Mathematics 1b Notes

Emil A. Overbeck (s246119)

6th May, 2025

Contents

1 Functions of Several Variables

Quadratic form	$q(x) = x^{\mathrm{T}}Ax + x^{\mathrm{T}}b + c$ where $x \in \mathbb{R}^n$	16
Vector function	$f\colon \operatorname{dom}(f) \to R^k$ where $\operatorname{dom}(f) \subseteq \mathbb{R}^n$	18
Parametrisations	• Helix: $r(t) = [\cos(t), \sin(t), t]^{\mathrm{T}}$ • Sphere $r(u,v) = [\cos(u)\cos(v), \sin(u)\cos(v), \sin(v)]^{\mathrm{T}}$	19
Function graph	$\{\, (x, f(x)) \in \mathbb{R}^{n+k} \mid x \in \operatorname{dom}(f) \,\}$	21
Level set	$\{\, x \in \operatorname{dom}(f) \mid f(x) = c \,\}$	23
Shape equations	• Sphere $(x-c_1)^2 + (y-c_2)^2 + (z-c_3)^2 = r^2$ • Circle: $(x-c_1)^2 + (y-c_2)^2 = r^2$ • Ellipsoid: $\dfrac{(x-c_1)^2}{a^2} + \dfrac{(y-c_2)^2}{b^2} = 1$ • Hyperbola: $\dfrac{(x-c_1)^2}{a^2} - \dfrac{(y-c_2)^2}{b^2} = 1$	23

2 Inner Product Spaces and the Spectral Theorem

Norm	$\lVert\cdot\rVert\colon V \to \mathbb{R}$ for normed space V such that: • Non-negativity: $\lVert x\rVert \in \mathbb{R}_{\geq 0}$ • Non-degeneracy: $\lVert x\rVert = 0 \iff x = 0$ • Scaling: $\lVert cx\rVert = \lvert c\rvert\lVert x\rVert$ • Triangle inequality: $\lVert x+y\rVert \leq \lVert x\rVert + \lVert y\rVert$	32
Inner product	$\langle\,\cdot\,,\,\cdot\,\rangle\colon V \times V \to \mathbb{R}$ for inner product space V such that: • Non-negativity: $\langle x, x\rangle \geq 0$ • Non-degeneracy: $\langle x, x\rangle = 0 \iff x = 0$ • Conjugate symmetry: $\langle x, y\rangle = \overline{\langle y, x\rangle}$ • Linearity: $\langle cx + dy, z\rangle = c\langle x, z\rangle + d\langle y, z\rangle$	33
Derived properties	• Conjugate linearity: $\langle x, cy + dz\rangle = \overline{c}\langle x, y\rangle + \overline{d}\langle x, z\rangle$ • Zero product: $\langle x, 0\rangle = \langle 0, y\rangle = 0$ • Derived norm: $\lVert x\rVert = \sqrt{\langle x, x\rangle}$ is a valid norm	34
Matrix properties	• $Ax \cdot y = x \cdot A^{*}y$ and $x \cdot Ax = A^{*}y \cdot y$ • $Ax \cdot y = x \cdot A^{\mathrm{T}}y$ and $x \cdot Ax = A^{\mathrm{T}}y \cdot y$	35
Polynomial product	$\langle p, q\rangle_{L^2} = \displaystyle\int_a^b p(x)\overline{q(x)}\,\mathrm{d}x$ where $p, q \in P_b([a,b])$	36
Unit vector	v such that $\lVert v\rVert = 1$	37
Unit vector constr.	$y/\lVert y\rVert$ is a unit vector pointing in the same direction as y	37
Orthogonality	x and y are orthogonal if $\langle x, y\rangle = 0$	37
Orthonormality	x and y are orthonormal if they are orthogonal unit vectors	37
Orthogonal compl.	$S^{\perp} = \{\, x \in V : \langle x, s\rangle = 0 \text{ for all } s \in S \,\}$ where $S \subseteq V$; if $S = \varnothing$, then $S^{\perp} = V$	37
Zero vector ortho.	If $\langle x, y\rangle = 0$ for all x, then $y = 0$	40

Pythagoras	Let inner product $\langle \cdot\,, \cdot \rangle$ and derived norm $\| \cdot \|$; if $\langle x, y\rangle = 0$, then $\|x+y\|^2 = \|x\|^2 + \|y\|^2$	40
Cauchy-Schwarz ineq.	Let inner product $\langle \cdot\,, \cdot \rangle$ and derived norm $\| \cdot \|$; $\lvert\langle x, y\rangle\rvert \leq \|x\|\|y\|$	41
Triangle inequality	$\|x+y\| = \|x\| + \|y\|$ iff x and y point in the same direction; $\|x+y\| \leq \|x\| + \|y\|$ otherwise	43
Open ball	$B(x_0, r) = \{\, x \in V \mid \|x - x_0\| < r \,\}$; in $\mathbb{R}^2$, a circle around x_0 of radius r	43
Open set	If for each $x_0 \in U$, there exists $\epsilon > 0$ such that $B(x_0, \epsilon) \subseteq U$, then U is open (boundary not in the set)	44
Closed set	U in V is closed if $V \setminus U$ is open	45
Set boundary	Let $M \subseteq V$; ∂M is the set of points $x_0 \in V$ for which each $B(x_0, \epsilon)$ contains points from M and $V \setminus M$	45
Set closure	$\overline{M} = M \cup \partial M$	46
Set interior	$M^\circ = M \setminus \partial M$	46
Bounded set	U is bounded if there is r such that $\|x\| \leq r$ for all $x \in U$	47
Orthogonal projection	$\operatorname{proj}_Y(x) = \dfrac{\langle x, y\rangle}{\langle y, y\rangle} y = \langle x, u\rangle u$ where $y \in Y$ and $u = \dfrac{y}{\|y\|}$	47
Orthonormal basis	A list of vectors which is orthonormal and a basis	50
Orthonormal indep.	Orthonormal vectors are linearly independent	51
Gram-Schmidt process	Let $v_1, \ldots, v_\ell$ be linearly independent vectors; to make orthonormal vectors $u_1, \ldots, u_\ell$ spanning the same subspace: 1. Set $w_1 := v_1$ and $u_1 := w_1 / \|w_1\|$ 2. For $k = 2, \ldots, \ell$, $w_k := v_k - \sum_{j=1}^{k-1} \operatorname{proj}_{u_j}(v_k)$ 3. $u_k := w_k / \|w_k\|$	52
Unitary matrix	A square matrix such that $U^*U = UU^* = I$, or $U^{-1} = U^*$	56
Real orthogonal matrix	A real unitary matrix (such that $Q^\mathrm{T}Q = QQ^\mathrm{T} = I$)	56
Unitary properties	The following are equivalent: • $U \in \mathbb{C}^{n\times n}$ is unitary ($U^{-1} = U^* = \overline{U}^\mathrm{T}$) • The rows & columns of U are an orthonormal basis of $\mathbb{C}^n$ • U^* is unitary • $Ux \cdot Uy = x \cdot y$ and $\|Ux\| = \|x\|$	57
Real orthogonal prop.	The following are equivalent: • $Q \in \mathbb{R}^{n\times n}$ is real orthogonal ($Q^{-1} = Q^* = Q^\mathrm{T}$) • The columns of Q are an orthonormal basis of $\mathbb{R}^n$	59
Diagonalisable matrix	A matrix A is: • Diagonalisable if S so $S^{-1}AS$ is diagonal • Unitarily diagonalisable if unitary U so U^*AU is diagonal • Orthog. diagonalisable if real orthog. Q so $Q^\mathrm{T}AQ$ is diag.	60
Hermitian eigenvalues	If A is Hermitian ($A = A^*$), all of its eigenvalues are real	63

Symmetric eigenvector	If A is real & symmetric ($A = A^{\mathrm{T}}$), it has a real eigenvector	63
Spectral theorem in $\mathbb{R}$	The following are equivalent: • $A \in \mathbb{R}^{n\times n}$ is a symmetric matrix • A is real orthogonally diagonalisable • A's eigenvectors are an orthonormal basis of $\mathbb{R}^n$	64
Spectral decomp. in $\mathbb{R}$	If $A \in \mathbb{R}^{n\times n}$ is symmetric, then $A = Q\Lambda Q^{\mathrm{T}}$ where Λ is a diagonal matrix whose diagonal is the eigenvalues of A and Q is a real orthogonal matrix whose columns are eigenvectors of A	66
Quadratic symmetry	Let quadratic form $q(x) = x^{\mathrm{T}}Ax + x^{\mathrm{T}}b + c$; $B = (A + A^{\mathrm{T}})/2$ is symmetric and $q(x) = x^{\mathrm{T}}Bx + x^{\mathrm{T}}b + c$	67
Spectral theorem in $\mathbb{C}$	The following are equivalent: • $A \in \mathbb{C}^{n\times n}$ is normal ($A^*A = AA^*$) • A is unitary diagonalisable • The eigenvectors of A are an orthonormal basis of $\mathbb{C}^n$	69
Spectral decomp. in $\mathbb{C}$	If $A \in \mathbb{C}^{n\times n}$ is symmetric, then $A = U\Lambda U^*$ where Λ is a diagonal matrix whose diagonal is the eigenvalues of A and Q is a unitary matrix whose columns are eigenvectors of A	69
Definite matrix	A square matrix A is: • Positive definite if $A = A^*$ and $\langle Ax, x\rangle > 0$ for $x \neq \mathbf{0}$ • Positive semi-definite $A = A^*$ and $\langle Ax, x\rangle \geq 0$ for $x \neq \mathbf{0}$ • Negative definite if $-A$ is positive definite • Negative semi-definite if $-A$ is positive semi-definite	70
Definite properties	The following are equivalent: • A is positive definite (positive semi-definite) • The eigenvalues of A are positive (non-negative) • There is $c > 0$ ($c \geq 0$) such that $\langle Ax, x\rangle \geq c\langle x, x\rangle$ for all x	71

3 Continuity and Differentiability

Continuity	f is continuous at x_0 if $f(x) \to f(x_0)$ whenever $x \to x_0$; if $\lvert f(x) - f(x_0)\rvert \to 0$ whenever $\lvert x - x_0\rvert \to 0$; if $\forall\epsilon > 0, \exists\delta > 0, \forall x \in I\colon \lvert x - x_0\rvert < \delta \Rightarrow \lvert f(x) - f(x_0)\rvert < \epsilon$	75
Differentiability	$f'(x_0) = \lim_{h\to 0} \dfrac{f(x_0 + h) - f(x_0)}{h}$ if the limit exists	78
Curve in $\mathbb{R}^k$	$\mathrm{im}(\boldsymbol{r}) = \{\, \boldsymbol{r}(t) \mid t \in I \,\}$ is a curve in $\mathbb{R}^k$	80
Tangent vector	$r'(t) = (r_1'(t), \ldots, r_k'(t))$ is the tangent vector at $r(t)$; if $r'(t) \neq 0$ for all $t \in I$, then r is regular	81
Continuity in $\mathbb{R}^k$	$f\colon A- > \mathbb{R}^k$ is continuous at x_0 if $\lVert f(x) - f(x_0)\rVert \to 0$ whenever $\lVert x - x_0\rVert \to 0$, or $f(x) \to f(x_0)$ whenever $x \to x_0$	82
Coordinate continuity	$f = (f_1, \ldots, f_k)$ is continuous at $\boldsymbol{x}_0$ iff all f_i are cont. at $\boldsymbol{x}_0$	83

Continuity operations	If $f, g\colon \mathbb{R}^n \to \mathbb{R}^k$, then the following are continuous: • $f+g$ and $f-g$ • $h \times f$ where h is a scalar function • $\lVert f \rVert$ and $\langle f, g \rangle$ • f/h where h is a non-zero scalar function • $g \circ j$ where $j\colon \mathbb{R}^m \to \mathbb{R}^n$ is continuous	84
Partial derivative	$\dfrac{\partial f}{\partial x_j}(x) = \lim_{h\to 0} \dfrac{f(\ldots, x_j + h, \ldots) - f(\ldots, x_j, \ldots)}{h}$ where $f\colon \mathbb{R}^n \to \mathbb{R}^n$	89
Gradient	$\nabla f(x) = \left(\dfrac{\partial f}{\partial x_1}(x), \dfrac{\partial f}{\partial x_2}(x), \ldots, \dfrac{\partial f}{\partial x_n}(x)\right)$	89
Directional derivative	$\nabla_v f(x) = \lim_{h\to 0} \dfrac{f(x+hu) - f(x)}{h} = u \cdot \nabla f(x)$ where $u = v/\lVert v \rVert$, or $u = v$ if v is a unit vector	91
Hessian matrix	$H_f(x) = \begin{bmatrix} \dfrac{\partial^2 f}{\partial x_1^2}(x) & \cdots & \dfrac{\partial^2 f}{\partial x_1\, \partial x_n}(x) \\ \cdots & \ddots & \cdots \\ \dfrac{\partial^2 f}{\partial x_n\, \partial x_1}(x) & \cdots & \dfrac{\partial^2 f}{\partial x_n^2}(x) \end{bmatrix}$ for $f\colon \mathbb{R}^n \to \mathbb{R}$	93
Jacobian matrix	$J_f(x) = \begin{bmatrix} \dfrac{\partial f_1}{\partial x_1}(x) & \cdots & \dfrac{\partial f_1}{\partial x_n}(x) \\ \cdots & \ddots & \cdots \\ \dfrac{\partial f_k}{\partial x_1}(x) & \cdots & \dfrac{\partial f_k}{\partial x_n}(x) \end{bmatrix}$ for $f\colon \mathbb{R}^n \to \mathbb{R}^k$	104
Partial differentiability	If $f\colon \mathbb{R}^n \to \mathbb{R}$ is differentiable at $\boldsymbol{x}_0$, then all partial derivatives exist at $\boldsymbol{x}_0$	98
Full partial diff.	If the partial derivatives for $f\colon \mathbb{R}^n \to \mathbb{R}$ exist at all points and are continuous, then f is differentiable for all $\mathbb{R}^n$	99
Continuous diff.	If f is differentiable at $\boldsymbol{x}_0$, then it is continuous at $\boldsymbol{x}_0$	99
Partial derivative order	$\dfrac{\partial^2 f}{\partial x_j\, \partial x_k}(x) = \dfrac{\partial^2 f}{\partial x_k\, \partial x_j}(x)$	100
Chain rule	$g(f(x))' = \langle f'(x), \nabla g(f(x)) \rangle$ where $g\colon \mathbb{R}^n \to \mathbb{R}$, $f\colon \mathbb{R} \to \mathbb{R}^n$	101
Linear differentiability	A linear map $L\colon \mathbb{R}^n \to \mathbb{R}^k, L(x) = Ax$ is differentiable for all $\mathbb{R}^n$ with Jacobian A	103
Coordinate diff.	$f = (f_1, \ldots, f_k)$ is differentiable at $\boldsymbol{x}_0$ if all f_i are differentiable at x_0	104
C^1 vector function	Differentiable function who's first derivative is continuous	104
Generalised chain rule	$J_{g\circ f}(x_0) = J_g(f(x_0)) J_f(x_0)$	105

4 Taylor Approximation

Taylor polynomial	$P_K(x) = \sum_{k=0}^{K} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k$ with expansion point x_0	110
Taylor's formula	There exists ξ between x and x_0 such that we have the remainder term $R_K(x) = f(x) - P_K(x) = \frac{f^{K+1}(\xi)}{(K+1)!}(x-x_0)^{K+1}$	114
Taylor's formula alt.	There exists $\varepsilon_K \colon \mathbb{R} \to \mathbb{R}$ where $\varepsilon_K(x-x_0) \to 0$ as $x - x_0 \to 0$ such that $R_K(x) = \varepsilon_K(x-x_0)(x-x_0)^K$	114
Taylor's theorem	If there is $C > 0$ such that $\lvert f^{(k)}(x) \leq C\rvert$, then for K, x, $\left\lvert f(x) - \sum_{k=0}^{K} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k \right\rvert \leq \frac{C}{(K+1)!}\lvert x-x_0\rvert^{K+1}$	115
Taylor polynom. in $\mathbb{R}^n$	$P_1(x) = f(x_0) + (x-x_0)\cdot\nabla f(x_0)$ where $x, x_0 \in \mathbb{R}^n$; $P_2(x) = f(x_0) + (x-x_0)\cdot\nabla f(x_0) + \frac{1}{2}(x-x_0)\cdot H_f(x_0)(x-x_0)$	118
Taylor polynom. in $\mathbb{R}^2$	$P_1(x,y) = f(x_0,y_0) + \frac{\partial f}{\partial x}(x_0,y_0)(x-x_0) + \frac{\partial f}{\partial y}(x_0,y_0)(y-y_0)$	118
Taylor formula in $\mathbb{R}^2$	$R_2(x) = f(x) - P_2(x) = \varepsilon(x-x_0)\lVert x-x_0\rVert^2$ for some $\varepsilon \colon \mathbb{R}^n \to \mathbb{R}$ where $\varepsilon(x-x_0) \to 0$ whenever $x \to x_0$	123

5 Local and Global Extrema of Functions

Continuous image set	Let continuous $f \colon [a,b] \to \mathbb{R}$ on bounded and closed $[a,b]$; $\operatorname{im}(f)$ is bounded and closed	128
Continuous extremum	Let continuous $f \colon [a,b] \to \mathbb{R}$; if f attains extremum at x_0, then one of the following is true: • $x_0 = a$ • $x_0 = b$ • f is not differentiable at x_0 • f is differentiable at x_0 and $f'(x_0) = 0$	130
Second derivative test	We can use $f''(x)$ to find the extremum of $f(x)$	131
Connected set	$B \in \mathbb{R}^n$ is connected if for each $x_1, x_2 \in B$, there is a continuous $r \colon [0,1] \to B$ such that $(r(0), r(1)) = (x_1, x_2)$	133
Connected set function	Let $B \in \mathbb{R}^n$ and continuous $f \colon B \to \mathbb{R}$; f has a minimum and maximum and, if B is a connected set, then $\operatorname{im}(f) = f(B) = [m, M]$ for some $m, M \in \mathbb{R}$	133
Extremum properties	If $f \colon A \to \mathbb{R}$ attains a maximum at x_0, one of the following is true; • $x_0 \in A \cap \partial A$ (x_0 is a boundary point) • f is not differentiable at $x_0 \in A^\circ$ • f is differentiable at $x_0 \in A^\circ$ and $\nabla f(x_0) = 0$	133
Stationary point	$f \colon A \to \mathbb{R}$ has stationary point $x_0 \in A^\circ$ if $\nabla f(x_0) = \mathbf{0}$	134

Stationary extremum	An extremum where f is differentiable is a stationary point	137
Saddle point	A stationary point that isn't an extremum is a saddle point	138
Second partial deriv.	• If $H_f(x_0)$ is positive definite, x_0 is a strict local min. • If $H_f(x_0)$ is negative definite, x_0 is a strict local max. • If $H_f(x_0)$ has pos. & neg. eigenvalues, x_0 is a saddle point	138

6 Integration

Riemann integral prop.	Let $f\colon [a,b] \to \mathbb{R}$; • If f is Riemann integrable, $\mathrm{im}(f)$ is a bounded set in $\mathbb{R}$ • If f is continuous, then it is Riemann integrable	143
Riemann linearity	$\int_a^b (cf(x) + dg(x))\,\mathrm{d}x = c\int_a^b f(x)\,\mathrm{d}x + d\int_a^b g(x)\,\mathrm{d}x$	143
Riemann monotonicity	$\int_a^b f(x)\,\mathrm{d}x \leq \int_a^b g(x)\,\mathrm{d}x$ if $f(x) \leq g(x)$	143
Riemann triangle ineq.	$\left\lvert\int_a^b f(x)\,\mathrm{d}x\right\rvert \leq \int_a^b \lvert f(x)\rvert\,\mathrm{d}x$	143
Riemann insertion	$\int_a^b f(x)\,\mathrm{d}x = \int_a^{x_0} f(x)\,\mathrm{d}x + \int_{x_0}^b f(x)\,\mathrm{d}x$ where $x_0 \in \,]a,b[$	143
Fundamental theorem	$F(x) = \int_{x_0}^x f(y)\,\mathrm{d}y$ is an anti-derivative of f with x_0 fixed	145
Integration	$\int_a^b f(x)\,\mathrm{d}x = [F(x)]_a^b = F(b) - F(a)$	146
Integration by parts	$\int f(x)g(x)\,\mathrm{d}x = F(x)g(x) - \int F(x)g'(x)\,\mathrm{d}x$ where f is continuous and g is continuously differentiable	147
Integration by subst.	$\int f(g(x))g'(x)\,\mathrm{d}x = F(g(x))$ where f is continuous and g is continuously differentiable with its range in f's domain	147
Integration over $\mathbb{R}^2$	Let $Q = \{\, (x,y) \in \mathbb{R}^2 \mid a_1 \leq x \leq b_1 \wedge a_2 \leq y \leq b_2 \,\}$; $\int_Q f(x,y)\,\mathrm{d}(x,y) = \int_{a_1}^{b_1}\left(\int_{a_2}^{b_2} f(x,y)\,\mathrm{d}y\right)\mathrm{d}x$	151
Polar integration	$\int_B f(x,y)\,\mathrm{d}(x,y) = \int_\alpha^\beta \int_{\varphi_1(\theta)}^{\varphi_2(\theta)} f(r\cos(\theta), r\sin(\theta))r\,\mathrm{d}r\,\mathrm{d}\theta$	164
Integration over $\mathbb{R}^n$	$\int_Q f(\boldsymbol{x})\,\mathrm{d}(\boldsymbol{x}) = \int_{a_n}^{b_n} \cdots \left(\int_{a_1}^{b_1} f(\boldsymbol{x})\,\mathrm{d}x_1\right) \cdots \mathrm{d}x_n$	165
Change of var. over $\mathbb{R}^n$	$\int_{r(\Gamma)} f(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} = \int_\Gamma f(r(\boldsymbol{u}))\lvert\det(J_r(\boldsymbol{u}))\rvert\,\mathrm{d}\boldsymbol{u}$ if $\det \neq 0$	167
Volume	$\mathrm{vol}_n(B) = \int_B 1\,\mathrm{d}X = \lvert\det(J_r)\rvert$	167

7 Vector fields

Mass	$M = \int_B f(x)\,\mathrm{d}x$	171
Centre of mass	$x^{\mathrm{CM}} = \frac{1}{M}\int_B x f(x)\,\mathrm{d}x$	171
Regular parametr.	$u \mapsto \sqrt{\det(J^{\mathrm{T}}J)}$ where $J = \det\Big((J_r(u))^{\mathrm{T}} J_r(u)\Big) \neq 0$	175
Cross product in $\mathbb{R}^3$	$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \times \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} x_2y_3 - x_3y_2 \\ x_3y_1 - x_1y_3 \\ x_1y_2 - x_2y_1 \end{bmatrix}$	176
Special case Jacobian	• Curve in $\mathbb{R}^n$: $u \mapsto \Vert r'(u)\Vert$ • Surface in $\mathbb{R}^3$: $u \mapsto \Vert r'_{u_1}(u) \times r'_{u_2}(u)\Vert$ • Subset of $\mathbb{R}^3$: $\sqrt{\det(J_r(u)^{\mathrm{T}} J_r(u))} = \vert\det(J_r(u))\vert$	177
Integral over m-fold	$\int_{\mathcal{D}} f(x)\,\mathrm{d}S = \int_{\Gamma} f(r(u))\sqrt{\det(J_r(u)^{\mathrm{T}} J_r(u))}\,\mathrm{d}u$	179
Graph surface	$\mathcal{S} = \{\,(u, v, h(u, v)) \in \mathbb{R}^3 \mid (u, v) \in B\,\}$	184
Gradient field	$V(x) = \left(\frac{\partial f}{\partial x_1}(x), \ldots, \frac{\partial f}{\partial x_n}(x)\right)$	186
Gradient field symm.	$\frac{\partial V_i}{\partial x_j}(x) = \frac{\partial V_j}{\partial x_i}(x)$ ($J_V(x)$ is symmetric)	187
Star-shaped set	A set with a centre x such that the line between x & any point in S is contained in S; such sets are simply connected	188
Jacobian symmetry	Let $V\colon U \to \mathbb{R}^n$ where U is simply connected; If $J_V(x)$ is symmetric for all $x \in U$, V is a gradient field	189
Vector field line int.	$\int_{\mathcal{C}} V \cdot \mathrm{d}s = \int_a^b V(r(u)) \cdot r'(u)\,\mathrm{d}u$	179
Circulation theorem	V is a gradient field iff. for any closed & piecewise C^1 curve in $\mathcal{C}$ it holds that $\int_{\mathcal{C}} V \cdot \mathrm{d}s = 0$	195
Vector field surface int.	$\int_{\mathcal{F}} V \cdot \mathrm{d}s = \int_{\Gamma} \langle V(r(u_1, u_2)), n_{\mathcal{F}}(u_1, u_2)\rangle\,\mathrm{d}(u_1, u_2)$	197