Root finding

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Link

Slides：

https://pan.baidu.com/s/1uqWdtLXJhJ9TvKN1CsK_9w

Access code ：math

Software:

https://pan.baidu.com/s/1iXhXryPJG-YNYF-RedTZ1Q

Access code : 57fs

Contents

Numerical solutions of nonlinear equations

- Secant method
  - Two point secant method
  - One point secant method
- Fix-point iteration
  - Ordinary iteration
  - Iteration theory

Secant Method – Derivation

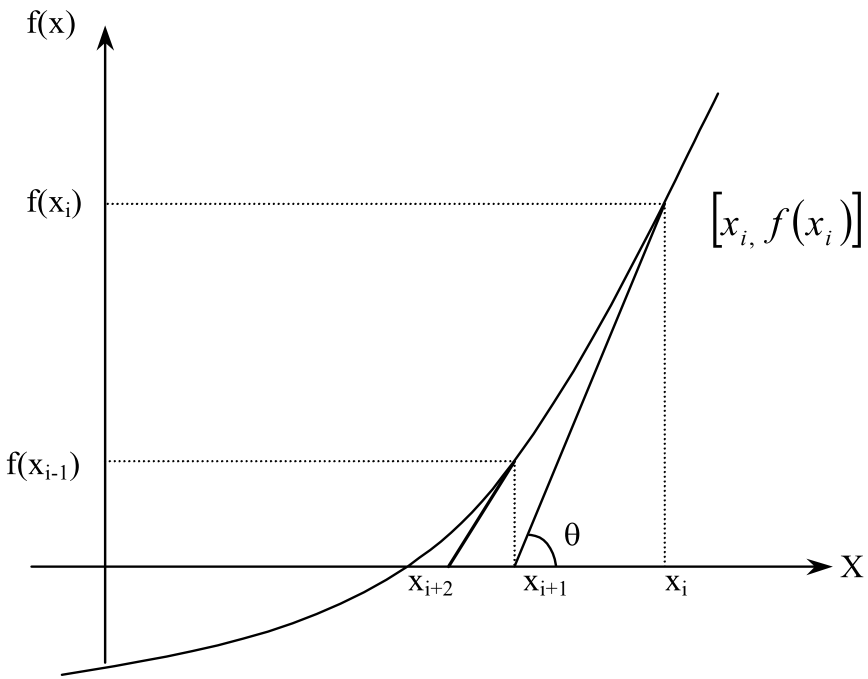


Figure 1 Geometrical illustration of the Newton-Raphson method.

Newton's Method

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)} \qquad (1)$$

Approximate the derivative

$$f'(x_i) = \frac{f(x_i) - f(x_{i-1})}{x_i - x_{i-1}} \qquad (2)$$

Substituting Equation (2) into Equation (1) gives the Secant method

$$x_{i+1} = x_i - \frac{f(x_i)(x_i - x_{i-1})}{f(x_i) - f(x_{i-1})}$$

Secant Method – Derivation

The secant method can also be derived from geometry:

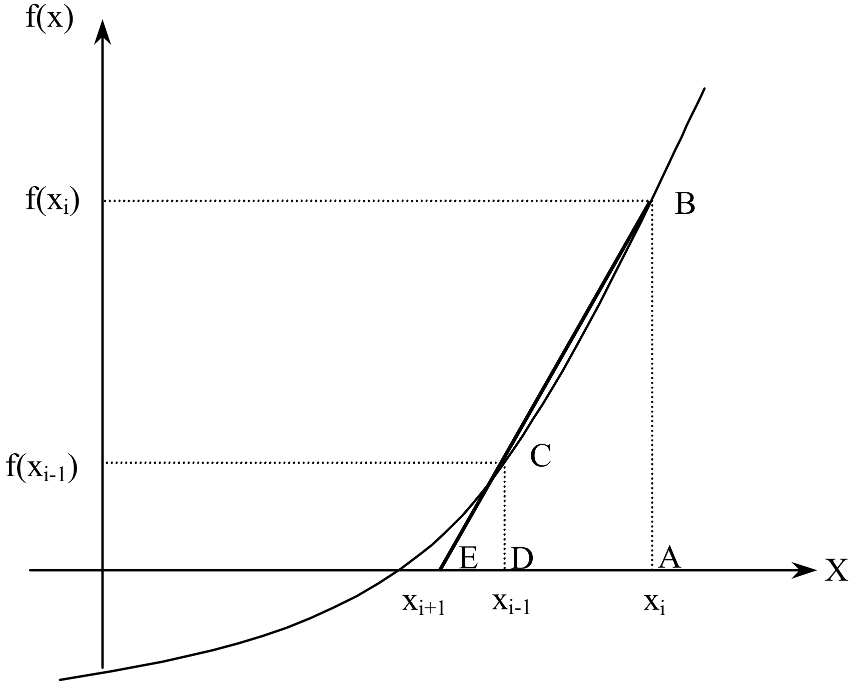


Figure 2 Geometrical representation of the Secant method.

The Geometric Similar Triangles

$$\frac{AB}{AE} = \frac{DC}{DE}$$

can be written as

$$\frac{f(x_i)}{x_i - x_{i+1}} = \frac{f(x_{i-1})}{x_{i-1} - x_{i+1}}$$

On rearranging, the secant method is given as

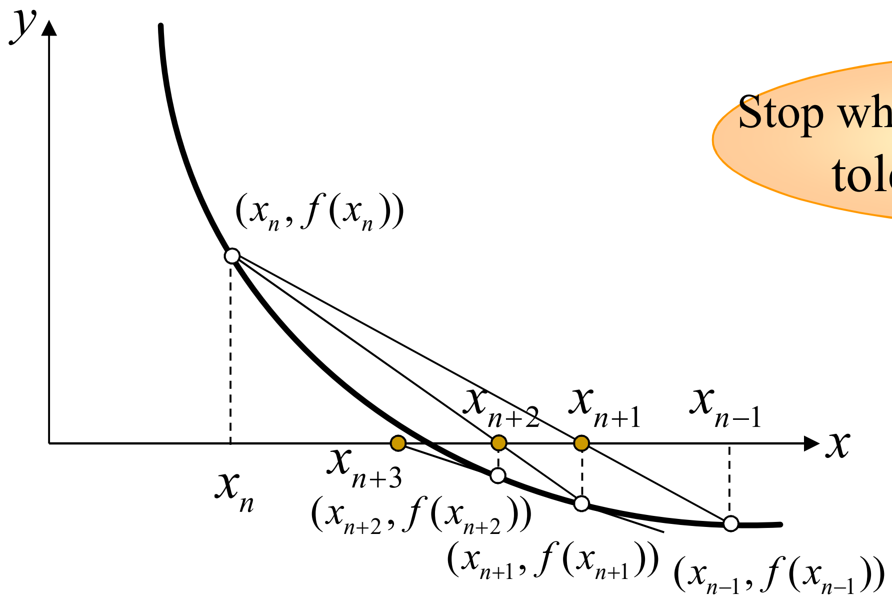
$$x_{i+1} = x_i - \frac{f(x_i)(x_i - x_{i-1})}{f(x_i) - f(x_{i-1})}$$

Two Point Secant Method

Substituting $\frac{f(x_n)-f(x_{n-1})}{x_n - x_{n-1}}$ for approximate $f'(x)$

Given x_0, x_1, for n=1,2,…, we have

$$\boldsymbol{x_{n+1} = x_n - \frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})} f(x_n)}$$



Stop when pre-specified error tolerance is satisfied

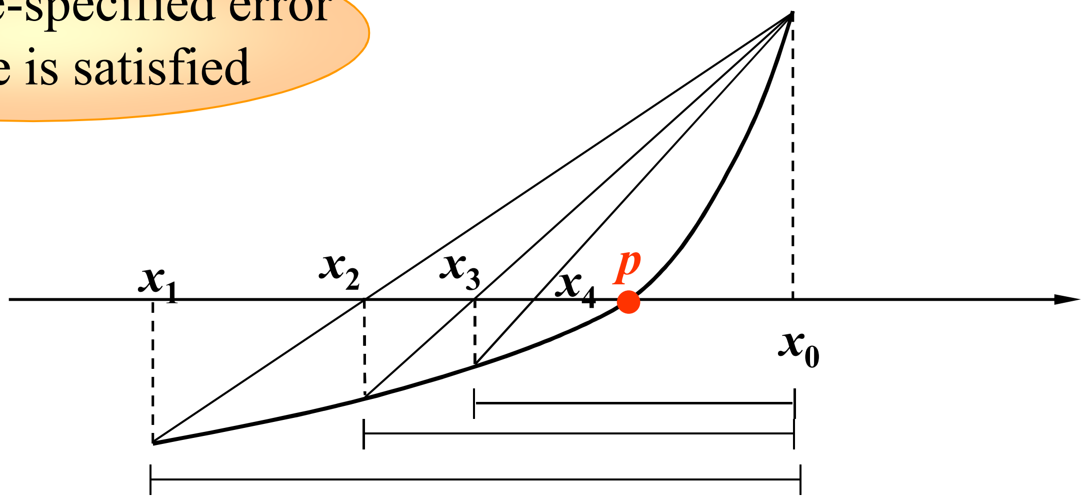
One Point Secant Method

$f(x_0)f(x_1) < 0$ must be guaranteed

Given x_0, x_1, for n=1,2,…, we have

$$x_{n+1} = x_n - \frac{x_n - x_0}{f(x_n) - f(x_0)} f(x_n)$$

Stop when pre-specified error tolerance is satisfied



Algorithm for Secant Method

Step 1

Calculate the next estimate of the root from two initial guesses

$$x_{i+1} = x_i - \frac{f(x_i)(x_i - x_{i-1})}{f(x_i) - f(x_{i-1})}$$

Find the absolute relative approximate error

$$\left|\varepsilon_a\right| = \left|\frac{x_{i+1} - x_i}{x_{i+1}}\right| \times 100$$

Step 2

Find if the absolute relative approximate error is greater than the prespecified relative error tolerance.

If so, go back to step 1, else stop the algorithm.

Also check if the number of iterations has exceeded the maximum number of iterations.

Example 1

You are working for 'DOWN THE TOILET COMPANY' that makes floats for ABC commodes. The floating ball has a specific gravity of 0.6 and has a radius of 5.5 cm. You are asked to find the depth to which the ball is submerged when floating in water.

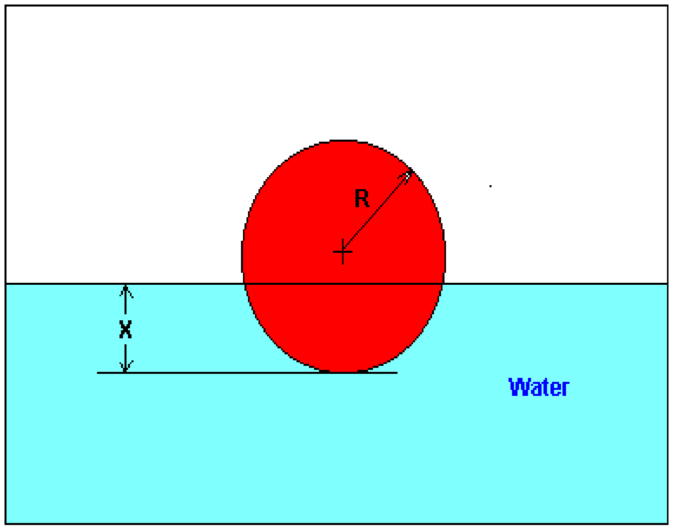


Figure 3 Floating Ball Problem.

Example 1 Cont.

The equation that gives the depth x to which the ball is submerged under water is given by

$$f(x) = x^3 - 0.165x^2 + 3.993 \times 10^{-4}$$

Use the Secant method of finding roots of equations to find the depth x to which the ball is submerged under water.

- Conduct three iterations to estimate the root of the above equation.
- Find the absolute relative approximate error and the number of significant digits at least correct at the end of each iteration.

Example 1 Cont.

Solution

To aid in the understanding of how this method works to find the root of an equation, the graph of f(x) is shown to the right,

where

$$f(x) = x^3 - 0.165x^2 + 3.993\times10^{-4}$$

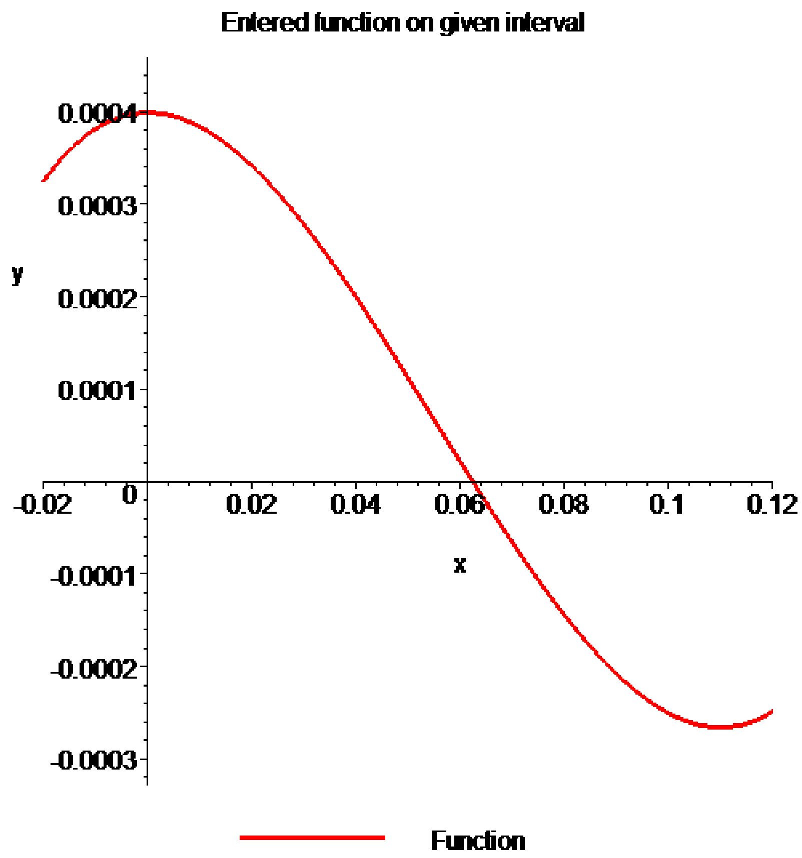


Figure 4 Graph of the function f(x).

Example 1 Cont.

Let us assume the initial guesses of the root of $f(x)=0$ as $x_{-1}=0.02$ and $x_0=0.05$.

Iteration 1

The estimate of the root is

$$x_1 = x_0 - \frac{f(x_0)(x_0 - x_{-1})}{f(x_0) - f(x_{-1})}$$

$$= 0.05 - \frac{\left(0.05^3 - 0.165(0.05)^2 + 3.993\times10^{-4}\right)(0.05 - 0.02)}{\left(0.05^3 - 0.165(0.05)^2 + 3.993\times10^{-4}\right) - \left(0.02^3 - 0.165(0.02)^2 + 3.993\times10^{-4}\right)}$$

$$= 0.06461$$

Example 1 Cont.

The absolute relative approximate error $|\varepsilon_a|$ at the end of **Iteration 1** is

$$|\varepsilon_a| = \left|\frac{x_1 - x_0}{x_1}\right| \times 100$$

$$= \left|\frac{0.06461 - 0.05}{0.06461}\right| \times 100$$

$$= 22.62\%$$

The number of significant digits at least correct is 0, as you need an absolute relative approximate error of 5% or less for one significant digits to be correct in your result.

Example 1 Cont.

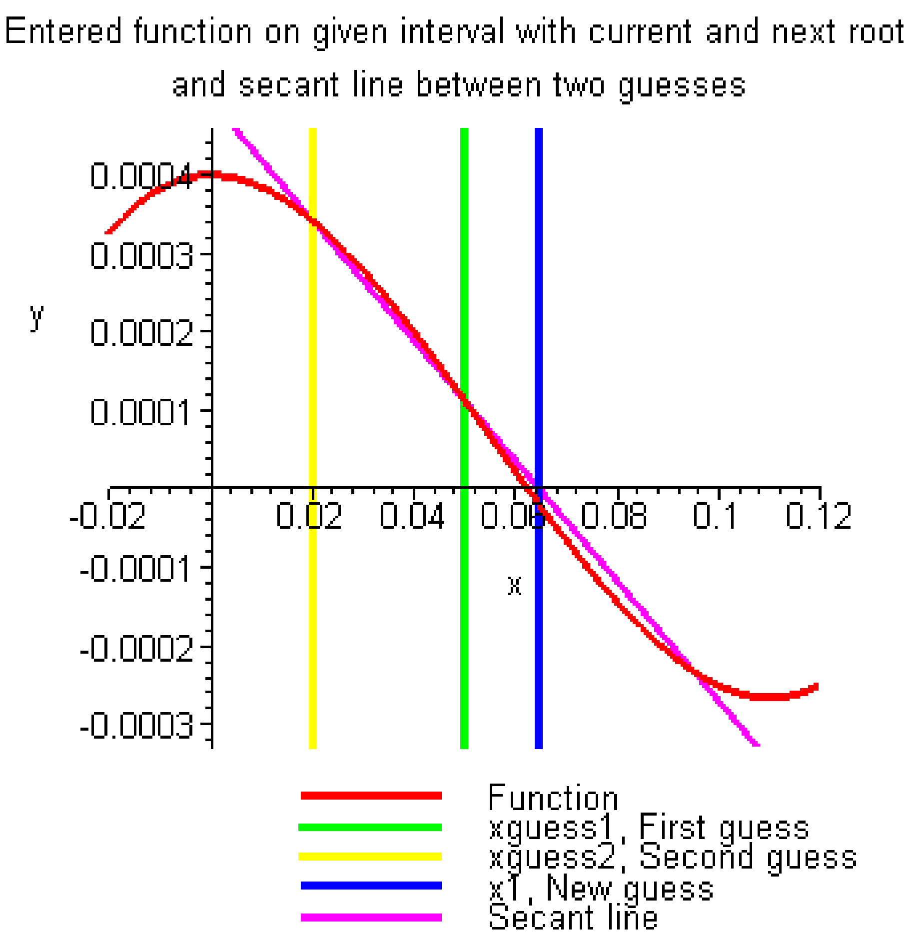


Figure 5 Graph of results of Iteration 1.

Example 1 Cont.

Iteration 2

The estimate of the root is

$$x_2 = x_1 - \frac{f(x_1)(x_1 - x_0)}{f(x_1) - f(x_0)}$$

$$= 0.06461 - \frac{\left(0.06461^3 - 0.165(0.06461)^2 + 3.993\times10^{-4}\right)(0.06461 - 0.05)}{\left(0.06461^3 - 0.165(0.06461)^2 + 3.993\times10^{-4}\right) - \left(0.05^3 - 0.165(0.05)^2 + 3.993\times10^{-4}\right)}$$

$$= 0.06241$$

Example 1 Cont.

The absolute relative approximate error $|\varepsilon_a|$ at the end of Iteration 2 is

$$|\varepsilon_a| = \left|\frac{x_2 - x_1}{x_2}\right| \times 100$$

$$= \left|\frac{0.06241 - 0.06461}{0.06241}\right| \times 100$$

$$= 3.525\%$$

The number of significant digits at least correct is 1, as you need an absolute relative approximate error of 5% or less.

Example 1 Cont.

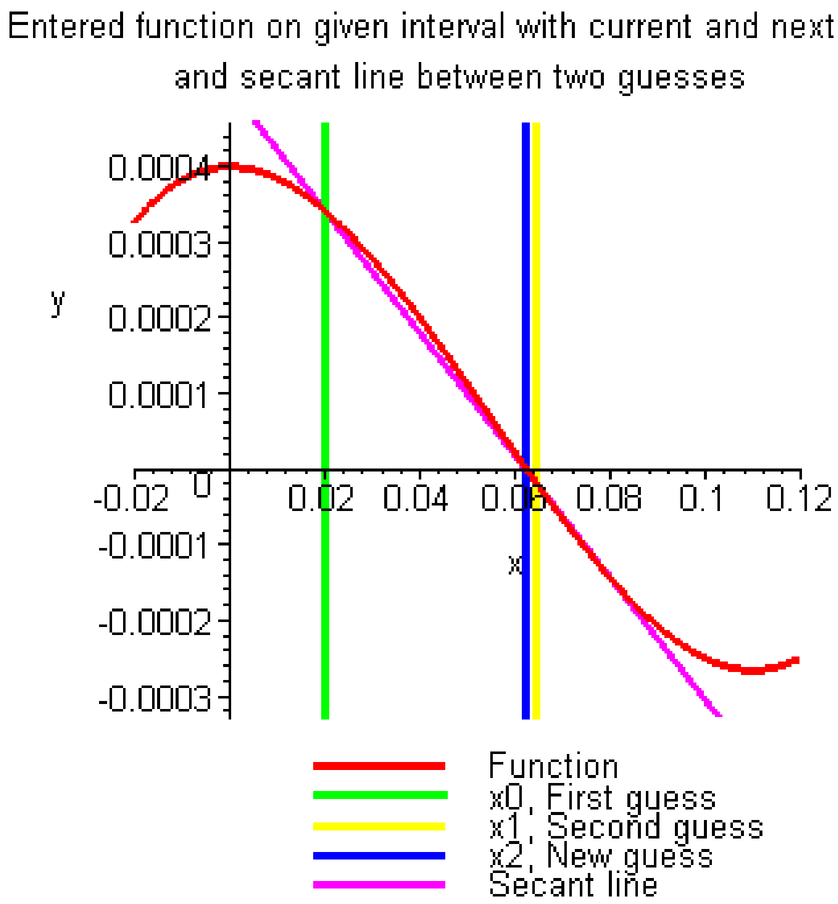


Figure 6 Graph of results of Iteration 2.

Example 1 Cont.

Iteration 3

The estimate of the root is

$$x_3 = x_2 - \frac{f(x_2)(x_2 - x_1)}{f(x_2) - f(x_1)}$$

$$= 0.06241 - \frac{\left(0.06241^3 - 0.165(0.06241)^2 + 3.993\times10^{-4}\right)(0.06241 - 0.06461)}{\left(0.06241^3 - 0.165(0.06241)^2 + 3.993\times10^{-4}\right) - \left(0.05^3 - 0.165(0.06461)^2 + 3.993\times10^{-4}\right)}$$

$$= 0.06238$$

Example 1 Cont.

The absolute relative approximate error $|\varepsilon_a|$ at the end of Iteration 3 is

$$|\varepsilon_a| = \left|\frac{x_3 - x_2}{x_3}\right| \times 100$$

$$= \left|\frac{0.06238 - 0.06241}{0.06238}\right| \times 100$$

$$= 0.0595\%$$

The number of significant digits at least correct is 5, as you need an absolute relative approximate error of 0.5% or less.

Iteration 3

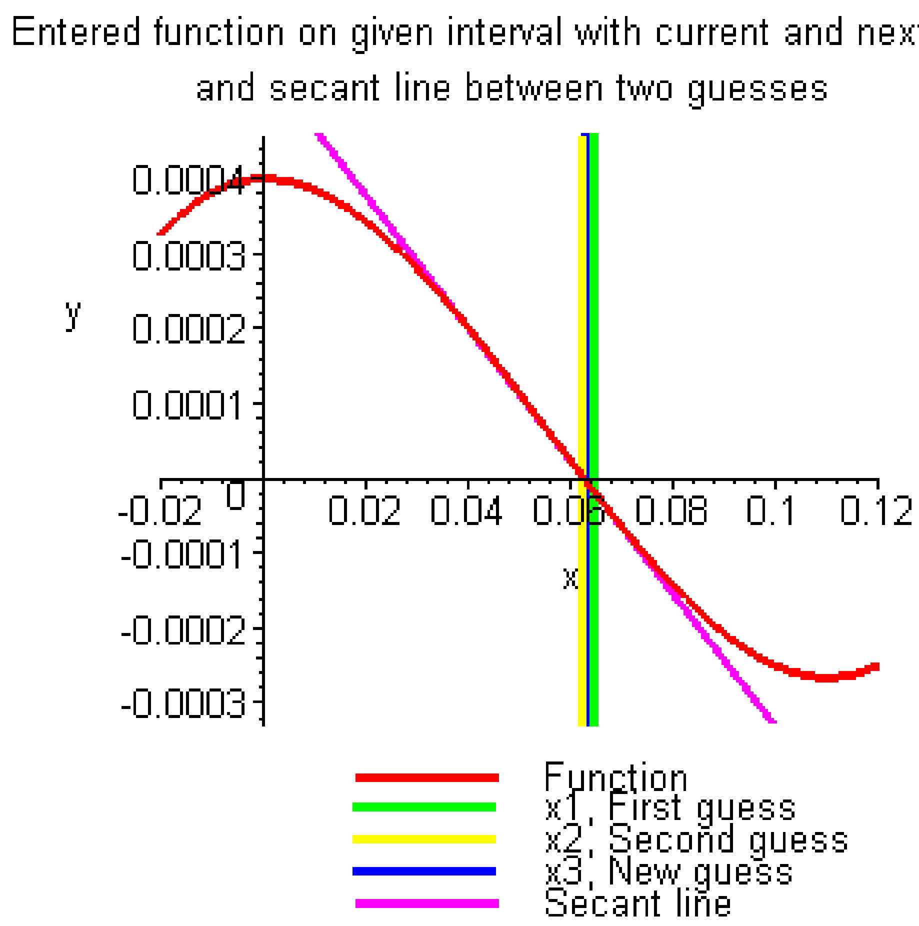


Figure 7 Graph of results of Iteration 3.

Example 2 **Applied one point and two point secant method to equation**

$$7x^5 - 13x^4 - 21x^3 - 12x^2 + 58x + 3 = 0$$

find root on [1, 2]， make sure the absolute relative error is less than $\varepsilon = 10^{-5}$ **.**

1. One point secant method

Solution: denotes $f(x) = 7x^5 - 13x^4 - 21x^3 - 12x^2 + 58x + 3$

assume $f(1) > 0$, $f(2) < 0$， so there exists a root on [1, 2].

Starting with an initial guess of $x_0 = 1, x_1 = 2$, since $x_{n+1} = x_n - \dfrac{x_n - x_0}{f(x_n) - f(x_0)} f(x_n)$

we have

$$x_3 = 2 - \frac{2-1}{f(2) - f(1)} f(2) = 1.21359$$

Example 2 **Applied one point and two point secant method to equation**

$$7x^5 - 13x^4 - 21x^3 - 12x^2 + 58x + 3 = 0$$

find root on [1, 2] ， make sure the absolute relative error is less than $\varepsilon = 10^{-5}$ **.**

1. One point secant method

Given $\varepsilon = 10^{-5}$, when $|x_n - x_{n-1}| < \varepsilon$ is satisfied, we find the approximate root

n	x_n	n	x_n
0	1	5	1.30813
1	2	6	1.30878
2	1.21359	7	1.30890
3	1.28755	8	1.30892
4	1.30474	9	1.30892

Example 2 **Applied one point and two point secant method to equation**

$$7x^5 - 13x^4 - 21x^3 - 12x^2 + 58x + 3 = 0$$

find root on [1, 2]， make sure the absolute relative error is less than $\varepsilon = 10^{-5}$.

2. Two point secant method

Starting with an initial guess $x_0 = 1, x_1 = 2$, since $\boldsymbol{x_{n+1} = x_n - \frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})} f(x_n)}$, given $\varepsilon = 10^{-5}$, when $|x_n - x_{n-1}| < \varepsilon$ is satisfied, we find the approximate root

n	x_n	n	x_n
0	1	4	1.31067
1	2	5	1.3089
2	1.21359	6	1.30892
3	1.28755	7	1.30892

Advantages

- Converges fast, if it converges
- Requires two guesses that do not need to bracket the root

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Iteration Method

Consider Newton's method as applied to *f(x)*, since $x_n = x_{n-1} - f(x_{n-1})/f'(x_{n-1})$, as $n \to \infty$, we know that $x_n \to p$, we can write x_n more abstractly as

$$x_n = \varphi(x_{n-1}), \mathrm{n} = 1, 2, \cdots \qquad (1)$$

where $\varphi \in C[a, b]$.

If the sequence $\{x_n\}$ of $x_n = \varphi(x_{n-1})$ converges, and suppose $\lim\limits_{n\to\infty} x_n = p$, then

$$p = \lim_{n\to\infty} x_n = \lim_{n\to\infty} \varphi(x_{n-1}) = \varphi(\lim_{n\to\infty} x_{n-1}) = \varphi(p)$$

Where p is the root of $f(x) = 0$

Iteration Method

Note that $f(p) = 0 \leftrightarrow p = \varphi(p)$

Because $p = \varphi(p)$ shows that $\varphi(p)$ "stays" at p, this kind of point is called *a fixed point* of the function φ, and an iteration of the form (1) is called *a fixed-point iteration* for φ.

Iteration Method

For a given function φ, a number of questions can be raised:

1. Under what conditions does a fixed point exist?
2. Under what conditions does the iteration (1) converge?
3. If the iteration converges, how fast does it converge?

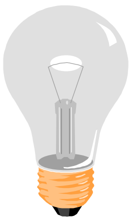
Fixed-point Iteration

equivalence transformation

$$f(x)=0 \longleftrightarrow x=\varphi(x)$$

Root of $f(x)$

Fixed point of $\varphi(x)$



Starting with an initial guess x_0, calculating $x_1=\varphi(x_0), x_2=\varphi(x_1), \cdots, x_n=\varphi(x_{n-1})$, **if** $\{x_n\}_{n=0}^{\infty}$ **converges, there exists** p *such that* $\lim\limits_{n\to\infty} x_n=p$, **and** φ **is continuous, since** $\lim\limits_{n\to 0} x_{n+1}=\lim\limits_{n\to 0}\varphi(x_n)$, **we have** $p=\varphi(p)$, **that is,** p **is the fixed point of** φ, **also is the root of** f.

Theorem 1 (Fixed-point Existence and Iteration Convergence Theory)

Let $\varphi \in C([a,b])$ *with* $a \leq \varphi(x) \leq b$ *for all* $x \in [a,b]$*;then:*

1.φ *has at least one fixed point* $p \in [a,b]$;

2. *If there exists a value* $L < 1$ *such that*

$$\|\varphi(x) - \varphi(y)| \leq L|x-y|, \forall x, y \in [a,b] \tag{2}$$

then:

(a) p *is unique;*

(b) The iteration $x_n = \varphi(x_{n-1})$ *converges to p for any initial guess* $x_0 \in [a,b]$*;*

(c) We have the error estimate

$$|p - x_n| \leq \frac{L^n}{1-L}|x_1 - x_0|. \tag{3}$$

3. *If* φ*is continuously differentiate on [a, b] with*

$$\max_{x\in[a,b]}|\varphi'(x)| = L < 1 \tag{4}$$

Theorem 1 (Fixed-point Existence and Iteration Convergence Theory)

then

(a) p is unique;

(b) The iteration $x_n = \varphi(x_{n-1})$ *converges to p far any initial guess* $x_0 \in [a, b]$*;*

(c) We have the error estimate

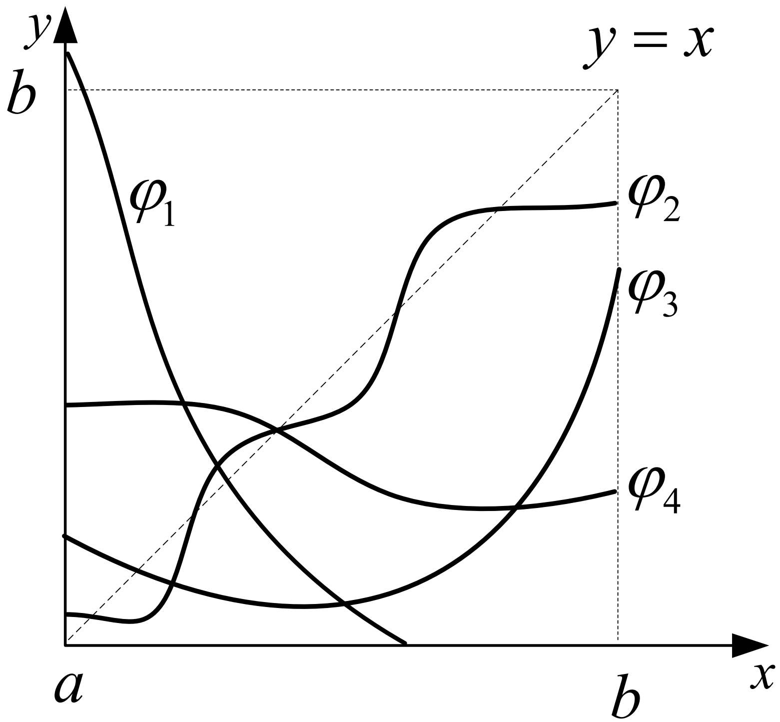
$$|x_n - p| \leq \frac{L^n}{1-L} |x_1 - x_0|$$

(d) The limit

$$\lim_{n\to\infty} \frac{x_{n+1} - p}{x_n - p} = \varphi'(p)$$

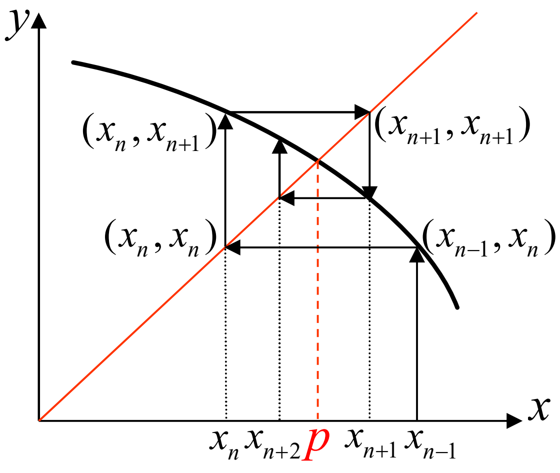
holds

Geometric Meaning of Fixed-point Iteration

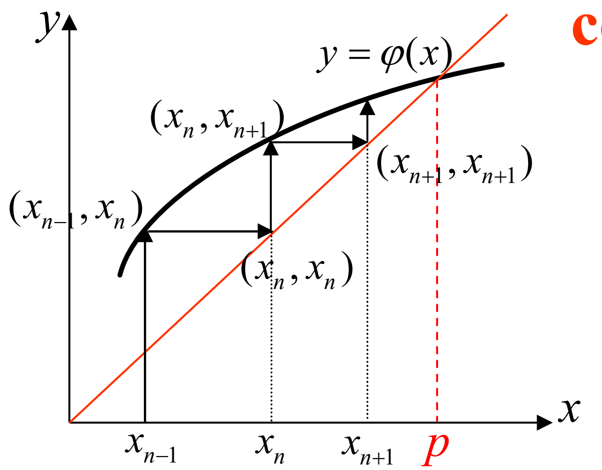


we already know as the root of f (the point where the graph of $y = f(x)$ crosses the x-axis), to be a point where the graph of the new function $y = \varphi(x)$ crosses the line $y = x$

Geometric Meaning of Fixed-point Iteration



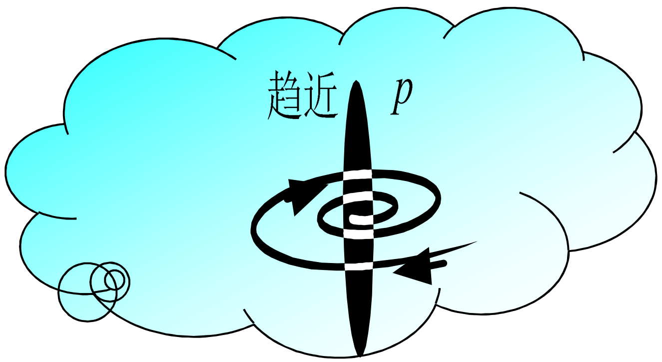
converge

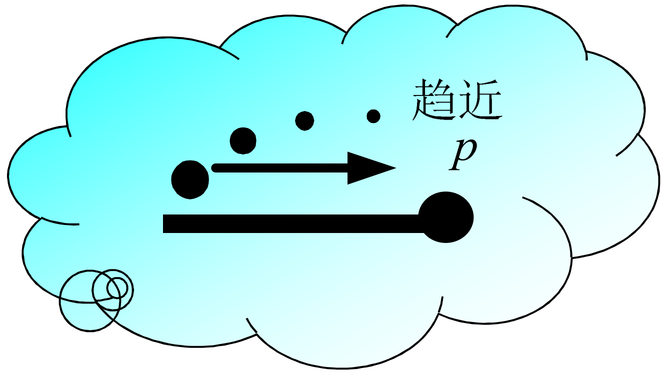


converge

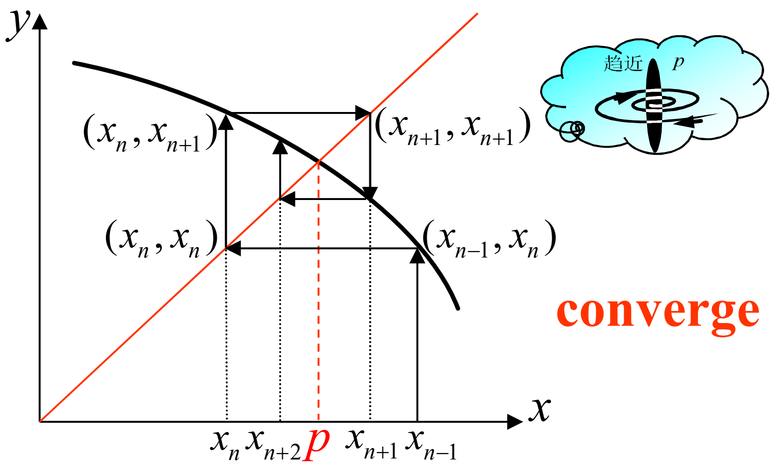
1) $-1 < \varphi'(x) < 0$

2) $0 < \varphi'(x) < 1$

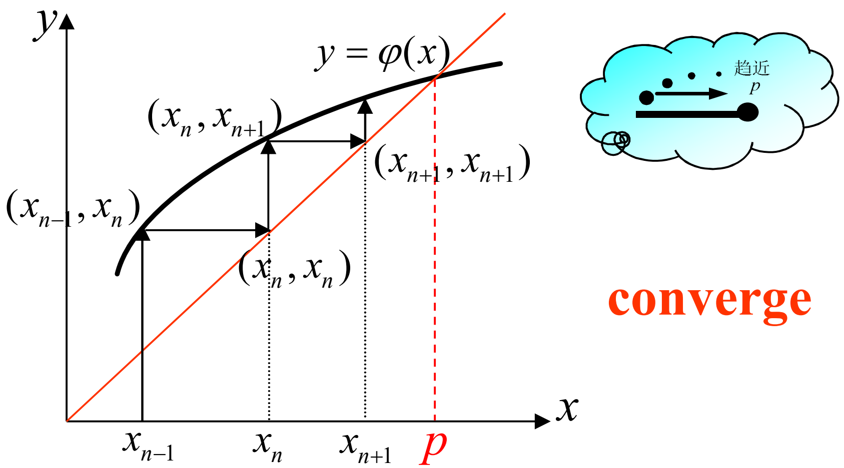




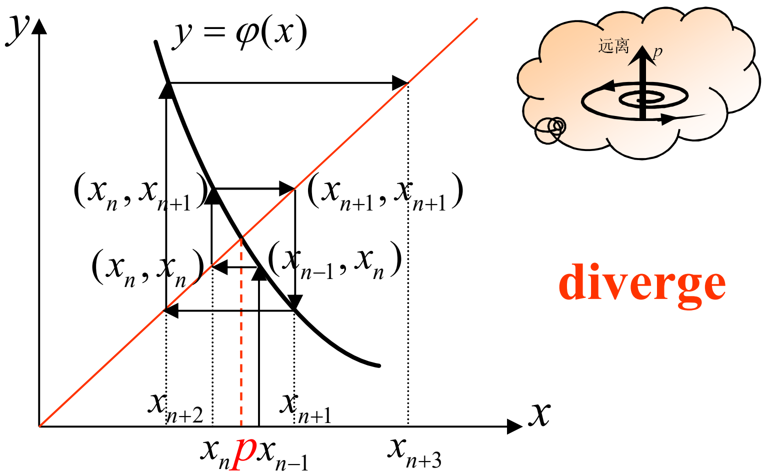
Geometric Meaning of Fixed-point Iteration



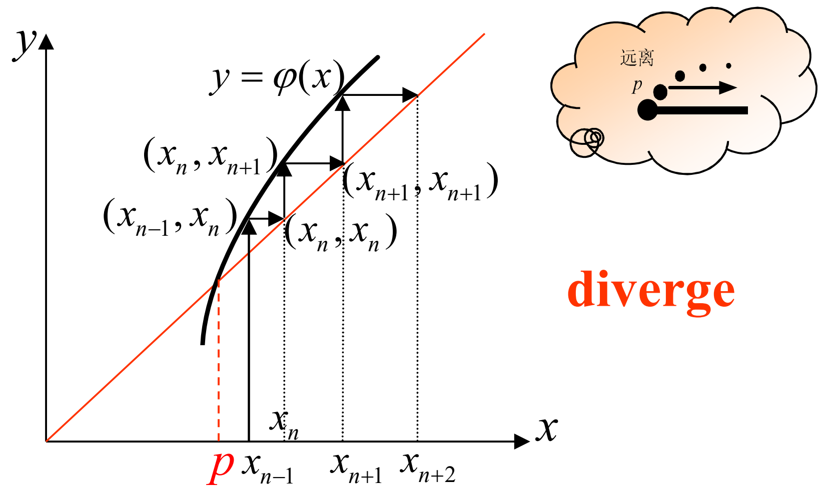
1) $-1 < \varphi'(x) < 0$



2) $0 < \varphi'(x) < 1$



3) $\varphi'(x) < -1$

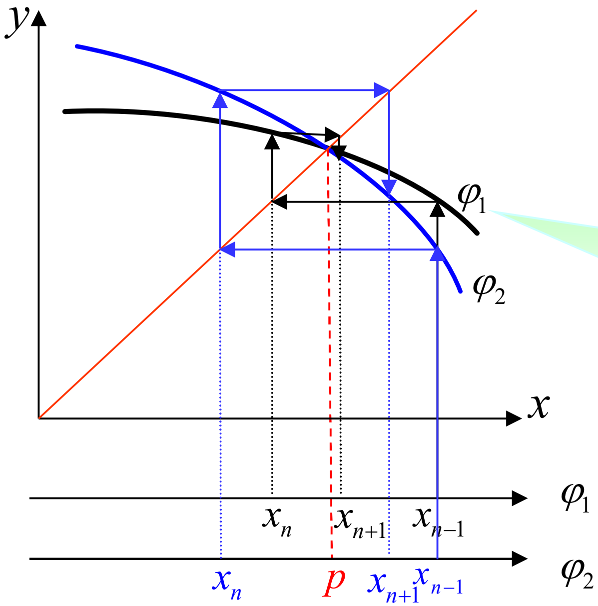


4) $\varphi'(x) > 1$

Comparison of iterative convergence speeds

Be different?

- **Starting with the same initial guess of x_{n-1} for two different iteration formats, thinking about the speed of convergence to the same fixed point p**



φ_1 converges faster

Example 3

•**Example 3 use various of iterative formula to solve equation**

$$7x^5 - 13x^4 - 21x^3 - 12x^2 + 58x + 3 = 0, \quad x \in [1,2]$$

Solution: 5 iterative formulas are given as

(1) $x = \varphi_1(x) = 7x^5 - 13x^4 - 21x^3 - 12x^2 + 59x + 3$

(2) $x = \varphi_2(x) = \left(\dfrac{13x^4 + 21x^3 + 12x^2 - 58x - 3}{7}\right)^{\frac{1}{5}}$

(3) $x = \varphi_3(x) = \dfrac{13 + \dfrac{21}{x} + \dfrac{12}{x^2} - \dfrac{58}{x^3} - \dfrac{3}{x^4}}{7}$

There is no unique fixed point under these 3 iterative equations

Example 3 Cont.

- **Example 3 use various of iterative formula to solve equation**

$$7x^5 - 13x^4 - 21x^3 - 12x^2 + 58x + 3 = 0\,,\quad x \in [1,2]$$

$$(4)\quad x = \varphi_4(x) = \left(\frac{12x^2 - 58x - 3}{7x^2 - 13x - 21}\right)^{\frac{1}{3}}$$

$$(5)\quad x = \varphi_5(x) = \left(\frac{-58x - 3}{7x^3 - 13x^2 - 21x - 12}\right)^{\frac{1}{2}}$$

There is an unique fixed point under these 2 iterative equations

Example 3 Cont.

- **Example 3 use various of iterative formula to solve equation**

$$7x^5 - 13x^4 - 21x^3 - 12x^2 + 58x + 3 = 0\,, \quad x \in [1,2]$$

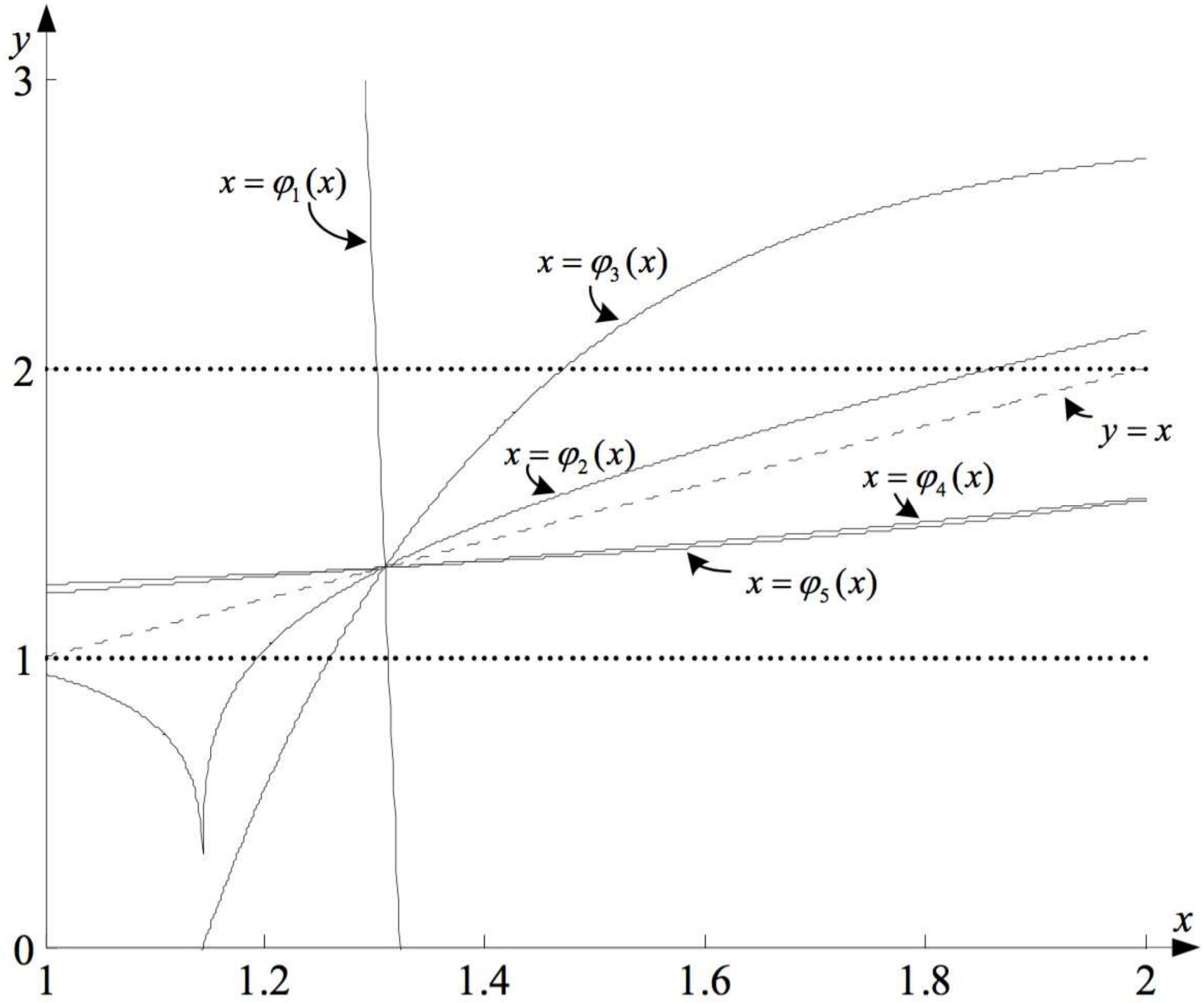
Set initial guess x_0=1.5, stop rule is

$$|x_n - x_{n-1}| < \varepsilon = 10^{-5}$$

the solution is 1.30893

n	(1)	(2)	(3)	(4)	(5)
0	1.5	1.5	1.5	1.5	1.5
1	-19.0313	1.60125	2.07937	1.36538	1.35354
2	-1.9e+007	1.7239	2.75186	1.32528	1.31852
3	-1.8e+037	1.85819	2.76861	1.31364	1.31095
4	-1.2e+187	1.9935	2.76664	1.31028	1.30935
5	…	2.12139	2.76687	1.30932	1.30901
6		2.23655	2.76685	1.30904	1.30894
7		2.33657	2.76685	1.30896	1.30893
8		2.42114		1.30893	1.30893
9		2.49122		1.30893	
10		…			

Example 3 Cont.



Homework

1. Program and verify the fixed-point iteration algorithm of Example 3.

2. Apply Bisection method, Newton-Raphson method, Secant method, Fixed-point iteration method to the equation of Example 3, list all results of each iteration in a table (Note that the same initial guess is best to be set), analyze these algorithms, then write a report and submit on CG.