AER1418: Assignment 4

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Part 1. Advection-diffusion equation

(a) We wish to show that $\exists \gamma < \infty$ such that

$$a_h(w,v) \equiv a(w,v) + (\tau \mathcal{L} w, \mathcal{L} v)_{L^2(\Omega)} \leq \gamma \|w\|_{\mathcal{V}_h} \|v\|_{\mathcal{V}_h}, \quad \forall w, v \in \mathcal{V}_h.$$

Since we already know that

$$a(w,v) \leq \left(\|\kappa\|_{L^\infty(\Omega)} + \|b\|_{L^\infty(\Omega)} + C_{\mathrm{tr}}^2 \|b\|_{L^\infty(\Gamma_N)} \right) \|w\|_{\mathcal{V}_h} \|v\|_{\mathcal{V}_h},$$

we seek some $\gamma' < \infty$ such that

$$(\tau \mathcal{L} w, \mathcal{L} v)_{L^2(\Omega)} \leq \gamma' \|w\|_{\mathcal{V}_h} \|v\|_{\mathcal{V}_h}, \quad \forall w, v \in \mathcal{V}_h.$$

We first apply the Cauchy-Schwarz inequality to the leas-squares term, obtaining

$$(\tau \mathcal{L} w, \mathcal{L} v)_{L^2(\Omega)} \leq \|\tau \mathcal{L} w\|_{L^2(\Omega)} \|\mathcal{L} v\|_{L^2(\Omega)}.$$

Now using the definition of $\mathcal{L}$, we can say that

$$(\tau \mathcal{L} w, \mathcal{L} v)_{L^2(\Omega)} \leq \|\tau(-\nabla \cdot (\kappa \nabla w) + \nabla \cdot (bw))\|_{L^2(\Omega)} \| -\nabla \cdot (\kappa \nabla v) + \nabla \cdot (bv)\|_{L^2(\Omega)}.$$

By applying the triangle inequality to the norms containing sums we arrive at

$$(\tau \mathcal{L} w, \mathcal{L} v)_{L^2(\Omega)} \leq (\| -\tau \nabla \cdot (\kappa \nabla w)\|_{L^2(\Omega)} + \|\tau \nabla \cdot (bw)\|_{L^2(\Omega)})(\| -\nabla \cdot (\kappa \nabla v)\|_{L^2(\Omega)} + \|\nabla \cdot (bv)\|_{L^2(\Omega)}).$$

If we assume that our advection and diffusion fields are constant we may reexpress the above as

$$(\tau \mathcal{L} w, \mathcal{L} v)_{L^2(\Omega)} \leq \tau(\kappa \|\nabla^2 w\|_{L^2(\Omega)} + b \cdot \|\nabla w\|_{L^2(\Omega)})(\kappa \|\nabla^2 v\|_{L^2(\Omega)} + b \cdot \|\nabla v\|_{L^2(\Omega)}).$$

Using the definitions of the $H^2(\Omega)$ and $H^1(\Omega)$ semi-norms we can say that

$$(\tau \mathcal{L} w, \mathcal{L} v)_{L^2(\Omega)} \leq \tau(\kappa |w|_{H^2(\Omega)} + b \cdot |w|_{H^1(\Omega)})(\kappa |v|_{H^2(\Omega)} + b \cdot |v|_{H^1(\Omega)}).$$

Since we have $\mathcal{V}_h \subset H^1(\Omega)$, $\overline{\Omega} = \bigcup_{K \in \mathcal{T}_h} \overline{K}$, and we assume the inverse esimtate estmate $|v|_{H^2(K)} \leq c_{\mathrm{inv}} h^{-1} \|v\|_{H^1(K)} \forall v \in \mathcal{V}_h$ our inequality now becomes

$$(\tau \mathcal{L} w, \mathcal{L} v)_{L^2(\Omega)} \leq \tau(\kappa c_{\mathrm{inv}} h^{-1} \|w\|_{H^1(K)} + b \cdot |w|_{H^1(\Omega)})(\kappa c_{\mathrm{inv}} h^{-1} \|v\|_{H^1(K)} + b \cdot |v|_{H^1(\Omega)}).$$

From the definition of the $H^1(\Omega)$ seminorm, $\|v\|_{H^1(\Omega)} \equiv \|v\|_{L^2(\Omega)} + |v|_{H^1(\Omega)}$, $\forall v \in H^1(\Omega)$, we note that by the non-negativity of $\|v\|_{L^2(\Omega)}$ implies that $\|v\|_{H^1(\Omega)} \geq |v|_{H^1(\Omega)}$, applying this to our inequality we arrive at

$$(\tau \mathcal{L} w, \mathcal{L} v)_{L^2(\Omega)} \leq \tau(\kappa c_{\mathrm{inv}} h^{-1} \|w\|_{H^1(K)} + b \cdot \|w\|_{H^1(\Omega)})(\kappa c_{\mathrm{inv}} h^{-1} \|v\|_{H^1(K)} + b \cdot \|v\|_{H^1(\Omega)}).$$

$$(\tau \mathcal{L} w, \mathcal{L} v)_{L^2(\Omega)} \leq \tau(\kappa c_{\mathrm{inv}} h^{-1} + b) \|w\|_{H^1(K)} (\kappa c_{\mathrm{inv}} h^{-1} + b) \|v\|_{H^1(K)}.$$

Since when we invoked the inverse estimate inequality, we can now express τ in the limit as $h \to 0$, which is $\tau = \dfrac{h^2}{12\kappa}$, this updates our inequality to become

$$(\tau \mathcal{L} w, \mathcal{L} v)_{L^2(\Omega)} \leq \frac{h^2}{12\kappa} (\kappa c_{\mathrm{inv}} h^{-1} + b)^2 \|w\|_{H^1(K)} \|v\|_{H^1(K)}.$$

Since $h \to 0$ the only term that survives is

$$(\tau \mathcal{L} w, \mathcal{L} v)_{L^2(\Omega)} \leq \frac{\kappa}{12} c_{\mathrm{inv}}^2 \|w\|_{H^1(K)} \|v\|_{H^1(K)}.$$

Therefore we say that the bilinear form of the advection-diffusion equation arising from the GLS discretization is continuous with a continuity constant

$$\gamma = \|\kappa\|_{L^\infty(\Omega)} + \|b\|_{L^\infty(\Omega)} + C_{\mathrm{tr}}^2 \|b\|_{L^\infty(\Gamma_N)} + \frac{\kappa}{12} c_{\mathrm{inv}}^2.$$

(b) Assuming a constant diffusion field, our problem is given by

$$-\kappa\left(\frac{\partial^2 u}{\partial x_1^2}+\frac{\partial^2 u}{\partial x_2^2}\right)+\frac{\partial u}{\partial x_1}=0.$$

We note that this problem is separable and express our solution as the product of monomial functions $u = X_1 X_2$ and obtain

$$-\kappa\frac{1}{X_1}\frac{\partial^2 X_1}{\partial x_1^2}+\frac{1}{X_1}\frac{\partial X_1}{\partial x_1}=\kappa\frac{1}{X_2}\frac{\partial^2 X_2}{\partial x_2^2}=m,$$

where m is some seperation constant. The ordinary differential equation in x_2,

$$\frac{\partial^2 X_2}{\partial x_2^2}=\frac{m}{\kappa}X_2,$$

is trivial to solve. The solution is given by

$$X_2(x_2)=c_1\exp\left(\sqrt{\frac{m}{\kappa}}x_2\right)+c_2\exp\left(-\sqrt{\frac{m}{\kappa}}x_2\right).$$

Our boundary conditions state that

$$\left.\frac{dX_2}{dx_2}\right|_{x_2=0}=0=c_1\sqrt{\frac{m}{\kappa}}-c_2\sqrt{\frac{m}{\kappa}}\implies c_1=c_2,$$

$$\left.\frac{dX_2}{dx_2}\right|_{x_2=1}=0=c_1\sqrt{\frac{m}{\kappa}}\exp\left(\sqrt{\frac{m}{\kappa}}\right)-c_1\sqrt{\frac{m}{\kappa}}\exp\left(-\sqrt{\frac{m}{\kappa}}\right)\implies c_1\vee m=0.$$

Part 2. Error estimation and adaptive mesh refinement

Part 3. Adaptive eigensolver

Appendix