# Math Semester 1 revision sheet

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Topic diversions from the teacher are in yellow, other topic diversions/extra additions are in green. Links to another part of the document are in blue

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# Complex numbers

## Basics

- Number sets:
  - $\mathbb{N}$ denotes the natural numbers 1, 2, 3, 4, etc.
  - $\mathbb{Z}$ denotes integers, both negative and positive, including 0
  - $\mathbb{Q}$ denotes rational numbers, numbers which can be expressed in the form $\frac{p}{q}$
    - This set is countable
  - $\mathbb{R}$ denotes all real numbers
    - This set is not countable

- A number *i* is defined as the square root of -1
- This creates a new number set of complex numbers $\mathbb{C}$. These are numbers of the form $a + ib$, where *a* and *b* are real numbers
    - $a + ib$ is called the *rectangular form* of a complex number
- Complex numbers can be plotted as points in a coordinate system, where the x-axis is the real axis and the y-axis is the imaginary axis. *a* is the *x*-coordinate and *b* is the *y*-coordinate
- In calculations, you can treat *i* like a variable, except for the fact that $i^2 = -1$
- For a complex number $z = a + ib$, *a* is called the *real part* and *b* is called the *imaginary part*
    - The real part can also be written as $Re(z)$ and the imaginary part $Im(z)$
    - Note that the imaginary part is a real number
- The absolute value of a complex number, also called the modulus, is the distance of the number from the origin. It is written as $|z|$
    - $|z| = \sqrt{a^2 + b^2}$
    - The absolute value of a difference (subtraction) between two complex numbers is the distance between the two numbers
    - The solution to the equation $|z - w| = q$ is a circle centered at *w* with radius *q*
        - If = is replaced with $\leq$ then the solution also includes the area inside the circle
        - An absolute value equation containing only real numbers can be solved as such ($\vee$ means "or"):

$$|x - a| = b$$
$$x - a = b \ \vee \ x - a = -b$$
$$x = b + a \ \vee \ x = -b + a$$

- The *conjugate* of a complex number, written $\bar{z}$, is $z$ with the sign of the imaginary part flipped. If $z = a + ib$, then $\bar{z} = a - ib$
    - The following rules apply to complex conjugates:

$$\overline{z_1 + z_2} = \bar{z_1} + \bar{z_2}$$
$$\overline{z_1 \cdot z_2} = \bar{z_1} \cdot \bar{z_2}$$
$$z \cdot \bar{z} = |z|^2$$

    - The last rule is useful for evaluating fractions with complex numbers. When there is a complex number in the denominator, expand the fraction (multiply both parts) by the conjugate of the denominator. This turns the denominator into a real number, letting you restate the whole number in rectangular form. For example:

$$\frac{1 - i}{1 + 2i} = \frac{(1 - i)(1 - 2i)}{(1 + 2i)(1 - 2i)} = \frac{1 - i - 2i + 2i^2}{|1 + 2i|^2} = \frac{-1 - 3i}{5} = -\frac{1}{5} - \frac{3}{5}i$$

- Diversion: Set notation
    - ]0,5] denotes the set with all real numbers from 0 to 5, excluding 0 and including 5
    - Set builder notation is another type of set notation. Here, you must specify the name of a variable, the number set that you start with, and then the conditions that limit the set. ]0,5] can be written in set builder notation as $\{x \in \mathbb{R} | 0 < x \leq 5\}$

## Polar and exponential forms

- Diversion: sin and cos of basic angles
    - sin and cos of important angles can be memorized in this way:
        - The order of the angles is $0, \frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}, \frac{\pi}{2}$
        - For these angles, the sin is $\frac{\sqrt{0}}{2}, \frac{\sqrt{1}}{2}, \frac{\sqrt{2}}{2}, \frac{\sqrt{3}}{2}$, and $\frac{\sqrt{4}}{2}$. $\frac{\sqrt{0}}{2}$ simplifies to 0, $\frac{\sqrt{1}}{2}$ simplifies to $\frac{1}{2}$, and $\frac{\sqrt{4}}{2}$ simplifies to 1.
        - The cosines of these angles are the same values, but in the opposite order
    - sin and cos of similar angles in other quadrants of the unit circle can be found using symmetry
- The *polar coordinates* of a complex number are given as ($|z|$, $v$) where $|z|$ is the absolute value and $v$ is the *argument*. The argument is the counterclockwise angle from the positive $x$-axis to the line going from the origin to $z$
    - The argument can be written as arg($z$)
    - Every complex number has infinitely many arguments, obtained by adding or subtracting $2\pi$
        - The *main argument* is the one in the range $]-\pi, \pi]$
- Arguments are found by solving the following two equations:

$$\cos(v) = \frac{a}{|z|}$$

$$\sin(v) = \frac{b}{|z|}$$

where $a$ and $b$ are the real and imaginary parts of the number respectively

    - Alternatively, you can solve only one of these equations and obtain two potential main arguments, then select the correct one based on which is in the correct quadrant
    - By rearranging these equations, you can convert from polar to rectangular form:

$$a = |z|\cos(v)$$
$$b = |z|\sin(v)$$

- Absolute values and arguments obey the following rules:

$$|z_1 \cdot z_2| = |z_1| \cdot |z_2|$$
$$\arg(z_1 \cdot z_2) = \arg(z_1) + \arg(z_2)$$

- The *exponential form* of a complex number is written as $z = |z|e^{iv}$ where, again, $|z|$ is the absolute value and $v$ is the argument
    - This can be thought of as a more useful way to write the polar form
    - When performing calculations using the exponential form, you can use all laws of exponents. This makes the rules for absolute values and arguments obvious. For example:

$$\frac{1}{2}e^{i\frac{\pi}{2}} \cdot 4e^{i\frac{\pi}{3}}$$
$$= 2(e^{i\frac{\pi}{2}} \cdot e^{i\frac{\pi}{3}})$$
$$= 2e^{i\frac{\pi}{2}+i\frac{\pi}{3}}$$

$$= 2e^{i\frac{5\pi}{6}}$$

- This form is made possible by Euler's formula, which states:
$$e^{iv} = \cos(v) + i\sin(v)$$

- *e* to the power of a complex number, called the complex exponential function, can be written in proper exponential form as such:
$$e^z = e^{a+ib} = e^a e^{ib}$$
which means that $e^a$ is the absolute value and *b* is the argument
  - For a complex number *z* and a real number *t*:
$$\overline{e^{zt}} = e^{\bar{z}t}$$

- An equation involving the complex exponential function can be solved by writing both sides in exponential form, and then equating the absolute values and arguments. However, you must include $+p2\pi$, where $p \in \mathbb{Z}$, in the argument on the right-hand side in order to account for all possible arguments. For example:
$$e^z = 2\sqrt{3} - 2i$$
...
$$e^x e^{iy} = 4e^{-i\frac{\pi}{6}+p2\pi}$$
$$e^x = 4 \text{ and } y = -\frac{\pi}{6} + p2\pi$$
$$z = x + iy = \ln(4) + i\left(-\frac{\pi}{6} + p2\pi\right),\ p \in \mathbb{Z}$$

- A binomial equation is an equation of the form $z^n = w$. These equations can be solved using the same general method as for equations with the complex exponential function. In this case, the absolute value on the left-hand side (LHS) will be to the power of *n*, and the argument on the LHS will be multiplied by *n*: $\left(|z|e^{iv}\right)^n = |z|^n e^{ivn}$
  - Diversion: When solving these equations, you may run into a situation where you have to solve something like $\cos(2x) = \frac{1}{2},\ 0 \le x \le 2\pi$
    - Here, you must state the domain restriction in terms of 2*x* by multiplying or dividing it on all sides like an equation
$$0 \le x \le 2\pi \Leftrightarrow 0 \le 2x \le 4\pi$$
    - Then, you find the solutions of 2*x* within this domain, and then divide all of them by 2 to find the solutions of x
  - It is also possible to solve binomial equations using rectangular form, in which case you must expand the brackets, then equate real and imaginary parts. However, this is only viable if *n* is a small number

## Polynomials

- A polynomial takes the form $P(z) = a_n z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0$
  - *z* can be a complex variable, and the *a*s can be complex constants
  - The highest exponent of *z* defines the degree of the polynomial
- A number $z_0$ is called a root if $P(z_0) = 0$
  - The multiplicity of a root is the number of times it appears when finding the roots
- The Fundamental Theorem of Algebra states that every polynomial of degree *n* has *n* roots when multiplicity is taken into account

- Given a polynomial P($z$) of degree $n > 2$ and a root $z_0$ for that polynomial, you can find a polynomial Q($z$) of degree $n$-1 such that $P(z) = (z - z_0)Q(z)$. This is done using the following method (the Theorem of Descent):

| Coefficients of P($z$) | Coefficients of Q($z$) |
|---|---|
| $a_n$ | |
| $a_{n-1}$ | $b_{n-1} = a_n$ |
| $a_{n-2}$ | $b_{n-2} = a_{n-1} + z_0 b_{n-1}$ |
| ... | ... |
| $a_k$ | $b_k = a_{k+1} + z_0 b_{k+1}$ |

  - This can be used to solve or factorize polynomials with degree higher than 2 if you already know $n$-2 roots

- For real polynomials (polynomials where all coefficients are real), roots come in conjugate pairs. So, if $z$ is a root, then $\bar{z}$ will also be a root. This does not apply if any of the coefficients are complex

## Differentiation

- An *epsilon function* $\varepsilon(x)$ is any function that fulfills $\varepsilon(0) = 0$ and tends to 0 as $x$ tends to 0
  - $\varepsilon(t - t_0)$ tends to 0 as $t$ tends to $t_0$
- The derivative of a function can be found "from first principles" using the following formula:

$$f'(x_0) = \lim_{x \to x_0} \left( \frac{f(x) - f(x_0)}{x - x_0} \right)$$

  where $\lim_{x \to x_0}$ means "the limit as $x$ tends to $x_0$"

- A function $f$ is differentiable at a point $t_0$ if it can be described in the following way:

$$f(t) = f(t_0) + c(t - t_0) + \varepsilon(t - t_0) \cdot (t - t_0)$$

  $c$ is the derivative at $t_0$, $f'(t_0)$

- A complex function of a real variable ($\mathbb{R} \to \mathbb{C}$) can be differentiated by differentiating the real and imaginary parts separately
- The following rules apply to derivatives:
  - $\big(f(x) \pm g(x)\big)' = f'(x) \pm g'(x)$
  - $\big(kf(x)\big)' = kf'(x)$
  - $\big(f(x)g(x)\big)' = f'(x)g(x) + f(x)g'(x)$ (Product rule)
  - $f\big(g(x)\big)' = f'\big(g(x)\big) \cdot g'(x)$ (Chain rule)
  - $\left(\frac{f(x)}{g(x)}\right)' = \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}$ (Quotient rule, follows from the product rule)
- The chain rule also applies to $i$ and to complex numbers in the complex exponential function:

$$\big(e^{ix}\big)' = ie^{ix}$$
$$(e^{zx})' = ze^{zx}$$

- Given $y = f(x)$, the following rule applies to the derivative of an inverse function:

$$f^{-1\prime}(y) = \frac{1}{f'(x)}$$

- A smooth function is a function that can be differentiated infinitely many times

## Taylor approximations

- The Taylor approximation of a function is given by:

$$P_n(x) = f(x_0) + f'(x_0) \cdot (x - x_0) + \frac{1}{2} f''(x_0) \cdot (x - x_0)^2 + \cdots + \frac{1}{n!} f^{(n)}(x_0) \cdot (x - x_0)^n$$

  - $x_0$ is the development point of the approximation. The approximation is only valid in a small area around this point
  - $n$ is the degree of the approximation. The first term in the formula is the 0th degree approximation, the first two terms constitute the first degree approximation, and so on. A higher degree increases the number of terms in the approximation and makes it more accurate

- The remainder function/error term of a Taylor approximation gives the difference between the real function value and the value of the Taylor approximation:

$$R_n(x) = f(x) - P_n(x)$$

  The remainder function is given by:

$$R_n(x) = \frac{1}{(n+1)!} f^{(n+1)}(c) \cdot (x - x_0)^{n+1}$$

  where $c$ is some number in the range $[x_0, x]$

  - By picking the value of $c$ that maximizes $f^{(n+1)}(c)$, you can find the maximum possible value of the remainder function for calculations

- Taylor's limit formula states that:

$$f(x) = P_n(x) + \varepsilon(x - x_0) \cdot (x - x_0)^n$$

  where $P_n(x)$ is the $n$th degree function approximation and $\varepsilon$ is an epsilon function

  - You can use this to calculate limits of functions of the form $\frac{f(x)}{x^n}$. You have to state the numerator as a Taylor limit function with development point at the limit you are calculating. For example:

$$\lim_{x \to 0} \left( \frac{\sin(x)}{x} \right) = \lim_{x \to 0} \left( \frac{x + \varepsilon(x - 0) \cdot (x - 0)}{x} \right)$$
$$= \lim_{x \to 0} \left( \frac{x + \varepsilon(x) \cdot x}{x} \right) = \lim_{x \to 0} (1 + \varepsilon(x)) = 1$$

  - You must use the degree of the approximation that matches the power that $x$ is raised to in the denominator, e.g. if the denominator is $x^3$, you must use the third degree approximation

# Matrices and vectors

## Systems of linear equations

- If you have a single equation with more than one unknown, you can pick one of the unknowns to be defined in terms of the others. The others become free parameters, which are variables that can take any value. When a variable becomes a free parameter, it gets a new name, usually *s* or *t*
    - The solution to an equation with free parameters can be written with vectors in a form called *parametric form*. For example, in the equation $x_1 = 4 + 2x_2 - 5x_3$, $x_2$ and $x_3$ become the free parameters *s* and *t*:
    $$x_1 = 4 + 2s - 5t$$
    $$x_2 = s$$
    $$x_3 = t$$
    The parametric form of this solution is
    $$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -5 \\ 0 \\ 1 \end{bmatrix}$$
- When you have a system of more than one equation, you can construct a coefficient matrix **A** and a right-hand-side vector **b**. The coefficient matrix is made up of the coefficients of the variables, with each column containing the coefficients of one variable, and each row representing one equation. The RHS vector has the right-hand-sides of all the equations. These two can be combined to form the *augmented matrix* **T**, usually with a dividing line between them. For example, given the following system of equations,
    $$-x_2 + x_3 = 2$$
    $$2x_1 + 4x_2 - 2x_3 = 2$$
    $$3x_1 + 4x_2 + x_3 = 9$$
    the coefficient matrix and RHS vector are
    $$\boldsymbol{A} = \begin{bmatrix} 0 & -1 & 1 \\ 2 & 4 & -2 \\ 3 & 4 & 1 \end{bmatrix}, \quad \boldsymbol{b} = \begin{bmatrix} 2 \\ 2 \\ 9 \end{bmatrix}$$
    and the augmented matrix is
    $$\boldsymbol{T} = \left[\begin{array}{ccc|c} 0 & -1 & 1 & 2 \\ 2 & 4 & -2 & 2 \\ 3 & 4 & 1 & 9 \end{array}\right]$$
- Once you have the augmented matrix, you can solve the system of equations by performing certain *row operations* on the matrix. These row operations do not change the solution to the system of equations. There are three of them:
    - Interchanging two rows, written as e.g. $R_1 \leftrightarrow R_2$
    - Multiplying a row by a number different from 0, written as e.g. $R_1 \cdot \frac{1}{2}$
    - Adding a multiple of one row to another, written as e.g. $R_1 - 2R_2$
- The goal of performing these row operations is to bring the matrix into *reduced row echelon form*. This is a form satisfying the following conditions:
    - The first number in a row that is not a 0 must be a 1. This is called the *leading one* of the row
    - For two consecutive rows with leading ones, the lower row's leading one must be further to the right than the one in the upper row
    - In a column with a leading 1, all the other numbers must be 0

- All rows containing only 0s must be at the bottom of the matrix

- After this is done, you can convert the matrix back into equations and read the answer. You may have to turn some variables into free parameters
- This whole process is called Gauss-Jordan elimination
- The number of leading ones in the reduced row echelon form is called the *rank* of the matrix, written as $\rho(\boldsymbol{T})$
    - If the rank of the coefficient matrix is less than the rank of the augmented matrix, or, in other words, if one of the leading ones in the augmented matrix is in the RHS vector, then the system of equations has no solutions. For example:
    $$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$$
    - If the rank of the coefficient matrix is the same as the rank of the augmented matrix (all the leading ones of the augmented matrix are in the coefficient matrix part, and there is a leading one for each column), then the system of equations has one unique solution. For example:
    $$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$
    - Otherwise, the rank can tell you the number of free parameters you will have. If the number of variables is *n*, then the number of free parameters is given by
    $$n - \rho(\boldsymbol{T})$$
- If a system of equations has only 0s in the right hand sides of the equations, the system is called *homogeneous*. If any of the right hand sides is different from 0, it is called *inhomogeneous*
    - When you solve an inhomogeneous system with free parameters and obtain the parametric form of the solution, the constant vector in the solution (the one not multiplied by a free parameter) is a *particular solution* to the system. The rest of the solution, consisting of vectors multiplied by free parameters, is the solution to the corresponding homogeneous system of equations. This is called the structural theorem, summarized by the following equation:
    $$L_{inhom} = x_0 + L_{hom}$$
    where $L_{inhom}$ is the solution to the inhomogeneous system, $x_0$ is a particular solution to the inhomogeneous system, and $L_{hom}$ is the solution to the homogeneous system

## Matrix algebra

- A matrix with *n* rows and *m* columns is said to be an $n \times m$ matrix
- Addition:
    - This can only be done if the matrices have the same dimensions
    - It is done by simply adding each element in the first matrix to the corresponding element in the second (similarly to vector addition)
- Multiplication by a scalar:
    - You simply multiply each element in the matrix by the scalar (similarly to vector multiplication by a scalar)

- Matrix-vector product:
  - This is done in the following way:
$$\begin{bmatrix} x_1 & x_2 & x_3 \\ x_4 & x_5 & x_6 \end{bmatrix} \cdot \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} x_1 a + x_2 b + x_3 c \\ x_4 a + x_5 b + x_6 c \end{bmatrix}$$
  - You can think of this as rotating the vector 90° counterclockwise, placing it on top of the matrix, and multiplying each element in the matrix by the vector element above it. Then, all the elements of each row are added together to form a vector
  - This works the same way with other dimensions of vectors and matrices. However, the number of columns in the matrix must always equal the number of rows in the vector
  - You always write the matrix first and the vector second
- Matrix-matrix product:
  - This is basically a version of the matrix-vector product where each column in the second matrix is treated like a separate vector. You evaluate the matrix-vector product for each of these vectors, then arrange the answers in columns in the final matrix. For example,
$$\begin{bmatrix} x_1 & x_2 & x_3 \\ x_4 & x_5 & x_6 \end{bmatrix} \cdot \begin{bmatrix} a & d \\ b & e \\ c & f \end{bmatrix} = \begin{bmatrix} x_1 a + x_2 b + x_3 c & x_1 d + x_2 e + x_3 f \\ x_4 a + x_5 b + x_6 c & x_4 d + x_5 e + x_6 f \end{bmatrix}$$
  - Similarly to the matrix-vector product, the number of columns in the first matrix must equal the number of rows in the second
  - This operation is not commutative, which means that it matters what order you write the two matrices in. Moreover, in algebra, it matters whether you multiply both sides of the equation "from the left" (premultiplying) or "from the right" (postmultiplying)
- Transpose:
  - The transpose of a matrix is the matrix with rows and columns interchanged, and it is written as $\boldsymbol{A}^T$. For example:
$$\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}^T = \begin{bmatrix} a & d \\ b & e \\ c & f \end{bmatrix}$$
  - The following law applies to the matrix transpose:
$$(\boldsymbol{A} \cdot \boldsymbol{B})^T = \boldsymbol{B}^T \cdot \boldsymbol{A}^T$$
- A matrix-vector equation of the form $\boldsymbol{Ax} = \boldsymbol{b}$, where **A** is a matrix and **b** is a vector, can be solved for the vector **x** using Gauss-Jordan elimination. If you write out the equation using the values of **A** and **b**, split **x** into variables, and evaluate the matrix-vector product, you can see that this is the same as a system of linear equations:
$$\begin{bmatrix} 1 & -1 \\ -2 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ -4 \end{bmatrix} \Leftrightarrow$$
$$x_1 - x_2 = 3$$
$$-2x_1 + 3x_2 = -4$$
  Therefore, if you create an augmented matrix where **A** is the coefficient matrix and **b** is the RHS, then find its reduced row echelon form, you will find the vector **x**.
- There are three special concepts that only apply to square matrices:
  - An identity matrix **I** is a square matrix with 1s in the main diagonal (the diagonal line going from top left to bottom right) and 0s everywhere else. This matrix is neutral with respect to matrix-vector and matrix-matrix multiplication

- The inverse matrix of a matrix **A** is written as $\mathbf{A}^{-1}$ and fulfills the following property:

$$\boldsymbol{A} \cdot \boldsymbol{A}^{-1} = \boldsymbol{I}$$

  - An inverse matrix can be found by performing Gauss-Jordan elimination on this equation. This is done in the same way as with a matrix-vector product equation, with the key difference being that the RHS is now a matrix instead of a vector. However, the process is the same. Your goal is to go from having an identity matrix on the RHS to having one on the LHS. For example:

$$\boldsymbol{T} = [\boldsymbol{A}|\boldsymbol{I}] = \left[\begin{array}{cc|cc} 1 & -1 & 1 & 0 \\ -2 & 3 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{cc|cc} 1 & 0 & 3 & 1 \\ 0 & 1 & 2 & 1 \end{array}\right] = [\boldsymbol{I}|\boldsymbol{A}^{-1}]$$

    where ~ symbolizes multiple row operations
  - There is a special formula for the inverse of a 2 × 2 matrix that uses the determinant
  - Not all square matrices have an inverse. For a matrix to have an inverse, it must have full rank, which means its reduced row form must have a leading one for each column. A matrix like this is called *regular*, and other matrices are called *singular*
  - $(\boldsymbol{A}^{-1})^{-1} = \boldsymbol{A}$
  - Matrix inverses provide a new method for solving equations of the form $\boldsymbol{Ax} = \boldsymbol{b}$, if the matrix is square. Simply multiply both sides of the equation by $\boldsymbol{A}^{-1}$ *from the left* (premultiply) to obtain the following:

$$\boldsymbol{A}^{-1}\boldsymbol{Ax} = \boldsymbol{A}^{-1}\boldsymbol{b}$$
$$\boldsymbol{x} = \boldsymbol{A}^{-1}\boldsymbol{b}$$

    This can also be done if **x** and **b** are matrices

- The determinant of a square matrix, written $\det(\boldsymbol{A})$, is a special number obtained in the following way:
  - $\det([a]) = a$
  - $\det\left(\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}\right) = a_1 b_2 - a_2 b_1$
  - For matrices with higher dimensions, there are two ways to find the determinant. One is to follow a process to write the determinant in terms of the determinants of 2×2 matrices, which you can then compute. The other is using Gauss-Jordan elimination and some properties of determinants. First, I will show the first method by defining the process, then giving an example and some hints. The process is as follows:
    - $\widehat{\boldsymbol{A}_{ij}}$ is a submatrix of **A**. It is the matrix obtained when you delete row *i* and column *j* from **A**
    - You can obtain the determinant in terms of the determinants of smaller matrices by "expanding" along an arbitrary row:

$$\det(\boldsymbol{A}) = \sum_{j=1}^{n} (-1)^{r+j} a_{rj} \det\left(\widehat{\boldsymbol{A}_{rj}}\right)$$

      where *r* is the row index, *j* is the column index, and $a_{rj}$ is the number at row *r* and column *j*

- You can also expand along an arbitrary column:

$$\det(\boldsymbol{A}) = \sum_{i=1}^{n} (-1)^{i+s} a_{is} \det\left(\widehat{\boldsymbol{A}_{is}}\right)$$

where *i* is the row index, *s* is the column index, and $a_{is}$ is the number at row *i* and column *s*

- This can be illustrated using the following example. Here, I expand along the first column, and I show which rows and columns I delete by striking them through:

$$\det\left(\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}\right)$$

$$= a_1 \det\left(\begin{bmatrix} \cancel{a_1} & \cancel{b_1} & \cancel{c_1} \\ \cancel{a_2} & b_2 & c_2 \\ \cancel{a_3} & b_3 & c_3 \end{bmatrix}\right) - a_2 \det\left(\begin{bmatrix} \cancel{a_1} & b_1 & c_1 \\ \cancel{a_2} & \cancel{b_2} & \cancel{c_2} \\ \cancel{a_3} & b_3 & c_3 \end{bmatrix}\right)$$

$$+ a_3 \det\left(\begin{bmatrix} \cancel{a_1} & b_1 & c_1 \\ \cancel{a_2} & b_2 & c_2 \\ \cancel{a_3} & \cancel{b_3} & \cancel{c_3} \end{bmatrix}\right.$$

$$= a_1 \det\left(\begin{bmatrix} b_2 & c_2 \\ b_3 & c_3 \end{bmatrix}\right) - a_2 \det\left(\begin{bmatrix} b_1 & c_1 \\ b_3 & c_3 \end{bmatrix}\right) + a_3 \det\left(\begin{bmatrix} b_1 & c_1 \\ b_2 & c_2 \end{bmatrix}\right)$$

- The sign of each term can be found from a diagram like this:

$$\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$$

- Always expand along the row or column that contains the most 0s in order to make the calculation easier
- The determinant has the following properties:
  - $\det(\boldsymbol{A}^T) = \det(\boldsymbol{A})$
  - If **A** has a row or column that contains only 0s, $\det(\boldsymbol{A}) = 0$
  - The determinant of a triangular matrix (a matrix where all the elements either above or below the main diagonal are 0) is the product of the elements in the main diagonal
  - $\det(\boldsymbol{I}) = 1$
  - Row operations modify the determinant:
    - If you swap two rows, the determinant changes sign
    - If a row is multiplied by a number, the determinant is multiplied by the same number
    - If a multiple of one row is added to another row, the determinant is unchanged
  - The determinant of a triangular matrix, along with the ways in which row operations modify the determinant, provide the second method for finding the determinant of a large matrix. Convert it into a triangular matrix (or the fully reduced form if necessary) using row operations, find the determinant by multiplying the elements in the main diagonal, write this determinant in terms of the original, and solve

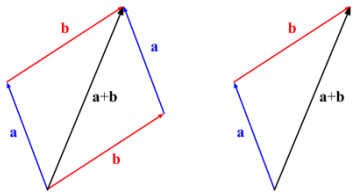
- If **A** is singular (not full rank), $\det(\boldsymbol{A}) = 0$
  If **A** is regular (full rank), $\det(\boldsymbol{A}) \neq 0$
- $\det(\boldsymbol{A} \cdot \boldsymbol{B}) = \det(\boldsymbol{A}) \det(\boldsymbol{B})$
- $\det(\boldsymbol{A}^{-1}) = \det(\boldsymbol{A})^{-1}$

▪ Diversion: For 2×2 matrices, there is a special formula for the inverse matrix that uses the determinant:

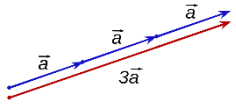
$$\boldsymbol{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$
$$\boldsymbol{A}^{-1} = \frac{1}{\det(\boldsymbol{A})} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

## Geometric vectors

- The vector between two points A and B in space is represented as $\overrightarrow{AB}$
- The origin O is a reference point for geometric vectors
  - The position vector of a point P is the vector from the origin to that point, $\overrightarrow{OP}$
- Vector addition is done geometrically by putting the tail of the second vector at the head of the first, then going from the tail of the first vector to the head of the second (see the figure)




- Multiplying a vector by a scalar is done by having a vector in the same direction, but with its length multiplied by the scalar. If the scalar is negative, the vector is in the opposite direction
- These two operations have the following properties:
  - $\boldsymbol{u} + \boldsymbol{v} = \boldsymbol{v} + \boldsymbol{u}$ (Commutative law)
  - $(\boldsymbol{u} + \boldsymbol{v}) + \boldsymbol{w} = \boldsymbol{u} + (\boldsymbol{v} + \boldsymbol{w})$ (Associative law)
  - $\boldsymbol{u} + \boldsymbol{0} = \boldsymbol{u}$ (Neutral element)
  - $\boldsymbol{u} + (-\boldsymbol{u}) = \boldsymbol{0}$ (Opposite element)
  - $k_1(k_2\boldsymbol{u}) = (k_1 k_2)\boldsymbol{u}$ (Associative law)
  - $(k_1 + k_2)\boldsymbol{u} = k_1\boldsymbol{u} + k_2\boldsymbol{u}$ (Distributive law)
  - $k(\boldsymbol{u} + \boldsymbol{v}) = k\boldsymbol{u} + k\boldsymbol{v}$ (Another distributive law)
  - $1 \cdot \boldsymbol{u} = \boldsymbol{u}$ (Neutral element)




- Vector coordinates are always with respect to a *basis*. Normally, this is the *standard basis*, $e = (\boldsymbol{i}, \boldsymbol{j})$, where **i** and **j** are called the *basis vectors*
  - **i** and **j** both have length 1
  - $\boldsymbol{j} = \hat{\boldsymbol{\imath}}$ (**j** is perpendicular to **i**)
  - The basis is written as a subscript on the left. For example, **v** with *e*-coordinates is written as ${}_e\boldsymbol{v}$
  - Vector coordinates can either be written as ${}_e\boldsymbol{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ or as ${}_e\boldsymbol{v} = (v_1, v_2)$ (tuple form)
  - In 3D space, a third base vector **k** and a third coordinate are needed

- Vector coordinates imply a *linear combination* of the basis vectors. This means that every basis vector is multiplied by the corresponding coordinate and added together:

$$_a\boldsymbol{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = v_1\boldsymbol{a}_1 + v_2\boldsymbol{a}_2 + v_3\boldsymbol{a}_3$$

  where $\boldsymbol{a}_1, \boldsymbol{a}_2$, and $\boldsymbol{a}_3$ are the basis vectors for the *a*-basis and $v_1, v_2$, and $v_3$ are the *a*-coordinates of **v**

  - If you have a basis, any vector in the space can be written with respect to that basis, or as a linear combination of the basis vectors

- Linear dependence and independence:
  - Two vectors in a plane are called *linearly independent* if they are not parallel. In space, three vectors are linearly independent if they are not coplanar
  - In a plane, you need two linearly independent vectors to make a basis. In space, you need three. In general, if *n* is the dimension of the space, you need *n* linearly independent vectors
  - You can use the coordinate matrix of a set of vectors (the matrix with the vectors arranged in columns) to find out whether the vectors are linearly independent:
    - If the coordinate matrix is regular, the determinant of the matrix is not 0 and the vectors are linearly independent, which means they can constitute a basis
    - If the coordinate matrix is singular, the determinant of the matrix is 0 and the vectors are linearly dependent, which means they cannot constitute a basis

- If you have a basis *e* and a basis $a = (\boldsymbol{a}_1, \boldsymbol{a}_2)$, and you want to change coordinates from one to the other, you need the *change of basis matrices*. One of these consists of the *e*-coordinates of the *a*-basis vectors, arranged in columns:

$$_e\boldsymbol{M}_a = [\boldsymbol{a}_1 \boldsymbol{a}_2]$$

  - Just like with vectors, the matrix $_e\boldsymbol{M}_a$ consists of *e*-coordinates, so the subscript *e* is on the left. The right subscript tells you what basis vectors are being represented
  - This matrix, when multiplied by a vector, converts from *a*-coordinates to *e*-coordinates. Subscripts next to each other cancel out:

$$_e\boldsymbol{M}_a \cdot {}_a\boldsymbol{v} = {}_e\boldsymbol{v}$$

  - However, you usually want to convert the other way around, from *e*-coordinates to *a*-coordinates. To do this, you need the other change of basis matrix $_a\boldsymbol{M}_e$. This is simply the inverse of $_e\boldsymbol{M}_a$:

$$_a\boldsymbol{M}_e = {}_e\boldsymbol{M}_a^{-1}$$

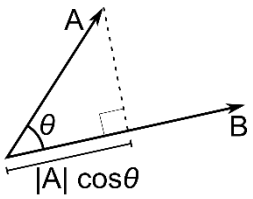
- If you are given a basis and a vector, you can find the coordinates of the vector in that basis either by using the basis shift matrix, or by solving a system of linear equations as follows:

$$k_1\boldsymbol{a}_1 + k_2\boldsymbol{a}_2 = \boldsymbol{v}$$
$$k_1\begin{bmatrix} 2 \\ 1 \end{bmatrix} + k_2\begin{bmatrix} 1 \\ -3 \end{bmatrix} = \begin{bmatrix} 7 \\ -7 \end{bmatrix}$$

  Obtaining an equation from each coordinate:

$$2k_1 + k_2 = 7$$
$$k_1 - 3k_2 = -7$$
$$\boldsymbol{T} = \left[\begin{array}{cc|c} 2 & 1 & 7 \\ 1 & -3 & -7 \end{array}\right] \sim \dots$$

- Parametric representations:
  - A line segment can be represented using vectors in a *parametric representation*. For example:
    $$\{\overrightarrow{OP} = \overrightarrow{OA} + t \cdot \boldsymbol{r}, t \in [-1,2]\}$$
    where P is a general point on the line, A is a particular point on the line, **r** is the *direction vector* of the line, and *t* is a free parameter. What this says is that you can obtain all the points on the line by starting from point A, and then going in the direction of the direction vector, or opposite to it, for as long as the constraints on the free parameter *t* allow
  - Generally, a line segment from point A to point B can be parametrized as:
    $$\{\overrightarrow{OP} = \overrightarrow{OA} + t \cdot \overrightarrow{AB}, t \in [0,1]\}$$
    This says that you can obtain all the points on the line by starting from A, and then following the vector $\overrightarrow{AB}$ until $t = 1$, which is when you have reached B
  - The same thing can be done with parallelograms in space. In this case, two direction vectors and two free parameters are required:
    $$\{\overrightarrow{OP} = \overrightarrow{OA} + s\boldsymbol{u} + t\boldsymbol{v}, s \in [0,1], t \in [0,2]\}$$
- The *magnitude* of a vector, or its length, is given by Pythagoras' theorem:
  $$|\boldsymbol{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2 + \cdots}$$
  where $a_1, a_2, a_3$ etc. are the coordinates of the vector with respect to the standard basis
- The *dot product* (also called *scalar product*) of two vectors is computed as follows:
  $$\boldsymbol{a} \cdot \boldsymbol{b} = a_1 b_1 + a_2 b_2 + a_3 b_3 + \cdots$$
  Note that the result of a dot product is a scalar, not a vector
  - The dot product fulfills the following:
    $$\boldsymbol{a} \cdot \boldsymbol{b} = |\boldsymbol{a}||\boldsymbol{b}|\cos(\theta)$$
    where $\theta$ is the angle between the two vectors. This equation can be used to find the angle if you know the coordinates of the vectors
  - The sign of the dot product can give you a general idea of the size of the angle:
    - $\boldsymbol{a} \cdot \boldsymbol{b} = 0 \Leftrightarrow \theta = \frac{\pi}{2}$
    - $\boldsymbol{a} \cdot \boldsymbol{b} > 0 \Leftrightarrow \theta < \frac{\pi}{2}$
    - $\boldsymbol{a} \cdot \boldsymbol{b} < 0 \Leftrightarrow \theta > \frac{\pi}{2}$
- The *projection* of a vector $\boldsymbol{a}$ onto a vector $\boldsymbol{b}$ is given by the following:
  $$\text{proj}(\boldsymbol{a}, \boldsymbol{b}) = \frac{\boldsymbol{a} \cdot \boldsymbol{b}}{|\boldsymbol{b}|^2} \cdot \boldsymbol{b}$$
  In the picture on the right, the projection is labeled with its length, $|\boldsymbol{A}|\cos(\theta)$



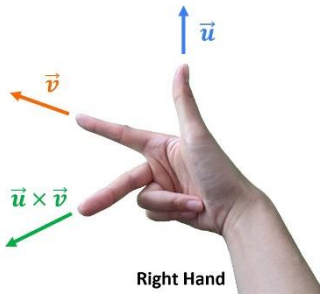

  - If the vector being projected onto is a unit (length 1) vector $\boldsymbol{e}$, this simplifies to
    $$\text{proj}(\boldsymbol{a}, \boldsymbol{e}) = (\boldsymbol{a} \cdot \boldsymbol{e})\boldsymbol{e}$$

- The *cross product* of two vectors in 3D space is computed as follows:

$$\boldsymbol{a} \times \boldsymbol{b} = \begin{bmatrix} \det\left(\begin{bmatrix} a_2 & b_2 \\ a_3 & b_3 \end{bmatrix}\right) \\ \det\left(\begin{bmatrix} a_3 & b_3 \\ a_1 & b_1 \end{bmatrix}\right) \\ \det\left(\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}\right) \end{bmatrix}$$

  Each coordinate of the cross product is computed using the next two coordinates of the original vectors, looping back around after the third. So, the first is computed with the second and third, the second is computed with the third and first, and the third is computed with the first and second.

  - Unlike the dot product, the cross product gives another vector
  - The cross product of two vectors is always perpendicular to both input vectors
  - The direction of a cross product can be found using the *right-hand rule*, as shown in the figure




  - Diversion: The magnitude of a cross product is related to the magnitudes of the input vectors by:

$$|\boldsymbol{a} \times \boldsymbol{b}| = |\boldsymbol{a}||\boldsymbol{b}||\sin(\theta)|$$

    where $\theta$ is the angle between the two vectors

- The area of a parallelogram can be found using vectors in two ways:
  - For vectors in space, it is the magnitude of the cross product of the two vectors that span it:

$$A = |\boldsymbol{a} \times \boldsymbol{b}|$$

  - It is also the determinant of the matrix containing the vectors that span it, arranged in columns. Remember to take the absolute value, as an area cannot be negative:

$$A = |\det([\boldsymbol{ab}])| = \left|\det\left(\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}\right)\right|$$

- The volume of a parallelepipedon, a kind of 3D parallelogram, can also be found in two ways:
  - For vectors in space, it is the absolute value of the *scalar triple product* of the three vectors that span it. The scalar triple product is just a cross product followed by a dot product with the third vector:

$$V = |(\boldsymbol{a} \times \boldsymbol{b}) \cdot \boldsymbol{c}|$$

  - It is also the absolute value of the determinant of the matrix containing the three vectors that span it:

$$V = |\det([\boldsymbol{abc}])| = \left|\det\left(\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}\right)\right|$$

## Vector spaces

- Vector spaces can be more abstract than geometric vectors. Any space of elements can be a vector space if it:
    - fulfills the properties of basic vector operations
    - is stable with respect to addition and multiplication by a scalar. This means that if you add two elements of the space, you will get another element of the space, and if you multiply an element of the space by a scalar, you will again get another element of the space. These are called the *stability requirements*
- A subset of a vector space is called a *subspace* if it is in itself a vector space
    - To prove that something is a subspace, you only need to show that it fulfills the stability requirements. To show the first one, add two generic vectors from the space and show that it results in another vector that fits in the space. To show the second one, multiply a variable by a generic vector from the space and show the same thing.
- A set of vectors is called linearly independent if none of them can be written as a linear combination of the others
    - Alternatively:
      $$k_1\boldsymbol{v}_1 + k_2\boldsymbol{v}_2 + k_3\boldsymbol{v}_3 + \cdots = 0$$
      is only fulfilled if all the constants $k_1, k_2, k_3$ etc. are 0
    - This leads to the condition for linear independence described previously, which is that the coordinate matrix of the vectors must be regular (have full rank). This also means that the coordinate matrix has a determinant different from 0
- The *span* of a set of vectors is the set of all possible linear combinations of those vectors
    - Every span is a subspace
- Vector bases:
    - For a set of vectors to be a basis for a space, the vectors must be linearly independent and the span of those vectors must be equal to the space
    - Every vector in the space can be written as a unique linear combination of the basis vectors
    - If a space has a basis with *n* vectors, then every basis will have *n* vectors. *n* is the dimension of the space, $\dim(V)$
- If you are given a vector span, you can find a basis for that span using the following method:
    - Find the reduced row form of the coordinate matrix containing the vectors
    - Compare the original matrix with the reduced form. For every column in the reduced form that does not have a leading 1, remove that column from the original
    - The vectors left in the original are linearly independent and constitute a basis for the span
    - The dimension of the span is equal to the rank of the coordinate matrix
- One type of abstract vector space is the polynomial space $P_n(\mathbb{R})$, where *n* is the order of the polynomials you are considering
    - The standard basis for this is called the *monomial basis*, written as $m = (1, x, x^2, x^3, \ldots, x^n)$

- $\dim\big(P_n(\mathbb{R})\big) = n + 1$
- For example, if you have the following three polynomials:

$$P_1(x) = 1 - x$$
$$P_2(x) = 2 + x + x^2$$
$$P_3(x) = -2x^2$$

they can be written as vectors with respect to the monomial basis by placing the coefficients in the correct places corresponding to the monomial basis. So, first the zeroth order coefficients, then the first order, then the second order, and so on:

$${}_mP_1(x) = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$
$${}_mP_2(x) = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$$
$${}_mP_3(x) = \begin{bmatrix} 0 \\ 0 \\ -2 \end{bmatrix}$$

In this case, since the highest order polynomial we have is second order, we are only using the first three monomial basis vectors, so the vectors have three coordinates

## Linear maps

- A map is a function for vectors. It maps vectors onto their *images* under that map
- A map is called linear if it satisfies the two *linearity requirements*:
  - $L_1$: $f(\boldsymbol{u} + \boldsymbol{v}) = f(\boldsymbol{u}) + f(\boldsymbol{v})$
  - $L_2$: $f(k \cdot \boldsymbol{u}) = k \cdot f(\boldsymbol{u})$
- All linear maps map the zero vector onto itself:

$$f(\mathbf{0}) = \mathbf{0}$$

- There are four vector spaces that maps work with:
  - The *domain* of a map is the space containing all possible inputs for the map
  - The *codomain* of a map is the space it maps into. However, it is not necessary for every vector in the codomain to be an image of a vector in the domain
    - If a map is shown as $f: V \to W$, that means that it maps from the domain V to the codomain W
  - The *image space* of a map consists of all vectors in the codomain that are actually images of vectors in the domain
    - If the map is called *f* and the domain is called V, the image space is written as $f(V)$
    - This is a subspace of the codomain
  - The *kernel* of a map consists of all vectors in the domain that map onto the zero vector
    - If the map is called *f*, the kernel is written as $\ker(f)$
    - This is a subspace of the domain
- The dimensions of the domain, kernel, and image space are related by the *dimension theorem*:

$$\dim(V) = \dim(\ker(f)) + \dim\big(f(V)\big)$$

- Linear maps are represented by *mapping matrices* of the form ${}_b\boldsymbol{F}_a$
  - $a$ is the basis being used in the domain, and $b$ is the basis being used in the codomain
  - When this matrix is multiplied by a vector in the domain, the result is the image vector
    $$ {}_b\boldsymbol{F}_a \cdot {}_a\boldsymbol{x} = {}_b\boldsymbol{y} $$
    As with change of base matrices, subscripts next to each other in the equation cancel out
  - The mapping matrix ${}_b\boldsymbol{F}_a$ consists of the $b$-coordinate images of the $a$-basis vectors, arranged in columns:
    $$ {}_b\boldsymbol{F}_a = [{}_bf(\boldsymbol{a}_1), {}_bf(\boldsymbol{a}_2), {}_bf(\boldsymbol{a}_3)] $$
    Since the matrix consists of vectors with $b$-coordinates, the subscript $b$ is on the left of the matrix. The right subscript tells you the input basis
  - The number of columns in a mapping matrix is the dimension of the domain, and the number of rows is the dimension of the codomain (not the image space)
  - A map is linear if and only if it can be represented by a mapping matrix
  - The kernel of a linear map can be found by solving the following equation:
    $$ \boldsymbol{F} \cdot \boldsymbol{x} = \boldsymbol{0} $$
    where $\boldsymbol{0}$ is the zero vector
    - This is done using the augmented matrix $\boldsymbol{T} = [\boldsymbol{F}|\boldsymbol{0}]$
    - The number of free parameters in the solution is the dimension of the kernel
  - The vectors in the mapping matrix span the image space. A basis for the image space can be found using the span reduction method described above
    - Because of this, the dimension of the image space is equal to the rank of the mapping matrix
  - A vector expression for a linear map can easily be converted into a mapping matrix:
    $$ f(x_1, x_2, x_3) = \begin{bmatrix} x_1 + x_2 - x_3 \\ 2x_2 - 2x_3 \\ 5x_1 + x_2 - 9x_3 \end{bmatrix} $$
    $$ \boldsymbol{F} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 - x_3 \\ 2x_2 - 2x_3 \\ 5x_1 + x_2 - 9x_3 \end{bmatrix} $$
    $$ \boldsymbol{F} = \begin{bmatrix} 1 & 1 & -1 \\ 0 & 2 & -2 \\ 5 & 1 & -9 \end{bmatrix} $$
- You can use change of basis matrices to change the bases that a mapping matrix works with. Multiply from the left to change the output basis, and multiply from the right to change the input basis. These operations take the following general forms:
  $$ {}_c\boldsymbol{F}_b = {}_c\boldsymbol{F}_a {}_a\boldsymbol{M}_b $$
  $$ {}_d\boldsymbol{F}_a = {}_d\boldsymbol{M}_c {}_c\boldsymbol{F}_a $$
  $$ {}_d\boldsymbol{F}_b = {}_d\boldsymbol{M}_c {}_c\boldsymbol{F}_a {}_a\boldsymbol{M}_b $$
  As always, subscripts next to each other cancel out

- You may be given a map defined by images of specific vectors. For example, for a map $f: P_2(x) \to P_1(x)$:

$$f(1 - x^2) = 1 - x$$
$$f(-1) = x$$
$$f(2 + x + x^2) = 3 + x$$

To find the mapping matrix, treat the domain vectors as a span and thin them down to a basis. State the mapping matrix that maps from the base you found in the domain to the standard base in the codomain. Then, use change of base matrices to find the mapping matrix with respect to the standard base on both sides

- The structural theorem applies to linear maps. Given a map *f*:
  - $f(\boldsymbol{x}) = \boldsymbol{0}$ is a homogeneous equation (which, when solved, gives the kernel of the map)
  - $f(\boldsymbol{x}) = \boldsymbol{b}$ is an inhomogeneous equation
  - The solution set to an inhomogeneous equation is given by:

$$L_{inhom} = x_0 + L_{hom} = x_0 + \ker(f)$$

where $x_0$ is a particular solution and $L_{hom}$ is the solution to the corresponding homogeneous equation, the same as $\ker(f)$

## Eigenvalues

- A linear map $f: V \to V$ (where the domain and codomain are the same, which also means that the mapping matrix is square) will have special vectors called *eigenvectors*. These are vectors that, when mapped, result in the original vector multiplied by a scalar. This scalar is called the *eigenvalue*. In other words, an eigenvector **v** and an eigenvalue λ fulfill the following equation:

$$\boldsymbol{A} \cdot \boldsymbol{v} = \lambda \cdot \boldsymbol{v}$$

  - If the eigenvalues and eigenvectors are real, the map constitutes a stretch in the direction of each eigenvector, with the corresponding eigenvalue being the scale factor

- You can find the eigenvalues for a map by solving a rearranged version of the equation above:

$$(\boldsymbol{A} - \lambda \boldsymbol{I})\boldsymbol{v} = 0$$

$(\boldsymbol{A} - \lambda \boldsymbol{I})$ is called the *characteristic matrix*, $\boldsymbol{K_A}$. It is obtained by subtracting $\lambda$ from every element in the main diagonal of **A**. This matrix must be singular, or else the only solution to the equation would be the **0** vector. For the matrix to be singular, the determinant must be 0:

$$\det(\boldsymbol{A} - \lambda \boldsymbol{I}) = 0$$

This is called the *characteristic equation* for **A**. When solved, it will yield a polynomial equation which will give you the eigenvalues

  - The multiplicity of each eigenvalue in the polynomial is called the *algebraic multiplicity* of that eigenvalue, $\mathrm{am}(\lambda)$

- You can find the eigenvectors for each eigenvalue by substituting the eigenvalues back into the equation $(\boldsymbol{A} - \lambda \boldsymbol{I})\boldsymbol{v} = 0$
  - This will give you a solution with one or more free parameters, where each free parameter is multiplied by an eigenvector
  - The solution set, which is the span of the eigenvectors, is a subspace called the *eigenspace* for that eigenvalue
  - The dimension of the eigenspace is called the *geometric multiplicity*, $\mathrm{gm}(\lambda)$
  - If one of the eigenvalues is 0, the eigenspace for the eigenvalue 0 is the same as the kernel

- For an eigenvalue $\lambda$, its algebraic multiplicity and its geometric multiplicity are related by:

$$1 \leq \text{gm}(\lambda) \leq \text{am}(\lambda)$$

- For a map $f: V \to V$:
    - The sum of the algebraic multiplicities of the eigenvalues is always equal to $\dim(V)$. These eigenvalues may be complex:

$$\Sigma \text{am}(\lambda) = \dim(V)$$

    - The sum of the geometric multiplicities of the eigenvalues, or the number of linearly independent eigenvectors, can at the most be $\dim(V)$:

$$\Sigma \text{gm}(\lambda) \leq \dim(V)$$

- Eigenvectors chosen from different eigenspaces are always linearly independent. It is also possible to have linearly independent vectors within the eigenspace, depending on the geometric multiplicity
- If a map $f: V \to V$ with $\dim(V) = n$ has $n$ different linearly independent eigenvectors, you can find a basis for *V* made entirely of eigenvectors for the map. This is called an *eigenbasis*
    - If $v = (v_1, v_2, \ldots v_n)$ is an eigenbasis, then the mapping matrix ${}_v\boldsymbol{F}_v$ will be a diagonal matrix, which is a matrix with only 0s both above and below the main diagonal. This is the only way for a mapping matrix to be diagonal
    - ${}_v\boldsymbol{F}_v$ is usually given the name $\boldsymbol{\Lambda}$ (capital lambda)
    - The numbers in the diagonal matrix will be the eigenvalues for the map, where the eigenvalue in the *i*th column has the *i*th basis vector as the corresponding eigenvector. This means that the mapping matrix will have the form:

$$\boldsymbol{\Lambda} = \begin{bmatrix} \lambda_1 & 0 & \ldots & 0 \\ 0 & \lambda_2 & \ldots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \ldots & \lambda_n \end{bmatrix}$$

    - The change of basis matrix ${}_e\boldsymbol{M}_v$, which contains the eigenvectors, is usually given the name $\boldsymbol{V}$
    - Therefore, the diagonal matrix is related to the original mapping matrix by the following equation:

$${}_v\boldsymbol{F}_v = {}_v\boldsymbol{M}_e \cdot {}_e\boldsymbol{A}_e \cdot {}_e\boldsymbol{M}_v$$
$$\boldsymbol{\Lambda} = \boldsymbol{V}^{-1} \cdot \boldsymbol{A} \cdot \boldsymbol{V}$$

      where **A** is the original mapping matrix
    - You can only diagonalize a map $f: V \to V$ if the sum of the geometric multiplicities for all the eigenvalues is equal to $\dim(V)$, or the sum of the algebraic multiplicities:

$$\dim(V) = \Sigma \text{gm}(\lambda)$$

        - If one λ exists such that $\text{gm}(\lambda) < \text{am}(\lambda)$, then the map cannot be diagonalized
    - Diagonalization provides a useful way to calculate matrix powers:

$$\boldsymbol{A}^n = \boldsymbol{V} \cdot \boldsymbol{\Lambda}^n \cdot \boldsymbol{V}^{-1}$$

      $\boldsymbol{\Lambda}^n$ can be found easily by raising all the elements in the diagonal to the *n*th power
- Eigenvalues can be complex numbers. For a matrix consisting of real numbers:
    - If $\lambda$ is a complex eigenvalue, then so is $\bar{\lambda}$
    - If $\boldsymbol{v}$ is an eigenvector for an eigenvalue $\lambda$, then $\overline{\boldsymbol{v}}$ is an eigenvector for $\bar{\lambda}$

- Two matrices **A** and **B** are said to be *similar* if a matrix **V** exists such that:
$$\boldsymbol{B} = \boldsymbol{V}^{-1} \cdot \boldsymbol{A} \cdot \boldsymbol{V}$$
    - If **A** is similar to **B** then **B** is similar to **A** (commutative law)
    - If **A** is similar to **B** and **B** is similar to **C**, then **A** is similar to **C** (transitive law)
    - If **A** and **B** are similar, they have identical eigenvalues with the same algebraic and geometric multiplicities. This is the most useful way to find out if two matrices are similar
    - Two matrices with the same dimensions are mapping matrices for the same map, but with respect to two different bases, if and only if the two matrices are similar

## Symmetric and orthogonal matrices

- The dot product of two vectors and the angle between two vectors are computed the same way for all vectors as they are for geometric vectors
- An *orthonormal basis* is a vector basis where all the basis vectors are *orthogonal* and have length 1. You can test if vectors are orthogonal by seeing whether their dot product is 0. An example of an orthonormal basis is the standard basis
- The projection of one vector onto another can be calculated for all types of vectors. If the vector being projected onto has length 1, the projection formula can be simplified:
$$\text{proj}(\boldsymbol{a}, \boldsymbol{e}) = (\boldsymbol{a} \cdot \boldsymbol{e}) \cdot \boldsymbol{e}$$
where $\boldsymbol{e}$ is a vector with length 1 and $\boldsymbol{a}$ is an arbitrary vector
- Any basis $u = (u_1, u_2, \dots)$ can be converted into an orthonormal basis $v = (v_1, v_2, \dots)$ using a process called the *Gram-Schmidt* algorithm:
    - The first orthonormal basis vector is the normalized version of the first old basis vector. You simply divide by its length in order to create a vector in the same direction, but with length 1:
$$\boldsymbol{v}_1 = \frac{\boldsymbol{u}_1}{|\boldsymbol{u}_1|}$$
    - For the second vector, we first define an intermediate vector $\boldsymbol{w}_2$ as:
$$\boldsymbol{w}_2 = \boldsymbol{u}_2 - (\boldsymbol{u}_2 \cdot \boldsymbol{v}_1)\boldsymbol{v}_1$$
The second orthonormal basis vector is then the normalized version of this intermediate vector:
$$\boldsymbol{v}_2 = \frac{\boldsymbol{w}_2}{|\boldsymbol{w}_2|}$$
    - For the third vector:
$$\boldsymbol{w}_3 = \boldsymbol{u}_3 - (\boldsymbol{u}_3 \cdot \boldsymbol{v}_1)\boldsymbol{v}_1 - (\boldsymbol{u}_3 \cdot \boldsymbol{v}_2)\boldsymbol{v}_2$$
$$\boldsymbol{v}_3 = \frac{\boldsymbol{w}_3}{|\boldsymbol{w}_3|}$$
    - For the *p*th vector:
$$\boldsymbol{w}_p = \boldsymbol{u}_p - (\boldsymbol{u}_p \cdot \boldsymbol{v}_1)\boldsymbol{v}_1 - (\boldsymbol{u}_p \cdot \boldsymbol{v}_2)\boldsymbol{v}_2 - \cdots - (\boldsymbol{u}_p \cdot \boldsymbol{v}_n)\boldsymbol{v}_n$$
$$\boldsymbol{v}_p = \frac{\boldsymbol{w}_p}{|\boldsymbol{w}_p|}$$

- An *ortho**gonal** matrix* is a matrix made up of ortho**normal** vectors, arranged in columns. Orthogonal matrices, usually written as **Q**, have the following properties:
  - $\boldsymbol{Q}^T = \boldsymbol{Q}^{-1}$
  - $\boldsymbol{Q}^T \cdot \boldsymbol{Q} = \boldsymbol{I}$ where **I** is an identity matrix
  - $\det(\boldsymbol{Q}) = 1$ for a positive orthogonal matrix
    $\det(\boldsymbol{Q}) = -1$ for a negative orthogonal matrix
  - You can switch a matrix between negative and positive orthogonal by changing the signs of all the elements in one column
- A *symmetric matrix* is a matrix that fulfills $\boldsymbol{A} = \boldsymbol{A}^T$**.** The line of symmetry is the main diagonal. Symmetric matrices have the following properties, compared to normal square matrices:

| Ordinary square matrix ($n \times n$) | Symmetric matrix ($n \times n$) |
|---|---|
| $n$ real or complex eigenvalues | $n$ real eigenvalues |
| Eigenvectors corresponding to different eigenvalues are linearly independent | Eigenvectors corresponding to different eigenvalues are orthogonal |
| $\text{gm}(\lambda) \leq \text{am}(\lambda)$ | $\text{gm}(\lambda) = \text{am}(\lambda)$ |
| If for all $\lambda$ $\text{gm}(\lambda) = \text{am}(\lambda)$ then the matrix can be diagonalized | Can always be diagonalized |
| If 4 is fulfilled then $\boldsymbol{\Lambda} = \boldsymbol{V}^{-1}\boldsymbol{A}\boldsymbol{V}$ where **A** is the original matrix | $\boldsymbol{\Lambda} = \boldsymbol{Q}^T\boldsymbol{A}\boldsymbol{Q}$ where **A** is the original matrix and **Q** is an orthogonal matrix. This is called the *spectral theorem* for symmetric matrices |

- Diversion: An equation for a plane can be created using one point in the plane and a vector that is normal to the plane. If the point is $(x_0, y_0, z_0)$ and the normal vector is $(a, b, c)$, the plane equation is:

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} \cdot \begin{bmatrix} x - x_0 \\ y - y_0 \\ z - z_0 \end{bmatrix} = 0$$

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

  - This can be converted to parametric form by transposing for one of the variables and treating the others as free parameters. For example:

$$1x + 2y - 3z = 6$$
$$x = -2y + 3z + 6$$
$$x = -2s + 3t + 6, \quad y = s, \quad z = t$$
$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}$$

  - You can convert in the other direction by setting the free parameters to whatever numbers are most convenient in order to find a single point on the plane, and then taking the cross product of the two spanning vectors to find a normal vector

- The *orthogonal complement* to a subspace $U$ in $\mathbb{R}^n$, written as $U^\perp$, is the set of all vectors in $\mathbb{R}^n$ that are orthogonal to all vectors in $U$
    - If $U$ has dimension $p$, then $U^\perp$ has dimension $n - p$
- There is a general method for finding an orthogonal complement to a vector space $U$:
    - A vector $\mathbf{x}$ in the orthogonal complement is orthogonal to all the vectors that span $U$. Therefore, the dot product between $\mathbf{x}$ and each spanning vector is 0. For example, if $\text{span}(U) = \{\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3\}$, $\boldsymbol{u}_1 = (1,1,1,1), \boldsymbol{u}_2 = (3,1,1,2), \boldsymbol{u}_3 = (2,0,0,2)$, then:
$$\boldsymbol{x} \cdot \boldsymbol{u}_1 = 0$$
$$\boldsymbol{x} \cdot \boldsymbol{u}_2 = 0$$
$$\boldsymbol{x} \cdot \boldsymbol{u}_3 = 0$$
$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = 0$$
$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 1 \\ 1 \\ 2 \end{bmatrix} = 0$$
$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 0 \\ 0 \\ 2 \end{bmatrix} = 0$$
    - Evaluating these dot products gives a system of linear equations. Solving this system of equations gives you the orthogonal complement to the subspace
    - This whole process is equivalent to finding ker $([\boldsymbol{u}_1\ \boldsymbol{u}_2\ \boldsymbol{u}_3]^T)$
- Every positive orthogonal 2 × 2 matrix, when treated as a mapping matrix, constitutes a rotation in the positive direction (counterclockwise). This type of matrix can be called a *rotation matrix*. The angle $u$ for this rotation can be found by comparison to the following general form, which all matrices of this type can be written in:
$$\begin{bmatrix} \cos(u) & -\sin(u) \\ \sin(u) & \cos(u) \end{bmatrix}$$
    - You can achieve a clockwise rotation by setting $u$ to a negative number
    - The inverse matrix is a rotation by the negative angle
    - Since every symmetric matrix $\boldsymbol{A}$ can be factorized into the form $\boldsymbol{Ax} = \boldsymbol{Q}(\boldsymbol{\Lambda}(\boldsymbol{Q}^T\boldsymbol{x}))$, every symmetric matrix constitutes:
        - A rotation ($\boldsymbol{Q}^T$),
        - followed by a stretch ($\boldsymbol{\Lambda}$),
        - followed by a rotation by the same angle, but in the opposite direction ($\boldsymbol{Q}$)
- Every positive orthogonal 3 × 3 matrix is also a rotation matrix. In particular, you can achieve a rotation about each of the three coordinate axes using three specific matrix functions:
    - For a rotation about the $x$-axis:
$$\boldsymbol{R}_x(u) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(u) & -\sin(u) \\ 0 & \sin(u) & \cos(u) \end{bmatrix}$$

- For a rotation about the *y*-axis:

$$\boldsymbol{R}_y(u) = \begin{bmatrix} \cos(u) & 0 & \sin(u) \\ 0 & 1 & 0 \\ -\sin(u) & 0 & \cos(u) \end{bmatrix}$$

- For a rotation about the *z*-axis (most commonly used):

$$\boldsymbol{R}_z(u) = \begin{bmatrix} \cos(u) & -\sin(u) & 0 \\ \sin(u) & \cos(u) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

# Differential equations

## First-order linear differential equations

- First order linear differential equations are equations of the form:

$$x'(t) + p(t)x(t) = q(t)$$

where $p(t)$ and $q(t)$ are known functions

- The structural theorem applies to these:

$$L_{inhom} = x_0 + L_{hom}$$

- Homogeneous first order linear differential equations take the form $x'(t) + p(t)x(t) = 0$. The solution to these is given by:

$$x(t) = ce^{-P(t)}$$

where *c* is a free parameter and $P(t)$ is an arbitrary antiderivative of $p(t)$

- If you have an inhomogeneous equation, you can find the solution to the corresponding homogeneous equation using the formula above, and then use the structural theorem to solve the inhomogeneous equation. For this, you need a particular solution, and this can be found using the *guess method*, which works as follows:
  - Using the form of the equation, guess the general form of the solution. For example, for the equations:

$$x'(t) - 2x(t) = e^{-t}$$
$$x'(t) - 2x(t) = -5\sin(t)$$
$$x'(t) - 2x(t) = 2t + 3$$

the general forms of the solutions are, respectively:

$$x_0(t) = ke^{-t}$$
$$x_0(t) = a\cos(t) + b\sin(t)$$
$$x_0(t) = at + b$$

  - Substitute the general form of the solution for $x(t)$ in the original equation, and solve for the constants in this general form. This gives you a particular solution
  - Solve the corresponding homogeneous equation, and use the structural theorem (add the particular solution to the homogeneous solution set)

- *Superposition principle:* Suppose you have different linear differential equations, all with the same $p(t)$ but different $q(t)$. If $x_1(t)$ is a solution when the RHS is $q_1(t)$, and $x_2(t)$ is a solution when the RHS is $q_2(t)$, then $x_1(t) + x_2(t)$ will be a solution when the RHS is $q_1(t) + q_2(t)$

- The general solution formula for inhomogeneous linear first order differential equations is:

$$x(t) = e^{-P(t)} \int e^{P(t)} q(t)\, dt + c e^{-P(t)}$$

where *c* is a free parameter. You can also state this in a factorized form:

$$x(t) = e^{-P(t)} \left( \int e^{P(t)} q(t)\, dt + c \right)$$

- You may have an *initial value condition* for the differential equation, which means that, for a specific *t*, it must assume a specific value. This condition narrows down the solution set to a single solution, with a single value of *c*. You can find this *c* simply by substituting the initial value condition into your solution set and solving for *c*
- You may have a differential equation that is not first order, but which has a restricted domain. The domain restriction might allow you to use matrix methods such as mapping matrices. For example, given the equation:

$$f\big(x(t)\big) = x''(t) - 2x'(t) - 3x(t)$$

with a domain restriction limiting you to the subspace with basis $a = (\cos(t), \sin(t))$, you can treat the equation as a linear map. Find the mapping matrix by finding the images of the basis vectors:

$$f(\cos(t)) = \cdots = -4\cos(t) + 2\sin(t)$$
$$f(\sin(\mathrm{t})) = \cdots = -2\cos(t) - 4\sin(t)$$
$${}_a\boldsymbol{F}_a = \begin{bmatrix} -4 & -2 \\ 2 & -4 \end{bmatrix}$$

## Systems of linear first-order differential equations

- A system of two homogeneous linear first-order differential equations takes the form:

$$x_1'(t) = a x_1(t) + b x_2(t)$$
$$x_2'(t) = c x_1(t) + d x_2(t)$$

- Just like with systems of linear equations, this system of differential equations can be written using vectors and matrices:

$$\boldsymbol{x}'(t) = \begin{bmatrix} {x_1}'(t) \\ {x_2}'(t) \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is called the *system matrix*

- The solution to this kind of system of equations is given by:

$$\boldsymbol{x}(t) = c_1 \boldsymbol{u}_1(t) + c_2 \boldsymbol{u}_2(t) + \cdots c_n \boldsymbol{u}_n(t) = c_1 e^{\lambda_1 t} \boldsymbol{v}_1 + c_2 e^{\lambda_2 t} \boldsymbol{v}_2 + \cdots + c_n e^{\lambda_n t} \boldsymbol{v}_n$$

where $\lambda$ and **v** are the eigenvalues and corresponding eigenvectors of the system matrix, and the *c* values are complex free parameters. A single eigenvalue may show up more than once if it has multiple eigenvectors

  - Since this is a vector equation, it can be split up into multiple scalar equations for $x_1(t), x_2(t), \ldots, x_n(t)$ by considering each vector coordinate separately
  - Just like with single differential equations, you can use initial value conditions to determine the *c* values. However, here it must be done with multiple scalar equations

- This type of solution can be converted to *matrix form* in the following way:

$$\boldsymbol{x}(t) = c_1 e^{\lambda_1 t}\boldsymbol{v}_1 + c_2 e^{\lambda_2 t}\boldsymbol{v}_2 + \cdots c_n e^{\lambda_n t}\boldsymbol{v}_n$$
$$= [\boldsymbol{v}_1 \quad \boldsymbol{v}_2 \quad \ldots \quad \boldsymbol{v}_n]\begin{bmatrix} c_1 e^{\lambda_1 t} \\ c_2 e^{\lambda_1 t} \\ \ldots \\ c_n e^{\lambda_n t} \end{bmatrix}$$

  where $[\boldsymbol{v}_1 \quad \boldsymbol{v}_2 \quad \ldots \quad \boldsymbol{v}_n]$ is a matrix/row vector with the eigenvectors as its columns

- If any eigenvalues are complex, the equation above will yield the complete complex solution. To find the *complete real solution*, a different method must be used. For every pair of complex conjugate eigenvalues $\lambda_1 = \alpha + i\beta$ and $\lambda_2 = \alpha - i\beta$, the corresponding **u** vectors in the above equation must be replaced with:

$$\boldsymbol{u}_1(t) = e^{\alpha t}\big(\cos(\beta t)\,\mathrm{Re}(\boldsymbol{v}) - \sin(\beta t)\,\mathrm{Im}(\boldsymbol{v})\big)$$
$$\boldsymbol{u}_2(t) = e^{\alpha t}\big(\sin(\beta t)\,\mathrm{Re}(\boldsymbol{v}) + \cos(\beta t)\,\mathrm{Im}(\boldsymbol{v})\big)$$

  where **v** is specifically the eigenvector for $\lambda_1$

  - However, this does not need to be used if the initial value conditions are real. Finding the complete complex solution and then using the initial value conditions will turn it into a real solution

- If an eigenvalue $\lambda$ has an algebraic multiplicity of 2 but a geometric multiplicity of 1 with the vector **v**, the corresponding **u** vectors in the equation must be replaced with:

$$\boldsymbol{u}_1(t) = e^{\lambda t}\boldsymbol{v}$$
$$\boldsymbol{u}_2(t) = te^{\lambda t}\boldsymbol{v} + e^{\lambda t}\boldsymbol{b}$$

  where **b** is a vector that must be determined by solving the following equation:

$$(\boldsymbol{A} - \lambda \boldsymbol{I})\boldsymbol{b} = \boldsymbol{v}$$

- For inhomogeneous systems, in this case systems with terms that do not contain an *x* function, the structural theorem applies. The particular solution must be found using the guess method described previously, with the only difference being that you must guess multiple solutions and thus solve a system of equations

  - Diversion: You may end up with equations like:

$$(a + 3c)t + a + b + 3d = 10t + 1$$
$$(-3a + c)t + c - 3b + d = -1$$

    where *a, b, c,* and *d* are the constants you must solve for. In this case, you must split each equation into two by equating the coefficients of *t* and equating the constant terms:

$$a + 3c = 10$$
$$a + b + 3d = 1$$
$$-3a + c = 0$$
$$c - 3b + d = -1$$

    Since there are now four equations to match the four constants, the constants can be found

## Second order linear differential equations

- A second order linear differential equation takes the form:
$$x''(t) + a_1 x'(t) + a_0 x(t) = q(t)$$
- To solve a homogeneous equation (where $q(t) = 0$) you must solve the *characteristic equation*, a quadratic equation with coefficients matching the coefficients in the differential equation:
$$\lambda^2 + a_1\lambda + a_0 = 0$$
The form of the solution then depends on the types of roots you found:
    - If you find two different real roots $\lambda_1$ and $\lambda_2$, the solution is:
$$x(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}$$
where $c_1$ and $c_2$ are free parameters
    - If you find two complex roots $\lambda = \alpha \pm \beta i$, the complete real solution is:
$$x(t) = c_1 e^{\alpha t}\cos(\beta t) + c_2 e^{\alpha t}\sin(\beta t)$$
    - If you find a double root $\lambda$, the solution is:
$$x(t) = c_1 e^{\lambda t} + c_2 t e^{\lambda t}$$
- To solve an inhomogeneous equation
$$x''(t) + a_1 x'(t) + a_0 x(t) = q(t)$$
you must, once again, use the structural theorem $L_{inhom} = x_0(t) + L_{hom}$ and guess the particular solution $x_0(t)$
    - If $q(t)$ is an *n*th degree polynomial, you should guess a polynomial of degree *n* + 2 at most
        - If a term with $x(t)$ in fact exists on the LHS ($a_0 \neq 0$), you should guess a polynomial of degree *n*
        - If there is no term with $x(t)$ ($a_0 = 0$) but there is a term with $x'(t)$ ($a_1 \neq 0$), the degree should be *n* + 1
        - If the LHS solely consists of $x''(t)$ ($a_1 = a_0 = 0$), the degree should be *n* + 2
    - If $q(t)$ contains trigonometric functions:
$$q(t) = a\cos(\omega t) + b\sin(\omega t)$$
then you should guess a particular solution with the same form:
$$x_0(t) = A\cos(\omega t) + B\sin(\omega t)$$
where $\omega$ is the same number but *A* and *B* must be determined
    - If $q(t)$ contains the exponential function:
$$q(t) = \beta e^{\alpha t}$$
then the form of your guess depends on whether $\alpha$ is a root in the characteristic equation
        - If $\alpha$ is not a root, you should guess a solution of the same form:
$$x_0(t) = \gamma e^{\alpha t}$$
where $\alpha$ is the same number and $\gamma$ must be determined. If $\alpha$ is a root in the characteristic equation and you try this, you will not be able to find $\gamma$ . You can always try this first, and then use the next method if it does not work
        - If $\alpha$ is a root, you must guess a solution of the form
$$x_0(t) = \gamma t e^{\alpha t}$$
where $\alpha$ is the same number and $\gamma$ must be determined

- The superposition principle applies to second order linear differential equations. Therefore, if you have a complicated $q(t)$, you can split the equation into many equations with the same LHS but different right hand sides that add up to the RHS of the original. Then, find particular solutions for each equation and add them together
- For a complicated $q(t)$ with trigonometric functions, you can also use a different method called the *complex guess method* (Method 18.18):
  - A $q(t)$ with the form
    $$q(t) = ae^{ut}\cos(\omega t) - be^{ut}\sin(\omega t)$$
    can be rewritten using Euler's form of complex numbers:
    $$q(t) = Re\left((a+bi)e^{(u+\omega i)t}\right)$$
    Either *a* or *b* may be 0 if $q(t)$ has only one trigonometric function. Note the minus sign in front of the sin term
  - Then, find a particular solution to the complex differential equation
    $$z''(t) + a_1 z'(t) + a_0 z(t) = (a+bi)e^{(u+\omega i)t}$$
    where $a_1$ and $a_0$ are the same coefficients as in the original equation. The guess should have the form $(c+di)e^{(u+\omega i)t}$, where *c* and *d* must be found
  - The particular solution to the original equation will then be the real part of the particular solution to the complex equation:
    $$x_0(t) = Re\big(z_0(t)\big)$$

- To find $c_1$ and $c_2$ in a second order differential equation, you need a double initial value condition:
  $$x(t_0) = x_0$$
  $$x'(t_0) = v_0$$
  These must be plugged into the solution set and the derivative of the solution set respectively
  - There is only one solution that fulfills both initial value conditions. This is called the *existence and uniqueness theorem*
- If you are given a solution set, you can work backwards and determine the equation. For example:
  $$L_{inhom} = 5e^{2t} - 4e^{t} + 2t + 1$$
  - The coefficients of *t* in the homogeneous solution set are the roots of the characteristic equation. Reverse their signs and plug them into the factorized form of the characteristic equation, then expand the brackets to find the coefficients of the differential equation. In this case, the roots are 2 and 1, so the factorized form is:
    $$(\lambda - 2)(\lambda - 1) = 0$$
    $$\lambda^2 - 3\lambda + 2 = 0$$
    This tells us that the LHS of the differential equation is:
    $$x''(t) - 3x'(t) + 2x(t)$$
  - To find the RHS ($q(t)$), simply plug the particular solution from the solution set into the LHS you just found:
    $$q(t) = (2t+1)'' - 3(2t+1)' + 2(2t+1) = \cdots = 4t - 4$$