AER1418 Problem Set 1

Due Tuesday, 6 February 2018, at 14:00.

A hard copy of the problem set is due in class at the specified time. Please adhere to the collaboration policy: the final write up must be prepared individually without consulting others. (See the syllabus for details.)

Part 1. Formulation and analysis

We wish to understand the insulation characteristics of a jacket worn by a person walking outside in a cold, windy day. In our simplified analysis, the person is a circle that acts as a heat source, and the jacket is a ring that surrounds the person. We wish to model the temperature distribution in the ring-shaped jacket and evaluate the temperature on the inner surface, which contacts the person.

We now provide a mathematical description of the problem. All variables are nondimensionalized. We introduce a two-dimensional annular domain

$$\Omega_{\text{annulus}} \equiv \{x \in \mathbb{R}^2 \mid r_{\text{in}} < \sqrt{x_1^2 + x_2^2} < r_{\text{out}}\},$$

where r_{in} and r_{out} are the inner and outer radius, respectively, so that $0 < r_{\text{in}} < r_{\text{out}}$. The associated inner and outer boundaries are

$$\Gamma_{\text{in}} \equiv \{x \in \mathbb{R}^2 \mid \sqrt{x_1^2 + x_2^2} = r_{\text{in}}\},$$
$$\Gamma_{\text{out}} \equiv \{x \in \mathbb{R}^2 \mid \sqrt{x_1^2 + x_2^2} = r_{\text{out}}\}.$$

The steady-state temperature distribution inside the annulus is modeled by the (steady-state) heat equation

$$-\nabla \cdot (k \nabla u) = 0 \quad \text{in } \Omega_{\text{annulus}},$$

where $k > 0$ is the thermal conductivity. The heat transfer at the inner wall Γ_{in} from the heat source is modeled by

$$-n \cdot (k \nabla u) = -g \quad \text{on } \Gamma_{\text{in}},$$

where n is the outward-pointing normal with respect to Ω, and $g > 0$ is the heat flux. The heat transfer at the outer wall Γ_{out} to the air is modeled by

$$-n \cdot (k \nabla u) = B_{\text{out}}(u - u_\infty) \quad \text{on } \Gamma_{\text{out}},$$

where $B_{\text{out}} > 0$ is the Biot number that characterizes the heat transfer to the surrounding air. Without loss of generality, we set $u_\infty = 0$; i.e., the temperature u is defined with respect to the ambient air temperature.

Answer the following questions:

(a) Derive a variational form of the two-dimensional heat conduction problem. Specifically, identify (i) the trial and test spaces, (ii) bilinear form, and (iii) linear form.

(b) The two-dimensional problem is axisymmetric. Reformulate the variational problem as a one-dimensional variational problem on the domain $\Omega \equiv (r_{\text{in}}, r_{\text{out}}) \subset \mathbb{R}^1$. Specifically, identify (i) the trial and test spaces, (ii) bilinear form, and (iii) linear form.

Note: the integrand of the bilinear form should depend on the coordinate $x \in \Omega \subset \mathbb{R}^1$.

For the following questions, the variational problem, bilinear form, and linear form refer to those obtained in (b).

(c) Show that the bilinear form is coercive.

(d) Show that the bilinear form is continuous.

(e) Show that the linear form is continuous.

(f) Prove or provide a counter example to the following statement: the variational problem has a unique solution.

(g) Suppose the Biot number is zero: $B_{\text{out}} = 0$. Prove that the bilinear form is not coercive. Prove or provide a counter example to the following statement: the variational problem has a unique solution.

Note: to prove that the bilinear from is not coercive, you need to find only one function for which the coercivity condition is violated.

(h) State a minimization form of the variational problem. Specifically, identify (i) the function space and (ii) energy functional.

Part 2. Implementation

We now wish to develop a finite-element code that approximates the solution to the one-dimensional variational problem obtained in Part 1. The code should use a piecewise linear finite element space with a uniform (but arbitrary) spacing $h > 0$.

(a) Write a finite-element code that solves the heat conduction problem.

Note. We need to evaluate the entries of the stiffness matrix $\hat{A}_h$ and load vector $\hat{f}_h$. For example, to evaluate the diagonal entries of $\hat{A}_h$, consider a hat-shaped piecewise-linear basis function delineated by points x_{i-1}, x_i, and x_{i+1} with the support (x_{i-1}, x_{i+1}) and the peak at x_i; then, express the diagonal entry as a function of the coordinates x_{i-1}, x_i, and x_{i+1}. Evaluate all other entries of the stiffness matrix and load vector in a similar manner.

(b) Plot the finite-element solution u_h for $r_{\text{in}} = 1/2$, $r_{\text{out}} = 2$, $k = 1$, $g = 1$, $B_{\text{out}} = 1$, and a uniform mesh spacing of $h = 1/4$.

(c) Consider the output $\ell^o(u) \equiv g r_{\text{in}} u(x = r_{\text{in}})$. Show that

$$\ell^o(u) - \ell^o(u_h) = a(e, e),$$

where $e \equiv u - u_h$ is the error in the finite element approximation u_h.

(d) Solve the finite element problem for $r_{\text{in}} = 1/2$, $r_{\text{out}} = 2$, $k = 1$, $g = 1$, $B_{\text{out}} = 1$, and uniform grid spacings of $h = 1/2, 1/4, 1/8, \dots, 1/64$. Report the reference value of the output for $h_{\text{ref}} \equiv 1/512$, $\ell^o(u_{h_{\text{ref}}})$. Plot the output error (with respect to $\ell^o(u_{h_{\text{ref}}})$) against the grid spacing h in log-log scale, and report the observed convergence rate. Does the observed convergence rate match your expectation based on the theory?