if a set of vectors are linearly independent, then a subset of the *m* vectors also has to be linearly independent.

Let this subset be

$$\vec{A}_{a1}, \vec{A}_{a2}, \ldots, \vec{A}_{ap}$$

where $p < m$

Then if this subset is linearly dependent, the linear combination

$$k_1\vec{A}_{a1} + k_2\vec{A}_{a2} + \ldots + k_p\vec{A}_{ap} = \vec{0}$$

Has a non-trivial solution.

Linearly Independent (cont.)

,

So

$$k_1\vec{A}_{a1} + k_2\vec{A}_{a2} + \ldots + k_p\vec{A}_{ap} + 0\vec{A}_{a(p+1)} + \ldots\ldots + 0\vec{A}_{am} = \vec{0}$$

also has a non-trivial solution too, where $\vec{A}_{a(p+1)},\ldots,\vec{A}_{am}$ are the rest of the $(m-p)$ vectors. However, this is a contradiction. Therefore, a subset of linearly independent vectors cannot be linearly dependent.

Linearly Dependent

,

Prove that if a set of vectors is linearly dependent, then at least one vector can be written as a linear combination of others.

Let $\vec{A}_1, \vec{A}_2, \ldots, \vec{A}_m$ be linearly dependent, then there exists a set of numbers $k_1, \ldots, k_m$ not all of which are zero for the linear combination

$$k_1\vec{A}_1 + k_2\vec{A}_2 + \ldots + k_m\vec{A}_m = \vec{0}$$

Let $k_p \neq 0$ to give one of the non-zero values of $k_i, i = 1, \ldots, m$, be for $i = p$, then

$$A_p = -\frac{k_2}{k_p}\vec{A}_2 - \ldots - \frac{k_{p-1}}{k_p}\vec{A}_{p-1} - \frac{k_{p+1}}{k_p}\vec{A}_{p+1} - \ldots - \frac{k_m}{k_p}\vec{A}_m.$$

and that proves the theorem.

Linearly Dependent

,

Prove that if the dimension of a set of vectors is less than the number of vectors in the set, then the set of vectors is linearly dependent.

Can you prove it?

How can vectors be used to write simultaneous linear equations?

If a set of *m* linear equations with *n* unknowns is written as

$$
\begin{aligned}
a_{11}x_1 + \ldots + a_{1n}x_n &= c_1 \\
a_{21}x_1 + \ldots + a_{2n}x_n &= c_2 \\
\vdots \qquad\qquad & \vdots \\
\vdots \qquad\qquad & \vdots \\
a_{m1}x_1 + \ldots + a_{mn}x_n &= c_n
\end{aligned}
$$

Linearly Dependent (cont.)

,

Where

$x_1, x_2, \ldots, x_n$ are the unknowns, then in the vector notation they can be written as

$$x_1\vec{A}_1 + x_2\vec{A}_2 + \ldots + x_n\vec{A}_n = \vec{C}$$

where

$$\vec{A}_1 = \begin{bmatrix} a_{11} \\ \vdots \\ a_{m1} \end{bmatrix}$$

Linearly Dependent (cont.)

,

Where

$$\vec{A}_1 = \begin{bmatrix} a_{11} \\ \vdots \\ a_{m1} \end{bmatrix}$$

$$\vec{A}_2 = \begin{bmatrix} a_{12} \\ \vdots \\ a_{m2} \end{bmatrix}$$

$$\vec{A}_n = \begin{bmatrix} a_{1n} \\ \vdots \\ a_{mn} \end{bmatrix}$$

$$\vec{C}_1 = \begin{bmatrix} c_1 \\ \vdots \\ c_m \end{bmatrix}$$

The problem now becomes whether you can find the scalars $x_1, x_2, \ldots\ldots, x_n$ such that the linear combination $x_1\vec{A}_1 + \ldots\ldots\ldots + x_n\vec{A}_n = \vec{C}$

Example 16

Write

$$25x_1 + 5x_2 + x_3 = 106.8$$

$$64x_1 + 8x_2 + x_3 = 177.2$$

$$144x_1 + 12x_2 + x_3 = 279.2$$

as a linear combination of vectors.

Example 16 (cont.)

Solution

$$\begin{bmatrix} 25x_1 & +5x_2 & +x_3 \\ 64x_1 & +8x_2 & +x_3 \\ 144x_1 & +12x_2 & +x_3 \end{bmatrix} = \begin{bmatrix} 106.8 \\ 177.2 \\ 279.2 \end{bmatrix}$$

$$x_1\begin{bmatrix} 25 \\ 64 \\ 144 \end{bmatrix} + x_2\begin{bmatrix} 5 \\ 8 \\ 12 \end{bmatrix} + x_3\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 106.8 \\ 177.2 \\ 279.2 \end{bmatrix}$$

What is the definition of the dot product of two vectors?

Example 16 (cont.)

Let $\vec{A} = [a_1, a_2, \ldots, a_n]$ and $\vec{B} = [b_1, b_2, \ldots, b_n]$ be two n-dimensional vectors. Then the dot product of the two vectors $\vec{A}$ and $\vec{B}$ is defined as

$$\vec{A} \cdot \vec{B} = a_1 b_1 + a_2 b_2 + \ldots + a_n b_n = \sum_{i=1}^{n} a_i b_i$$

A dot product is also called an inner product or scalar.

Example 17

Find the dot product of the two vectors $\vec{A}$ = (4, 1, 2, 3) and $\vec{B}$ = (3, 1, 7, 2).

Solution

$$\vec{A} \cdot \vec{B} = (4,1,2,3) \cdot (3,1,7,2)$$

$$= (4)(3)+(1)(1)+(2)(7)+(3)(2)$$

$$= 33$$

Example 18

A product line needs three types of rubber as given in the table below.

Rubber Type	Weight (lbs)	Cost per pound ($)
A	200	20.23
B	250	30.56
C	310	29.12

Use the definition of a dot product to find the total price of the rubber needed.

Example 18 (cont.)

Solution

The weight vector is given by

$$\vec{W} = (200,250,310)$$

and the cost vector is given by

$$\vec{C} = (20.23,30.56,29.12)$$

The total cost of the rubber would be the dot product of $\vec{W}$ and $\vec{C}$

$$\vec{W} \cdot \vec{C} = (200,250,310) \cdot (20.23,30.56,29.12)$$

$$= (200)(20.23) + (250)(30.56) + (310)(29.12)$$

$$= 4046 + 7640 + 9027.2$$

$$= \$20713.20$$

Key Terms:

- *Vector*
- *Addition of vectors*
- *Rank*
- *Dot Product*
- *Subtraction of vectors*
- *Unit vector*
- *Scalar multiplication of vectors*
- *Null vector*
- *Linear combination of vectors*
- *Linearly independent vectors*

Naïve Gaussian Elimination

A method to solve simultaneous linear equations of the form

$[A][X]=[C]$

Two steps

1. Forward Elimination
2. Back Substitution

Forward Elimination

The goal of forward elimination is to transform the coefficient matrix into an upper triangular matrix

$$\begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 106.8 \\ 177.2 \\ 279.2 \end{bmatrix}$$

$$\downarrow$$

$$\begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -1.56 \\ 0 & 0 & 0.7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 106.8 \\ -96.21 \\ 0.735 \end{bmatrix}$$

Forward Elimination

A set of n equations and n unknowns

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \ldots + a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \ldots + a_{2n}x_n = b_2$$

$$\vdots \qquad \vdots$$

$$a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 + \ldots + a_{nn}x_n = b_n$$

(n-1) steps of forward elimination

Forward Elimination

Step 1

For Equation 2, divide Equation 1 by a_{11} and multiply by a_{21}.

$$\left[\frac{a_{21}}{a_{11}}\right](a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \ldots + a_{1n}x_n = b_1)$$

$$a_{21}x_1 + \frac{a_{21}}{a_{11}}a_{12}x_2 + \ldots + \frac{a_{21}}{a_{11}}a_{1n}x_n = \frac{a_{21}}{a_{11}}b_1$$

Forward Elimination

Subtract the result from Equation 2.

$$
\begin{array}{rl}
 & a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \ldots + a_{2n}x_n = b_2 \\
- & a_{21}x_1 + \dfrac{a_{21}}{a_{11}}a_{12}x_2 + \ldots + \dfrac{a_{21}}{a_{11}}a_{1n}x_n = \dfrac{a_{21}}{a_{11}}b_1 \\
\hline
\end{array}
$$

$$
\left(a_{22} - \frac{a_{21}}{a_{11}}a_{12}\right)x_2 + \ldots + \left(a_{2n} - \frac{a_{21}}{a_{11}}a_{1n}\right)x_n = b_2 - \frac{a_{21}}{a_{11}}b_1
$$

or $$a'_{22}x_2 + \ldots + a'_{2n}x_n = b'_2$$

Forward Elimination

Repeat this procedure for the remaining equations to reduce the set of equations as

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \ldots + a_{1n}x_n = b_1$$

$$a'_{22}x_2 + a'_{23}x_3 + \ldots + a'_{2n}x_n = b'_2$$

$$a'_{32}x_2 + a'_{33}x_3 + \ldots + a'_{3n}x_n = b'_3$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$a'_{n2}x_2 + a'_{n3}x_3 + \ldots + a'_{nn}x_n = b'_n$$

End of Step 1

Forward Elimination

Step 2

Repeat the same procedure for the 3rd term of Equation 3.

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \ldots + a_{1n}x_n = b_1$$

$$a'_{22}x_2 + a'_{23}x_3 + \ldots + a'_{2n}x_n = b'_2$$

$$a''_{33}x_3 + \ldots + a''_{3n}x_n = b''_3$$

$$\vdots \qquad \vdots$$

$$a''_{n3}x_3 + \ldots + a''_{nn}x_n = b''_n$$

End of Step 2

Forward Elimination

At the end of (n-1) Forward Elimination steps, the system of equations will look like

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \ldots + a_{1n}x_n &= b_1 \\
a'_{22}x_2 + a'_{23}x_3 + \ldots + a'_{2n}x_n &= b'_2 \\
a''_{33}x_3 + \ldots + a''_{3n}x_n &= b''_3 \\
\vdots \qquad & \quad \vdots \\
a_{nn}^{(n-1)}x_n &= b_n^{(n-1)}
\end{aligned}
$$

End of Step (n-1)

Matrix Form at End of Forward Elimination

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & a'_{22} & a'_{23} & \cdots & a'_{2n} \\ 0 & 0 & a''_{33} & \cdots & a''_{3n} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & 0 & a_{nn}^{(n-1)} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b'_2 \\ b''_3 \\ \vdots \\ b_n^{(n\text{-}1)} \end{bmatrix}$$

Back Substitution

Solve each equation starting from the last equation

$$\begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -1.56 \\ 0 & 0 & 0.7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 106.8 \\ -96.21 \\ 0.735 \end{bmatrix}$$

Example of a system of 3 equations

Back Substitution Starting Eqns

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \ldots + a_{1n}x_n = b_1$$

$$a'_{22}x_2 + a'_{23}x_3 + \ldots + a'_{2n}x_n = b'_2$$

$$a''_{33}x_3 + \ldots + a''_{n}x_n = b''_3$$

$$\vdots \qquad \vdots$$

$$a_{nn}^{(n-1)}x_n = b_n^{(n-1)}$$

Back Substitution

Start with the last equation because it has only one unknown

$$x_n = \frac{b_n^{(n-1)}}{a_{nn}^{(n-1)}}$$

Back Substitution

$$x_n = \frac{b_n^{(n-1)}}{a_{nn}^{(n-1)}}$$

$$x_i = \frac{b_i^{(i-1)} - a_{i,i+1}^{(i-1)} x_{i+1} - a_{i,i+2}^{(i-1)} x_{i+2} - \ldots - a_{i,n}^{(i-1)} x_n}{a_{ii}^{(i-1)}} \text{ for } i = n-1,\ldots,1$$

$$x_i = \frac{b_i^{(i-1)} - \sum_{j=i+1}^{n} a_{ij}^{(i-1)} x_j}{a_{ii}^{(i-1)}} \text{ for } i = n-1,\ldots,1$$

Naïve Gauss Elimination Example

Example 1

The upward velocity of a rocket is given at three different times

Table 1 Velocity vs. time data.

Time, t (s)	**Velocity,** v (m/s)
5	106.8
8	177.2
12	279.2

The velocity data is approximated by a polynomial as:

$$v(t) = a_1 t^2 + a_2 t + a_3 \ , \qquad 5 \le t \le 12.$$

Find the velocity at t=6 seconds .

Example 1 Cont.

Assume

$$v(t) = a_1 t^2 + a_2 t + a_3 , \quad 5 \le t \le 12.$$

Results in a matrix template of the form:

$$\begin{bmatrix} t_1^2 & t_1 & 1 \\ t_2^2 & t_2 & 1 \\ t_3^2 & t_3 & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$$

Using data from Table 1, the matrix becomes:

$$\begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 106.8 \\ 177.2 \\ 279.2 \end{bmatrix}$$

Example 1 Cont.

$$\begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 106.8 \\ 177.2 \\ 279.2 \end{bmatrix} \Rightarrow \begin{bmatrix} 25 & 5 & 1 & \vdots & 106.8 \\ 64 & 8 & 1 & \vdots & 177.2 \\ 144 & 12 & 1 & \vdots & 279.2 \end{bmatrix}$$

1. Forward Elimination
2. Back Substitution

Forward Elimination

Number of Steps of Forward Elimination

Number of steps of forward elimination is

$(n-1)=(3-1)=2$

Forward Elimination: Step 1

$$\begin{bmatrix} 25 & 5 & 1 & \vdots & 106.8 \\ 64 & 8 & 1 & \vdots & 177.2 \\ 144 & 12 & 1 & \vdots & 279.2 \end{bmatrix}$$

Divide Equation 1 by 25 and multiply it by 64, $\frac{64}{25} = 2.56$

$$[25 \quad 5 \quad 1 \quad \vdots \quad 106.8] \times 2.56 = [64 \quad 12.8 \quad 2.56 \quad \vdots \quad 273.408]$$

Subtract the result from Equation 2

$$\begin{array}{r} [64 \quad 8 \quad 1 \quad \vdots \quad 177.2] \\ -[64 \quad 12.8 \quad 2.56 \quad \vdots \quad 273.408] \\ \hline [0 \quad -4.8 \quad -1.56 \quad \vdots \quad -96.208] \end{array}$$

Substitute new equation for Equation 2

$$\begin{bmatrix} 25 & 5 & 1 & \vdots & 106.8 \\ 0 & -4.8 & -1.56 & \vdots & -96.208 \\ 144 & 12 & 1 & \vdots & 279.2 \end{bmatrix}$$

Forward Elimination: Step 1 (cont.)

$$\begin{bmatrix} 25 & 5 & 1 & \vdots & 106.8 \\ 0 & -4.8 & -1.56 & \vdots & -96.208 \\ 144 & 12 & 1 & \vdots & 279.2 \end{bmatrix}$$

Divide Equation 1 by 25 and multiply it by 144, $\frac{144}{25} = 5.76$

$$[25 \quad 5 \quad 1 \quad \vdots \quad 106.8] \times 5.76 = [144 \quad 28.8 \quad 5.76 \quad \vdots \quad 615.168]$$

Subtract the result from Equation 3

$$\begin{array}{r} [144 \quad 12 \quad 1 \quad \vdots \quad 279.2] \\ -[144 \quad 28.8 \quad 5.76 \quad \vdots \quad 615.168] \\ \hline [0 \quad -16.8 \quad -4.76 \quad \vdots \quad -335.968] \end{array}$$

Substitute new equation for Equation 3

$$\begin{bmatrix} 25 & 5 & 1 & \vdots & 106.8 \\ 0 & -4.8 & -1.56 & \vdots & -96.208 \\ 0 & -16.8 & -4.76 & \vdots & -335.968 \end{bmatrix}$$

Forward Elimination: Step 2

$$\begin{bmatrix} 25 & 5 & 1 & \vdots & 106.8 \\ 0 & -4.8 & -1.56 & \vdots & -96.208 \\ 0 & -16.8 & -4.76 & \vdots & -335.968 \end{bmatrix}$$

Divide Equation 2 by −4.8

and multiply it by −16.8,

$$\frac{-16.8}{-4.8} = 3.5$$

$$\begin{bmatrix} 0 & -4.8 & -1.56 & \vdots & -96.208 \end{bmatrix} \times 3.5 = \begin{bmatrix} 0 & -16.8 & -5.46 & \vdots & -336.728 \end{bmatrix}$$

Subtract the result from Equation 3

$$\begin{array}{r} \begin{bmatrix} 0 & -16.8 & -4.76 & \vdots & 335.968 \end{bmatrix} \\ -\begin{bmatrix} 0 & -16.8 & -5.46 & \vdots & -336.728 \end{bmatrix} \\ \hline \begin{bmatrix} 0 & 0 & 0.7 & \vdots & 0.76 \end{bmatrix} \end{array}$$

Substitute new equation for Equation 3

$$\begin{bmatrix} 25 & 5 & 1 & \vdots & 106.8 \\ 0 & -4.8 & -1.56 & \vdots & -96.208 \\ 0 & 0 & 0.7 & \vdots & 0.76 \end{bmatrix}$$

Back Substitution

Back Substitution

$$\begin{bmatrix} 25 & 5 & 1 & \vdots & 106.8 \\ 0 & -4.8 & -1.56 & \vdots & -96.2 \\ 0 & 0 & 0.7 & \vdots & 0.7 \end{bmatrix} \Rightarrow \begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -1.56 \\ 0 & 0 & 0.7 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 106.8 \\ -96.208 \\ 0.76 \end{bmatrix}$$

Solving for a_3

$$0.7a_3 = 0.76$$

$$a_3 = \frac{0.76}{0.7}$$

$$a_3 = 1.08571$$

Back Substitution (cont.)

$$\begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -1.56 \\ 0 & 0 & 0.7 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 106.8 \\ -96.208 \\ 0.76 \end{bmatrix}$$

Solving for a_2

$$-4.8a_2 - 1.56a_3 = -96.208$$

$$a_2 = \frac{-96.208 + 1.56a_3}{-4.8}$$

$$a_2 = \frac{-96.208 + 1.56 \times 1.08571}{-4.8}$$

$$a_2 = 19.6905$$

Back Substitution (cont.)

$$\begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -1.56 \\ 0 & 0 & 0.7 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 106.8 \\ -96.2 \\ 0.76 \end{bmatrix}$$

Solving for a_1

$$25a_1 + 5a_2 + a_3 = 106.8$$

$$\begin{aligned} a_1 &= \frac{106.8 - 5a_2 - a_3}{25} \\ &= \frac{106.8 - 5 \times 19.6905 - 1.08571}{25} \\ &= 0.290472 \end{aligned}$$

Naïve Gaussian Elimination Solution

$$\begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 106.8 \\ 177.2 \\ 279.2 \end{bmatrix}$$

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 0.290472 \\ 19.6905 \\ 1.08571 \end{bmatrix}$$

Example 1 Cont.

Solution

The solution vector is

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 0.290472 \\ 19.6905 \\ 1.08571 \end{bmatrix}$$

The polynomial that passes through the three data points is then:

$$\begin{aligned} v(t) &= a_1 t^2 + a_2 t + a_3 \\ &= 0.290472t^2 + 19.6905t + 1.08571, \quad 5 \le t \le 12 \end{aligned}$$

$$\begin{aligned} v(6) &= 0.290472(6)^2 + 19.6905(6) + 1.08571 \\ &= 129.686 \text{ m/s.} \end{aligned}$$

Naïve Gauss Elimination Pitfalls

Pitfall#1. Division by zero

$$10x_2 - 7x_3 = 3$$
$$6x_1 + 2x_2 + 3x_3 = 11$$
$$5x_1 - x_2 + 5x_3 = 9$$

$$\begin{bmatrix} 0 & 10 & -7 \\ 6 & 2 & 3 \\ 5 & -1 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 11 \\ 9 \end{bmatrix}$$

Is division by zero an issue here?

$$
\begin{aligned}
12x_1 + 10x_2 - 7x_3 &= 15 \\
6x_1 + 5x_2 + 3x_3 &= 14 \\
5x_1 - x_2 + 5x_3 &= 9
\end{aligned}
$$

$$
\begin{bmatrix} 12 & 10 & -7 \\ 6 & 5 & 3 \\ 5 & -1 & 5 \end{bmatrix}
\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}
=
\begin{bmatrix} 15 \\ 14 \\ 9 \end{bmatrix}
$$

Is division by zero an issue here? YES

$$\begin{aligned}12x_1 + 10x_2 - 7x_3 &= 15\\ 6x_1 + 5x_2 + 3x_3 &= 14\\ 24x_1 - x_2 + 5x_3 &= 28\end{aligned}$$

$$\begin{bmatrix}12 & 10 & -7\\ 6 & 5 & 3\\ 24 & -1 & 5\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix} = \begin{bmatrix}15\\ 14\\ 28\end{bmatrix} \longrightarrow \begin{bmatrix}12 & 10 & -7\\ 0 & 0 & 6.5\\ 12 & -21 & 19\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix} = \begin{bmatrix}15\\ 6.5\\ -2\end{bmatrix}$$

Division by zero is a possibility at any step of forward elimination

Pitfall#2. Large Round-off Errors

$$\begin{bmatrix} 20 & 15 & 10 \\ -3 & -2.249 & 7 \\ 5 & 1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 45 \\ 1.751 \\ 9 \end{bmatrix}$$

Exact Solution

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Pitfall#2. Large Round-off Errors

$$\begin{bmatrix} 20 & 15 & 10 \\ -3 & -2.249 & 7 \\ 5 & 1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 45 \\ 1.751 \\ 9 \end{bmatrix}$$

Solve it on a computer using 6 significant digits with chopping

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0.9625 \\ 1.05 \\ 0.999995 \end{bmatrix}$$

Pitfall#2. Large Round-off Errors

$$\begin{bmatrix} 20 & 15 & 10 \\ -3 & -2.249 & 7 \\ 5 & 1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 45 \\ 1.751 \\ 9 \end{bmatrix}$$

Solve it on a computer using 5 significant digits with chopping

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0.625 \\ 1.5 \\ 0.99995 \end{bmatrix}$$

Is there a way to reduce the round off error?

Avoiding Pitfalls

Increase the number of significant digits

- Decreases round-off error
- Does not avoid division by zero

Avoiding Pitfalls

Gaussian Elimination with Partial Pivoting

- Avoids division by zero
- Reduces round off error

Gauss Elimination with Partial Pivoting

Pitfalls of Naïve Gauss Elimination

- Possible division by zero
- Large round-off errors

Avoiding Pitfalls

Increase the number of significant digits

- Decreases round-off error
- Does not avoid division by zero

Avoiding Pitfalls

Gaussian Elimination with Partial Pivoting

- Avoids division by zero
- Reduces round off error

What is Different About Partial Pivoting?

At the beginning of the k^{th} step of forward elimination, find the maximum of

$$\left|a_{kk}\right|, \left|a_{k+1,k}\right|, \ldots\ldots\ldots\ldots\ldots, \left|a_{nk}\right|$$

If the maximum of the values is $\left|a_{pk}\right|$

in the p^{th} row, $k \le p \le n$, then switch rows p and k.

Matrix Form at Beginning of 2nd Step of Forward Elimination

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & a'_{22} & a'_{23} & \cdots & a'_{2n} \\ 0 & a'_{32} & a'_{33} & \cdots & a'_{3n} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & a'_{n2} & a'_{n3} & a'_{n4} & a'_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b'_2 \\ b'_3 \\ \vdots \\ b'_n \end{bmatrix}$$

Example (2^{nd} step of FE)

$$\begin{bmatrix} 6 & 14 & 5.1 & 3.7 & 6 \\ 0 & -7 & 6 & 1 & 2 \\ 0 & 4 & 12 & 1 & 11 \\ 0 & 9 & 23 & 6 & 8 \\ 0 & -17 & 12 & 11 & 43 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 5 \\ -6 \\ 8 \\ 9 \\ 3 \end{bmatrix}$$

Which two rows would you switch?

Example (2^{nd} step of FE)

$$\begin{bmatrix} 6 & 14 & 5.1 & 3.7 & 6 \\ 0 & -17 & 12 & 11 & 43 \\ 0 & 4 & 12 & 1 & 11 \\ 0 & 9 & 23 & 6 & 8 \\ 0 & -7 & 6 & 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 5 \\ 3 \\ 8 \\ 9 \\ -6 \end{bmatrix}$$

Switched Rows

Gaussian Elimination with Partial Pivoting

A method to solve simultaneous linear equations of the form [A][X]=[C]

Two steps

1. Forward Elimination
2. Back Substitution

Forward Elimination

Same as naïve Gauss elimination method except that we switch rows before **each** of the (n-1) steps of forward elimination.

Example: Matrix Form at Beginning of 2nd Step of Forward Elimination

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & a'_{22} & a'_{23} & \cdots & a'_{2n} \\ 0 & a'_{32} & a'_{33} & \cdots & a'_{3n} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & a'_{n2} & a'_{n3} & a'_{n4} & a'_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b'_2 \\ b'_3 \\ \vdots \\ b'_n \end{bmatrix}$$

Matrix Form at End of Forward Elimination

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & a'_{22} & a'_{23} & \cdots & a'_{2n} \\ 0 & 0 & a''_{33} & \cdots & a''_{3n} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & 0 & a_{nn}^{(n-1)} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b'_2 \\ b''_3 \\ \vdots \\ b_n^{(n-1)} \end{bmatrix}$$

Back Substitution Starting Eqns

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \ldots + a_{1n}x_n &= b_1 \\
a'_{22}x_2 + a'_{23}x_3 + \ldots + a'_{2n}x_n &= b'_2 \\
a''_{33}x_3 + \ldots + a''_{n}x_n &= b''_3 \\
&\vdots \\
a_{nn}^{(n-1)}x_n &= b_n^{(n-1)}
\end{aligned}
$$

Back Substitution

$$x_n = \frac{b_n^{(n-1)}}{a_{nn}^{(n-1)}}$$

$$x_i = \frac{b_i^{(i-1)} - \sum_{j=i+1}^{n} a_{ij}^{(i-1)} x_j}{a_{ii}^{(i-1)}} \text{ for } i = n-1, \ldots, 1$$

Gauss Elimination with Partial Pivoting Example

Example 2

Solve the following set of equations by Gaussian elimination with partial pivoting

$$\begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 106.8 \\ 177.2 \\ 279.2 \end{bmatrix}$$

Example 2 Cont.

$$\begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 106.8 \\ 177.2 \\ 279.2 \end{bmatrix} \Rightarrow \begin{bmatrix} 25 & 5 & 1 & \vdots & 106.8 \\ 64 & 8 & 1 & \vdots & 177.2 \\ 144 & 12 & 1 & \vdots & 279.2 \end{bmatrix}$$

1. Forward Elimination
2. Back Substitution

Forward Elimination

Number of Steps of Forward Elimination

Number of steps of forward elimination is $(n-1)=(3-1)=2$

Forward Elimination: Step 1

- Examine absolute values of first column, first row and below.

$$|25|, |64|, |144|$$

- Largest absolute value is 144 and exists in row 3.
- Switch row 1 and row 3.

$$\begin{bmatrix} 25 & 5 & 1 & \vdots & 106.8 \\ 64 & 8 & 1 & \vdots & 177.2 \\ 144 & 12 & 1 & \vdots & 279.2 \end{bmatrix} \Rightarrow \begin{bmatrix} 144 & 12 & 1 & \vdots & 279.2 \\ 64 & 8 & 1 & \vdots & 177.2 \\ 25 & 5 & 1 & \vdots & 106.8 \end{bmatrix}$$

Forward Elimination: Step 1 (cont.)

$$\begin{bmatrix} 144 & 12 & 1 & \vdots & 279.2 \\ 64 & 8 & 1 & \vdots & 177.2 \\ 25 & 5 & 1 & \vdots & 106.8 \end{bmatrix}$$

Divide Equation 1 by 144 and multiply it by 64, $\frac{64}{144} = 0.4444$

$$[144 \quad 12 \quad 1 \quad \vdots \quad 279.2] \times 0.4444 = [63.99 \quad 5.333 \quad 0.4444 \quad \vdots \quad 124.1]$$

Subtract the result from Equation 2

$$\begin{array}{r} [64 \quad 8 \quad 1 \quad \vdots \quad 177.2] \\ -[63.99 \quad 5.333 \quad 0.4444 \quad \vdots \quad 124.1] \\ \hline [0 \quad 2.667 \quad 0.5556 \quad \vdots \quad 53.10] \end{array}$$

Substitute new equation for Equation 2

$$\begin{bmatrix} 144 & 12 & 1 & \vdots & 279.2 \\ 0 & 2.667 & 0.5556 & \vdots & 53.10 \\ 25 & 5 & 1 & \vdots & 106.8 \end{bmatrix}$$

Forward Elimination: Step 1 (cont.)

$$\begin{bmatrix} 144 & 12 & 1 & \vdots & 279.2 \\ 0 & 2.667 & 0.5556 & \vdots & 53.10 \\ 25 & 5 & 1 & \vdots & 106.8 \end{bmatrix}$$

Divide Equation 1 by 144 and multiply it by 25, $\frac{25}{144} = 0.1736$

$$[144 \quad 12 \quad 1 \quad \vdots \quad 279.2] \times 0.1736 = [25.00 \quad 2.083 \quad 0.1736 \quad \vdots \quad 48.47]$$

Subtract the result from Equation 3

$$\begin{array}{r} [25 \quad 5 \quad 1 \quad \vdots \quad 106.8] \\ -[25 \quad 2.083 \quad 0.1736 \quad \vdots \quad 48.47] \\ \hline [0 \quad 2.917 \quad 0.8264 \quad \vdots \quad 58.33] \end{array}$$

Substitute new equation for Equation 3

$$\begin{bmatrix} 144 & 12 & 1 & \vdots & 279.2 \\ 0 & 2.667 & 0.5556 & \vdots & 53.10 \\ 0 & 2.917 & 0.8264 & \vdots & 58.33 \end{bmatrix}$$

Forward Elimination: Step 2

- Examine absolute values of second column, second row and below.

$$|2.667|, |2.917|$$

- Largest absolute value is 2.917 and exists in row 3.
- Switch row 2 and row 3.

$$\begin{bmatrix} 144 & 12 & 1 & \vdots & 279.2 \\ 0 & 2.667 & 0.5556 & \vdots & 53.10 \\ 0 & 2.917 & 0.8264 & \vdots & 58.33 \end{bmatrix} \Rightarrow \begin{bmatrix} 144 & 12 & 1 & \vdots & 279.2 \\ 0 & 2.917 & 0.8264 & \vdots & 58.33 \\ 0 & 2.667 & 0.5556 & \vdots & 53.10 \end{bmatrix}$$

Forward Elimination: Step 2 (cont.)

$$\begin{bmatrix} 144 & 12 & 1 & \vdots & 279.2 \\ 0 & 2.917 & 0.8264 & \vdots & 58.33 \\ 0 & 2.667 & 0.5556 & \vdots & 53.10 \end{bmatrix}$$

Divide Equation 2 by 2.917 and multiply it by 2.667, $\dfrac{2.667}{2.917} = 0.9143.$

$$\begin{bmatrix} 0 & 2.917 & 0.8264 & \vdots & 58.33 \end{bmatrix} \times 0.9143 = \begin{bmatrix} 0 & 2.667 & 0.7556 & \vdots & 53.33 \end{bmatrix}$$

Subtract the result from Equation 3

$$\begin{array}{r} \begin{bmatrix} 0 & 2.667 & 0.5556 & \vdots & 53.10 \end{bmatrix} \\ -\begin{bmatrix} 0 & 2.667 & 0.7556 & \vdots & 53.33 \end{bmatrix} \\ \hline \begin{bmatrix} 0 & 0 & -0.2 & \vdots & -0.23 \end{bmatrix} \end{array}$$

Substitute new equation for Equation 3

$$\begin{bmatrix} 144 & 12 & 1 & \vdots & 279.2 \\ 0 & 2.917 & 0.8264 & \vdots & 58.33 \\ 0 & 0 & -0.2 & \vdots & -0.23 \end{bmatrix}$$

Back Substitution

Back Substitution

$$\begin{bmatrix} 144 & 12 & 1 & \vdots & 279.2 \\ 0 & 2.917 & 0.8264 & \vdots & 58.33 \\ 0 & 0 & -0.2 & \vdots & -0.23 \end{bmatrix} \Rightarrow \begin{bmatrix} 144 & 12 & 1 \\ 0 & 2.917 & 0.8264 \\ 0 & 0 & -0.2 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 279.2 \\ 58.33 \\ -0.23 \end{bmatrix}$$

Solving for a_3

$$-0.2a_3 = -0.23$$

$$a_3 = \frac{-0.23}{-0.2}$$

$$= 1.15$$

Back Substitution (cont.)

$$\begin{bmatrix} 144 & 12 & 1 \\ 0 & 2.917 & 0.8264 \\ 0 & 0 & -0.2 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 279.2 \\ 58.33 \\ -0.23 \end{bmatrix}$$

Solving for a_2

$$2.917a_2 + 0.8264a_3 = 58.33$$

$$\begin{aligned} a_2 &= \frac{58.33 - 0.8264a_3}{2.917} \\ &= \frac{58.33 - 0.8264 \times 1.15}{2.917} \\ &= 19.67 \end{aligned}$$

Back Substitution (cont.)

$$\begin{bmatrix} 144 & 12 & 1 \\ 0 & 2.917 & 0.8264 \\ 0 & 0 & -0.2 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 279.2 \\ 58.33 \\ -0.23 \end{bmatrix}$$

Solving for a_1

$$144a_1 + 12a_2 + a_3 = 279.2$$

$$\begin{aligned} a_1 &= \frac{279.2 - 12a_2 - a_3}{144} \\ &= \frac{279.2 - 12 \times 19.67 - 1.15}{144} \\ &= 0.2917 \end{aligned}$$

Gaussian Elimination with Partial Pivoting Solution

$$\begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 106.8 \\ 177.2 \\ 279.2 \end{bmatrix}$$

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 0.2917 \\ 19.67 \\ 1.15 \end{bmatrix}$$

Gauss Elimination with Partial Pivoting Another Example

Partial Pivoting: Example

Consider the system of equations

$$10x_1 - 7x_2 = 7$$

$$-3x_1 + 2.099x_2 + 6x_3 = 3.901$$

$$5x_1 - x_2 + 5x_3 = 6$$

In matrix form

$$\begin{bmatrix} 10 & -7 & 0 \\ -3 & 2.099 & 6 \\ 5 & -1 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7 \\ 3.901 \\ 6 \end{bmatrix}$$

Solve using Gaussian Elimination with Partial Pivoting using five significant digits with chopping

Partial Pivoting: Example

Forward Elimination: Step 1

Examining the values of the first column

|10|, |-3|, and |5| or 10, 3, and 5

The largest absolute value is 10, which means, to follow the rules of Partial Pivoting, we switch row1 with row1.

Performing Forward Elimination

$$\begin{bmatrix} 10 & -7 & 0 \\ -3 & 2.099 & 6 \\ 5 & -1 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7 \\ 3.901 \\ 6 \end{bmatrix} \Rightarrow \begin{bmatrix} 10 & -7 & 0 \\ 0 & -0.001 & 6 \\ 0 & 2.5 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7 \\ 6.001 \\ 2.5 \end{bmatrix}$$

Partial Pivoting: Example

Forward Elimination: Step 2

Examining the values of the first column

|-0.001| and |2.5| or 0.0001 and 2.5

The largest absolute value is 2.5, so row 2 is switched with row 3

Performing the row swap

$$\begin{bmatrix} 10 & -7 & 0 \\ 0 & -0.001 & 6 \\ 0 & 2.5 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7 \\ 6.001 \\ 2.5 \end{bmatrix} \Rightarrow \begin{bmatrix} 10 & -7 & 0 \\ 0 & 2.5 & 5 \\ 0 & -0.001 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7 \\ 2.5 \\ 6.001 \end{bmatrix}$$

Partial Pivoting: Example

Forward Elimination: Step 2

Performing the Forward Elimination results in:

$$\begin{bmatrix} 10 & -7 & 0 \\ 0 & 2.5 & 5 \\ 0 & 0 & 6.002 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7 \\ 2.5 \\ 6.002 \end{bmatrix}$$

Partial Pivoting: Example

Back Substitution

Solving the equations through back substitution

$$\begin{bmatrix} 10 & -7 & 0 \\ 0 & 2.5 & 5 \\ 0 & 0 & 6.002 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7 \\ 2.5 \\ 6.002 \end{bmatrix}$$

$$x_3 = \frac{6.002}{6.002} = 1$$

$$x_2 = \frac{2.5 - 5x_3}{2.5} = -1$$

$$x_1 = \frac{7 + 7x_2 - 0x_3}{10} = 0$$

Partial Pivoting: Example

Compare the calculated and exact solution

The fact that they are equal is coincidence, but it does illustrate the advantage of Partial Pivoting

$$[X]_{calculated} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \qquad [X]_{exact} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

Determinant of a Square Matrix Using Naïve Gauss Elimination Example

Theorem of Determinants

If a multiple of one row of $[A]_{nxn}$ is added or subtracted to another row of $[A]_{nxn}$ to result in $[B]_{nxn}$ then

$\det(A)=\det(B)$

Theorem of Determinants

The determinant of an upper triangular matrix $[A]_{nxn}$ is given by

$$\det(\mathrm{A}) = a_{11} \times a_{22} \times \ldots \times a_{ii} \times \ldots \times a_{nn}$$

$$= \prod_{i=1}^{n} a_{ii}$$

Forward Elimination of a Square Matrix

Using forward elimination to transform $[A]_{nxn}$ to an upper triangular matrix, $[U]_{nxn}$.

$$[A]_{n\times n} \rightarrow [U]_{n\times n}$$

$$\det(A) = \det(U)$$

Example

Using naïve Gaussian elimination find the determinant of the following square matrix.

$$\begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix}$$

Forward Elimination

Forward Elimination: Step 1

$$\begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix}$$

Divide Equation 1 by 25 and multiply it by 64, $\frac{64}{25} = 2.56$

$$\begin{bmatrix} 25 & 5 & 1 \end{bmatrix} \times 2.56 = \begin{bmatrix} 64 & 12.8 & 2.56 \end{bmatrix}$$

Subtract the result from Equation 2

$$\begin{array}{r} \begin{bmatrix} 64 & 8 & 1 \end{bmatrix} \\ -\begin{bmatrix} 64 & 12.8 & 2.56 \end{bmatrix} \\ \hline \begin{bmatrix} 0 & -4.8 & -1.56 \end{bmatrix} \end{array}$$

Substitute new equation for Equation 2

$$\begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -1.56 \\ 144 & 12 & 1 \end{bmatrix}$$

Forward Elimination: Step 1 (cont.)

$$\begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -1.56 \\ 144 & 12 & 1 \end{bmatrix}$$

Divide Equation 1 by 25 and multiply it by 144, $\frac{144}{25} = 5.76$

$$\begin{bmatrix} 25 & 5 & 1 \end{bmatrix} \times 5.76 = \begin{bmatrix} 144 & 28.8 & 5.76 \end{bmatrix}$$

Subtract the result from Equation 3

$$\begin{array}{r} \begin{bmatrix} 144 & 12 & 1 \end{bmatrix} \\ -\begin{bmatrix} 144 & 28.8 & 5.76 \end{bmatrix} \\ \hline \begin{bmatrix} 0 & -16.8 & -4.76 \end{bmatrix} \end{array}$$

Substitute new equation for Equation 3

$$\begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -1.56 \\ 0 & -16.8 & -4.76 \end{bmatrix}$$

Forward Elimination: Step 2

$$\begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -1.56 \\ 0 & -16.8 & -4.76 \end{bmatrix}$$

Divide Equation 2 by −4.8

and multiply it by −16.8,

$$\frac{-16.8}{-4.8} = 3.5$$

$$\left(\begin{bmatrix} 0 & -4.8 & -1.56 \end{bmatrix}\right) \times 3.5 = \begin{bmatrix} 0 & -16.8 & -5.46 \end{bmatrix}$$

Subtract the result from Equation 3

$$\begin{array}{r} \begin{bmatrix} 0 & -16.8 & -4.76 \end{bmatrix} \\ -\begin{bmatrix} 0 & -16.8 & -5.46 \end{bmatrix} \\ \hline \begin{bmatrix} 0 & 0 & 0.7 \end{bmatrix} \end{array}$$

Substitute new equation for Equation 3

$$\begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -1.56 \\ 0 & 0 & 0.7 \end{bmatrix}$$

Finding the Determinant

After forward elimination

$$\begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -1.56 \\ 0 & 0 & 0.7 \end{bmatrix}$$

$$\begin{aligned} \det(\mathrm{A}) &= u_{11} \times u_{22} \times u_{33} \\ &= 25 \times (-4.8) \times 0.7 \\ &= -84.00 \end{aligned}$$

Summary

-Forward Elimination

-Back Substitution

-Pitfalls

-Improvements

-Partial Pivoting

-Determinant of a Matrix

Homework

Using a computer and round up to 3 decimal places (报留3位小数)

$$\begin{cases} 2.51x_1 + 1.48x_2 + 4.53x_3 = 0.05 \\ 1.48x_1 + 0.93x_2 - 1.3x_3 = 1.03 \\ 2.68x_1 + 3.04x_2 - 1.48x_3 = -0.53 \end{cases}$$

① Solve this equations by Naïve Gauss elimination；

② Solve this equations by Gaussian elimination with partial pivoting；

③ Compare these two solutions and find which one is closer to the exact solution (x_1=1.4531, x_2=-1.589195, x_3=-0.2748947).

Submit to CG in pdf format and in English