Math 1 Semester 2 revision sheet

By: Ioannis Karras

Version: Release 1.1

Link to revision folder

Topic diversions from the teacher are in yellow, other topic diversions/extra additions are in green. Links to another part of the document are in blue

Table of Contents (contains links)

Functions of two variables

Basics

- A function of two variables maps from two values x and y to an output value written as $f(x, y)$. The fact that it has two inputs and one output can be written as $f: \mathbb{R}^2 \to \mathbb{R}$
- Geometrically, this output value can be interpreted as the height z above the xy plane. To plot a function of two variables, you need a 3D graph
- A *level curve* K_c is the two-dimensional curve you obtain when considering all the inputs that give a specific output c
    - Geometrically, if you have a plane parallel to the xy plane and set its height to c, the level curve is the intersection between this plane and the graph
    - An expression for a level curve can be found simply by setting $f(x, y) = c$. You may be able to rearrange this into a function of either x or y, but it will most likely be a relation which cannot be expressed as a standard function, like $x^2 + y^2 = 1$

- Functions of two variables do not have a single derivative. Rather, they have two *partial derivatives*, one with respect to each variable
    - These are computed by differentiating with respect to one variable, while treating the other one as a constant
    - The partial derivative with respect to x can either be written as
    $$f'_x(x,y)$$
    or as
    $$\frac{\partial f}{\partial x}(x,y)$$
    - You can then find second partial derivatives by differentiating the two first partial derivatives with respect to either x or y. Because of this, there are four second partial derivatives, $f''_{xx}(x,y)$, $f''_{xy}(x,y)$, $f''_{yy}(x,y)$, and $f''_{yx}(x,y)$. These can alternatively be written as e.g.
    $$f''_{xx}(x,y) = \frac{\partial^2 f}{\partial x \partial x}(x,y)$$
    $$f''_{xy}(x,y) = \frac{\partial^2 f}{\partial x \partial y}(x,y)$$
    - The two *mixed derivatives* $f''_{xy}(x,y)$ and $f''_{yx}(x,y)$ are always equal
- The *gradient* of a function is a vector containing the two partial derivatives:
$$\boldsymbol{\nabla} f(x,y) = \begin{bmatrix} f'_x(x,y) \\ f'_y(x,y) \end{bmatrix}$$
    - This is usually drawn starting from the point (x,y)
- f'_x is the slope in the *x*-direction and f'_y is the slope in the *y*-direction. You can also find slopes in any other direction, in which case it is called a *directional derivative*. The directional derivative at a point in the direction given by a unit vector $\boldsymbol{e}$ is calculated by taking the dot product of the gradient at that point with $\boldsymbol{e}$:
$$f'\big((x,y),\boldsymbol{e}\big) = \boldsymbol{\nabla} f(x,y) \cdot \boldsymbol{e}$$
- Differentiability can be defined for functions of two variables:
    - Epsilon functions of two variables fulfill $\varepsilon(0,0) = 0$ and tend to zero as *x* and *y* tend to zero
        - Similarly to functions of one variable, you can have epsilon functions around any point. $\varepsilon(x - x_0, y - y_0)$ tends to 0 as x tends to x_0 and y tends to y_0
    - The distance function ρ is a function that gives the distance from a reference point (x_0, y_0) to an input point (x,y):
    $$\rho_{(x_0,y_0)}(x,y) = \sqrt{(x-x_0)^2 + (y-y_0)^2}$$
    - A function f is differentiable at a point (x_0, y_0) if it can be written in the following way:
    $$f(x,y) = f(x_0,y_0) + a(x-x_0) + b(y-y_0) + \rho_{(x_0,y_0)}(x,y) \cdot \varepsilon(x-x_0, y-y_0)$$
    a and b are the partial derivatives at (x_0, y_0) with respect to *x* and *y*
- Diversion: Point sets
    - A set of points in a plane can have *interior points*, *exterior points*, and *boundary points*, points which are right on the boundary of the set

- Sets do not always include their boundary points:
  - An *open set* is a set that only has interior points
  - A *closed set* has its boundary included
  - A set that is *neither open nor closed* has some of the boundary included, but not all of it
- A *bounded set* is a limited set, or one that can be enclosed in a circular disc centered at the origin. An *unbounded set* cannot be enclosed in this manner
- $A°$ denotes the set of interior points of A
- ∂A denotes the boundary of A
- $\bar{A}$ denotes the *closure* of A, which is the union of the interior points and the boundary:

$$\bar{A} = A° \cup \partial A$$

- The domain of a function, $Dm(f)$, is the largest set of points where the function is actually defined and can be evaluated. To find the domain:
  - Figure out what restrictions need to be placed on *x* and *y* so that the function can be evaluated. For example, you cannot take the ln of 0 or a negative number, so for ln $(x - y)$, the restriction would be $x - y > 0$
  - Replace the inequality sign with an equals sign to obtain the boundary of the domain
  - Test points on both sides of the boundary to decide which side is the domain
  - Decide whether the boundary is included or not based on the inequality sign
  - It may be necessary to do this for more than one part of the expression. For example,

$$\frac{\sqrt{1-y^2}}{\sqrt{1-x^2}}$$

    requires two restrictions

$$1 - y^2 \geq 0$$
$$1 - x^2 > 0$$

    and results in a domain that is neither open nor closed

- A 2D parametric curve given by $\boldsymbol{r}(u)$ can be elevated to the graph of a function $f(x, y)$ by adding a *z* coordinate to the parametric representation. The *z* coordinate will be a *height function* given by the composite function $f(\boldsymbol{r}(u))$. For example, if the 2D curve is

$$\boldsymbol{r}(u) = \begin{bmatrix} u \cdot \cos(u) \\ u \cdot \sin(u) \end{bmatrix}$$

  the elevated curve is

$$\boldsymbol{R}(u) = \begin{bmatrix} u \cdot \cos(u) \\ u \cdot \sin(u) \\ f(\boldsymbol{r}(u)) \end{bmatrix} = \begin{bmatrix} u \cdot \cos(u) \\ u \cdot \sin(u) \\ f(u \cdot \cos(u), u \cdot \sin(u)) \end{bmatrix}$$

Taylor approximations

- The *first degree approximating polynomial* for a function of two variables is given by
$$P_1(x,y) = f(x_0,y_0) + f_x'(x_0,y_0)\cdot(x-x_0) + f_y'(x_0,y_0)\cdot(y-y_0)$$
where (x_0, y_0) is a development point
    - This can be rewritten and interpreted geometrically as the equation for the tangent plane at (x_0, y_0):
$$z = z_0 + a(x-x_0) + b(y-y_0)$$
$$-a(x-x_0) - b(y-y_0) + 1(z-z_0) = 0$$
    - This plane has a normal vector given by
$$\boldsymbol{N} = \begin{bmatrix} -a \\ -b \\ 1 \end{bmatrix} = \begin{bmatrix} -f_x'(x_0,y_0) \\ -f_y'(x_0,y_0) \\ 1 \end{bmatrix}$$
- The *second degree approximating polynomial* for a function of two variables is given by
$$P_2(x,y) = f(x_0,y_0) + f_x'(x_0,y_0)(x-x_0) + f_y'(x_0,y_0)(y-y_0) + \frac{1}{2}f_{xx}''(x_0,y_0)(x-x_0)^2$$
$$+\frac{1}{2}f_{yy}''(x_0,y_0)(y-y_0)^2 + f_{xy}''(x_0,y_0)(x-x_0)(y-y_0)$$
- The Hessian matrix for a function at a point (x_0, y_0) is a matrix containing all four second derivatives at that point:
$$\boldsymbol{H}_f(x_0,y_0) = \begin{bmatrix} f_{xx}''(x_0,y_0) & f_{xy}''(x_0,y_0) \\ f_{yx}''(x_0,y_0) & f_{yy}''(x_0,y_0) \end{bmatrix}$$
Since $f_{xy}''(x_0,y_0) \equiv f_{yx}''(x_0,y_0)$, this matrix is always symmetric
- The second degree approximation can be written more simply using the Hessian matrix and the gradient:
$$f(x,y) = f(x_0,y_0) + \boldsymbol{\nabla} f(x_0,y_0)\cdot\begin{bmatrix} x-x_0 \\ y-y_0 \end{bmatrix} + \begin{bmatrix} x-x_0 & y-y_0 \end{bmatrix}\cdot\frac{1}{2}\boldsymbol{H}_f(x_0,y_0)\cdot\begin{bmatrix} x-x_0 \\ y-y_0 \end{bmatrix}$$
    - The matrix $\frac{1}{2}\boldsymbol{H}_f(x_0,y_0)$ is sometimes given the name $\boldsymbol{A}$

Conic sections and base changes

- Level curves for functions of two variables will often fall into a special class of curves called *conic sections*. There are three types: parabolas, ellipses, and hyperbolas:
    - Parabolas have the general form
$$y = ax^2$$
for a vertical parabola, or
$$x = ay^2$$
for a horizontal parabola
        - The turning point of a parabola is called its *vertex*. Both of these parabolas have their vertices at the origin
    - Ellipses have the general form
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$
where a and b are the semi-axes of the ellipse. This ellipse has its center at the origin
        - Circles are special ellipses, given by the form
$$x^2 + y^2 = R^2$$
where R is the radius of the circle

- Hyperbolas have the general form
$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$
or
$$\frac{y^2}{b^2} - \frac{x^2}{a^2} = 1$$
where a and b are the semi-axes. These hyperbolas have their vertices at the origin

- Each of these conic sections can be shifted so that they have their centers/vertices somewhere other than the origin. To displace a conic section by the vector (c_1, c_2), replace x with $x - c_1$ and y with $y - c_2$
- To bring a level curve into one of these standard forms, you often have to *complete the square*:
  - Find the quadratic and linear terms for each variable. For example, in the curve
$$x^2 + 4y^2 - 6x + 16y + 9 = 0$$
you must consider $x^2 - 6x$ and $4y^2 + 16y$
  - Factorize these parts so that the coefficient of the quadratic term is 1. In this case, the x expression can stay as it is, but the y expression becomes
$$4(y^2 + 4y)$$
  - For each of these two parts, find a term that, when squared, will give you the quadratic and linear term you have, and then correct for the constant term by adding/subtracting a number:
$$x^2 - 6x = (x - 3)^2 - 9$$
because $(x - 3)^2 = x^2 - 6x + 9$
$$4(y^2 + 4y) = 4((y + 2)^2 - 4) = 4(y + 2)^2 - 16$$
because $(y + 2)^2 = y^2 + 4y + 4$
  - Put the modified terms back into the equation, and bring the equation into the standard form by dividing by the appropriate number. For ellipses and hyperbolas, make sure you have a 1 on the right hand side:
$$(x - 3)^2 - 9 + 4(y + 2)^2 - 16 + 9 = 0$$
$$(x - 3)^2 + 4(y + 2)^2 = 16$$
$$\frac{(x - 3)^2}{16} + \frac{4(y + 2)^2}{16} = 1$$
$$\frac{(x - 3)^2}{4^2} + \frac{(y + 2)^2}{2^2} = 1$$
- Bringing level curves into standard form becomes more complicated when there is a *mixed term* in the equation, like xy. In this case, you have to perform a change of base:
  - A *quadratic form* is the part of a quadratic equation involving x^2, y^2, and xy. It has the general form $ax^2 + by^2 + cxy$
  - A quadratic form can be written using vectors and matrices. In the case of the general form:
$$ax^2 + by^2 + cxy = \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} a & c/2 \\ c/2 & b \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x & y \end{bmatrix} \cdot \boldsymbol{A} \cdot \begin{bmatrix} x \\ y \end{bmatrix}$$

- To get rid of the mixed term, you must diagonalize $\boldsymbol{A}$ to get rid of the $c/2$ elements. Since $\boldsymbol{A}$ is symmetric, you can find a diagonal matrix $\boldsymbol{\Lambda}$ and a positive orthogonal matrix $\boldsymbol{Q}$ such that

$$\boldsymbol{\Lambda} = \boldsymbol{Q^T} \cdot \boldsymbol{A} \cdot \boldsymbol{Q}$$

  You do this by finding the eigenvalues and eigenvectors of $\boldsymbol{A}$. You also need to check that $\boldsymbol{Q}$ is positive orthogonal. If it is not, you must make it so by changing the signs of all the elements in one column. The coordinates of the eigenbasis you create are given the symbols $\tilde{x}$ and $\tilde{y}$

- Then, you can get rid of the mixed term in the quadratic form. You have that

$$\begin{bmatrix} x \\ y \end{bmatrix} = \boldsymbol{Q} \cdot \begin{bmatrix} \tilde{x} \\ \tilde{y} \end{bmatrix}$$

  and, taking the transpose,

$$[x \quad y] = [\tilde{x} \quad \tilde{y}] \cdot \boldsymbol{Q^T}$$

  Therefore,

$$\begin{aligned} & [x \quad y] \cdot \boldsymbol{A} \cdot \begin{bmatrix} x \\ y \end{bmatrix} \\ & = [\tilde{x} \quad \tilde{y}] \cdot \boldsymbol{Q^T} \cdot \boldsymbol{A} \cdot \boldsymbol{Q} \cdot \begin{bmatrix} \tilde{x} \\ \tilde{y} \end{bmatrix} \\ & = [\tilde{x} \quad \tilde{y}] \cdot \boldsymbol{\Lambda} \cdot \begin{bmatrix} \tilde{x} \\ \tilde{y} \end{bmatrix} \\ & = \lambda_1 \tilde{x}^2 + \lambda_2 \tilde{y}^2 \end{aligned}$$

- The constant terms in the quadratic equation do not need to be changed, but you do need to deal with the linear terms. This is done by writing the linear part of the equation with vectors, and then using $\boldsymbol{Q}$ to change the coordinate base. For example:

$$-40x + 30y = [-40 \quad 30] \begin{bmatrix} x \\ y \end{bmatrix} = [-40 \quad 30] \cdot \boldsymbol{Q} \cdot \begin{bmatrix} \tilde{x} \\ \tilde{y} \end{bmatrix}$$

  You then evaluate $[-40 \quad 30] \cdot \boldsymbol{Q}$ to find the new linear coefficients. Using an example for $\boldsymbol{Q}$,

$$\begin{aligned} & [-40 \quad 30] \cdot \boldsymbol{Q} \cdot \begin{bmatrix} \tilde{x} \\ \tilde{y} \end{bmatrix} \\ & = [-40 \quad 30] \cdot \begin{bmatrix} \frac{3}{5} & -\frac{4}{5} \\ \frac{4}{5} & \frac{3}{5} \end{bmatrix} \cdot \begin{bmatrix} \tilde{x} \\ \tilde{y} \end{bmatrix} \\ & = [0 \quad 50] \begin{bmatrix} \tilde{x} \\ \tilde{y} \end{bmatrix} \\ & = 0\tilde{x} + 50\tilde{y} \\ & = 50\tilde{y} \end{aligned}$$

- After all this, you should have the conic section written in new coordinates without a mixed term. You can find the general form, but remember that the center/vertex and lines of symmetry that you find will be in the new coordinates, which you might have to convert to the old ones by premultiplying with $\boldsymbol{Q}$

Optimization

- A function of two variables has a *stationary point* wherever its gradient is the **0** vector, which means both the partial derivatives are 0:

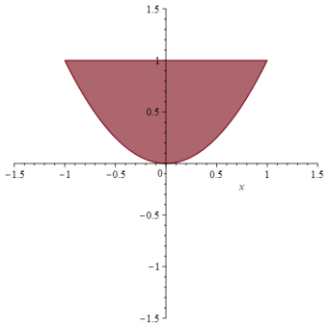
$$\nabla f(x_0, y_0) = \mathbf{0}$$
$$f'_x(x_0, y_0) = 0 \text{ and } f'_y(x_0, y_0) = 0$$

- A *local minimum point* is a point where all the points in a small neighborhood around it have a function value that is either higher than or equal to the function value at the point. The function value at the point is called a *local minimum value*
    - If all the points around it have a strictly higher (not equal) function value, the function value at the point is a *proper* local minimum value
    - Local maximum points, local maximum values, and proper local maximum values are defined similarly
- In most cases, you can determine the nature of a stationary point by finding the Hessian matrix at that point, and then finding its eigenvalues λ_1 and λ_2. The type of stationary point depends on the signs of the eigenvalues:
    - If $\lambda_1 > 0$ and $\lambda_2 > 0$, it is a proper local minimum point
    - If $\lambda_1 < 0$ and $\lambda_2 < 0$, it is a proper local maximum point
    - If λ_1 and λ_2 have opposite signs, it is a *saddle point*, which means it curves upwards in one direction and downwards in another, like a saddle
    - If either λ_1 or λ_2 is 0, the Hessian method is inconclusive as to the nature of the stationary point
- To find *global minima and maxima* on a compact (closed and bounded) set, you do not need to determine the nature of any stationary points. Instead, you must find a list of candidate function values, and then compare them to see which is the global minimum ($G_{\min}$) and which is the global maximum ($G_{\max}$)
    - These candidates are attained at:
        - exception points (where the function is not differentiable; these usually do not exist)
        - stationary points
        - the boundary of the set
    - In practice, to check the boundary of the set, you must restrict the function to each boundary line and find the 1-dimensional stationary points on each line. Then, check the corners of the boundary as well
        - For boundary lines parallel to the *x* or *y* axis, simply set *x* or *y* in the function to the appropriate value. For example, if one of the boundary lines is the line from $(3,1)$ to $(3,4)$, set *x* to 3:

          $$f(3, y)$$

          This results in a function of one variable *y*. Find stationary points from $y = 1$ to $y = 4$ and their function values, using the normal 1D method with first derivatives. You do not need to consider stationary points at exactly $y = 1$ or $y = 4$, since those *y* values correspond to the corners of the boundary, which you have to check later separately anyway

- For boundary lines that are not parallel to the *x* or *y* axes, you might be able to find a function for the boundary line, or it might be given in that form in the question. For example, for the set (shown in the picture)



$$M = \{(x, y) | -1 \leq x \leq 1 \text{ and } x^2 \leq y \leq 1\}$$

the lower boundary is given by $y = x^2$. Therefore, the function restricted to this line is

$$f(x, y) = f(x, x^2)$$

which is a function of one variable *x*. In this case, you have to find stationary points from $x = -1$ to $x = 1$

- If all else fails (or if you just want a single method for all situations), find a parametric representation $\boldsymbol{r}(u)$ for the boundary line and apply the function to it, creating a composite function, and then find the stationary points of this composite function. Remember to convert from the parametric variable *u* at a stationary point to the *x* and *y* coordinates, using the original parametric representation
- After all this, remember to check the corners of the boundary, and add those function values to the list

- Once you have the function values at the exception points, regular stationary points, boundary stationary points, and boundary corner points, the highest one will be the global maximum, and the lowest will be the global minimum
- If the set is connected (continuous), the range of the function on the set will be $[G_{\min}, G_{\max}]$

Riemann integrals

Basics and methods

- The area under a graph in a region $[a, b]$ can be approximated by dividing the region into *n* segments of equal length δu, and then creating a rectangle at every segment with width δu and height equal to the function value at the left end of the segment. The sum of the areas of these rectangles is called a *left sum*:

$$\sum_{u=1}^{n} f(u_i)\delta u$$

- As *n* tends to infinity, this becomes a perfect measure of the area under the graph. This limit value is called the *definite integral* or *Riemann integral*, and it is denoted

$$\int_a^b f(x)\, dx$$

- The Fundamental Theorem of Calculus states that this integral can be computed in the following way:

$$\int_a^b f(x)\, dx = [F(x)]_a^b = F(b) - F(a)$$

where $F(x)$ is an *indefinite integral/antiderivative* of $f(x)$, usually denoted

$$\int f(x)\, dx$$

- A similar thing can be done for functions of two variables, resulting in a *double integral*. Double integrals can be computed as two consecutive single integrals:

$$\int_c^d \int_a^b f(u,v)\,du\,dv = \int_c^d \left(\int_a^b f(u,v)\,du \right) dv$$

- Given a curve *K* in space and a function $f(x,y,z)$ defined at every point in space, the *curve integral* of this function across this curve is denoted as

$$\int_K f(x,y,z)\,d\mu$$

The function f is usually interpreted as a *mass density function*, in which case this integral gives the total mass of the curve. However, μ is not a real variable, so this is not a true integral, but rather only a symbol for the curve integral. $d\mu$ tells you that you must do three things:

1. Restrict f to the parametric representation of the curve. Parametric representations are usually called $\boldsymbol{r}(u)$, so the restricted function is $f\big(\boldsymbol{r}(u)\big) = f\big(r_1(u), r_2(u), r_3(u)\big)$
2. Multiply the integrand by the *Jacobi function*
3. Integrate across the bounds of the parametric representation

The Jacobi function is a function that describes how the parameter field is warped by the parametric representation, and it changes depending on the type of integral being calculated. For a curve integral, it is given by the magnitude of the derivative of the parametric representation:

$$Jacobi = |\boldsymbol{r}'(u)|$$

Applying these three steps, the integral takes the following form:

$$\int_K f(x,y,z)\,d\mu = \int_a^b f\big(\boldsymbol{r}(u)\big) \cdot Jacobi\,du = \int_a^b f\big(\boldsymbol{r}(u)\big) \cdot |\boldsymbol{r}'(u)|\,du$$

where *a* and *b* are the parameter bounds. f is a <u>linear</u> mass density function, in units like kg/m.

To find the length of the curve, find the integral without a mass density function (set the mass density function to 1):

$$l = \int_K d\mu = \int_a^b |\boldsymbol{r}'(u)|\,du$$

- To compute the integral of a product, you can use a method called *integration by parts*. In general,

$$\int f(x)g(x)\,dx = F(x)g(x) - \int F(x)g'(x)\,dx$$

where $F(x)$ is an antiderivative of $f(x)$

  - To do this effectively, you must choose the right $f(x)$ and $g(x)$ such that $\int F(x)g'(x)dx$ is easier to calculate than the original integral
  - For definite integrals, integration by parts takes the form

$$\int_a^b f(x)g(x)\,dx = [F(x)g(x)]_a^b - \int_a^b F(x)g(x)dx$$

- To compute the integral of a composite function, you can use a method called *integration by substitution*. In general, when you have an integral of the form

$$\int f\big(g(x)\big) \cdot g'(x)\, dx$$

you can change the integration variable dx. First find the derivative of $g(x)$, and write it in the following way:

$$\frac{d\big(g(x)\big)}{dx} = g'(x)$$

Then, you can use algebra to obtain dx in terms of $d\big(g(x)\big)$:

$$dx = \frac{d\big(g(x)\big)}{g'(x)}$$

Substitute this back into the integral:

$$\int f\big(g(x)\big) \cdot g'(x) \cdot dx$$
$$= \int f\big(g(x)\big) \cdot g'(x) \cdot \frac{d\big(g(x)\big)}{g'(x)}$$
$$= \int f\big(g(x)\big)\, d\big(g(x)\big)$$

which can then be computed

  - This is best shown by example. To find

$$\int \frac{4x}{x^2+1} dx$$

do the following:

$$\frac{d(x^2+1)}{dx} = 2x$$
$$dx = \frac{d(x^2+1)}{2x}$$

$$\int \frac{4x}{x^2+1} dx = \int \frac{4x}{x^2+1} \cdot \frac{d(x^2+1)}{2x} = \int \frac{2}{x^2+1} d(x^2+1) = 2 \cdot \ln\,(x^2+1)$$

No guessing required.

  - Sometimes, it is simpler to call $g(x)$ by another name, like *u*, and carry that name forward. However, you must remember to substitute the original variable back before inserting the limits of a definite integral
  - The relationship between $d\big(g(x)\big)$ and dx can also be found in the opposite way. If we call $g(x)$ by the name u, you can find $\frac{du}{dx} = \frac{d(g(x))}{dx}$ as above, but you can also find $\frac{dx}{du}$. This is useful for certain integrals where you define the substitution in reverse, and the substitution you should use is nowhere near obvious. For example, to find

$$\int \frac{\sqrt{x}}{1+x} dx$$

you can use the "substitution"

$$x = u^2$$

where

$$\frac{dx}{du} = 2u$$
$$dx = 2u \cdot du$$

so that

$$\int \frac{\sqrt{x}}{1+x}\,dx$$
$$= \int \frac{\sqrt{x}}{1+x} \cdot 2u \cdot du$$
$$= \int \frac{u}{1+u^2} \cdot 2u \cdot du$$
$$= 2\int \frac{u^2}{1+u^2} \cdot du$$

Then, pulling some other shenanigans to obtain the derivative of arctan,

$$= 2\int \frac{u^2+1-1}{1+u^2}\,du$$
$$= 2\int \left(1 - \frac{1}{1+u^2}\right) du$$
$$= 2(u - \arctan(u))$$

Since $u = \sqrt{x}$,

$$= 2\left(\sqrt{x} - \arctan\left(\sqrt{x}\right)\right)$$

Plane and surface double integrals

- Areas in the plane can be parametrized using parametric representations of two variables. For example, to parametrize the area under the graph $f(x) = x^2$ between -1 and 1, the first parameter should take you along the *x* axis:

  $$\boldsymbol{r}(u,v) = \begin{bmatrix} u \\ 0 \end{bmatrix} + \cdots, \quad u \in [-1,1]$$

  and the second parameter should take you from $(u, 0)$ up to the point on the graph vertically above $(u, 0)$, which is (u, u^2):

  $$\boldsymbol{r}(u,v) = \begin{bmatrix} u \\ 0 \end{bmatrix} + v\left(\begin{bmatrix} u \\ u^2 \end{bmatrix} - \begin{bmatrix} u \\ 0 \end{bmatrix}\right), \quad u \in [-1,1], v \in [0,1]$$
  $$= \begin{bmatrix} u \\ v \cdot u^2 \end{bmatrix}, \quad u \in [-1,1], v \in [0,1]$$

- Just like how you can integrate over a curve using a single integral, you can also integrate over an area in the plane or a surface in space using a double integral. The same notation using $d\mu$ is used, and the same three steps must be taken to compute the integral. The Jacobi differs depending on whether the area is in the plane or in space
  - For an area in the plane, the Jacobi function you need is given by:

    $$Jacobi = |\det([\boldsymbol{r}'_u(u,v)\ \boldsymbol{r}'_v(u,v)])|$$

    where $[\boldsymbol{r}'_u(u,v)\ \boldsymbol{r}'_v(u,v)]$ is a matrix with the two partial derivatives of $\boldsymbol{r}(u,v)$ arranged in columns.

    Since the parametric representation has two variable inputs, you must integrate over two parameter ranges, so you must use a double integral. Without a mass density function, the integral yields the area of the plane region

    $$A = \int_K d\mu = \int_c^d \int_a^b Jacobi\, du\, dv$$

    where *K* is the name of the plane region, *a* and *b* are the parameter bounds on *u*, and *c* and *d* are the parameter bounds on *v*.

    With a mass density function, you get the total mass of the plane region. Like with the curve integral, you must restrict the function to the parametric representation of the area:

$$M = \int_K f(x,y)\, d\mu = \int_c^d \int_a^b f\big(\boldsymbol{r}(u,v)\big) \cdot Jacobi\, du\, dv$$

The mass density function has to be a 2D mass density function, in units like kg/m^2

- To integrate over a surface in 3D space, the same logic applies, except now the Jacobi function is given by

$$Jacobi = |\boldsymbol{r}'_u(u,v) \times \boldsymbol{r}'_v(u,v)|$$

You still use a double integral, as the parametric representation still only has two variable inputs. The mass density function now has three variable inputs *x*, *y*, and *z*, but it still has 2D units, and you still restrict it to the parametric representation in the same way

- Given a plane region *B* with parametric representation $\boldsymbol{r}(u,v)$, the coordinates of its *mass midpoint* (c_1, c_2) are given by

$$c_1 = \frac{1}{M}\int_B r_1(u,v) \cdot f(x,y)\, d\mu$$
$$c_2 = \frac{1}{M}\int_B r_2(u,v) \cdot f(x,y)\, d\mu$$

where *M* is the total mass of the region, f is the mass density function, and r_1 and r_2 are the first and second coordinates of the parametric representation. $d\mu$ means you must restrict f to $\boldsymbol{r}(u,v)$, multiply by the Jacobi function, and integrate over the parameter bounds, like before.

Similarly, for a surface in space,

$$c_1 = \frac{1}{M}\int_B r_1(u,v) \cdot f(x,y,z)\, d\mu$$
$$c_2 = \frac{1}{M}\int_B r_2(u,v) \cdot f(x,y,z)\, d\mu$$
$$c_3 = \frac{1}{M}\int_B r_3(u,v) \cdot f(x,y,z)\, d\mu$$

- There are three special types of surfaces in 3D space that are defined in special ways:
  - A *graph surface* is defined by a *height function* $h(u,v)$. As the name implies, this is simply the surface of a function graph. These are parametrized in the following way:

$$\boldsymbol{r}(u,v) = \begin{bmatrix} u \\ v \\ h(u,v) \end{bmatrix},\ u \in [a,b], v \in [c,d]$$

where *a*, *b*, *c*, and *d* are the limits of the region in the (*x*, *y*) plane above and/or below which the surface is located

  - A *cylindrical surface* is defined by a curve in a plane called a *directrix* and the bounds on the coordinate perpendicular to that plane defining how far the directrix should be extended. For a directrix $\boldsymbol{r}_d(u) = (r_{d1}(u), r_{d2}(u), 0)$ in the (*x*, *y*) plane, the parametric representation of the cylindrical surface is

$$\boldsymbol{r}(u,v) = \begin{bmatrix} r_{d1}(u) \\ r_{d2}(u) \\ v \end{bmatrix},\ u \in [a,b], v \in [c,d]$$

where *a* and *b* are the bounds on the directrix and *c* and *d* are the bounds on how much it should be extended. In other words, you simply attach a third coordinate with defined bounds to the directrix

- A *surface of revolution* is defined by a *profile curve* in a plane and the bounds on the angle it should be rotated by around an axis. The surface is the area that the profile curve sweeps as it rotates around the axis. For a profile curve $\boldsymbol{r}_p(u) = \big(r_{p1}(u), 0, r_{p2}(u)\big)$ in the (x, z) plane, the parametric representation of the surface of revolution is

$$\boldsymbol{r}(u,v) = \begin{bmatrix} r_{p1}(u)\cdot\cos(v) \\ r_{p1}(u)\cdot\sin(v) \\ r_{p2}(u) \end{bmatrix},\ u \in [a,b], v \in [c,d]$$

  In other words, "split" one of the coordinates of the profile curve across two coordinates by multiplying by $\cos(v)$ and $\sin(v)$. For a full revolution, set $c = 0$ and $d = 2\pi$

Volume triple integrals

- Solids in space can be parametrized using parametric representations of three variables
- You can integrate over a solid in space using a triple integral. Given a parametric representation $\boldsymbol{r}(u,v,w)$, the Jacobi function is given by

$$Jacobi = |\det([\boldsymbol{r}'_u(u,v,w)\ \boldsymbol{r}'_v(u,v,w)\ \boldsymbol{r}'_w(u,v,w)])|$$

  where the matrix contains the three partial derivatives of $\boldsymbol{r}(u,v,w)$ arranged in columns. Triple integrating the Jacobi by itself over the three parameter ranges yields the volume of the solid:

$$V = \int_\Omega d\mu = \int_e^f \int_c^d \int_a^b Jacobi\ du\ dv\ dw$$

  where Ω is the name usually given to a solid, and the letters are the parameter bounds.

  You can also find the mass of the solid by integrating a volume mass density function (in e.g. $\mathrm{kg/m^3}$) over the solid:

$$M = \int_\Omega f\ d\mu = \int_e^f \int_c^d \int_a^b f\big(\boldsymbol{r}(u,v,w)\big)\cdot Jacobi\ du\ dv\ dw$$

- The center of mass of a solid is found similarly to the center of mass of a surface. In vector form,

$$\boldsymbol{c} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \frac{1}{M}\int_\Omega \begin{bmatrix} r_1(u,v,w) \\ r_2(u,v,w) \\ r_3(u,v,w) \end{bmatrix}\cdot f(x,y,z)\,d\mu$$

  where $d\mu$, again, means you have to restrict f to $\boldsymbol{r}(u,v,w)$, multiply by the Jacobi, and integrate over the parameter bounds

- There exist space-filling versions of graph surfaces, cylindrical surfaces, and surfaces of revolution, and they are created in very similar ways:
  - A solid bounded by a graph surface is defined by a height function $h(u,v)$. These are parametrized in the following way:

$$\boldsymbol{r}(u,v,w) = \begin{bmatrix} u \\ v \\ w\cdot h(u,v) \end{bmatrix},\ u \in [a,b], v \in [c,d], w \in [0,1]$$

    where a, b, c, and d are the limits of the region in the (x, y) plane above and/or below which the solid is located
  - A cylindrical solid is defined by a surface in a plane which is extended perpendicular to that plane. Given a surface $\boldsymbol{r}_s = (r_{s1}(u,v), r_{s1}(u,v), 0)$ in the (x, y) plane, the

parametric representation of the cylinder is

$$\boldsymbol{r}(u,v,w) = \begin{bmatrix} r_{s1}(u,v) \\ r_{s2}(u,v) \\ w \end{bmatrix}, \quad u \in [a,b], v \in [c,d], w \in [e,f]$$

where a, b, c, and d are the bounds on the surface and e and f are the bounds on how much it should be extended

- A solid of revolution is defined by a profile *surface* and the bounds on the angle it should be rotated by around an axis. The solid is the volume that the surface sweeps as it is rotated. For a profile surface $\boldsymbol{r}_p(u,v) = \left(r_{p1}(u,v), 0, r_{p2}(u,v)\right)$ in the (x, z) plane, the parametric representation of the solid of revolution is

$$\boldsymbol{r}(u,v,w) = \begin{bmatrix} r_{p1}(u,v)\cos(w) \\ r_{p1}(u,v)\sin(w) \\ r_{p2}(u,v) \end{bmatrix}, \quad u \in [a,b], v \in [c,d], w \in [e,f]$$

For a full revolution, set $e = 0$ and $f = 2\pi$

- Diversion: Parametrization of triangles and tetrahedrons
  - To parametrize a triangle, first parametrize one of its sides. Then, add a second parameter multiplied by a vector taking you from the opposite corner to that side. In other words, for a triangle with sides (a,b), (c,d), and (e,f), you can parametrize the line from (a,b) to (c,d) as

$$\boldsymbol{r}_{line}(u) = \begin{bmatrix} a \\ b \end{bmatrix} + u\left(\begin{bmatrix} c \\ d \end{bmatrix} - \begin{bmatrix} a \\ b \end{bmatrix}\right), \quad u \in [0,1]$$

Then, you can parametrize the triangle as:

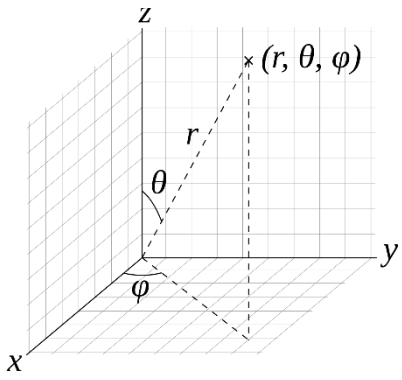
$$\boldsymbol{r}_{triangle}(u,v) = \begin{bmatrix} e \\ f \end{bmatrix} + v\left(\boldsymbol{r}_{line}(u) - \begin{bmatrix} e \\ f \end{bmatrix}\right), \quad u \in [0,1], v \in [0,1]$$

  - To parametrize a tetrahedron, parametrize one of its edges, which is a triangle, and then add a third parameter multiplied by a vector taking you from the opposite corner to that edge. Adding the point (g,h) to the previous example, the tetrahedron spanned by the four points can be parametrized by

$$\boldsymbol{r}_{tetra} = \begin{bmatrix} g \\ h \end{bmatrix} + w\left(\boldsymbol{r}_{triangle}(u,v) - \begin{bmatrix} g \\ h \end{bmatrix}\right), \quad u \in [0,1], v \in [0,1], w \in [0,1]$$

Spheres and ellipsoids

- Spherical coordinates:
  - Spherical coordinates are an alternative to standard Cartesian coordinates
  - Points in space are defined by three values:
    - A *radial distance* r
    - A *polar angle* θ
    - An *azimuthal angle* ϕ
  - To convert from spherical coordinates to standard coordinates,



$$x = r\sin(\theta)\cos(\phi)$$
$$y = r\sin(\theta)\sin(\phi)$$
$$z = r\cos(\theta)$$

- The general equation for a sphere centered at the origin is

$$x^2 + y^2 + z^2 = R^2$$

  where R is the radius of the sphere. To move its center to (c_1, c_2, c_3), replace x with $(x - c_1)$, y with $(y - c_2)$, and z with $(z - c_3)$

- The conversion from spherical coordinates to regular coordinates can be used to parametrize sphere surfaces and solid spheres
  - For a sphere surface with center at the origin:

$$\boldsymbol{r}(u, v) = \begin{bmatrix} R\sin(u)\cos(v) \\ R\sin(u)\sin(v) \\ R\cos(u) \end{bmatrix}, \quad u \in [0, \pi], v \in [0, 2\pi]$$

    where R is the radius of the sphere
  - For a solid sphere with center at the origin:

$$\boldsymbol{r}(u, v, w) = \begin{bmatrix} Ru\sin(v)\cos(w) \\ Ru\sin(v)\sin(w) \\ Ru\cos(v) \end{bmatrix}, \quad u \in [0,1], v \in [0, \pi], w \in [0, 2\pi]$$

    where R is again the radius of the sphere. This time, it is multiplied by a parameter, so every point in the interior of the sphere is also included
  - To move the sphere somewhere else, just add a displacement vector

- Ellipsoids are generalized versions of spheres. The general equation for an ellipsoid centered at the origin is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

  where *a*, *b*, and *c* are the semi-axes of the ellipsoid. To move its center somewhere else, make the same substitutions as with the sphere

- Ellipsoids are parametrized similarly to spheres
  - For an ellipsoid surface with center at the origin,

$$\boldsymbol{r}(u, v) = \begin{bmatrix} a\sin(u)\cos(v) \\ b\sin(u)\sin(v) \\ c\cos(u) \end{bmatrix}, \quad u \in [0, \pi], v \in [0, 2\pi]$$

    where *a*, *b*, and *c* are again the semi-axes
  - For a solid ellipsoid with center at the origin,

$$\boldsymbol{r}(u, v, w) = \begin{bmatrix} au\sin(v)\cos(w) \\ bu\sin(v)\sin(w) \\ cu\cos(v) \end{bmatrix}, \quad u \in [0,1], v \in [0, \pi], w \in [0, 2\pi]$$

  - To move the ellipsoid somewhere else, add a displacement vector

Vector fields

Basics

- A vector field attaches a vector to every point in the plane or in space. In the plane, this is written as

$$\boldsymbol{V}(x,y) = \begin{bmatrix} V_1(x,y) \\ V_2(x,y) \end{bmatrix}$$

and in space,

$$\boldsymbol{V}(x,y,z) = \begin{bmatrix} V_1(x,y,z) \\ V_2(x,y,z) \\ V_3(x,y,z) \end{bmatrix}$$

- The gradient of a function of two variables can be thought of as forming a *gradient vector field* in the plane. Similarly, the gradient of a function of three variables ($\mathbb{R}^3 \to \mathbb{R}$) forms a gradient vector field in space

- The *tangential curve integral* is best explained with a physics analogy as the work done by a vector field along a curve. This is written as

$$\int_K (\boldsymbol{V} \cdot \boldsymbol{e})\, d\mu$$

where *K* is the name of the curve, $\boldsymbol{V}$ is the vector field, and $\boldsymbol{e}$ is a unit vector in the direction of motion. If you take the usual steps, the integral simplifies to a more useful form:

$$\int_K (\boldsymbol{V} \cdot \boldsymbol{e})\, d\mu = \int_a^b \boldsymbol{V}\big(\boldsymbol{r}(u)\big) \cdot \boldsymbol{r}'(u)\, du$$

where *a* and *b* are the parameter bounds on *u*, and $\boldsymbol{V}\big(\boldsymbol{r}(u)\big)$ is the vector field applied on $\boldsymbol{r}(u)$, or $\boldsymbol{V}\big(r_1(u), r_2(u), r_3(u)\big)$

- A tangential curve integral can be found much more easily if the vector field being used is a gradient vector field. If $\boldsymbol{V} = \boldsymbol{\nabla} f$ and the curve starts at the point *P* and ends at the point *Q*,

$$\int_K (\boldsymbol{V} \cdot \boldsymbol{e})\, d\mu = f(Q) - f(P)$$

The function f can be called an antiderivative of the vector field. Just like with regular antiderivatives, you can add a constant to obtain another antiderivative

  - The tangential curve integrals of any two curves with the same start and end points will be the same:

$$\int_{K_1} (\boldsymbol{V} \cdot \boldsymbol{e})\, d\mu = \int_{K_2} (\boldsymbol{V} \cdot \boldsymbol{e})\, d\mu$$

  - The tangential curve integral of a curve starting and ending at the same point will be 0:

$$\oint_K (\boldsymbol{V} \cdot \boldsymbol{e})\, d\mu = 0$$

    The special integral sign shows that the curve starts and ends at the same point. This type of integral is called a *circulation*

  - Again, for these to apply, the vector field must be a gradient vector field

- To find out whether a vector field is a gradient vector field, and to find an antiderivative if it is, you can use the *stair line* method

  - To find an antiderivative of a gradient vector field, you can find the tangential curve integral from the origin to a general point (x, y). Since you are assuming that you are working with a gradient vector field, you can integrate along any path that leads to this

point. The most convenient path is first along the *x* axis to the point $(x, 0)$, and then along the *y* axis to the point (x, y). Similarly, in space, you go from $(x, 0, 0)$, to $(x, y, 0)$, and then to (x, y, z)

- In the plane, this is done in the following way:

$$f(x,y) = \int_{\text{stair}} \boldsymbol{V} \cdot \boldsymbol{e}\, d\mu = \int_0^x V_1(u,0)\, du + \int_0^y V_2(x,u)\, du$$

- In space,

$$f(x,y,z) = \int_{\text{stair}} \boldsymbol{V} \cdot \boldsymbol{e}\, d\mu = \int_0^x V_1(u,0,0)\, du + \int_0^y V_2(x,u,0)\, du + \int_0^z V_3(x,y,u)\, du$$

- After doing this, if you do not know whether $\boldsymbol{V}$ is a gradient vector field, you must always find the gradient of f to verify that you get $\boldsymbol{V}$. If you do not, then $\boldsymbol{V}$ is not a gradient vector field

- The *nabla* operator $\boldsymbol{\nabla}$ used for gradients can also be used to find two quantities for a vector field in 3D space, its *divergence* and its *curl*. The operator is defined as

$$\boldsymbol{\nabla} = \begin{bmatrix} \dfrac{\partial}{\partial x} \\ \dfrac{\partial}{\partial y} \\ \dfrac{\partial}{\partial z} \end{bmatrix}$$

  - The gradient is found when this is multiplied by a scalar function:

$$\boldsymbol{\nabla} f(x,y,z) = \begin{bmatrix} \dfrac{\partial}{\partial x} \\ \dfrac{\partial}{\partial y} \\ \dfrac{\partial}{\partial z} \end{bmatrix} \cdot f(x,y,z) = \begin{bmatrix} \dfrac{\partial f}{\partial x} \\ \dfrac{\partial f}{\partial y} \\ \dfrac{\partial f}{\partial z} \end{bmatrix}$$

  - The divergence is the dot product of the nabla and a vector field:

$$\text{div}\,(\boldsymbol{V}) = \boldsymbol{\nabla} \cdot \boldsymbol{V} = \begin{bmatrix} \dfrac{\partial}{\partial x} \\ \dfrac{\partial}{\partial y} \\ \dfrac{\partial}{\partial z} \end{bmatrix} \cdot \begin{bmatrix} V_1(x,y,z) \\ V_2(x,y,z) \\ V_3(x,y,z) \end{bmatrix} = \frac{\partial V_1}{\partial x} + \frac{\partial V_2}{\partial y} + \frac{\partial V_3}{\partial z}$$

  - The curl is the cross product of the nabla and a vector field:

$$\textbf{cur l}(\boldsymbol{V}) = \boldsymbol{\nabla} \times \boldsymbol{V} = \begin{bmatrix} \dfrac{\partial}{\partial x} \\ \dfrac{\partial}{\partial y} \\ \dfrac{\partial}{\partial z} \end{bmatrix} \times \begin{bmatrix} V_1(x,y,z) \\ V_2(x,y,z) \\ V_3(x,y,z) \end{bmatrix} = \begin{bmatrix} \dfrac{\partial V_3}{\partial y} - \dfrac{\partial V_2}{\partial z} \\ \dfrac{\partial V_1}{\partial z} - \dfrac{\partial V_3}{\partial x} \\ \dfrac{\partial V_2}{\partial x} - \dfrac{\partial V_1}{\partial y} \end{bmatrix}$$

- In space, all gradient vector fields have zero curl (the zero vector), and all vector fields with zero curl are gradient vector fields:

$$\textbf{curl}(\boldsymbol{V}) = \boldsymbol{0} \Leftrightarrow \boldsymbol{V} = \boldsymbol{\nabla} f$$

So, if you are working in space, you do not need to use the stair line method to check whether a

vector field is a gradient vector field. You cannot use this for vector fields in the plane, as curl is not defined in the plane

- The divergence of a curl is always 0:

$$\text{div}(\mathbf{curl}(\boldsymbol{V})) = 0$$

- A flow curve is the path a point follows when it flows along with a vector field. It starts at the point and always has the vector field as its tangent vector:

$$\boldsymbol{r}'(u) = \boldsymbol{V}(\boldsymbol{r}(u)) \Leftrightarrow \begin{bmatrix} x'(u) \\ y'(u) \end{bmatrix} = \begin{bmatrix} V_1(\boldsymbol{r}(u)) \\ V_2(\boldsymbol{r}(u)) \end{bmatrix}$$

$$\boldsymbol{r}(0) = (x_0, y_0) \Leftrightarrow \begin{bmatrix} x(0) \\ y(0) \end{bmatrix} = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$$

The first condition is a system of differential equations, and the second is an initial value condition. By solving the differential equations and using the initial value condition, you can find $x(u)$ and $y(u)$, the coordinates of the parametric representation of the flow curve

Flux and Gauss' divergence theorem

- Flux is a measure of the flow rate of a vector field through a surface in space. It is given as the surface integral of the vector field dotted with the unit normal vector to the surface:

$$\text{Flux}(\boldsymbol{V}, F) = \int_F \boldsymbol{V} \cdot \boldsymbol{n}\, d\mu$$

where *F* is the name of the surface and $\boldsymbol{n}$ is the unit normal vector. However, similarly to the tangential curve integral, this formula becomes much more practical when we substitute the Jacobi and simplify. The unit normal vector times the Jacobi is simply the cross product of the two partial derivatives:

$$\text{Flux}(\boldsymbol{V}, F) = \int_c^d \int_a^b \boldsymbol{V}(\boldsymbol{r}(u,v)) \cdot \boldsymbol{r}'_u \times \boldsymbol{r}'_v \, du\, dv$$

$$= \int_c^d \int_a^b \boldsymbol{V}(\boldsymbol{r}(u,v)) \cdot \boldsymbol{N}(u,v)\, du\, dv$$

where $\boldsymbol{r}(u,v)$ is the parametric representation of the surface and *a*, *b*, *c*, and *d* are the parameter bounds. $\boldsymbol{N}(u,v)$ is the name usually given to the cross product normal vector

  - $\boldsymbol{N}(u,v)$ can point in either direction relative to the surface, so you always have to make sure it is pointing in the appropriate direction for the flux you are calculating. If it is pointing in the wrong direction, either just multiply it by -1 or swap the two parameters in the parametric representation and recalculate the normal vector

- Gauss' Divergence Theorem states that, given a vector field $\boldsymbol{V}$ and a solid in space Ω, the volume integral of the divergence throughout the solid is equal to the total outward flux through the boundary of the solid:

$$\int_\Omega \text{div}\,(\boldsymbol{V})\, d\mu = \text{Flux}(\boldsymbol{V}, \partial\Omega)$$

The integral on the left is a triple/volume integral, while the flux on the right is a double/surface integral. This theorem can be used to find the total flux through any *closed surface* by imagining that it is the boundary of a solid.

If $\boldsymbol{r}(u,v,w)$ is the parametric representation of the solid and $\boldsymbol{r}_b(u,v)$ is the parametric representation of its boundary, div $(\boldsymbol{V})$ must be restricted to $\boldsymbol{r}(u,v,w)$ just like any other function, and the equation becomes

$$\int_e^f \int_c^d \int_a^b \text{div}\,(\boldsymbol{V})(\boldsymbol{r}(u,v,w)) \cdot Jacobi_S\, du\, dv\, dw = \int_r^s \int_p^q \boldsymbol{V}(\boldsymbol{r}_b(u,v)) \cdot \boldsymbol{N}(u,v)\, du\, dv$$

where the letters are parameter bounds and $Jacobi_S$ is the Jacobi function for the solid. Since the right-hand side is the outward flux, $\boldsymbol{N}(u, v)$ must be the outward pointing cross product vector

- When the boundary of a solid is made of many surfaces that have to be parametrized separately, the flux on the right hand side is the sum of all the fluxes through all the surfaces. Because of this, it is usually easier to find the divergence integral, hence the usefulness of this theorem
- Sometimes, the divergence integral can be useful even for open surfaces. In this case, you have to add extra surface segments to create a closed surface, find the divergence integral in the volume enclosed by that surface, and then subtract the flux through the extra segments you added
- When a vector field is divergence free, the total flux through any closed surface is 0
  - Since the divergence of a curl is always 0, the total flux through any closed surface in a curl vector field is 0
- When a vector field has constant divergence *c*, the divergence integral simplifies to the divergence times the volume of the solid:

$$\int_{\Omega} \text{div}\,(\boldsymbol{V})\, d\mu = c \int_{\Omega} d\mu = c \cdot \text{Vol}(\Omega)$$

- When a surface flows along the flow curves for a vector field, it sweeps a spatial region with a volume that can be stated as a function of time. The rate of change of this volume at time 0 is equal to the positive flux through the surface:

$$\text{Vol}'(0) = \text{Flux}\,(\boldsymbol{V}, F)$$

- Similarly, when a solid flows along the flow curves for a vector field, the rate of change of its volume at time 0 is equal to the total outward flux through its boundary, or the divergence integral:

$$\text{Vol}'(0) = \text{Flux}(\boldsymbol{V}, \partial\Omega) = \int_{\Omega} \text{div}\,(\boldsymbol{V})\, d\mu$$

Stokes' theorem

- Stokes' theorem states that the flux of the curl of a vector field on a surface is equal to the tangential curve integral of the vector field along the boundary of the surface. This tangential curve integral will always be a circulation:

$$\text{Flux}(\textbf{curl}(\boldsymbol{V})\,, F) = \text{Circ}(\boldsymbol{V}, \partial F)$$

$$\int_F \textbf{curl}(\boldsymbol{V}) \cdot \boldsymbol{n}_F\, d\mu = \oint_{\partial F} \boldsymbol{V} \cdot \boldsymbol{e}_{\partial F}\, d\mu$$

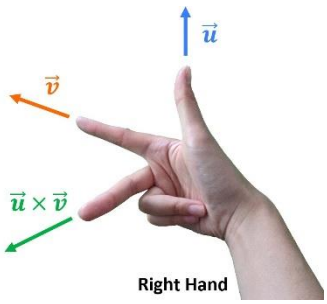
If $\boldsymbol{r}(u, v)$ is the parametric representation of the surface and $\boldsymbol{r}_b(t)$ is the one for its boundary, $\textbf{curl}(\boldsymbol{V})$ must be restricted to $\boldsymbol{r}(u, v)$ just like any other function, and the equation becomes

$$\int_c^d \int_a^b \textbf{curl}(\boldsymbol{V})\big(\boldsymbol{r}(u, v)\big) \cdot \boldsymbol{N}(u, v)\, du\, dv = \int_e^f \boldsymbol{V}\big(\boldsymbol{r}_b(t)\big) \cdot \boldsymbol{r}_b'(t)\, dt$$

where the letters are parameter bounds and $\boldsymbol{N}(u, v)$ is a cross product normal vector to the surface, as is always used to compute a flux.

The orientation of the parametric representations MUST be chosen such that the cross product $\boldsymbol{e} \times \boldsymbol{n}$ points away from the surface along the boundary. Since $\boldsymbol{r}_b'(t)$ points in the same direction as $\boldsymbol{e}$ and $\boldsymbol{N}(u, v)$ points in the same direction as $\boldsymbol{n}$, this is equivalent to the cross

product $\boldsymbol{r}_b'(t) \times \boldsymbol{N}(u,v)$ pointing away from the surface along the boundary. The direction of the cross product vector can be found using the right-hand rule; see the figure



- Similarly to Gauss' theorem, when the boundary of a surface is made of many curves that have to be parametrized separately, the tangential curve integral on the right-hand side is the sum of all the tangential curve integrals through all the curves. The parametric representations of all the curve segments must point in the same direction relative to the surface, either clockwise or counterclockwise
- Unlike Gauss' theorem, there are infinitely many surfaces that can be used for the same boundary curve. All of these will have the same curl flux
- If you are computing the cross product $\boldsymbol{r}_b' \times \boldsymbol{N}$ to check that the orientations are correct, you must make sure that the parameters in $\boldsymbol{r}_b$ are either completely different to the ones in $\boldsymbol{r}$, or that they match the ones in $\boldsymbol{r}$ in terms of their meaning in space. Otherwise, you are essentially using the same variable name in two different contexts, and the cross product will be meaningless. For example:

$$\boldsymbol{r}(u,v) = \begin{bmatrix} u\cos(v) \\ u\sin(v) \\ 0 \end{bmatrix}, \quad u \in [0,1], v \in [0,2\pi]$$

This gives a disc in space. The boundary should either use a new parameter, or it should use the parameter v in the same way $\boldsymbol{r}(u,v)$ uses it as an angle that takes you around the circle. In the latter case, $\boldsymbol{r}_b$ would have the form

$$\boldsymbol{r}_b(v) = \begin{bmatrix} \cos(v) \\ \sin(v) \\ 0 \end{bmatrix}, \quad v \in [0,2\pi]$$

Note how this was found by fixing u to 1, which is one of its boundary values. More on this in the next point

- Given the parametric representation for a surface, there is a method for finding potential representations for its boundary with matching parameters. This is done by restricting each parameter to each of the boundary values for that parameter in turn. In other words, for a surface

$$\boldsymbol{r}(u,v), \quad u \in [a,b], v \in [c,d]$$

the four following functions might be parametric representations for different segments of the surface boundary:

$$\boldsymbol{r}_{b1}(u) = \boldsymbol{r}(u,c), \quad u \in [a,b] \text{ (going from } \boldsymbol{r}(a,c) \text{ to } \boldsymbol{r}(b,c))$$
$$\boldsymbol{r}_{b2}(v) = \boldsymbol{r}(b,v), \quad v \in [c,d] \text{ (going from } \boldsymbol{r}(b,c) \text{ to } \boldsymbol{r}(b,d)$$
$$\boldsymbol{r}_{b3}(v) = \boldsymbol{r}(a,v), \quad v \in [c,d] \text{ (going from } \boldsymbol{r}(a,c) \text{ to } \boldsymbol{r}(a,d))$$
$$\boldsymbol{r}_{b4}(u) = \boldsymbol{r}(u,d), \quad u \in [a,b] \text{ (going from } \boldsymbol{r}(a,d) \text{ to } \boldsymbol{r}(b,d))$$

  - This guarantees that the parameters of the surface and the boundary match, so that the cross product is meaningful
  - Note that the first two point in one direction ($\boldsymbol{r}(a,c) \to \boldsymbol{r}(b,c) \to \boldsymbol{r}(b,d)$) and the last two point in the opposite direction ($\boldsymbol{r}(a,c) \to \boldsymbol{r}(a,d) \to \boldsymbol{r}(b,d)$). In order to use all four of them, you have to create new parameters so that they all point either clockwise or anticlockwise. However, you will rarely have to use all four of them; for example, for discs, only the one that restricts the radius to its maximum value will be a boundary curve, and it will be the only boundary curve (as in the example above)

Changelog

r1.1: Minor changes in the first section