AER1418: Assignment 3

Geoff Donoghue April 1, 2019

Part 1. Heat equation with discontinuous conductivity

We consider a one-dimensional heat equation over the unit domain $\Omega \equiv (0,1)$, in the trial space $\mathcal{V} \equiv \{v \in H^1(\Omega) | v(x=0) = 0\}$ we seek some solution $u \in \mathcal{V}$ such that

$$\int_0^1 \kappa(x) \frac{dv}{dx}\frac{du}{dx} dx = v(x=1) \quad \forall v \in \mathcal{V},$$

where we have a discontinuous heat conductivity given by

$$\kappa(x) = \begin{cases} 1 & x \in (0, 1/2) \\ 2 & x \in (1/2, 1). \end{cases}$$

(a) In order to analytically compute the weak solution to this heat conduction problem we will first split the domain in half and integrate by parts to obtain

$$-\int_0^{\frac{1}{2}} v \frac{d^2u}{dx^2} dx + 2v\frac{du}{dx}\bigg|_{x=1} - v\frac{du}{dx}\bigg|_{x=1/2} - 2\int_{\frac{1}{2}}^1 v\frac{d^2u}{dx^2} dx = v(x=1).$$

If we hope to enforce the above equation in a point-wise sense we must satisfy the following equations

$$\kappa(x)\frac{d^2u}{dx^2} = 0 \quad x \in (0,1), \qquad \frac{du}{dx}\bigg|_{x=1} = \frac{1}{2}, \qquad \frac{du}{dx}\bigg|_{x=\frac{1}{2}} = 0, \quad u(x=0) = 0,$$

where the last equation is a Dirichlet boundary condition enforced by the trial space in the weak form of the problem. Immediately we see that the solution will take the form of a piecewise linear polynomial, by applying the conditions at the left and right boundaries we obtain

$$u = \begin{cases} c_L x & x \in (0, 1/2) \\ \frac{1}{2}x + c_R & x \in (1/2, 1). \end{cases}$$

To enforce our interface condition (and also to ensure our solution is $\in \mathcal{V}$) we will say that

$$\frac{du}{dx}\bigg|_{x=\frac{1}{2}} = 0 \implies \lim_{\epsilon \to 0} u(\frac{1}{2} + \epsilon) = \lim_{\epsilon \to 0} u(\frac{1}{2} - \epsilon).$$

Applying this condition yields $c_R = \dfrac{2c_L - 1}{4}$. We can say that

$$\int_0^1 \kappa(x)\frac{dv}{dx}\frac{du}{dx}dx = 1\int_0^{\frac{1}{2}} c_L \frac{dv}{dx}dx + 2\int_{\frac{1}{2}}^1 \frac{1}{2}\frac{dv}{dx}dx = c_L v(\frac{1}{2}) + v(1) - v(\frac{1}{2}),$$

and note that $c_L = 1$ will satisfy the weak problem $\forall v \in \mathcal{V}$. The solution is therefore given by

$$u = \begin{cases} x & x \in (0, 1/2) \\ \frac{1}{2}x + \frac{1}{4} & x \in (1/2, 1). \end{cases}$$

Furthermore, since we have a coercive and continuous bilinear form and a continuous linear form, by the Lax-Milgram theorem this solution is unique. For completeness we can verify that our solution is $\in \mathcal{V}$. The only nontrivial part of this involves proving the existence and boundedness of the weak derivative of u. It can be shown that the first weak derivative exists, but the second does not so the solution is $\in H^1(\Omega)$, but $\notin H^2\Omega$.

(b) Since we have $\Omega \subset \mathbb{R}$ as a Lipschitz domain, $H_0^1(\Omega) \subset \mathcal{V} \subset H^1(\Omega)$, our bilinear form is coercive and continuous, our linear form is continuous, and $\mathcal{V}_h \equiv \{v \in V | v \in \mathbb{P}^1(K), K \in \mathcal{T}_h^{\text{even}}\} \subset \mathcal{V}$ we can use Céa's lemma

$$\|u - u_h\|_{\mathcal{V}} \leq \frac{\gamma}{\alpha} \inf_{w_h \in \mathcal{V}_h} \|u - w_h\|_{\mathcal{V}}.$$

Where γ and α are the continuity and coercivity constants for the bilinear form, respectively. We note that $\|\cdot\|_{\mathcal{V}} = \|\cdot\|_{H^1(\Omega)}$, and that since our analytical solution u can be expressed as an element of our approximation space (i.e. $u \in \mathcal{V}_h$), we can therefore say that

$$\inf_{w_h \in \mathcal{V}_h} \|u - w_h\|_{\mathcal{V}} = 0.$$

We therefore say that $\|u - u_h\|_{H^1(\Omega)} = 0$.

(c) We now update our approximation space to consist of quadratic piecewise polynomials, and note that $\mathcal{V}_h' \equiv \{v \in \mathcal{V} | v \in \mathbb{P}^2(K), K \in \mathcal{T}_h^{\text{even}}\} \supset \mathcal{V}_h$. Since any approximation in $\mathcal{V}_h$ exists in $\mathcal{V}_h'$ the infinimum is again zero; furthermore, the conditions required to invoke Céa's lemma are again satisfied, and we can again state that $\|u - u_h\|_{H^1(\Omega)} = 0$.

(d) We no longer have $u \in \mathcal{V}_h$, we note that on every element except for the one located in the middle of the domain (K_{mid}) the solution can be approximated exactly and therefore note that

$$\|u - u_h\|_{H^1(\Omega)} = \|u - u_h\|_{H^1(K_{\text{mid}})} \geq |u - u_h|_{H^1(K_{\text{mid}})} \geq \inf_{w_h \in \mathcal{V}_h} |u - w_h|_{H^1(K_{\text{mid}})}.$$

We can deduce the derivative of the best-fit solution to be $\frac{dw_h}{dx} = \frac{3}{4}$ and state that

$$\begin{aligned}
\inf_{w_h \in \mathcal{V}_h} |u - w_h|_{H^1(K_{\text{mid}})} &= \inf_{w_h \in \mathcal{V}_h} \left(\int_{K_{\text{mid}}} \left(\frac{du}{dx} - \frac{dw_h}{dx} \right)^2 dx \right)^{1/2} \\
&= \left[\int_{\frac{1-h}{2}}^{\frac{1}{2}} (1 - \frac{3}{4})^2 dx + \int_{\frac{1}{2}}^{\frac{1+h}{2}} (\frac{1}{2} - \frac{3}{4})^2 dx \right]^{1/2} \\
&= \left[\frac{x}{16} \bigg|_{\frac{1-h}{2}}^{\frac{1}{2}} + \frac{x}{16} \bigg|_{\frac{1}{2}}^{\frac{1+h}{2}} \right]^{1/2} \\
&= \frac{1}{4} h^{1/2}.
\end{aligned}$$

We can therefore state that the smallest r such that

$$\|u - u_h\|_{H^1(\Omega)} \geq C h^r = \frac{1}{4} h^{1/2}$$

is $r = 1/2$.

(e) Similarly to the previous problem, we now deduce the derivative of the best-fit solution to be $\frac{dw_h}{dx} = mx + b$, with $m = -\frac{1}{2h}$, and $b = \frac{3h+1}{4h}$. We can substitute this into our infinimum expression to obtain

$$\begin{aligned}
\inf_{w_h \in \mathcal{V}_h} |u - w_h|_{H^1(K_{\text{mid}})} &= \inf_{w_h \in \mathcal{V}_h} \left(\int_{K_{\text{mid}}} \left(\frac{du}{dx} - \frac{dw_h}{dx} \right)^2 dx \right)^{1/2} \\
&= \left[\int_{\frac{1-h}{2}}^{\frac{1}{2}} (1 + \frac{x}{2h} - \frac{3h+1}{4h})^2 dx + \int_{\frac{1}{2}}^{\frac{1+h}{2}} (\frac{1}{2} + \frac{x}{2h} - \frac{3h+1}{4h})^2 dx \right]^{1/2} \\
&= \left[\frac{h}{96} + \frac{h}{96} \right]^{1/2} \\
&= \frac{1}{\sqrt{48}} h^{1/2}
\end{aligned}$$

Where I have evaluated the integral using wolfram alpha. We see that our value for r is once again $\dfrac{1}{2}$. This goes to show that since our solution is not sufficiently regular in $H^{s+1}(\mathcal{T}_h)$. We expect $r \equiv \min\{s, p\}$, and can deduce that $s = \frac{1}{2}$.

Part 2. Verification: method of manufactured solution

(a) For the $\mathbb{P}^1$ and $\mathbb{P}^2$ finite element approximations we expect that if our solution is smooth our $H^1(\Omega)$ error norm will converge at a rate of p. A convergence plot of the error against h is shown in Figue 1 below.

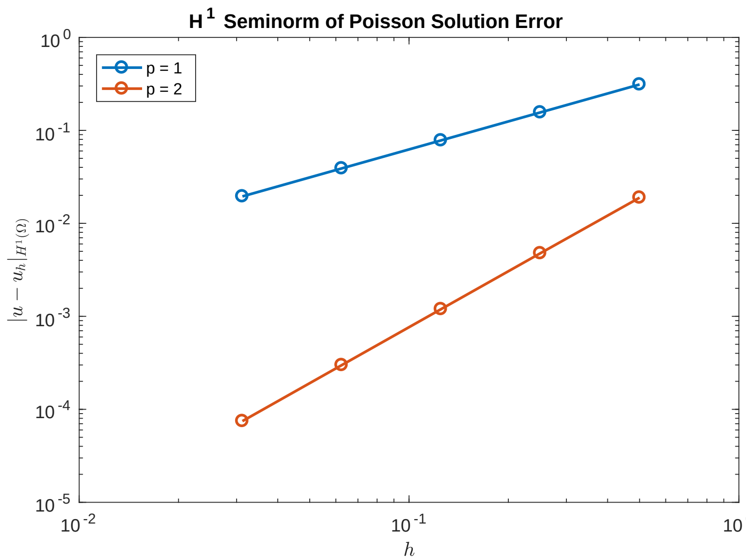

Figure 1: Caption

The convergence rates were found to be 1.0000 and 1.9998 for the $\mathbb{P}^1$ and $\mathbb{P}^2$ approximations, respectively; these rates agree very well with the theory.

(b) For the $\mathbb{P}^1$ and $\mathbb{P}^2$ finite element approximations we expect that if our solution is smooth our $H^1(\Omega)$ error norm will converge at a rate of $p + 1$. A convergence plot of the error against h is shown in Figure 2 below.

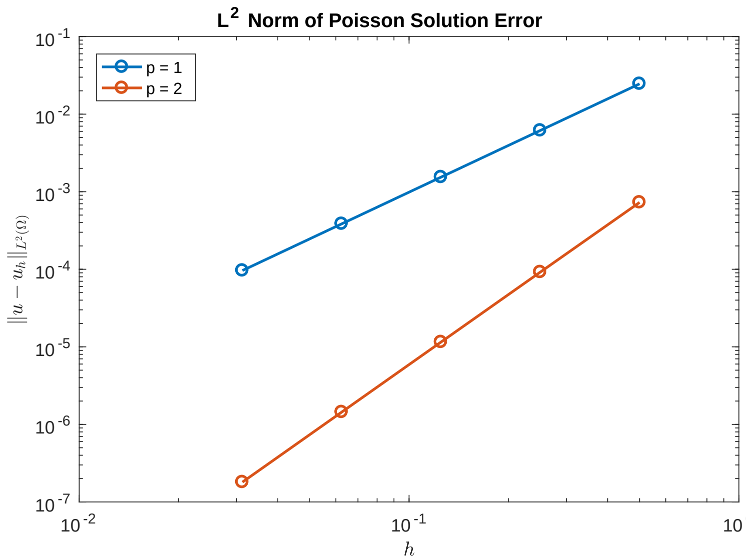

Figure 2: Caption

The convergence rates were found to be 2.0000 and 2.9998 for the $\mathbb{P}^1$ and $\mathbb{P}^2$ approximations, respectively; these rates agree very well with the theory.

(c) For the $\mathbb{P}^1$ and $\mathbb{P}^2$ finite element approximations we may naïvely expect that if our solution is smooth our output error $|\ell^{\mathrm{o}}(u) - \ell^{\mathrm{o}}(u_h)|$ will converge at a rate of $2p$. We can however note that since our problem satisfies assumptions 6.1, 6.2, and 6.3 from the notes, and our primal and adjoint solutions are $\in H^1(\Omega) \cap H^{s^{(\prime)}+1}(\mathcal{T}_h)$ we have

$$|\ell^{\mathrm{o}}(u) - \ell^{\mathrm{o}}(u_h)| \lesssim h^{r+r'} |u|_{H^{r+1}(\mathcal{T}_h)} |\psi|_{H^{r'+1}(\mathcal{T}_h)}$$

with $r \equiv \min\{s, p\}$ and $r' \equiv \min\{s', p\}$. Our adjoint equation is given by: find $\psi \in \mathcal{V}$ such that

$$a(w, \psi) = \ell^{\mathrm{o}}(w) \quad \forall w \in \mathcal{V}.$$

We will solve this equation by first converting it into its strong form by integrating by parts

$$w(x=1) \left.\frac{d\psi}{dx}\right|_{x=1} - \cancel{w(x=0)} \left.\frac{d\psi}{dx}\right|_{x=0} = \int_\Omega w(1 + \frac{d^2\psi}{dx^2}) dx.$$

We note from here that the strong form of this equation is

$$-\frac{d^2\psi}{dx^2} = 1, \quad \psi(x=0) = 0, \quad \left.\frac{d\psi}{dx}\right|_{x=1} = 0,$$

the solution of which is

$$\psi = x - \frac{x^2}{2},$$

i.e. a second order polynomial, for which $|\psi|_{H^{r'+1}(\mathcal{T}_h)} = 0$, where $r' = p$.

A convergence plot of the error against h is shown in Figue 3 below.

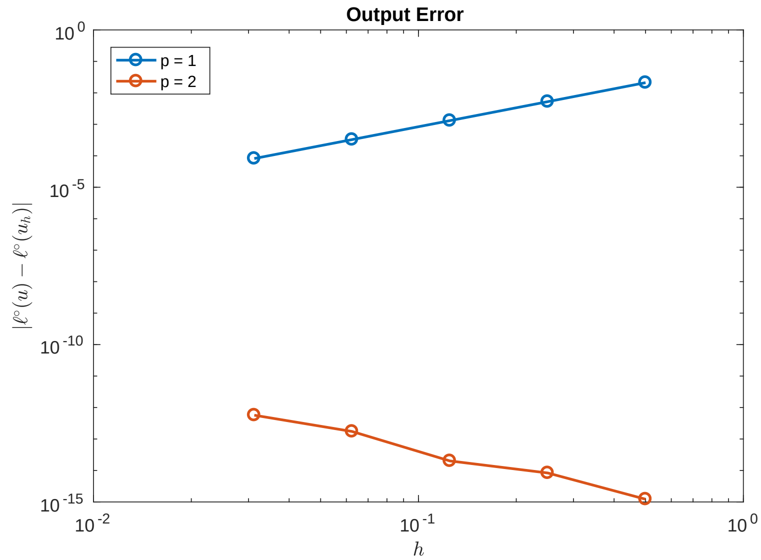

Figure 3: Caption

The convergence rates were found to be 2.0001 and -1.7175 for the $\mathbb{P}^1$ and $\mathbb{P}^2$ approximations, respectively. The second convergence rate is meaningless and a consequence of quadrature error (we use $p_{quad} = 4p$), which will be proportional to the number of elements.

Part 3. Linear Elasticity

We can describe our problem with the following equations

$$-\nabla \cdot \sigma(u) = 0 \text{ in } \Omega, \quad u = u^B \text{ on } \Gamma_D, n \cdot \sigma(u) = g \text{ on } \Gamma_N.$$

We have $\Omega \equiv (0,2) \times (0,1/2)$, $\Gamma_N \equiv \Gamma_{\text{right}} = \{2\} \times (0,1/2)$, $\Gamma_D = \partial\Omega / \Gamma_{\text{right}}$, and $g = [g_{\text{pull}}\ 0]^T$. Multiplying by a test function and integrating by parts yields the weak form of our problem

$$\int_\Omega \nabla v : \sigma(u) dx = \int_{\Gamma_{\text{right}}} v \cdot g ds$$

By substituting in our expression for stress in terms of strain and noting that because $\epsilon(u)$ is symmetric, we have $\nabla v : \epsilon(u) = \epsilon(v) : \epsilon(u)$ we can rewrite our weak form as

$$\int_\Omega 2\mu\epsilon(v) : \epsilon(u) + \lambda \text{tr}(\epsilon(v))(\epsilon(u)) dx = \int_{\Gamma_{\text{right}}} v \cdot g ds$$

(a) Twice the strain energy density integrated over the domain is

$$\begin{aligned}
\int_\Omega W(u)dx =& 2\int_\Omega \frac{1}{2}\epsilon(u) : \sigma(u)dx && \text{(Def. of } W(u)) \\
=& \int_\Omega \epsilon(u) : [2\mu\epsilon(u) + \lambda \mathrm{tr}(\epsilon(u))I]\, dx && \text{(Def. of } \sigma(u)) \\
=& \int_\Omega 2\mu\epsilon(u) : \epsilon(u) + \lambda \mathrm{tr}(\epsilon(u))(\epsilon(u))dx && \\
=& a(u,u) && \text{(Def. of } a(u,u)) \\
=& \ell(u) && \text{(Def. of } u) \\
=& \int_{\Gamma_{\mathrm{right}}} u \cdot g ds && \text{(Def. of } \ell(u)) \\
=& \int_{\Gamma_{\mathrm{right}}} u_1 g_{\mathrm{pull}} ds && \text{(Def. of } g(u)) \\
=& \ell^{\mathrm{o}}(u) && \text{(Def. of } \ell^{\mathrm{o}}(u)),
\end{aligned}$$

which is exactly our compliance ouput.

(b) Our problem satisfies all the requirements for a minimization formulation with an energy functional

$$u = \arg\min_{w \in \mathcal{V}} J(w), \quad J(v) \equiv \frac{1}{2}a(v,v) - \ell(v) \quad \forall v \in \mathcal{V}.$$

Likewise we have a finite elemenet approximation of the above

$$u_h = \arg\min_{w_h \in \mathcal{V}_h} J(w_h), \quad J(v) \equiv \frac{1}{2}a(v,v) - \ell(v) \quad \forall v \in \mathcal{V}_h.$$

We note that since $\mathcal{V}_h \subset \mathcal{V}$ we can say that

$$J(u) \le J(u_h), \implies \frac{1}{2}a(u,u) - \ell(u) \le \frac{1}{2}a(u_h,u_h) - \ell(u_h).$$

The definitions of u and u_h imply that $a(u,u) = \ell(u)$ and $a(u_h,u_h) = \ell(u_h)$, therefore

$$\frac{1}{2}\ell(u) - \ell(u) \le \frac{1}{2}\ell(u_h) - \ell(u_h).$$

Since we have $\ell(u) = \ell^{\mathrm{o}}(u)$ and $\ell(u_h) = \ell^{\mathrm{o}}(u_h)$ we can say

$$-\frac{1}{2}\ell^{\mathrm{o}}(u) \le -\frac{1}{2}\ell^{\mathrm{o}}(u_h).$$

This implies that

$$\ell^{\mathrm{o}}(u_h) \le \ell^{\mathrm{o}}(u).$$

(c) Comment code and use matlab prettifier here.

(d) Figure 4 shows the solution for which the reference output is calculated. It uses 28649 elements ($h = 0.008$) and yields a compliance output of $\ell^{\mathrm{o}}(u_{\mathrm{ref}}) = 0.00456145212$

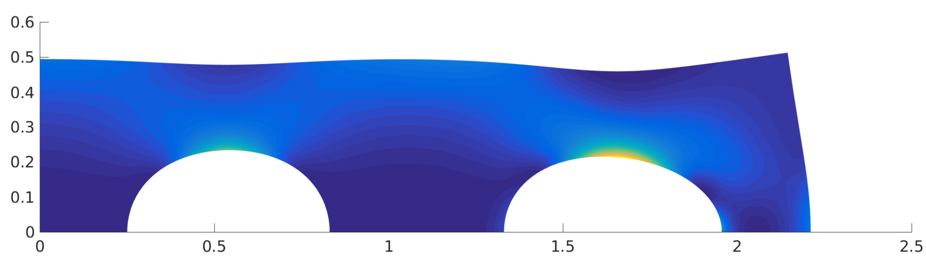

Figure 4: Caption.

(e) Reference outputs for each family of approximations are summarized in Table 1

	A1	A2	A3
$\ell^{\text{o}}(u_h)$	0.0045588902	0.0045610541	0.0045614521

Table 1: Caption.

(f) Figure 5 shows a semilog plot of $\ell^{\text{o}}(u_h)$ against the number of degrees of freedom for each of our approximation families; the reference output is also shown. From part (b) we expect that $\ell^{\text{o}}(u_h) < \ell^{\text{o}}(u_{\text{ref}})$ for each solution.

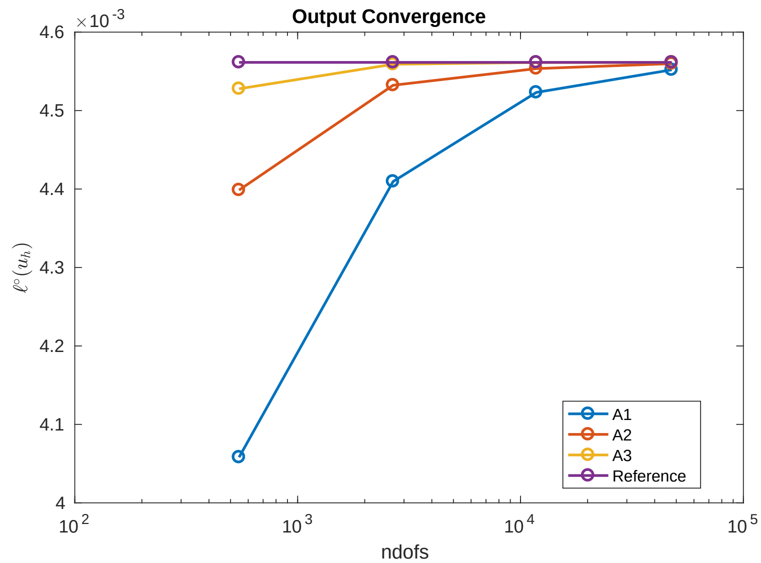

Figure 5: Caption.

We note that as expected, $\ell^{\text{o}}(u_h) < \ell^{\text{o}}(u_{\text{ref}})$, for each solution.

(g) A plot showing a log-log relation between $|\ell^{\text{o}}(u_h) - \ell^{\text{o}}(u_{\text{ref}})|$ against the number of degrees of freedom for each approximation family is shown in Figure 6 below.

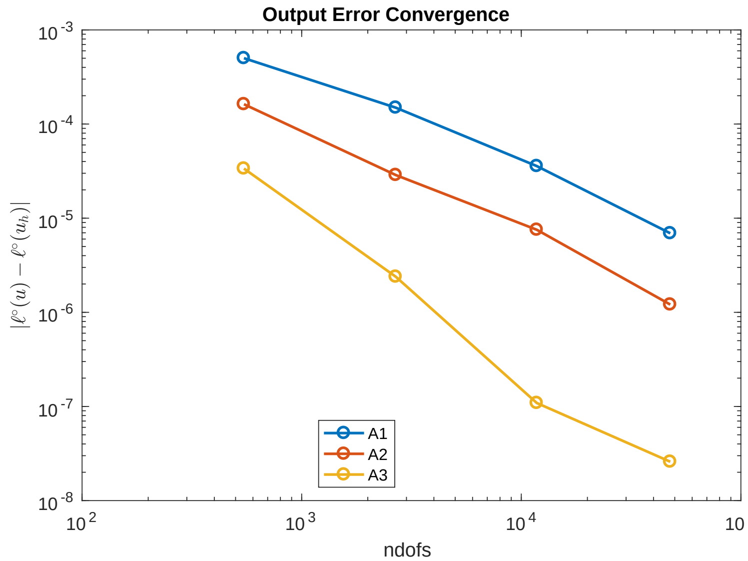

Figure 6: Caption.

(h) The slopes of the above plot are:

	A1	A2	A3
m	-0.8596	-1.0001	-1.8673

We may also compute the convergence rates with respect to h, these are:

	A1	A2	A3
r	1.9042	2.2153	4.1364

Which are approximately twice the negative convergence rates with respect to the number of degrees of freedom, this is expected as ndofs $\sim 1/h^2$.

Furthermore, we note that if $u \in H^1(\Omega) \cap H^{s+1}(\mathcal{T}_h)$ and $\psi \in H^1(\Omega) \cap H^{s'+1}(\mathcal{T}_h)$ then we have

$$|\ell^{\text{o}}(u_h) - \ell^{\text{o}}(u_{\text{ref}})| \lesssim h^{r+r'} |u|_{H^{r+1}(\mathcal{T}_h)} |\psi|_{H^{r'+1}(\mathcal{T}_h)},$$

where $r \equiv \min\{s, p\}, r' \equiv \min\{s', p\}$. Since we can achieve $r = 2p$ convergence we can conclude that $s > 2$ or that $u \in H^{k \geq 3}(\Omega)$.