Error Accumulation and Propagation

GongYi龚怡

2167570874@qq.com

Link

Slides：

https://pan.baidu.com/s/1uqWdtLXJhJ9TvKN1CsK_9w

Access code ：math

Software:

https://pan.baidu.com/s/1iXhXryPJG-YNYF-RedTZ1Q

Access code : 57fs

Butterfly Effect

- The butterfly effect is the idea that small, seemingly trivial events may ultimately result in something with much larger consequences.

- In other words, they have non-linear impacts on very complex systems.

Butterfly Effect

- For instance, when a butterfly flaps its wings in India, that tiny change in air pressure could eventually cause a tornado in Iowa.

Butterfly Effect

Butterfly effect in numerical method

Computer arithmetic is generally inexact, and while the errors that are made are very small, they can accumulate under some circumstances and make the calculation solution is far from the theory solution.

This phenomenon is called ill-conditioning of numerical method.

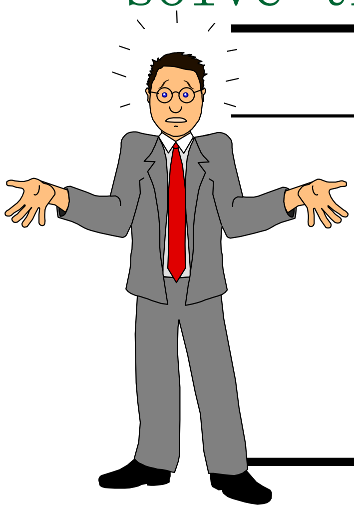
Error Accumulation and Propagation

- **Example: Solve definite integral equations**

$$I_n = \int_0^1 \frac{x^n}{x+100}\mathrm{d}x \qquad n = 0,1,2,\cdots$$

Note that this formula holds precisely

Based on recursive equation $I_n = \frac{1}{n} - 100 I_{n-1}$ we have the initial guess $I_0^* = 0.995033\times 10^{-2}$, and the absolute error limit is $|e(I_0^*)| = |I_0^* - I_0| \le 5\times 10^{-9}$, we can solve this recursion to get:



n	I_n^*	n		I_n^*
0	0.995033×10^{-2}	4	?!	-0.833333×10^{-1}
1	0.496700×10^{-2}	5	?!	8.53333
2	0.330000×10^{-2}	6	?!	-853.167
3	0.333333×10^{-2}			

$$I_6^* = 0.141618\times 10^{-2}$$

Error Accumulation and Propagation

❖ The reason for the error:

$$\left|I_1 - I_1^*\right| = \left|(1-100I_0) - (1-100I_0^*)\right| = 100\left|I_0 - I_0^*\right|$$

$$\left|I_2 - I_2^*\right| = \left|(\frac{1}{2}-100I_1) - (\frac{1}{2}-100I_1^*)\right| = 100\left|I_1 - I_1^*\right| = 100^2\left|I_0 - I_0^*\right|$$

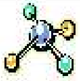
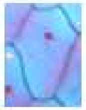
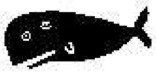
……

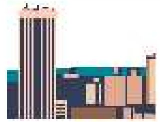
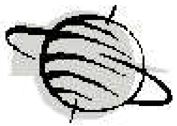
$$\left|I_n - I_n^*\right| = 100\left|I_{n-1} - I_{n-1}^*\right| = 100^n\left|I_0 - I_0^*\right|$$

The error increases exponentially by 100 ！ ！

Error Accumulation and Propagation

- Even if the error of I_0^* is very small， with the increase of n ， the error will be too large as so as astronomical figures (in this example, meter is used as the unit)

n	error		
n=0	10E-9		Molecular diameter
n=1	10E-7		Virus diameter
n=2	10E-5		Cell diameter
n=3	10E-3		Thread diameter
n=4	0.1		Leaf length
n=5	10		Whale length
n=6	1000		Park diameter

n	error		
n=7	10E5		Super city diameter
n=8	10E7		Terrestrial planet diameter
n=9	10E9		Earth-Moon system diameter
n=10	10E11		Sun-Earth distance

Error Accumulation and Propagation

- Suppose there is a initial guess $I_6^* = 0.141618 \times 10^{-2}$, and the recursive equation is $I_n^* = \frac{1}{100}(\frac{1}{n+1} - I_{n+1}^*)$, $n = 5,4,3,2,1,0$, we can get:

n	I_n^*	n	I_n^*
6	0.141618×10^{-2}	2	0.330853×10^{-2}
5	0.165250×10^{-2}	1	0.496691×10^{-2}
4	0.198347×10^{-2}	0	0.995033×10^{-2}
3	0.248017×10^{-2}		

Error Accumulation and Propagation

❖ **The reason for error decrease:**

$$\left|I_5 - I_5^*\right| = \frac{1}{100}\left|I_6 - I_6^*\right|$$

$$\left|I_4 - I_4^*\right| = \left(\frac{1}{100}\right)^2\left|I_6 - I_6^*\right|$$

……

$$\left|I_0 - I_0^*\right| = \left(\frac{1}{100}\right)^6\left|I_6 - I_6^*\right|$$

In this way, the error is much smaller！ ！

From examples above, errors are inevitable in future learning. To algorithms, the stability is great important.

Error Estimation of Four Operations

- The absolute error limit of the addition or subtraction does not exceed the sum of the absolute error limits

$$\varepsilon(x^* \pm y^*) = \varepsilon(x^*) + \varepsilon(y^*)$$

$$\begin{aligned} \left|e(x^* + y^*)\right| &= \left|(x + y) - (x^* + y^*)\right| = \left|(x - x^*) + (y - y^*)\right| \\ &\le \left|x - x^*\right| + \left|y - y^*\right| = \left|e(x^*)\right| + \left|e(y^*)\right| \\ &\le \varepsilon(x^*) + \varepsilon(y^*) \end{aligned}$$

Error Estimation of Four Operations

- The absolute error limit of the addition or subtraction does not exceed the sum of the absolute error limits

$$\varepsilon(x^* \pm y^*) = \varepsilon(x^*) + \varepsilon(y^*)$$

- When two numbers have the same sign, the relative error limit of the sum does not exceed the relative error limit of the most incorrect of the term.
- When the numbers on both sides of the minus sign differ greatly, the relative error of the large number plays a decisive role
- Subtraction of two similar numbers should be avoided as much as possible
- Example: Let x = 18.496, y = 18.493, take four significant digits, analyze the absolute and relative errors after subtracting the approximate values of x and y

Error Estimation of Four Operations

- **The relative error of the multiplication is the sum of the relative errors of the multipliers.**
- **The absolute error limit of the multiplication**

$$\varepsilon(x^* y^*) = \left|x^*\right| \varepsilon(y^*) + \left|y^*\right| \varepsilon(x^*)$$

$$\begin{aligned}
\left|e(x^* y^*)\right| &= \left|xy - x^* y^*\right| = \left|(x^* + x - x^*)(y^* + y - y^*) - x^* y^*\right| \\
&= \left|(x^* + e(x^*))(y^* + e(y^*)) - x^* y^*\right| \\
&= \left|x^* y^* + x^* e(y^*) + y^* e(x^*) + e(x^*) e(y^*) - x^* y^*\right| \\
&\le \left|x^* e(y^*)\right| + \left|y^* e(x^*)\right| + \left|e(x^*) e(y^*)\right| \\
&\le \left|x^*\right| \varepsilon(y^*) + \left|y^*\right| \varepsilon(x^*) + \underline{\varepsilon(x^*) \varepsilon(y^*)} \text{ Too small to be ignored}
\end{aligned}$$

Error Estimation of Four Operations

- The relative error of the quotient is the difference between the relative error of the dividend and the divisor

- The absolute error limit of the quotient

$$\varepsilon\left(\frac{x^*}{y^*}\right) = \frac{\left|x^*\right|\varepsilon(y^*) + \left|y^*\right|\varepsilon(x^*)}{\left|y^*\right|^2}, \quad y \neq 0, \ y^* \neq 0$$

Error Estimation of Four Operations

- In general, the relative error limit of any number of consecutive multiplications and divisions does not exceed the sum of the relative error limits of the terms participating in the operation.

- In actual calculation, one or two more significant digits can be taken to ensure the accuracy of the calculation

Error Estimation of Four Operations

- Example: we have three sides' lengths of the cuboid 10 ± 0.1, 20 ± 0.1, 50 ± 0.2, try to estimate the absolute error limit of the surface area and volume of the cuboid
- Solution: suppose three sides' lengths are x , y , z , and the approximation are $x^* = 10$, $y^* = 20$, $z^* = 50$, the absolute error limit is $\varepsilon(x^*) = 0.1$, $\varepsilon(y^*) = 0.1$, $\varepsilon(z^*) = 0.2$.

  The absolute error limit for the surface area of the cuboid can be estimated to be

$$
\begin{aligned}
\varepsilon(2x^*y^* + 2x^*z^* + 2y^*z^*) &= 2[\varepsilon(x^*y^*) + \varepsilon(x^*z^*) + \varepsilon(y^*z^*)] \\
&= 2[|x^*|\varepsilon(y^*) + |y^*|\varepsilon(x^*) + |x^*|\varepsilon(z^*) + |z^*|\varepsilon(x^*) + |y^*|\varepsilon(z^*) + |z^*|\varepsilon(y^*)] \\
&= 2(10 \times 0.1 + 20 \times 0.1 + 10 \times 0.2 + 50 \times 0.1 + 20 \times 0.2 + 50 \times 0.1) \\
&= 38
\end{aligned}
$$

Error Estimation of Four Operations

- Similarly, the absolute error limit for the volume of a cuboid can be estimated

$$
\begin{aligned}
\varepsilon(x^* y^* z^*) &= \varepsilon[(x^* y^*) z^*] = \left|x^* y^*\right| \varepsilon(z^*) + \left|z^*\right| \varepsilon(x^* y^*) \\
&= \left|x^* y^*\right| \varepsilon(z^*) + \left|z^*\right| [\left|x^*\right| \varepsilon(y^*) + \left|y^*\right| \varepsilon(x^*)] \\
&= \left|x^* y^*\right| \varepsilon(z^*) + \left|x^* z^*\right| \varepsilon(y^*) + \left|y^* z^*\right| \varepsilon(x^*) \\
&= 200 \times 0.2 + 500 \times 0.1 + 1000 \times 0.1 \\
&= 190
\end{aligned}
$$

Principles for designing calculation methods

- Evaluation criteria for the calculation method

Stability

Accuracy

Volume of computer calculations

Storage capability

Logical complexity

Good calculation method

When designing calculation methods, it is often difficult to take into account the above requirements, so it is necessary to make trade-offs according to the actual situation.

Principles for designing calculation methods

(1) Avoid subtracting two numbers that are close

The following transformation formula is often used:

$$\sqrt{x+\varepsilon}-\sqrt{x}=\frac{\varepsilon}{\sqrt{x+\varepsilon}+\sqrt{x}},\quad \varepsilon\to 0$$

$$\ln(x+\varepsilon)-\ln x=\ln(1+\frac{\varepsilon}{x}),\quad \varepsilon\to 0\text{且}|x|>>0$$

$$1-\cos x=2\sin^2\frac{x}{2},\qquad x\to 0$$

$$e^x-1=(1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots)-1=x(1+\frac{x}{2}+\frac{x^2}{6}+\cdots),\quad x\to 0$$

$$\begin{array}{r} 123.456 \\ -123.455 \\ \hline 0.001 \end{array}$$

6 significant digits

Only 1 significant digit !

1 significant digit

4 significant digits

Example: Calculate $10^7\times(1-\cos 2^\circ)$.

Solution: $10^7\times(1-\cos 2^\circ)=10^7\times(1-0.9994)=6\times 10^3$

let $\sin 1^\circ=0.0175$, we get

$10^7\times(1-\cos 2^\circ)=10^7\times 2\times(0.0175)^2=6.125\times 10^3$

Principles for designing calculation methods

(2) Prevent large number from "eating" small number

$$\begin{array}{rl} & 1.000000000 \quad \times 10^{10} \\ + & 0.000000000\boxed{1} \quad \times 10^{10} \\ \hline & 1.000000000 \quad \times 10^{10} \end{array} \Rightarrow 10^{10}+1$$

Example: Solve on a computer with 10 decimal significant digits

$x^2-(10^{10}+1)x+10^{10}=0$ **(Exact solution** $x_1=10^9, x_2=1$**)**

✍ Algorithm 1： Solving with root-finding formulas

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

Large number "eats" small number，b=10^{10}

$$\Rightarrow x_1=\frac{-b+\sqrt{b^2-4ac}}{2a}=10^9, x_2=\frac{-b-\sqrt{b^2-4ac}}{2a}=0$$

Principles for designing calculation methods

(2) Prevent large numbers from "eating" small numbers

$$
\begin{array}{r}
1.000000000 \quad \times 10^{10} \\
+\ 0.000000000\boxed{1} \quad \times 10^{10} \\
\hline
1.000000000 \quad \times 10^{10}
\end{array}
\Rightarrow 10^{10}+1
$$

Example: Solve on a computer with 10 decimal significant digits

$x^2-(10^{10}+1)x+10^{10}=0$ **(Exact solution** $x_1=10^9, x_2=1$**)**

✍ Algorithm 2: Transform the rooting formula

Get exact solution !

$$
\begin{cases}
x_1 = \dfrac{-2c}{b-\operatorname{sgn}(-b)\sqrt{b^2-4ac}} \\
x_2 = \dfrac{c}{ax_1}
\end{cases}
,\operatorname{sgn}(-b)=\begin{cases} 1, & -b\ge 0 \\ -1, & -b<0 \end{cases}
$$

Principles for designing calculation methods

(2) <u>Prevent large numbers from "eating" small numbers</u>

Example: Solve on a computer with 10 decimal significant digits

$$10^{10}+1+2+3+4$$

Solution: calculate directly

$$\begin{aligned}10^{10}+1+2+3+4 &= 10^{10}+0.000000000\times10^{10}+2+3+4\\ &= 10^{10}+0.000000000\times10^{10}+3+4\\ &= 10^{10}+0.000000000\times10^{10}+4\\ &= 10^{10}+0.000000000\times10^{10}\\ &= 10^{10}\end{aligned}$$

Note: When summing, add from small number to large number to reduce the error.

$$\begin{aligned}1+2+3+4+10^{10} &= 10+10^{10}\\ &= 0.000000001\times10^{10}+10^{10}\\ &= 1.000000001\times10^{10}\end{aligned}$$

Principles for designing calculation methods

(3) <u>Avoid divisor with small absolute value</u>

When a number with a small absolute value is used as the denominator, it is easy to produce floating-point overflow, and it may also cause large numbers to eat small number, resulting in large errors

Principles for designing calculation methods

(4) <u>Simplify the calculation steps and reduce the number of calculations</u>

- Good calculation methods should have acceptable time complexity and storage capability.
- Simplifying the calculation steps can not only effectively reduce the operation time, but also reduce the accumulation of round-off errors, which is the principle that must be followed in the design of calculation methods. In general, the computing speed of the following operations processed on a computer is

$$(+\,,-)>(\times\,,\div)>(\mathbf{exp})$$

- Therefore, when designing calculation methods, Priority will be given to addition and subtraction operations.

Principles for designing calculation methods

- **Example:** Try to design a calculation method to reduce the number of operations to solve the following polynomial as much as possible

$$P_n(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$$

Solution: If $a_0, a_1 x, a_2 x^2, \cdots, a_n x^n$ are calculated separately, and added together, then there are

$$0 + 1 + 2 + \cdots + n = \frac{1}{2} n(n+1)$$

multiplications and n additions are required

Principles for designing calculation methods

- **Example:** Try to design a calculation method to reduce the number of operations to solve the following polynomial as much as possible

$$P_n(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$$

Solution: If it is calculated as follows

$$\begin{cases} S_n = a_n \\ S_k = xS_{k+1} + a_k, \quad k = n-1, n-2, \cdots, 1, 0 \\ P_n(x) = S_0 \end{cases}$$

there are only n additions and n multiplications

Principles for designing calculation methods

(5) <u>Control the error propagation to ensure the stability in calculation method</u>

- Generally, during the operation process, an algorithm with the gradual attenuation of error are regarded as a stable algorithm, and an algorithm that accumulates more and more errors are regarded as an unstable algorithm.

- When designing the calculation method, it is necessary to study the accumulation and propagation of error, and reduce the error as much as possible, adopt the stability algorithm, and avoid the use of “ill-conditioned" and unstable calculation methods

Principles for designing calculation methods

(5) <u>Control the error propagation to ensure the stability in calculation method</u>

Especially, when computing by using the recurrence formula, we must first analyze the error accumulation in the recurrence formula, and the recurrence formula with increasing error during the recurrence process cannot be selected, but the one with the gradual decrease of error should be used.

Brief summary

Calculation methods: the numerical approximation method for solving mathematical problems on a computer

Features:

① Rigorousness

② Practical

③ Approximation

④ Structural

Brief summary

Sources of error: 1. Observation error 2. Model error 3. Chopping error 4. Rounding error

The basic concept of error:

- Absolute error: the difference between the exact value and the approximate value. $e(x^*) = x - x^*$

- Absolute error limit: the upper limit of the absolute value of the absolute error. $|e(x^*)| = |x - x^*| \leq \varepsilon(x^*)$

- relative error: Ratio of absolute error to exact value (usually replaced by the ratio of absolute error to approximate value).

$$e_r(x^*) = \frac{e(x^*)}{x} = \frac{x - x^*}{x}$$

Sources of error: 1. Observation error 2. Model error 3. Chopping error 4. Rounding error

The basic concept of error:

- Relative error limit: The upper limit of the absolute value of the relative error. $$\left|e_r(x^*)\right| = \frac{\left|x - x^*\right|}{|x|} \le \frac{\varepsilon(x^*)}{|x|} = \varepsilon_r(x^*)$$

- Significant figures and errors:

x^* has n significant digits

Absolute error limit $|e(x^*)| \le \varepsilon(x^*) = \frac{1}{2} \times r^{m-n}$

Relative error limit $|e_r(x^*)| \le \varepsilon_r(x^*) = \frac{1}{2a_1} \times r^{-(n-1)}$

Relative error limit $|e_r(x^*)| \le \frac{1}{2(a_1+1)} \times r^{-(n-1)}$ $\Rightarrow$ x^* has n significant digits

Brief summary

Error Propagation

Error Propagation of Four Operations

Addition and subtraction: $\varepsilon(x^* \pm y^*) = \varepsilon(x^*) + \varepsilon(y^*)$

Multiplication: $\varepsilon(x^* y^*) = \left|x^*\right| \varepsilon(y^*) + \left|y^*\right| \varepsilon(x^*)$

Division: $$\varepsilon\left(\frac{x^*}{y^*}\right) = \frac{\left|x^*\right| \varepsilon(y^*) + \left|y^*\right| \varepsilon(x^*)}{\left|y^*\right|^2}, \quad y \neq 0, \ y^* \neq 0$$

Brief summary

Principles for designing calculation methods

① Avoid subtracting two numbers that are close

② Prevent large number from “eating” small number

③ Avoid divisor with small absolute value

④ Simplify the calculation steps and reduce the number of calculations

⑤ Control the error propagation to ensure the stability in calculation method

Homework

Solving $ax^2 + bx + c = 0$ by using root-finding formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

directly may lost the significant digits. Write a solution that avoids the loss of significant digits.