Math 2 revision sheet

By: Ioannis Karras

Version: Final (post-exam)

Link to revision folder

Named theorems from the notes are in gray, self-contained prerequisite topics are in yellow, and extra information, mostly given by the TAs during exercise classes, is in green. Links to another part of the document are in blue

Table of Contents (contains links)

Differential equations

nth order differential equations

- An *nth order linear differential equation* with constant coefficients takes the form
$$a_0 y^{(n)}(t) + a_1 y^{(n-1)}(t) + \cdots + a_{n-1} y'(t) + a_n y(t) = u(t)$$
  - $y(t)$ is a function that depends on t. This is the function sought for
  - $u(t)$ is another function that depends on t, usually called the *forcing function*. If there is no $u(t)$ ($u(t) = 0$), the differential equation is *homogeneous*. Otherwise, it is *inhomogeneous*
  - D_n is sometimes used as a shorthand for the left-hand side of a differential equation. Using this, the equation above can be written as
$$D_n(y) = u$$
- There are a few different notations for derivatives:
$$y' = \dot{y} = y^{(1)} = \frac{dy}{dt}$$
$$y'' = \ddot{y} = y^{(2)} = \frac{d^2y}{dt^2}$$
...
- To find the solutions to a homogeneous differential equation, you must construct the *characteristic equation* for the differential equation. This is done by replacing all the y functions with a variable λ raised to the power equal to the order of the derivative of the y function. For *y* itself, you get $\lambda^0 = 1$. In general,
$$a_0 y^{(n)}(t) + a_1 y^{(n-1)}(t) + \cdots + a_{n-1} y'(t) + a_n y(t) = 0$$
becomes
$$a_0 \lambda^n + a_1 \lambda^{n-1} + \cdots + a_{n-1}\lambda + a_n = 0$$
The left-hand side of the characteristic equation is called the *characteristic polynomial*
- The solutions to an *n*th order homogeneous equation will form an *n*-dimensional vector space. If $y_1, y_2 \ldots y_n$ are linearly independent solutions, every solution can be written as a linear combination of these solutions:
$$y = c_1 y_1 + c_2 y_2 + \cdots + c_n y_n$$
where the *c*s are either real or complex constants. (Theorem 1.4)
- The solutions to a homogeneous equation are found by first finding the roots of the characteristic equation. Depending on the roots and whether you want a real solution or a complex solution, you need to apply slightly different methods (Theorem 1.14):
  - General complex solution:
    - For every root λ of the characteristic equation, there is a solution $y(t) = e^{\lambda t}$, where t is the variable of the differential equation
    - If a root λ has an algebraic multiplicity p greater than 1 (if it is a double root, triple root, or higher), there also exist the solutions
$$y(t) = te^{\lambda t},\ y(t) = t^2 e^{\lambda t},\ \ldots,\ y(t) = t^{p-1} e^{\lambda t}$$
For a double root, you go up to the solution with t as a coefficient; for a triple root you go up to the solution with t^2; and so on. For an *n*th order root, you need to get n solutions in total including the basic $e^{\lambda t}$
    - The general complex solution is then found by combining all these solutions in a linear combination with complex coefficients

- General real solution:
  - For every real root of the characteristic equation, use the same method as for the general complex solution
  - Roots that are not real will come in complex conjugate pairs $\alpha \pm i\omega$. For these roots, use the following solutions instead:
    $$y(t) = e^{\alpha t}\cos(\omega t) \quad \text{and} \quad y(t) = e^{\alpha t}\sin(\omega t)$$
    Since you are working with two roots in a conjugate pair, you get two solutions to match
  - If $\alpha \pm i\omega$ has algebraic multiplicity p greater than 1, you apply a similar procedure as for the complex solution, where you multiply the previous solutions with t until you have enough to match the multiplicities:
    $$y(t) = te^{\alpha t}\cos(\omega t) \quad \text{and} \quad y(t) = te^{\alpha t}\sin(\omega t)$$
    $$y(t) = t^2e^{\alpha t}\cos(\omega t) \quad \text{and} \quad y(t) = t^2e^{\alpha t}\sin(\omega t)$$
    ...
    $$y(t) = t^{p-1}e^{\alpha t}\cos(\omega t) \quad \text{and} \quad y(t) = t^{p-1}e^{\alpha t}\sin(\omega t)$$
    For two double roots in a pair you go up to the solutions with t as coefficients, for two triple roots up to t^2, and so on
  - The general real solution is then found by combining all these solutions in a linear combination, this time with real coefficients

- The solution to an inhomogeneous equation $D_n(y) = u$ always has the form
  $$y(t) = y_p(t) + y_{\text{hom}}(t)$$
  where $y_p(t)$ is a *particular solution* to the inhomogeneous equation and $y_{\text{hom}}(t)$ is the solution to the *corresponding homogeneous equation* $D_n(y) = 0$ (Theorem 1.19)

- One way to find the particular solution is through the *guess and test method*. You state a general form of the solution with variables, plug this into the differential equation, and find the variables. The form you should guess depends on the form of the *u* function:
  - If $u(t)$ is an exponential function
    $$u(t) = be^{st}$$
    then guess a solution of the form
    $$y(t) = ce^{st}$$
    with the same s
    - Extra (from the exercises): If that does not work, it may work to guess
      $$y(t) = tce^{st}$$
  - If $u(t)$ is a trigonometric function
    $$u(t) = \sin(kt) \quad \text{or} \quad u(t) = \cos(kt)$$
    then guess a solution of the form
    $$y(t) = A\cos(kt) + B\sin(kt)$$
    with the same k
  - If $u(t)$ is a kth degree polynomial
    $$u(t) = c_0t^k + c_1t^{k-1} + \cdots + c_{k-1}t + c_k$$
    then guess a solution in the form of a polynomial of degree $n + k$, where n is the order of the differential equation
    $$y(t) = d_0t^{n+k} + d_1t^{n+k-1} + \cdots + d_{n+k-1}t + d_{n+k}$$

- For example, for the differential equation
$$y''(t) + 2y'(t) + y = \cos(2t)$$
you should guess $y(t) = A\cos(2t) + B\sin(2t)$ and plug this into the differential equation. Eventually, you get
$$(-3A + 4B)\cos(2t) + (-4A - 3B)\sin(2t) = \cos(2t)$$
Since this equation must be valid for all values of t, you can solve it by comparing the coefficients of cos and sin:
$$-3A + 4B = 1 \text{ and } -4A - 3B = 0$$

- *The principle of superposition* states that if y_1 is a solution to $D_n(y) = u_1$ and y_2 is a solution to $D_n(y) = u_2$, then
$$y = c_1y_1 + c_2y_2$$
is a solution to
$$D_n(y) = c_1u_1 + c_2u_2$$
(Theorem 1.21). This allows you to find the solutions to an equation with a right-hand side that has an addition. Split it up into smaller *u* functions for each term in the addition, take away the coefficients, solve $D_n(y) = u$ for each of those *u* functions, and then add the solutions with each multiplied by the same *c* coefficient that the corresponding *u* function had in the equation

- If an inhomogeneous differential equation has a *u* function that is either an exponential function or a trigonometric function, you can find a particular solution for it using a *transfer function*. The solution found is called the *stationary solution*. The equation must have the following form:
$$a_0y^{(n)}(t) + a_1y^{(n-1)}(t) + \cdots + a_ny(t) = b_0u^{(m)}(t) + b_1u^{(m-1)}(t) + \cdots + b_mu(t)$$
  - The transfer function is made as such:
$$H(s) = \frac{b_0s^m + b_1s^{m-1} + \cdots + b_{m-1}s + b_m}{a_0s^n + a_1s^{n-1} + \cdots + a_{n-1}s + a_n}$$
    This is the characteristic polynomial for the *u* functions divided by the characteristic polynomial for the *y* functions, but using s instead of λ
    - The transfer function is not defined for values of s where the characteristic polynomial for the y functions evaluates to 0. These s values a and b always have to be found, and the domain restriction $s \neq a, b$ always has to be stated together with the function expression
  - If $u(t)$ is an exponential function
$$u(t) = e^{st}$$
then the stationary solution is found as such:
$$y(t) = H(s)e^{st}$$
with the same s as in $u(t)$ (Theorem 1.23)
  - If $u(t)$ is a trigonometric function, the process is more complicated:
    - If $u(t)$ is a cos function, $u(t) = \cos(\omega t)$, then the stationary solution is
$$y(t) = \mathrm{Re}\big(H(i\omega)e^{i\omega t}\big)$$
$$= |H(i\omega)|\cos\big(\omega t + \mathrm{Arg}\big(H(i\omega)\big)\big)$$
If $u(t)$ is a sin function, $u(t) = \sin(\omega t)$, then the stationary solution is
$$y(t) = \mathrm{Im}\big(H(i\omega)e^{i\omega t}\big)$$
$$= |H(i\omega)|\sin\big(\omega t + \mathrm{Arg}\big(H(i\omega)\big)\big)$$
with the same ω as in $u(t)$ in both cases (Theorem 1.26)

        - For both trigonometric functions, the first expression is easier to use in Maple if you provide the right `assuming` statement, usually `assuming t::real`. You may need to use the `simplify` function with this assumption
        - Two special functions, called *frequency characteristics*, appear in the second expression in both cases:
            - $A(\omega) = |H(i\omega)|$ is called the *amplitude characteristic*
            - $\phi(\omega) = \text{Arg}\big(H(i\omega)\big)$ is called the *phase characteristic*

        (Definition 1.27)
    - If there is a coefficient in front of e^{st}, $\cos(\omega t)$, or $\sin(\omega t)$, that coefficient should not, in theory, be considered part of $u(t)$. It should instead play a part in the expression for the transfer function. However, if you find a transfer function for an equation where the right-hand side is $u(t)$ by itself and then need to solve the equation where the right-hand side is $au(t)$ by itself, just multiplying $au(t)$ by the old transfer function has the same effect as creating a new transfer function that takes a into account, and then multiplying $u(t)$ by it
    - Transfer functions allow systems of equations to be seen as a "black boxes" which are given the input $u(t)$ and return the output $y(t)$

Systems of first-order differential equations

- A *homogeneous system of first-order linear differential equations* takes the following form:

$$x_1'(t) = a_{11}x_1(t) + a_{12}x_2(t) + \cdots + a_{1n}x_n(t)$$
$$x_2'(t) = a_{21}x_1(t) + a_{22}x_2(t) + \cdots + a_{2n}x_n(t)$$
$$\ldots$$
$$x_n(t) = a_{n1}x_1(t) + a_{n2}x_2(t) + \cdots + a_{nn}x_n(t)$$

  It is a series of differential equations that are solved together, where the LHS (left-hand side) of each equation is the derivative of one x function, and the RHS (right-hand side) is a linear combination of all the x functions. These x functions are the functions sought for. A system like this can be written much more simply in matrix form:

$$\boldsymbol{x}'(t) = \boldsymbol{A}\boldsymbol{x}(t)$$

  where $\boldsymbol{A}$ is a matrix containing all the coefficients, called the *system matrix.* The solution in this case is the vector $\boldsymbol{x}(t)$, which contains all the scalar x functions as components. An inhomogeneous system has the form

$$\boldsymbol{x}'(t) = \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{u}(t)$$

  where $\boldsymbol{u}(t)$ is a vector with functions unrelated to the *x* functions. You can usually write this vector as a scalar function $u(t)$ multiplied by a vector of constants $\boldsymbol{b}$:

$$\boldsymbol{x}'(t) = \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{b}u(t)$$

    - A single first order differential equation can be thought of as a system of first-order differential equations where all the matrices and vectors are 1×1

- Any *n*th order differential equation can be converted to a system of *n* first-order equations. To do this, you must create a separate function for each derivative in the *n*th order system, and then write all the relations as a system of first-order equations:
    - For simplicity, you always use the form where the highest order derivative in the *n*th order equation has coefficient 1 (no visible coefficient). This can be achieved simply by dividing the equation on both sides by whatever coefficient the highest order derivative has

- The *n*th order equation will thus have the form
$$y^{(n)}(t) + b_1 y^{(n-1)}(t) + \cdots + b_{n-1} y'(t) + b_n y(t) = u(t)$$
- You must define the *x* functions as
$$x_1(t) = y(t),\ \ x_2(t) = y'(t),\ \ \ldots,\ \ x_n(t) = y^{(n-1)}(t)$$
- The system must have the derivatives of these functions on the LHS, not the functions themselves. However, it can be seen that
$$x_1'(t) = x_2(t),\ \ x_2'(t) = x_3(t), \ldots\ \ , x_{n-1}'(t) = x_n(t)$$
- For $y^{(n)}(t)$, we have that
$$y^{(n)}(t) = -b_n y(t) - b_{n-1} y'(t) - \cdots - b_2 y^{(n-2)}(t) - b_1 y^{(n-1)}(t) + u(t)$$
We also have that $y^{(n)}(t) = x_n'(t)$. We can then write, using the new functions,
$$x_n'(t) = -b_n x_1(t) - b_{n-1} x_2(t) - \cdots - b_2 x_{n-1}(t) - b_1 x_n(t) + u(t)$$
- In the end, the system will have the form
$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \\ \ldots \\ x_{n-1}'(t) \\ x_n'(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & . & 0 & 0 \\ 0 & 0 & 1 & . & 0 & 0 \\ . & . & . & . & . & . \\ 0 & 0 & 0 & . & 0 & 1 \\ -b_n & -b_{n-1} & -b_{n-2} & . & -b_2 & -b_1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ \ldots \\ x_{n-1}(t) \\ x_n(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \ldots \\ 0 \\ 1 \end{bmatrix} u(t)$$
- The first coordinate function of the solution, $x_1(t)$, will be the solution to the *n*th order differential equation

- The solution to an inhomogeneous system is again a particular solution plus the homogeneous solution:
$$\boldsymbol{x}(t) = \boldsymbol{x}_{\mathrm{p}}(t) + \boldsymbol{x}_{\mathrm{hom}}(t)$$
- The solutions to a homogeneous system of *n* first order differential equations will form an *n*-dimensional vector space, similarly to *n*th order equations. If $\boldsymbol{x}_1, \boldsymbol{x}_2 \ldots \boldsymbol{x}_n$ are linearly independent solutions, all the solutions can be written as a linear combination of these solutions:
$$\boldsymbol{x} = c_1 \boldsymbol{x}_1 + c_2 \boldsymbol{x}_2 + \cdots + c_n \boldsymbol{x}_n$$
where the *c*s are either real or complex constants
- To find the general solution to a homogeneous system, you have to find the eigenvalues and eigenvectors of the system matrix. If you look for the eigenvalues by hand, you will solve a similar characteristic equation to the one seen for *n*th order equations. The same concept of algebraic multiplicity applies (which Maple also tells you if you use the `eigenvectors` command), but there is now also a *geometric multiplicity*. This is the number of linearly independent eigenvectors corresponding to an eigenvalue (which you can also easily see in Maple just by counting the eigenvectors).

  Depending on the eigenvalues, their multiplicities, and whether you want a real solution or a complex solution, you need to apply different methods:

  - General complex solution:
    - For every eigenvalue λ of the characteristic matrix and corresponding eigenvector $\boldsymbol{v}$, the homogeneous system has the solution
$$\boldsymbol{x}(t) = e^{\lambda t}\boldsymbol{v}$$
(Theorem 2.2)

- If an eigenvalue has multiple associated eigenvectors, but the geometric multiplicity is equal to the algebraic multiplicity, you can simply use the above equation many times, once with each eigenvector and with the same eigenvalue
- If the geometric multiplicity is less than the algebraic multiplicity p, then there exist the following solutions to the system:

$$\boldsymbol{x}_1(t) = \boldsymbol{b}_{11}e^{\lambda t}$$
$$\boldsymbol{x}_2(t) = \boldsymbol{b}_{21}e^{\lambda t} + t\boldsymbol{b}_{22}e^{\lambda t}$$
$$\ldots$$
$$\boldsymbol{x}_p(t) = \boldsymbol{b}_{p1}e^{\lambda t} + t\boldsymbol{b}_{p2}e^{\lambda t} \ldots + t^{p-1}\boldsymbol{b}_{pp}e^{\lambda t}$$

  where the $\boldsymbol{b}$ vectors must be found. (Theorem 2.9)
  In the simple case where the algebraic multiplicity is 2, we only use the first two equations, and we have that

$$\boldsymbol{b}_{11} = \boldsymbol{b}_{22} = \boldsymbol{v}$$

  where $\boldsymbol{v}$ is a nonzero eigenvector corresponding to λ. $\boldsymbol{b}_{21}$ is found by solving the equation

$$(\boldsymbol{A} - \lambda\boldsymbol{I})\boldsymbol{b}_{21} = \boldsymbol{v}$$

  where $\boldsymbol{A}$ is the system matrix and $\boldsymbol{I}$ is a 2×2 identity matrix
- The general complex solution is then found by combining all these solutions in a linear combination, with complex coefficients

o General real solution:

- Real eigenvalues correspond to real eigenvectors. Complex eigenvalues come in conjugate pairs, and correspond to conjugate pairs of eigenvectors
- For every real eigenvalue, use the same method as for the general complex solution (Theorem 2.5)
- For every conjugate pair of eigenvalues $a \pm i\omega$, take the eigenvalue $\lambda = a + i\omega$ and its eigenvector $\boldsymbol{v}$. Instead of the solution from the other method, you now get the real and imaginary parts of that as your solutions:

$$\boldsymbol{x}(t) = \mathrm{Re}\left(e^{\lambda t}\boldsymbol{v}\right) \quad \text{and} \quad \boldsymbol{x}(t) = \mathrm{Im}\left(e^{\lambda t}\boldsymbol{v}\right)$$

  (Theorem 2.5). Since you are working with two eigenvalues in a conjugate pair, you get two solutions to match
- The same multiplicity issues as in the general complex solution apply here. Again, if the geometric multiplicity is equal to the algebraic multiplicity, simply apply the regular solution multiple times, once for each eigenvector

- For every conjugate pair of eigenvalues $a \pm i\omega$ with geometric multiplicity less than their algebraic multiplicity, take the eigenvalue $\lambda = a + i\omega$. Then, take the real and imaginary parts of the solutions detailed above:

$$\boldsymbol{x}_1(t) = \mathrm{Re}\left(\boldsymbol{b}_{11}e^{\lambda t}\right)$$
$$\boldsymbol{x}_2(t) = \mathrm{Re}\left(\boldsymbol{b}_{21}e^{\lambda t} + t\boldsymbol{b}_{22}e^{\lambda t}\right)$$
$$\ldots$$
$$\boldsymbol{x}_p(t) = \mathrm{Re}\left(\boldsymbol{b}_{p1}e^{\lambda t} + t\boldsymbol{b}_{p2}e^{\lambda t} \ldots + t^{p-1}\boldsymbol{b}_{pp}e^{\lambda t}\right)$$
$$\boldsymbol{x}_{p+1}(t) = \mathrm{Im}\left(\boldsymbol{b}_{11}e^{\lambda t}\right)$$
$$\boldsymbol{x}_{p+2}(t) = \mathrm{Im}\left(\boldsymbol{b}_{21}e^{\lambda t} + t\boldsymbol{b}_{22}e^{\lambda t}\right)$$
$$\ldots$$
$$\boldsymbol{x}_{2p}(t) = \mathrm{Im}\left(\boldsymbol{b}_{p1}e^{\lambda t} + t\boldsymbol{b}_{p2}e^{\lambda t} \ldots + t^{p-1}\boldsymbol{b}_{pp}e^{\lambda t}\right)$$

  where the $\boldsymbol{b}$ vectors must be found (Theorem 2.9). See above for a method of finding the $\boldsymbol{b}$ vectors when the algebraic multiplicity is 2

- The general real solution is then found by combining all these solutions in a linear combination, with real coefficients

- A *fundamental matrix* is a matrix composed of linearly independent solutions to a homogeneous system, all arranged in columns:

$$\boldsymbol{\Phi}(t) = [\boldsymbol{x}_1(t) \quad \boldsymbol{x}_2(t) \quad \ldots \quad \boldsymbol{x}_n(t)]$$

  (Definition 2.12)

  - You can use this to write the general solution in a more compact way:

$$\boldsymbol{x}_{\mathrm{hom}}(t) = \boldsymbol{\Phi}(t)\begin{bmatrix} c_1 \\ \ldots \\ c_n \end{bmatrix}$$

    (Theorem 2.15)

  - The particular solution for which $\boldsymbol{x}(t_0) = \boldsymbol{x}_0$ is given by

$$\boldsymbol{x}(t) = \boldsymbol{\Phi}(t) \cdot \boldsymbol{\Phi}(t_0)^{-1} \cdot \boldsymbol{x}_0$$

    If $t_0 = 0$, then the condition $\boldsymbol{x}(t_0) = \boldsymbol{x}_0$ is called an initial condition. (Theorem 2.15)

- The solution to an inhomogeneous system is given by a formula using the fundamental matrix:

$$\boldsymbol{x}(t) = \boldsymbol{\Phi}(t)\left(\int_{t_0}^{t} \boldsymbol{\Phi}(\tau)^{-1} \cdot \boldsymbol{u}(\tau)\, d\tau + \boldsymbol{c}\right)$$

  where $\boldsymbol{c} = \begin{bmatrix} c_1 \\ \ldots \\ c_n \end{bmatrix}$ and t_0 can be chosen arbitrarily. $t_0 = 0$ usually gives the simplest solution. (Theorem 2.18)

  - Extra (from the exercises): Maple may give you a solution like

$$\boldsymbol{x}(t) = c_1 e^{-4t}\begin{bmatrix} -\frac{1}{3} \\ 1 \end{bmatrix} + c_2 e^{8t}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + e^{8t}\begin{bmatrix} \frac{1}{4} \\ \frac{1}{4} \end{bmatrix} + e^{-4t}\begin{bmatrix} -\frac{1}{4} \\ \frac{3}{4} \end{bmatrix}$$

    You may be required to simplify this. Here, the two vectors multiplied by e^{-4t} are multiples of each other, and so can be factorized. Similarly for the vectors multiplied by e^{8t}. You can perform these factorizations and then use the free parameters to absorb unnecessary numbers, since a free parameter added to or multiplied by a constant is the same as a free parameter by itself:

$$\boldsymbol{x}(t) = e^{-4t}\left(c_1\begin{bmatrix}-\frac{1}{3}\\1\end{bmatrix}+\begin{bmatrix}-\frac{1}{4}\\\frac{3}{4}\end{bmatrix}\right)+e^{8t}\left(c_2\begin{bmatrix}1\\1\end{bmatrix}+\begin{bmatrix}\frac{1}{4}\\\frac{1}{4}\end{bmatrix}\right)$$
$$= e^{-4t}\left(\begin{bmatrix}-1\\3\end{bmatrix}\left(\frac{1}{3}c_1+\frac{1}{4}\right)\right)+e^{8t}\left(\begin{bmatrix}1\\1\end{bmatrix}\left(c_2+\frac{1}{4}\right)\right)$$
$$= e^{-4t}c_1\begin{bmatrix}-1\\3\end{bmatrix}+e^{8t}c_2\begin{bmatrix}1\\1\end{bmatrix}$$

- Just like how there is a transfer function for inhomogeneous *n*th order equations, there is also a transfer function for inhomogeneous systems of first order equations
    - Here, in addition to the system, you need to know exactly which scalar *x* function in the system you are looking for, or which linear combination of *x* functions you are looking for. This *output function* containing a linear combination of *x* functions is called $y(t)$. For an inhomogeneous system with the form
    $$\boldsymbol{x}'(t) = \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{b}u(t)$$
    the output function has the form
    $$y(t) = \boldsymbol{d}^T\boldsymbol{x}(t)$$
    $$= [d_1 \quad \dots \quad d_n]\begin{bmatrix}x_1(t)\\\dots\\x_n(t)\end{bmatrix}$$
    $$= d_1x_1(t) + \cdots + d_nx_n(t)$$
    The output function is specified using a horizontal vector of constants $\boldsymbol{d}^T$ with *n* columns, where *n* is the number of equations in the system. This is multiplied by the vertical $\boldsymbol{x}(t)$ vector of solutions. Since you are multiplying a horizontal vector with *n* columns by a vertical vector with *n* rows, you get a scalar result, which is in this case a linear combination of the scalar *x* functions. If you are looking for a single one of the *x* functions, set the corresponding value in $\boldsymbol{d}^T$ to 1 and the rest to 0
    - Once you have a scalar output function, the transfer function for systems is given by
    $$H(s) = -\boldsymbol{d}^T(\boldsymbol{A} - s\boldsymbol{I})^{-1}\boldsymbol{b}$$
    (Theorem 2.20). However, like the *n*th order transfer function, this always has a domain restriction. The matrix $\boldsymbol{A} - s\boldsymbol{I}$ has to be invertible, so $\det(\boldsymbol{A} - s\boldsymbol{I})$ cannot equal 0. The domain restriction can simply be stated as $\det(\boldsymbol{A} - s\boldsymbol{I}) \neq 0$, or you can find the specific values of s that are not allowed
        - Remember the minus sign in front in the transfer function expression
    - For $u(t) = e^{st}$, the stationary solution is then given by
    $$y(t) = H(s)u(t) = H(s)e^{st}$$
    - Similarly to the *n*th order case, if there is a coefficient in front of e^{st}, that coefficient should not, in theory, be considered part of $u(t)$. It should instead be absorbed into the $\boldsymbol{b}$ vector and contribute to the transfer function. However, if you find a transfer function for an equation where the right-hand side is $u(t)$ by itself and then need to solve the equation where the right-hand side is $au(t)$ by itself, just multiplying $au(t)$ by the old transfer function has the same effect as creating a new transfer function that takes a into account, and then multiplying $u(t)$ by it

Stability

- A homogeneous system of first-order differential equations is said to be stable if the solution vector $\boldsymbol{x}(t)$ remains bounded within a certain distance from the origin, no matter how high t gets

- There are three situations:
  - A system is *unstable* if the solution is not bounded, i.e. it keeps going further from the origin as t tends to infinity
  - A system is *asymptotically stable* if the solution tends to the origin (all coordinate functions tend to 0) as t tends to infinity
  - A system is *marginally stable* if the solution is bounded, but does not tend to the origin

  (Definition 2.25)
- You can determine whether a system is stable using the eigenvalues of the system matrix $\boldsymbol{A}$. The real parts of the eigenvalues determine stability:
  - If **any** of the eigenvalues has a positive real part, the system is unstable (Theorem 2.31)
  - If all of the eigenvalues have negative real parts, the system is asymptotically stable (Theorem 2.35)
  - If all of the eigenvalues have real parts that are either negative or 0, you need to examine further. If every eigenvalue with zero real part has the same algebraic and geometric multiplicity, the system is marginally stable. Otherwise, it is unstable (Theorem 2.31)
- The *Routh-Hurwitz* criterion provides a method for determining the stability properties of a system without explicitly calculating the eigenvalues. Eigenvalues are found from a characteristic equation, and this method works directly with the coefficients of that equation. First, the characteristic equation must be in a form where the highest-order coefficient is 1. If it is not already 1, divide both sides of the equation by that coefficient. The equation then has the form

$$P(\lambda) = \lambda^n + a_1\lambda^{n-1} + \cdots + a_{n-1}\lambda + a_n$$

  The criterion says that the roots of this equation, which are the eigenvalues, will have negative real parts if and only if:
  - all the coefficients are greater than 0
  - all determinants of some special matrices are greater than 0

  The matrices required are constructed as such:

$$\begin{bmatrix} a_1 & a_3 & a_5 & a_7 & . & . \\ 1 & a_2 & a_4 & . & . & . \\ 0 & a_1 & a_3 & a_5 & . & . \\ 0 & 1 & a_2 & . & . & . \\ 0 & 0 & a_1 & . & . & . \\ . & . & . & . & . & . \end{bmatrix}$$

  You must use a number of $k \times k$ matrices formed in this way, starting from the top left, where k takes values in the range $2, \ldots, n-1$:

$$\det\left(\begin{bmatrix} a_1 & a_3 \\ 1 & a_2 \end{bmatrix}\right), \quad \det\left(\begin{bmatrix} a_1 & a_3 & a_5 \\ 1 & a_2 & a_4 \\ 0 & a_1 & a_3 \end{bmatrix}\right), \quad \ldots$$

  If the matrix asks for a coefficient that does not exist in the equation, like a_5 for a 4th degree polynomial, use 0 instead
  (Theorem 2.38)

- For a second-order polynomial, $n-1=1$, but the range for k starts at 2. Therefore, you do not need to consider any determinants. In this case, you only need to check

$$a_1 > 0 \text{ and } a_2 > 0$$

- For a third-order polynomial, $n-1=2$, so the range for k consists of only the number 2. Here, you need to check the three coefficients and one determinant:

$$a_1 > 0,\ a_2 > 0,\ a_3 > 0,\ \det\left(\begin{bmatrix} a_1 & a_3 \\ 1 & a_2 \end{bmatrix}\right) > 0$$

- For a fourth-order polynomial, $n-1=3$, so k takes the values 2 and 3. You need to check the four coefficients, and two determinants:

$$a_1 > 0,\ a_2 > 0,\ a_3 > 0,\ a_4 > 0,\ \det\left(\begin{bmatrix} a_1 & a_3 \\ 1 & a_2 \end{bmatrix}\right) > 0,\ \det\left(\begin{bmatrix} a_1 & a_3 & 0 \\ 1 & a_2 & a_4 \\ 0 & a_1 & a_3 \end{bmatrix}\right) > 0$$

Again, since a_5 does not exist here, we use 0 instead in the last matrix

- Stability is necessarily defined differently for inhomogeneous systems than it is for homogeneous systems
  - An inhomogeneous system is said to be asymptotically stable if the difference between any two particular solutions tends to 0 as t tends to infinity, for any forcing function $\boldsymbol{u}(t)$:

$$\boldsymbol{x}_i(t) - \boldsymbol{x}_j(t) \to 0 \text{ as } t \to \infty$$

(Definition 2.43)

  - An inhomogeneous system is asymptotically stable if and only if the corresponding homogeneous system is asymptotically stable (Theorem 2.44)
  - Another concept for inhomogeneous systems is BIBO-stability, which means that the solution $\boldsymbol{x}(t)$ is bounded as long as the forcing function $\boldsymbol{u}(t)$ is bounded. BIBO stands for "Bounded Input – Bounded Output" (Definition 2.45)
  - A system is BIBO-stable if and only if the corresponding homogeneous system is asymptotically stable (Theorem 2.46). Therefore, BIBO-stability, asymptotic stability, and asymptotic stability of the corresponding homogeneous system are all equivalent

Infinite series

Limits, infinite integrals

- A function $f(x)$ is said to have a *limit L* as x tends to infinity if the function can get as close as you want to the limit when *x* is high enough, and it does not go further away after that. More formally, for every number ε greater than 0, you can find an x_0 such that $|f(x) - L| < \varepsilon$ for all $x > x_0$
  - Limits can be written in a few different ways:

$$\lim_{x\to\infty}\big(f(x)\big) = L$$

$$f(x) \to L \text{ as } x \to \infty$$

$$f(x) \xrightarrow{x\to\infty} L$$

- Limits can distribute themselves over the basic operations:
  - The limit of a sum is the sum of the limits:

$$\lim_{x\to\infty}\big(f(x)+g(x)\big)=\lim_{x\to\infty}\big(f(x)\big)+\lim_{x\to\infty}\big(g(x)\big)$$

  - The limit of a product is the product of the limits:

$$\lim_{x\to\infty}\big(f(x)\cdot g(x)\big)=\lim_{x\to\infty}\big(f(x)\big)\cdot\lim_{x\to\infty}\big(g(x)\big)$$

  - For quotients, there are three situations:
    - If $\lim_{x\to\infty}\big(g(x)\big)\neq 0$, then

$$\lim_{x\to\infty}\left(\frac{f(x)}{g(x)}\right)=\frac{\lim_{x\to\infty}\big(f(x)\big)}{\lim_{x\to\infty}\big(g(x)\big)}$$

    - If $\lim_{x\to\infty}\big(g(x)\big)=0$ and $\lim_{x\to\infty}\big(f(x)\big)\neq 0$, then

$$\lim_{x\to\infty}\left(\frac{f(x)}{g(x)}\right)=\pm\infty$$

    - If both limits are 0, then it needs more analysis
- Limits do nothing when applied to constants or unrelated variables

$$\lim_{x\to\infty} n=n$$

- There is a hierarchy of functions when it comes to the rate at which they approach infinity. Logarithmic functions like $\ln(x)$ are the slowest, then come polynomials, and then exponential functions. When a slower function is divided by a faster one, the limit is 0, and when a faster function is divided by a slower one, the limit is $\pm\infty$:

$$\lim_{x\to\infty}\left(\frac{\ln(x)}{x^n}\right)=0$$

$$\lim_{x\to\infty}\left(\frac{e^x}{x^n}\right)=\infty$$

- Extra (from the exercises): The limit of a constant divided by a function that tends to infinity is 0:

$$\lim_{x\to\infty}\left(\frac{1}{x^2}\right)=0$$

This can be used to evaluate limits of fractions with polynomials. Reduce the fraction (divide both numerator and denominator) by the highest power, and then use the limit rules to find the limit. For example,

$$\lim_{n\to\infty}\left(\frac{3n^4+45n^2+217n-1015}{4n^4+3000n+5}\right)=\lim_{n\to\infty}\left(\frac{3+\frac{45}{n^2}+\frac{217}{n^3}-\frac{1015}{n^4}}{4+\frac{3000}{n^3}+\frac{5}{n^4}}\right)=\frac{3+0+0-0}{4+0+0}=\frac{3}{4}$$

- An integral can have ∞ as its upper bound, and this is dealt with using limits. Set the upper bound to a variable, and then take the limit of the integral as that variable tends to infinity:

$$\int_a^{\infty} f(x)\,dx = \lim_{t\to\infty}\left(\int_a^{t} f(x)\,dx\right)$$

  - If the integral $\int_a^t f(x)\,dx$ has a finite limit as $t \to \infty$, the infinite integral is *convergent*
  - If the integral $\int_a^t f(x)\,dx$ does not have a limit (it tends to $\pm\infty$), then the infinite integral is *divergent*. Informally, you might say that the infinite integral is equal to ∞ or $-\infty$

(Definition 4.1)

Sequences, series, and convergence tests

- A sequence is a list of numbers. It is written as

$$\{s_n\}_{n=a}^{b}$$

where s_n is a single element, *a* is the lower bound and *b* is the upper bound. For example, if we define $s_n = n^2$, then

$$\{s_n\}_{n=1}^{4} = \{1,4,9,16\}$$

  - Here, $s_1 = 1$, $s_2 = 4$, etc. You can think of s_n as the discrete version of a function, mapping integers to real numbers
  - Sequences are said to *sample* a continuous function at integer values. Here, the function $f(x) = x^2$ is sampled

- A sequence can have ∞ as its upper bound, in which case it is called an *infinite sequence*. An infinite sequence is said to be convergent if there exists a finite number *L* such that

$$\lim_{n\to\infty} s_n = L$$

L is called the limit of the sequence. A sequence that is not convergent is divergent
(Definition 4.5)

- An *infinite series* is the sum of all the elements in an infinite sequence:

$$\sum_{n=0}^{\infty} s_n = s_1 + s_2 + s_3 + \cdots$$

The *N*th partial sum, written as S_N, is the sum of the first *N* elements of the infinite sequence. If there is a sequence with elements s_n, the first few partial sums of the corresponding infinite series are

$$S_1 = s_1$$
$$S_2 = s_1 + s_2$$
$$S_3 = s_1 + s_2 + s_3$$

In general,

$$S_N = \sum_{n=0}^{N} s_n = s_1 + s_2 + \cdots + s_N$$

The elements of the sequence become the *terms* of the series.
Similarly to infinite integrals, infinite series are dealt with using limits. Take the limit of the partial sum as *N* tends to infinity:

$$\sum_{n=0}^{\infty} s_n = \lim_{N\to\infty}\left(\sum_{n=0}^{N} s_n\right) = \lim_{N\to\infty} S_N$$

This limit value is sometimes called S_∞

  - If the partial sum has a finite limit as $N \to \infty$, the infinite series is convergent

- If the partial sum does not have a limit (it tends to $\pm\infty$), then the infinite series is divergent. Informally, you might say that the infinite series is equal to ∞ or $-\infty$

(Definition 4.15)

- Even though an infinite series is, by definition, a sum, the limit value is still sometimes called "the sum of the series"
- An infinite series $\sum_{n=0}^{\infty} s_n$ is said to be *absolutely convergent* if the series with the absolute value function applied to all the terms, $\sum_{n=0}^{\infty} |s_n|$, is convergent (Definition 4.26)
  - If a series is absolutely convergent, it is also convergent (Theorem 4.27)
  - If a series is convergent but not absolutely convergent ($\sum_{n=0}^{\infty} s_n$ converges but $\sum_{n=0}^{\infty} |s_n|$ does not), it is called *conditionally convergent* (Definition 4.28)
- If an infinite sequence is convergent, that does not necessarily mean the corresponding infinite series is also convergent. To determine convergence of infinite series, you need to use *convergence tests:*
  - The nth term test (Theorem 4.19):
    If the corresponding infinite sequence does not tend to 0, the infinite series is divergent

$$\text{If } s_n \nrightarrow 0, \text{ then } \sum_{n=0}^{\infty} s_n \text{ is divergent}$$

    Intuitively, the series can only "slow down" and approach a finite value if the terms of the series constantly get smaller. If the terms do not get smaller and approach 0, the series must be divergent
  - The comparison test (Theorem 4.20):
    Assume that $0 \leq a_n \leq b_n$ for all $n \geq K$, where $K \in \mathbb{N}$
    - If $\sum_{n=1}^{\infty} b_n$ is convergent, then $\sum_{n=1}^{\infty} a_n$ is also convergent
    - If $\sum_{n=1}^{\infty} a_n$ is divergent, then $\sum_{n=1}^{\infty} b_n$ is also divergent

    The inequality $0 \leq a_n \leq b_n$ only has to be satisfied above a certain integer K, as finite sums up to a certain point do not impact convergence
  - The equivalence test:
    Definition 4.23:
    Two infinite series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are *equivalent* if there exists a constant *C* such that

$$\frac{a_n}{b_n} \to C \text{ as } n \to \infty$$

    In other words, the quotient of the terms needs to have a finite limit
    Proposition 4.24:
    If two series have positive terms and are equivalent, then they have the same convergence properties. If one of them is convergent, so is the other, and if one is divergent, so is the other

- The quotient test (Theorem 4.30):
  For a series $\sum_{n=1}^{\infty} s_n$, $s_n \neq 0$, if there is a number q such that
  $$\left|\frac{s_{n+1}}{s_n}\right| \to q \text{ as } n \to \infty$$
  then
  - if $q > 1$, the series is divergent
  - if $q < 1$, the series is absolutely convergent
  - if $q = 1$, the quotient test does not yield any information

  The quotient test is generally much simpler to apply than the previous ones, since you do not need to come up with another series to make a comparison. However, it is often inconclusive
- The integral test (Theorem 4.33):
  Given a function f that is continuous and constantly decreasing, the infinite integral $\int_1^{\infty} f(x)\,dx$ and the infinite series $\sum_{n=1}^{\infty} f(n)$ have the same convergence properties. If one is convergent, so is the other, and vice versa. It is easier to determine convergence for an integral, hence the usefulness of this method. This is generally the most powerful convergence test (and the easiest to cheese with Maple)

Truncation error, alternating series

- It is rarely possible to calculate the exact sum of an infinite series, and you usually have to make an approximation. There are two ways of doing this:
  - Calculate the partial sum with a large value of *N*. When taking part of an infinite series using a partial sum (truncating), the error is called the *truncation error*. When doing this, it can be important to know the maximum possible error being made
  - Use an infinite integral along with the partial sum to get closer to the true value than you would with the partial sum alone. Here, it is also important to know the maximum possible approximation error

  These two paths correspond to Methods 1 and 2 below. Method 1 has two versions, one from the book and one from the lecture (for some reason)

  - Method 1 (Book version) (Corollary 4.35):
    Given an infinite series
    $$\sum_{1}^{\infty} f(n)$$
    the truncation error for a partial sum is bounded as follows:
    $$|S_\infty - S_N| \leq \int_{N+1}^{\infty} f(x)\,dx + f(N+1)$$
    Therefore, if you have a specified tolerance ϵ, you must choose *N* such that
    $$\int_{N+1}^{\infty} f(x)\,dx + f(N+1) \leq \epsilon$$
    To clarify, this is because you cannot directly solve the equation $|S_N - S_\infty| \leq \epsilon$, so you have to find a value of *N* that makes the truncation error less than or equal to something, which is less than or equal to ϵ:
    $$|S_\infty - S_N| \leq \int_{N+1}^{\infty} f(x)\,dx + f(N+1) \leq \epsilon$$
    This same reasoning applies to the next two points

- Method 1 (Lecture version) (Theorem 1 from "The integral test" document):
  Given an infinite series
  $$\sum_{1}^{\infty} f(n)$$
  the truncation error for a partial sum is bounded as follows:
  $$|S_\infty - S_N| \leq \int_N^\infty f(x)\,dx$$
  Therefore, if you have a specified tolerance ϵ, you must choose *N* such that
  $$\int_N^\infty f(x)\,dx \leq \epsilon$$
- Method 2 (Corollary 4.35):
  Given an infinite series
  $$\sum_{1}^{\infty} f(n)$$
  you can estimate its sum using a value A_N given by
  $$A_N = S_N + \int_{N+1}^\infty f(x)\,dx$$
  $$= \sum_{1}^{N} f(n) + \int_{N+1}^\infty f(x)\,dx$$
  The approximation error of this value is bounded as follows:
  $$|S_\infty - A_N| \leq f(N+1)$$
  Therefore, if you have a specified tolerance ϵ, you must choose *N* such that
  $$f(N+1) \leq \epsilon$$

  Method 2 always gives a better approximation for the same *N*. However, you still need method 1 if you are asked for the value of *N* that brings the partial sum by itself within a certain tolerance of the infinite sum

- The truncation error from Method 1 can be stated as a sum. It is simply the rest of the infinite series, apart from the partial sum:
  $$|S_\infty - S_N| = \left|\sum_{n=1}^{\infty} a_n - \sum_{n=1}^{N} a_n\right| = \left|\sum_{n=N+1}^{\infty} a_n\right|$$
  Since this is also an infinite series, it cannot be directly evaluated, and so it is not immediately useful. However, this way of writing the truncation error does come in handy in some situations

- Extra: Inequality simplifications
  You sometimes have to simplify your expression with your own upper bounds before using the upper bound provided by Method 1. For example, suppose we want to know the maximum possible error being made when taking a partial sum of the series
  $$\sum_{n=1}^{\infty} \left|\frac{2(-1+\cos(n\pi))}{\pi n^2}\right|$$
  Using the lecture version of Method 1,
  $$|S_\infty - S_N| \leq \int_N^\infty \left|\frac{2(-1+\cos(n\pi))}{\pi n^2}\right| dx$$
  Maple gives a very complicated expression when evaluating this integral, which we cannot use. We thus have to make a simplification. We have that
  $$|2(-1+\cos(n\pi))| \leq 4 \text{ for all } n$$

This is because $\cos(n\pi)$ fluctuates between -1 and 1, so the highest magnitude value that $(-1+\cos(n\pi))$ takes is -2, and the highest magnitude value $2(-1+\cos(n\pi))$ takes is -4. Since we are taking the absolute value, the expression is always less than or equal to 4. We can then write the truncation error in the sum form and use this result:

$$|S_\infty - S_N| = \sum_{n=N+1}^{\infty} \left|\frac{2(-1+\cos(n\pi))}{\pi n^2}\right| \leq \sum_{n=N+1}^{\infty} \left|\frac{4}{\pi n^2}\right|$$

Since the truncation error of the old sum is less than or equal to the truncation error of the new sum, you can just use Method 1 on the new sum, and the upper bound given by that method will also hold for the old sum:

$$|S_\infty - S_N| = \sum_{n=N+1}^{\infty} \left|\frac{2(-1+\cos(n\pi))}{\pi n^2}\right| \leq \sum_{n=N+1}^{\infty} \left|\frac{4}{\pi n^2}\right| \leq \int_N^{\infty} \left|\frac{4}{\pi n^2}\right|$$

$$\Rightarrow |S_\infty - S_N| \leq \int_N^{\infty} \left|\frac{4}{\pi n^2}\right|$$

The new sum gives a much simpler integral, which can actually be used. Since we made this simplification, the value of N that we obtain will be a crude overestimate, but in this case, it was the only option. There are cases where you have to apply more than one simplification, building up a chain of $\leq$ signs. There are also cases where you can apply different simplifications, which then lead to different values of N. All that matters is that the method used to get to a value of N is valid

- An *alternating series* is a series with alternating positive and negative terms. In general, these have one of two forms:

$$\sum_{n=1}^{\infty} (-1)^{n-1} b_n = b_1 - b_2 + b_3 - b_4 + \cdots$$

$$\sum_{n=1}^{\infty} (-1)^{n} b_n = -b_1 + b_2 - b_3 + b_4 - \cdots$$

where $b_n \geq 0$ for all n. (Definition 4.37)

  - Note that, in these forms, the numbers b_n themselves are either positive or 0, and the coefficient in front is what makes the series alternating

- Finding out whether an alternating series is convergent and bounding the truncation error is much easier than with other series. Consider the two forms shown in the point above. For the series to be convergent, it must satisfy the following criteria:
  - the numbers b_n are positive for all *n* (strictly greater than 0)
  - b_n decreases monotonically ($b_1 \geq b_2 \geq b_3 \geq \cdots$)
  - $b_n \to 0$ as $n \to \infty$

  This is called the Leibniz test. Also, for any convergent alternating series, the truncation error is bounded as follows:

$$|S_\infty - S_N| \leq b_{N+1}$$

  If you have a specified tolerance ϵ, you much choose *N* such that

$$b_{N+1} \leq \epsilon$$

  (Theorem 4.38)

Geometric and power series

- A geometric series has the form

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots$$

where the number x is called the *quotient* (Definition 5.1)

- A geometric series is convergent if and only if $|x| < 1$, in which case the sum is

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$$

(Theorem 5.2)

  - If the index does not start at 0, the sum is

$$\sum_{n=N}^{\infty} x^n = \frac{x^N}{1-x}$$

(Corollary 5.5)

- Power series are a generalization of geometric series. They have the form

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \cdots$$

where the c_n coefficients depend on n, but not on x (Definition 5.12)

- All power series satisfy one of three convergence properties. A power series is either:

  i. only convergent for $x = 0$

  ii. absolutely convergent for all x

  iii. absolutely convergent for $|x| < \rho$, and divergent for $|x| > \rho$

  The number ρ is called the *radius of convergence*. In case iii, at the boundary values $x = \pm\rho$, there may or may not be convergence. In case (i), it is said that $\rho = 0$, and in (ii), $\rho = \infty$ (Theorem 5.13)

  - The best way to find the radius of convergence is usually with the quotient test, as it leads directly to an inequality with an obvious radius of convergence. For example, consider the series

$$\sum_{n=0}^{\infty} \frac{x^{2n}}{3^n(n^2+1)}$$

Using the quotient test,

$$\left|\frac{a_{n+1}}{a_n}\right| = \cdots = \left|\frac{x^2\left(1+\frac{1}{n^2}\right)}{3\left(1+\frac{2}{n}+\frac{2}{n^2}\right)}\right| \xrightarrow{n\to\infty} \frac{x^2}{3}$$

According to the quotient test, this has to be less than 1 for the series to be absolutely convergent:

$$\frac{x^2}{3} < 1$$
$$x^2 < 3$$
$$-\sqrt{3} < x < \sqrt{3}$$

From this, you can clearly tell that the radius of convergence is $\sqrt{3}$. If you want to go further, you can also write this inequality in the form $|x| < \rho$ as $|x| < \sqrt{3}$

- Because the terms in a power series depend on a variable x, every term can be considered a function of x, and so power series are an example of *infinite series of functions.* The infinite sum of the series and every partial sum are also functions, called $S(x)$ and $S_N(x)$ respectively, where N is the number of terms being used:

$$S(x) = \sum_{n=0}^{\infty} c_n x^n, \quad |x| < \rho$$

$$S_N(x) = \sum_{n=0}^{N} c_n x^n$$

However, the $S(x)$ notation for the infinite sum is rarely used. Normally, you just define a function $f(x)$ as equal to the series:

$$f(x) = \sum_{n=0}^{\infty} c_n x^n, \quad |x| < \rho$$

$S(x)$ and $f(x)$ are only defined over the interval where the power series is convergent, i.e. within the radius of convergence, stated as $|x| < \rho$

  - Since power series are a generalization of geometric series, this also applies to geometric series

- If you define a function using a power series, then you know that the function will be infinitely often differentiable. The first derivative is given by

$$f'(x) = \sum_{n=1}^{\infty} c_n n x^{n-1}, \quad |x| < \rho$$

In general, the *k*th derivative is given by

$$f^{(k)}(x) = \sum_{n=k}^{\infty} c_n n(n-1) \dots (n-k+1) x^{n-k}, \quad |x| < \rho$$

(Theorem 5.17)

  - Notice the change of summation index in the derivative. This is not something you should always do. The index is changed when the first term of $f(x)$ is constant, because it does not need to be included in the derivative, and it may cause trouble when taking higher order derivatives. For example, consider the general power series starting at 0:

$$f(x) = \sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \cdots$$

Here, the first term c_0 does not depend on x, so the summation index $n = 0$ should be changed to $n = 1$ in the derivative, as shown above. However, if the first term of the series is not a constant, then the index should stay as it is. For example,

$$f(x) = \sum_{n=1}^{\infty} x^n = x + x^2 + x^3 + \cdots$$

The first term depends on x, so the summation index should not be changed when taking the first derivative:

$$f'(x) = \sum_{n=1}^{\infty} n x^{n-1} = 1 + 2x + 3x^2 + \cdots$$

If you now want to take the second derivative, the first term of the first derivative is a constant, so you will have to change the index. In general, you should always check the first term when taking the derivative of a series

- Taylor series are an example of power series equal to a function. In fact, if you have a series with radius of convergence $\rho > 0$, and you set a function $f(x)$ equal to that series,

$$f(x) = \sum_{n=0}^{\infty} c_n x^n$$

then the series is **necessarily** the Taylor series for $f(x)$ with point of expansion $x_0 = 0$. Therefore, you have that

$$c_n = \frac{f^{(n)}(0)}{n!}$$

(Theorem 5.20)

  - The Taylor series for a function is sometimes called the *power series representation* of that function
  - However, not every function has a power series representation

Uniform convergence and majorant series

- Power series are one example of infinite series of functions, but you can generalize this to infinite series of any kind of function, and again have a function defined as the sum of the series:

$$f(x) = \sum_{n=0}^{\infty} f_n(x) = f_0(x) + f_1(x) + f_2(x) + \cdots$$

where $f_n(x)$, the expression inside the sum sign, can be thought of as a function of two variables, n and x. However, n can only assume integer values, and it is fully controlled by the sum sign

  - For a fixed x, this just becomes a series of numbers
  - Do not confuse $f(x)$ with $f_1(x)$. $f(x)$ is the function given by the sum of the series, while $f_1(x)$ is one of the terms (an instance of $f_n(x)$)
  - The domain of $f(x)$ is the interval where the series is convergent. This does not necessarily have anything to do with the domains of the $f_n(x)$ functions
  - As before, you can also have a partial sum function:

$$S_N(x) = \sum_{n=0}^{N} f_n(x)$$

- The convergence considered so far is more precisely called *pointwise convergence*; this is where for any x, you can pick N such that the partial sum S_N gets within a certain range of S_∞. For this type of convergence, N will depend on the specific x you are considering
- However, there is another kind of convergence, called *uniform convergence*. This is where you can pick N to get within a certain range of S_∞ **for all** x. The partial sum will then be a function that gets within the specified tolerance away from the true function $f(x)$ no matter the input
  - More formally, an infinite series function $f(x) = \sum_{n=1}^{\infty} f_n(x)$ with domain I is uniformly convergent if for each $\epsilon > 0$, you can find an $N_0 \in \mathbb{N}$ such that

$$\left| f(x) - \sum_{n=1}^{N} f_n(x) \right| \leq \epsilon \text{ for all } x \in I \text{ and all } N \geq N_0$$

(Definition 5.28)

- A *majorant series* for a series of functions $\sum_{n=1}^{\infty} f_n(x)$ is a series of numbers $\sum_{n=1}^{\infty} k_n$ such that
$$|f_n(x)| \leq k_n$$
for every term n and for all x. In other words, every term k_n of the majorant series has to be greater than or equal to the absolute value of the equivalent term in the function series, no matter what input x the function term takes. A series that satisfies this and is also convergent is called a *convergent majorant series* for the function series (Definition 5.31)
    - For example, in the series
    $$\sum_{n=1}^{\infty} \frac{\cos(nx)}{n^4}$$
    the $\cos(nx)$ expression only assumes values between -1 and 1. After taking the absolute value, it goes between 0 and 1. Therefore,
    $$\left|\frac{\cos(nx)}{n^4}\right| \leq \frac{1}{n^4}$$
    and the series has a majorant series formed from these upper bound terms:
    $$\sum_{n=1}^{\infty} \frac{1}{n^4}$$
    This majorant series is convergent
- If a series of continuous functions
$$\sum_{n=1}^{\infty} f_n(x)$$
has a **convergent** majorant series, then the series is uniformly convergent. In addition, if you set a function $f(x)$ equal to the sum of the series,
$$f(x) = \sum_{n=1}^{\infty} f_n(x)$$
then $f(x)$ is continuous (Theorem 5.33)
- If some properties related to uniform convergence are satisfied, you can more easily find the integral or derivative of an infinite series:
    - If a series of continuous functions is uniformly convergent, then you can integrate the series by integrating inside the sum sign:
    $$\int_a^b \sum_{n=1}^{\infty} f_n(x)\,dx = \sum_{n=1}^{\infty} \int_a^b f_n(x)\,dx$$
    (Theorem 5.34)
    - Suppose you have a function given by a series of functions:
    $$f(x) = \sum_{n=1}^{\infty} f_n(x)$$
    In order to do something similar to the integral method above, and find the derivative of this function by differentiating inside the sum sign,
    $$f'(x) = \sum_{n=1}^{\infty} f_n'(x)$$
    the requirements are different. Two requirements must be met:
        - The $f_n(x)$ functions must be differentiable with a continuous derivative

- The series of the **derivatives** of the $f_n(x)$ functions has to be uniformly convergent. This series has the form

$$\sum_{n=1}^{\infty} f_n'(x)$$

You do not need to show that the original series is uniformly convergent, but that this new series made up of the function derivatives is uniformly convergent. To do this, as always, you need to show that the derivative series has a convergent majorant series:

$$|f_n'(x)| \leq k_n, \quad \sum_{n=1}^{\infty} k_n \text{ is convergent}$$

If these two requirements are met, it also immediately tells you that $f(x)$ is differentiable
(Theorem 5.35)

  - If $f(x)$ is given by a power series, and all you need to do is show that it is differentiable, Theorem 5.17 already tells you that it is differentiable without needing these requirements

- Extra: Suppose you have an infinite series **of functions** and you want to put a bound on the truncation error for all x. This is only possible if the series is uniformly convergent according to the definition, and it can be achieved with the majorant series. The truncation error of a series will always be less than or equal to the same truncation error of the corresponding majorant series. We use the sum expression of the truncation error to show this:

$$|S(x) - S_N(x)| = \left| \sum_{n=N+1}^{\infty} f_n(x) \right| \leq \sum_{n=N+1}^{\infty} |f_n(x)| \leq \sum_{n=N+1}^{\infty} k_n$$

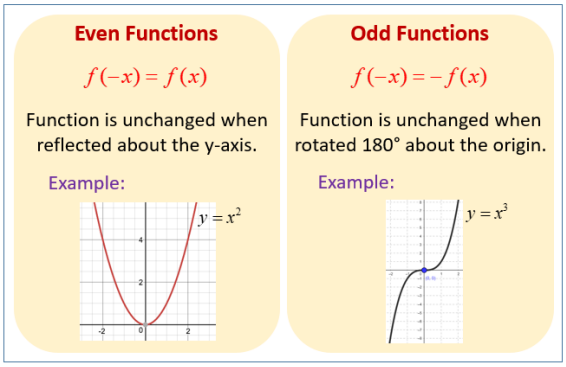
You can then use either version of Method 1 to put a bound on the truncation error of the majorant series. To be precise, we are able to put the absolute value sign inside the sum in the second step because of the *triangle inequality* (Appendix E), which states that

$$|a + b| \leq |a| + |b|$$

  - This is very similar to the inequality simplification method. That method is needed when the expression is too complex, whereas a majorant series upper bound is needed simply because of the definition of uniform convergence. Of course, if the majorant series used for this method is too complex, you can simplify further using that method

Fourier series

- Prerequisite: Even and odd functions
  - A function is called *even* if $f(x) = f(-x)$, which means that the function graph is symmetrical about the *y*-axis
  - A function is called *odd* if $f(-x) = -f(x)$, or $f(x) = -f(-x)$, which means that the function graph has rotational symmetry about the origin



  - For even functions, any integral over a region symmetrical about the *y*-axis is equal to double the integral over the positive part of that region:

$$\int_{-a}^{a} f(x)\,dx = 2 \cdot \int_{0}^{a} f(x)\,dx$$

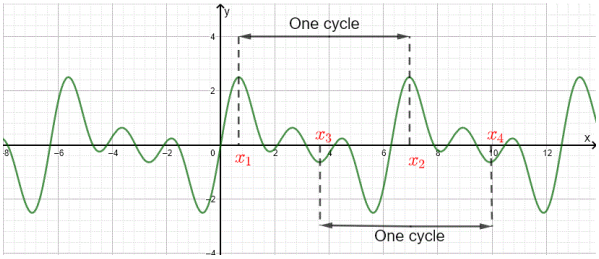
For odd functions, any integral over a region symmetrical about the *y*-axis is 0:

$$\int_{-a}^{a} f(x)\,dx = 0$$

(Lemma 6.2)

  - The property of being even or odd is called *parity*

- Prerequisite: Periodic functions
  - A function is called *periodic* with period T if it repeats every period, or $f(x+T) = f(x)$. For example, the trigonometric functions
  - If you integrate a periodic function over a full period, the integral will always be the same, regardless of start and end point:



$$\int_{0}^{T} f(x)\,dx = \int_{-T/2}^{T/2} f(x)\,dx = \int_{-T/4}^{3T/4} f(x)\,dx = \cdots$$

(Lemma 6.2)

  - Because of this, an integral over a full period is given a new notation, only specifying the length of the period:

$$\int_{T} f(x)\,dx$$

  - We can work solely with periods of 2π without loss of generality
  - Periodic functions are normally defined by stating that the function is periodic, and giving its expression within a single period
  - Periodic functions can also be defined by stating that the function is periodic and either even or odd, and then giving its expression within half a period, like $0 \leq x \leq \pi$. You can use the information about whether it is even or odd to extrapolate to the other half of the period
  - In general, it is always a good idea to sketch a periodic function before using it

- The *Fourier series* associated with a 2π-periodic function f is the function series given by

$$f \sim \frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos(nx) + b_n \sin(nx))$$

where

$$a_n = \frac{1}{\pi}\int_{2\pi} f(x)\cos(nx)\,dx$$
$$b_n = \frac{1}{\pi}\int_{2\pi} f(x)\sin(nx)\,dx$$

The tilde $\sim$ means "is associated with". a_n and b_n are called *Fourier coefficients* (Definition 6.1)

  - Note that a_n and b_n are only functions of n, not x. The xs in the expressions for a_n and b_n are dummy variables for the integral
  - The Fourier coefficients are easier to find when the function f is even or odd, because of the rules relating to integrals of even and odd functions:
    - If f is even,

$$b_n = 0 \text{ for all } n$$
$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos(nx)\,dx$$

    - If f is odd,

$$a_n = 0 \text{ for all } n$$
$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin(nx)\,dx$$

    (Theorem 6.3)
  - Most of the time, the general expression you find for a_n cannot be used to find a_0, as it contains a division by n to some power. For example,

$$a_n = \frac{2}{\pi}\int_0^{\pi} x^2\cos(nx)\,dx = \left[\frac{n^2x^2\sin(nx) - 2\sin(nx) + 2nx\cos(nx)}{n^3}\right]_0^{\pi}$$

    In these cases, you have to find a_0 separately by inserting 0 into the integral:

$$a_0 = \frac{2}{\pi}\int_0^{\pi} x^2\cos(0x)\,dx = \frac{2}{\pi}\int_0^{\pi} x^2\,dx = \cdots$$

  - The $\frac{1}{2}a_0$ term in the Fourier series is the average value of the function, since the infinite sum averages out to 0

- A function is said to be *piecewise differentiable* if
  - you can isolate finitely many points of discontinuity or points where the derivative is not defined
  - the function is given by a subfunction f_i in every **open** (exclusive) interval between these points
  - every subfunction is differentiable with a continuous derivative in the **closed** (inclusive) interval between the points

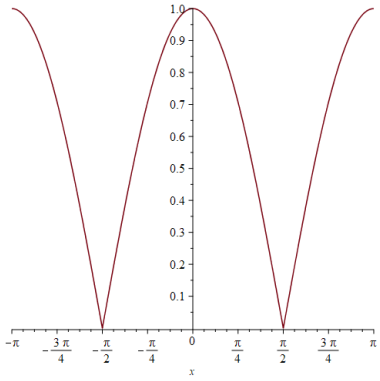
  More formally, a function is piecewise differentiable if there exist finitely many points x_1, x_2, x_3, ... and functions $f_i: [x_i, x_{i+1}] \to \mathbb{R}$, such that each f_i is differentiable with a continuous derivative, and

  $$f(x) = f_i(x), \quad x \in]x_i, x_{i+1}[$$

  (Definition 6.10)

  - For example, the function $f(x) = |\cos(x)|, \ x \in [-\pi, \pi[$ is piecewise differentiable, as you can split it into subfunctions with open intervals and finite points as follows:

$$f(x) = \begin{cases} 1, & x = -\pi \\ -\cos(x), & x \in]-\pi, -\frac{\pi}{2}[\\ 0, & x = -\frac{\pi}{2} \\ \cos(x), & x \in]-\frac{\pi}{2}, \frac{\pi}{2}[\\ 0, & x = \frac{\pi}{2} \\ -\cos(x), & x \in]\frac{\pi}{2}, \pi[\end{cases}$$



    The function has an undefined derivative at $-\frac{\pi}{2}$ and $\frac{\pi}{2}$. However, this strict adherence to the formulation of the definition makes the piecewise function very long. It can be stated more simply as

$$f(x) = \begin{cases} -\cos(x), & x \in [-\pi, -\frac{\pi}{2}[\\ \cos(x), & x \in [-\frac{\pi}{2}, \frac{\pi}{2}[\\ -\cos(x), & x \in [\frac{\pi}{2}, \pi[\end{cases}$$

    Here, the function is split into subfunctions in intervals that are closed on one side and open on the other. This does not matter for the definition, since the open interval (e.g. $]-\pi, -\frac{\pi}{2}[$) is always a subset of the half-open interval (e.g. $[-\pi, -\frac{\pi}{2}[$). No matter which formulation of the piecewise function you use, you can see that each of the subfunctions is differentiable with a continuous derivative on the closed intervals $\left[-\pi, -\frac{\pi}{2}\right]$, $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, and $\left[\frac{\pi}{2}, \pi\right]$ respectively, and so the function is piecewise differentiable

  - On the other hand, the function $f(x) = \sqrt{|x|}, \ x \in [-\pi, \pi[$ is not piecewise differentiable. You can split it into subfunctions as follows:

$$f(x) = \begin{cases} \sqrt{x}, & x \in [0, \pi[\\ \sqrt{-x}, & x \in [-\pi, 0[\end{cases}$$

    However, this time, $\sqrt{x}$ is not differentiable on the closed interval $[0, \pi]$ since it is not

differentiable at 0, and $\sqrt{-x}$ is not differentiable on the closed interval $[-\pi, 0]$ since it is also not differentiable at 0. Therefore, the function is not piecewise differentiable

- The function value at the discontinuity/undefined derivative points does not matter. For example, the function

$$f(x) = \begin{cases} x^2, & x < 2 \\ 167, & x = 2 \\ \log(x), & x > 2 \end{cases}$$

is piecewise differentiable

- If a function is piecewise differentiable, then its Fourier series will converge for all $x \in \mathbb{R}$, and
  - if f is continuous at a point x_0, then the series will converge to $f(x_0)$ at that point
  - if f has a discontinuity at x_0, then the series will converge to the midpoint of the function limits from both sides, given by

$$\frac{1}{2}\left(\lim_{x \to x_0^-}\big(f(x)\big) + \lim_{x \to x_0^+}\big(f(x)\big)\right)$$

where $\lim_{x \to x_0^-}\big(f(x)\big)$ is the limit of $f(x)$ as x tends to x_0 from the left, and $\lim_{x \to x_0^+}\big(f(x)\big)$ is the limit as x tends to x_0 from the right. The value thus does not depend on the actual function value at the discontinuity, only on the limits of the continuous functions on either side

(Theorem 6.12)

- Because of this, if a function is piecewise differentiable and continuous everywhere, we can replace the association ~ between the function and the Fourier series with an equals sign:

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos(nx) + b_n \sin(nx)$$

Also, the series will converge uniformly, and the truncation error will be bounded by

$$|f(x) - S_N(x)| \leq \frac{1}{\sqrt{N}}\frac{1}{\sqrt{\pi}}\sqrt{\int_{2\pi} |f'(x)|^2\, dx}$$

(Corollary 6.13)

Therefore, if you have a specified tolerance ϵ, you must choose N such that

$$\frac{1}{\sqrt{N}}\frac{1}{\sqrt{\pi}}\sqrt{\int_{2\pi} |f'(x)|^2\, dx} \leq \epsilon$$

Transposing for N, you get

$$N \geq \frac{\int_{2\pi} |f'(t)|^2\, dt}{\pi\epsilon^2}$$

(Corollary 6.16)

- However, there is a much more accurate way to truncate a Fourier series within a specified tolerance ϵ. Given that the Fourier coefficients are known, the truncation error is bounded by

$$|f(x) - S_N(x)| \leq \sum_{n=N+1}^{\infty} (|a_n| + |b_n|)$$

(Theorem 6.17). This sum is an infinite sum, and so cannot be evaluated directly. However, it is equal to the truncation error from truncating the series

$$\sum_{n=1}^{\infty} (|a_n| + |b_n|)$$

with the partial sum S_N. Therefore, you can use either the book version or the lecture version of Method 1 to put a bound on this value as well. Here, I will show the lecture version, as it is simpler:

$$|f(x) - S_N(x)| \leq \sum_{n=N+1}^{\infty} (|a_n| + |b_n|) \leq \int_N^{\infty} |a_n| + |b_n| \, dn$$

In this case, you must choose N such that

$$\int_N^{\infty} |a_n| + |b_n| \, dn \leq \epsilon$$

Fourier series in complex form, Parseval's theorem

- Complex exponential functions provide an alternative way of writing the Fourier series associated with a function. This is called a Fourier series in complex form, and it is given by

$$f \sim \sum_{n=-\infty}^{\infty} c_n e^{inx}$$

where

$$c_n = \frac{1}{2\pi} \int_{2\pi} f(x) e^{-inx} \, dx$$

(Definition 6.21)

  - Unlike the real version, these series are an infinite sum in both directions, with an infinite start point as well as an infinite end point. Partial sums are obtained by setting the start point to $-N$ and the end point to N:

$$S_N(x) = \sum_{n=-N}^{N} c_n e^{inx}$$

(Lemma 6.20)

  - Even though the expression inside the sum sign contains complex numbers, the sum is real-valued as all the imaginary parts cancel each other out
  - Extra (from Mini-project 2): As with a_0 in the real form, you usually need to find c_0 separately. When this happens, c_0 cannot be part of the summation, and so the summation has to skip $n = 0$ while the term with coefficient c_0 remains outside the sum sign. This term is $c_0 e^{i0x}$, and since $e^{i0x} = e^0 = 1$, it simplifies to just c_0. The skipping is also given a special notation, and the whole thing looks as follows:

$$f \sim c_0 + \sum_{n=\infty,\ n\neq 0}^{\infty} c_n e^{inx}$$

- You can convert between the a_n and b_n coefficients of the real form and the c_n coefficient of the complex form:

$$c_0 = \frac{1}{2}a_0, \quad c_n = \frac{a_n - ib_n}{2}, \quad c_{-n} = \frac{a_n + ib_n}{2}$$

$$a_0 = 2c_0, \quad a_n = c_n + c_{-n}, \quad b_n = i(c_n - c_{-n})$$

(Lemma 6.22)

  - A separate expression is needed for c_{-n} because c_n is defined for negative values of n, whereas a_n and b_n are not

- For real-valued functions (in this course, always), it is true that

$$c_{-n} = \overline{c_n}$$

where $\overline{c_n}$ denotes the complex conjugate of c_n (Lemma 6.23)

- Extra (mostly): Truncating Fourier series in complex form

Directly truncating Fourier series in complex form is more complicated than in real form. When doing this, you are subtracting an infinite series which is infinite in both directions, by a partial sum of terms with values within a distance N from 0 in both directions. Because of this, you need two sums in standard form to state the truncation error, one from $-\infty$ to $-(N+1)$, and one from $N+1$ to ∞:

$$|f(x) - S_N(x)| = \left| \sum_{n=-\infty}^{\infty} c_n e^{inx} - \sum_{n=-N}^{N} c_n e^{inx} \right| = \left| \sum_{n=-\infty}^{-(N+1)} c_n e^{inx} + \sum_{n=N+1}^{\infty} c_n e^{inx} \right|$$

This sum can be stated with a new notation, saying that the sum covers all n where the absolute value of n is greater than N:

$$|f(x) - S_N(x)| = \sum_{|n|>N} c_n e^{inx}$$

Unfortunately, we were not given any concrete theorems for dealing with these sums specifically, but there are a few useful methods:

  - Of course, you may just be able to convert to the real form and truncate from there. If you cannot do that, see the next points

  - Firstly, the notation with two sums is more practically useful in most situations. This is because of the previously mentioned triangle inequality for absolute values, which states that

$$|a + b| \leq |a| + |b|$$

(Appendix E). This means that you can split the expression above as follows, using a less than or equal to sign:

$$\left| \sum_{n=-\infty}^{-(N+1)} c_n e^{inx} + \sum_{n=N+1}^{\infty} c_n e^{inx} \right| \leq \left| \sum_{n=-\infty}^{-(N+1)} c_n e^{inx} \right| + \left| \sum_{n=N+1}^{\infty} c_n e^{inx} \right|$$

  - From here, you may find a way to merge the two sums together. Then, as mentioned with truncating the real form, you can use the fact that a series of the form $\sum_{n=N+1}^{\infty} a_n$ is equal to the truncation error from truncating the series

$$\sum_{n=1}^{\infty} a_n$$

with the partial sum S_N. This lets you apply either version of Method 1 to put an upper bound on the value of the series $\sum_{n=N+1}^{\infty} a_n$

- You can also apply the triangle inequality to the sums themselves, in order to put the absolute value signs inside the sum signs:

$$\left|\sum_{n=-\infty}^{-(N+1)} c_n e^{inx}\right| + \left|\sum_{n=N+1}^{\infty} c_n e^{inx}\right| \leq \sum_{n=-\infty}^{-(N+1)} \left|c_n e^{inx}\right| + \sum_{n=N+1}^{\infty} \left|c_n e^{inx}\right|$$

- There are two more identities that may come in handy, which state that the absolute value can distribute over products and quotients:

$$|ab| = |a||b|$$
$$\left|\frac{a}{b}\right| = \frac{|a|}{|b|}$$

- The absolute value of e^{inx} is always 1, due to how the complex exponential function is defined. This, combined with the previous point, allows for the following:

$$\sum_{n=N+1}^{\infty} \left|c_n e^{inx}\right| = \sum_{n=N+1}^{\infty} |c_n|\left|e^{inx}\right| = \sum_{n=N+1}^{\infty} |c_n|$$

- Parseval's theorem states that, for a function f with Fourier coefficients a_n, b_n in real form and c_n in complex form, the following holds:

$$\frac{1}{2\pi}\int_{2\pi} |f(x)|^2\, dx$$
$$= \frac{1}{4}|a_0|^2 + \frac{1}{2}\sum_{n=1}^{\infty}(|a_n|^2 + |b_n|^2)$$
$$= \sum_{n=-\infty}^{\infty} |c_n|^2$$

(Theorem 6.25)

- This can be used to obtain new results for values of infinite series, in addition to values obtainable from plain Fourier series
- The value $\frac{1}{2\pi}\int_{2\pi}|f(x)|^2\, dx$ is called the *power* of a series

- All the theorems for Fourier series stated here have assumed a function period of 2π. However, with a few modifications, the theorems can also be stated for an arbitrary period T. These generalized forms can be found in Appendix C of the book

Advanced differential equation methods

Fourier's method

- Fourier's method is a method that allows Fourier series to be used in solving systems of differential equations:
  Consider a transfer function $H(s)$ associated with an asymptotically stable system of differential equations

$$\boldsymbol{x}'(t) = \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{b}u(t)$$
$$y(t) = \boldsymbol{d}^T\boldsymbol{x}(t)$$

  If the forcing function $u(t)$ is 2π-periodic, piecewise differentiable, continuous, and given by the Fourier series

$$u(t) = \sum_{n=-\infty}^{\infty} c_n e^{int}$$

  then the system has the solution

$$y(t) = \sum_{n=-\infty}^{\infty} c_n H(in) e^{int}$$

  (Theorem 7.8)

  - A similar result applies if the forcing function is defined by a partial sum. If $u(t)$ is given by

$$u(t) = \sum_{n=-N}^{N} c_n e^{int}$$

    then the solution is given by

$$y(t) = \sum_{n=-N}^{N} c_n H(in) e^{int}$$

    (Lemma 7.7)

  - Extra (from Mini-project 2): When the c_0 term is outside the summation, you still multiply the expression inside the sum sign by the transfer function, but you also multiply c_0 by the transfer function applied at $n = 0$, which is $H(0)$:

$$u(t) = c_0 + \sum_{n=-\infty,\ c_0 \neq 0}^{\infty} c_n e^{int}$$
$$y(t) = c_0 H(0) + \sum_{n=-\infty,\ c_0 \neq 0}^{\infty} c_n H(in) e^{int}$$

    You can think of this as using the transfer function on $c_0 e^0$ as well as on the sum. You can also think of it as putting c_0 inside the sum sign by defining c_n piecewise,

$$u(t) = \sum_{n=-\infty}^{\infty} c_n e^{int}, \quad c_n = \begin{cases} c_0, & n = 0 \\ \dots, & n \neq 0 \end{cases}$$

    then using the transfer function as usual, and then "unpacking" the sum sign once more

Power series method

Please note that the power series method is the most difficult part of this course, so there is a higher chance of mistakes in this section

- Prerequisite: Change of index
  You can change the starting index of an infinite sum to any number you want, as long as you change all instances of the variable inside the sum expression to match, so that all the terms of the sum remain the same. If you shift the starting point up by a number a, you have to change the variable n to $(n-a)$ and if you shift it down by a, you have to replace n with $(n+a)$. For example:
  $$\sum_{n=0}^{\infty} f(n) = \sum_{n=1}^{\infty} f(n-1) = f(0) + f(1) + f(2) + \cdots$$
  $$\sum_{n=2}^{\infty} f(n) = \sum_{n=0}^{\infty} f(n+2) = f(2) + f(3) + f(4) + \cdots$$
- Power series are unique, which means that if a power series has a certain sum, there is only one choice of coefficients that can lead to that sum. This has two important implications. If two power series with the same starting point have the same sum,
  $$\sum_{n=0}^{\infty} c_n t^n = \sum_{n=0}^{\infty} d_n t^n$$
  then they must also have the same coefficients ($c_n = d_n$ for all n). Moreover, if you have a power series equal to 0,
  $$\sum_{n=0}^{\infty} c_n t^n = 0$$
  then $c_n = 0$ for all n, because these are the only coefficients that make the sum 0. (Corollary 5.21)
- The *power series method* is a method that allows for solving of *n*th order differential equations with coefficients that are not constant, but instead are polynomial functions of the variable t:
  $$a_0(t)y^{(n)}(t) + a_1(t)y^{(n-1)}(t) + \cdots + a_{n-1}(t)y'(t) + a_n(t)y(t) = u(t)$$
  The solution $y(t)$ will be a power series, of the form
  $$y(t) = \sum_{n=0}^{\infty} c_n t^n$$
- I will show the basic method with an example. Consider the differential equation
  $$\frac{d^2y}{dt^2} + \frac{2}{t}\frac{dy}{dt} + y = 0$$
  - The coefficients are not polynomials. However, they can be made into polynomials by multiplying both sides of the equation by t:
    $$t\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + ty = 0$$

- We then want to plug the power series form of $y(t)$ into this equation. The equation contains $y(t)$ itself, as well as the first and second derivatives. The derivatives can be found using Theorem 5.17:
$$y = \sum_{n=0}^{\infty} c_n t^n, \quad \frac{dy}{dt} = \sum_{n=1}^{\infty} c_n n t^{n-1}, \quad \frac{d^2y}{dt^2} = \sum_{n=2}^{\infty} c_n n(n-1) t^{n-2}$$
- Plugging these results into the equation, we get
$$t\sum_{n=2}^{\infty} c_n n(n-1)t^{n-2} + 2\sum_{n=1}^{\infty} c_n n t^{n-1} + t\sum_{n=0}^{\infty} c_n t^n = 0$$
- We then simplify by absorbing the lone t coefficients into the sums:
$$\sum_{n=2}^{\infty} c_n n(n-1)t^{n-1} + 2\sum_{n=1}^{\infty} c_n n t^{n-1} + \sum_{n=0}^{\infty} c_n t^{n+1} = 0$$
- It is then useful to write the first few terms of each sum:
$$(c_2 2t + c_3 3 \cdot 2t^2 + \cdots) + 2(c_1 + c_2 2t + \cdots) + (c_0 t + c_1 t^2 + \cdots) = 0$$
The only term here that does not have coefficient t is the c_1 in the second sum. We will later merge the sums together, and this c_1 will be the only contributor to the t^0 coefficient. Since the right-hand side of the equation is 0, using the uniqueness theorem, we can already conclude that c_1 must be 0
- Knowing that the first term of the second sum is 0, we can increase the index of the second sum from 1 to 2 without changing anything else:
$$\sum_{n=2}^{\infty} c_n n(n-1)t^{n-1} + 2\sum_{n=2}^{\infty} c_n n t^{n-1} + \sum_{n=0}^{\infty} c_n t^{n+1} = 0$$
We could also keep c_1 in the equation, but take it out of the sum
- We then use the change of index rule to make all the sums start at 0:
$$\sum_{n=0}^{\infty} c_{n+2}(n+2)(n+1)t^{n+1} + 2\sum_{n=0}^{\infty} c_{n+2}(n+2)t^{n+1} + \sum_{n=0}^{\infty} c_n t^{n+1} = 0$$
- Now all the sums have the same starting point and the same power of t, so we can merge them together and factor out t^{n+1}:
$$\sum_{n=0}^{\infty} [(n+2)(n+1)c_{n+2} + 2(n+2)c_{n+2} + c_n]t^{n+1} = 0$$
Note that the square brackets mean the same thing as parentheses in this case, and are only used to make the equation clearer
- Since the right-hand side of the equation is 0, using the uniqueness theorem, we know that the coefficient of t^{n+1} inside the sum has to be 0:
$$(n+2)(n+1)c_{n+2} + 2(n+2)c_{n+2} + c_n = 0$$
Simplifying, we get
$$c_{n+2}(n+2)(n+1+2) + c_n = 0$$
$$c_{n+2}(n+2)(n+3) + c_n = 0$$
- This is an equation relating c_n to c_{n+2}. We can rewrite it as follows:
$$c_{n+2} = \frac{-c_n}{(n+2)(n+3)}$$
This is called a *recursion relation*. If you know what c_n is, this formula tells you what c_{n+2} is

- Since we know that $c_1 = 0$ from earlier, we can use the recursion relation to conclude that $c_n = 0$ for all odd numbers n
- For the even numbers, all we can do is try to find a general formula for c_n in terms of c_0:

$$c_2 = -\frac{c_0}{2 \cdot 3}, \quad c_4 = -\frac{c_2}{4 \cdot 5} = \frac{c_0}{2 \cdot 3 \cdot 4 \cdot 5}, \dots$$

$$c_{2k} = \frac{c_0(-1)^k}{(2k+1)!}, \quad k = 1,2,3,4, \dots$$

where the new variable k was used to emphasize that this expression only holds for even inputs to c. However, a new variable name is not actually needed

- The solution is then

$$y(t) = \sum_{k=0}^{\infty} \frac{c_0(-1)^k}{(2k-1)!} t^{2k}$$

where c_0 is a free parameter. We could have also used the name n instead of k

- If, during the phase where you write out the first few terms of the sum, you see that a certain term cancels itself out:

$$(c_1 t + 2c_2 t^2 + \cdots) - (c_0 + c_1 t + c_2 t^2 + \cdots) = \cdots$$

you lose information about that term. In the solution, the term will be outside the sum sign by itself, and the coefficient inside the term, in this case c_1, will be a free parameter:

$$y(t) = c_1 t + \sum_{k=1}^{\infty} c_{2k} t^{2k}, \quad c_1 \in \mathbb{R}$$

- When solving inhomogeneous equations, you must replace the forcing function $u(t)$ with its Fourier series form. You then use a similar method, but with the other statement of the uniqueness theorem, relating the coefficients of two sums. For example, in this equation,

$$\sum_{n=1}^{\infty} c_n(n-1)t^n = \sum_{k=0}^{\infty} \frac{(-1)^{k+1}}{(2k+1)!} t^{2k+2}$$

the right-hand side (RHS) only has even powers of t, which tells you that all odd coefficients on the LHS must be 0. For the even powers, you can write

$$\sum_{k=1}^{\infty} c_{2k}(2k-1)t^{2k} = \sum_{k=1}^{\infty} \frac{(-1)^k}{(2k+1)!} t^{2k}$$

The index shift method was used on the RHS sum

- If the RHS sum does not have a t^p term for some number p, you can use a similar 0-locating method as with homogeneous equations. If you find only one term c_i multiplied by t^p on the LHS, and the RHS does not have a t^p term, c_i must be 0 according to the uniqueness theorem

- Extra: Another transformation that may be useful is shifting the starting index down when you know it will not change the sum. For example:

$$\sum_{n=1}^{\infty} c_n n x^n = \sum_{n=0}^{\infty} c_n n x^n$$

since the extra term in the second sum is 0, and so does not affect the value of the sum. You can also shift the index up when you know it will not change the sum, which was already discussed in the example

- Extra: You may have to change the "displayed input" of a recursion relation. For example:
$$a_{n+3} = \frac{a_{n-1}}{n+3}$$
This is an expression for a_{n+3}. To change it into an expression for a_n, you can substitute all occurrences of n by an expression which, when substituted, will transform $n+3$ into n. In this case, that expression is $n-3$, since substituting $n-3$ for n in $n+3$ leads to $(n-3)+3=n$. Performing this substitution in the relation, we get
$$a_{(n-3)+3} = \frac{a_{(n-3)-1}}{(n-3)+3}$$
Simplifying, we then get the expression for a_n as planned
$$a_n = \frac{a_{n-4}}{n}$$
This is not to be confused with the change of index method for sums, which works differently

- The radius of convergence of a series found using the power series method can sometimes be determined using the recursion relation and the quotient test. The quotient test does not require any individual term of the series, only the relation between one term and the next, $\frac{s_{n+1}}{s_n}$, which is easily found if the recursion relation relates s_{n+1} to s_n

- You may also be given *initial conditions* that the power series solution needs to satisfy. These are values that the function, or some derivative, must have at $x=0$. The solution function and its derivatives at $x=0$ correspond to the c_n coefficients as follows:
$$y(0)=c_0,\ \ y'(0)=c_1,\ \ y''(0)=2c_2,\ \ y'''(0)=6c_3,\ldots\ \ y^{(n)}(0)=n!\,c_n$$
Therefore, initial conditions restrict free parameters, constraining them to a specific value. If an initial condition restricts a non-free coefficient, and says that it should be something other than what was found from the power series method, then the equation has no solutions that satisfy the given initial conditions

- If you have a **second-order** homogeneous differential equation with variable coefficients
$$\frac{d^2y}{dt^2} + a_1(t)\frac{dy}{dt} + a_2(t)y = 0,\ \ t \in I$$
and you have a single, particular solution to it, you can use that to find the general solutions to both the homogeneous equation, and most corresponding inhomogeneous equations. Assume that a function $y_1(t)$ is a solution to this homogeneous equation, and that $y_1(t) \neq 0$ for all $t \in I$. We define new functions as follows:
$$A_1(t) = \int a_1(t)\,dt, \quad \Omega(t) = e^{A_1(t)}$$
The integral is an indefinite integral, but we discard the arbitrary constant of integration. Then, the following hold:
  - The homogeneous equation has another solution
$$y_2(t) = y_1(t)\left(\int \frac{1}{y_1(t)^2\Omega(t)}\,dt\right)$$
and the functions y_1 and y_2 are linearly independent
  - The general solution to the inhomogeneous equation is
$$y(t) = c_1y_1(t) + c_2y_2(t)$$
where c_1 and c_2 are free parameters
  - The inhomogeneous equation with a function $u(t)$ on the right-hand side has the particular solution
$$y_{\text{par}}(t) = y_1(t)\left(\int \frac{1}{y_1(t)^2\Omega(t)}\left(\int y_1(t)\Omega(t)u(t)\,dt\right)dt\right)$$

- The general solution to the inhomogeneous equation is
$$y(t) = y_{\text{par}}(t) + c_1 y_1(t) + c_2 y_2(t)$$
where c_1 and c_2 are again free parameters

(Theorem 1.30)

Fourier transforms

- The vector space L^1 contains all the functions that have a finite integral from negative infinity to infinity:
$$L^1 = \left\{ f \middle| \int_{-\infty}^{\infty} |f(t)|\, dt < \infty \right\}$$
- The *Fourier transform* of a function in L^1-space is defined as follows:
$$\hat{f}(\omega) = \int_{-\infty}^{\infty} f(t) e^{-i2\pi\omega t}\, dt$$
  - Another notation that is used is
$$\hat{f} = \mathcal{F}(f)$$
- The *inverse Fourier transform* converts back from the Fourier function to the regular function:
$$f(t) = \int_{-\infty}^{\infty} \hat{f}(\omega) e^{i2\pi\omega t}\, d\omega$$
  - The other notation is
$$f = \mathcal{F}^{-1}\left(\hat{f}\right)$$
- The Fourier transform and its inverse are linear maps from L^1 space to L^1 space. This means that a Fourier transform of a linear combination of functions is equal to the linear combination of the Fourier-transformed functions:
$$\widehat{\alpha f(t) + \beta g(t)} = \alpha \hat{f}(\omega) + \beta \hat{g}(\omega)$$
- Fourier-transformed functions are generally complex, so you may need to take the real part in order to plot them
- The Fourier transform has the following properties:
  i. If f is even, then
$$\hat{f}(\omega) = 2 \int_0^{\infty} f(t) \cos(2\pi\omega t)\, dt$$
  and $\hat{f}(\omega)$ is real-valued
  ii. If f is odd, then
$$\hat{f}(\omega) = -2i \int_0^{\infty} f(t) \cos(2\pi\omega t)\, dt$$
  and $\hat{f}(\omega)$ is purely imaginary
  iii. If $f(t) = g(t - a)$, then
$$\hat{f}(\omega) = \hat{g}(\omega) e^{-2\pi i a\omega}$$
  iv. If $f(t) = g(t) e^{2\pi i\theta t}$, then
$$\hat{f}(\omega) = \hat{g}(\omega - \theta)$$
  v. If f is differentiable, then
$$\widehat{f'}(\omega) = 2\pi i\omega \hat{f}(\omega)$$

(Theorem 6.29)

- Fourier transforms are usually used in signal analysis, but property v from the theorem above allows them to be used to find particular solutions to *n*th order differential equations. Take the Fourier transform of both sides of the equation, transpose for $\hat{y}(\omega)$, and then use the inverse transform. For example, consider the equation

$$\frac{d^3y}{dt^3} + 4\frac{dy}{dt} - y = f$$

We want to take the Fourier transform of both sides of this equation. The right-hand side is a single function, and the left-hand side is a linear combination of functions. Because Fourier transforms are linear maps, we can take the transform on the left-hand side just by transforming each function individually:

$$\widehat{\frac{d^3y}{dt^3}} + 4\widehat{\frac{dy}{dt}} - \hat{y} = \hat{f}$$

Then, using property v, we get

$$(2\pi i\omega)^3\hat{y} + 4(2\pi i\omega)\hat{y} - \hat{y} = \hat{f}$$
$$\hat{y}((2\pi i\omega)^3 + (2\pi i\omega) - 1) = \hat{f}$$

We can then solve for $\hat{y}$:

$$\hat{y} = \frac{\hat{f}}{((2\pi i\omega)^3 + (2\pi i\omega) - 1)}$$

We would then find the Fourier transform of $\hat{f}$ and plug it into this. Finally, we would use the inverse Fourier transform on this expression for $\hat{y}$ to find y:

$$y(t) = \int_{-\infty}^{\infty} \hat{y}(\omega)e^{i2\pi\omega t}\, d\omega$$